HYPERSPACE TOPOLOGIES

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In this paper hyperspace topologies on metric spaces are studied. We investigate the Hausdorff topology and the Wijsman topology. Necessary and sufficient conditions are given for when a particular pseudo-metric is a metric in the Wijsman topology. The metric properties of the two topologies are compared and contrasted to show which also hold in the respective topologies. We then look at the metric space $\mathbf{R}^n$, and build two residual sets. One residual set is the collection of uncountable, closed subsets of $\mathbf{R}^n$ and the other residual set is the collection of closed subsets of $\mathbf{R}^n$ having Lebesgue measure zero. We conclude with the intersection of these two sets being a residual set representing the collection of uncountable, closed subsets of $\mathbf{R}^n$ having Lebesgue measure zero.

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CHAPTER 1

INTRODUCTION

Let (X, d) be a metric space. We denote the collection of nonempty, closed subsets of X by $Cl(X)$. We will investigate topologies on $Cl(X)$ generated by the metric d. Such topologies are called hyperspace topologies. Of particular interest is the case where the metric space is $\mathbf{R}^n$ with the usual metric. We obtain all the necessary basic results about metric spaces in Chapter 2. This includes a brief discussion of the Baire Category Theorem.

In Chapter 3, we will first define the Hausdorff metric on $Cl(X)$, and study various properties of this hyperspace topology. We then define the Wijsman Topology on $Cl(X)$. It is shown that if (X, d) is a separable metric, then $(Cl(X), \tau_W)$ is metrizable. The properties of the Wijsman hyperspace topology are investigated and compared with those of the Hausdorff metric topology.

In Chapter 4, we focus our attention on the separable space $\mathbf{R}^n$ with the ρ metric. By using the Baire Category Theorem, it is shown that a "typical" closed subset of $\mathbf{R}^n$ is uncountable and has Lebesgue measure zero.

CHAPTER 2

METRIC SPACES

In order to discuss metric spaces, we first define the appropriate terms.

Definition 2.1 *Let X be a nonempty set. A real-valued function $d : X \times X \to [0, \infty)$ is a $pseudo-metric$ if*

1. $d(x, x) = 0$ *for all $x \in X$.*

2. $d(x, y) = d(y, x)$ *for all $x, y \in X$.*

3. $d(x, y) \leq d(x, z) + d(z, y)$ *for all $x, y, z \in X$.*

If, in addition, we have $d(x, y) = 0$ if and only if $x = y$, then d is called a metric and (X, d) is a metric space.

Lemma 2.2 *If d is a pseudo-metric, then $\dfrac{d}{1+d}$ is a pseudo-metric.*

Proof. Assume d is a pseudo-metric. Since $d \geq 0$, then $\dfrac{d}{1+d} \geq 0$. Let $x \in X$ be arbitrary. Note that

$$\frac{d(x,x)}{1+d(x,x)} = \frac{0}{1+0} = 0$$

Since x was arbitrary, this is true for all $x \in X$. Let $x, y \in X$ be arbitrary. Then

$$\frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)}.$$

Since x, y were arbitrary, this is true for all $x, y \in X$.

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \quad \text{for all } x,y,z \in X \\ &\leq d(x,y) + d(y,z) + 2d(x,y)d(y,z) + d(x,y)d(y,z)d(x,z). \end{aligned}$$

Thus

$$\begin{aligned} d(x,z) &+ d(x,y)d(x,z) + d(y,z)d(x,z) + d(x,y)d(y,z)d(x,z) \\ &\leq d(x,y) + 2d(x,y)d(y,z) + d(x,y)d(x,z) \\ &+ 2d(x,y)d(y,z)d(x,z) + d(y,z) + d(y,z)d(x,z). \end{aligned}$$

By factoring both sides we obtain

$$\begin{aligned} d(x,z)[1 &+ d(x,y) + d(y,z) + d(x,y)d(y,z)] \\ &\leq d(x,y)[1 + d(y,z) + d(x,z) + d(y,z)d(x,z)] \\ &+ d(y,z)[1 + d(x,y) + d(x,z) + d(x,y)d(x,z)]. \end{aligned}$$

Hence

$$\begin{aligned} d(x,z)[1 &+ d(x,y)][1+d(y,z)] \\ &\leq d(x,y)[1+d(x,z)][1+d(y,z)] \\ &+ d(y,z)[1+d(x,y)][1+d(x,z)]. \end{aligned}$$

Thus

$$\frac{d(x,z}{1+d(x,z)} \leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \text{ for all } x,y,z \in X.$$

Hence $\dfrac{d}{1+d}$ is a pseudo-metric. □

Lemma 2.3 *If* $\{d_n : n \in \mathbf{N}\}$ *is a uniformly bounded family of pseudo-metrics, then* $\sum_{n=1}^{\infty} \frac{1}{2^n} d_n$ *is a pseudo-metric.*

Proof. Assume the hypothesis. Since $d_n \geq 0$, $\sum_{n=1}^{\infty} \frac{1}{2^n} d_n \geq 0$. Let $x \in X$ be arbitrary. We have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x,x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot (0) = 0.$$

Since x was arbitrary, this is true for all $x \in X$. Let $x, y \in X$ be arbitrary. Thus

$$\sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(y,x).$$

Since x, y were arbitrary, this is true for all $x, y \in X$. Let $x, y, z \in X$ be arbitrary. Since d_n is a pseudo-metric we know $d_n(x,z) \leq d_n(x,y) + d_n(y,z)$ for every $n \in \mathbf{N}$.

Furthermore,

$$\frac{1}{2^n}d_n(x,z) \;\leq\; \frac{1}{2^n}d_n(x,y)+\frac{1}{2^n}d(y,z).$$

Since this is true for every $n \in \mathbf{N}$,

$$\sum_{n=1}^{\infty}\frac{1}{2^n}d_n(x,z) \;\leq\; \sum_{n=1}^{\infty}\frac{1}{2^n}d_n(x,y)+\sum_{n=1}^{\infty}\frac{1}{2^n}d_n(y,z).$$

Therefore $\sum_{n=1}^{\infty}\frac{1}{2^n}d_n$ is a pseudo-metric. □

A real number x is called a *point of closure* of a set X if for every $\delta > 0$ there is a y in X such that $d(x,y) < \delta$. We denote the set of points of closure of X by $\overline{X}$, and say X is closed if $X = \overline{X}$. A subset A of X is said to be *dense* if $\overline{A} = X$. We will denote the open ball (or more generally the open set) centered at x in the d metric by $B_d(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$ for $\epsilon > 0$. A set O of real numbers is called *open* in a metric space (X,d) if for each $x \in O$ there is a $\delta > 0$ such that $B_d(x,\delta) \subseteq O$. We say that a collection U of open sets in a metric space is an *open covering* for a set X if X is contained in the union of the sets in U. A G_δ *set* in a space X is a set A that equals a countable intersection of open sets of X. A sequence (x_n) from a metric space is called a *Cauchy sequence*, if given $\epsilon > 0$, there is an N such that for all n and m larger that N we have $d(x_n,x_m) < \epsilon$. A sequence $(x_n) \in X$ is called *convergent* if there exists $x \in X$ such that for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $d(x_n,x) < \epsilon$ for $n \geq N$. A metric space (X,d) is *complete* if every Cauchy sequence in X converges to an element in X. A metric space (X,d) is said to be *totally bounded* if for each $\epsilon > 0$ there exists a finite collection of points $\{x_1, \ldots, x_n\}$ such that for every $x \in X$ there exists i where $1 \leq i \leq n$ such that $d(x, x_i) < \epsilon$. A metric space X is *compact* if every open covering U of X has a finite subcovering.

Theorem 2.4 *A metric space (X, d) is compact if and only if every sequence in X has a convergent subsequence.*

Proof. ($\subseteq$) Suppose there exists a sequence (x_n) in X which has no convergent subsequence. We claim that for every $x \in X$ there exists an $\epsilon > 0$ such that

$$\{n \in \mathbf{N} : x_n \in B_d(x, \epsilon)\}$$

is finite. For if some $x \in X$ has the property such that every open ball $B_d(x, \epsilon)$ contains x_n for infinitely many $n \in \mathbf{N}$, it would readily follow that some subsequence of (x_n) converges to x. Hence for each $x \in X$ there exists $\epsilon_x > 0$ such that $\{n \in \mathbf{N} : x_n \in B_d(x, \epsilon_x)\}$ is finite. Now $U = \{B_d(x, \epsilon_x) : x \in X\}$ is an open cover of X having no finite subcover. Therefore (X, d) is not compact.

($\supseteq$) Now suppose every sequence (x_n) in X has a convergent subsequence. Let $\epsilon > 0$. We claim that there exist $x_1, \ldots, x_n$ in X such that $X = \bigcup_{i=1}^{n} B_d(x_i, \epsilon)$. To see this, let $x_1 \in X$ and consider $B_d(x_1, \epsilon)$. If this equals X we are done. If not there exists $x_2 \in X \setminus B_d(x_1, \epsilon)$ and thus $d(x_2, x_1) > \epsilon$. If $X = B_d(x_1, \epsilon) \cup B_d(x_2, \epsilon)$ then we are done. Otherwise there exists $x_3 \in X$ such that $d(x_3, x_1) \geq \epsilon$ and $d(x_3, x_2) \geq \epsilon$. This process must stop after a finite number of steps. For otherwise we obtain a sequence (x_n) such that $d(x_n, x_m) \geq \epsilon$ if $n \neq m$. Such a sequence cannot have a convergent subsequence. Hence we have the claim.

We will now show that if $\mathcal{U}$ is an open cover of X then there exists $\epsilon > 0$ such that if $x \in X$, $B_d(x, \epsilon) \subseteq U$ for some $U \in \mathcal{U}$. Suppose this conclusion is false. Then for every $\epsilon > 0$ there exists $x \in X$ such that $B_d(x, \epsilon) \nsubseteq U$ for any $U \in \mathcal{U}$. Taking $\epsilon = \frac{1}{n}$ there exists an $x_n \in X$ such that $B_d(x_n, \frac{1}{n}) \nsubseteq U$ for any $U \in \mathcal{U}$. By our hypothesis, there exists an $x \in X$ such that $\lim_{k\to\infty} x_{n_k} = x$ for some subsequence of (x_n). Let $U \in \mathcal{U}$ be such that $x \in U$. Let $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$. Choose $k \in \mathbf{N}$ such that $d(x_{n_k}, x) < \frac{\epsilon}{2}$ and $\frac{1}{n_k} < \frac{\epsilon}{2}$. We will now show that $B_d(x_{n_k}, \frac{1}{n_k}) \subseteq U$, which would be a contradiction. Let $y \in B_d(x_{n_k}, \frac{1}{n_k})$. Then $d(x_{n_k}, y) < \frac{1}{n_k}$. Now

$$d(y, x) \leq d(y, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + \frac{\epsilon}{2} < \epsilon.$$

Thus $y \in U$.

Let $\mathcal{U}$ be an open cover of X. Thus there exists $\epsilon > 0$ such that if $x \in X$, $B_d(x, \epsilon) \subseteq U$ for some $U \in \mathcal{U}$. By the above claim, there exists $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} B_d(x_i, \epsilon)$. For each $1 \leq i \leq n$, choose $U_i \in \mathcal{U}$ such that $B_d(x_i, \epsilon) \subseteq U_i$. Thus $X = \bigcup_{i=1}^{n} U_i$. Hence $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$ and thus X is compact.□

Lemma 2.5 *A metric space X is compact if and only if X is complete and totally bounded.*

Proof. ($\Rightarrow$) Suppose X is compact. Let $\epsilon > 0$. $U = \{B_\epsilon(x) \mid x \in X\}$. U is an open cover of X. Since X is compact, there exists $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} B_\epsilon(x_i)$. Therefore X is totally bounded.

Let (x_n) be a Cauchy sequence in X. Since X is compact, there exists a subsequence (x_{n_j}) of (x_n) and $x \in X$ such that $(x_{n_j}) \to x$. We let $\epsilon > 0$. There exists $J \in \mathbf{N}$ such that if $j \geq J$ then $d(x_{n_j}, x) < \frac{\epsilon}{2}$. Also there exists $N \in \mathbf{N}$ such that if $n, m \geq N$ then $d(x_n, x_m) < \frac{\epsilon}{2}$ since (x_n) is Cauchy. Let $k = \max\{J, N\}$ so $n_k \geq k$. Let $m \geq k$; then

$$\begin{aligned} d(x_m, x) &\leq d(x_m, x_{n_k}) + d(x_{n_k}, x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence $(x_n) \to x$. Thus every Cauchy sequence converges and X is complete.

($\Leftarrow$) Now suppose X is complete and totally bounded. Let (x_n) be a sequence in X. Since X is totally bounded, we may obtain $y_1^1, y_2^1, \ldots, y_{m_1}^1 \in X$ such that $X = \bigcup_{i=1}^{m_1} B_{\frac{1}{2}}(y_i^1)$. Now there exists $i \in 1 \leq i \leq m_1$ such that $\left\{n \mid x_n \in B_{\frac{1}{2}}(y_i^1)\right\}$ is infinite. Thus there exists a subsequence (x_n^1) of (x_n) such that

$$x_n^1 \in B_{\frac{1}{2}}(y_i^1) \text{ and } d(x_n^1, x_m^1) < 1 \quad \text{for all } n, m \in \mathbf{N}.$$

Let $y_1^2, y_2^2, \ldots, y_{m_2}^2 \in X$ such that $X = \bigcup_{i=1}^{m_2} B_{\frac{1}{4}}(y_i^2)$. As before there exists a subsequence (x_n^2) of (x_n^1) such that $x_n^2 \in B_{\frac{1}{4}}(y_i^2)$ for some $1 \leq i \leq m_2$. Therefore

$$d(x_n^2, x_m^2) < \frac{1}{2} \quad \text{for all } m, n \in \mathbf{N}.$$

Continue this process inductively. At the end of the kth stage we have a subsequence (x_n^k) such that $d(x_n^k, x_m^k) < \frac{1}{k}$. Now choose a subsequence(x_n^{k+1}) of (x_n^k) such that $d(x_n^{k+1}, x_m^{k+1}) < \frac{1}{k+1}$. Consider $(x_n^n)_{n=1}^{\infty}$, a diagonal subsequence of (x_n). Let $\epsilon > 0$. Choose $N \in \mathbf{N}$ such that $\epsilon > \frac{1}{N}$. If $m, n \geq N$, then $\mathrm{d}(x_m^m, x_n^n) < \frac{1}{N} < \epsilon$. Since X is complete there exists $x \in X$ such that $\lim_{n\to\infty} x_n^n = x$. Therefore every sequence has a convergent subsequence in X. Hence X is compact. □

Lemma 2.6 *If (X, d) is a compact metric space, then (X, d) is separable.*

Proof. Suppose (X, d) is a compact metric space. Then X is totally bounded and for all $N \in \mathbf{N}$ and $x_1^N, \ldots, x_{i_N}^N \in X$ we fix $N \in \mathbf{N}$ and let $X = \bigcup_{j=1}^{i_N} B_{\frac{1}{N}}(x_j^N)$. Now we let $Y = \bigcup_{N=1}^{\infty} (x_j^N)$ such that $1 \leq j \leq i_N$. Clearly Y is a countable, dense subset of X. Therefore (X, d) is separable. □

The following lemma will be useful in the next chapter.

Lemma 2.7 *If (A_n) is a Cauchy sequence in metric space (X, d) and (A_n) has a convergent subsequence, then (A_n) converges.*

Proof. Assume (A_n) is a Cauchy sequence in (X, d) and let (A_{n_k}) be the convergent subsequence of (A_n). Let $\epsilon > 0$. Let $A = \lim_{k\to\infty}(A_{n_k})$. We know there exists $N_1 \in \mathbf{N}$ such that for $k \geq N_1$ we have $d(A_{n_k}, A) < \frac{\epsilon}{2}$ by the fact that (A_{n_k}) is a convergent subsequence. Also there exists $N_2 \in \mathbf{N}$ such that for $m, n \geq N_2$ we have $d(A_n, A_m) <$

$\frac{\epsilon}{2}$ since (A_n) is Cauchy. Let $N = \max\{N_1, N_2\}$ and let $k = N$. Now for all $m \geq N$ it is true that

$$\begin{aligned} d(A_m, A) &\leq d(A_m, A_{n_k}) + d(A_{n_k}, A) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore $(A_n) \to A$. □

In Chapter 4, we will be doing work with the Baire Category Theorem. The necessary definitions for that chapter are provided here. If X is a metric space and $A \subseteq X$, then A is said to be *nowhere dense in* X if $\overline{(X \setminus A)} = X$. Now a closed set is nowhere dense if and only if its complement is dense. If X is a metric space, then a subset A of X is said to be of *first category in* X if A is a countable union of nowhere dense sets, and A is said to be of *second category in* X if A is not of first category in X. A set A is said to be a *residual* subset of X if $X \setminus A$ is of first category. The following theorem is key to the proof of the Baire Category Theorem.

Theorem 2.8 *Let (U_n) be a sequence of dense open subsets in a complete metric space X. Then $\bigcap_{n=1}^{\infty} U_n$ is dense.*

Proof. Let $x \in X$ be arbitrary and $\epsilon > 0$. We must find $y \in \bigcap_{n=1}^{\infty} U_n$ such that $d(x, y) < \epsilon$. Since U_1 is dense in X, there exists $y_1 \in U_1 \cap B_\epsilon(x)$. This set is open so there exists $\epsilon_1 > 0$ such that $B_{\epsilon_1}(y_1) \subseteq U_1 \cap B_\epsilon(x)$. Let $\delta_1 = \min\left\{\frac{\epsilon_1}{2}, 1\right\}$. Since U_2 is dense in X, there exists $y_2 \in U_2 \cap B_{\delta_1}(y_1)$ which is an open set so there exists $\epsilon_2 > 0$

such that $B_{\epsilon_2}(y_2) \subseteq U_2 \cap B_{\delta_1}(y_1)$. Continue inductively and let $\delta_2 = \min\left\{\frac{\epsilon_2}{2}, \frac{1}{2}\right\}$. Choose $y_3 \in U_3 \cap B_{\delta_2}(y_2)$. Then there exists $\epsilon_3 > 0$ such that $B_{\epsilon_3}(y_3) \subseteq U_3 \cap B_{\delta_2}(y_2)$. Let $\delta_3 = \min\left\{\frac{\epsilon_3}{2}, \frac{1}{3}\right\}$. We continue this process. We then have a sequence (y_n) and open balls $\{B_{\epsilon_n}(y_n)\}$ and $\{B_{\delta_n}(y_n)\}$ such that

$$B_{\epsilon_{n+1}}(y_{n+1}) \subseteq U_{n+1} \cap B_{\delta_n}(y_n),$$

where $\delta_n = \min\left\{\frac{\epsilon_n}{2}, \frac{1}{n}\right\}$ for all $n \in \mathbf{N}$. Now $B_{\delta_{n+1}}(y_{n+1}) \subseteq B_{\epsilon_{n+1}}(y_{n+1})$. Therefore $B_{\delta_{n+1}}(y_{n+1}) \subseteq B_{\delta_n}(y_n)$. Hence these balls are nested for all $n \in \mathbf{N}$. Now if $m \geq n$, we have $y_m \in B_{\delta_n}(y_n)$ and since $\delta_n \leq \frac{1}{n}$ we have $d(y_m, y_n) < \frac{1}{n}$ for all $m \geq n$. Therefore (y_n) is a Cauchy sequence in X. Since X is complete by hypothesis, there exists $y \in X$ such that $\lim_{n\to\infty} y_n = y$. So

$$\begin{aligned} y \in \overline{B_{\delta_n}}(y_n) &\subseteq \overline{B_{\frac{\epsilon_n}{2}}}(y_n) \\ &\subseteq B_{\epsilon_n}(y_n) \subseteq U_n \text{ for all } n \in \mathbf{N}. \end{aligned}$$

Therefore $y \in U_n$ and $y \in \bigcap_{n=1}^{\infty} U_n$. Now $y \in B_{\epsilon_1}(y_1) \subseteq B_\epsilon(x)$. In particular we have $y \in B_\epsilon(x)$. Therefore $d(x, y) < \epsilon$ and $y \in \bigcap_{n=1}^{\infty} U_n$. Since $x \in X$ was arbitrary, this is true for all $x \in X$. Therefore $\bigcap_{n=1}^{\infty} U_n$ is dense in X. □

Theorem 2.9 *If X is a nonempty, complete metric space, then X is of second category in X.*

Proof. By way of contradiction suppose X is of first category in X. Then $X = \bigcup_{n=1}^{\infty} A_n$ where each A_n is nowhere dense. Hence each $\overline{A_n}$ is nowhere dense. We have

$$X = \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \overline{A_n} \subseteq X.$$

Therefore $\bigcap X \setminus \overline{A_n} = \emptyset$. Since $\overline{A_n}$ is nowhere dense, then $X \setminus \overline{A_n}$ is open and dense. But $\bigcap X \setminus \overline{A_n}$ is dense in X. This is a contradiction since $\emptyset$ cannot be dense. Therefore X is not of first category in X and X is of second category in X. □

Lemma 2.10 *Let X be a complete, countable metric space. Then X has an isolated point.*

Proof. We know $X = \{x_n : n \in \mathbf{N}\} = \bigcup_{n=1}^{\infty} \{x_n\}$ where each x_n is closed. As a consequence of the Baire Category Theorem, at least one set has to have a non-empty interior which implies that it is open. Hence there exists $m \in \mathbf{N}$ such that $\{x_m\}^{\circ} \neq \emptyset$. Therefore X has an isolated point. □

Theorem 2.11 *Let X be a metric space and let $A \subseteq X$ be of first category. Then $X \setminus A$ contains a dense G_δ set.*

Proof. Let $A \subseteq X$ be of first category. Then there exists (F_n) of nowhere dense sets such that $A = \bigcup_{n=1}^{\infty} F_n$. Hence $A \subseteq \bigcup_{n=1}^{\infty} \overline{F_n}$ where each $\overline{F_n}$ is a closed, nowhere dense set. Thus

$$\bigcap (X \setminus \overline{F_n}) \subseteq \bigcap (X \setminus F_n) = X \setminus A.$$

Therefore $X \setminus A$ contains a dense G_δ set. $\square$

Corollary 2.12 *$\bigcup_{n=1}^{\infty} F_n$ is a countable union of nowhere dense sets and is therefore of first category.*

Definition 2.13 *Let (f_n) be a sequence of real-valued functions defined on a set X, and let f be a real-valued function defined on X. We say that (f_n) converges to f uniformly on X if for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that if $n \geq N$ then $| f_n(x) - f(x) | < \epsilon$ for every $x \in X$.*

Definition 2.14 *A metric space (X, d) is separable if there exists a countable dense subset of X (i.e. there exists (x_n) such that $\overline{\{x_n \mid n \in \mathbf{N}\}} = X$).*

We conclude this chapter with the following definitions. A topological space (X, τ) is *metrizable* if there exists a metric d on X such that the topology induced by d is the same as τ. A topological space is said to be a *Polish space* if the topology is equivalent to a complete, separable metric.

CHAPTER 3

HYPERSPACE TOPOLOGIES

Let (X, d) be a metric space. The collection of nonempty, closed subsets of X will be denoted by $Cl(X)$. We will investigate two topologies on $Cl(X)$ determined by d. Such topologies are called hyperspace topologies. There are many such hyperspace topologies that have been studied extensively. (see....) In this chapter we investigate the properties of two of these hyperspace topologies: the Hausdorff metric topology, τ_H, and the Wijsman topology, τ_W.

If (X, d) is a metric space, then we may define another metric, d', on X by $d'(x, y) = \min\{d(x, y), 1\}$ for all $x, y \in X$. It is readily seen that d and d' are equivalent metrics. (Two metrics, d and d', on a set X are said to be equivalent if the corresponding metric topologies are the same.) This is equivalent to saying that a sequence (x_n) converges to x in (X, d) if and only if (x_n) converges to x in (X, d'). Thus, without loss of generality, we will assume throughout that (X, d) is a bounded metric space.

We begin with the definition of the Hausdorff metric on $Cl(X)$ where (X, d) is a bounded metric space.

Definition 3.1 *Let (X, d) be a bounded metric space. For all $A \subseteq X$, we have*

$U(A,\epsilon) = \bigcup_{a\in A} B_d(a,\epsilon)$. *If* $A, B \in Cl(X)$ *then define*

$$D(A,B) \;=\; inf\,\{\epsilon > 0 :\; A \subseteq U(B,\epsilon) \text{ and } B \subseteq U(A,\epsilon)\}$$

It will now be shown that $(Cl(X), D)$ forms a metric space called the Hausdorff metric space determined by (X, d).

Theorem 3.2 *Let* (X, d) *be a bounded metric space, and let* $Cl(X)$ *be the collection of all nonempty, closed subsets of* X. *Let* $D(A, B)$ *be defined as above. Then* D *is a metric on* $Cl(X)$.

Proof. We will show the following:

1. $D : Cl(X) \times Cl(X) \to [0, \infty)$.

2. $D(A, B) = 0$ if and only if $A = B$.

3. $D(A, B) = D(B, A)$ for all $A, B \in Cl(X)$.

4. $D(A, C) \leq D(A, B) + D(B, C)$ for all $A, B, C \in Cl(X)$.

For arbitrary $A, B \in Cl(X)$ we define

$$E_{AB} \;=\; \{\epsilon > 0 :\; A \subseteq U(B,\epsilon) \text{ and } B \subseteq U(A,\epsilon)\}.$$

First we show $E_{AB} \neq \emptyset$. Since A, B are nonempty and D is bounded, there exists $M > 0$ such that if $x, y \in A$ and $v, w \in B$ then $d(x, y) < M$ and $d(v, w) < M$. Choose

$x_0 \in A$ and $y_0 \in B$. Define $\epsilon_0 = M + d(x_0, y_0)$. Let $x \in A$ be arbitrary. Now

$$\begin{aligned} d(x, y_0) &\leq d(x, x_0) + d(x_0, y_0) \\ &< M + d(x_0, y_0) \\ &= \epsilon_0. \end{aligned}$$

So $x \in B_d(y_0, \epsilon_0) \subseteq U(B, \epsilon_0)$. Hence $A \subseteq U(B, \epsilon_0)$. Similarly $B \subseteq U(A, \epsilon_0)$. Therefore $\epsilon_0 \in E_{AB}$ and $E_{AB} \neq \emptyset$. We have that $D(A, B)$ is well defined and $D : Cl(X) \times Cl(X) \to [0, \infty)$. Now for every $\epsilon > 0$ we have that $A \subseteq U(A, \epsilon)$. Hence $D(A, A) \leq \epsilon$ for all $\epsilon > 0$. So $D(A, A) = 0$. Suppose $A, B \in Cl(X)$ and $D(A, B) = 0$. This gives $A \subseteq U(B, \epsilon)$ and $B \subseteq U(A, \epsilon)$ for all $\epsilon > 0$. Let $x \in A$ and $x \notin B$. Then $x \in X \setminus B$. Since B is closed, $X \setminus B$ is open. Now there exists $\epsilon > 0$ such that $B_d(x, \epsilon) \bigcap B = \emptyset$. So for every $y \in B$ we have $d(x, y) \geq \epsilon$, or $x \notin U(B, \epsilon)$. This is a contradiction.

Thus $x \in B$, and $A \subseteq B$. By a similar argument, $B \subseteq A$. Therefore if $D(A, B) = 0$ then $A = B$.

For (3) we have simply

$$\begin{aligned} D(A, B) &= \inf \{\epsilon > 0 : \; A \subseteq U(B, \epsilon) \text{ and } B \subseteq U(A, \epsilon)\} \\ &= \inf \{\epsilon > 0 : \; B \subseteq U(A, \epsilon) \text{ and } A \subseteq U(B, \epsilon)\} \\ &= D(B, A). \end{aligned}$$

For the triangle inequality, let $A, B, C \in Cl(X)$ be arbitrary. Choose $\epsilon_{AB} \in E_{AB}$ and

$\epsilon_{BC} \in E_{BC}$. Let $\epsilon = \epsilon_{AB} + \epsilon_{BC}$ and $x \in A$. Since $x \in U(B, \epsilon_{AB})$, there exists $b \in B$ such that $x \in B_d(b, \epsilon_{AB})$.

Now $d(x,b) < \epsilon_{AB}$, and $b \in U(C, \epsilon_{BC})$. Thus there exists $c \in C$ such that $b \in B_d(c, \epsilon_{BC})$ and $d(b,c) < \epsilon_{BC}$. Hence

$$\begin{aligned} d(x,c) &\leq d(x,b) + d(b,c) \\ &< \epsilon_{AB} + \epsilon_{BC} \\ &= \epsilon. \end{aligned}$$

Thus $x \in B_d(c, \epsilon)$ and $x \in U(C, \epsilon)$. Therefore $A \subseteq U(C, \epsilon)$.

Let $x \in C$. Now $x \in U(B, \epsilon_{BC})$. There exists $b \in B$ such that $x \in B_d(b, \epsilon_{BC})$ which implies $d(x,b) < \epsilon_{BC}$. But $b \in U(A, \epsilon_{AB})$ implies there exists $a \in A$ such that $b \in B_d(a, \epsilon_{AB})$. So $d(a,b) < \epsilon_{AB}$. Thus

$$\begin{aligned} d(x,a) &\leq d(x,b) + d(b,a) \\ &< \epsilon_{BC} + \epsilon_{AB} = \epsilon \end{aligned}$$

and $x \in B_d(a, \epsilon) \subseteq U(A, \epsilon)$. Hence $C \subseteq U(A, \epsilon)$. Now $D(A,C) \leq \epsilon = \epsilon_{AB} + \epsilon_{BC}$ for all $\epsilon_{BC} \in E_{BC}$. So $D(A,C) \leq \epsilon_{AB} + D(B,C)$ for all $\epsilon_{AB} \in E_{AB}$. Thus $D(A,C) \leq D(A,B) + D(B,C)$. □

A second definition of the Hausdorff metric exists as stated in the following:

$$d_H(A, B) = \sup_{x \in X} |\, d(x, A) - d(x, B)\, |$$

where $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. Before we proceed, we shall prove that the two definitions are equivalent.

Lemma 3.3 *$D(A, B) = d_H(A, B)$ for all $A, B \in Cl(X)$.*

Proof. Let $A, B \in Cl(X)$ be arbitrary. Let $x_0 \in X$ be arbitrary. Without loss of generality, we assume that

$$d(x_0, B) - d(x_0, A) \geq 0.$$

Let $\delta > 0$. Choose $a_0 \in A$ such that $d(x_0, A) > d(a_0, x_0) - \delta$. Let $\epsilon > 0$ be such that $A \subseteq \bigcup_{b \in B} B_d(b, \epsilon)$ and $B \subseteq \bigcup_{a \in A} B_d(a, \epsilon)$. Choose b_0 such that $d(a_0, b_0) < \epsilon$. Note that

$$\begin{aligned} d(x_0, b_0) &\leq d(x_0, a_0) + d(a_0, b_0) \\ &< d(x_0, a_0) + \epsilon. \end{aligned}$$

Therefore, $d(x_0, b_0) - d(x_0, a_0) < \epsilon$. Hence

$$\begin{aligned} |\, d(x_0, B) - d(x_0, A)\, | &= d(x_0, B) - d(x_0, A) \\ &< d(x_0, B) - d(x_0, a_0) + \delta \\ &\leq d(x_0, b_0) - d(x_0, a_0) + \delta \\ &< \epsilon + \delta. \end{aligned}$$

As this holds for all $\delta > 0$, we have

$$| \, d(x_0, B) - d(x_0, A) \, | \quad \leq \quad \epsilon.$$

Since $x_0 \in X$ was arbitrary, we have that

$$\begin{aligned} d_H(A, B) &= \sup_{x \in X} | \, d(x, B) - d(x, A) \, | \\ &\leq \epsilon \end{aligned}$$

which holds for all ϵ that satisfy our conditions. Therefore $d_H(A, B) \leq D(A, B)$.

Now let $\gamma > d_H(A, B)$. Thus, for all $x \in X$, $| \, d(x, A) - d(x, B) \, | < \gamma$. In particular, if $a \in A$ and $b \in B$ then

$$| \, d(a, B) \, | = | \, d(a, A) - d(a, B) \, | < \gamma \text{ and } \; | \, d(b, A) \, | = | \, d(b, A) - d(b, B) \, | < \gamma.$$

Thus if $a \in A$, there exists $b \in B$ such that $d(a, b) < \gamma$. Similarly, if $b \in B$, there exists $a \in A$ such that $d(a, b) < \gamma$. Hence $A \subseteq \bigcup_{b \in B} B_\gamma(b)$ and $B \subseteq \bigcup_{a \in A} B_\gamma(a)$. Therefore $\gamma > D(A, B)$. Since this holds true for all $\gamma > d_H(A, B)$, we have

$$D(A, B) \quad \leq \quad d_H(A, B).$$

The above holds for all $A, B \in Cl(X)$. Therefore, $d_H(A, B) = D(A, B)$ for all $A, B \in Cl(X)$. □

The next definition allows us to obtain another characterization of convergence of sequences in the Hausdorff metric.

Definition 3.4 *If A is a closed subset of X, define $f_A : X \to \mathbf{R}$ by the rule $f_A(x) = d(x, A)$. If $f : X \to \mathbf{R}$ is bounded, we let $\|f\|_{\sup} = \sup_{x \in X} | f(x) |$.*

Lemma 3.5 *A sequence (A_n) of closed subsets of X converges in d_H to A if and only if $(f_{A_n}) \to f_A$ uniformly on X.*

Proof. ($\Rightarrow$) Let (A_n) be a sequence of closed subsets of X, and $A \subseteq X$ with $(A_n) \to A$ in the Hausdorff metric. Let f_{A_n} and f_A be as in the above definition.

Let $\epsilon > 0$. There exists $N \in \mathbf{N}$ such that if $n \geq N$ then $d_H(A_n, A) < \epsilon$. Hence

$$\sup_{x \in X} | d(x, A_n) - d(x, A) | < \epsilon \text{ and}$$
$$\sup_{x \in X} | f_{A_n}(x) - f_A(x) | < \epsilon.$$

If this is true for the supremum, then it is true for all $x \in X$. Therefore

$$| f_{A_n}(x) - f_A(x) | \quad < \quad \epsilon \text{ for } n \geq N \in \mathbf{N}.$$

Since this is true for all $\epsilon > 0$, we have $(f_{A_n}) \to f_n$ uniformly.

($\Leftarrow$) Now assume $(f_{A_n}) \to f_A$ uniformly on X. Let $\epsilon > 0$. There exists $N \in \mathbf{N}$ such that $| f_A(x) - f_{A_n}(x) | < \epsilon$ for all $n \geq N$. Substituting we get $| d(x, A) - d(x, A_n) | < \epsilon$ for all $x \in X$. Hence

$$\sup_{x \in X} | d(x, A) - d(x, A_n) | \quad \leq \quad \epsilon$$

and $d_H(A, A_n) < \epsilon$. Therefore $(A_n) \to A$ in the Hausdorff metric. $\square$

It is interesting to note what metric space properties that (X, d) possesses are inherited by $(Cl(X), d_H)$. For instance we have the following.

Theorem 3.6 *If X is totally bounded, then $(Cl(X), d_H)$ is totally bounded.*

Proof. Let X be totally bounded. Let $\epsilon > 0$. Choose $\{x_1, \ldots, x_n\} \in X$ such that $X \subseteq \bigcup_{k=1}^{n} B_\epsilon(x_k)$. Let $\mathbf{F} = \{F : F \subseteq \{x_1, \ldots, x_n\}$ and $F \neq \emptyset\}$. Now $Cl(X) \subseteq \bigcup_{F \in \mathbf{F}} B_\epsilon(F)$.
Let $A \subseteq Cl(X)$ be arbitrary and let $\hat{F} = \{x_i : B_\epsilon(x_i) \cap A \neq \emptyset\}$. Hence $A \subseteq \bigcup_{x_i \in \hat{F}} B_\epsilon(x_i)$ and $\hat{F} \subseteq \bigcup_{x \in A} B_\epsilon(x)$. We now have $\mid d(x, A) - d(x, \hat{F}) \mid < \epsilon$. This must also be true for the supremum; hence $\sup_{x \in X} \mid d(x, A) - d(x, \hat{F}) \mid < \epsilon$.
Therefore $(Cl(X), d_H)$ is totally bounded. □

Theorem 3.7 *If (X, d) is complete, then $(Cl(X), d_H)$ is complete.*

Proof. Let (X, d) be a complete metric space. $(Cl(X), d_H)$ represents the Hausdorff metric on the nonempty, closed subsets of X. Let (A_n) be a Cauchy sequence in $Cl(X)$. Now by Theorem 2.10, it suffices to show that if (A_n) has a convergent subsequence, (A_n) converges. Let $n_1 \in \mathbf{N}$ such that if $m, n \geq n_1$, then $d_H(A_m, A_n) < \frac{1}{2}$ and let $n_2 > n_1 \in \mathbf{N}$ such that if $m, n \geq n_2$ then $d_H(A_m, A_n) < \frac{1}{4}$. Continuing this process, we get $n_{k+1} > n_k \in \mathbf{N}$ such that if $m, n \geq n_{k+1}$, then $d_H(A_m, A_n) < \frac{1}{2^{k+1}}$.

We need to show there exists A such that $(A_{n_k}) \to A$. Let $B_k = A_{n_k}$ for all k. We have defined $f_A(x) = d(x, A)$. We know $f_{B_k} \to f$ uniformly by Lemma 3.5. We now show that $\{x : f(x) = 0\} \neq \emptyset$. Let $x_1 \in B_1$. Then $| d(x_1, B_1) - d(x_1, B_2) | < \frac{1}{2}$. Since $d(x_1, B_1) = 0$ we have $| d(x_1, B_2) | < \frac{1}{2}$. Let $x_2 \in B_2$ such that $d(x_1, x_2) < \frac{1}{2}$, then $| d(x_2, B_2) - d(x_2, B_3) | < \frac{1}{4}$. This results in $| d(x_2, B_3) | < \frac{1}{4}$. Let $x_3 \in B_3$ such that $d(x_2, x_3) < \frac{1}{4}$, then suppose $x_1, \ldots, x_n$ have been chosen such that $x_i \in B_i$ for $1 \leq i \leq n$ and $d(x_i, x_{i+1}) < \frac{1}{2^i}$ for all $i = 1, \ldots, n-1$. Now $| d(x_n, B_n) - d(x_n, B_{n+1}) | < \frac{1}{2^n}$ where $d(x_n, B_n) = 0$. Therefore there exists $x_{n+1} \in B_{n+1}$ such that $d(x_n, x_{n+1}) < \frac{1}{2^n}$. Let $\epsilon > 0$. Choose $N \in \mathbf{N}$ such that $\frac{1}{2^{N-1}} < \epsilon$. Then if $m < n$ and $m \geq N$ then

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=m}^{n-1} d(x_i, x_{i+1}) \\ &< \sum_{i=m}^{n-1} \frac{1}{2^i} \\ &\leq \sum_{i=N}^{\infty} \frac{1}{2^i} \\ &= \frac{1}{2^{N-1}} \\ &< \epsilon. \end{aligned}$$

Hence (x_n) is a Cauchy sequence. So there exists $x \in X$ such that $(x_n) \to x$. Let $\epsilon > 0$. Choose $\delta > 0$ such that if $y \in B_\delta(x)$ then $| f(x) - f(y) | < \frac{\epsilon}{2}$. Let $N \in \mathbf{N}$ such that if $n \geq N$ then $x_n \in B_\delta(x)$ and $\| f - f_{B_n} \|_{\sup} < \frac{\epsilon}{2}$. Let $n \geq N$. Therefore

$$\begin{aligned} | f(x) | &= | f(x) - f(x_n) + f(x_n) | \\ &< | \frac{\epsilon}{2} + f(x_n) | \\ &= \frac{\epsilon}{2} + | f(x_n) - f_{B_n}(x_n) | \\ &< \epsilon. \end{aligned}$$

This is true for all $\epsilon > 0$, so $f(x) = 0$ and $\{x : f(x) = 0\} \neq \emptyset$. Now we let

$A = \{x \in X : f(x) = 0\}$. Thus $A \in Cl(X)$. We need to show that $f = f_A$. Let $y \in X$. Let $\epsilon > 0$. Choose $N_1 \in \mathbf{N}$ such that if $n \geq N_1$ then

$$\sup_{x \in X} \mid f_{B_n}(x) - f(x) \mid < \frac{\epsilon}{3}.$$

As this holds for the supremum, we have

$$\mid f_{B_n}(y) - f(y) \mid < \frac{\epsilon}{3}.$$

Choose $N_2 \in \mathbf{N}$ such that $\frac{1}{2^{N_2-1}} < \frac{\epsilon}{3}$. Let $N = \max\{N_1, N_2\}$. Let $x_N \in B_N$ such that $\mid d(y, B_N) - d(y, x_N) \mid < \frac{\epsilon}{3}$. Construct a sequence $(x_n)_{n=N}^{\infty}$ as before where $x_{N+1} \in B_{N+1}$ such that $d(x_N, x_N + 1) < \frac{1}{2^N}$ and $\mid d(x_N, B_N) - d(x_N, B_{N+1}) \mid < \frac{1}{2^N}$. So $x_N \to x \in A$. And

$$\begin{aligned} \mid d(y, x_N) - d(y, x) \mid &\leq \sum_{i=N}^{\infty} \mid d(y, x_i) - d(y, x_{i+1}) \mid \\ &< \frac{1}{2^{N-1}} \\ &\leq \frac{\epsilon}{3}. \end{aligned}$$

Now

$$\begin{aligned} \mid f(y) - f_A(y) \mid &= \mid f(y) - d(y, A) \mid \\ &\leq \mid f(y) - f_{B_n}(y) \mid + \mid f_{B_n}(y) - d(y, x_N) \mid + \mid d(y, x_N) - d(y, x) \mid \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \text{ for all } \epsilon > 0. \end{aligned}$$

Hence $f(y) = f_A(y)$ for all $y \in X$. So $f = f_A$. Hence $f_{B_k} = f_{A_{n_k}} \to f_A$ uniformly and so (A_{n_k}) converges in d_H to $A \in Cl(X)$. Since a subsequence of (A_n) converges to $A \in Cl(X)$, (A_n) converges to $A \in Cl(X)$. Therefore $(Cl(X), d_H)$ is complete. □

Corollary 3.8 *If X is compact, then $(Cl(X), d_H)$ is compact.*

Proof. Let X be compact. Then X is complete and totally bounded. By the previous two theorems, $(Cl(X), d_H)$ is complete and totally bounded. Therefore $(Cl(X), d_H)$ is compact. □

Not all metric spaces properties are inherited by $(Cl(X), d_H)$ as the following indicates.

Theorem 3.9 *If (X, d) is not totally bounded, then $(Cl(X), d_H)$ is not separable.*

Proof. Assume that X is not totally bounded. Then there exists $\epsilon > 0$ and $(x_n) \in X$ such that $d(x_n, x_m) > \epsilon$ for all $m \neq n$. Let $A = \{F \subseteq \{x_n \mid n \in \mathbf{N}\};\ \ F \neq \emptyset\}$. The set A is clearly uncountable. If $F, G \in A$ and $F \neq G$, then $d_H(F, G) \geq \frac{\epsilon}{2}$. Therefore, d_H is not a separable metric on X. □

We will now define the Wijsman Topology and develop a series of pseudo-metrics on the nonempty, closed subsets of X.

Definition 3.10 *Define the Wijsman Topology, τ_W, on $Cl(X)$ by first defining a basis for τ_W. For $\epsilon > 0$, $x \in X$ and $A \in Cl(X)$ set*

$$U_{(A,x,\epsilon)} = \{B \in Cl(X) : \mid d(x, A) - d(x, B) \mid < \epsilon\}.$$

Basis elements of τ_W *are of the form* $\bigcap_{i=1}^{n} U_{(A,x_i,\epsilon_0)}$.

Theorem 3.11 *Let* (X, d) *be a bounded metric space and* $x \in X$*. Define* $\tilde{\rho}$ *on* $Cl(X)$ *by* $\tilde{\rho}(A, B) = \mid d(x, A) - d(x, B) \mid$ *where* $A, B \in Cl(X)$*. Then* $\tilde{\rho}$ *is a pseudo-metric.*

Proof. Let $A \in Cl(X)$ and $x \in X$ be arbitrary. Then

$$\tilde{\rho}(A, A) = \mid d(x, A) - d(x, A) \mid = 0.$$

Since A was arbitrary, this is true for all $A \in Cl(X)$.

Let $A, B \in Cl(X)$ be arbitrary. Let $x \in X$. Then

$$\begin{aligned} \tilde{\rho}(A, B) &= \mid d(x, A) - d(x, B) \mid \\ &= \mid d(x, B) - d(x, A) \mid \\ &= \tilde{\rho}(B, A) \text{ for all } A, B \in Cl(X). \end{aligned}$$

Since A, B were arbitrary, this is true for all $A, B \in Cl(X)$.

For the triangle inequality, let $A, B, C \in Cl(X)$ be arbitrary. Let $x \in X$. Now

$$\begin{aligned} \tilde{\rho}(A, B) &= \mid d(x, A) - d(x, B) \mid \\ &= \mid d(x, A) - d(x, C) + d(x, C) - d(x, B) \mid \\ &\leq \mid d(x, A) - d(x, C) \mid + \mid d(x, C) - d(x, B) \mid \\ &= \tilde{\rho}(A, C) + \tilde{\rho}(C, B) \text{ for all } A, B, C \in Cl(X). \end{aligned}$$

Therefore $\tilde{\rho}$ is a pseudo-metric on $Cl(X)$. □

Theorem 3.12 *$(A_n) \to A$ in $(Cl(X), \tau_W)$ if and only if $f_{A_n} \to f_A$ pointwise.*

Proof. ($\Rightarrow$) A basic open set in τ_W is of the form $\bigcap_{i=1}^{n} U_{(A,x_i,\epsilon)}$. Assume $(A_n) \to A$ in $(Cl(X), \tau_W)$. Let $x \in X$ and $\epsilon > 0$. There exists $N \in \mathbf{N}$ such that if $n \geq N$ then $A_n \in U_{(A,x,\epsilon)}$. Therefore if $n \geq N$, $\mid d(x, A_n) - d(x, A) \mid < \epsilon$.

($\Leftarrow$) Assume $f_{A_n} \to f_A$ pointwise. We know $(d(x, A_n) - d(x, A)) \to 0$ for each $x \in X$. Therefore there exists $N \in \mathbf{N}$ such that if $n \geq N$ then $\mid d(x_i, A_n) - d(x_i, A) \mid < \epsilon$ for all $i = 1, \ldots, n$. Thus $A_n \in U$ for all $n \geq N$. □

Theorem 3.13 *If (X, d) is a separable metric space, then $(Cl(X), \tau_W)$ is metrizable.*

Proof. Let X be separable, and let $D = \{x_n : n \in \mathbf{N}\}$ be dense in X. For each $n \in \mathbf{N}$ define the pseudo-metric $\tilde{\rho}_n(A, B) = \mid d(x_n, A) - d(x_n, B) \mid$. For each $n \in \mathbf{N}$ let $\hat{\rho}_n(A, B) = \frac{\tilde{\rho}_n(A,B)}{1+\tilde{\rho}_n(A,B)}$. Finally, we define ρ on $Cl(X)$ by

$$\rho(A, B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{\rho_n}.$$

By Lemma 2.3, ρ is a pseudo-metric on $Cl(X)$. To show that ρ is a metric, we need only show if $\rho(A, B) = 0$, then $A = B$ for all $A, B \in Cl(X)$.

Assume $\rho(A, B) = 0$. Then

$$\begin{aligned} \rho(A, B) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(A, B)}{1 + \rho_n(A, B)} \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mid d(x_n, A) - d(x_n, B) \mid}{1 + \mid d(x_n, A) - d(x_n, B) \mid} = 0. \end{aligned}$$

We then have $\sum_{n=1}^{\infty} \mid d(x_n, A) - d(x_n, B) \mid = 0$ and $d(x_n, A) = d(x_n, B)$ for all $n \in \mathbf{N}$. We need to show if $d(x_n, A) = d(x_n, B)$, then $A = B$. By using the contrapositive, we will assume $A \neq B$. Without loss of generality, there exists $y \in A \setminus B$ and $\delta > 0$ such that $B_\delta(y) \subseteq X \setminus B$.

Now let $n \in \mathbf{N}$ such that $x_n \in B_{\frac{\delta}{3}}(y)$. We have

$$d(x_n, A) \leq d(x_n, y) < \frac{\delta}{3}.$$

By the reverse triangle inequality, we have

$$\begin{aligned} d(x_n, B) &\geq d(y, B) - d(y, x_n) \\ &> \delta - \frac{\delta}{3} = \frac{2\delta}{3}. \end{aligned}$$

And this gives $d(x_n, A) \neq d(x_n, B)$ and the contrapositive is proven. Hence $A = B$ and ρ is a metric.

It is left to show that this metric induces the Wijsman topology. We first show that every ρ-open ball is open in τ_W. Let $A \in Cl(X)$ and let $\epsilon > 0$. Let $B \in B_\rho(A, \epsilon)$. Then

$$\begin{aligned} \rho(A, B) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{\rho_n}(A, B) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\tilde{\rho}_n(A, B)}{1 + \tilde{\rho}_n(A, B)} \\ &< \epsilon. \end{aligned}$$

Choose $N \in \mathbf{N}$ such that $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2}$.

Suppose $\tilde{\rho}_n(A,B) < \frac{\epsilon}{2}$. For $1 \le n \le N$, we get

$$\begin{aligned} \rho(A,B) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{\rho}_n(A,B) \\ &\le \sum_{n=1}^{N} \frac{1}{2^n} \frac{\rho_n(A,B)}{1+\rho_n(A,B)} \quad + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Let $U = \bigcap_{n=1}^{N} U_{(A, x_n, \frac{\epsilon}{2})}$. Then $U \subseteq B_\rho(A,\epsilon)$. Hence $B_\rho(A,\epsilon)$ is open in τ_W.

Now consider $U = \bigcap_{n=1}^{N} U_{(A, y_n, \epsilon)}$, a basic τ_W open set. We need to find $\delta > 0$ such that $B_\rho(A,\delta) \subseteq U$. That is, if $\rho(A,B) < \delta$ then $B \in U_{(A,y_n,\epsilon)}$. For each $n \in \{1,2,\ldots,N\}$ choose $x_{k_n} \in D$ such that $d(x_{k_n}, y_n) < \frac{\epsilon}{3}$. Let $V = \bigcap_{n=1}^{N} U_{(A, x_{k_n}, \frac{\epsilon}{3})}$, and let $C \in V$. Then

$$| \, d(x_{k_n}, C) - d(x_{k_n}, A) \, | \; < \; \frac{\epsilon}{3}.$$

By the triangle inequality we have

$$\begin{aligned} &| \, d(y_n, C) - d(y_n, A) \, | \\ &\le \; | \, d(y_n, C) - d(x_{k_n}, C) \, | + | \, d(x_{k_n}, C) - d(x_{k_n}, A) \, | + | \, d(x_{k_n}, A) - d(y_n, A) \, | \\ &< \; \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \; \epsilon. \end{aligned}$$

Therefore $V \subseteq U = \bigcap_{n=1}^{N} U_{(A, y_n, \epsilon)}$.

Now let $0 < \eta < 1$. We choose $m \in \mathbf{N}$ such that $k_n \le m$ for all $n \in \mathbf{N}$. Let $\delta = \frac{\eta}{2^m}$.

If $\rho(A,B) < \delta$, then $\rho(A,B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{\rho_n}(A,B) < \frac{\eta}{2^m}$. But

$$\begin{aligned} \frac{1}{2^m} \hat{\rho_n}(A,B) &\leq \sum_{n=1}^{m} \frac{1}{2^n} \hat{\rho_n}(A,B) \\ &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{\rho_n}(A,B) \\ &< \frac{\eta}{2^m}. \end{aligned}$$

Hence if $\rho(A,B) < \delta$, then $\hat{\rho_n}(A,B) < \eta$. Observe that the function $h : [0,\infty) \to [0,1)$ defined by $h(x) = \dfrac{x}{1+x}$ is continuous, strictly increasing, $h(0) = 0$ and bounded above by 1. Now we choose η such that if $\dfrac{\rho_n(A,B)}{1+\rho_n(A,B)} < \eta$ then $\rho_n(A,B) < \frac{\epsilon}{3}$. Therefore

$$B_\rho(A,\delta) \subseteq V \subseteq U \text{ where } \delta = \frac{\eta}{2^m}.$$

Hence τ_W is metrizable. □

Theorem 3.14 *If X is separable, then $(Cl(X), \tau_W)$ is separable.*

Proof. Let X be separable. Let (x_n) be dense in X, ρ be the metric defined in the previous theorem, and $Z = \{x_n : n \in \mathbf{N}\}$. Then Z is a countable dense subset of X. We know $\overline{Z} = X$. Define

$$\mathbf{F} = \{F \neq \emptyset : F \text{ is a finite subset of } Z\}.$$

Clearly $\mathbf{F}$ is countable. We need to show $\overline{\mathbf{F}} = Cl(X)$.

Let $A \in Cl(X)$ be arbitrary. We need to show for every $\epsilon > 0$ there exists $F \in \overline{\mathbf{F}}$ such that $\rho(A, F) < \epsilon$. We know

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mid d(x_n, A) - d(x_n, F) \mid}{1 + \mid d(x_n, A) - d(x_n, F) \mid} \leq \sum_{n=1}^{\infty} \mid d(x_n, A) - d(x_n, F) \mid .$$

Choose $N \in \mathbf{N}$ such that $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2}$. Now

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{\mid d(x_n, A) - d(x_n, F) \mid}{1 + \mid d(x_n, A) - d(x_n, F) \mid} \quad < \quad \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2}.$$

We now have the following relationship.

$$\begin{aligned} \rho(A, F) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mid d(x_n, A) - d(x_n, F) \mid}{1 + \mid d(x_n, A) - d(x_n, F) \mid} \\ &= \sum_{n=1}^{N} \frac{1}{2^n} \frac{\mid d(x_n, A) - d(x_n, F) \mid}{1 + \mid d(x_n, A) - d(x_n, F) \mid} + \frac{\epsilon}{2} \\ &\leq \sum_{n=1}^{N} \mid d(x_n, A) - d(x_n, F) \mid + \frac{\epsilon}{2}. \end{aligned}$$

Now choose $y_n \in A \in Cl(X)$ such that

$$\sum_{n=1}^{N} \mid d(x_n, A) - d(x_n, y_n) \mid < \frac{\epsilon}{8}.$$

Now choose $z_n \in Z$ such that

$$\sum_{n=1}^{N} \mid d(x_n, y_n) - d(x_n, z_n) \mid < \frac{\epsilon}{8}.$$

Let $F = \{z_n : \ n = 1, \ldots, N\}$. This gives us

$$\begin{aligned} \rho(A, F) &< \sum_{n=1}^{N} \mid d(x_n, A) - d(x_n, y_n) \mid + \mid d(x_n, y_n) - d(x_n, z_n) \mid \\ &+ \mid d(x_n, z_n) - d(x_n, F) \mid + \frac{\epsilon}{2}. \end{aligned}$$

And $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) \leq \frac{\epsilon}{4}$. We also have $d(x_n, F) \leq \frac{\epsilon}{4}$ so $| \ d(x_n, z_n) - d(x_n, F) \ | \leq \frac{\epsilon}{4}$ and $\rho(A, F) < \frac{\epsilon}{8} + \frac{\epsilon}{8} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon$. Hence $A \in \overline{\mathbf{F}}$ and $Cl(X) \subseteq \overline{\mathbf{F}}$. Thus $Cl(X) = \overline{\mathbf{F}}$. □

Lemma 3.15 *Let (X,d) be a complete, separable metric space. Suppose that (A_n) is a sequence in Cl(X) and $A \in Cl(X)$. Suppose $(A_n) \to A$ in τ_W. Then $A = \{x \in X :$ there exists (x_n) with $x_n \in A_n$ for all n and $(x_n) \to x\}$.*

Proof. ($\subseteq$) Let $x \in A$. Since $(A_n) \to A$ in τ_W, $d(x, A_n) \to d(x, A) = 0$. For each $n \in \mathbf{N}$, let $x_n \in A_n$ be such that

$$d(x, A_n) \leq d(x, x_n) \leq d(x, A_n) + \frac{1}{2^n}.$$

It then follows that $(d(x, x_n)) \to 0$, or $(x_n) \to x$.

($\supseteq$) Let $x \in \{x \in X :$ there exists (x_n) with $x_n \in A_n$ for all n and $(x_n) \to x\}$, and let (x_n) be such that $x_n \in A_n$ and $(x_n) \to x$. Clearly, $(d(x, A_n)) \to 0$. Thus $(d(x, A_n)) \to d(x, A) = 0$, or $x \in A$. □

Definition 3.16 *A metric space X is boundedly compact if the closed, bounded sets of X are compact.*

An example of this is property is the space $\mathbf{R}^n$. Clearly boundedly compact implies complete.

Theorem 3.17 *Let X be a separable metric space and let $\{x_n : n \in \mathbf{N}\}$ be dense in X. $(Cl(X), \rho)$ is complete if and only if X is boundedly compact.*

Proof. ($\Leftarrow$) Suppose X is boundedly compact. Let (A_n) be a Cauchy sequence in τ_W. We define A to be $\{x \in X :$ there exists a subsequence (A_{n_k}) of (A_k) such that there exists $x_k \in A_{n_k}$ for all k and $(x_k) \to x\}$. Let $y \in X$. Now for all $n \in \mathbf{N}$, let $x_n \in A_n$ such that $d(y, A_n) = d(y, x_n)$. Since (A_n) is Cauchy, then the sequence $(d(y, x_n))$ is a Cauchy sequence. Therefore (x_n) is bounded. Since X is boundedly compact, there exists a subsequence (x_{n_j}) of (x_n) and $x \in X$ such that $(x_{n_j}) \to x \in A$. Thus

$$\begin{aligned}
\lim_{n\to\infty} d(y, A_n) &= \lim_{j\to\infty} d(y, A_{n_j}) \\
&= \lim_{j\to\infty} d(y, x_j) \\
&= d(y, x) \\
&\geq d(y, A).
\end{aligned}$$

Now let $z \in A$. By definition there exists a subsequence (A_{n_k}) of (A_n) and $x_k \in A_{n_k}$ for every k and $(x_k) \to z$. Now

$$\begin{aligned}
\lim_{n\to\infty} d(y, A_n) &= \lim_{k\to\infty} d(y, A_{n_k}) \\
&\leq \lim_{k\to\infty} d(y, x_k) \\
&= d(y, z) \text{ for all } z \in A.
\end{aligned}$$

Hence $\lim_{n\to\infty} d(y, A_n) \leq d(y, A)$ and $(A_n) \to A$ in τ_W. Therefore $(Cl(X), \tau_W)$ is complete.

($\Rightarrow$) Now assume (X, d) is not boundedly compact. Then there exists a bounded sequence (z_n) in X such that (z_n) has no convergent subsequence. So there exists $\epsilon > 0$ such that $d(z_n, z_m) > \epsilon$ for all $n \neq m \in \mathbf{N}$. Note that for all $k \in \mathbf{N}$ the sequence $(d(x_k, z_n))_{n=1}^{\infty}$ is bounded. Choose a subsequence $(z_{(n,1)})$ of (z_n) such that $d(x_1, z_{(n,1)})$ converges. Choose a subsequence $(z_{(n,2)})$ of $(z_{(n,1)})$ such that $d(x_2, z_{(n,2)})$ converges. Inductively we now choose $(z_{(n,k+1)})$ of $(z_{(n,k)})$ such that $d(x_{k+1}, z_{(n,k+1)})$ converges. Now let $y_n = z_{(n,n)}$ for each $n \in \mathbf{N}$. If $k \in \mathbf{N}$, then $\{y_n\}_{n=k}^{\infty}$ is a subsequence of $z_{(n,k)}$. Thus the sequence $(d(x_n, y_k))_{k=1}^{\infty}$ converges for all $n \in \mathbf{N}$. Consider $\{ \{y_k\} : k \in \mathbf{N}\} \subseteq Cl(X)$. Since $(d(x_n, y_k))_{k=1}^{\infty}$ converges, $(\{y_k\})_{k=1}^{\infty}$ is a Cauchy sequence in ρ. To complete the proof, we need to show at least one Cauchy sequence does not converge. By contradiction, assume $\{y_k\} \to A \neq \emptyset$ where $A \in Cl(X)$. Let $x \in A$. Then there exists a sequence $w_k \in \{y_k\}$ such that $(w_k) \to x$. But then we would have $y_k \to x$. This is a contradiction since (z_n) has no convergent subsequence. Therefore $\{y_k\}$ does not converge. $\square$

Theorem 3.18 *If $A \in Cl(X)$ and $y \in X$ where X is a boundedly compact set, then there exists $x \in A$ such that $d(y, A) = d(y, x)$.*

Proof. Let $A \in Cl(X)$ and $y \in X$ where X is a boundedly compact set. By definition we have $f_A(y) = d(y, A)$ and $d(y, A) < \infty$. So there exists $L \in \mathbf{R}^+$ such that $d(y, A) = L$. Hence $A \cap B_{L+1}(y) \neq \emptyset$ is compact. We now have f_A is a continuous function and $A \cap B_{L+1}(y)$ is compact, therefore there exists $x \in A \cap B_{L+1}(y)$ such that $f_A(x) = L$. Hence $d(y, x) = d(y, A)$ for some $x \in A \cap B_{L+1}(y) \subseteq A$. $\square$

We conclude this chapter with the following result. If (X, d) is a complete, separable metric space, then it is shown in Theorem 4.3 of [Wi1] that $(Cl(X), \tau_W)$ is always a Polish space.

CHAPTER 4

TYPICAL CLOSED SUBSETS OF $\mathbf{R}^n$

Theorem 4.1 *Let X be a complete, separable metric space. Let $n \in \mathbf{N}$. Then $F_n = \{A \in Cl(X) :\mid A \mid \leq n\}$ is closed in $(Cl(X), \tau_W)$. (Where $\mid A \mid =$ the cardinality of A.)*

Proof. Let $n \in \mathbf{N}$. Let (A_k) be a sequence in F_n such that $(A_k) \to A$ in $(Cl(X), \tau_W)$. By way of contradiction, assume $\mid A \mid > n$. Let $y^1, y^2, \ldots, y^{n+1} \in A$. By previous work, we know there exists for each $i \in \{1, \ldots, n+1\}$ a sequence $(x_k^i)_{k=1}^{\infty} \to y^i$ and $x_k^i \in A_k$ for all $k \in \mathbf{N}$. Let $\epsilon > 0$ such that $B_\epsilon(y^i) \cap B_\epsilon(y^j) = \emptyset$ if $j \neq i$. Then there exists $n \in \mathbf{N}$ such that if $k \geq N$ then

$$d(x_k^i, y^i) < \epsilon \text{ for all } i \in \{1, \ldots, n+1\}.$$

Hence the elements $x_k^1, \ldots, x_k^{n+1} \in A_k$ are distinct. This is a contradiction since $\mid A_k \mid \leq n$. Hence $\mid A \mid \leq n$ and $A \in F_n$. Therefore F_n is closed in $(Cl(X), \tau_W)$. □

Definition 4.2 *A set X is said to be perfect if it is closed and has no isolated points, i.e., if A is equal to the set of its own limit points.*

Theorem 4.3 *Let X be a complete, separable, perfect metric space. $F_m = \{A \in Cl(X) :\mid A \mid \leq m\}$. Then F_m is nowhere dense.*

Proof. Let X be a complete, separable, perfect metric space. Let Θ be any open subset of $(Cl(X), \tau_W)$. Let $m \in \mathbf{N}$. We need to show there exists $A \in \Theta$ such that $| A | > m$. Let $A \in \Theta$. Choose $U = \bigcap_{n=1}^{k} U_{(A,x_n,\epsilon)} \subseteq \Theta$. Let

$$\begin{aligned} U' &= \bigcap_{n=1}^{k} U_{(A,x_n,\frac{\epsilon}{2})} \\ &= \bigcap_{n=1}^{k} \left\{ B \in Cl(X) : | d(x_n, A) - d(x_n, B) | < \frac{\epsilon}{4} \right\}. \end{aligned}$$

Now for all $1 \leq n \leq k$, choose $y_n \in A$ such that $d(x_n, A) > d(x_n, y_n) - \frac{\epsilon}{2}$. Therefore, if $1 \leq n \leq k$, then

$$d(x_n, y_n) - \frac{\epsilon}{2} < d(x_n, A) \leq d(x_n, y_n).$$

Let $Y = \{y_i : 1 \leq i \leq k\}$. We then have that $d(x_n, Y) \leq d(x_n, y_n) < d(x_n, A) + \frac{\epsilon}{2}$. Since $Y \subseteq A$, then $d(x_n, A) \leq d(x_n, Y)$. Therefore

$$| d(x_n, Y) - d(x_n, A) | < \frac{\epsilon}{2} \text{ for all } n \in \{1, \ldots, k\}.$$

Hence $Y \in U'$. Let $\hat{Y} = B_{\frac{\epsilon}{2}}(y_1) \cup \{y_2, \ldots, y_k\}$. We will now show that $\hat{Y} \in U$. Since $Y \subseteq \hat{Y}$, we also have that $d(x_n, \hat{Y}) \leq d(x_n, Y)$. Now

$$d(x_n, \hat{Y}) = \min \left\{ d(x_n, y_2), \ldots, d(x_n, y_k), d(x_n, B_{\frac{\epsilon}{2}}(y_1)) \right\}.$$

Let $z \in B_{\frac{\epsilon}{2}}(y_1)$ and $1 \leq n \leq k$. Then

$$\begin{aligned} d(x_n, z) &\geq d(x_n, y_1) - d(y_1, z) \\ &> d(x_n, y_1) - \frac{\epsilon}{2} \\ &\geq d(x_n, Y) - \frac{\epsilon}{2}. \end{aligned}$$

Therefore $d(x_n, \hat{Y}) \geq d(x_n, Y) - \frac{\epsilon}{2}$, or $\frac{\epsilon}{2} \geq d(x_n, Y) - d(x_n, \hat{Y})$ for every $1 \leq n \leq k$. Hence $\mid d(x_n, \hat{Y}) - d(x_n, Y) \mid \leq \frac{\epsilon}{2}$ for every $1 \leq n \leq k$. Thus if $1 \leq n \leq k$,

$$\mid d(x_n, \hat{Y}) - d(x_n, A) \mid \leq \mid d(x_n, \hat{Y}) - d(x_n, Y) \mid + \mid d(x_n, Y) - d(x_n, A) \mid < \epsilon.$$

Thus $\hat{Y} \in U$. Since X is perfect, $\hat{Y}$ is uncountable. Therefore $\mid \hat{Y} \mid > m$. So $Cl(X) \backslash F_m$ is dense and F_m is nowhere dense. □

For the remainder of our work, we will consider the complete, separable metric space $\mathbf{R}^n$ where $n \in \mathbf{N}$ is fixed.

Theorem 4.4 *The set* $I = \{X \in Cl(\mathbf{R}^n) : X$ *has an isolated point*$\}$ *is of first category.*

Proof. Let B be an open ball in $\mathbf{R}^n$. Define

$$I_B = \{X \in Cl(\mathbf{R}^n) : \text{ there exists } \hat{x} \in X \text{ such that } X \cap B = \{\hat{x}\}\}.$$

We need to show I_B is nowhere dense. Let $A \in \overline{I_B}$. Let U be open in τ_W such that $A \subseteq U$. Then $I_B \cap U \neq \emptyset$. Let $D \in I_B \cap U$. There exists $x_0 \in \mathbf{R}$ such that $D \cap B = \{x_0\}$. Let $\bigcap_{i=1}^{n} U_{(D, x_i, \epsilon)} \subseteq U$.

$$\bigcap_{i=1}^{n} U_{(D, x_i, \epsilon)} = \bigcap_{i=1}^{n} \{E \in Cl(\mathbf{R}^n) : \mid d(x_i, D) - d(x_i, E) \mid < \epsilon\}.$$

Let $F = D \cup \overline{B_{\frac{\epsilon}{2}}(x_0)}$. Since $A \in \overline{I_B}$ there exists $A_n \in I_B$ such that $(A_n) \to A$ where each A_n has exactly one isolated point in B. Now suppose $x \in A \cap B$. There exists

$\delta > 0$ such that $B_\delta(x) \subseteq B$. Let (x_n) be a sequence such that $x_n \in A_n$ and $(x_n) \to x$. (Recall that $(A_n) \to A$ if and only if there exists $x_n \in A_n$ such that $x_n \to x$ for all $x \in A$.) Let $N \in \mathbf{N}$ such that if $n \geq N$ then $\mid x_n - x \mid < \delta$. Therefore if $n \geq N$, then $x_n \in B$. Hence for all $x \in A \cap B$, the sequence $(x_n) \to x$ where $\{x_n\} = A_n \cap B$. Therefore $\mid A \cap B \mid \leq 1$.

We need to show $\mid d(x_i, D) - d(x_i, F) \mid < \epsilon$. We know $D \subseteq F$, so for $1 \leq i \leq n$,

$$d(x_i, F) \leq d(x_i, D) < d(x_i, D) + \frac{\epsilon}{2}$$
$$d(x_i, F) - d(x_i, D) < \frac{\epsilon}{2}.$$

Now $d(x_i, F) = \min\left\{d(x_i, D), d(x_i, \overline{B_{\frac{\epsilon}{2}}(x_0)}\right\}$. Let $z \in \overline{B_{\frac{\epsilon}{2}}(x_0)}$. Then for $1 \leq i \leq n$,

$$\begin{aligned} d(x_i, z) &\geq d(x_i, x_0) - d(x_0, z) \\ &> d(x_i, x_0) - \frac{\epsilon}{2} \\ &\geq d(x_i, D) - \frac{\epsilon}{2}. \end{aligned}$$

Hence $\mid d(x_i, D) - d(x_i, F) \mid < \frac{\epsilon}{2} < \epsilon$. Thus $Cl(\mathbf{R}^n) \setminus \overline{I_B}$ is dense and I_B is nowhere dense for $1 \leq i \leq n$.

Since $\mathbf{R}^n$ is separable, we can choose (B_i) to be a countable base for $\mathbf{R}^n$. Now

$$I_{B_i} = \{X \in Cl(\mathbf{R}^n) : X \cap B_i = \{\hat{x}\}\}$$

for $i \in \mathbf{N}$ and each I_{B_i} is nowhere dense. Therefore $I = \bigcup_{i=1}^{\infty} I_{B_i}$ is of first category, since it is a countable union of nowhere dense sets. □

Now, we have shown in Lemma 2.10 that every nonempty, countable, closed subset of X has an isolated point. Thus the following is easily obtained from the above.

Corollary 4.5 *The collection of countable closed subsets of* $\mathbf{R}^n$ *is of first category in* $(Cl(\mathbf{R}^n), \tau_W)$*. Hence the collection of uncountable, closed subsets of* $\mathbf{R}^n$*,* C*, is residual in* $(Cl(\mathbf{R}^n), \tau_W)$*.*

Our next goal is to show that the collection of nonempty, closed subsets of $\mathbf{R}^n$ having (n-dimensional) Lebesgue measure zero is also residual in $(Cl(\mathbf{R}^n), \tau_W)$. In order to do this, we need the following theorem.

Theorem 4.6 *Let* O *be an open subset of* $\mathbf{R}^n$ *and* $k > 0$*. Then*

$$U(O, k) = \{A \in Cl(\mathbf{R}^n) : A \cap kB \subseteq O \text{ where } B \text{ is the closed unit ball in } \mathbf{R}^n\}$$

is open in $(Cl(\mathbf{R}^n), \tau_W)$*.*

Proof. Let O be open in $\mathbf{R}^n$ and $k > 0$ and $A \in U(O, k)$. Let $d(A \cap kB, \mathbf{R}^n \setminus O) = \delta$ where $A \cap kB$ is compact since kB is closed and totally bounded.

Let $\delta > 0$. Let $\{x_i\}_{i=1}^p \subseteq kB$ such that $kB \subseteq \bigcup_{i=1}^{p} B_{\frac{\delta}{3}}(x_i)$. If $A \cap kB = \emptyset$, then let $W = \emptyset$ otherwise let $W = \bigcap_{i=1}^{p} \left\{Y : | d(x_i, Y) - d(x_i, A) | < \frac{\delta}{3}\right\}$. W is now a τ_W-neighborhood of A.

We need to show that $W \subseteq U(O,k)$. Let $Y \in W$ and $y \in Y \cap kB$. Choose an appropriate i such that $d(x_i, y) < \frac{\delta}{3}$. Thus $(x_i, Y) < \frac{\delta}{3}$ and

$$\begin{aligned} d(x_i, A) &\leq d(x_i, Y) + d(Y, A) \\ &< \frac{\delta}{3} + \frac{\delta}{3} = \frac{2\delta}{3}. \end{aligned}$$

Hence $d(y, A) \leq d(x_i, y) + d(x_i, A) < \frac{\delta}{3} + \frac{2\delta}{3} = \delta$. Thus $y \in O$ and $W \subseteq U(O,k)$. Therefore $U(O,k)$ is open in $(Cl(\mathbf{R}^n), \tau_W)$. □

Theorem 4.7 *The collection of all closed subsets of $\mathbf{R}^n$ having (n-dimensional) Lebesgue measure zero is residual in $(Cl(\mathbf{R}^n), \tau_W)$.*

Proof. For each $m, k \in \mathbf{N}$ let $V_m^k = \bigcup \left\{ U(O,k) : O \subseteq \mathbf{R}^n \text{ is open and } \lambda(O) < \frac{1}{m} \right\}$ where λ is (n-dimensional) Lebesgue measure on $\mathbf{R}^n$. For each m, k we have V_m^k is open. V_m^k is also dense in $Cl(\mathbf{R}^n)$, since it contains the finite sets which are dense in $Cl(\mathbf{R}^n)$. Also if $A \in Cl(\mathbf{R}^n)$, then $\lambda(A) \leq \frac{1}{m}$ if and only if $A \in \bigcap_{k=1}^{\infty} V_m^k$. Hence $\lambda(A) = 0$ if and only if $A \in \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} V_m^k$. Since $\bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} V_m^k$ is a dense G_δ set, it is residual. Therefore the collection of all closed subsets of $\mathbf{R}^n$ having Lebesgue measure zero is residual in $(Cl(\mathbf{R}^n), \tau_W)$. □

Corollary 4.8 *The collection of all uncountable, closed subsets of $\mathbf{R}^n$ that have Lebesgue measure zero is residual in $(Cl(\mathbf{R}^n), \tau_W)$.*

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