THREE TOPICS IN DESCRIPTIVE SET THEORY

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This dissertation deals with three topics in descriptive set theory. First, the order topology is a natural topology on ordinals. In Chapter 2, a complete classification of order topologies on ordinals up to Borel isomorphism is given, answering a question of Benedikt Löwe. Second, a map between separable metrizable spaces X and Y preserves complete metrizability if Y is completely metrizable whenever X is; the map is resolvable if the image of every open (closed) set in X is resolvable in Y. In Chapter 3, it is proven that resolvable maps preserve complete metrizability, generalizing results of Sierpiński, Vainštein, and Ostrovsky. Third, an equivalence relation on a Polish space has the Laczkovich–Komjáth property if the following holds: for every sequence of analytic sets such that the limit superior along any infinite set of indices meets uncountably many equivalence classes, there is an infinite subsequence such that the intersection of these sets contains a perfect set of pairwise inequivalent elements. In Chapter 4, it is shown that every coanalytic equivalence relation has the Laczkovich–Komjáth property, extending a theorem of Balcerzak and Głąb.

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CONTENTS

CHAPTER 1

INTRODUCTION

Set theory started with Cantor's discovery that the reals $\mathbb{R}$ are uncountable. In particular, $|\mathbb{N}| < |\mathbb{R}|$. Cantor's continuum hypothesis (CH) is the statement that for every $X \subseteq \mathbb{R}$ either $|X| \leq |\mathbb{N}|$ or $|X| = |\mathbb{R}|$. Of course, we now know that the continuum hypothesis (CH) is independent of the usual axioms of mathematics, Zermelo–Fraenkel set theory ZFC. In a way, modern set theory grew out of the undecidability of CH. At the same time, descriptive set theory stems from Cantor's program to settle CH in another direction.

Nowadays, the uncountability of $\mathbb{R}$ is often proved using Cantor's diagonal argument. However, Cantor originally gave a more topological argument which shows that every nonempty open set has the cardinality of the continuum. The first real theorem in descriptive set theory deals with closed sets:

THEOREM 1.1 (Cantor–Bendixson). *A closed subset of the real line is either countable or else has the cardinality of the continuum.*

This result illustrates the philosophy of descriptive set theory: statements which are undecidable in ZFC for arbitrary sets often become decidable for 'definable' sets. Here, 'definable' is taken to be topologically simple: open and closed sets, and sets build from these, such as the Borel sets. The following theorem firmly established the field of descriptive set theory:

THEOREM 1.2 (Alexandrov, Hausdorff). *Every Borel set is either countable or else has the cardinality of the continuum.*

An account of the origins of descriptive set theory is given in [14]. An exciting feature of modern descriptive set theory is that it mixes topology, set theory, and computability theory. This combination is visible in the following three chapters, which deal with three different topics in descriptive set theory, broadly defined. The rest of this chapter provides an overview of the results contained in this dissertation.

1.1. A Borel Classification of Ordinals

Ordinals were introduced by Cantor to keep track of a sequence of derivatives of a set. Starting with a closed set $P \subseteq \mathbb{R}$, Cantor defined its *derived set* P' as the set of all limit points of P. Repeatedly taking the derivative results in an infinite sequence:

$$P^{(1)}, P^{(2)}, \ldots$$

Cantor noticed that for some sets of reals the intersection $\bigcap_{n=1}^{\infty} P^{(n)}$ was nonempty. He set $P^{(\infty)} = \bigcap_{n=1}^{\infty} P^{(n)}$ and then continued the process into the transfinite:

$$P^{(\infty+1)}, P^{(\infty+2)}, \ldots$$

Eventually, these considerations led Cantor to introduce the concepts of *ordinal numbers* and (a little later) *cardinal numbers*, both fundamental to modern set theory.

We consider ordinals as topological spaces in Chapter 2. A fundamental problem in topology is the classification of all topological spaces of a certain class up to some notion of equivalence. For example, we can consider the classification of all topological spaces up to homeomorphism or all Polish (i.e., separable, completely metrizable) spaces up to Borel isomorphism. The *order topology* is a natural topology on ordinals. A classification of ordinal topologies up to homeomorphism was known [1], using the Cantor normal form for ordinals. In Chapter 2 we consider the classification of ordinals up to Borel isomorphism.

It is easy to see that not all countable ordinals are homeomorphic. For example, ω is not compact while $\omega + 1$ is; hence, ω and $\omega + 1$ are not homeomorphic. On the other hand, every subset of a countable ordinal is Borel (in fact, F_σ) and therefore all countable ordinals (in particular, ω and $\omega + 1$) are Borel isomorphic. Hence, Borel isomorphism is a genuinely more general notion of equivalence than homeomorphism for ordinals.

It turns out that complete invariants for Borel isomorphism are not related to the Cantor normal form. To state our main result precisely, define for an ordinal α a cardinal $\kappa(\alpha)$ as follows. Let $\kappa(\alpha) = 0$ if $|\alpha|$ is singular or countable; otherwise, let $\kappa(\alpha)$ be the largest cardinal such that $|\alpha| \cdot \kappa(\alpha) \leq \alpha$. Of course, in order to be Borel isomorphic, two ordinals α and β must have the same cardinality. We will prove that the cardinality of α together with $\kappa(\alpha)$ suffices:

THEOREM 1.3. *Two ordinals α and β are Borel isomorphic if and only if $|\alpha| = |\beta|$ and $\kappa(\alpha) = \kappa(\beta)$.*

In other words, the data $|\alpha|$ and $\kappa(\alpha)$ together form a complete Borel isomorphism invariant for the order topology of α.

1.2. Resolvable Maps Preserve Complete Metrizability

The earliest results in descriptive set theory were stated specifically for subsets of $\mathbb{R}^n$. However, it was soon realized that the theory could be generalized to arbitrary separable, completely metrizable spaces. In honor of the many contributions to the field made by Polish mathematicians, these spaces are now known as *Polish spaces.*

Polish spaces are the natural setting for analysis and descriptive set theory. It is therefore of interest to find criteria that will ensure that a separable, metrizable space is in fact completely metrizable. We study one aspect of this problem in Chapter 3. Specifically, we consider the following question: let X be a Polish space, Y a separable metrizable space, and $f\colon X \to Y$ a continuous surjection. When is Y completely metrizable?

Some restrictions on the map f are necessary, since there is a continuous surjection from ω^ω to $\mathbb{Q}$. The need to put restrictions on f leads us to consider Hausdorff's difference hierarchy, a transfinite hierarchy of sets defined from the open sets using set-theoretic difference and intersections. The sets in the differency hierarchy are sometimes called *resolvable sets.* The extend of the difference hierarchy is an interesting problem in itself. Lavrentiev proved that for uncountable Polish spaces the difference hierarchy does not collapse. His proof does not work for the rationals. We give a direct construction to show the following:

THEOREM 1.4. *The difference hierarchy over the open sets of the rationals does not collapse.*

Returning to the question of metrizability, we say that a map $f\colon X \to Y$ is *open-resolvable* (*closed-resolvable*) if the image under f of every open (closed) set is resolvable. We say that f is *resolvable* if f is either open-resolvable or closed-resolvable. In Chapter 3 we will prove the following theorem, generalizing earlier results by Sierpiński [25], Vainštein [27, 28], and Ostrovsky [23]:

THEOREM 1.5. *Let X be a Polish space, Y a separable metrizable space, and $f\colon X \to Y$ a continuous surjection. If f is resolvable, then Y is Polish.*

1.3. The Laczkovich–Komjáth Property

An equivalence relation E on a Polish space X is said to be Borel (analytic, coanalytic, etc.) if E is Borel (analytic, coanalytic, etc.) as a subset of $X \times X$. Definable equivalence relations have been a topic of intense study in descriptive set theory for the last three decades. We consider coanalytic equivalence relations and limit superiors of analytic sets in Chapter 4.

Given a sequence $(A_n)_{n\in\omega}$ of subsets of a Polish space X, the limit superior of $(A_n)_{n\in\omega}$ relative to an infinite set $H \in [\omega]^\omega$ is defined by

$$x \in \limsup_{n\in H} A_n \Leftrightarrow \forall m\, \exists n \geq m\, (n \in H \wedge x \in A_n).$$

Laczkovich raised the following question: if $\limsup_{n\in H} A_n$ is uncountable for every $H \in [\omega]^\omega$, is there an $H \in [\omega]^\omega$ such that $\bigcap_{n\in\omega} A_n$ is uncountable? He and Komjáth proved that this is true when the sets A_n are respectively Borel or analytic subsets of a Polish space. Balcerzak and Głab [2] extended these results of Laczkovich and Komjáth to equivalence relations in the following way:

DEFINITION 1.6. Let E be an equivalence relation on a Polish space X. We say that E has the *Laczkovich–Komjáth property* if for every sequence $(A_n)_{n\in\omega}$ of analytic subsets of X such that $\limsup_{n\in H} A_n$ meets uncountably many E-equivalence classes for every $H \in [\omega]^\omega$, there is an $H \in [\omega]^\omega$ such that $\bigcap_{n\in H} A_n$ contains a perfect set of pairwise E-inequivalent elements.

Balcerzak and Głab have shown that every F_σ equivalence relation has the Laczkovich–Komjáth property. In turn, we generalize this to coanalytic equivalence relations:

THEOREM 1.7. *Every coanalytic equivalence relation has the Laczkovich–Komjáth property.*

Our proof uses the techniques of effective descriptive set theory, which is based on computability theory.

CHAPTER 2

A BOREL CLASSIFICATION OF ORDINALS

The *order topology* on a linearly ordered set $(X, <)$ is generated by the subbase of open rays $(x, \rightarrow) = \{y \in X : x < y\}$ and $(\leftarrow, y) = \{x \in X : x < y\}$ for $x, y \in X$. Perhaps the most familiar example of an order topology is the usual topology on the real line $\mathbb{R}$. The order topology is also a natural topology on ordinals. When we consider an ordinal as a topological space we will always assume it has the order topology.

A typical problem in topology is the classification of all spaces in a certain class up to some notion of equivalence. A complete classification of ordinals up to homeomorphism is known ([1]; an independent proof was given in [16]). Specifically, given an arbitrary ordinal a complete homeomorphism invariant for its order topology can be computed from its *Cantor normal form.* Recall that every nonzero ordinal α can uniquely be written in Cantor normal form as

$$\alpha = \omega^{\alpha_0} \cdot k_0 + \cdots + \omega^{\alpha_n} \cdot k_n,$$

where $\alpha \geq \alpha_0 > \cdots > \alpha_n$ and $0 < k_i < \omega$ for $0 \leq i \leq n$. Define the *limit complexity* of α as $\mathrm{lc}(\alpha) := \alpha_0$, the *coefficient* of α as $\mathrm{c}(\alpha) := k_0$, and the *purity* of α as

$$\mathrm{p}(\alpha) := \begin{cases} 0 & \text{if } \alpha = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha), \text{ and} \\ \omega^{\alpha_n} & \text{otherwise.} \end{cases}$$

It turns out that these three data provide a complete homeomorphic invariant for ordinal topologies, that is, $\alpha \cong \beta$ if and only if

$$\langle \mathrm{lc}(\alpha), \mathrm{c}(\alpha), \mathrm{p}(\alpha) \rangle = \langle \mathrm{lc}(\beta), \mathrm{c}(\beta), \mathrm{p}(\beta) \rangle .$$

Benedikt Löwe proposed to study the similar classification problem for ordinals up to Borel isomorphism (see Section 3 for the definition of Borel isomorphism). He asked whether the Cantor normal form still provides a complete invariant.

Example 2.1. It is easy to see that not all countable ordinals are homeomorphic. For example, ω is not compact while $\omega + 1$ is; hence, ω and $\omega + 1$ are not homeomorphic. On the other hand, every subset of a countable ordinal is Borel (in fact, F_σ) and therefore all countable ordinals are Borel isomorphic.

The preceeding example shows that that Borel isomorphism is a genuinely more general notion of equivalence than homeomorphism. In this chapter we give a complete classification of all ordinals up to Borel isomorphism. It turns out that the complete invariants are not related to the Cantor normal form of the ordinals, and are in fact somewhat simpler. To state our main theorem precisely, define a cardinal $\kappa(\alpha)$ for any given ordinal α as follows.

Definition 2.2. For an ordinal α, let $\kappa(\alpha) = 0$ if $|\alpha|$ is singular or countable; otherwise, let $\kappa(\alpha)$ be the largest cardinal such that $|\alpha| \cdot \kappa(\alpha) \leq \alpha$.

Of course, ordinals necessarily need to have the same cardinality in order to be Borel isomorphic. We will show that the cardinality of α together with $\kappa(\alpha)$ constitute a complete invariant up to Borel isomorphism:

Theorem 2.3. *Two ordinals α and β are Borel isomorphic iff $|\alpha| = |\beta|$ and $\kappa(\alpha) = \kappa(\beta)$.*

This chapter is organized as follows. In Section 1 we review some preliminaries on the Borel structures generated by the order topologies on ordinals. In particular we give a characterization of Borelness for subsets of ordinals which will be useful in further research. In Section 2 we review material on Borel isomorphisms. Finally, we give the proof of Theorem 2.3 in Section 3.

2.1. Borel Structures on Ordinals

The *Borel structure* of any topological space is the σ-algebra generated by the open sets, that is, the smallest σ-algebra that contains all open sets and is closed under complements and countable unions. All Borel sets appear in a stratified *Borel hierarchy*, which can be defined by induction as follows:

$$\Sigma^0_1 = \text{all open sets},$$

$$\Pi^0_\alpha = \text{all complements of } \Sigma^0_\alpha \text{ sets},$$

$$\Sigma^0_\alpha = \text{all countable unions } \bigcup_{n\in\mathbb{N}} A_n, \text{ where } A_n \in \Pi^0_{\alpha_n} \text{ for some } \alpha_n < \alpha,$$

$$\Delta^0_\alpha = \Sigma^0_\alpha \cap \Pi^0_\alpha.$$

The collection of all *Borel sets* of a topological space X then is defined as

$$\mathrm{B}(X) = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha.$$

The following proposition records the basic facts about the levels of the Borel hierarchy which are true in any topological space.

PROPOSITION 2.4. *In any topological space the following hold:*

(i) $\Sigma^0_\alpha \subseteq \Pi^0_\beta$ *and* $\Pi^0_\alpha \subseteq \Sigma^0_\beta$ *for* $\alpha < \beta$.

(ii) Σ^0_α *is closed under countable unions and* Π^0_α *under countable intersections.*

(iii) Σ^0_α *is closed under finite intersections and* Π^0_α *under finite unions for all* $\alpha \neq 3$.

(iv) *If* $2 \leq \alpha \leq \beta$, *then* $\Sigma^0_\alpha \subseteq \Sigma^0_\beta$ *and* $\Pi^0_\alpha \subseteq \Pi^0_\beta$.

PROOF. All of the statements are immediate from the definitions except perhaps the closure of Σ^0_α under finite intersections for $\alpha \neq 3$. To see this, suppose $A, B \in \Sigma^0_\alpha$, say $A = \bigcup_n A_n$, $B = \bigcup_m B_m$, where $A_n \in \Pi^0_{\alpha_n}$, $B_m \in \Pi^0_{\beta_m}$ and $\alpha_n, \beta_m < \alpha$. Then $A \cap B = \bigcup_{n,m}(A_n \cap B_m)$. If $\alpha \geq 4$, then A_{α_n} B_{β_m} both lie in Π^0_δ where $\delta = \max\{\alpha_n, \beta_m, 3\}$. This is because $\Pi^0_\alpha \subseteq \Pi^0_\beta$ for $2 \leq \alpha \leq \beta$ and $\Pi^0_1 \subseteq \Sigma^0_2 \subseteq \Pi^0_3$. Since Π^0_δ is closed under intersections, $A_{\alpha_n} \cap B_{\beta_m} \in \Pi^0_\delta$, and so $A \cap B \in \Sigma^0_\alpha$. If $\alpha = 1$, the result is immediate from the definition of a topology, and if $\alpha = 2$ the result follows from the fact that each A_{α_n}, B_{β_m} will be Π^0_1, and thus so will be $A_{\alpha_n} \cap B_{\beta_m}$. $\square$

If the underlying space is metrizable, then its Borel hierarchy has the usual additional properties such as the following.

PROPOSITION 2.5. *In any metrizable space the following hold:*

(i) $\boldsymbol{\Sigma}^0_\alpha \subseteq \boldsymbol{\Sigma}^0_\beta$ *and* $\boldsymbol{\Pi}^0_\alpha \subseteq \boldsymbol{\Pi}^0_\beta$ *for* $\alpha < \beta$,

(ii) $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$ *are closed under finite unions and finite intersections,*

(iii) *every* $\boldsymbol{\Sigma}^0_{\alpha+1}$ *set is a countable union* $\bigcup_n A_n$ *where each* $A_n \in \boldsymbol{\Pi}^0_\alpha$. $\square$

However, these additional facts are no longer true for ordinal spaces. In general $\boldsymbol{\Sigma}^0_1 \not\subseteq \boldsymbol{\Sigma}^0_2$ and $\boldsymbol{\Pi}^0_1 \not\subseteq \boldsymbol{\Pi}^0_2$. The following observation shows that $\boldsymbol{\Sigma}^0_3$ is not closed under finite intersections if the underlying space is an ordinal $\geq \omega_2$. For its proof, recall the following standard set-theoretic terminology: a set $A \subseteq \alpha$ is *unbounded* if for every $\beta < \alpha$ there is a $\gamma \in A$ such that $\beta < \gamma$; *club* if A is closed and unbounded; *stationary* if $A \cap C \neq \varnothing$ for every club $C \subseteq \alpha$; *nonstationary* if A is not stationary; and *costationary* if $\alpha - A$ is stationary.

PROPOSITION 2.6. *There exists an open set* $U \subseteq \omega_2$ *and a closed set* $F \subseteq \omega_2$ *such that* $U \cap F$ *is not* $\boldsymbol{\Sigma}^0_3$.

PROOF. Let

$$U = \omega_2 - \{\omega_1 \cdot \alpha \colon \alpha < \omega_2\}$$

and let F be the set of all limit ordinals below ω_2. Clearly, U is open and F is closed. Suppose $U \cap F$ is $\boldsymbol{\Sigma}^0_3$, say

$$U \cap F = \bigcup_{n \in \omega} A_n \cup \bigcup_{n \in \omega} B_n,$$

where each A_n is $\boldsymbol{\Pi}^0_1$ and each B_n is $\boldsymbol{\Pi}^0_2$. Since U misses a club in ω_2, $U \cap F$ is nonstationary, which in turn implies that each A_n is bounded in ω_2. Now that the union $\bigcup_{n \in \mathbb{N}} A_n$ is also bounded in ω_2, let β be an upper bound. Let α_0 be the least ordinal such that $\omega_1 \cdot \alpha_0 \geq \beta$.

Now consider the copy of ω_1 consisting of ordinals in the interval $I = (\omega_1 \cdot \alpha_0, \omega_1 \cdot \alpha_0 + \omega_1)$. Our assumption implies that the limit ordinals in I can be written as $\bigcup_n (B_n \cap I)$. It follows that the limit ordinals in ω_1 can be written as $\bigcup_n C_n$ where each C_n is $\boldsymbol{\Pi}^0_2$. Since the limit ordinals in ω_1 form a club, one of the C_n must be stationary. We claim that a stationary $\boldsymbol{\Pi}^0_2$ subset of ω_1 must contain a tail, and this is a contradiction.

Suppose $G = \bigcap_{n\in\omega} G_n$ is a stationary $\mathbf{\Pi}^0_2$ in ω_1, with all G_n open. Each G_n is also stationary, and therefore it contains a tail. Since $\mathrm{cof}(\omega_1) > \omega$, a countable intersection of tails is still a tail. Hence, G contains a tail. $\square$

The Borel structures on ordinals have been studied before, see for example [20, 24]. We summarize the known results as well as present the techniques used in the study of this topic. For the convenience of the reader we include some proofs of previously known results here.

LEMMA 2.7 (Rao–Rao [24]). *Every Borel subset of a limit ordinal either contains or misses a club.*

PROOF. Clearly, every subset of a limit ordinal of cofinality ω either contains or misses a club. In case of uncountable cofinality, a countable intersection of clubs is still a club. Hence, the collection of all sets which contain or miss a club is a σ-algebra containing all closed sets and therefore contains all the Borel sets. $\square$

In particular, a stationary and costationary subset of a limit ordinal is not Borel. A subset of ω_1 is Borel if and only if it either contains or misses a club [24]. Another characterization of Borel subsets of ω_1 was also given in [24], and was completely generalized by Mauldin [20] as follows.

THEOREM 2.8 (Mauldin [20]). *Every Borel subset of an ordinal can be expressed as a union of countably many sets, each of which is the intersection of an open set and a closed set.* $\square$

Mauldin's theorem shows that the Borel hierarchy on any ordinal collapses at a rather low level: every Borel subset of an ordinal is in fact $\mathbf{\Delta}^0_4$. In view of Proposition 2.6 this is optimal.

Below we give another characterization of Borelness of subsets of ordinals. We state the result in a way that encompasses the results in [20] and [24], and provide a self-contained proof. It should be noted, however, that the main ideas and techniques used in the proof are the same as those presented in [20] and [24]. We will use the following lemma repeatedly throughout this chapter; its proof is a straigtforward induction on ξ.

LEMMA 2.9. *Let X be an arbitrary topological space. Suppose $X = \bigcup_{i\in I} U_i$, where $\{U_i\}_{i\in I}$ is a family of pairwise disjoint open subsets. Let $\xi < \omega_1$ and $B \subseteq X$. Then B is Σ^0_ξ (or Π^0_ξ) if and only if for every $i \in I$, $B \cap U_i$ is Σ^0_ξ (respectively Π^0_ξ) in U_i.* $\square$

The following theorem characterizes Borel subsets of ordinals.

THEOREM 2.10. *Let α be an ordinal. Then the following are equivalent:*

(1) *$B \subseteq \alpha$ is Borel.*

(2) *$B = \bigcup_{n\in\mathbb{N}}(U_n \cap F_n)$, where each U_n is open and each F_n is closed.*

(3) *For every limit ordinal $\beta \leq \alpha$, B contains or misses a club in β.*

(4) *For every limit ordinal $\beta \leq \alpha$ and every club C in β, B contains or misses a club of C.*

PROOF. The implication (1) $\Rightarrow$ (4) is immediate from Lemma 2.7. The implications (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) are trivial. It suffices to show (3) $\Rightarrow$ (2). We use induction on α. For the base case and the successor case there is nothing to do. Assume α is a limit ordinal. By condition (3) B contains or misses a club in α. For definiteness assume that B misses a club C in α. In this case let α_i, $i < \eta = \mathrm{cof}(\alpha)$, enumerate the elements of C in the increasing order. Without loss of generality we may assume $\alpha_0 = 0$. Then let $U_i = (\alpha_i, \alpha_{i+1})$ for $i < \eta$. Thus we get that $\alpha - C = \bigcup_{i<\eta} U_i$. Note that condition (3) is still true for each interval U_i. Since each U_i is a copy of an ordinal $< \alpha$, the inductive hypothesis gives that $B \cap U_i$ is a union of countably many sets, each of which is the intersection of an open set with a closed set. Now Lemma 2.9 implies that

$$B = B \cap (\alpha - C) = \bigcup_{n\in\mathbb{N}} (U_n \cap F_n)$$

for relatively open U_n in $\alpha - C$ and relatively closed F_n in $\alpha - C$. Let C_n be the closure of F_n in α, $U_{-1} = \alpha - C$ and $V_n = U_n \cap U_{-1}$. Then each V_n is open in α, C_n is closed in α, $F_n = C_n \cap U_{-1}$ and

$$B = \bigcup_{n\in\mathbb{N}} (U_n \cap F_n) = \bigcup_{n\in\mathbb{N}} (U_n \cap C_n \cap U_{-1}) = \bigcup_{n\in\mathbb{N}} (V_n \cap C_n).$$

This finishes the proof of the case that B misses a club C in α. Suppose alternatively B contains a club C in α, then $B - C$ continues to satisfy (3) and the same argument shows that $B - C$ is a union of the form in (2). It follows that B is of the same form since $B = (B - C) \cup C$. $\square$

As another application of the same technique we note that every Borel subset of ω_1 is $\mathbf{\Delta}^0_3$.

PROPOSITION 2.11. *Every Borel subset of ω_1 is $\mathbf{\Delta}^0_3$.*

PROOF. It suffices to show that every Borel subset of ω_1 is $\mathbf{\Sigma}^0_3$. In view of Theorem 2.10 it is enough to show that the intersection of an open set U and a closed set F is $\mathbf{\Sigma}^0_3$. If $U \cap F$ is bounded, then this intersection is countable and easily seen to be $\mathbf{\Sigma}^0_3$. Assume $U \cap F$ is unbounded. In particular, both U and F are unbounded. If $\omega_1 - U$ is bounded, then the bounded part of $U \cap F$ is relatively $\mathbf{\Sigma}^0_2$, the unbounded part is relatively closed, thus relatively $\mathbf{\Sigma}^0_2$, hence by Lemma 2.9, $U \cap F$ is $\mathbf{\Sigma}^0_2$ in ω_1.

If $\omega_1 - U$ is unbounded, write $U = \bigcup I_\gamma$, where the I_γ are maximal disjoint open intervals. Each I_γ is homeomorphic to a countable ordinal, hence $U \cap F$ is $\mathbf{\Pi}^0_2$ in I_γ, thus in U. Hence, $U \cap F$ is the intersection of an open and a $\mathbf{\Pi}^0_2$ set in ω_1, hence $\mathbf{\Pi}^0_2$. $\square$

In view of the collapse of the Borel hierarchy our basic Lemma 2.9 can be restated as the following convenient fact for subsets of ordinals. For obvious reasons we will refer to it as the gluing lemma.

LEMMA 2.12 (The gluing lemma). *Let α be an ordinal, $\{U_i\}_{i\in I}$ a family of pairwise disjoint open sets in α, and C the closed set such that $\alpha - C = \bigcup_{i\in I} U_i$. Then $B \subseteq \alpha - C$ is Borel in α if and only if $B \cap U_i$ is Borel in U_i for every $i \in I$.* $\square$

2.2. Borel Isomorphisms

We now turn to a review of Borel isomorphisms between topological spaces. Let X and Y be arbitrary topological spaces. A map $f\colon X \to Y$ is called *Borel measurable* or simply

Borel if the pre-image $f^{-1}(U)$ of any open subset U of Y is a Borel subset of X. This easily implies that the pre-image $f^{-1}(U)$ of any Borel subset of Y is Borel in X.

A *Borel isomorphism* is a bijection f such that both f and f^{-1} are Borel. If there is a Borel isomorphism from X onto Y, then we say that X and Y are *Borel isomorphic*. We denote this by $X \cong_B Y$. Recall that if X and Y are both Polish spaces, then $X \cong_B Y$ if and only if there is a Borel injection from X into Y and a Borel injection from Y into X. Here a Borel injection is merely an injective Borel map. The proof is a repetition of the proof of the classical Cantor–Bernstein theorem. However, the reason it runs smoothly in this context is because of the important theorem of Luzin–Suslin that a Borel injection from a Polish space to another preserves Borelness of subsets; every Borel injection between Polish spaces is a Borel embedding.

In our context the following definition is needed. A Borel injection $f\colon X \to Y$ is called a *Borel embedding* if the image of a Borel set under f is Borel. Now the proof of the classical Cantor–Bernstein theorem can be repeated to show that if there exist Borel embeddings $f\colon X \to Y$ and $g\colon Y \to X$, then X and Y are Borel isomorphic.

PROPOSITION 2.13. *Let X and Y be topological spaces. If there exist Borel embeddings $f\colon X \to Y$ and $g\colon Y \to X$, then X and Y are Borel isomorphic.* □

We adopt the notation $f\colon X \hookrightarrow_B Y$ to denote that f is a Borel embedding from X into Y, and write $X \hookrightarrow_B Y$ or simply $X \hookrightarrow Y$ if there exists $f\colon X \hookrightarrow_B Y$.

The following simple observations on Borel isomorphism and embeddability of ordinals will be useful. Let $\alpha < \beta$ be ordinals. Note that the canonical injection (that is, the identity map) from α into β is a Borel embedding (in fact a homeomorphic one). It follows that for $\alpha < \beta$ we have $\alpha \cong_B \beta$ if and only if $\beta \hookrightarrow \alpha$. The following lemma is our main tool to show that β Borel embeds into $\alpha < \beta$.

LEMMA 2.14. *Let $\alpha < \beta$ be ordinals, $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be pairwise disjoint open sets in α and in β respectively, and C and D be closed subsets of α and β respectively with $\alpha - C = \bigcup_{i\in I} U_i$ and $\beta - D = \bigcup_{j\in J} V_j$. Suppose that there is a $k \in I$ such that $\psi\colon D \hookrightarrow_B U_k$*

and there is an injection $\pi\colon J \to I - \{k\}$ *such that for every* $j \in J$ *there is* $\psi_j\colon V_j \hookrightarrow_B U_{\pi(j)}$. *Then* β *Borel embeds into* $\alpha - C$, *thus into* α, *and* $\beta \cong_B \alpha$.

PROOF. Let $\phi\colon \beta \to \alpha - C$ be the piecewise defined map from ψ and the ψ_j's. Clearly, ϕ is injective. If $B \subseteq \beta$ is Borel, then $B \cap D$ is Borel in D and $B \cap V_j$ is Borel for each $j \in J$. Hence, ϕ"B is Borel in each U_j. By the gluing lemma, ϕ"B is Borel in $\alpha - C$. Similarly, if $B \subseteq \alpha - C$ is Borel, then $\phi^{-1}(B \cap U_k)$ is Borel in D and for any $l \in J - \{k\}$, $\phi^{-1}(B \cap U_l)$ is Borel in $V_{\pi^{-1}(l)}$, hence $\phi^{-1}B$ is Borel in β again by the gluing lemma. □

Under the hypotheses of the above lemma a particularly easy way to guarantee $D \hookrightarrow U_k$ for some k is to make sure that $\mathrm{ot}(D) \leq \mathrm{ot}(U_k)$. Note that the lemma is still meaningful even if the ordinals are the same. Specifically, if $\alpha \geq \omega$ and $C \subseteq \kappa \cdot \alpha$ is closed with order type $\leq \kappa$, then the lemma gives that $\kappa \cdot \alpha - C \cong_B \kappa \cdot \alpha$.

2.3. The Classification up to Borel Isomorphism

In this section we classify all ordinals up to Borel isomorphism. Recall the following definition from the introduction:

DEFINITION 2.15. For an ordinal α, let $\kappa(\alpha) = 0$ if $|\alpha|$ is singular or countable; otherwise, let $\kappa(\alpha)$ be the largest cardinal such that $|\alpha| \cdot \kappa(\alpha) \leq \alpha$.

We will show that $\alpha \cong_B \beta$ iff $|\alpha| = |\beta|$ and $\kappa(\alpha) = \kappa(\beta)$.

Since all countable ordinals are Borel isomorphic and $\alpha \not\cong_B \beta$ whenever $|\alpha| \neq |\beta|$, we can restrict ourselves to ordinals α and β such that $\kappa \leq \alpha < \beta < \kappa^+$ for some uncountable cardinal κ. As remarked before, in order to show that $\alpha \cong_B \beta$ it suffices to find a Borel embedding of β into α. We split the proof of Theorem 2.3 into three parts. First, we show that all ordinals greater than or equal to $\kappa \cdot \mathrm{cof}(\kappa)$ are Borel isomorphic to $\kappa \cdot \mathrm{cof}(\kappa)$. Second, we show that for singular cardinals κ, $\kappa \cdot \mathrm{cof}(\kappa)$ is Borel isomorphic to κ. Finally, we identify the Borel isomorphism types between κ and $\kappa^2 = \kappa \cdot$ for regular κ.

For the first part, we need the following lemma.

LEMMA 2.16. *If* $\omega \leq \alpha \leq \kappa$, *then* $\kappa \cdot \alpha^2 \cong_B \kappa \cdot \alpha$.

PROOF. We first show $\kappa \cdot \alpha^2 \hookrightarrow \kappa \cdot \alpha \cdot 2$. Let $C = \{\kappa \cdot \xi \colon \xi < \alpha^2\}$. Then C is a club in $\kappa \cdot \alpha^2$ and $\kappa \cdot \alpha^2 - C$ consists of $|\alpha^2| = |\alpha|$ many maximal disjoint open intervals each of which is a copy of the ordinal κ. We refer to these maximal open intervals as *κ-blocks.*

For $\kappa \cdot \alpha \cdot 2$ we let $D = \{\kappa \cdot \alpha + \kappa \cdot \xi \colon \xi < \alpha\}$. Then $\kappa \cdot \alpha \cdot 2 - D$ consists of a copy of $\kappa \cdot \alpha$ and $|\alpha|$ many κ-blocks. Now since $\mathrm{ot}(C) \leq \kappa \cdot \alpha$, C can be Borel embedded into the copy of $\kappa \cdot \alpha$. Since there are the same number of κ-blocks in the remaining parts of the two ordinals, they can be paired off. Lemma 2.14 gives the desired Borel embedding.

Second, we show $(\kappa \cdot \alpha) \cdot 2 \hookrightarrow \kappa \cdot \alpha$. Let $C_1 = \{\kappa \cdot \xi \colon \xi < \alpha\}$ and let $C_2 = \{\kappa \cdot \alpha + \kappa \cdot \xi \colon \xi < \alpha\}$. Since $\mathrm{ot}(C_1) = \mathrm{ot}(C_2) = \alpha \leq \kappa$, we can embed C_1 into the first κ-block of $\kappa \cdot \alpha$, and C_2 into the second κ-block of $\kappa \cdot \alpha$. Now we are in a position to apply Lemma 2.14 again, since there are again the same number $|\alpha \cdot 2| = |\alpha|$ of κ-blocks in the remaining part of the two ordinals. $\square$

THEOREM 2.17. *If $\kappa \cdot \mathrm{cof}(\kappa) \leq \alpha < \kappa^+$, then $\alpha \cong_B \kappa \cdot \mathrm{cof}(\kappa)$.*

PROOF. We prove by induction that α can be partitioned into countably many Borel subsets $A_0, A_1, \ldots$ such that each A_n embeds into $\kappa \cdot \mathrm{cof}(\kappa)$. This gives a Borel embedding of α into $\kappa \cdot \mathrm{cof}(\kappa) \cdot \omega$, which embeds into $\kappa \cdot \mathrm{cof}(\kappa)^2$ and hence in $\kappa \cdot \mathrm{cof}(\kappa)$ by the preceding lemma.

The statement is certainly true for $\alpha = \kappa \cdot \mathrm{cof}(\kappa)$. The successor case is also easy. We assume α is a limit ordinal. Let $C = \{x_\beta \colon \beta < \mathrm{cof}(\alpha)\}$ be a club in α, with $x_0 = 0$. Since $\mathrm{cof}(\alpha) \leq \kappa$ (because $\alpha < \kappa^+$), C can be embedded into κ and thus in $\kappa \cdot \mathrm{cof}(\kappa)$. For each $\beta < \mathrm{cof}(\alpha)$ let $I_\beta = (x_\beta, x_{\beta+1})$. The I_β's are pairwise disjoint open subsets of α such that $\alpha - C = \bigcup_{\beta < \mathrm{cof}(\alpha)} I_\beta$. Also for each $\beta < \mathrm{cof}(\alpha)$, I_β is a copy of an ordinal $< \alpha$. Thus by the inductive hypothesis, for every $\beta < \mathrm{cof}(\alpha)$ there is a pairwise disjoint family $\{A'_{\beta,n} \colon n \in \mathbb{N}\}$ such that $I_\beta = \bigcup_{n<\omega} A'_{\beta,n}$, every $A'_{\beta,n}$ is Borel in I_β and there is a Borel embedding $\varphi_{\beta,n} \colon A'_{\beta,n} \to \kappa \cdot \mathrm{cof}(\kappa)$.

Define $A'_n := \bigcup_{\beta < \mathrm{cof}(\alpha)} A'_{\beta,n}$. Since each $A'_n \cap I_\beta = A'_{\beta,n}$ is Borel in I_β, A'_n is Borel in α by the gluing lemma. Also for every $n < \omega$, $A'_n = \bigcup_{\beta < \mathrm{cof}(\alpha)} A'_{\beta,n}$ is Borel embeddable in

$\kappa \cdot \mathrm{cof}(\kappa) \cdot \mathrm{cof}(\kappa)$, and thus A'_n embeds into $\kappa \cdot \mathrm{cof}(\kappa)$ by the preceding lemma. Then $A_0 = C$, $A_{n+1} = A'_n$ is the required decomposition of α. $\square$

Thus between any cardinal κ and its successor κ^+ there are no new isomorphism types after $\kappa \cdot \mathrm{cof}(\kappa)$. For singular κ, there is in fact only one isomorphism type:

THEOREM 2.18. *If κ is singular and $\kappa \leq \alpha < \kappa^+$, then $\alpha \cong_B \kappa$.*

PROOF. In view of Theorem 2.17 it suffices to prove that $\kappa \cdot \mathrm{cof}(\kappa) \cong_B \kappa$. Fix a club-in-$\kappa$ sequence $\langle \lambda_\zeta \colon \zeta < \mathrm{cof}(\kappa) \rangle$ of cardinals such that $\mathrm{cof}(\kappa)^2 \leq \lambda_\zeta < \kappa$. Let

$$C = \{\kappa \cdot \xi \colon \xi < \mathrm{cof}(\kappa)\} \cup \bigcup_{\xi < \mathrm{cof}(\kappa)} \{\kappa \cdot \xi + \lambda_\zeta \colon \zeta < \mathrm{cof}(\kappa)\}.$$

This is a club in $\kappa \cdot \mathrm{cof}(\kappa)$ of order type $\mathrm{cof}(\kappa)^2$. Again $\kappa \cdot \mathrm{cof}(\kappa) - C$ can be written as a union of $|\mathrm{cof}(\kappa)^2| = \mathrm{cof}(\kappa)$ many maximal disjoint open intervals, or *blocks*, each of which is a copy of some λ_ζ. Moreover, for each $\zeta < \mathrm{cof}(\kappa)$ there are exactly $\mathrm{cof}(\kappa)$ many λ_ζ-blocks.

On the other hand, $D = \{\lambda_\zeta \colon \zeta < \mathrm{cof}(\kappa)\}$ is a club in κ of order type $\mathrm{cof}(\kappa)$, and $\kappa - D$ is the union of $\mathrm{cof}(\kappa)$ many blocks each of which is a copy of some λ_ζ. However, for each $\zeta < \mathrm{cof}(\kappa)$ there is exactly one λ_ζ-block in $\kappa - D$, which we denote by B_ζ.

We now define a Borel embedding from $\kappa \cdot \mathrm{cof}(\kappa)$ into κ in view of Lemma 2.14. First note that C embed into B_0 since $\lambda_0 \geq \mathrm{cof}(\kappa)^2$. Then for each $\zeta < \mathrm{cof}(\kappa)$ we let all $\mathrm{cof}(\kappa)$ many λ_ζ-blocks in $\kappa \cdot \mathrm{cof}(\kappa)$ embed into the $\lambda_{\zeta+1}$-block $B_{\lambda_{\zeta+1}}$ of κ. This is possible since $\lambda_{\zeta+1} > \lambda_\zeta, \mathrm{cof}(\kappa)$ is a cardinal. $\square$

Finally, we consider ordinals between κ and κ^2 when κ is a regular uncountable cardinal. Any such ordinal can be written as $\kappa \cdot \alpha + \beta$ with $0 < \alpha \leq \kappa$ and $0 \leq \beta < \kappa \cdot \alpha$.

LEMMA 2.19. *If $\beta < \kappa \cdot \alpha$, then $\kappa \cdot \alpha + \beta \cong_B \kappa \cdot \alpha$.*

PROOF. This is immediate when β is finite, so assume β is infinite. In this case $\kappa \cdot \alpha + \beta = \kappa \cdot \alpha + 1 + \beta$ is the disjoint union of the open sets $[0, \kappa \cdot \alpha + 1)$ and $(\kappa \cdot \alpha, \kappa \cdot \alpha + \beta)$. In other words, $\kappa \cdot \alpha + \beta$ is homeomorphic to the direct sum $(\kappa \cdot \alpha + 1) \oplus \beta$. Replacing β with the

Borel isomorphic $\beta + 1$, we are allowed to transpose the disjoint open parts:

$$(\kappa \cdot \alpha + 1) \oplus \beta \cong_B (\kappa \cdot \alpha + 1) \oplus (\beta + 1) \cong (\beta + 1) \oplus (\kappa \cdot \alpha + 1).$$

Finally, $(\beta + 1) \oplus (\kappa \cdot \alpha + 1) \cong \beta + 1 + \kappa \cdot \alpha + 1 = \kappa \cdot \alpha + 1 \cong_B \kappa \cdot \alpha$. □

We can therefore restrict our attention to ordinals of the form $\kappa \cdot \alpha$ for $0 < \alpha \leq \kappa$. It follows immediately from Lemma 2.14 that $\kappa \cdot \alpha \cong_B \kappa \cdot \beta$ whenever $|\alpha| = |\beta|$. To motivate the converse, suppose towards a contradiction that θ is a Borel isomorphism between $\omega_1 \cdot 2$ and ω_1. The larger ordinal $\omega_1 \cdot 2$ consists of two copies B_1, B_2 of ω_1 (and a limit point), while the smaller ordinal ω_1 has only one block. Each of the copies is Borel in $\omega_1 \cdot 2$ and therefore so are their images $\theta``B_1$ and $\theta``B_2$. By Lemma 2.7, both images either contain or omit a club. Since $\theta``B_1$ and $\theta``B_2$ are disjoint, and any two clubs meet, one of the images, say $\theta``B_1$ must omit a club $C$. This closed set splits $\omega_1$ into open blocks. One can construct a stationary and costationary $S \subseteq B_1$ such that $\theta``S$ contains at most one point in each block. Hence, $\theta``S$ is Borel in ω_1 by the gluing lemma, but S is not Borel in B_1 and hence not in $\omega_1 \cdot 2$, a contradiction. The argument in the proof of the following theorem is a generalization of this idea.

THEOREM 2.20. *Let κ be a regular uncountable cardinal and let $\alpha < \beta \leq \kappa$. If $|\alpha| \neq |\beta|$, then $\kappa \cdot \alpha \ncong_B \kappa \cdot \beta$.*

PROOF. We may assume without loss of generality that $|\alpha| < |\beta|$. Let $C_0 = \{\kappa \cdot \xi \colon \xi < \alpha\}$ and $D_0 = \{\kappa \cdot \xi \colon \xi < \beta\}$. It follows from Lemma 2.14 that $\kappa \cdot \alpha \cong_B \kappa \cdot \alpha - C_0$ and $\kappa \cdot \beta \cong_B \kappa \cdot \beta - D_0$. Thus it suffices to show that $\kappa \cdot \alpha - C_0 \ncong_B \kappa \cdot \beta - D_0$.

Toward a contradiction we assume that $\theta \colon \kappa \cdot \beta - D_0 \to \kappa \cdot \alpha - C_0$ is a Borel isomorphism. As before $\kappa \cdot \alpha - C_0$ consists of $|\alpha|$ many κ-blocks, which we denote in increasing order by A_ζ for $\zeta < \alpha$. Similarly $\kappa \cdot \beta - D_0$ consists of $|\beta|$ many κ-blocks, which we denote in increasing order by B_ξ for $\xi < \beta$.

CLAIM 2.21. *There is a $\xi < \beta$ such that for every $\zeta < \alpha$, $A_\zeta \cap \theta``B_\xi$ is nonstationary in A_ζ.*

PROOF. Note that the κ-blocks A_ζ, B_ξ are open. For every $\xi < \beta$, $\theta``B_\xi$ is Borel in $\kappa \cdot \alpha$, thus for every $\zeta < \alpha$, $A_\zeta \cap \theta``B_\xi$ is Borel in $\kappa \cdot \alpha$ and thus in A_ζ. But A_ζ is a copy of the regular κ, hence $A_\zeta \cap \theta``B_\xi$ must either contain or miss a club in $A_\zeta$ by Lemma 2.7. Since two clubs necessarily meet, for every $\zeta < \alpha$ there can be at most one $\xi < \beta$ such that $A_\zeta \cap \theta``B_\xi$ contains a club in A_ζ. Because $|\alpha| < |\beta|$, there must be a $\xi < \beta$ such that for every $\zeta < \alpha$, $A_\zeta \cap \theta``B_\xi$ is nonstationary in A_ζ. □

CLAIM 2.22. *There is a stationary $S \subseteq B_\xi$ such that $\theta``S \subseteq A_\zeta$ for some $\zeta < \alpha$.*

PROOF. For each $\zeta < \alpha$ let $B_{\xi,\zeta} = B_\xi \cap \theta^{-1}(A_\zeta)$. Then $B_\xi = \bigcup_{\zeta<\alpha} B_{\xi,\zeta}$. Since B_ξ is a copy of κ and $\alpha < \kappa$, it follows from the regularity of κ that B_ξ is not the union of $|\alpha|$ many nonstationary sets. Hence, there must be a $\zeta < \alpha$ such that $B_{\xi,\zeta}$ is stationary. This stationary set $S = B_{\xi,\zeta}$ has the required property. □

We now have a stationary set $S \subseteq B_\xi$ such that $\theta``S$ is entirely contained in $A_\zeta$. Since $\theta``B_\xi$ is nonstationary on every κ-block of $\kappa \cdot \alpha$, $\theta``S$ is nonstationary in A_ζ. Note that both B_ξ and A_ζ are copies of the regular cardinal κ.

Let C be a club in A_ζ such that $\theta``S \cap C = \varnothing$. Then $A_\zeta - C$ can be written as the disjoint union of maximal open intervals, say $A_\zeta - C = \bigcup_{i\in\kappa} U_i = \bigcup_{i\in\kappa}(\gamma_i, \gamma_{i+1})$. Note that $\theta``S \subseteq \bigcup_{i\in\kappa} U_i$.

CLAIM 2.23. *There is an $S_1 \subseteq S$ which is stationary and costationary in B_ξ such that $\theta``S_1 \cap U_i$ is Borel in U_i for every $i \in \kappa$.*

PROOF. For any $x \in S$, denote by $\mathrm{block}(x) \in \kappa$ the index of the block that $\theta(x)$ is in, that is, $\theta(x) \in U_{\mathrm{block}(x)}$. We will construct a club D such that for $x, y \in D \cap S$ with $x \neq y$, $\mathrm{block}(x) \neq \mathrm{block}(y)$. Then $S_0 := D \cap S$ is a stationary set such that $|\theta``S_0 \cap U_i| \leq 1$. This trivially implies that $\theta``S_0 \cap U_i$ is Borel in U_i for every $i \in \kappa$. Furthermore, let $S_1 \subseteq S_0$ be any stationary and costationary subset. Then $\theta``S_1 \cap U_i$ is Borel in U_i for every $i \in \kappa$.

To construct this club D, we define a function $g\colon B_\xi \to B_\xi$ and then let D be the set of closure points of g, that is, $D = \{\alpha \in B_\xi \colon \forall \beta < \alpha\ (g(\beta) < \alpha)\}$. Let $x \in B_\xi$ be arbitrary. Let $B = \{\text{block}(z) \colon z \in S \wedge z \leq x\}$. Since κ is regular, B is bounded in κ. Let $g(x) = \sup\{x' \in B_\xi \cap S \colon \text{block}(x') \in B\}$. Since κ is regular and θ is one-to-one, $g(x) \in B_\xi$. To see this works, suppose $x, y \in S_0 = D \cap S$ with $x < y$. Since $y \in D$, $g(x) < y$. Thus, $\text{block}(y) \notin \{\text{block}(z) \colon z \leq x \wedge z \in S\}$, a set which includes $\text{block}(x)$. $\square$

Since $\theta``S_1 \cap U_i$ is Borel in $U_i$ for every $i < \kappa$, $\theta``S_1$ is Borel in $\bigcup_{i<\kappa} U_i$ by the gluing lemma, and hence $\theta``S_1$ is Borel in A_ζ and also in $\kappa \cdot \alpha$. But S_1 is not Borel in B_ξ by Lemma 2.7 and thus not Borel in $\kappa \cdot \beta$. This contradicts the assumption that θ is a Borel embedding. $\square$

This completes the proof of Theorem 2.3: if κ is singular or countable, all ordinals between κ and κ^+ are Borel isomorphic by Theorems 2.17 and 2.18, and if κ is regular and uncountable, the Borel isomorphism types are precisely $\kappa \cdot \lambda$ for cardinals $1 \leq \lambda \leq \kappa$ by Theorems 2.17 and 2.20.

CHAPTER 3

RESOLVABLE MAPS PRESERVE COMPLETE METRIZABILITY

A *Polish space* is a separable, completely metrizable space. Polish spaces are abundant in mathematics. Familiar examples are separable Banach spaces (e.g. $\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, ℓ^p), the Baire space ω^ω, and the Cantor space 2^ω. These spaces are the natural setting for analysis and descriptive set theory. It is therefore of interest to find criteria which imply that a separable metrizable space is in fact completely metrizable.

Consider a continuous surjection $f\colon X \to Y$ between separable metrizable spaces X and Y. We say that f *preserves complete metrizability* if Y is completely metrizable whenever X is completely metrizable. A natural question to ask is which maps preserve complete metrizability.

Some restrictions need to be imposed on the map $f\colon X \to Y$, since there exist continuous surjections from Polish spaces onto separable metrizable, but not completely metrizable spaces.

Example 3.1. Enumerate $\mathbb{Q}$ as $q_0, q_1, \dots$ and define $f\colon \omega^\omega \to \mathbb{Q}$ by $f(x) = q_{x(0)}$. Clearly, f is a continuous surjection which does not preserve metrizability.

In fact, Michael and Stone [21] proved that if there is a continuous surjection from ω^ω onto a metrizable space X, then there also is a quotient map from ω^ω onto X. In other words, quotient maps need not preserve complete metrizability.

On the positive side, Sierpiński [25] (see also Hausdorff [11]) showed that open maps preserve complete metrizability. Similarly, Vainštein [27, 28] proved that closed maps preserve complete metrizability. We provide alternative proofs of these theorems (see Theorems 3.16 and 3.17). There has been much work on other kinds of maps. Recently, Ostrovsky [23] obtained the following result: if the image of every open set or every closed set is the union of an open and a closed set, then the map preserves complete metrizability. He raised the question whether the same is true when the images are the intersection of an open set and a closed set. In this chapter we will answer Ostrovsky's question in the affirmative by proving a generalization of his result (see Theorem 3.44). In some technical sense (to be made precise), this theorem is the best possible result along these lines.

The chapter is organized as follows. In Section 3.1 we review the well-known fact that the range of a continuous surjection which sends open (closed) sets to Borel sets is an absolute Borel space (see Proposition 3.4), in particular a coanalytic space. In Section 3.2 we prove Hurewicz's classical results that complete Baireness and complete metrizability coincide for coanalytic spaces (see Theorem 3.14), and that a coanalytic space is completely Baire if and only if it contains no closed subspace homeomorphic to $\mathbb{Q}$ (see Theorem 3.8). This provides a useful criterion for proving that a maps preserves complete metrizability. In Section 3.3 we use this criterion to derive Sierpiński's and Vainštein's results (see Theorems 3.16 and 3.17).

In order to prove our main result, we study Hausdorff's difference hierarchy. In Section 3.4 we review its basic structure. In Section 3.5 we explictly link the finite levels of the difference hierarchy to Ostrovsky's question. In Section 3.6 we characterize the nonresolvable subsets of $\mathbb{Q}$ and prove that the difference hierarchy over the open sets is proper. Finally, we turn to our main result. In Section 3.7 we show that there cannot be a continuous surjection $f\colon \omega^\omega \to \mathbb{Q}$ which maps clopen sets to resolvable sets. In Section 3.8 we prove that every resolvable continuous surjection preserves complete metrizability (see Theorem 3.44).

While preparing the results contained in Sections 3.6, 3.7, and 3.8 for publication, we obtained a preprint by Holický and Pol [12] who independently proved a result corresponding to Theorem 3.44.

3.1. Absolute Borel Spaces

A metrizable space X is an *absolute Borel space* if for every metrizable space Y and every homeomorphic embedding $j\colon X \to Y$, $j(X)$ is a Borel subset of Y. Every completely metrizable space is absolute G_δ. To see this, let X be a completely metrizable space, Y a metrizable space, and $j\colon X \to Y$ a homeomorphic embedding. Then $j(X)$ is a completely metrizable subspace of a metric space Y, and hence is G_δ in Y. Conversely, if a metrizable space X is absolute G_δ, then X is completely metrizable. To see this, fix a compatible metric d on X and let $\hat{X}$ be the completion of X with respect to d. The map $j\colon X \to \hat{X}$ given by $j(x) = (x)_{n\in\omega}$ is a homeomorphic embedding. By assumption, X is G_δ in $\hat{X}$, and G_δ subspaces of completely metrizable spaces are completely metrizable.

The following lemma provides several conditions equivalent to absolute Borelness for separable metrizable spaces.

LEMMA 3.2. *Let (X, d) be a separable metric space and $\hat{X}$ its completion. The following are equivalent:*

(i) *X is an absolute Borel space;*

(ii) *X is Borel in $\hat{X}$;*

(iii) *There is a Polish space Z and a continuous bijection $\theta\colon Z \to X$;*

(iv) *There is a Polish space Z and a Borel bijection $\theta\colon Z \to X$.*

(v) *There is a separable absolute Borel space Z and a Borel bijection $\theta\colon Z \to X$.*

PROOF. The implications (i)⇒(ii) and (iii)⇒(iv)⇒(v) are obvious. For (ii)⇒(iii), note that $\hat{X}$ is Polish and $X \subseteq \hat{X}$. By the change of topology technique, there is a finer Polish topology τ on $\hat{X}$ so that X is τ-clopen. Thus $\tau \restriction X$ is Polish. Let Z be X equipped with $\tau \restriction X$ and $\theta\colon Z \to X$ be the identity map. Then θ is continuous. We next show (iv)⇒(i). For this let Y be a metric space and $j\colon X \to Y$ a homeomorphic embedding. Let $\hat{Y}$ be the completion of Y and S the closure of $j(X)$ in $\hat{Y}$. Since X is second countable, so are $j(X)$ and S; therefore S is a Polish space. Now $j \circ \theta\colon Z \to S$ is a Borel injection from a Polish space Z into S with image $j(X)$. It follows from the Luzin-Suslin theorem that $j(X)$ is Borel in S. Since S is closed in $\hat{Y}$, it follows that $j(X)$ is Borel in $\hat{Y}$ and hence in Y. We have established the equivalence of (i) through (iv). To finish the proof, it suffices to show (v)⇒(iv). Assume Z is a separable absolute Borel space and $\theta\colon Z \to X$ a Borel bijection. Applying (i)⇒(iv) for Z, we obtain a Polish space W and a Borel bijection $\eta : W \to Z$. Then $\theta \circ \eta$ is a Borel bijection from W onto X. $\square$

We say that a map $f\colon X \to Y$ is *open-Borel* if $f(U)$ is Borel in Y for every open $U \subseteq X$. We will show that continuous images of Polish spaces under open-Borel maps are absolute Borel spaces. We will make use of the Effros Borel structure. For a Polish space X, let $F^*(X)$ denote the set of all nonempty closed subsets of X. The *Effros Borel structure* on $F^*(X)$ is the σ-algebra generated by the sets

$$\{F \in F(X) \colon F \cap U \neq \varnothing\},$$

where $U \subseteq X$ is open. Beer [3] has shown that $F^*(X)$ with the Effros Borel structure is a standard Borel space, i.e. there is a Polish topology on $F^*(X)$ whose Borel structure is precisely the Effros Borel structure. The following selection theorem is a basic fact about $F^*(X)$ with the Effros Borel space.

THEOREM 3.3 (Kuratowski–Ryll-Nardzewski). *There is a Borel function $\sigma\colon F^*(X) \to X$ such that $\sigma(F) \in F$ for every $F \in F^*(X)$.* $\square$

In fact, Kuratowski–Ryll-Nardzewski proved a stronger result, see [15, Theorem 12.13].

PROPOSITION 3.4. *Let X be a Polish space, Y a separable metrizable space, and $f\colon X \to Y$ a continuous surjection. If the image under f of every open set in X is Borel in Y, then Y is an absolute Borel space.*

PROOF. Let E be the equivalence relation on X defined by $x_1 \; E \; x_2$ iff $f(x_1) = f(x_2)$. Since f is continuous, every E-equivalence class is closed. The condition is equivalent to the statement that the E-saturation of every open set is Borel, since $[U]_E = f^{-1}(f(U))$. Consider the map $\theta\colon X \to F^*(X)$ defined by $\theta(x) = [x]_E = f^{-1}(f(x))$. Then θ is Borel since for any nonempty open $U \subseteq X$,

$$\theta(x) \cap U \neq \varnothing \iff x \in [U]_E.$$

Let $\sigma\colon F^*(X) \to X$ be the Borel selector function given by the Kuratowski–Ryll-Nardzewski theorem. Then $\sigma \circ \theta\colon X \to X$ is a Borel selector for E. Let $A = \sigma \circ \theta(X)$. Then A is a Borel transversal for E; it is Borel since $x \in A$ iff $\sigma \circ \theta(x) = x$. Thus A is a separable absolute Borel space and $f \restriction A\colon A \to Y$ is a continuous bijection. It follows that Y is an absolute Borel space. $\square$

We say that a map $f\colon X \to Y$ is *closed-Borel* if $f(U)$ is Borel in Y for every closed $U \subseteq X$. To prove Proposition 3.4 for closed-Borel maps, we use the following result of Engelking [6]: for every Polish space X there is a closed continuous surjection from ω^ω onto X.

PROPOSITION 3.5. *Let X be a Polish space, Y a separable metrizable space, and $f\colon X \to Y$ a continuous surjection. If the image under f of every closed set in X is Borel in Y, then Y is an absolute Borel space.*

PROOF. Let $g\colon \omega^\omega \to X$ be a closed continuous surjection from ω^ω onto X. The composition $f \circ g\colon \omega^\omega \to Y$ is continuous, and the image under $f \circ g$ of every closed (in particular, clopen) set in ω^ω is Borel in Y. This implies that the image under $f \circ g$ of every open set in ω^ω is Borel in Y, since every open set in ω^ω is a countable union of clopen sets. Applying Proposition 3.4 to $f \circ g$, we conclude that Y is an absolute Borel space. $\square$

We have thus shown that the continuous image of a Polish space under an open-Borel or closed-Borel map is an absolute Borel space.

3.2. Completely Baire Spaces

A topological space is *Baire* iff every open set is nonmeager in itself. The Baire Category Theorem is the statement that completely metrizable spaces and locally compact Hausdorff spaces are Baire; the rationals $\mathbb{Q}$ are an example of a separable metrizable space which is not Baire. It is easy to see that an open subspace of a Baire space is a Baire space in the subspace topology. However, a closed subspace of a Baire space is not necessarily Baire. When every closed subspace of X is Baire, we say that X is *completely Baire*. Clearly, every completely Baire space is Baire. For an example of a Baire space which is not completely Baire, consider

$$X = \{(x, y) \in \mathbb{R}^2 \colon (y = 0 \text{ and } x \in \mathbb{Q}) \text{ or } (0 \leq x \leq 1 \text{ and } 0 < y \leq 1)\}$$

with the subspace topology inherited from $\mathbb{R}^2$. The space has a closed subset homeomorphic to $\mathbb{Q}$, so it is not completely Baire. It is easy to check that X is Baire. The following theorem gives several characterizations of complete Baireness for separable metrizable spaces.

THEOREM 3.6 (Hurewicz [13]). *Let X be a separable metrizable space. The following are equivalent:*

(a) *X is completely Baire.*

(b) *Every closed subset of X is nonmeager in itself.*

(c) *Every G_δ subset of X is nonmeager in itself.*

PROOF. The implications (c) $\Rightarrow$ (a) $\Rightarrow$ (b) are clear. We show that (b) $\Rightarrow$ (c), and therefore (a), (b), and (c) are all equivalent. Assume that there is a G_δ set Y which is meager in itself. We claim that $\overline{Y}$ is meager in itself. Without loss of generality we may assume that $\overline{Y} = X$. Write $Y = \bigcup_{n\in\omega} Y_n$, where each Y_n is closed nowhere dense in Y.

CLAIM 3.7. *Y is meager in X.*

PROOF. Since $Y \subseteq \bigcup_{n\in\omega} \overline{Y}_n$, it suffices to check that each $\overline{Y}_n$ is nowhere dense in $\overline{Y}$. Suppose that $W \subseteq \overline{Y}$ is open in $\overline{Y}$ such that $W \subseteq \overline{Y}_n$ for some n. Then

$$W \cap Y \subseteq \overline{Y}_n \cap Y = Y_n,$$

contradicting the assumption that Y_n is nowhere dense in Y. □

Now to show that X is meager in itself, it suffices to show that $X - Y$ is meager in X. Since Y is G_δ, we can write

$$Y = \bigcap_{n\in\omega} U_n$$

for $U_n \subseteq X$ open. Now

$$X - Y = X - \bigcap_{n\in\omega} U_n = \bigcup_{n\in\omega} (X - U_n).$$

Since each U_n is open dense (dense because it contains a dense subset Y), each $X - U_n$ is closed nowhere dense. This shows that $X - Y$ is meager, and completes the proof of (b) $\Rightarrow$ (c). $\square$

Fréchet and Sierpiński (c.f. [15, Exercise 7.12]) have proved that every countable perfect space is homeomorphic to $\mathbb{Q}$. A separable metrizable space which contains a closed subspace homeomorphic to $\mathbb{Q}$ is certainly not completely Baire. In fact, this turns out to be the canonical obstruction to being completely Baire.

THEOREM 3.8 (Hurewicz [13]). *A separable metrizable space X is completely Baire if and only if X does not contain a countable perfect subset (i.e. a closed subspace homeomorphic to $\mathbb{Q}$).*

PROOF. The forward direction is trivial, since a closed subspace homeomorphic to $\mathbb{Q}$ is not Baire. We prove the backward direction by contraposition. Suppose X is not completely Baire. Then X has a closed subset that is meager in itself. Without loss of generality we may assume that X is the closed subset that is meager in itself. Let

$$X = \bigcup_{n\in\omega} F_n,$$

where each F_n is closed nowhere dense. We construct $(p_s)_{s\in\omega^{<\omega}}$ and $(U_s)_{s\in\omega^{<\omega}}$ such that the following properties hold:

(i) $p_s \in U_s$,

(ii) $\operatorname{diam}(U_s) < 2^{-\operatorname{lh}(s)}$,

(iii) $U_{s^\frown n} \subseteq \overline{U}_{s^\frown n} \subseteq U_s$,

(iv) $\lim_{n\to\infty} p_{s^\frown n} = p_s$,

(v) if $\operatorname{lh}(s) = n + 1$, then $\overline{U}_s \cap (F_0 \cup F_1 \cup \cdots \cup F_n) = \varnothing$.

Without loss of generality we may assume $\mathrm{diam}(X) < 1$. Let $U_\varnothing = X$ and $p_\varnothing \in X$ be arbitrary. Suppose p_s and U_s are defined for some $s \in \omega^{<\omega}$ with $\mathrm{lh}(s) = k$. Since $\bigcup_{m>k} F_m$ is dense (because $F_0 \cup \cdots \cup F_k$ is closed nowhere dense) and $p_s \in U_s$, we may find a sequence $q_n \in U_s \cap \bigcup_{m>k} F_m$ so that $\lim q_n = p_s$. Let $p_{s^\frown n} = q_n$, and let $U_{s^\frown n}$ be an open set such that

$$p_{s^\frown n} \in U_{s^\frown n} \subseteq \overline{U}_{s^\frown n} \subseteq U_s$$

$\mathrm{diam}(U_{s^\frown n}) < 2^{-(k+1)}$, and $\overline{U}_{s^\frown n} \cap (F_0 \cup \cdots \cup F_k) = \varnothing$. This finishes the construction.

Now let $Q = \{p_s \colon s \in \omega^{<\omega}\}$. Then Q is countable and has no isolated points. It suffices to verify that D is closed. Suppose $(x_n)_{n\in\omega}$ is a sequence in Q with $\lim_{n\to\infty} x_n = x_\infty$. Consider the tree

$$T = \{s \in \omega^{<\omega} \colon \exists t \supseteq s\ \exists n\ x_n = p_t\}.$$

We consider two cases.

Case 1: T is finitely splitting at every node. In this case, by König's lemma, T has an infinite branch. That is to say, there is a subsequence of x_n and $z \in \omega^\omega$ such that $x_n = p_{z\restriction l_n}$ for some increasing sequence $l_n \in \omega$. It follows that for any k the subsequence of x_n given by $p_{z\restriction l_n}$ where $l_n \geq k+2$ satisfies that for $s = z \restriction k+1$, $x_n \in \overline{U}_s \subseteq X - (F_0 \cup \cdots \cup F_k)$. It follows that $x_\infty \notin F_0 \cup \cdots \cup F_k$ for all k. But $\bigcap_k (X - (F_0 \cup \cdots \cup F_k)) = \varnothing$, so this is a contradiction.

Case 2: T is infinitely splitting at some node, i.e., there is an $s \in T$ such that $s^\frown n \in T$ for infinitely many n. It follows that for an subsequence of $(x_n)_{n\in\omega}$ we have $x_n \in U_{s^\frown m_n}$. By the construction of the open sets $U_{s^\frown n}$ any sequence $(t_n)_{n\in\omega}$ with $t_n \in U_{s^\frown n}$ converges to p_s. Thus, we know that $\lim_{n\to\infty} x_n = p_s$, and thus $p_s = x_\infty$. This shows that $x_\infty \in Q$. □

The above proof can be slightly modified to work for any first countable, regular space X. Note that for a first countable, regular space a countable perfect subset is in fact separable metrizable and therefore homeomorphic to $\mathbb{Q}$. Debs [4], apparently unaware that Hurewicz's original proof can be easily adapted, gave a proof of this using Choquet games. The Dutch mathematician Van Douwen [5] also published a proof of this result; at that time Van Douwen was at the University of North Texas.

Every separable, metrizable, completely metrizable space (i.e., every Polish space) is completely Baire, but the converse is not true in general. Hurewicz [13] gives the following example, using the Axiom of Choice (AC). Recall that using AC one can construct an uncountable set $A \subseteq \mathbb{R}$ such that A does not contain any perfect subset.

THEOREM 3.9 (Bernstein, AC). *There exists an uncountable set $A \subseteq \mathbb{R}$ such that neither A nor $\mathbb{R} - A$ contains a perfect subset.*

PROOF. Let P_α ($\alpha < 2^{\aleph_0}$) enumerate all perfect subsets of $\mathbb{R}$. Each P_α is uncountable, and in fact has cardinality $2^{\aleph_0}$. Define two sets A_α and B_α by induction on $\alpha < 2^{\aleph_0}$. Set $A_0 = B_0 = \varnothing$. For a general α suppose A_β and B_β have been defined for every $\beta < \alpha$ such that $|A_\beta|, |B_\beta| < 2^{\aleph_0}$ and $A_\beta \subseteq A_\gamma$, $B_\beta \subseteq B_\gamma$ whenever $\beta < \gamma < \alpha$. Since $|P_\alpha| = 2^{\aleph_0}$, there exist $p, q \in P_\alpha - \bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)$. Define $A_\alpha = \bigcup_{\beta<\alpha} A_\beta \cup \{p\}$ and $B_\alpha = \bigcup_{\beta<\alpha} B_\beta \cup \{q\}$. Finally, let $A = \bigcup_{\alpha<2^{\aleph_0}} A_\alpha$ and $B = \bigcup_{\alpha<2^{\aleph_0}} B_\alpha$. Then both A and B are uncountable sets which do not contain a perfect subset. $\square$

A set as in Theorem 3.9 is called a *Bernstein set.* In particular, neither A nor $\mathbb{R} - A$ is analytic, since any uncountable analytic set contains a perfect subset.

THEOREM 3.10 (Hurewicz, AC). *There exists a separable, metrizable, completely Baire space which is not completely metrizable.*

PROOF. Consider $X = \mathbb{R} - A$, where A is a Bernstein set. Clearly, X is separable metrizable. We claim that X is completely Baire, but not completely metrizable. To see that it is not completely metrizable, we argue as follows. If X were completely metrizable, then it is Polish, and therefore a G_δ subset of $\mathbb{R}$; it follows that A would be an F_σ subset of $\mathbb{R}$, a contradiction. It only remains to check that X is completely Baire. We use Theorem 3.8. Assume that X contains a countable perfect subset P. Then $\mathrm{Cl}_{\mathbb{R}}(P)$ is a perfect subset of $\mathbb{R}$, and therefore it has cardinality $2^{\aleph_0}$. It follows that $\mathrm{Cl}_{\mathbb{R}}(P) - P$ is an uncountable G_δ subset, and therefore it contains a perfect subset of $\mathbb{R}$. Since A does not contain any perfect subset of $\mathbb{R}$, we have that $X \cap (\mathrm{Cl}_{\mathbb{R}}(P) - P) \neq \varnothing$. Let $x_0 \in X \cap (\mathrm{Cl}_{\mathbb{R}}(P) - P)$. Then on the one hand, x_0 is a limit point of P (in the topology of $\mathbb{R}$, and so also in the topology of X), and on the other hand, $x_0 \in X - P$, where P is closed in X, a contradiction. $\square$

We have seen that

$$\text{completely metrizable} \Longrightarrow \text{completely Baire} \Longrightarrow \text{Baire}$$

and that these implications cannot be reversed. However, Hurewicz has shown that for definable subsets of Polish spaces, being completely Baire does imply being completely metrizable: if A is a coanalytic subset of a Polish space, then A is completely Baire iff A is completely metrizable. The proof is a modification of the construction in the proof of Theorem 3.8. We need the following definition.

DEFINITION 3.11. Let $M, N \subseteq X$ be subsets of a separable metrizable space X. We say that M is *closed with respect to* N iff $\overline{M} \cap N \subseteq M$.

Since this is not a standard notion, let us compare it to $M \cap N$ being relatively closed in N. If M is closed with respect to N, then $\overline{M \cap N} \cap N \subseteq \overline{M} \cap N \subseteq M \cap N$. Hence, $M \cap N$ is relatively closed in N. When $M \subseteq N$, the two notions coincide, but in general the converse is not true. For example, take $M = [0,1)$ and $N = [1,2)$. Since $M \cap N = \varnothing$, $M \cap N$ is relatively closed in N. However, $\overline{M} \cap N = \{1\} \not\subseteq M$, so M is not closed with respect to N.

DEFINITION 3.12. We say that M is F_σ *with respect to* N iff M is the countable union of sets closed with respect to N. We write $M \equiv F_\sigma$ (wrt N) in this case.

Note that if M is F_σ in X, then $M \equiv F_\sigma$ (wrt N) for any $N \subseteq X$. Also, a countable union $\bigcup_{n \in \omega} M_n$ with each $M_n \equiv F_\sigma$ (wrt N) is again F_σ (wrt N), and so is a finite intersection $M_1 \cap \cdots \cap M_n$. The following is the main technical lemma.

LEMMA 3.13. *Let X be a separable metrizable space, $M, N \subseteq X$ arbitrary subsets, and $U \subseteq X$ an open subset. If $M \cap U \not\equiv F_\sigma$ (wrt N), then there is a $p \in N \cap U$ such that $M \cap V \not\equiv F_\sigma$ (wrt N) for every basic nbhd V of p.*

PROOF. Suppose towards a contradiction for every $p \in N \cap U$ we can pick a basic nbhd V such that $M \cap V \equiv F_\sigma$ (wrt N). Since X is second countable, there are only countably many such nbhds. Enumerate these as $V_1, V_2, \ldots$, and let $V = \bigcup_{n\in\omega} V_n$. Note that $N \cap U \subseteq V$ and

$$M \cap V = \bigcup_{n\in\omega} M \cap V_n \equiv F_\sigma \text{ (wrt } N).$$

Since U is open, U is F_σ in X, thus $U \equiv F_\sigma$ (wrt N). Thus, $U \cap M \cap V \equiv F_\sigma$ (wrt N). Now,

$$U \cap M = (U \cap M \cap V) \cup ((U \cap M) - (U \cap M \cap V)).$$

We claim that

$$(U \cap M) - (U \cap M \cap V) \equiv F_\sigma \text{ (wrt } N).$$

Write $U = \bigcup_{n\in\omega} F_n$, where each F_n is closed in X. Then

$$(U \cap M) - (U \cap M \cap V) = U \cap (M - (M \cap V)) = \bigcup_{n\in\omega} F_n \cap (M - (M \cap V)).$$

Now

$$\overline{F_n \cap (M - (M \cap V))} \subseteq \overline{F_n \cap (X - V)} = F_n \cap (X - V) \subseteq X - V.$$

Since $U \cap N \subseteq V$, we have

$$\overline{F_n \cap (M - (M \cap V))} \cap N = \varnothing.$$

This proves the claim. $\square$

We remark that if $M \cap V \not\equiv F_\sigma$ (wrt N) for every nbhd V of p, then $p \in \overline{M}$. For suppose $p \notin \overline{M}$, then there is a nbhd V of p such that $\overline{M} \cap V = \varnothing$, thus $M \cap V \equiv F_\sigma$ (wrt N).

THEOREM 3.14 (Hurewicz [13]). *Let $N \subseteq X$ be a coanalytic subset of a separable metrizable space X. If N is not G_δ, then N contains a countable perfect set.*

PROOF. Since $M = X - N$ is analytic, there is a Suslin scheme $(A_s)_{s\in\omega^{<\omega}}$ of closed sets such that

$$M = \bigcup_{x\in\omega^\omega} \bigcap_{n\in\omega} A_{x\restriction n}.$$

For $s \in \omega^{<\omega}$, let

$$M_s = \bigcup_{s \subseteq x} \bigcap_{n \in \omega} A_{x \restriction n}.$$

We will recursively define points p_s, integers i_s, and nonempty open sets U_s for $s \in \omega^{<\omega}$ with the following properties:

(i) $p_s \in N \cap U_s$,

(ii) for every nbhd V of p_s, $M_{i_{s_0}, i_{s_0, s_1}, \ldots, i_{s_0, s_1, \ldots, s_k}} \cap V \not\equiv F_\sigma$ (wrt N).

(iii) $\operatorname{diam}(U_s) < 2^{-\operatorname{lh}(s)}$,

(iv) $U_{s^\frown n} \subseteq \overline{U}_{s^\frown n} \subseteq U_s$.

(v) $\lim_{n \to \infty} p_{s^\frown n} = p_s$,

First, note that since N is not G_δ in X, M is not F_σ in X. This implies that $M \not\equiv F_\sigma$(wrt N). To see this, suppose $M = \bigcup_{n \in \omega} M_n$ with $\overline{M}_n \cap N \subseteq M_n$. Then

$$\overline{M_n} = \overline{M}_n \cap X = \overline{M_n} \cap (M \cup N) = \overline{M}_n \cap M \cup \overline{M}_n \cap N \subseteq M \cup M = M.$$

Thus, $M = \bigcup_{n \in \omega} \overline{M}_n$ is F_σ in X, a contradiction.

We start the construction. Without loss of generality we may assume that $\operatorname{diam}(X) < 1$. By Lemma 3.13 with $U = X$, there is a $p \in N$ such that for every basic nbhd V of p, $M \cap V \not\equiv F_\sigma$(wrt N). Let $U_\varnothing = X$ and $p_\varnothing = p$. Let us explicitly do the second step in the construction. By (ii), for each $n \in \omega$,

$$M \cap B(p_\varnothing, 2^{-n}) \not\equiv F_\sigma \text{ (wrt N)},$$

where $B(p_\varnothing, 1/n)$ is a basic nbhd of $p_\varnothing$ with radius $\leq 2^{-n}$. For each $n \in \omega$, because

$$M \cap B(p_\varnothing, 2^{-n}) = \bigcup_{i \in \omega} M_i \cap B(p_\varnothing, 2^{-n}),$$

there must be an index i_n such that

$$M_{i_n} \cap B(p_\varnothing, 2^{-n}) \not\equiv F_\sigma \text{ (wrt } N).$$

Apply Lemma 3.13 with $U = B(p_\varnothing, 2^{-n})$ for each $n \in \omega$ to find a $p_n \in N \cap B(p_\varnothing, 2^{-n})$ such that for every basic nbhd V of p_n, $M_{i_n} \cap V \not\equiv F_\sigma$(wrt N). Note that we have $p = \lim_{n \to \infty} p_n$. Let $U_n = B(p_\varnothing, 2^{-n})$. The general step in the construction is similar.

Let $Q = \{p_s : s \in \omega^{<\omega}\}$. Clearly, $Q \subseteq N$ is countable and has no isolated points. It suffices to verify that Q is closed in N. Suppose $(x_n)_{n\in\omega}$ is a sequence in Q with $\lim_{n\to\infty} x_n = x_\infty \in N$. As in the proof of Theorem 3.8, consider the tree

$$T = \{s \in \omega^{<\omega} : \exists t \supseteq s\ \exists n\ x_n = p_t\}$$

and the following two cases.

Case 1: T is finitely splitting at every node. In this case, by König's lemma, T has an infinite branch. This means that there is a subsequence $(x_{n_i})_{i\in\omega}$ of $(x_n)_{n\in\omega}$ and a $z \in \omega^\omega$ such that $d(x_{n_i}, p_{z\restriction k_i}) < 2^{-i}$ for some increasing sequence $k_i \in \omega$. Note that by (ii), each $p_s \in \overline{M}_{i_{s_0}, i_{s_0,s_1}, \ldots, i_{s_0,s_1,\ldots,s_k}}$ (see the remark after Lemma 3.13). It follows that $d(x_{n_i}, z \restriction k_i) < 2^{-i}$ and $p_{z\restriction k_i} \to x_\infty$ as $i \to \infty$. Writing $z = (z_0, z_1, \ldots)$, we have

$$p_{z\restriction k_i} \in \overline{M}_{i_{z_0}, i_{z_0,z_1}, \ldots, i_{z_0,z_1,\ldots,z_{k_i}}} \subseteq \overline{A}_{i_{z_0}, i_{z_0,z_1}, \ldots, i_{z_0,z_1,\ldots,z_{k_i}}} = A_{i_{z_0}, i_{z_0,z_1}, \ldots, i_{z_0,z_1,\ldots,z_{k_i}}}.$$

Since each A_s is closed, we have

$$x_\infty \in \bigcap_{k\in\omega} A_{i_{z_0}, i_{z_0,z_1}, \ldots, i_{z_0,z_1,\ldots,z_{k_i}}} \subseteq M = X - N,$$

a contradiction.

The argument in the second case is identical to that in the proof of Theorem 3.8. We repeat it here for the convenience of the reader.

Case 2: T is infinitely splitting at some node, i.e., there is an $s \in T$ such that $s^\frown n \in T$ for infinitely many n. It follows that for an subsequence of $(x_n)_{n\in\omega}$ we have $x_n \in U_{s^\frown m_n}$. By the construction of the open sets $U_{s^\frown n}$ any sequence $(t_n)_{n\in\omega}$ with $t_n \in U_{s^\frown n}$ converges to p_s. Thus, we know that $\lim_{n\to\infty} x_n = p_s$, and thus $p_s = x_\infty$. This shows that $x_\infty \in Q$. $\square$

An alternative proof of this theorem can be found in [15] (Theorem 21.18). It was shown to follow from a general separation result of Kechris–Louveau–Woodin. Hurewicz's Theorem 3.14 in particular applies to all absolute Borel spaces. We state the following theorem as a summary.

THEOREM 3.15. *Let X be a separable absolute Borel space. Then the following are equivalent:*

(i) *X is Polish.*

(ii) *X is completely metrizable.*

(iii) *X is absolute G_δ.*

(iv) *X is completely Baire.*

(v) *X does not contain a countable perfect subset.* □

3.3. Open or Closed Continuous Surjections

In this section we use Hurewicz's criterion to give alternative proofs of the theorems of Sierpiński [25] and Vainštein [27, 28]. Recall that a map $f\colon X \to Y$ is *open* if $f(U)$ is open in Y for every open $U \subseteq X$. Similarly, $f\colon X \to Y$ is *closed* if $f(U)$ is closed in Y for every closed $U \subseteq X$. Both open and closed continuous surjections preserve complete metrizability.

THEOREM 3.16 (Sierpiński [25]). *Let X be a Polish space, Y a separable metrizable space, and $f\colon X \to Y$ a continuous surjection. If f is open, then Y is Polish.*

PROOF. In view of Hurewicz's theorem and Proposition 3.4, it suffices to show that Y is completely Baire, or equivalently, that Y does not contain a countable perfect subset.

For this assume $Q \subseteq Y$ is a countable perfect subset of Y. Let $P = f^{-1}(Q)$. Then $P \subseteq X$ is closed and $f \restriction P\colon P \to Q$ is continuous and open. Now P is a Baire space and Q is countable, and it follows that there exists $y \in Q$ so that $f^{-1}(y)$ has a nonempty interior U. Now $f(U) = \{y\}$, implying that y is isolated, a contradiction to the assumption that Q is perfect. □

Another proof can be found in [8, Theorem 2.2.9]. Using the strong Choquet property, Sierpiński's theorem also follows easily from Choquet's characterization of Polish spaces (see [8, Exercise 4.1.5] or [15, Theorem 8.19]). In fact [25] only dealt with the case of Euclidean spaces, and Hausdorff (responding to a request of Kuratowski) gave a proof of the general case in [11]. This is why the theorem is sometimes attributed to Hausdorff.

The following theorem was announced in [27] and proved in [28]. Later, Engelking [7] gave another proof. We use Hurewicz's criterion again; the proof starts out the same as the previous proof.

THEOREM 3.17 (Vainštein [27, 28]). *Let X be Polish, Y a separable metrizable space, and $f\colon X \to Y$ a continuous surjection. If f is closed, then Y is Polish.*

PROOF. In view of Hurewicz's theorem and Proposition 3.4, it suffices to show that Y is completely Baire, or equivalently, that Y does not contain a countable perfect subset.

For this assume $Q \subseteq Y$ is a countable perfect subset of Y. Let $P = f^{-1}(Q)$. Then $P \subseteq X$ is closed and $f \restriction P\colon P \to Q$ is continuous and closed. Now P is a Baire space and Q is countable, and it follows that there exists $y \in Q$ so that $f^{-1}(y)$ has a nonempty interior. Let

$$W = \bigcup_{q \in Q} \operatorname{Int} f^{-1}(q).$$

Then W is nonempty open in P and $f \restriction (P - W)\colon (P - W) \to Q$ is closed.

CLAIM 3.18. *$f \restriction (P - W)$ is onto Q.*

PROOF. Assume that $q \in Q$ is not in $f(P - W)$. Then $\operatorname{Int} f^{-1}(q) = f^{-1}(q)$ and thus $f^{-1}(q)$ is clopen. In particular, $P - f^{-1}(q)$ is closed. Since f is closed, we have that $f(P - f^{-1}(q)) = Q - \{q\}$ is closed. This means that q is isolated in Q, a contradiction to the assumption that Q is perfect. □

Now let $P_0 = P$, $W_0 = W$, $P_1 = P_0 - W_0$, $f_0 = f \restriction P_0$, and $f_1 = f \restriction P_1$. By repeating the above argument we may define P_α, W_α, and f_α for any ordinal α as follows. When α is a successor ordinal the definition is similar to above. When α is a limit ordinal, we let $P_\alpha = \bigcap_{\beta < \alpha} P_\beta$ and $f_\alpha = f \restriction P_\alpha$. Then P_α is closed and $f_\alpha\colon P_\alpha \to Q$ is closed.

CLAIM 3.19. *$f_\alpha \restriction P_\alpha$ is onto Q.*

PROOF. Let $q \in Q$ and consider $\operatorname{Fr}_X f^{-1}(q) := f^{-1}(q) - \operatorname{Int} f^{-1}(q)$. We claim that $\operatorname{Fr}_X f^{-1}(q)$ is compact. From this it follows that $f_\beta^{-1}(q)$ $(\beta < \alpha)$ is a decreasing sequence of compact subsets of X, and hence

$$f_\alpha^{-1}(q) = \bigcap_{\beta < \alpha} f_\beta^{-1}(q) \neq \varnothing.$$

To see that $F := \mathrm{Fr}_X f^{-1}(q)$ is compact, let $(x_n)_{n\in\omega}$ be a sequence in F. Since f is continuous, there is a sequence $(y_n)_{n\in\omega}$ in $X - f^{-1}(q)$ such that $d(x_n, y_n) < 2^{-n}$ and $d(f(y_n), q) < 2^{-n}$. Hence, $(y_n)_{n\in\omega}$ has an accumulation point $y \in f^{-1}(q)$. This implies that y is an accumulation point of $(x_n)_{n\in\omega}$ and that $y \in F$. Thus, F is compact. □

We have thus shown that for all ordinal α the set P_α is closed, $f_\alpha \colon P_\alpha \to Q$ a continuous closed surjection, and W_α is nonempty open in P_α. In particular, we obtain a decreasing ω_1-sequence of closed subsets $(P_\alpha)_{\alpha<\omega_1}$ of P, which contradicts the second countability of P. □

We remark that the following inverse of Sierpiński's theorem is false:

> Let X, Y be separable metrizable spaces and $f \colon X \to Y$ an open continuous surjection. If Y is Polish, then X is Polish.

For a counterexample, consider $X = \mathbb{R} \times \mathbb{Q}$, $Y = \mathbb{R}$, and $f \colon X \to Y$ the projection to the first coordinate. Then f is an open continuous surjection, Y is Polish, X is absolute Borel but not completely Baire (since it obviously contains a countable perfect subset), and hence not Polish.

Extending the results of Sierpiński and Vainštein, Ostrovsky [23] proved the following theorem:

THEOREM 3.20 (Ostrovsky [23]). *Let X be a Polish space, Y a separable metrizable space, and $f \colon X \to Y$ a continuous surjection. If the image under f of every open set or every closed set in X is the union of an open and a closed set in Y, then Y is Polish.* □

He raised the question whether the same is true when the images are the intersection of an open set and a closed set. We will answer this question in Section 3.8. For this, we need several facts about the difference hierarchy, which we study in the next three sections.

3.4. The Difference Hierarchy

An intersection of an open set and a closed set can also be written as the difference of two open sets; a union of an open set and a closed set is the complement of such a difference. Indeed, sets of these forms constitute the second level of the *difference hierarchy* introduced by Hausdorff. More complicated combinations of open and closed sets occur at higher levels (see Section 3.5). The sets in the difference hierarchy are also known as *resolvable sets.* In this section, we review the structure of the difference hierarchy.

Every ordinal θ can be uniquely written as $\theta = \lambda + n$, where $\lambda \leq \theta$ is a limit ordinal and $n \in \omega$. By definition, the *parity* of θ is the parity of n. Let $\theta \geq 1$ be an ordinal and $(A_\eta)_{\eta<\theta}$ an increasing sequence of subsets of a set X. Define the set $D_\theta((A_\eta)_{\eta<\theta})$ by

$$x \in D_\theta((A_\eta)_{\eta<\theta}) \Leftrightarrow x \in \bigcup_{\eta<\theta} A_\eta \text{ and the least } \eta < \theta \text{ with}$$

$$x \in A_\eta \text{ has parity opposite to that of } \theta$$

For example, $D_1(A_0) = A_0$, $D_2(A_0, A_1) = A_1 - A_0$, and $D_3(A_0, A_1, A_2) = (A_2 - A_1) \cup A_0$. Of course, we do allow transfinite sequences:

$$D_\omega((A_n)_{n<\omega}) = \bigcup_{n<\omega} (A_{2n+1} - A_{2n})$$

and

$$D_{\omega+1}((A_n)_{n\leq\omega}) = A_0 \cup \bigcup_{n<\omega} (A_{2n+2} - A_{2n+1}) \cup A_\omega - \bigcup_{n<\omega} A_n.$$

LEMMA 3.21. *Let $(A_\eta)_{\eta<\theta}$ be a sequence of subsets of X. Then*

$$X - D_\theta((A_\eta)_{\eta<\theta}) = D_{\theta+1}((A_\eta)_{\eta<\theta}, X).$$

PROOF. If $x \notin D_\theta((A_\eta)_{\eta<\theta})$, then either $x \notin \bigcup_{\eta<\theta} A_\eta$ (and hence $x \in X - \bigcup_{\eta<\theta} A_\eta$) or else the least $\eta < \theta$ such that $x \in A_\eta$ has the same parity as θ, i.e. the parity opposite of $\theta + 1$. In both cases $x \in D_{\theta+1}((A_\eta)_{\eta<\theta}, X)$. Conversely, if $x \in D_{\theta+1}((A_\eta)_{\eta<\theta}, X)$, then the least $\eta < \theta$ such that $x \in A_\eta$ has parity opposite to that of $\theta + 1$, i.e. the same parity as θ. Hence, $x \notin D_\theta((A_\eta)_{\eta<\theta})$. □

Let X be a topological space. For $1 \leq \xi, \theta < \omega_1$, let

$$D_\theta(\boldsymbol{\Sigma}^0_\xi)(X) = \{D_\theta((A_\eta)_{\eta<\theta}) \colon A_\eta \in \boldsymbol{\Sigma}^0_\xi(X) \text{ for all } \eta < \theta\}$$

and

$$\breve{D}_\theta(\boldsymbol{\Sigma}^0_\xi)(X) = \{A \subseteq X \colon X - A \in D_\theta(\boldsymbol{\Sigma}^0_\xi)\}.$$

When the ambient space is understood, we also write $D_\theta(\boldsymbol{\Sigma}^0_\xi)$ and $\breve{D}_\theta(\boldsymbol{\Sigma}^0_\xi)$. The previous lemma gives us the following picture:

$$\begin{array}{cccc} \boldsymbol{\Sigma}^0_\xi = D_1(\boldsymbol{\Sigma}^0_\xi) & D_2(\boldsymbol{\Sigma}^0_\xi) & D_\theta(\boldsymbol{\Sigma}^0_\xi) & D_\eta(\boldsymbol{\Sigma}^0_\xi) \\ & \cdots & \cdots & \cdots \\ \boldsymbol{\Pi}^0_\xi = \breve{D}_1(\boldsymbol{\Sigma}^0_\xi) & \breve{D}_2(\boldsymbol{\Sigma}^0_\xi) & \breve{D}_\theta(\boldsymbol{\Sigma}^0_\xi) & \breve{D}_\eta(\boldsymbol{\Sigma}^0_\xi) \end{array}$$

where $\theta \leq \eta$ and every class is contained in every class to the right of it. The *difference hierarchy* over $\boldsymbol{\Sigma}^0_\xi$ is

$$\mathrm{DH}(\boldsymbol{\Sigma}^0_\xi)(X) = \bigcup_{\theta<\omega_1} D_\theta(\boldsymbol{\Sigma}^0_\xi)(X).$$

PROPOSITION 3.22. *For a metrizable space X,* $\mathrm{DH}(\boldsymbol{\Sigma}^0_\xi)(X) \subseteq \boldsymbol{\Delta}^0_{\xi+1}(X)$

PROOF. We have $D_\theta(\boldsymbol{\Sigma}^0_\xi) \subseteq \boldsymbol{\Sigma}^0_{\xi+1}$ and $\breve{D}_\theta(\boldsymbol{\Sigma}^0_\xi) \subseteq D_{\theta+1}(\boldsymbol{\Sigma}^0_\xi)$. Therefore, if $A \in D_\theta(\boldsymbol{\Sigma}^0_\xi)(X)$, then $A \in \boldsymbol{\Sigma}^0_{\xi+1}$ and

$$X - A \in \breve{D}_\theta(\boldsymbol{\Sigma}^0_\xi)(X) \subseteq D_{\theta+1}(\boldsymbol{\Sigma}^0_\xi)(X) \subseteq \boldsymbol{\Sigma}^0_{\xi+1}.$$

Hence, $A \in \boldsymbol{\Delta}^0_{\xi+1}(X) = \boldsymbol{\Sigma}^0_{\xi+1}(X) \cap \boldsymbol{\Pi}^0_{\xi+1}(X)$.. □

Hausdorff and Kuratowski showed that the reverse inclusion holds for Polish spaces, i.e. in these spaces the sets in $\mathrm{DH}(\boldsymbol{\Sigma}^0_\xi)$ are exactly the $\boldsymbol{\Delta}^0_{\xi+1}$ sets. Following the proof of this result found in [15, p. 176 – 177], we start by analyzing $\mathrm{DH}(\boldsymbol{\Sigma}^0_1)$.

LEMMA 3.23. *Let $A \subseteq X$ be a subset of a metrizable space X. Then $A \in \mathrm{DH}(\boldsymbol{\Sigma}^0_1)$ if and only if*

$$A = \bigcup_{\eta<\theta} (F_\eta - H_\eta)$$

for some $\theta < \omega_1$ and some decreasing sequence

$$F_0 \supseteq H_0 \supseteq F_1 \supseteq H_0 \supseteq \cdots \supseteq F_\eta \supseteq H_\eta \supseteq \cdots$$

of closed subsets of X.

PROOF. Suppose $A = \bigcup_{\eta<\theta}(F_\eta - H_\eta)$ for $\theta = \lambda + 1$. Let $\theta^* = \lambda + 2n$ and define $A_{\omega\cdot\xi+2k} = X - F_{\omega\cdot\xi+k}$ and $A_{\omega\cdot\xi+2k+1} = X - H_{\omega\cdot\xi+k}$. Then $A = D_{\theta^*}((A_\xi)_{\xi<\theta^*})$. Conversely, if $A = D_{\theta^*}((A_\eta)_{\eta<\theta^*})$ we may assume $\theta^* = \lambda + 2n$ and define F_η, H_η for $\eta < \theta = \lambda + n$ by the same formulas as before. Then $A = \bigcup_{\eta<\theta}(F_\eta - H_\eta)$. □

Let X be a separable metrizable space, $A \subseteq X$ an arbitrary subset, and $F \subseteq X$ a closed subset. The *boundary* of $A \cap F$ in F is given by

$$\partial_F(A) = \overline{(A \cap F)} \cap \overline{((X - A) \cap F)}.$$

Clearly, $\partial_F(A)$ is closed and $\partial_F(\partial_F(A)) \subseteq \partial_F(A)$. Define by transfinite recursion a sequence (F_η) as follows:

$$\begin{aligned} F_0 &= X, \\ F_{\eta+1} &= \partial_{F_\eta}(A), \\ F_\lambda &= \bigcap_{\eta<\lambda} F_\eta \quad \text{for } \lambda \text{ a limit ordinal.} \end{aligned}$$

Since (F_η) is a decreasing sequence of closed sets in a separable space, there is a least $\theta < \omega_1$ such that $F_\theta = F_{\theta+1}$. We call $(F_\eta)_{\eta\leq\theta}$ the *boundary sequence* of A.

PROPOSITION 3.24. *Let X be a separable metrizable space, $A \subseteq X$ an arbitrary subset, and $(F_\eta)_{\eta\leq\theta}$ the boundary sequence of A. If $F_\theta = \varnothing$, then*

$$A = \bigcup_{\eta<\theta}(F_\eta - H_\eta),$$

where $H_\eta = \overline{((X - A) \cap F_\eta)}$, and hence $A \in \mathrm{DH}(\Sigma^0_1)(X)$.

PROOF. Assume $x \in A$. Let $\eta < \theta$ be such that $x \in F_\eta - F_{\eta+1}$. If $x \in H_\eta$, then $x \in \overline{(X - A) \cap F_\eta} \cap \overline{(A \cap F_\eta)} \subseteq F_{\eta+1}$, which is impossible. Hence, $x \in F_\eta - H_\eta$. Conversely, if $x \in F_\eta - H_\eta$ for some $\eta < \theta$ but $x \notin A$, then $x \in (X - A) \cap F_\eta \subseteq \overline{(X - A) \cap F_\eta} = H_\eta$, a contradiction. □

Note that A is dense, codense in F_θ. In other words, in a separable metrizable space X either $A \in \mathrm{DH}(\Sigma^0_1)(X)$ or else A is dense, codense in some closed subset of X.

PROPOSITION 3.25. *Let X be a Polish space, $A \subseteq X$ an arbitrary subset, and $(F_\eta)_{\eta\leq\theta}$ the boundary sequence of A. If $A \in \mathbf{\Delta}^0_2(X)$, then $F_\theta = \varnothing$ and therefore $A \in \mathrm{DH}(\mathbf{\Sigma}^0_1)(X)$.*

PROOF. Suppose $F_\theta \neq \varnothing$. Then $F_\theta \subseteq X$ is a Polish space and $A \cap F_\theta$ is $\mathbf{\Delta}^0_2(F_\theta)$. The boundary of $A \cap F_\theta$ in F_θ is $\partial_{F_\theta}(A) = F_{\theta+1} = F_\theta = \overline{A \cap F_\theta} \cap \overline{(X - A) \cap F_\theta}$. Hence, $A \cap F_\theta$ and $F_\theta - A$ are two disjoint dense G_δ sets, a contradiction. □

The preceeding propositions show that for Polish spaces, $\mathbf{\Delta}^0_2 = \mathrm{DH}(\mathbf{\Sigma}^0_1)$. Using the change of topology technique, this implies the general result.

THEOREM 3.26 (Hausdorff, Kuratowski). *For a Polish space X,*

$$\mathbf{\Delta}^0_{\xi+1}(X) = \bigcup_{\theta<\omega_1} D_\theta(\mathbf{\Sigma}^0_\xi)(X)$$

PROOF. Let (X, τ) be a Polish space. Assume $A \in \mathbf{\Delta}^0_{\xi+1}(X, \tau)$. Then there are $A_n \in \mathbf{\Delta}^0_\xi(X, \tau)$ with $A = \lim_n A_n$ by [15, Theorem 22.17]. By [15, Theorem 22.18], there is a finer Polish topology $\sigma \supseteq \tau$ so that $A_n \in \mathbf{\Delta}^0_1(X, \sigma)$ and $\sigma \subseteq \mathbf{\Sigma}^0_\xi(X, \tau)$. Then $A \in \mathbf{\Delta}^0_2(X, \sigma)$ by [15, Theorem 22.17] and thus $A \in \mathrm{DH}(\mathbf{\Sigma}^0_1)(X, \sigma)$. Since $\sigma \subseteq \mathbf{\Sigma}^0_\xi(X, \tau)$, we have $A \in \mathrm{DH}(\mathbf{\Sigma}^0_\xi)(X, \tau)$. □

Lavrentiev showed that in uncountable Polish spaces the difference hierarchy over $\mathbf{\Sigma}^0_\xi$ is proper, i.e. $D_\theta(\mathbf{\Sigma}^0_\xi) \neq \check{D}_\theta(\mathbf{\Sigma}^0_\xi)$ for every $\theta < \omega_1$. Recall that a set $U \subseteq X \times Y$ is *universal* for a pointclass Γ if $U \in \Gamma(X \times Y)$ and for every $A \subseteq Y$, $A \in \Gamma(Y)$ iff $A = U_x$ for some $x \in X$.

PROPOSITION 3.27 (Lavrentiev). *Let X be a separable metrizable space. There exists a universal set $U \subseteq 2^\omega \times X$ for $D_\theta(\mathbf{\Sigma}^0_\xi)(X)$.* □

The usual diagonal argument now shows that $D_\theta(\mathbf{\Sigma}^0_\xi) \neq \check{D}_\theta(\mathbf{\Sigma}^0_\xi)$ in any uncountable Polish space, as follows.

THEOREM 3.28 (Lavrentiev). *Let X be an uncountable Polish space. Then the difference hierarchy $\mathrm{DH}(\mathbf{\Sigma}^0_\xi)(X)$ is proper.*

PROOF. Since X is an uncountable Polish space, we may assume $2^\omega \subseteq X$ [15, Theorem 6.2, p. 31]. Suppose $D_\theta(\mathbf{\Sigma}^0_\xi)(X) = \check{D}_\theta(\mathbf{\Sigma}^0_\xi)(X)$ for some $1 \leq \theta < \omega_1$. This then implies that

$D_\theta(\boldsymbol{\Sigma}^0_\xi)(2^\omega) = \breve{D}_\theta(\boldsymbol{\Sigma}^0_\xi)(2^\omega)$. Let $U \subseteq 2^\omega \times 2^\omega$ be universal for $D_\theta(\boldsymbol{\Sigma}^0_\xi)(2^\omega)$. Define $A \subseteq 2^\omega$ by $x \in A \Leftrightarrow (x, x) \notin U$. Since $U \in D_\theta(\boldsymbol{\Sigma}^0_\xi)(2^\omega \times 2^\omega)$, $A \in \breve{D}_\theta(\boldsymbol{\Sigma}^0_\xi)(2^\omega) = D_\theta(\boldsymbol{\Sigma}^0_\xi)(2^\omega)$. By universality of U, there is an $x \in 2^\omega$ such that $A = U_x$. But then

$$(x, x) \in U \Leftrightarrow x \in U_x \Leftrightarrow x \in A \Leftrightarrow (x, x) \notin U,$$

a contradiction. □

Therefore, we have the following picture for Polish spaces:

$$\begin{array}{ccccccc} \boldsymbol{\Sigma}^0_\xi = D_1(\boldsymbol{\Sigma}^0_\xi) & D_2(\boldsymbol{\Sigma}^0_\xi) & & D_\theta(\boldsymbol{\Sigma}^0_\xi) & & D_\eta(\boldsymbol{\Sigma}^0_\xi) & \\ & & \cdots & & \cdots & & \cdots \\ \boldsymbol{\Pi}^0_\xi = \breve{D}_1(\boldsymbol{\Sigma}^0_\xi) & \breve{D}_2(\boldsymbol{\Sigma}^0_\xi) & & \breve{D}_\theta(\boldsymbol{\Sigma}^0_\xi) & & \breve{D}_\eta(\boldsymbol{\Sigma}^0_\xi) & \end{array}$$

where $\theta \leq \eta$ and every class is properly contained in every class to the right of it. We address the question whether the difference hierarchy over the open sets is proper in arbitrary separable, metrizable spaces in Section 3.6. Finally, two observations on the difference hierarchy of a subspace.

LEMMA 3.29. *Let $X \subseteq Y$ be separable metrizable spaces. Then*

$$D_\theta(\boldsymbol{\Sigma}^0_\xi)(X) = D_\theta(\boldsymbol{\Sigma}^0_\xi)(Y) \restriction X = \{A \cap X \colon A \in D_\theta(\boldsymbol{\Sigma}^0_\xi)(X)\}.$$

In particular, if Y is Polish, then

$$\mathrm{DH}(\boldsymbol{\Sigma}^0_\xi)(X) = \boldsymbol{\Delta}^0_{\xi+1}(Y) \restriction X.$$

PROOF. The first equality follows immediately from the fact that $\boldsymbol{\Sigma}^0_\xi(X) = \boldsymbol{\Sigma}^0_\xi(Y) \restriction X$. Assume Y is a Polish space. If $A \in \mathrm{DH}(\boldsymbol{\Sigma}^0_\xi)(X)$, then $A = B \cap X$ for some $B \in \mathrm{DH}(\boldsymbol{\Sigma}^0_\xi)(Y)$ and $B \in \boldsymbol{\Delta}^0_{\xi+1}(Y)$ by Proposition 3.22. Conversely, if $A = B \cap X$ for some $B \in \boldsymbol{\Delta}^0_{\xi+1}(Y)$, then $B \in \mathrm{DH}(\boldsymbol{\Sigma}^0_\xi)(Y)$ by Theorem 3.26. Hence, $A \in \mathrm{DH}(\boldsymbol{\Sigma}^0_\xi)(Y) \restriction X = \mathrm{DH}(\boldsymbol{\Sigma}^0_\xi)(X)$. □

3.5. The Finite Levels

Fix an ambient separable metrizable space X. The difference of two open sets can always be written as the intersection of an open set and a closed set. Hence, if $A \in$

$D_{2n}(A_0, \ldots, A_{2n-1})$, then

$$A = \bigcup_{i=1}^{n}(O_i \cap C_i),$$

where $O_i = A_{2i-1}$ is open and $C_i = X - A_{2i-2}$ is closed. In this section we will show that conversely every set of the form $\bigcup_{i=1}^{n}(O_i \cap C_i)$, where each O_i is open and each C_i is closed, is an element of $D_{2n}(\mathbf{\Sigma}_1^0)$. This explicitly connects the finite levels of the Hausdorff difference hierarchy to Ostrovsky's question.

We need two lemmas on the complexity of finite unions. The pointclass $D_n(\mathbf{\Sigma}_1^0)$ is not closed under arbitrary finite unions. For example, when $A_0 \subset A_1 \subset B_0 \subset B_1$ are properly nested open sets, $A = A_1 - A_0$ and $B = B_1 - B_0$ are both $D_2(\mathbf{\Sigma}_1^0)$, but their union $A \cup B$ is properly $D_4(\mathbf{\Sigma}_1^0)$. However, the union of an open set and a set in $D_{2n}(\mathbf{\Sigma}_1^0)$ is always an element of $D_{2n+1}(\mathbf{\Sigma}_1^0)$.

Lemma 3.30. *Let O be open and $B \in D_{2n}(\mathbf{\Sigma}_1^0)$. Then $O \cup B \in D_{2n+1}(\mathbf{\Sigma}_1^0)$.*

Proof. Let $B = D_{2n}(B_0, \ldots, B_{2n-1})$, where $B_0 \subseteq \cdots \subseteq B_{2n-1}$ are open. Define sets $C_0, \ldots, C_{2n}$ by

$$C_0 = O \cap B_0,$$

$$C_1 = B_0, \text{ and}$$

$$C_i = O \cup B_{i-1} \text{ for } 2 \leq i \leq 2n.$$

Clearly, $C_0 \subseteq \cdots \subseteq C_{2n}$ is an increasing sequence of open sets. We verify that

$$O \cup B = D_{2n+1}(C_0, \ldots, C_{2n}) = C_0 \cup \bigcup_{i=1}^{n}(C_{2i} - C_{2i-1}).$$

Suppose $x \in O \cup B$. If $x \in O$, then either $x \in C_0$ (when $x \in B_0$) or $x \in C_2 - C_1$ (when $x \notin B_0$). If $x \in B - O$, then $x \in B_{2i+1} - B_{2i}$ for some $0 \leq i < n$. Hence, $x \in C_{2i+2} - C_{2i+1}$. Conversely, suppose $x \in D_{2n+1}(C_0, \ldots, C_{2n})$. If $x \in C_0$, then $x \in O$. Otherwise, $x \in C_{2i} - C_{2i-1}$ for some $1 \leq i \leq n$. If $i = 1$, then possibly $x \in O$. In all other cases, $x \in B_{2i-1} - B_{2i-2}$. □

The previous lemma shows that the union of a set in $D_1(\Sigma_1^0)$, i.e. an open set, and a set in $D_{2n}(\Sigma_1^0)$ is an element of $D_{2n+1}(\Sigma_1^0)$. Similarly, the union of a set in $D_2(\Sigma_1^0)$ and a set in $D_{2n}(\Sigma_1^0)$ is always an element of $D_{2n+2}(\Sigma_1^0)$.

LEMMA 3.31. *Let $A \in D_2(\Sigma_1^0)$ and $B \in D_{2n}(\Sigma_1^0)$. Then $A \cup B \in D_{2n+2}(\Sigma_1^0)$.*

PROOF. Let $A = A_1 - A_0$ and $B = D_{2n}(B_0, \ldots, B_{2n-1})$, where $A_0 \subseteq A_1$ and $B_0 \subseteq \cdots \subseteq B_{2n-1}$ are open. Define sets $C_0, \ldots, C_{2n+1}$ as follows. Let

$$C_0 = B_0 \cap A_0,$$

$$C_1 = B_1 \cap A_1.$$

For $1 \leq i < n$, let

$$C_{2i} = (B_{2i} \cap A_0) \cup B_{2i-2} \cup (B_{2i-1} \cap A_1),$$

$$C_{2i+1} = (B_{2i+1} \cap A_1) \cup B_{2i-1}.$$

Finally, let

$$C_{2n} = B_{2n-2} \cup A_0, \text{ and}$$

$$C_{2n+1} = B_{2n-1} \cup A_1.$$

Each C_i is open and $C_0 \subseteq \cdots \subseteq C_{2n+1}$. We now verify that $A \cup B = D_{2n+2}(C_0, \ldots, C_{2n+2})$. First, suppose $x \in A \cup B$ and consider the following two cases.

Case 1: Assume $x \in A = A_1 - A_0$. If $x \in B_1$, then $x \in C_1 - C_0$. If $x \in B_3 - B_1$, then $x \in C_3 - C_2$. If $x \in B_{2i+1} - B_{2i-1}$ for $1 \leq i < n$, then $x \in C_{2i+1} - C_{2i}$. Finally, if $x \notin B_{2n-2}$, then $x \in C_{2n+1} - C_{2n}$.

Case 2: Assume $x \in B - A$. Let $x \in B_{2i-1} - B_{2i-2}$ for $1 \leq i < n$. Either $x \notin A_1$ or $x \in A_0$. If $x \notin A_1$, then $x \in C_{2i+1} - C_{2i}$. If $x \in A_0$, then $x \in C_{2i-1} - C_{2i-2}$.

This shows that $A \cup B \subseteq D_{2n+2}(C_0, \ldots, C_{2n+2})$. For the converse, suppose $x \in C_{2i+1} - C_{2i}$ for $1 \leq i < n$. If $x \in B_{2i-1} - B_{2i-2}$, then $x \in B$. Suppose $x \in A_1 \cap B_{2i+1}$. If $x \notin B_{2i}$, then $x \in B$. Assume $x \in B_{2i}$. Then $x \notin A_0$, otherwise $x \in C_{2i}$. Thus, $x \in A$. The reasoning when $i = 0$ or $i = n$ is similar. $\square$

We can now connect finite unions and intersections of open and closed sets to the finite levels of the difference hierarchy.

PROPOSITION 3.32. *Let $A \subseteq X$.*

(i) *$A \in D_{2n}(\Sigma_1^0)$ if and only if $A = \bigcup_{i=1}^n (O_i \cap C_i)$, where each O_i is open and each C_i is closed.*

(ii) *$A \in \breve{D}_{2n}(\Sigma_1^0)$ if and only if $A = \bigcap_{i=1}^n (O_i \cap C_i)$, where each O_i is open and each C_i is closed.*

(iii) *$A \in D_{2n+1}(\Sigma_1^0)$ if and only if $A = O \cup \bigcup_{i=1}^n (O_i \cap C_i)$, where each O_i is open and each C_i is closed.*

(iv) *$A \in \breve{D}_{2n+1}(\Sigma_1^0)$ if and only if $A = C \cap \bigcup_{i=1}^n (O_i \cap C_i)$, where each O_i is open and each C_i is closed.*

PROOF. Statements (ii) and (iv) follow from (i) and (iii) by DeMorgan's laws. The forward directions of (i) and (iii) are clear. We prove the backward direction of (i) by induction on n; the backward direction of (iii) follows from Lemma 3.30 in a similar way.

If $A = O \cap C$, then $A = D_2(O \cap (X - C), O)$. Assume that every set of the form $\bigcup_{i=1}^n (O_i \cap C_i)$ is an element of $D_{2n}(\Sigma_1^0)$. If $A = \bigcup_{i=1}^{n+1} (O_i \cap C_i)$, then A is the union of a set in $D_2(\Sigma_1^0)$ and a set in $D_{2n}(\Sigma_1^0)$. Hence, $A \in D_{2n+2}(\Sigma_1^0)$ by Lemma 3.31. $\square$

Therefore, the sets on levels $2n, 2n+1, 2n+2$ of the difference hierarchy can be represented as follows:

$$\ldots \quad \bigcup_{i=1}^n (O_i \cap C_i) \quad O \cup \bigcup_{i=1}^n (O_i \cap C_i) \quad \bigcup_{i=1}^{n+1} (O_i \cap C_i) \quad \ldots$$

$$\ldots \quad \bigcap_{i=1}^n (O_i \cup C_i) \quad C \cap \bigcap_{i=1}^n (O_i \cup C_i) \quad \bigcap_{i=1}^{n+1} (O_i \cup C_i) \quad \ldots$$

3.6. The Difference Hierarchy of the Rationals

We have seen that $\mathrm{DH}(\Sigma_1^0)(X) \subseteq \Delta_2^0(X)$ in an arbitrary metrizable space X. Moreover, if X is Polish, then $\mathrm{DH}(\Sigma_1^0)(X) = \Delta_2^0(X)$ by the Hausdorff–Kuratowski Theorem 3.26. This equality does not necessarily hold in an arbitrary separable metrizable space. For example, consider the rationals $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, every subset of $\mathbb{Q}$ is F_σ and therefore every subset is $\mathbf{\Delta}^0_2$. However, not every subset of $\mathbb{Q}$ is resolvable; the following proposition characterizes the nonresolvable subsets of $\mathbb{Q}$.

PROPOSITION 3.33. *Let $A \subseteq \mathbb{Q}$. The following are equivalent:*

(i) *A is not resolvable;*

(ii) *A is not relatively $\mathbf{\Delta}^0_2$, i.e., there is no $\mathbf{\Delta}^0_2$ subset B of $\mathbb{R}$ such that $B \cap \mathbb{Q} = A$;*

(iii) *A is dense, codense in a closed $F \subseteq \mathbb{Q}$;*

(iv) *A is dense, codense in a perfect $F \subseteq \mathbb{Q}$;*

(v) *A is dense, codense in a homeomorphic copy of $\mathbb{Q}$ (inside $\mathbb{Q}$).*

PROOF. The equivalence of (i) and (ii) is Lemma 3.29. In fact, if A is resolvable in $\mathbb{Q}$, then its representation in the difference hierarchy of $\mathbb{Q}$ can be lifted to the difference hierarchy of $\mathbb{R}$ to obtain a set $B \subseteq \mathbb{R}$ resolvable in $\mathbb{R}$ so that $B \cap \mathbb{Q} = A$. Since B is $\mathbf{\Delta}^0_2$, A is relatively $\mathbf{\Delta}^0_2$. Conversely, if $B \subseteq \mathbb{R}$ is $\mathbf{\Delta}^0_2$ and $B \cap \mathbb{Q} = A$, then B is resovable, and the restriction to $\mathbb{Q}$ of its representation in the difference hierarchy of $\mathbb{R}$ gives a representation of A in the difference hierarchy of $\mathbb{Q}$.

(i) $\Rightarrow$ (iii) is a consequence of the proof of the Hausdorff–Kuratowski Theorem 3.26. Suppose $A \subseteq \mathbb{Q}$. Let $(F_\eta)_{\eta<\theta}$ be the boundary sequence of A. If $F_\theta = \varnothing$, then A is resolvable by Proposition 3.24. Otherwise, A is dense, codense in $F_\theta = F_{\theta+1} = \partial_{F_\theta}(A)$.

(iii) $\Rightarrow$ (iv) Assume A is dense, codense in a closed set $F \subseteq \mathbb{Q}$. Suppose $x \in F$ is an isolated point. Then $\{x\}$ is open in F, hence $x \in A$ because A is dense. Similarly, $x \in F - A$ because A is codense. This is a contradiction.

(iv) $\Rightarrow$ (v) If $F \subseteq \mathbb{Q}$ is perfect, then F is homeomorphic to $\mathbb{Q}$ by the Fréchet–Sierpiński result (c.f. [15, Exercise 7.12]).

(v) $\Rightarrow$ (ii) We may assume that A is dense, codense in $\mathbb{Q}$ itself. Suppose $A = B \cap \mathbb{Q}$ where B is a $\mathbf{\Delta}^0_2$ subset of $\mathbb{R}$. Then B and $\mathbb{R} - B$ are both dense G_δ in $\mathbb{R}$. But B and $\mathbb{R} - B$ are disjoint, a contradiction to the fact that $\mathbb{R}$ is a Baire space. $\square$

In Section 3.4 we have seen that the difference hierarchy over $\mathbf{\Sigma}^0_\xi$ in an uncountable Polish space is proper (see Lavrentiev's Theorem 3.28). Since $\mathbb{Q}$ is countable, Lavrentiev's theorem does not apply to $\mathbb{Q}$. We will use a direction construction to show that the difference hierarchy over the open sets in $\mathbb{Q}$ is indeed proper.

Denote the relatively open interval $\mathbb{Q} \cap (1/2^{n+1}, 1/2^n)$ by I_n. By the Fréchet–Sierpiński theorem, I_n is homeomorphic to $\mathbb{Q}$. Also, note that if $O \subseteq X$ is open and $A = D_\theta((A_\eta)_{\eta<\theta}) \in D_\theta(\Sigma_1^0)(X)$, then $O \cap A$ is still an element of $D_\theta(\Sigma_1^0)(X)$, since $O \cap A = D_\theta((O \cap A_\eta)_{\eta<\theta})$.

LEMMA 3.34. *Let $(A_n)_{n\in\omega}$ be a sequence of subsets of $\mathbb{Q}$ such that $A_n \subseteq I_n$ and $A_n \in D_\theta(\Sigma_1^0)(I_n)$ for every $n \in \omega$. Then $A = \bigcup_{n\in\omega} A_n \in D_\theta(\Sigma_1^0)(\mathbb{Q})$.*

PROOF. By Lemma 3.29, for every $n \in \omega$ there is an $B_n \in D_\theta(\Sigma_1^0)(\mathbb{Q})$ such that $A_n = I_n \cap B_n$. Since I_n is open, $I_n \cap B_n$ is an element of $D_\theta(\Sigma_1^0)(\mathbb{Q})$, say $I_n \cap B_n = D_\theta((B_\eta^n)_{\eta<\theta})$, where $(B_\eta^n)_{\eta<\theta}$ is an increasing sequence of open sets for each $n \in \omega$. Replacing each B_η^n with $I_n \cap B_\eta^n$, we may assume that $B_\eta^n \cap B_\xi^m = \varnothing$ whenever $n \neq m$ or $\eta \neq \xi$. Let $C_\eta = \bigcup_{n\in\omega} B_\eta^n$. Clearly, $(C_\eta)_{\eta<\theta}$ is an increasing sequence of open sets. We verify that $A = D_\theta((C_\eta)_{\eta<\theta})$.

Suppose $x \in A = \bigcup_{n\in\omega} A_n$, say $x \in A_n \subseteq I_n$. Then there is some $\eta < \theta$ with parity opposite to that of θ such that $x \in B_\eta^n$ and $x \notin B_\xi^n$ for any $\xi < \eta$. Obviously, $x \in C_\eta$. Since B_ξ^m is disjoint from B_η^n whenever $n \neq m$ or $\eta \neq \xi$, we have $x \notin C_\xi$ for $\xi < \eta$. Hence, $x \in D_\theta((C_\eta)_{\eta<\theta})$.

Conversely, if $x \in D_\theta((C_\eta)_{\eta<\theta})$, then there is some $\eta < \theta$ with parity opposite to that of θ such that $x \in C_\eta$ and $x \notin C_\xi$ for any $\xi < \eta$. Then $x \in B_\eta^n$ for some $n \in \omega$, and since $x \notin B_\xi^n$ for all $\xi < \eta$, we have $x \in D_\theta((B^n)_{\eta<\theta}) = I_n \cap B_n = A_n$. □

Using Lemma 3.34, we can stitch together sets of increasing complexity to create a set on the next level in the difference hierarchy. We first present the easier case when we want to construct such a set at a limit level.

Let Γ be any pointclass. We say that a set A is *genuinely* Γ if $A \in \Gamma - \check{\Gamma}$.

PROPOSITION 3.35. *Let $\lambda < \omega_1$ be a limit ordinal and $(\lambda_n)_{n<\omega}$ a cofinal sequence in λ. Assume $(B_n)_{n\in\omega}$ is a sequence of sets such that B_n is genuinely $\check{D}_{\lambda_n}(\Sigma_1^0)(I_n)$. Then $A = \bigcup_{n\in\omega} B_n$ is genuinely $D_\lambda(\Sigma_1^0)(\mathbb{Q})$.*

PROOF. Since $\breve{D}_{\lambda_n}(\Sigma_1^0)(I_n) \subseteq D_\lambda(\Sigma_1^0)(I_n)$, we have $A \in D_\lambda(\Sigma_1^0)(\mathbb{Q})$ by Lemma 3.34. Suppose that $\mathbb{Q} - A = D_\lambda((A_\eta)_{\eta<\lambda})$ for some increasing sequence $(A_\eta)_{\eta<\lambda}$ of open subsets of $\mathbb{Q}$. Since $0 \in \mathbb{Q} - A$, there is an even ordinal $\eta < \lambda$ such that $0 \in A_{\eta+1} - A_\eta$. Since $A_{\eta+1}$ is open, there is an $\varepsilon > 0$ such that $(0, \varepsilon) \subseteq A_{\eta+1}$. This implies that $(0, \varepsilon) \subseteq A_\theta$ for all $\theta \geq \eta + 1$. Let N be large enough so that $B_N \subseteq (0, \varepsilon)$ and $\lambda_N > \eta + 1$. We have $I_N \cap (A_{\theta+1} - A_\theta) = \varnothing$ for all $\theta > \eta + 1$. Now,

$$B_N = I_N \cap A = I_N \cap \bigcup_{\alpha<\lambda \text{ even}} A_{\alpha+1} - A_\alpha = I_N \cap \bigcup_{\alpha<\eta+1 \text{ even}} A_{\alpha+1} - A_\alpha.$$

Thus, $B_N \in D_{\eta+1}(I_N)$, a contradiction. □

The argument for the successor case is very similar but slightly different to the argument for the limit case.

PROPOSITION 3.36. *Let $\theta < \omega_1$. Assume $(B_n)_{n\in\omega}$ is a sequence of sets such that $B_n \in D_{\theta+1}(\Sigma_1^0)(I_n)$ and $B_n \notin D_\theta(\Sigma_1^0)(I_n) \cup \breve{D}_\theta(\Sigma_1^0)(I_n)$. Then $A = \bigcup_{n\in\omega} B_n$ is genuinely $D_{\theta+1}(\Sigma_1^0)(\mathbb{Q})$.*

PROOF. As in Proposition 3.35, we have $A \in D_{\theta+1}(\Sigma_1^0)(\mathbb{Q})$ by Lemma 3.34. Suppose $\mathbb{Q} - A = D_{\theta+1}((A_\eta)_{\eta\leq\theta})$ for some increasing sequence $(A_\eta)_{\eta\leq\theta}$ of open subsets of $\mathbb{Q}$. Note that $0 \notin A = \bigcup_{n\in\omega} A_n$, because $A_n \subseteq I_n = \mathbb{Q} \cap (1/2^{n+1}, 1/2^n)$. Hence, $0 \in D_{\theta+1}((A_\eta)_{\eta\leq\theta})$. Let $\xi \leq \theta$ be least such that $0 \in A_\xi$. The parity of ξ is equal to the parity of θ. Since A_ξ is open, there is an $\varepsilon > 0$ such that $(0, \varepsilon) \subseteq A_\xi$. Let N be large enough so that $I_N \subseteq (0, \varepsilon)$. Since the sequence $(A_\eta)_{\eta\leq\theta}$ is increasing, $I_N \subseteq A_\eta$ for all $\eta \geq \xi$. We now consider two cases.

Case 1: $\xi < \theta$. We will show that $B_N \in \breve{D}_\zeta(\Sigma_1^0)(I_N)$ for some $\zeta < \theta$, which is a contradiction. Since ξ and θ have the same parity, there is a ζ of the opposite parity strictly between ξ and θ. Now,

$$\mathbb{Q} - B_N = I_N \cap A = I_N \cap D_{\theta+1}((A_\eta)_{\eta\leq\theta})$$

and we only need to show that

$$I_N \cap D_{\theta+1}((A_\eta)_{\eta\leq\theta}) = I_N \cap D_\zeta((A_\eta)_{\eta<\zeta}).$$

Since ζ and $\theta + 1$ have the same parity,

$$I_N \cap D_\zeta((A_\eta)_{\eta<\zeta}) \subseteq I_N \cap D_{\theta+1}((A_\eta)_{\eta\leq\theta}).$$

Conversely, if $x \in I_N \cap D_{\theta+1}((A_\eta)_{\eta\leq\theta})$, then the least $\alpha \leq \theta$ such that $x \in A_\alpha$ must be less than or equal to ξ, since $I_N \subseteq A_\xi$. Thus, we have $x \in D_\zeta((A_\eta)_{\eta<\zeta})$.

Case 2: $\xi = \theta$. We will show that $I_N - B_N = D_\theta((A_\eta)_{\eta<\theta})$, i.e., $B_N \in \check{D}_\theta(\Sigma^0_1)(I_N)$, which is a contradiction. Suppose $x \in I_N - B_N$. Since $B_N = I_N \cap D_{\theta+1}((A_\eta)_{\eta\leq\theta})$, this implies that the least $\alpha \leq \theta$ such that $x \in A_\alpha$ has parity equal to $\theta + 1$, that is, parity opposite to that of θ. (Note that since $I_N \subseteq A_\theta$, there always is an $\alpha \leq \theta$ such that $x \in A_\theta$). Thus, $x \in D_\theta((A_\eta)_{\eta<\theta})$.

Conversely, suppose $x \in I_N \cap D_\theta((A_\eta)_{\eta<\theta})$. Then there is an $\alpha < \theta$ such that $x \in A_\alpha$, $x \notin A_\beta$ for $\beta < \alpha$, and α has parity opposite to that of θ, that is, α has the same parity as $\theta + 1$. This implies that $x \notin D_{\theta+1}((A_\eta)_{\eta\leq\theta})$. Thus, $x \notin B_N$.

□

Using transfinite induction, we can now prove that for all $\theta < \omega_1$,

$$D_\theta(\Sigma^0_1)(\mathbb{Q}) \neq \check{D}_\theta(\Sigma^0_1)(\mathbb{Q}).$$

THEOREM 3.37. *The difference hierarchy over the open sets of the rationals is proper.*

PROOF. The induction basis is the observation that

$$D_1(\Sigma^0_1)(\mathbb{Q}) = \Sigma^0_1(\mathbb{Q}) \neq \Pi^0_1(\mathbb{Q}) = \check{D}_1(\Sigma^0_1)(\mathbb{Q}).$$

Noting that each I_n is homeomorphic to $\mathbb{Q}$, the limit case follows from Proposition 3.35. For the successor case, let A be a genuinely $D_\theta(\Sigma^0_1)(\mathbb{Q})$ set. Without loss of generality we may assume that $A \subseteq (0, 1)$. Let B be $(0, 1) - A$, translated to the interval $(1, 2)$. Now, $A \cup B$ is neither $D_\theta(\Sigma^0_1)(\mathbb{Q})$ nor $\check{D}_\theta(\Sigma^0_1)(\mathbb{Q})$, while $A \cup B \in D_{\theta+1}(\Sigma^0_1)(\mathbb{Q})$. The successor case now follows from Proposition 3.36. □

COROLLARY 3.38 (Gao). *Let X be a separable metrizable space with uncountable completion $\hat{X}$. Then the difference hierarchy over the open sets is proper.*

PROOF. If X is Polish, this is Lavrentiev's Theorem 3.28. Assume X is not Polish. Let $D \subseteq X$ be a countable dense subset of X. By Hurewicz's Theorem 3.14, D contains a closed subspace homeomorphic to $\mathbb{Q}$. Using Lemma 3.29, $\mathrm{DH}(\Sigma_1^0)(D)$ is proper by Theorem 3.37. This implies that $\mathrm{DH}(\Sigma_1^0)(X)$ is proper. □

The question whether the difference hierarchy over Σ_ξ^0 is proper when $\xi > 1$ remains open.

3.7. Continuous Surjections from ω^ω Onto $\mathbb{Q}$

A map $f\colon X \to Y$ is *clopen-resovable* iff $f(U)$ is resolvable in Y for every clopen set $U \subseteq X$. In this section we will show that there cannot be a clopen-resolvable continuous surjection from a closed subset $P \subseteq \omega^\omega$ onto $\mathbb{Q}$.

First, we review some standard notation and terminology. For any $s = (s_0, \ldots, s_k) \in \omega^{<\omega}$ let N_s denote the set $\{x \in \omega^\omega \colon s \subseteq x\}$. If $n \in \omega$, then we denote the sequence $(s_0, \ldots, s_k, n)$ by $s^\frown n$. A *tree* on ω is a set $T \subseteq \omega^{<\omega}$ of finite sequences of natural numbers such that if $(t_0, \ldots, t_n) \in T$, then $(t_0, \ldots, t_m) \in T$ for all $m \leq n$. The *body* of a tree T is the set $[T] = \{x \in \omega^\omega \colon \forall n \in \omega\ (x \restriction n \in T)\}$. A tree T is *pruned* if $N_s \cap [T] \neq \varnothing$ for any $s \in T$. When $P \subseteq \omega^\omega$ is closed, there is a unique pruned tree T on ω such that $P = [T]$ (c.f. [15, Sections 2.A and 2.B]). For notational simplicity, when $P \subseteq \omega^\omega$ and $f\colon P \to \mathbb{Q}$, we write $f(N_s)$ for $f(N_s \cap P)$.

Let $P \subseteq \omega^\omega$ be a closed set and $f\colon P \to \mathbb{Q}$ a continuous surjection. Using the unique pruned tree T with $P = [T]$, we identify a sufficient condition such that the image $f(U)$ of some clopen set $U \subseteq P$ is nonresolvable.

LEMMA 3.39. *Let $P = [T]$ be the body of a pruned tree T on ω and $f\colon P \to \mathbb{Q}$ a continuous surjection. Suppose there is an $s \in T$ and a nonempty open set $O \subseteq \mathbb{Q}$ such that*

(i) *$f(N_{s^\frown n})$ is nowhere dense in O for all $n \in \omega$, and*

(ii) *$f(N_s) = \bigcup_{n\in\omega} f(N_{s^\frown n})$ is dense in O.*

Then there is a clopen $U \subseteq P$ such that $f(U)$ is not resolvable.

PROOF. To prove the lemma it suffices to define a clopen $U \subseteq P$ such that $f(U)$ is dense, codense in O. Since O is homeomorphic to $\mathbb{Q}$, $f(U)$ will then be nonresolvable by Proposition 3.33. We recursively define a sequence N_n ($n \in \omega$) of basic clopen sets and take $U = \bigcup_n N_n$. In the construction, we use the following observation.

CLAIM 3.40. *If $S \subseteq \omega$ is cofinite, then $\bigcup_{n \in S} f(N_{s^\frown n})$ is dense in O.*

PROOF. Assume $\omega - S = \{n_0, \ldots, n_k\}$ and let $B \subseteq O$ be a basic open set. Since $f(N_{s^\frown n_0})$ is nowhere dense in O, there is a nonempty open $B_0 \subseteq B$ such that $f(N_{s^\frown n_0}) \cap B_0 = \varnothing$. Similarly, there is a nonempty open $B_1 \subseteq B_0$ such that $f(N_{s^\frown n_1}) \cap B_1 = \varnothing$, etc. We thus get a nonempty open $B_k \subseteq O$ such that for all $i = 0, \ldots, k$, $f(N_{s^\frown n_i}) \cap B_k = \varnothing$. Since $\bigcup_{n \in \omega} f(N_{s^\frown n})$ is dense in O, for some $n \in \omega$, $f(N_{s^\frown n}) \cap B_k \neq \varnothing$. Hence, $n \in S$. $\square$

Enumerate all basic open sets contained in O as $B_0, B_1, \ldots$. Let $\mathrm{lh}(s) = k$ be the length of s. At stage 0, pick $x_0, y_0 \supseteq s$ such that $f(x_0), f(y_0) \in B_0$ and $f(x_0) \neq f(y_0)$. Let $p_0 = x_0(k)$ and $q_0 = y_0(k)$. Then let $N_0 \subseteq N_{s^\frown p_0}$ be a basic clopen nbhd of x_0 such that $f(y_0) \notin f(N_0)$. It is possible to pick such an N_0 since f is continuous. Note that $f(N_0) \subseteq f(N_{s^\frown p_0})$ is nowhere dense in O. By the claim, the set $\bigcup_{m \neq p_0, q_0} f(N_{s^\frown m})$ is still dense in O.

At stage 1 we first pick an $x_1 \supseteq s$ such that $x_1(k) \neq p_0, q_0$ and $f(x_1) \in B_1 - \{f(x_0), f(y_0)\}$. Next pick $y_1 \supseteq s$ such that $y_1(k) \neq p_0, q_0$ and $f(y_1) \in B_1 - f(N_0) - \{f(x_1)\}$. Let $p_1 = x_1(k)$ and $q_1 = y_1(k)$. Then let $N_1 \subseteq N_{s^\frown p_1}$ be a basic clopen nbhd of x_1 such that $f(y_0), f(y_1) \notin f(N_1)$. Now $f(y_0), f(y_1) \notin f(N_0) \cup f(N_1)$ and $f(N_0) \cup f(N_1)$ is still nowhere dense in O. Again the set $\bigcup_{m \neq p_0, q_0, p_1, q_1} f(N_{s^\frown m})$ is still dense in O.

In general, at stage n, pick $x_n \supseteq s$ such that $x_n(k) \neq x_i(k), y_i(k)$ for all $0 \leq i < n$ and $f(x_n) \in B_n - \{f(x_i), f(y_i) \colon 0 \leq i < n\}$. Pick $y_n \supseteq s$ such that $y_n(k) \neq x_i(k), y_i(k)$ for all $0 \leq i < n$ and $f(y_n) \in B_n - \bigcup_{0 \leq i < n} f(N_i) - \{f(x_n)\}$. Then let $N_n \subseteq N_{s^\frown x_n(k)}$ be a basic clopen nbhd of x_n such that $f(y_i) \notin f(N_n)$ for all $0 \leq i < n$. We actually have $f(y_i) \notin f(N_0) \cup \cdots \cup f(N_n)$ for $0 \leq i \leq n$, and that $f(N_0) \cup \cdots \cup f(N_n)$ is nowhere dense in O.

This finishes the definition of the sequence of basic clopen sets N_n for $n \in \omega$. Let $U = \bigcup_{n \in \omega} N_n$. By the construction, U is clopen. Also, $f(U)$ is dense in O, since $f(x_n) \in f(U)$ for each $n \in \omega$, and $f(x_n) \in B_n$. Similarly, $f(U)$ is codense in O, since $f(y_n) \notin f(U)$ but $f(y_n) \in B_n$. □

The following lemma is more general but the proof is the same as above. We state it without proof.

LEMMA 3.41. *Let T be a pruned tree, $P = [T]$, and $f \colon P \to \mathbb{Q}$ be a continuous surjection. Suppose there are $s, t_n \in T$, $n \in \omega$, and a nonempty open set $O \subseteq \mathbb{Q}$ such that*

(0) *$s \subseteq t_n$ for all $n \in \omega$, $N_{t_n} \cap N_{t_m} = \varnothing$ for $n \neq m$, and $N_s \cap P = \bigcup_n N_{t_n} \cap P$,*

(1) *$f(N_{t_n})$ is nowhere dense in O for all $n \in \omega$, and*

(2) *$f(N_s) = \bigcup_n f(N_{t_n})$ is dense in O.*

Then there is a clopen $U \subseteq P$ such that $f(U)$ is not resolvable. □

We now show that there cannot be a clopen-resolvable continuous surjection $f \colon P \to \mathbb{Q}$, because we can always find nodes as in the lemma above.

PROPOSITION 3.42. *Let $P \subseteq \omega^\omega$ be closed and $f \colon P \to \mathbb{Q}$ be a continuous surjection. Then there is a clopen $U \subseteq P$ such that $f(U)$ is not resolvable.*

PROOF. Let $T \subseteq \omega^{<\omega}$ be the unique pruned tree with $P = [T]$. We describe a search algorithm to find $s, t_n \in T$ $(n \in \omega)$ and $O \subseteq \mathbb{Q}$ satisfying the assumptions of Lemma 3.41. The search will be conducted by induction on the lengths of the nodes in T and produces a sequence $T \supseteq T_0 \supseteq T_1 \supseteq \ldots$ of subtrees of T, where some of the nodes have been labeled with nonempty open subsets of $\mathbb{Q}$. We will maintain the following properties for all $l \in \omega$:

(i) $T_l \subseteq T$ contains all labeled nodes,

(ii) $[T_l]$ is clopen in $[T]$,

(iii) if $s \in T_l$ is labeled with O, then $f(N_s \cap [T_l])$ is dense in O, and

(iv) if $s \in T_l$ with $\mathrm{lh}(s) = l$ is labeled, then no nodes $t \supsetneq s$ are labeled.

When considering a particular length l, we will define for each $s \in T_l$ with $\mathrm{lh}(s) = l$ a pruned tree $S_s \subseteq T_l$ such that $[S_s] \subseteq N_s \cap [T_l]$ and $[S_s]$ is clopen in $[T_l]$, label certain nodes in S_s with nonempty open sets $O \subseteq \mathbb{Q}$, and take T_{l+1} to be the union of all the subtrees S_s.

To start, let $T_0 = T$ with the root labeled with $\mathbb{Q}$. Assume a partially labeled tree $T_l \subseteq$ has been defined satisfying conditions (i) through (iv). For each $s \in T_l$ with $\mathrm{lh}(s) = l$ define $S_s \subseteq T_l$ as follows. If s is not labeled with any open set, then we do nothing: let S_s be the unique pruned tree with $[S_s] = N_s \cap [T_l]$ and label no extensions of s by open sets. Assume that s has been labeled with a nonempty open set $O \subseteq \mathbb{Q}$. Let $t_n \in T_l$ $(n \in \omega)$ be such that $t_n \supseteq s$ for all $n \in \omega$, $N_{t_n} \cap N_{t_m} = \varnothing$ for $n \neq m$, and $N_s \cap [T_l] = \bigcup_n [N_{t_n}] \cap [T_l]$. Such t_n exist since $N_s \cap [T_l]$ is clopen in $[T]$. One of the following four cases must occur:

Case 1(a). For all $n \in \omega$, $f(N_{t_n} \cap [T_l])$ is nowhere dense in O. We have found the desired $s, t_n \in T$ and $O \subseteq \mathbb{Q}$, since by applying Lemma 3.41 a clopen subset $U \subseteq [T_l]$ can be found with $f(U \cap [T_l])$ nonresolvable, and $U \cap [T_l]$ is indeed clopen in $P = [T]$.

Case 1(b). There is exactly one $k \in \omega$ for which $f(N_{t_k} \cap [T_l])$ is somewhere dense in O but not dense in O, and for all $n \neq k$, $f(N_{t_n})$ is nowhere dense in O. Pick a nonempty open $O' \subseteq O$ such that $f(N_{t_k}) \cap O' = \varnothing$. Now each $f(N_{t_n} \cap [T_l])$ is nowhere dense in O', while $f(N_s \cap [T_l])$ is dense in O', so again we have found the required s, t_n and O'.

The search algorithm terminates with success in Cases 1(a) and 1(b).

Case 2. There is exactly one $k \in \omega$ such that $f(N_{t_k})$ is dense in O, and for all $n \neq k$, $f(N_{t_n})$ is nowhere dense in O. In this case let S_s be the unique pruned tree with $[S_s] = N_{t_k} \cap [T_l]$ and label t_k with O.

Case 3. There are at least two distinct $k_1, k_2 \in \omega$ such that each of $f(N_{t_{k_1}} \cap [T_l])$ and $f(N_{t_{k_2}} \cap [T_l])$ is respectively dense in some nonempty open $O_1, O_2 \subseteq O$. For notational simplicity, and without loss of generality, we may assume $k_1 = 1$ and $k_2 = 2$. By shrinking O_1 and O_2 if necessary, we may also assume that O_1 and O_2 are disjoint clopen sets in $\mathbb{Q}$.

Now let S_s be the unique pruned tree with

$$[S_s] = \left(N_{t_1} \cap [T_l] \cap f^{-1}(O_1) \right) \cup \left(N_{t_2} \cap [T_l] \cap f^{-1}(O_2) \right).$$

Then $[S_s]$ is clopen in $[T_l]$, and $t_1, t_2 \in S_s$. We then label t_1 and t_2 with O_1 and O_2, respectively.

We claim that this search algorithm always terminates in Case 1(a) or 1(b) after finitely many steps. Suppose this is not the case. Then we obtain a pruned tree $T_\infty = \bigcap_l T_l$ which contains all labeled nodes. If below every labeled node in T_∞ there is a split as in Case 3, then T_∞ has uncountably (in fact, $2^{\aleph_0}$) many branches. Since $f([T_\infty]) \subseteq \mathbb{Q}$ is countable, there are distinct branches $x \neq y \in [T_\infty]$ such that $f(x) = f(y)$. Let $s \subseteq x, y$ be the longest labeled node, t_1 and t_2 are labeled nodes such that $s \subsetneq t_1 \subseteq x$ and $s \subsetneq t_2 \subseteq y$. Then Case 3 occurs when s is considered, and t_1 and t_2 are respectively labeled with disjoint clopen sets O_1 and O_2. Let $l = \mathrm{lh}(s)$. By our construction $f(x) \in f(N_{t_1} \cap [T_{l+1}]) \subseteq O_1$ and $f(y) \in f(N_{t_2} \cap [T_{l+1}]) \subseteq O_2$. Since $O_1 \cap O_2 = \varnothing$, $f(x) \neq f(y)$, a contradiction. Hence, there is an $s \in T_\infty$ with label O such that all labeled nodes in T_∞ extending s are obtained from Case 2. Therefore, there is $x \in [T_\infty]$ such that $x \restriction n$ has label O for infinitely many $n \in \omega$. By our construction, $f(N_{x \restriction n})$ is dense in O for every $n \in \omega$. On the other hand, f is continuous; thus, we could pick a nbhd $O' \subsetneq O$ of $f(x)$ strictly smaller than O and an $N_{x \restriction n}$ such that $f(N_{x \restriction n}) \subseteq O'$. Then $f(N_{x \restriction n})$ is not dense in O, a contradiction. □

3.8. Resolvable Continous Surjections

We now derive the main theorem of this chapter from Proposition 3.42. The arguments in the following proofs are essentially the same as those in [23].

LEMMA 3.43. *Let Y be a separable metrizable space and $f \colon \omega^\omega \to Y$ a clopen-resolvable continuous surjection. Suppose $Q \subseteq Y$ is a countable perfect set and $P = f^{-1}(Q)$. Then $f \restriction P \colon P \to Q$ is a clopen-resolvable continuous surjection.*

PROOF. Suppose $U \subseteq P$ is clopen in P. We need to show that $f(U)$ is resolvable in Q. For every $x \in U$, pick a basic clopen nbhd $U_x = N_s \subseteq \omega^\omega$ for some $s \subseteq x$ with

$2^{-\operatorname{lh}(s)} = \operatorname{diam} U_x < d(x, P - U)$, where d is the usual metric on ω^ω. Similarly, for every $x \in P - U$, pick a basic clopen nbhd $U_x \subseteq \omega^\omega$ with $\operatorname{diam} U_x < d(x, U)$. Finally, for every $x \in \omega^\omega - P$, pick a basic clopen nbhd U_x with $U_x \subseteq \omega^\omega - P$. The collection $\{U_x \colon x \in \omega^\omega\}$ is an open cover of ω^ω, and has a countable subcover since ω^ω is second countable, in particular Lindelöf. Let $U_0, U_1, \ldots$ enumerate the elements of this countable subcover. For each $n \in \omega$ let $V_n = U_n - \bigcup_{m<n} U_m$. Since each U_n is clopen, we get that each V_n is clopen. Thus $\{V_n \colon n \in \omega\}$ is an open refinement of $\{U_x \colon x \in \omega^\omega\}$ consisting of disjoint clopen sets such that each V_n is a subset of some U_x. Let

$$V = \bigcup \{V_n \colon V_n \cap U \neq \varnothing\}.$$

Then V is clopen in ω^ω and $V \cap P = U$. Hence, $f(U) = Q \cap f(V)$. Since $f(V)$ is resolvable in Y, $f(U)$ is resolvable in Q. □

We say that $f \colon X \to Y$ is *open-resolvable* if $f(U)$ is resolvable in Y for every open $U \subseteq X$. Similarly, $f \colon X \to Y$ is *closed-resolvable* if $f(U)$ is resolvable in Y for every closed $U \subseteq X$. Finally, we say that f is *resolvable* if f is either open-resolvable or closed-resolvable.

THEOREM 3.44. *Resolvable maps preserve complete metrizability.*

PROOF. Let X be a Polish space, Y a separable metrizable space, and $f \colon X \to Y$ a resolvable continuous surjection. Suppose towards a contradiction that Y is not completely metrizable. By Propositions 3.4 and 3.5, and Hurewicz's Theorem 3.14, Y contains a countable perfect subset $Q \subseteq Y$ homeomorphic to $\mathbb{Q}$.

Assume first that f is open-resolvable. We use a classical result of Hausdorff [11] that there is a continuous open surjection $g \colon \omega^\omega \to X$. The composition $f \circ g$ is now clopen-resolvable, and by Lemma 3.43 so is $f \circ g \restriction P \colon P \to Q$, where $P = (f \circ g)^{-1}(Q)$. This contradicts Proposition 3.42. If f is closed-resolvable, we use a continuous closed surjection $g \colon \omega^\omega \to X$ given by the theorem of Engelking [6] (c.f. proof of Proposition 3.5) and obtain a contradiction in a similar fashion. □

We note that not every map which preserves complete metrizability is necessarily resolvable. Consider any Polish space (X, σ) and any Borel set $B \subseteq X$ which is not resolvable, that is, $B \notin \mathbf{\Delta}^0_2$. There is a finer Polish topology $\tau \supseteq \sigma$ on X such that B is clopen in τ [15, Theorem 13.1]. The identity map between (X, τ) and (X, σ) is a continuous surjection between completely metrizable spaces but is not resolvable.

CHAPTER 4

THE LACZKOVICH–KOMÁTH PROPERTY

Let $(A_n)_{n\in\omega}$ be a sequence of sets and $K \in [\omega]^\omega$ an infinite subset of ω. The limit superior $\limsup_{n\in K} A_n$ is the set of all elements which belong to A_n for infinitely many $n \in K$. Laczkovich [18] showed that for every sequence $(A_n)_{n\in\omega}$ of Borel sets in a Polish space, if $\limsup_{n\in K} A_n$ is uncountable for every $K \in [\omega]^\omega$, then there exists a $K \in [\omega]^\omega$ such that $\bigcap_{n\in K} A_n$ is uncountable. Komjáth [17] generalized this result to the case where the sets $(A_n)_{n\in\omega}$ are analytic. Note that by the perfect set property of analytic sets, if $\bigcap_{n\in K} A_n$ is uncountable, then it contains a perfect set. Balcerzak and Głąb [2] extended these results to F_σ equivalence relations in the following way.

Definition 4.1. An equivalence relation E on a Polish space X is has the *Laczkovich-Komjáth property* if for every sequence $(A_n)_{n\in\omega}$ of analytic subsets of X such that $\limsup_{n\in K} A_n$ meets uncountably many E-equivalence classes for every $K \in [\omega]^\omega$, there exists a $K \in [\omega]^\omega$ such that $\bigcap_{n\in K} A_n$ contains a perfect set of pairwise E-inequivalent elements.

In this terminology, Komjáth has shown that the identity relation $=$ has the Laczkovich–Komjáth property. Balcerzak and Głąb [2] proved that every F_σ equivalence relation has the Laczkovich–Komjáth property. In this chapter, we generalize this to coanalytic equivalence relations:

Theorem 4.2. *Every coanalytic equivalence relation on a Polish space has the Laczkovich–Komjáth property.*

A fundamental result on coanalytic equivalence relations is Silver's theorem: a coanalytic equivalence relation either has only countably many equivalence classes, or else there exists a perfect set of pairwise inequivalent elements. Silver's original proof [26] used forcing. Harrington (unpublished) later gave a simpler (forcing) proof using effective descriptive set theory, which nowadays is usually cast in terms of the Gandy–Harrington topology. We will use similar methods and assume familiarity with effective descriptive set theory throughout this chapter. An introduction to effective descriptive set theory is given in [19], where the reader can also find the topological version of Harrington's proof. The review in [10] provides details on the Gandy–Harrington topology and strong Choquet games. Instead of strong Choquet games, we will make use of the set of low elements, which is a Polish space in the Gandy–Harrington topology. We will summarize the technical facts we use later on. Further details can be found in [8], which also provides another source on effective descriptive set theory.

This chapter is organized as follows. We review the original results of Laczkovich and Komjáth on limit superiors of sequences of sets in Section 4.1. In Section 4.2 we briefly consider their results in the context of definable sets, and then introduce the generalization of Laczkovich's and Komjáth's work to definable equivalence relations. The rest of the chapter is devoted to the proof of Theorem 4.2. In Section 4.3 we review a well-known coding mechanism for Π^1_1 and Δ^1_1 sets, mainly to fix notation and establish the uniformity of a diagonal intersection operator. In Section 4.4, we provide details on canonical cofinal sequences as developed in [9]. We use these sequences in Section 4.5 to prove our main technical result. Finally, we prove Theorem 4.2 in Section 4.6, where we also derive a parametric version of the theorem, corresponding to a result of Balcerzak and Głąb [2].

4.1. Limit Superiors of Sequences of Sets

The limit superior of a sequence $(A_n)_{n\in\omega}$ of sets is the collection of all points which are elements of infinitely many A_n. In other words, the limit superior is defined by

$$x \in \limsup_{n\in\omega} A_n \Leftrightarrow \forall m \exists n \geq m (x \in A_n).$$

For any $H \subseteq \omega$, let $[H]^\omega$ denote the collection of all infinite subsets of H. Then

$$\limsup_{n\in\omega} A_n = \bigcup_{H\in[\omega]^\omega} \bigcap_{n\in H} A_n.$$

This formula inspired Laczkovich to raise the following question: if $\limsup_{n\in\omega} A_n$ is large (say, uncountable), is there always an $H \in [\omega]^\omega$ such that $\bigcap_{n\in H} A_n$ is large as well (say, infinite)? The following example shows this is not the case.

EXAMPLE 4.3 (Laczkovich). Let $(A_n)_{n\in\omega}$ enumerate the following sequence of closed intervals contained in the unit interval $[0,1]$:

$$\left[\frac{i-1}{j}, \frac{i}{j}\right]$$

with $i \leq j < \omega$. Then $\limsup_{n\in\omega} A_n = [0,1]$ has cardinality $2^{\aleph_0}$ but $\left|\bigcap_{n\in H} A_n\right| \leq 1$ for every $H \in [\omega]^\omega$.

Hence, we need to change the question. By definition,

$$\limsup_{n\in H} A_n = \bigcup_{K\in[H]^\omega} \bigcap_{n\in K} A_n.$$

Laczkovich asked the following reformulated question: if $\limsup_{n\in H} A_n$ is large for every $H \in [\omega]^\omega$, is there then always an $H \in [\omega]^\omega$ such that $\bigcap_{n\in H} A_n$ is large? O6f course, we need to make precise what we mean by large. We do this in the following definition.

DEFINITION 4.4. For cardinals $\kappa \geq \lambda$, let $\mathsf{LK}(\kappa, \lambda)$ denote the following combinatorial statement: for every sequence $(A_n)_{n\in\omega}$ of sets such that

$$\left|\limsup_{n\in H} A_n\right| \geq \kappa$$

for every $H \in [\omega]^\omega$, there exists an $H \in [\omega]^\omega$ such that

$$\left|\bigcap_{n\in H} A_n\right| \geq \lambda.$$

Laczkovich provided the following two straightforward observations.

PROPOSITION 4.5 (Laczkovich). $\forall m \in \omega\, (\mathsf{LK}(m, m))$.

PROOF. Let m be a natural number and $(A_n)_{n\in\omega}$ a sequence of sets such that $\limsup_{n\in H} A_n$ has cardinality at least m for every $H \in [\omega]^\omega$. Suppose every intersection $\bigcap_{n\in H} A_n$ has cardinality less than m. Pick an H such that the cardinality of $\bigcap_{n\in H} A_n$ is maximal. We claim that $\limsup_{n\in H} A_n = \bigcap_{n\in H} A_n$ from which the proposition follows.

Certainly, $\bigcap_{n\in H} A_n \subseteq \limsup_{n\in H} A_n$. In order to see that the other inclusion holds, suppose $x \in \limsup_{n\in H} A_n$. Then there is a $K \in [H]^\omega$ such that $\bigcap_{n\in K} A_n$. Clearly, $\bigcap_{n\in K} A_n \supseteq \bigcap_{n\in H} A_n$. Since the cardinality of $\bigcap_{n\in H} A_n$ is maximal, $\bigcap_{n\in K} A_n = \bigcap_{n\in H} A_n$. Hence, $x \in \bigcap_{n\in H} A_n$. This is a contradiction. $\square$

The following example of Laczkovich's shows that $\mathsf{LK}(\aleph_0, \aleph_0)$ is false.

EXAMPLE 4.6. The sets $A_n = \{1, \dots, n\}$ are a counterexample to $\mathsf{LK}(\aleph_0, \aleph_0)$. Clearly, for every $H \in [\omega]^\omega$ the limit superior $\limsup_{n\in H} A_n = \omega$ is infinite, but the intersection $\bigcap_{n\in H} A_n = A_{\min(H)}$ is finite.

On the positive side, $\mathsf{LK}(\aleph_1, \aleph_0)$ does hold. To prove this, we introduce the following terminology.

DEFINITION 4.7. Let $(A_n)_{n\in\omega}$ be a sequence of sets. A set Y is *good* with respect to $H \in [\omega]^\omega$ if and only if $Y \cap \limsup_{n\in K} A_n$ is uncountable for every $K \in [H]^\omega$. We call a sequence $(A_n)_{n\in\omega}$ good if and only if $\bigcup_{n\in\omega} A_n$ is ω-good.

Note that if Y is H-good, then Y is K-good for any $K \in [H]^\omega$. The following lemma is the key combinatorial fact about good sets.

LEMMA 4.8 (Laczkovich). *If $Y = \bigcup_{n\in\omega} Y_n$ is H-good, then there is a $k \in \omega$ and a $H' \in [H]^\omega$ such that Y_k is H'-good.*

PROOF. Suppose towards a contradiction that for every $k \in \omega$ and every $H' \in [H]^\omega$ there is an $K \in [H]^\omega$ such that $Y_k \cap \limsup_{n\in K} A_n$ is countable. We can then pick a $K_1 \in [H]^\omega$ such that $Y_1 \cap \limsup_{n\in K_1} A_n$ is countable. Similarly, when $K_i \in [H]^\omega$ has been chosen, we can choose a $K_{i+1} \in [K_i]^\omega$ such that $Y_{i+1} \cap \limsup_{n\in K_{i+1}}$ is countable. Finally, pick a strictly increasing sequence $n_i \in K_i$. Then $K = \{n_1, n_2, \dots\}$ is almost contained in every K_i, which implies that for every $i \in \omega$, $\limsup_{n\in K} A_n \subseteq \limsup_{n\in K_i} A_n$ Hence, $Y_i \cap \limsup_{n\in K} A_n$ is countable for every $i \in \omega$. Therefore,

$$\bigcup_{i\in\omega}(Y_i \cap \limsup_{n\in K} A_n) = (\bigcup_{i\in\omega} Y_i) \cap \limsup_{n\in K} A_n = Y \cap \limsup_{n\in K} A_n$$

is countable, a contradiction. $\square$

We are ready for the proof of the theorem:

THEOREM 4.9 (Laczkovich). $\mathsf{LK}(\aleph_1, \aleph_0)$.

PROOF. Assume $(A_n)_{n\in\omega}$ is a good sequence. By the lemma, there is a $k_0 \in \omega$ and a $H_0 \in [\omega]^\omega$ such that A_{k_0} is H_0-good. Pick $x_0 \in A_{k_0} \cap \limsup_{n\in H_0} A_n$, and then $K_0 \in [H_0]^\omega$ such that $x_0 \in A_{k_0} \cap \bigcap_{n\in K_0} A_n$. Note that A_{k_0} is K_0-good.

Suppose $k_0 < \cdots < k_i$, $x_0, \ldots, x_i$, and $K_i \in [\omega]^\omega$ are given such that

(i) $x_0, \ldots, x_i \in A_{k_0} \cap \cdots \cap A_{k_i} \cap \bigcap_{n\in K_i} A_n$

(ii) $\bigcap_{n=1}^{i} A_{k_n}$ is K_i-good, i.e.

$$\bigcup_{n>k_i, n\in K_i} (A_n \cap A_{k_0} \cap \cdots \cap A_{k_i})$$

is K_i-good.

By the lemma, there is a $k_{i+1} > k_i$, $k_{i+1} \in K_i$, and a $H_{i+1} \in [K_i]^\omega$ such that $A_{k_0} \cap \cdots \cap A_{k_i} \cap A_{k_{i+1}}$ is H_{i+1}-good. We can pick $x_{i+1} \neq x_0, \ldots, x_i$ and $K_{i+1} \in [H_{i+1}]^\omega$ such that

$$x_{i+1} \in A_{k_0} \cap \cdots \cap A_{k_{i+1}} \cap \bigcap_{n\in K_{i+1}} A_n.$$

Note that $x_0, \ldots, x_i$ are in this set as well by the first assumption. In this way we get two sequences $(A_{k_n})_{n\in\omega}$ and $(x_n)_{n\in\omega}$ such that

$$\{x_0, \ldots, x_n\} \subseteq A_{k_0} \cap \cdots \cap A_{k_n}$$

for every $n \in \omega$. Hence, $\bigcap_{n\in\omega} A_{k_n}$ is infinite. $\square$

Clearly, $\mathsf{LK}(\aleph_1, \aleph_0)$ implies $\mathsf{LK}(\kappa, \aleph_0)$ for all $\kappa \geq \aleph_1$. A reasonable next goal to prove would be $\mathsf{LK}(\aleph_1, \aleph_1)$. However, this fails when the continuum hypothesis (CH) holds.

THEOREM 4.10 (Laczkovich). $\mathsf{CH} \vdash \neg\mathsf{LK}(\aleph_1, \aleph_1)$.

PROOF. The continuum hypothesis implies the existence of a Sierpiński set: an uncountable set $S \subseteq [0,1]$ such that $S \cap N$ is countable for every $N \subseteq [0,1]$ with Lebesgue measure $\mu(N) = 0$. Note that S meets every set of full measure in uncountably many points (To see this, if F has full measure, then $[0,1] - F$ has measure 0. Thus, $S \cap [0,1] - F$ is countable, and $S \cap F$ is uncountable). Let

$$B_n = \bigcup_{k=1}^{2^n-1}(\frac{2k-2}{2^n}, \frac{2k-1}{2^n}).$$

Then $\mu(\bigcap_{n\in H} B_n) = 0$ for every $H \in [\omega]^\omega$. Moreover,

$$\mu(\limsup_{n\in H} B_n) = \mu(\bigcap_{k=1}^{\infty} \bigcup_{n\in H, n\geq k} B_n) = \lim_{k\to\infty} \mu(\bigcup_{n\in H, n\geq k} B_n) = 1.$$

Let S be a Sierpiński set and put $A_n = S \cap B_n$. Since $\limsup_{n\in H} B_n$ has full Lebesgue measure,

$$\limsup_{n\in H} A_n = \limsup_{n\in H}(S \cap B_n) = S \cap \limsup_{n\in H} B_n$$

is uncountable. At the same time, $\bigcap_{n\in H} B_n$ has Lebesgue measure 0, and therefore

$$\bigcap_{n\in H} A_n = \bigcap_{n\in H}(S \cap B_n) = S \cap \bigcap_{n\in H} B_n$$

is countable. □

On the other hand, Komjáth derived $\mathsf{LK}(\aleph_1, \aleph_1)$ by strengthening ZFC with Martin's axiom $\mathsf{MA}(\aleph_1)$. Only the following consequence of $\mathsf{MA}(\aleph_1)$ is used. A family $\mathscr{A}$ of sets is called *centered* if $A_1 \cap \cdots \cap A_n$ is infinite for all $A_1, \ldots, A_n \in \mathscr{A}$.

LEMMA 4.11 (Solovay). *Assume* $\mathsf{MA}(\aleph_1)$. *If* $\mathscr{A}$ *is a centered family of subsets of* ω *with* $|\mathscr{A}| \leq \aleph_1$, *then there is an* $H \in [\omega]^\omega$ *such that* $H \subseteq^* A$ *for every* $A \in \mathscr{A}$. □

THEOREM 4.12 (Komjáth). $\mathsf{MA}(\aleph_1) \vdash \mathsf{LK}(\aleph_1, \aleph_1)$.

PROOF. Let $(A_n)_{n\in\omega}$ be a good sequence. For $x \in \bigcup_{n\in\omega} A_n$, let $H_x = \{n \in \omega \colon x \in A_n\}$.

CLAIM 4.13. *There exists an uncountable set* $S \subseteq \bigcup_{n\in\omega} A_n$ *such that* $\{H_x \colon x \in S\}$ *is centered, i.e.* $H_{x_0} \cap \cdots \cap H_{x_n}$ *is infinite for all* $x_0, \ldots, x_n \in S$.

PROOF. Suppose towards a contradiction that there does not exists an uncountable centered family. Choose a maximal centered family

$$\{H_x : x \in S\}.$$

By assumption S is countable. Hence, we can get an $H \in [\omega]^\omega$ such that $H \subseteq^* H_{x_0} \cap \cdots \cap H_{x_n}$ for all $x_0, \ldots, x_n \in S$. We claim that

$$\limsup_{n \in H} A_n \subseteq S,$$

which is a contradiction, because the limit superior is uncountable.

Suppose $x \notin S$. Since S is a maximal centered family, there are $x_0, \ldots, x_n \in S$ such that

$$H_x \cap H_{x_0} \cap \cdots \cap H_{x_n}$$

is finite. Since $H \subseteq^* H_{x_0} \cap \cdots \cap H_{x_n}$, we have $H_x \cap H$ is finite, too. Thus, there are only finitely many $n \in H$ such that $x \in A_n$, that is, $x \notin \limsup_{n \in H} A_n$. □

Let S be an uncountable set such that $\{H_x : x \in S\}$ is centered by the claim. By Solovay's Lemma, there is an $H \in [\omega]^\omega$ such that $H \subseteq^* H_x$ for every $x \in S$. Hence,

$$S = \bigcup_{k \in \omega} \{x \in S : H - H_x \subseteq \{0, 1, \ldots, k\}\}$$

Since an uncountable set is not a countable union of countable sets, there is an $k \in \omega$ such that $S' := \{x \in S : H - H_x \subseteq \{0, 1, \ldots, k\}\}$ is uncountable. Now note that

$$S' \subseteq \bigcap_{n \in H, n > k} A_n$$

and hence $K := \{n \in H : n > k\}$ is the required set so that $\bigcap_{n \in K} A_n$ is uncountable. □

4.2. Definable Sets and Equivalence Relations

We have seen that $\mathsf{CH} \vdash \neg\mathsf{LK}(\aleph_1, \aleph_1)$. This leaves open the possibility that in ZFC, for some $\kappa > \aleph_1$ we do have $\mathsf{LK}(\kappa, \aleph_1)$ for some $\kappa > \aleph_1$. However, Komjáth has shown that by adding κ Cohen reals to a model of GCH we get $\neg\mathsf{LK}(\kappa, \aleph_1)$.

In summary, consider the combinatorial statement $\mathsf{LK}(\aleph_1, \aleph_1)$. Theorem 4.10 shows that it is consistently false, while Theorem 4.12 shows it is consistently true. In other words,

$\mathsf{LK}(\aleph_1, \aleph_1)$ is independent of ZFC. In order to avoid this obstacle, we could restrict our attention to definable sets of Polish spaces. In particular, we may require the sets $(A_n)_{n\in\omega}$ to be Borel, analytic, or coanalytic subsets of a Polish space.

Definition 4.14. For a pointclass Γ, let $\mathsf{LK}_\Gamma(\kappa, \lambda)$ denote the statement $\mathsf{LK}(\kappa, \lambda)$ restricted to sequences $(A_n)_{n\in\omega}$ with each $A_n \in \Gamma$.

Denote the pointclasses of the Borel, analytic, and coanalytic sets by $\mathbf{B}$, $\mathbf{A}$, and $\mathbf{CA}$, respectively. Laczkovich [18] proved $\mathsf{LK}_\mathbf{B}(\aleph_1, \aleph_1)$ and Komjáth [17] generalized this to $\mathsf{LK}_\mathbf{A}(\aleph_1, \aleph_1)$. Besides a direct construction to prove this, Komjáth also gave a forcing argument to derive $\mathsf{LK}_\mathbf{A}(\aleph_1, \aleph_1)$ from Theorem 4.12. His proofs can be found in [17]. We note here that our main result subsumes $\mathsf{LK}_\mathbf{A}(\aleph_1, \aleph_1)$, providing yet another proof. Komjáth did show that we cannot take the sets $(A_n)_{n\in\omega}$ to be coanalytic.

Theorem 4.15 (Komjáth). $\mathbf{V} = \mathbf{L} \vdash \neg\mathsf{LK}_\mathbf{CA}(\aleph_1, \aleph_1)$.

Proof. The axiom of constructibility $\mathbf{V} = \mathbf{L}$ implies the existance of a Lusin set: an uncountable set $T \subseteq \mathbb{R}$ such that every nowhere dense subset of T is countable. Moreover, T is a continuous image of a coanalytic set, $T = f(S)$. By the Novikoff-Kondo uniformization theorem, we can assume that f is injective on S. For $n \in \omega$ let B_n denote the set of reals with 1 a nth binary digit after the "decimal" point, $A_n = S \cap f^{-1}(B_n)$. For $H \in [\omega]^\omega$, $\bigcap_{n\in H} A_n$ is nowhere dense. The complement of $\limsup_{n\in H} A_n$ is nonmeager. By the Lusin property,

$$\bigcap_{n\in H} A_n = S \cap \bigcap_{n\in H} f^{-1}(B_n) = T \cap \bigcap_{n\in H} B_n$$

is countable and

$$\limsup_{n\in H} A_n = T \cap \limsup_{n\in H} B_n$$

is uncountable. $\square$

We now consider a generalization of Laczkovich's original question in another direction.

DEFINITION 4.16. Let E be an equivalence relation on a Polish space. We say that E has the *Laczkovich–Komjáth property* if for every sequence $(A_n)_{n\in\omega}$ such that $\limsup_{n\in H} A_n$ meets uncountably many E-equivalence classes for every $H \in [\omega]^\omega$, there is an $H \in [\omega]^\omega$ such that $\bigcap_{n\in H} A_n$ contains a perfect set of pairwise E-inequivalent elements.

In this terminology, Laczkovich and Komjáth showed that the equality relation $=$ has the Laczkovich–Komjáth property. Balcerzak and Głab [2] proved the following generalization:

THEOREM 4.17 (Balcerzak–Głab). *Every F_σ equivalence relation has the Laczkovich–Komjáth property.*

We will generalize their result to coanalytic equivalence relations. The rest of this chapter is devoted to the proof of this generalization, stated earlier as Theorem 4.2.

4.3. Coding Π^1_1 and Δ^1_1 sets

In this section we review a well-known coding mechanism for Π^1_1 and Δ^1_1 sets, mainly to fix notation. A good introduction can be found in [10, Section 3.2], where the notion of uniformity is also discussed. We will need the uniformity of a diagonal intersection operation. Since this operation is not canonical, we provide a little more of the details.

A *product space* is any $X = X_0 \times \cdots \times X_n$ (with the product topology), where each factor is either ω or ω^ω. For every product space X there is a $U^X \subseteq \omega \times X$ such that $U^X \in \Pi^1_1$ and for any $A \subseteq X$, $A \in \Pi^1_1$ if and only if $\exists n(A = U^X_n)$. Such a set U^X is called a *universal Π^1_1 set.* A *Π^1_1 code* for $A \subseteq X$ is any $n \in \omega$ such that $A = U^X_n$. There exists a collection $\{U^X\}$ of universal Π^1_1 sets with the following additional property: for any $m \in \omega$ and any product space X there is a recursive function $S^{m,X} : \omega^{m+1} \to \omega$ such that

$$(e, k_1, \ldots, k_m, x) \in U^{\omega^m \times X} \Leftrightarrow (S^{m,X}(e, k_1, \ldots, k_m), x) \in U^X.$$

Such a collection is called a *good universal system.* For the rest of this chapter, fix a good universal system $\{U^X\}$ for Π^1_1. This good universal system can be used to code Δ^1_1 subsets, as we now describe. This coding is always relative to a particular product space X. When there is no danger of confusion, we will drop the superscript in U^X. For every $k \in \omega$, fix a recursive bijection $(n_1, \ldots, n_k) \mapsto \langle n_1, \ldots, n_k \rangle$ between ω^k and ω. Define

$$(\langle m, n \rangle, x) \in U_0 \Leftrightarrow (m, x) \in U,$$

$$(\langle m, n \rangle, x) \in U_1 \Leftrightarrow (n, x) \in U.$$

Then $U_0, U_1 \in \Pi^1_1$. By the reduction property for Π^1_1 sets, there are Π^1_1 sets $U_0^*, U_1^* \subseteq \omega \times X$ such that $U_0^* \cup U_1^* = U_0 \cup U_1$ and $U_0^* \cap U_1^* = \varnothing$. Let $P = U_0^*$ and $S = (\omega \times X) - U_1^*$. Let

$$\langle m, n \rangle \in C \Leftrightarrow \forall x \in X((\langle m, n \rangle, x) \in U_0^* \vee (\langle m, n \rangle, x) \in U_1^*).$$

Then $C \in \Pi^1_1$ and for all $n \in C$, $P_n = S_n := D_n$.

A Δ^1_1 *code* for $A \subseteq X$ is any $n \in C$ such that $A = D_n$. In that case, $(n)_0$ is a Π^1_1 code for A and $(n)_1$ is a Π^1_1 code for $X - A$. Conversely, if $m, n \in \omega$ are Π^1_1 codes for A and $X - A$, respectively, then $\langle m, n \rangle$ is a Δ^1_1 code for A. It is important that the set C of Δ^1_1 codes is Π^1_1 and that set-theoretic operations are effective in the codes, in the following way.

EXAMPLE 4.18. Given Δ^1_1 codes $m, n \in C$ for $A, B \subseteq X$, we can effectively compute a Δ^1_1 code for $A - B$. To see this, define

$$(m, n, x) \in Z_0 \Leftrightarrow x \in D_m \wedge x \notin D_n,$$

$$(m, n, x) \in Z_1 \Leftrightarrow x \notin D_m \vee x \in D_n.$$

Clearly, $Z_0, Z_1 \in \Pi^1_1$. Let e_0, e_1 be their respective Π^1_1 codes. Then for $i = 0, 1$,

$$(m, n, x) \in Z_i \Leftrightarrow (e_i, m, n, x) \in U^{\omega^2 \times X} \Leftrightarrow (S^{2,X}(e_i, m, n), x) \in U^X.$$

Also, $Z_0 = (\omega^2 \times X) - Z_1$. Thus,

$$\langle S^{2,X}(e_0, m, n), S^{2,X}(e_1, m, n) \rangle$$

is a Δ^1_1 code for $A - B$.

A similar property (often called *uniformity*) holds for all basic set-theoretic operations. We will need the uniformity of a diagonal intersection operator, which we define next. Recall that when $H, K \in [\omega]^\omega$, we write $H \subseteq^* K$ to denote that H is almost contained in K, i.e. $K - H$ is finite.

DEFINITION 4.19. For a (finite or infinite) sequence (K_n) of infinite subsets of ω with $K_n \subseteq^* K_m$ for $n > m$, define $\triangle K_n$ by $m \in \triangle K_n$ if and only if there exists $m_0 < m_1 < \cdots < m_k = m$ such that m_0 is the least element of K_0, m_1 is the least element of $K_0 \cap K_1$ such that $m_1 > m_0$, $\ldots$, m_k is the least element of $K_0 \cap \cdots \cap K_k$ such that $m_k > m_{k-1}$.

Note that $\triangle K_n \subseteq^* K_m$ for all m. To obtain the desired uniformity for this diagonal intersection operation, we need to assume that the sequence of Δ^1_1 codes for (K_n) is effective. One way to formalize this is to let $n \in C^*$ if and only if

(i) $n \in C^\omega$,
(ii) D^ω_n is infinite,
(iii) $\forall m(m \in D^\omega_n \Rightarrow (m)_1 \in C^\omega)$, and
(iv) $\forall i \exists! m(m \in D^\omega_n \wedge (m)_0 = i)$.

Informally, $n \in C^*$ if and only if n is a Δ^1_1 code for an infinite subset of ω of the form $\{\langle i, n_i \rangle : i \in \omega, n_i \in C\}$. Clearly, $C^* \in \Pi^1_1$.

LEMMA 4.20. *There is a function* Diag: $\omega \to \omega$ *which is* Δ^1_1 *on* C^* *such that whenever* $n \in C^*$ *is a code for an infinite* Δ^1_1 *subset* $\{\langle i, n_i \rangle : i \in \omega, n_i \in C\}$ *of* ω, Diag(n) *is a* Δ^1_1 *code for* $\triangle D^\omega_{n_i}$.

PROOF. It suffices to find Π^1_1 codes e_0 and e_1 for $\triangle D^\omega_{n_i}$ and $\omega - \triangle D^\omega_{n_i}$, respectively, because $\langle e_0, e_1 \rangle$ will then be a Δ^1_1 code for $\triangle D^\omega_{n_i}$. We need the following three auxiliary functions:

(i) There is a recursive function $u \colon \omega \to \omega$ such that whenever $n = \langle n_0, \ldots, n_k \rangle$ is a finite sequence of Δ^1_1 codes, $u(n)$ is a Δ^1_1 code for $D^\omega_{n_0} \cap \cdots \cap D^\omega_{n_k}$.
(ii) There is a recursive function $i \colon \omega \times \omega \to \omega$ such that whenever $n \in C^*$, $i(n, j)$ is the (unique) $m \in \omega$ such that $\langle j, m \rangle \in D^\omega_n$.

(iii) There is a Δ^1_1 on the codes function $\mu\colon \omega \times \omega \to \omega$ such that whenever n is a Δ^1_1 code for an infinite subset of ω, $\mu(n, j)$ is the least element of D^ω_n greater than or equal to j.

Now define

$$(n, m) \in Z_0 \Leftrightarrow n \in C^* \wedge \exists \langle m_0, \ldots, m_k \rangle \,(m_0 < \cdots < m_k \wedge m_k = m \wedge$$
$$m_0 = \mu(u(\langle i(n,0)\rangle), 0) \wedge m_1 = \mu(u(\langle i(n,0), i(n,1)\rangle), m_0 + 1) \wedge$$
$$\cdots \wedge m_k = \mu(u(\langle i(n,0), \ldots, i(n,k)\rangle), m_{k-1} + 1)).$$

Then $Z_0 \in \Pi^1_1$. Pick a Π^1_1 code e_0 for Z_0. Similarly, we can write down a Π^1_1 definition for $Z_1 = C^* - Z_0$ and pick a Π^1_1 code e_1. The rest of the argument is as in the Example. $\square$

Now that we have established the uniformity of this diagonal intersection operator, we will use it implicitly. Finally, for codes $h, k \in C^\omega$, we write $h \subseteq^* k$ if and only if the set coded by h is almost contained in the set coded by k. Writing out the definitions, we see that $h \subseteq^* k$ is Δ^1_1 on the set C^ω of codes.

4.4. Canonical Cofinal Sequences

For $w \in 2^\omega$, define a binary relation $<_w$ on a subset of ω by

$$m <_w n \Leftrightarrow w(\langle m, n\rangle) = 1.$$

The domain of $<_w$ is the set

$$\mathrm{dom}(<_w) = \{n \in \omega\colon \exists m \in \omega(m <_w n \text{ or } n <_w m)\}.$$

Let LO denote the set of all $w \in 2^\omega$ such that $<_w$ is a linear order, and let LO^* denote the set of all $w \in \mathsf{LO}$ such that $<_w$ has a least element and every $n \in \mathrm{dom}(<_w)$ has an immediate successor $n^+_{<_w}$. For $w \in \mathsf{LO}$, let $|<_w|$ denote the order type of $<_w$. The next lemma shows that in a uniform way, we can effectively obtain a canonical cofinal sequence in $<_w$ given $w \in \mathsf{LO}^*$.

Lemma 4.21 (Gao–Jackson–Laczkovich–Mauldin [9]). *There is a Δ^1_1 function*

$$\mathrm{Cof}\colon \{(w,n,j) \in \mathsf{LO}^* \times \omega^2 \colon n \in \mathrm{dom}(<_w)\} \to \omega$$

such that

(i) *if $w \in \mathsf{LO}^*$, $n \in \mathrm{dom}(<_w)$ and $j \in \omega$, then $\mathrm{Cof}(w,n,j) \in \mathrm{dom}(<_w)$ and $\mathrm{Cof}(w,n,j) <_w n$, unless n is the $<_w$-least element;*

(ii) *if $w \in \mathsf{LO}^*$ and $n \in \mathrm{dom}(<_w)$ has an immediate predecessor in $<_w$, then $\mathrm{Cof}(w,n,j)^+_w = n$ for all $j \in \omega$;*

(iii) *if $w \in \mathsf{LO}^*$, $n \in \mathrm{dom}(<_w)$ is not $<_w$-least and n does not have an immediate predecessor in $<_w$, then*

(a) *if $j < j'$, then $\mathrm{Cof}(w,n,j) <_w \mathrm{Cof}(w,n,j')$, and*

(b) *for any $q \in \mathrm{dom}(<_w)$ with $q <_w n$ there is a $j \in \omega$ such that $q <_w \mathrm{Cof}(w,n,j)$.*

□

We also need a variation of this lemma for Π^1_1 norms, whose proof uses the same ideas. Recall that a Π^1_1-norm on a pointset $P \in \Pi^1_1$ is a function φ from P into the ordinals On such that there exist binary relations $<^*_\varphi$ and $\leq^*_\varphi$ in Π^1_1 with the following properties:

$$x \leq^*_\varphi y \Leftrightarrow P(x) \wedge (\neg P(y) \vee \varphi(x) \leq \varphi(y)),$$

$$x <^*_\varphi y \Leftrightarrow P(x) \wedge (\neg P(y) \vee \varphi(x) < \varphi(y)).$$

Recall that WO denotes the set of all $w \in \mathsf{LO}$ such that $<_w$ is a well-order. Every Π^1_1 pointset admits a Π^1_1-norm $\varphi\colon P \to \omega_1^{\mathrm{CK}}$, where

$$\omega_1^{\mathrm{CK}} = \sup\{|<_w| : w \in \mathsf{WO} \text{ is recursive}\},$$

see for example [22, Section 4B].

Lemma 4.22. *Let φ be a Π^1_1-norm on a Π^1_1 set $P \subseteq \omega$. There is a Π^1_1 function $\mathrm{Cof}\colon \omega \to \omega$ such that*

(i) *for all $j \in \omega$, $\mathrm{Cof}(j) \in P$;*

(ii) *if* $j < j'$, *then* $\mathrm{Cof}(j) <^*_\varphi \mathrm{Cof}(j')$;

(iii) *for any* $q \in P$, *there is an* $j \in \omega$ *such that* $q <^*_\varphi \mathrm{Cof}(j)$ *unless* q *is* $<_\varphi$*-maximal.*

PROOF. We define the function Cof by induction on j. Let $p_0 = \mathrm{Cof}(0)$ be the least integer in P. Assume we have defined $p_j = \mathrm{Cof}(j)$. Let $p_{j+1} = \mathrm{Cof}(j+1)$ be the smallest integer in P such that $p_j < p_{j+1}$ and $p_j <^*_\varphi p_{j+1}$. Since $n = p_{j+1}$ if and only if $n \in P$ and $p_j < n$ and $p_j <^*_\varphi n$ and $\forall m(p_j < m < n \Rightarrow m \leq^*_\varphi p_j)$, this defines a Π^1_1 function. To see that (3) holds, let $q \in P$ be a nonmaximal element. Since the sequence $(p_j)_{j\in\omega}$ is strictly increasing in the natural order $<$ on ω, there is a least integer j such that $p_j \leq q < p_{j+1}$. Because p_{j+1} is the least integer larger than p_j such that $p_j <^*_\varphi p_{j+1}$, we cannot have $p_j <^*_\varphi q$. Hence, $q \leq^*_\varphi p_j <^*_\varphi p_{j+1}$. $\square$

4.5. A Completely Good Pair

Suppose E is a Π^1_1 equivalence relation on ω^ω. A key idea in Harrington's proof of Silver's dichotomy is to consider the set

$$W = \{x \in \omega^\omega \colon \text{there is no } \Delta^1_1 \text{ set } D \text{ such that } x \in D \subseteq [x]_E\}.$$

A computation shows that W is Σ^1_1. Moreover, when E has uncountably many equivalence classes, $W \neq \varnothing$ and every nonempty Σ^1_1 subset $X \subseteq W$ meets uncountably many E-equivalence classes. In fact, a nonempty Σ^1_1 subset $X \subseteq \omega^\omega$ meets uncountably many E-equivalence classes if and only if $X \cap W \neq \varnothing$. We will establish the following corresponding result in our context.

PROPOSITION 4.23. *Let E be a Π^1_1 equivalence relation on ω^ω and $(A_n)_{n\in\omega}$ a sequence of uniformly Σ^1_1 subsets of ω^ω. If $\limsup_{n\in K} A_n$ meets uncountably many E-equivalence classes for every $K \in [\omega]^\omega$, then there exists a nonempty Σ^1_1 set $V \subseteq \omega^\omega$ and a Δ^1_1 set $H \in [\omega]^\omega$ such that for every nonempty Σ^1_1 set $X \subseteq V$ and every Δ^1_1 set $K \in [H]^\omega$ the set $X \cap \limsup_{n\in K} A_n$ meets uncountably many E-equivalence classes.*

We call such a pair (V, H) *completely good.* The rest of this section is devoted to the proof of Proposition 4.23 and a further refinement. In contrast with Harrington's proof, we need a recursive construction of transfinite length in which we remove all possible bad pairs one by one.

DEFINITION 4.24. We say that $n = \langle y, k\rangle \in \omega$ is a *bad pair* if the following properties hold:

(i) $y \in C^{\omega^\omega}$ and $k \in C^\omega$,

(ii) $D^\omega_k \in [\omega]^\omega$, and

(iii) $D^{\omega^\omega}_y \cap \limsup_{n\in D^\omega_k} A_n$ meets only countably many E-equivalence classes, i.e. $D^{\omega^\omega}_y \cap W \cap \limsup_{n\in D^\omega_k} A_n = \varnothing$.

It is clear from this definition that the set $P \subseteq \omega$ of all bad pairs is Π^1_1. Let $\varphi\colon P \to \omega_1^{\mathrm{CK}}$ be a Π^1_1-norm on P. Define a well-order on P by

$$m <_\varphi n \Leftrightarrow \varphi(m) < \varphi(n) \vee (\varphi(m) = \varphi(n) \wedge m < n)$$

and let $\leq^*_\varphi$ be the Π^1_1 relation given by

$$m \leq^*_\varphi n \Leftrightarrow P(m) \wedge (\neg P(n) \vee \varphi(m) \leq \varphi(n))$$

Denote by C^ω_∞ the set of all $n \in C^\omega$ such that $D^\omega_n \in [\omega]^\omega$. Then C^ω_∞ is Π^1_1. Given an $h \in C^\omega_\infty$, we define the *next bad pair relative to* h to be the $<_\varphi$-least $\langle y, k\rangle \in P$ such that $k \subseteq^* h$. Set $R(h, \langle y, k\rangle)$ if and only if $\langle y, k\rangle$ is the next bad pair relative to h.

LEMMA 4.25. *The relation $R \subseteq \omega \times \omega$ is Π^1_1. Moreover, R is a Δ^1_1 function on the set $B = \{h \in \omega\colon h \in C^\omega_\infty \wedge \exists n(R(h, n))\}$.*

PROOF. We have $R(h, \langle y, k\rangle)$ if and only if

$$h \in C^\omega_\infty \wedge \langle y, k\rangle \in P \wedge k \subseteq^* h \wedge \forall y', k' \in \omega\,(\langle y, k\rangle \not\leq^*_\varphi \langle y', k'\rangle \Rightarrow k' \not\subseteq^* h)).$$

This is a Π^1_1 definition. If $R(h, n)$ holds, then n is the unique such integer. Thus, for $h \in B$, $\neg R(h, n) \Leftrightarrow \exists m(R(h, m) \wedge n \neq m)$, which is Π^1_1. Hence, R is Δ^1_1 on B. □

Initial segments of the recursive construction can be coded by reals, as follows. Recall that $\mathsf{WO}_\alpha = \{w \in \mathsf{WO}\colon |<_w| = \alpha\}$ and for $\alpha < \omega_1^{\mathrm{CK}}$, we have $\mathsf{WO}_\alpha \in \Delta^1_1$.

DEFINITION 4.26. Let $\alpha < \omega_1^{\mathrm{CK}}$. A real $z \in \omega^\omega$ is *α-adequate* if $z = \langle w, v, h\rangle$, where $w \in 2^\omega$, $v \in \omega^\omega$, and $h \in \omega^\omega$, and the following conditions are satisfied:

(i) $w \in \mathsf{WO}_\alpha$,

(ii) if $n \notin \mathrm{dom}(<_w)$, then $v(n) = h(n) = 0$,

(iii) the $<_w$-least element is the $<_\varphi$-least element,

(iv) if $n \in \mathrm{dom}(<_w)$ is a $<_w$-successor (say $n = m^+_{<_w}$), then the following holds:

(a) $n = \langle y, k\rangle$ is the next bad pair relative to $h(m)$ such that $\langle y, k\rangle \notin \mathrm{dom}(<_w) \upharpoonright n$,

(b) $v(n)$ is a canonical code for $D^{\omega^\omega}_{v(m)} - D^{\omega^\omega}_y$,

(c) $h(n) = k$.

(v) if $n \in \mathrm{dom}(<_w)$ is a $<_w$-limit, then with v' the canonical code for

$$\bigcap_{j\in\omega} D^{\omega^\omega}_{v(\mathsf{Cof}(w,n,j))}$$

and h' the canonical code for $\triangle_{j\in\omega} D^{\omega}_{h(\mathsf{Cof}(w,n,j))}$, the following holds:

(a) $n = \langle y, k\rangle$ is the next pair relative to h' such that $\langle y, k\rangle \notin \mathrm{dom}(<_w) \upharpoonright n$,

(b) $v(n)$ is the canonical code for $D^{\omega^\omega}_{v'} - D^{\omega^\omega}_y$, and

(c) $h(n) = k$.

Some comments on these conditions: (1) says that z represents the construction up to stage α, (2) is needed only to ensure that there can be at most one α-adequate real for every $\alpha < \omega_1^{\mathrm{CK}}$, (3), (4a), and (5a) state that $<_w$ represents the order in which the bad pairs are picked in our construction and that we pick a new bad pair at each stage, and conditions (4b,c) and (5b,c) require $v(n)$ and $h(n)$ to be codes for the correct sets whenever $n \in \mathrm{dom}(<_w)$. We call a real *adequate* if it is α-adequate for some $\alpha < \omega_1^{\mathrm{CK}}$.

LEMMA 4.27. *The set of all adequate reals is Π^1_1.*

PROOF. Replace condition (1) above with condition (1)$'$ $w \in \mathsf{WO}$, which is Π^1_1. Condition (2) is arithmetical. Condition (3) is equivalent both to

$$n \in \mathrm{dom}(<_w) \wedge \forall m(m \in \mathrm{dom}(<_w) \Rightarrow n \leq_w m) \Rightarrow \forall m(n \leq^*_\varphi m)$$

and to

$$n \in \mathrm{dom}(<_w) \wedge \forall m(m \in \mathrm{dom}(<_w) \Rightarrow n \leq_w m) \Rightarrow \forall m(m \not<^*_\varphi n)$$

which shows that (3) is both Π^1_1 and Σ^1_1, i.e. Δ^1_1. For (4), n is a $<_w$-successor, $n = (m)^+_{<_w}$, and (4b, c) are arithmetical predicates, while (4a) is Π^1_1. Thus, (4) is Π^1_1. Similarly, (5) is Π^1_1. □

LEMMA 4.28. *Every adequate real is* Δ^1_1.

PROOF. Assume $z = \langle w, v, h\rangle$ is α-adequate for some $\alpha < \omega_1^{\mathrm{CK}}$. Conditions (1), (2) and (3) are Δ^1_1. Conditions (4) and (5) are Π^1_1, because (4a) and (5a) contain a predicate $R(n, h)$, i.e. n is the next bad pair relative to h (where $h = h(m)$ in 4a and $h = h'$ in 5a). However, since z is given, we know that this h is an element of $B = \{h \in \omega \colon h \in C^\omega_\infty \wedge \exists n(R(h, n))\}$. By Lemma 4.25, R is Δ^1_1 on B. Thus, conditions (4) and (5) are Δ^1_1. □

LEMMA 4.29. *For each* $\alpha < \omega_1^{\mathrm{CK}}$, *if there is an* α*-adequate real, then this is the unique* Δ^1_1 *real* $z_\alpha \in \omega^\omega$ *which is* α*-adequate.*

PROOF. This is immediate from the definition of α-adequate and the previous lemma. □

Finally, we define $V \subseteq \omega^\omega$ and $H \in [\omega]^\omega$ as follows. Let $x \in V$ if and only if

$$\forall z \in \Delta^1_1(z = \langle w, v, h\rangle \text{ adequate} \Rightarrow \forall n(n \in \mathrm{dom}(<_w) \Rightarrow x \in S^{\omega^\omega}_{v(n)}))$$

and $n \in H$ if and only if

$$\exists z \in \Delta^1_1(z = \langle w, v, h\rangle \text{ is adequate} \wedge$$
$$\forall j \leq n(\mathrm{Cof}(j) \in \mathrm{dom}(<_w) \Rightarrow n \in \triangle_{j \leq n} h(\mathrm{Cof}(j))).$$

Equivalently by Lemma 4.29, $n \in H$ if and only if

$$\forall z \in \Delta^1_1(z = \langle w, v, h\rangle \text{ is adequate} \wedge$$
$$\forall j \leq n(\mathrm{Cof}(j) \in \mathrm{dom}(<_w) \Rightarrow n \in \triangle_{j \leq n} h(\mathrm{Cof}(j))).$$

LEMMA 4.30. $V \in \Sigma^1_1$ *and* $H \in \Delta^1_1$. *Moreover,* $V \neq \varnothing$.

Proof. By Kleene's restricted quantification theorem (see for example [22, Theorem 4D.3]), $V \in \Sigma^1_1$. (Note that if the construction stops below ω_1^{CK}, then V is actually Δ^1_1. We will not need that fact.) Similarly, the first definition of H is Π^1_1 and the second definition is Σ^1_1. Therefore, $H \in [\omega]^\omega$ is Δ^1_1. We show that $V \neq \varnothing$. Suppose towards a contradiction that $V = \varnothing$. Then for every $x \in \omega^\omega$ there is an $\alpha < \omega_1^{\mathrm{CK}}$ and a $k \in \omega$ such that for $z_\alpha = \langle w_\alpha, v_\alpha, h_\alpha \rangle$, we have $k \in \mathrm{dom}(<_{w_\alpha})$ and $x \notin D^{\omega^\omega}_{v_\alpha(k)}$. For $k \in \mathrm{dom}(<_{w_\alpha})$, denote by $y_\alpha(k)$ the code for the set removed at that stage. By assumption,

$$\omega^\omega = \bigcup_{\alpha<\omega_1^{\mathrm{CK}}} \bigcup_{k\in\mathrm{dom}(<_{w_\alpha})} D^{\omega^\omega}_{y_\alpha(k)}.$$

Since $H \subseteq^* D^\omega_{h_\alpha(k)}$ for every $k \in \mathrm{dom}(<_{w_\alpha})$,

$$\limsup_{n\in H} A_n \subseteq \limsup_{n\in D^\omega_{h_\alpha(k)}} A_n.$$

In particular for every $k \in \mathrm{dom}(<_{w_\alpha})$,

$$D^{\omega^\omega}_{y_\alpha(k)} \cap \limsup_{n\in H} A_n \subseteq D^{\omega^\omega}_{y_\alpha(k)} \cap \limsup_{n\in D^\omega_{h_\alpha(k)}} A_n.$$

Hence,

$$\begin{aligned}\limsup_{n\in H} A_n &= \bigcup_{\alpha<\omega_1^{\mathrm{CK}}} \bigcup_{k\in\mathrm{dom}(<_{w_\alpha})} D^{\omega^\omega}_{y_\alpha(k)} \cap \limsup_{n\in H} A_n \\ &\subseteq \bigcup_{\alpha<\omega_1^{\mathrm{CK}}} \bigcup_{k\in\mathrm{dom}(<_{w_\alpha})} D^{\omega^\omega}_{y_\alpha(k)} \cap \limsup_{n\in D^\omega_{h_\alpha(k)}} A_n\end{aligned}$$

meets only countably many E-equivalence classes, a contradiction. Thus, $V \neq \varnothing$. □

We now verify that the pair (V, H) is indeed completely good. In the proof of the next lemma we use the following observation. Let $z = \langle w, v, h \rangle$ be an adequate real. If $m <_w n$, then $m, n \in P$ and $\varphi(m) < \varphi(n)$. This is the case, because whenever $\langle y, k \rangle$ is a bad pair such that $k \subseteq^* h(n)$, also $k \subseteq^* h(m)$, since $h(n) \subseteq^* h(m)$.

Lemma 4.31. *If $X \subseteq V$ is a nonempty Σ^1_1 set and $K \in [H]^\omega$ a Δ^1_1 set, then $X \cap \limsup_{n\in K} A_n$ meets uncountably many E-classes.*

PROOF. Suppose $X \cap \limsup_{n \in K} A_n$ meets only countably many E-equivalence classes, i.e. $X \cap \limsup_{n \in K} A_n \cap W = \varnothing$. By Σ^1_1 separation, there is a Δ^1_1 set $Y \subseteq \omega^\omega$ such that $X \subseteq Y$ and $Y \cap \limsup_{n \in K} A_n \cap W = \varnothing$. Let y, k be a code for Y, K, respectively. Clearly, $\langle y, k \rangle$ is a bad pair.

First, suppose the construction halted at stage $\alpha < \omega_1^{\mathrm{CK}}$. Let $z = \langle w, v, h \rangle$ be the unique α-adequate real. The construction stops only if there does not exists a next bad pair which we have not picked already. Since $\langle y, k \rangle$ is a bad pair such that $k \subseteq^* h(n)$ for every $n \in \mathrm{dom}(<_{w_\alpha})$, there must be an $n \in \mathrm{dom}(<_{w_\alpha})$ such that $n = \langle y, k \rangle$, i.e. we picked $\langle y, k \rangle$ at that stage (otherwise, we can extend the construction by picking it now). But then $D^{\omega^\omega}_{v(n)} \cap D^{\omega^\omega}_{y} = \varnothing$, which implies $V \cap Y = \varnothing$ and so $V \cap X = \varnothing$.

Second, suppose the construction continued all the way up to ω_1^{CK}. Then there exists an $\alpha < \omega_1^{\mathrm{CK}}$ such that $\alpha > \varphi(\langle y, k \rangle)$. Let $z = \langle w, v, h \rangle$ be α-adequate. By the observation above, the pair $\langle y, k \rangle$ was considered, hence $n \in \mathrm{dom}(<_{w_\alpha})$ such that $n = \langle y, k \rangle$. Again, this implies $V \cap X = \varnothing$. $\square$

This finishes the proof of Proposition 4.23. We now derive a further refinement. A second key element of Harrington's proof is that E is meager on $W \times W$, when W is given the (subspace) Gandy–Harrington topology τ_{GH}. This is the topology on ω^ω generated by the Σ^1_1 sets. Although ω^ω with the Gandy–Harrington topology is not metrizable, it is strong Choquet and this enables one to redo the familiar construction of a perfect set of inequivalent elements, using a winning strategy for the second player. While this approach would also work in our case, we will use the set X_{low} of low elements instead. This makes the construction in the proof of the main theorem more transparent, at the cost of some technicalities which we now summarize. Let $X_{\mathrm{low}} = \{x \in \omega^\omega \colon \omega_1^{\mathrm{CK}(x)} = \omega_1^{\mathrm{CK}}\}$. We will use the following facts about W, X_{low}, and τ_{GH}:

(i) W and X_{low} are both nonempty Σ^1_1 sets,

(ii) X_{low} is dense in τ_{GH} and $(X_{\mathrm{low}}, \tau_{\mathrm{GH}})$ is a Polish space, and

(iii) a nonempty Σ^1_1 set $A \subseteq \omega^\omega$ meets uncountably many E-equivalence classes if and only if $A \cap W \neq \varnothing$ if and only if $A \cap W \cap X_{\mathrm{low}} \neq \varnothing$.

Proofs of these facts can be found in [8].

PROPOSITION 4.32. *Let E be a Π^1_1 equivalence relation on ω^ω and $(A_n)_{n\in\omega}$ a sequence of uniformly Σ^1_1 subsets of ω^ω. If $\limsup_{n\in K} A_n$ meets uncountably many E-equivalence classes for every $K \in [\omega]^\omega$, then there exists a completely good pair (V, H) such that V is a Polish space in the Gandy–Harrington topology τ_{GH} and E is meager on $V \times V$ (with the product topology $\tau_{\mathrm{GH}} \times \tau_{\mathrm{GH}}$).*

PROOF. Let (V, H) be the completely good pair given by Proposition 4.23. Using the facts stated above, it is easy to see that $(V \cap W \cap X_{\mathrm{low}}, H)$ is a completely good pair with the required additional properties. $\square$

4.6. Proof of the Main Theorem

We now prove an effective version of Theorem 4.2. By the usual relativization and transfer arguments, this implies our main result.

THEOREM 4.33. *Let E be a Π^1_1 equivalence relation on ω^ω and $(A_n)_{n\in\omega}$ a sequence of uniformly Σ^1_1 subsets of ω^ω. If $\limsup_{n\in K} A_n$ meets uncountably many E-equivalence classes for every $K \in [\omega]^\omega$, then there exists a $K \in [\omega]^\omega$ such that $\bigcap_{n\in K} A_n$ contains a perfect set of pairwise E-inequivalent elements.*

PROOF. Let (V, H) be the completely good pair given by Proposition 4.32. Since E is meager on $V \times V$ in the Gandy-Harrington topology τ_{GH}, we can fix an increasing sequence $(F_n)_{n\in\omega}$ of τ_{GH}-closed nowhere dense sets such that $E \subseteq \bigcup_{n\in\omega} F_n$. We may assume that the diagonal $\{(x,x)\colon x \in V\}$ is contained in F_0. We will recursively define a strictly increasing sequence $j_0 < j_1 < \cdots$ of natural numbers and a Cantor scheme $(X_s)_{s\in 2^{<\omega}}$ of nonempty Σ^1_1 subsets of V such that for all $s, t \in 2^{<\omega}$,

(i) $\overline{X}_{s^\frown 0}, \overline{X}_{s^\frown 1} \subseteq X_s$, $\overline{X}_{s^\frown 0} \cap \overline{X}_{s^\frown 1} = \varnothing$, and $\operatorname{diam}(X_s) \leq 2^{-\operatorname{lh}(s)}$,

(ii) if $s \neq t \in 2^{n+1}$, then $X_s \times X_t \cap F_n = \varnothing$, and

(iii) if $s \in 2^n$, then $X_s \subseteq A_{j_0} \cap \cdots \cap A_{j_n}$.

Once this construction is completed, let $K = \{j_0, j_1, \dots\}$ and

$$P = \bigcup_{\sigma \in 2^\omega} \bigcap_{n \in \omega} X_{\sigma \restriction n}.$$

It is easy to see that $P \subseteq \bigcap_{n \in K} A_n$ is nonempty perfect set of pairwise E-inequivalent elements.

Without loss of generality we may assume that $A_0 = \omega^\omega$. Start the construction with $j_0 = 0$ and $X_\varnothing = \omega^\omega$. Suppose we have defined natural numbers $j_0 < \dots < j_n$ and nonempty Σ^1_1 sets $X_s \subseteq A_{j_0} \cap \dots \cap A_{j_n}$ for $s \in 2^n$ satisfying the requirements above. By intersecting with sufficiently small basic open neigbhorhoods, we can split each X_s into disjoint nonempty Σ^1_1 sets $X_{s^\frown 0}$ and $X_{s^\frown 1}$ satisfying requirement (1). Since F_n is closed nowhere dense, given any pair $s \neq t \in 2^{n+1}$ we can shrink X_s and X_t so that $X_s \times X_t \cap F_n = \varnothing$. After finitely many iterations, we have defined X_s for $s \in 2^{n+1}$ satisfying requirements (1) and (2).

Claim 4.34. *There is an $j > j_n$ such that $X_s \cap A_j \neq \varnothing$ for all $s \in 2^{n+1}$.*

Proof. Suppose towards a contradiction that for every $j > j_n$ there is an $s \in 2^{n+1}$ such that $X_s \cap A_j = \varnothing$. Define a binary relation $R \subseteq \omega \times 2^{n+1}$ by $R(j, s) \Leftrightarrow X_s \cap A_j = \varnothing$. Since R is Π^1_1, there is a Δ^1_1 uniformizing function $f \colon \omega \to 2^{n+1}$. By the pigeonhole principle, there is an $s \in 2^{n+1}$ such that $\{j \in \omega \colon f(j) = s\} \cap H$ is infinite. Pick such an $s \in 2^{n+1}$. Then $K = \{j \in \omega \colon j \in H \text{ and } f(j) = s\}$ is Δ^1_1, $K \in [H]^\omega$, and $X_s \cap \bigcup_{n \in K} A_n = \varnothing$. This implies that $X_s \cap \limsup_{n \in K} A_n = \varnothing$, contradicting the fact that (V, H) is a completely good pair. □

To complete this step in the construction, let $j_{n+1} = j$ and intersect each X_s with $A_{j_{n+1}}$. This finishes the proof of Theorem 4.33. □

The following parametric version of the Laczkovich–Komjáth property was also considered by Balcerzak and Głąb.

Definition 4.35. An equivalence relation E on a Polish space Y has the *parametric Laczkovich–Komjáth property* if for every uncountable Polish space X and every sequence $(A_n)_{n \in \omega}$ of analytic subsets of $X \times Y$, if $\limsup_{n \in K} A_n(x)$ meets uncountably many E-equivalence classes for every $x \in X$ and $K \in [\omega]^\omega$, then there exists a $K \in [\omega]^\omega$ and a perfect set $P \subseteq X$ such that $\bigcap_{n \in K} A_n(x)$ meets perfectly many E-equivalence classes for each $x \in P$.

THEOREM 4.36 (Balcerzak–Głąb [2]). *If E has the Laczkovich–Komjáth property and for every analytic set $A \subseteq X \times X$, the set*

$$\{x \in X \colon A_x \textit{ meets uncountably many } E\textit{-equivalence classes}\}$$

is analytic, then E has the parametric Lackovich–Komjáth property. □

PROPOSITION 4.37. *Every coanalytic equivalence relation on a Polish space has the parametric Laczkovic–Komjáth property.*

PROOF. Let E be a coanalytic equivalence relation on a Polish space X and $A \subseteq X \times X$ an analytic subset. Without loss of generality we may assume E is a Π^1_1 equivalence relation on $X = \omega^\omega$ and $A \subseteq \omega^\omega \times \omega^\omega$ is Σ^1_1. Since A is Σ^1_1, each section A_x is Σ^1_1 as well. Hence, A_x meets uncountably many E-equivalence classes if and only if $A_x \cap W \neq \varnothing$. Thus,

$$\{x \in \omega^\omega \colon A_x \text{ meets uncountably many } E\text{-equivalence classes}\}$$

is Σ^1_1. Hence, E has the parametric Laczkovich–Komjáth property by Theorem 4.2 and Theorem 4.36. □

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