CENTERS OF INVARIANT DIFFERENTIAL OPERATOR ALGEBRAS
FOR JACOBI GROUPS OF HIGHER RANK

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Let $G$ be a Lie group acting on a homogeneous space $G/K$. The center of the universal enveloping algebra of the Lie algebra of $G$ maps homomorphically into the center of the algebra of differential operators on $G/K$ invariant under the action of $G$. In the case that $G$ is a Jacobi Lie group of rank 2, we prove that this homomorphism is surjective and hence that the center of the invariant differential operator algebra is the image of the center of the universal enveloping algebra. This is an extension of work of Bringmann, Conley, and Richter in the rank 1 case.
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CHAPTER 1

INTRODUCTION

Let $G$ be a real Lie group, $K$ a Lie subgroup of $G$, and $V$ a complex representation of $K$. Then one has the $G$-vector bundle $G \times_K V$ over the homogeneous space $G/K$. The algebra $\mathbb{D}(G \times_K V)$ of $G$-invariant differential operators on $G \times_K V$ is of interest in many areas of mathematics, for example number theory and representation theory [1].

In general $\mathbb{D}(G \times_K V)$ is not commutative and one would like to know its center. There is always a homomorphism to $\mathbb{D}(G \times_K V)$ from the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of $G$, but it is not necessarily either injective or surjective. In this thesis we use Helgason’s algebraic description of $\mathbb{D}(G \times_K V)$ [6] to prove that this homomorphism is surjective for the Jacobi group of rank 2. The same result was proven for the Jacobi group of rank 1 in [3]. In a forthcoming work we will prove the result for arbitrary rank.
A Lie algebra \( \mathfrak{g} \) is a vector space equipped with a bilinear operation \([\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\), satisfying the following conditions:

1. Skew-symmetry: \([X, Y] = -[Y, X]\) for all \(X, Y \in \mathfrak{g}\).
2. The Jacobi identity: \([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\) for all \(X, Y, Z \in \mathfrak{g}\).

The binary operation \([\cdot, \cdot]\) is called the Lie bracket. For example, the space \(\mathfrak{gl}_n(\mathbb{C})\) of \(n \times n\) matrices with bracket \([X, Y] = XY - YX\) is a Lie algebra. The space \(\mathfrak{sl}_2(\mathbb{C})\) of \(2 \times 2\) traceless matrices is an important subalgebra of \(\mathfrak{gl}_2(\mathbb{C})\) which we now describe. We will frequently write simply \(\mathfrak{gl}_2\) and \(\mathfrak{sl}_2\) to denote \(\mathfrak{gl}_2(\mathbb{C})\) and \(\mathfrak{sl}_2(\mathbb{C})\), respectively. Set

\[
E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Then \(\{E, H, F\}\) is a basis of \(\mathfrak{sl}_2\) and the brackets among the basis elements are as follows:

\([E, F] = H, \quad [H, E] = 2E, \quad \text{and} \quad [H, F] = -2F\).

2.1. Representations of Lie Algebras

Let \(V\) be a vector space. A representation of a Lie algebra \(\mathfrak{g}\) on \(V\) is a linear map \(\pi: \mathfrak{g} \to \text{End}(V)\) satisfying

\[\pi([X, Y]) = [\pi(X), \pi(Y)], \text{ for all } X, Y \in \mathfrak{g}.\]

The adjoint representation \(\text{ad}: \mathfrak{g} \to \text{End}(\mathfrak{g})\) given by \(\text{ad}(X)(Y) = [X, Y]\) for all \(X, Y \in \mathfrak{g}\) is an example of a representation of a Lie algebra \(\mathfrak{g}\) on itself. Let \(\pi\) be a representation of \(\mathfrak{g}\) on \(V\). A subspace \(W\) of \(V\) is called a subrepresentation if \(\pi(\mathfrak{g})W \subseteq W\). The representation \(V\) is called irreducible if it has no subrepresentations except 0 and \(V\) itself. Suppose \(\pi\) and \(\pi'\) are representations of \(\mathfrak{g}\) on \(V\) and \(V'\), respectively. A linear map \(\phi: V \to V'\) is called an intertwining map from \(V\) to \(V'\) if for all \(X \in \mathfrak{g}\),

\[\pi'(X) \circ \phi = \phi \circ \pi(X).\]
If the intertwining map $\phi$ is also a vector space isomorphism then it is called an \textit{equivalence}
and the representations $\pi$ and $\pi'$ are called \textit{equivalent}.

2.2. The Symmetric Algebra

Let $V$ be a finite dimensional vector space over $\mathbb{C}$. Let us write

$$T^0(V) = \mathbb{C}, \text{ and } T^k(V) = V \otimes V \otimes \cdots \otimes V (k \text{ times}).$$

Define $T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$, an associative algebra over $\mathbb{C}$ with product

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_m)(w_1 \otimes w_2 \otimes \cdots \otimes w_n) = v_1 \otimes v_2 \otimes \cdots \otimes v_m \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_n,$$

the tensor algebra of $V$. Let $I$ be the two sided ideal in $T(V)$ generated by all relations

$$x \otimes y - y \otimes x \text{ for } x,y \in V.$$ 

Then $S(V) = T(V)/I$ is a commutative graded algebra called the \textit{symmetric algebra} on $V$. If $(x_1, x_2, \cdots, x_n)$ is a basis of $V$, then $S(V)$ is isomorphic to the polynomial algebra over $\mathbb{C}$ in $n$ variables, $x_1, x_2, \cdots, x_n$, and \( \{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} : i_1, i_2, \cdots, i_n \in \mathbb{N} \} \)

is a basis of $S(V)$. We denote the image of $T^k(V)$ in $S(V)$ by $S^k(V)$.

If $\pi$ is a representation of $\mathfrak{g}$ on a space $V$, then there is a representation $\pi$ of $\mathfrak{g}$ on $S(V)$ acting by derivations

$$\pi(X)(Y_1 Y_2 \cdots Y_n) = \sum_{i=1}^{n} \{ Y_1 Y_2 \cdots (\pi(X)Y_i) \cdots Y_n \} \text{ for } X \in \mathfrak{g} \text{ and } (Y_1 Y_2 \cdots Y_n) \in S(V).$$

2.3. The Universal Enveloping Algebra

Let $\mathfrak{g}$ be a Lie algebra, let $T(\mathfrak{g})$ be its tensor algebra and let $J$ be the ideal generated
by all relations $x \otimes y - y \otimes x - [x, y]$ in $T(\mathfrak{g})$ with $X, Y \in \mathfrak{g}$. Then $\mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/J$ is an
associative algebra called the \textit{universal enveloping algebra} of $\mathfrak{g}$. The grading of $T(\mathfrak{g})$ imposes
a filtration $\mathfrak{U}_n(\mathfrak{g})$ on $\mathfrak{U}(\mathfrak{g})$.

Given a representation $\pi$ of $\mathfrak{g}$ on $V$, then it extends to representation of $\mathfrak{U}(\mathfrak{g})$ by

$$\pi(X_1 X_2 \cdots X_n) \cdot v = \pi(X_1)\pi(X_2) \cdots \pi(X_n)v.$$

Also, ad extends to a representation of $\mathfrak{g}$ on $\mathfrak{U}(\mathfrak{g})$ by derivations:

$$\text{ad}(Y)(X_1 X_2 \cdots X_n) = YX_1 X_2 \cdots X_n - X_1 X_2 \cdots X_n Y.$$
For $\Omega \in \mathfrak{U}(\mathfrak{g})$, we define the degree of $\Omega$ to be the smallest integer $m$ such that $\Omega \in \mathfrak{U}_m$ and we write $\text{deg}(\Omega) = m$. The symbol of $\Omega$ is denoted by $\text{symb}(\Omega)$ and is defined by

$$\text{symb}(\Omega) = \Omega + \mathfrak{U}_{m-1}.$$ 

**Definition.** For any Lie algebra $\mathfrak{g}$, the map $\lambda : \mathcal{S}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ defined by

$$\lambda(X_1X_2 \cdots X_n) = \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)}X_{\sigma(2)} \cdots X_{\sigma(n)}$$

is called the *symmetrizer map* which is a degree preserving vector space isomorphism and is an algebra isomorphism up to symbol, i.e., if $\Omega_1 \in \mathcal{S}^m(\mathfrak{g})$ and $\Omega_2 \in \mathcal{S}^n(\mathfrak{g})$, then

$$\lambda(\Omega_1\Omega_2) - \lambda(\Omega_1)\lambda(\Omega_2) \in \mathfrak{U}_{m+n-1}.$$

2.4. Representations of $\mathfrak{sl}_2$

Let $\pi$ be a representation of $\mathfrak{sl}_2$ on a finite dimensional vector space $V$. Since $H$ is semisimple, it acts diagonally on $V$ [8]. So the vector space $V$ can be decomposed into a direct sum of eigenspaces $V_\lambda = \{ v \in V : \pi(H)v = \lambda v \}$ of $H$. The eigenspace $V_\lambda$ is called the $\lambda$-weight space and any non-zero vector $v \in V_\lambda$ is called a weight vector with weight $\lambda$. The following lemma is elementary:

**Lemma 2.1.** $\pi(E) : V_\lambda \to V_{\lambda+2}$ and $\pi(F) : V_\lambda \to V_{\lambda-2}$.

Note that if $V$ is an $\mathfrak{sl}_2$ representation with $v_1, v_2, \ldots, v_n$ weight vectors with weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, then $v_1^{i_1}v_2^{i_2} \cdots v_n^{i_n}$ is a weight vector in $\mathcal{S}(V)$ with weight $\lambda_1i_1 + \lambda_2i_2 + \cdots + \lambda_ni_n$.

A vector $v \in V_\lambda$ is called a maximal vector if $Ev = 0$. Let $V$ be an irreducible $\mathfrak{sl}_2$ representation and let $v_0$ be a maximal vector with weight $\lambda$. Let $v_{-1} = 0$, and for $i \geq 0$, let $v_i = \frac{1}{i!}F^iv_0$, so that $Fv_i = (i + 1)v_{i+1}$. By Lemma 2.1, $Hv_i = (\lambda - 2i)v_i$, and we can use induction to see that

$$Ev_i = (\lambda - i + 1)v_{i-1}. \quad (1)$$

Since non-zero $v_i$’s are independent and $V$ is finite dimensional, there exists $m$ such that $v_m \neq 0$ but $v_{m+1} = 0$. But $\text{Span}\{v_0, v_1, \ldots, v_m\}$ is a subrepresentation of the irreducible
representation $V$ and hence must be $V$ itself. Hence the dimension of $V$ is $m + 1$, and taking $i = m + 1$ in (1) we get $\lambda = m$. Thus we conclude that:

**Lemma 2.2.** Any irreducible representation $V$ of $\mathfrak{sl}_2$ can be written as a direct sum of 1-dimensional weight spaces:

$$V = \bigoplus_{i=0}^{m} V_{m-2i},$$

where the dimension of $V$ is $m + 1$.

The basis $\{v_0, v_1, \cdots, v_m\}$ of $V$ gives the following well-known lemma:

**Lemma 2.3.** All $m + 1$ dimensional irreducible representations of $\mathfrak{sl}_2$ are equivalent.

In particular all $m + 1$ dimensional irreducible representations of $\mathfrak{sl}_2$ are equivalent to the irreducible representation $\pi_m : \mathfrak{sl}_2 \to \text{End}(\mathbb{C}^{m+1})$ given by

$$\pi_m(E) = \begin{pmatrix} 0 & m & 0 & 0 & \cdots & 0 \\ 0 & 0 & m-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \pi_m(H) = \begin{pmatrix} m & 0 & 0 & 0 & \cdots & 0 \\ 0 & m-2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

$$\pi_m(F) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & m & 0 \end{pmatrix}.$$
If \( \pi \) and \( \pi' \) are representations of an arbitrary Lie algebra \( g \) on \( V \) and \( V' \), respectively, then
\[
(\pi \otimes \pi')(X) := \pi(X) \otimes 1 + 1 \otimes \pi'(X)
\]
defines a representation of \( g \) on \( V \otimes V' \). Note that if \( g = \mathfrak{sl}_2 \) and \( v_\lambda \in V \) and \( v'_{\lambda'} \in V' \) are weight vectors with weights \( \lambda \) and \( \lambda' \), respectively, then \( v_\lambda \otimes v'_{\lambda'} \in V \otimes V' \) is a weight vector with weight \( \lambda + \lambda' \). We leave the following lemma as an exercise to the reader.

**Lemma 2.4.** For \( n \geq m \),

\[
L_n \otimes L_m = L_{n+m} \oplus L_{n+m-2} \oplus \cdots \oplus L_{n-m}.
\]

**Definition.** Let \( \pi \) be a representation of \( \mathfrak{sl}_2 \) on \( V \). Then the action of the element \( H^2 + 2H + 4FE \) of \( \mathfrak{u}(\mathfrak{sl}_2) \) on \( V \) commutes with the action of \( \mathfrak{u}(\mathfrak{sl}_2) \) on \( V \). This element is called the Casimir element of \( \mathfrak{sl}_2 \).
CHAPTER 3

INvariant Differential Operator Algebras

This chapter is a review of the algebraic description of invariant differential operator algebras developed by Helgason [6]. We use the notation of Bringmann-Conley-Richter [3].

Fix a Lie algebra $g$, a subalgebra $\mathfrak{k}$, and any complement $\mathfrak{m}$ of $\mathfrak{k}$ in $g$, so $g = \mathfrak{k} \oplus \mathfrak{m}$. Let $\pi$ be a representation of $\mathfrak{k}$ on a finite dimensional vector space $V$ and let us define the following structures:

\[ \mathcal{E}_\pi = \mathfrak{U}(g) \otimes \text{End}(V), \]
\[ \mathcal{I}_\pi = \mathcal{E}_\pi \{ Y \otimes 1 + 1 \otimes \pi(Y) : Y \in \mathfrak{k} \}, \]
\[ \mathcal{A}_\pi = \{ P \in \mathcal{E}_\pi : \mathcal{I}_\pi P \subseteq \mathcal{I}_\pi \}, \]
\[ \mathcal{D}_\pi = \mathcal{A}_\pi / \mathcal{I}_\pi. \]

Then $\mathcal{E}_\pi$ is an associative algebra, $\mathcal{I}_\pi$ is a left ideal in $\mathcal{E}_\pi$, and $\mathcal{A}_\pi$ is a subalgebra of $\mathcal{E}_\pi$ such that $\mathcal{I}_\pi \subseteq \mathcal{A}_\pi$. Moreover, $\mathcal{I}_\pi$ is a two sided ideal in $\mathcal{A}_\pi$ and therefore $\mathcal{D}_\pi$ is an associative algebra.

$\mathcal{D}_\pi$ is related to invariant differential operator (IDO) algebras as follows. Suppose that $G$ is a real Lie group with complexified Lie algebra $g$, $K$ is a Lie subgroup with complexified Lie algebra $\mathfrak{k}$, and $\pi$ is a representation of $K$ on $V$. Let $G \times_K V$ be the $G$-vector bundle over $G/K$ of fiber $V$, and as in [3], write $\mathcal{D}(G \times_K V)$ for the algebra of IDOs on the space of sections of this bundle. The following theorem was proven by Helgason in the 1950s (see [6]) in the case that $\mathfrak{m}$ is $\mathfrak{k}$-invariant. For the general result see [3].

**Theorem 3.1.** $\mathcal{D}(G \times_K V)$ is isomorphic to $\mathcal{D}_\pi$.

In light of this theorem we now consider $\mathcal{D}_\pi$ in detail. For any $\mathfrak{k}$-module $W$, define

\[ \mathcal{J} = \text{Span}\{ \Omega(Y \otimes w - 1 \otimes Yw) : \Omega \in \mathfrak{U}(\mathfrak{k}), Y \in \mathfrak{k}, w \in W \}. \]

**Lemma 3.2.** $\mathcal{J} = \text{Span}\{ \Omega \otimes w - 1 \otimes \Omega w : \Omega \in \mathfrak{U}(\mathfrak{k}), w \in W \}$. 

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Proof. Let 

\[ J' = \text{Span}\{\Omega \otimes w - 1 \otimes \Omega w : \Omega \in \mathfrak{U}(\mathfrak{f}), w \in W\}. \]

We will show that \( J = J' \). An arbitrary element in \( J \) is a sum of elements of the form \( \Omega(Y \otimes w - 1 \otimes Yw) \), where \( \Omega \in \mathfrak{U}(\mathfrak{f}) \), \( Y \in \mathfrak{f} \), and \( w \in W \). Since \( \Omega Y \in \mathfrak{U}(\mathfrak{f}) \) and \( Yw \in W \),

\[ \Omega(Y \otimes w - 1 \otimes Yw) = \Omega Y \otimes w - \Omega \otimes Yw \]

\[ = (\Omega Y \otimes w - 1 \otimes \Omega Yw) - (\Omega \otimes Yw - 1 \otimes \Omega Yw) \in J'. \]

Hence \( J \subseteq J' \).

Next, let \( S = \{\Omega \in \mathfrak{U}(\mathfrak{f}) : \Omega \otimes w - 1 \otimes \Omega w \in J \text{ for all } w \in W\} \). Then for \( Y \in \mathfrak{f} \),

\[ Y\Omega \otimes w - 1 \otimes Y\Omega w = Y(\Omega \otimes w - 1 \otimes \Omega w) + (Y \otimes \Omega w - 1 \otimes Y\Omega w). \]

But \( Y \in \mathfrak{f} \) and \( \Omega w \in W \), so \( (Y \otimes \Omega w - 1 \otimes Y\Omega w) \in J \) and since \( \Omega \in S \), \( Y \in \mathfrak{f} \), we must have \( Y(\Omega \otimes w - 1 \otimes \Omega w) \in J \). Thus \( Y\Omega \otimes w - 1 \otimes Y\Omega w \in J \) and therefore \( Y\Omega \in S \), i.e., \( S \) is closed under left multiplication by elements of \( \mathfrak{f} \). Since \( \mathfrak{f} \subseteq S \) by the definition of \( J \), we have \( S = \mathfrak{U}(\mathfrak{f}) \). Therefore \( J' \subseteq J \) and hence \( J = J' \).

Lemma 3.3. \( \mathfrak{U}(\mathfrak{f}) \otimes W = (1 \otimes W) \oplus J \).

Proof. Let \( \{Y_1, Y_2, \cdots, Y_n\} \) be a basis of \( \mathfrak{f} \). Then

\[ \{Y^K = Y_1^{k_1}Y_2^{k_2} \cdots Y_n^{k_n} : k_1, k_2, \cdots, k_n \in \mathbb{N}\} \]

is a basis of \( \mathfrak{U}(\mathfrak{f}) \), and if \( \{w_i\}_{i \in I} \) is a basis of \( W \), then \( \{Y^K \otimes w_i\}_{K,i} \), and \( \{1 \otimes w_i\}_{i \in I} \) are bases of \( \mathfrak{U}(\mathfrak{f}) \otimes W \), and \( (1 \otimes W) \), respectively. It is clear from Lemma 3.2 that \( \{Y^K \otimes w_i - 1 \otimes Y^K w_i\}_{K \neq 0,i} \) spans \( J \). We now show that \( \{Y^K \otimes w_i - 1 \otimes Y^K w_i\}_{K \neq 0,i} \cup \{1 \otimes w_i\}_{i \in I} \) is a basis of \( \mathfrak{U}(\mathfrak{f}) \otimes W \).

Span: Let \( Y^K \otimes w_i \) be any basis element of \( \mathfrak{U}(\mathfrak{f}) \otimes W \). If \( K = 0 \), then \( Y^K \otimes w_i = 1 \otimes w_i \).

If \( K \neq 0 \), then

\[ Y^K \otimes w_i = (Y^K \otimes w_i - 1 \otimes Y^K w_i) + (1 \otimes Y^K w_i) \]

\[ = (Y^K \otimes w_i - 1 \otimes Y^K w_i) + \sum a_i(1 \otimes w_i) \]
because $1 \otimes Y^K w_i \in 1 \otimes W$, which has a basis $\{1 \otimes w_i\}_i$. This shows that every basis element of $\mathfrak{U}(\mathfrak{t}) \otimes W$ can be written as a linear combination of elements of the set

$$\{Y^K \otimes w_i - 1 \otimes Y^K w_i\}_{K \neq 0, i} \cup \{1 \otimes w_i\}_{i \in I}.$$

Thus $\mathfrak{U}(\mathfrak{t}) \otimes W$ is spanned by $\{Y^K \otimes w_i - 1 \otimes Y^K w_i\}_{K \neq 0, i} \cup \{1 \otimes w_i\}_{i \in I}$.

**Linear independence:** Suppose

$$\sum_{i \in I} a_i (1 \otimes w_i) + \sum_{K \neq 0, K, i} b_{K, i} (Y^K \otimes w_i - 1 \otimes Y^K w_i) = 0.$$

Since $\{Y^K \otimes w_i\}_{K, i}$ is a basis of $\mathfrak{U}(\mathfrak{t}) \otimes W$, $b_{K, i} = 0$. Hence $\sum_{i \in I} a_i (1 \otimes w_i) = 0$. So $a_i = 0$. Thus $\{Y^K \otimes w_i - 1 \otimes Y^K w_i\}_{K \neq 0, i} \cup \{1 \otimes w_i\}_{i \in I}$ is linearly independent.

Hence $\{Y^K \otimes w_i - 1 \otimes Y^K w_i\}_{K \neq 0, i} \cup \{1 \otimes w_i\}_{i \in I}$ is a basis of $\mathfrak{U}(\mathfrak{t}) \otimes W$. Here we also proved that $\{Y^K \otimes w_i - 1 \otimes Y^K w_i\}_{K \neq 0, i}$ is a basis of $\mathfrak{J}$. Therefore $\mathfrak{U}(\mathfrak{t}) \otimes W = (1 \otimes W) \oplus \mathfrak{J}$. □

**Lemma 3.4.** $\mathfrak{U}(\mathfrak{g}) \otimes W = [\lambda(S(m)) \otimes W] \oplus \mathcal{I}$, where

$$\mathcal{I} = \mathfrak{U}(\mathfrak{g}) \{Y \otimes w - 1 \otimes Y w : Y \in \mathfrak{t}, w \in W\},$$

and $\lambda$ is the symmetrizer map.

**Proof.** This follows easily from the fact that the multiplication map

$$\lambda(S(m)) \otimes \mathfrak{U}(\mathfrak{t}) \rightarrow \mathfrak{U}(\mathfrak{g})$$

is a vector space isomorphism, which in turn follows from the PBW theorem. □

With the help of these lemmas we now prove some important facts about $\mathcal{E}_\pi, \mathcal{I}_\pi, \mathcal{A}_\pi$, and $\mathcal{D}_\pi$.

**Corollary 3.5.** $\mathcal{E}_\pi = \mathcal{I}_\pi \oplus [\lambda(S(m)) \otimes \text{End}(V)]$.

**Proof.** Take $W$ to be $\text{End}(V)$ in Lemma 3.4. □
Note that \( \text{ad} \otimes \text{ad}_\pi : \mathfrak{k} \longrightarrow \text{End}(\mathcal{E}_\pi) \),

\[
(\text{ad} \otimes \text{ad}_\pi)(X) := \text{ad}(X) \otimes 1 + 1 \otimes \text{ad}_\pi(X),
\]
is a representation which acts by derivations on the algebra \( \mathcal{E}_\pi \).

**Lemma 3.6.** \( \mathcal{I}_\pi \) and \( \mathcal{A}_\pi \) are \( \mathfrak{k} \)-invariant subspaces of \( \mathcal{E}_\pi \), and

\[
\mathcal{D}_\pi = (\mathcal{E}_\pi / \mathcal{I}_\pi)^\mathfrak{k}.
\]

**Proof.** To prove that \( \mathcal{I}_\pi \) and \( \mathcal{A}_\pi \) are \( \mathfrak{k} \)-invariant, we prove that \( (\text{ad} \otimes \text{ad}_\pi)(Y) \) maps \( \mathcal{A}_\pi \) into \( \mathcal{I}_\pi \) for all \( Y \in \mathfrak{k} \). For any \( P \in \mathcal{A}_\pi \),

\[
(\text{ad} \otimes \text{ad}_\pi)(Y)(P) = (\text{ad}(Y) \otimes 1 + 1 \otimes \text{ad}_\pi(Y))(P)
\]

\[
= (Y \otimes 1 + 1 \otimes \pi(Y))(P) - P(Y \otimes 1 + 1 \otimes \pi(Y)) \in \mathcal{I}_\pi,
\]
because \( P \in \mathcal{A}_\pi \). Thus \( \mathcal{I}_\pi \) and \( \mathcal{A}_\pi \) are \( \mathfrak{k} \)-invariant, and \( \mathfrak{k} \) acts trivially on \( \mathcal{D}_\pi \).

Next, let us write \( (\text{ad} \otimes \text{ad}_\pi)_\mathfrak{k} \) for the quotient \( \mathfrak{k} \)-action on \( \mathcal{E}_\pi / \mathcal{I}_\pi \), and let \( Q \in \mathcal{E}_\pi \) be such that \( Q + \mathcal{I}_\pi \) is in \( (\mathcal{E}_\pi / \mathcal{I}_\pi)^{(\text{ad} \otimes \text{ad}_\pi)}_\mathfrak{k} \). Deduce that

\[
(X \otimes 1 + 1 \otimes \pi(X))Q \in \mathcal{I}_\pi
\]
for all \( X \in \mathfrak{k} \). This leads to \( \mathcal{I}_\pi Q \subseteq \mathcal{I}_\pi \), i.e., \( Q \in \mathcal{A}_\pi \). Thus

\[
(\mathcal{E}_\pi / \mathcal{I}_\pi)^{(\text{ad} \otimes \text{ad}_\pi)}_\mathfrak{k} \subseteq \mathcal{A}_\pi / \mathcal{I}_\pi.
\]
But we have already seen in the first paragraph of the proof that

\[
\mathcal{A}_\pi / \mathcal{I}_\pi \subseteq (\mathcal{E}_\pi / \mathcal{I}_\pi)^{(\text{ad} \otimes \text{ad}_\pi)}_\mathfrak{k}.
\]
Therefore \( \mathcal{D}_\pi = (\mathcal{E}_\pi / \mathcal{I}_\pi)^{(\text{ad} \otimes \text{ad}_\pi)}_\mathfrak{k} \).

**Lemma 3.7.** If \( \mathfrak{m} \) is \( \mathfrak{k} \)-invariant, then

\[
\mathcal{D}_\pi = \mathcal{E}_\pi / \mathcal{I}_\pi = [\lambda(\mathcal{S}(\mathfrak{m})) \otimes \text{End}(V)]^\mathfrak{k},
\]
where \( \lambda \) is the symmetrizer map.
Proof. In the case that \( m \) is \( \mathfrak{t} \)-invariant, \( \mathcal{I}_\pi \oplus (\lambda(S(m)) \otimes \text{End}(V)) \) is a \( \mathfrak{t} \)-splitting of \( \mathcal{E}_\pi \), so \( \mathcal{E}_\pi / \mathcal{I}_\pi \) is \( \mathfrak{t} \)-equivalent to \( \lambda(S(m)) \otimes \text{End}(V) \). The result follows from Lemma 3.6. \( \square \)

Remark. Let \( \mathfrak{z}(\mathfrak{g}) \) be the center of \( \mathfrak{u}(\mathfrak{g}) \). Observe that \( \mathfrak{z}(\mathfrak{g}) \otimes 1 \) is in \( \mathcal{A}_\pi \), and its image is central in \( \mathcal{D}_\pi \). We remark that \( \mathcal{D}_\pi \) is not always commutative, and the image of \( \mathfrak{z}(\mathfrak{g}) \otimes 1 \) is not always all of the center \( Z(\mathcal{D}_\pi) \) of \( \mathcal{D}_\pi \).
CHAPTER 4

THE JACOBI LIE ALGEBRA \( \mathfrak{g}_J^N \)

In the theory of Jacobi forms one is interested in a certain action of the Jacobi Lie group

\[
G_J^N := \text{SL}(2, \mathbb{R}) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^N) \sim \text{Mat}_N^{\text{Sym}}(\mathbb{R}),
\]

where \( \sim \) denotes a central extension. The stabilizer of this action is the subgroup

\[
\hat{K}_J^N := \text{SO}_2 \times \{0\} \times \text{Mat}_N^{\text{Sym}}(\mathbb{R}).
\]

Our main result, Theorem 4.4, describes the centers of the IDO algebras of certain line bundles over \( G_J^N/\hat{K}_J^N \). Since we will work only with the complexified Lie algebras \( \mathfrak{g}_J^N \) and \( \hat{k}_J^N \) of \( G_J^N \) and \( \hat{K}_J^N \), we will not give the precise definition of the product in \( G_J^N \); it may be found in [5].

4.1. Definitions

The complexified Lie algebra \( \mathfrak{g}_J^N \) is

\[
\mathfrak{g}_J^N := \{(M, X, \kappa) : M \in \mathfrak{sl}_2(\mathbb{C}), X \in M_{N,2}(\mathbb{C}), \kappa \in \text{Mat}_N^{\text{Sym}}(\mathbb{C})\},
\]

where \( M_{N,2}(\mathbb{C}) \) denotes the space of \( N \times 2 \) matrices over \( \mathbb{C} \) and \( \text{Mat}_N^{\text{Sym}}(\mathbb{C}) \) denotes the space of \( N \times N \) symmetric matrices over \( \mathbb{C} \). The bracket in \( \mathfrak{g}_J^N \) is given by

\[
[(M, X, \kappa), (\tilde{M}, \tilde{X}, \tilde{\kappa})] = ([M, \tilde{M}], XM - \tilde{X}M, \tilde{X}J_2X^T - XJ_2\tilde{X}^T),
\]

where \( J_2 = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \).

Recall the basis \( \{E, H, F\} \) of \( \mathfrak{sl}_2(\mathbb{C}) \). Denote the matrix with \((i, j)^{th}\) entry 1 and all other entries 0 by \( \epsilon_{ij} \), where the size of the matrix is determined by context. Then \( \{\epsilon_{i1}, \epsilon_{i2} : 1 \leq i \leq N\} \) is a basis of \( M_{N,2}(\mathbb{C}) \) and \( \{\frac{1}{2}(\epsilon_{ij} + \epsilon_{ji}) : 1 \leq i \leq j \leq N\} \) is a basis of \( \text{Mat}_N^{\text{Sym}}(\mathbb{C}) \). Let us write \( f_i \) for \( \epsilon_{i1} \), \( e_i \) for \( \epsilon_{i2} \), and \( Z_{ij} \) for \( \frac{1}{2}(\epsilon_{ij} + \epsilon_{ji}) \). Then

\[
\{(E, 0, 0), (H, 0, 0), (F, 0, 0), (0, e_i, 0), (0, f_i, 0) : 1 \leq i \leq N; (0, 0, Z_{ij}) : 1 \leq i \leq j \leq N\}
\]
is a basis of \( g_N \). We will write it as

\[
\{ E, H, F, e_i, f_i : 1 \leq i \leq N; Z_{ij} : 1 \leq i \leq j \leq N \}.
\]

The dimension of \( g_N \) is \( \frac{(N+2)(N+3)}{2} \). The brackets are as follows:

\[
\begin{align*}
[H,e_i] &= e_i, & [H,f_i] &= -f_i, & [E,e_i] &= 0, \\
[F,f_i] &= 0, & [E,f_i] &= -e_i, & [F,e_i] &= -f_i, \\
[e_i,f_j] &= -2Z_{ij}, & [X,Z_{ij}] = 0 & \text{for all } X \in g_N.
\end{align*}
\]

The Lie subalgebra \( \hat{k}_N \) of the subgroup \( \hat{K}_N \) of \( G_N \) is

\[
\hat{k}_N = \text{Span}\{F - E, Z_{ij} : 1 \leq i \leq j \leq N\}.
\]

This subalgebra is conjugate to

\[
k_N = \text{Span}\{H, Z_{ij} : 1 \leq i \leq j \leq N\}.
\]

Therefore results proven for the pair \( (g_N, k_N) \) also hold for \( (g_N, \hat{k}_N) \). Since it is more convenient to work with \( k_N \), we will use it in place of \( \hat{k}_N \) from here on. The following lemma is elementary.

**Lemma 4.1.** (1) The center \( z_N \) of \( g_N \) is \( \text{Span}\{Z_{ij} : 1 \leq i \leq j \leq N\} \).

(2) There is a \( k_N \)-invariant complement \( m_N \) of \( k_N \) in \( g_N \), defined by

\[
m_N := \text{Span}\{F, E, e_i, f_i : 1 \leq i \leq N\}.
\]

4.2. The Casimir Element

Let us write \( e \) and \( f \) for the column vectors with entries \( e_i \) and \( f_i \), respectively, and let \( Z \) denote the \( N \times N \) symmetric matrix with entries \( Z_{ij} \). Since the \( Z_{ij} \) are central, we may formally adjoin \( \det(Z)^{-1} \) to \( \mathcal{U}(g_N) \). In the resulting algebra \( \mathcal{U}(g_N)[\det(Z)^{-1}] \), \( Z \) is invertible.
and we have the following elements

\[
\begin{align*}
    f^T Z^{-1} e &:= \sum_{i,j} f_i (Z^{-1})_{ij} e_j, \\
    e^T Z^{-1} f &:= \sum_{i,j} e_i (Z^{-1})_{ij} f_j,
\end{align*}
\]

Define the Casimir element \( \Omega_N \) of \( \mathfrak{U}(\mathfrak{g}_N^J) \) to be the following element of \( \mathfrak{U}(\mathfrak{g}_N^J)[\det(Z)^{-1}] \).

\[
\Omega_N := \det(Z) \left[ H^2 - (N + 2)H + 4EF - \frac{1}{2} (N + 3)(e^T Z^{-1} f)^2 + EF f^T Z^{-1} f - e^T Z^{-1} eF + \frac{1}{4} e^T (e^T Z^{-1} e)(f^T Z^{-1} f) \right].
\]

The following theorem is proven for \( N = 1 \) in [3], and in general in [5].

**Theorem 4.2.** \( \Omega_N \) is in fact in \( \mathfrak{U}(\mathfrak{g}_N^J) \), and it is central. The center \( \mathfrak{Z}(\mathfrak{g}_N^J) \) of \( \mathfrak{U}(\mathfrak{g}_N^J) \) is the polynomial algebra \( \mathbb{C}[\Omega_N, Z_{ij} : 1 \leq i \leq j \leq N] \).

### 4.3. IDO Algebras Over \( G_N^J/\hat{K}_N^J \)

For \( k \in \mathbb{C} \) and \( L \) an \( N \times N \) symmetric matrix with entries in \( \mathbb{C} \), let \( \pi_{k,L} \) be the representation of \( \mathfrak{t}_N^J \) on \( \mathbb{C} \) given by

\[
\pi_{k,L}(H) = -k, \quad \pi_{k,L}(Z_{ij}) = L_{ij}; 1 \leq i \leq j \leq N.
\]

We will denote \( \mathbb{C} \) with this action by \( \mathbb{C}_{k,L} \). For \( \pi_{k,L} \), the associative algebra \( \mathfrak{E}_\pi \) defined in Chapter 2 is

\[
\mathfrak{E}_{\pi_{k,L}} = \mathfrak{U}(\mathfrak{g}_N^J) \otimes \text{End}(\mathbb{C}_{k,L}) \cong \mathfrak{U}(\mathfrak{g}_N^J),
\]

and the ideal \( \mathcal{I}_\pi \) reduces to

\[
\mathcal{I}_{\pi_{k,L}} = \mathfrak{U}(\mathfrak{g}_N^J) \{ H - k, Z_{ij} + L_{ij} : 1 \leq i \leq j \leq N \}.
\]

By Lemmas 3.6 and 3.7, the algebra \( \mathcal{D}_{\pi_{k,L}} \) is

\[
\mathcal{D}_{\pi_{k,L}} = \mathfrak{U}(\mathfrak{g}_N^J)^{t_N^J} / \mathfrak{U}(\mathfrak{g}_N^J)^{t_N^J} \{ H - k, Z_{ij} + L_{ij} \} = \lambda(S(m_N^J))^{t_N^J} = \lambda(S(m_N^J)^{t_N^J}).
\]

Henceforth we will write \( \mathfrak{E}_{k,L}, \mathcal{I}_{k,L}, \mathcal{A}_{k,L}, \mathcal{D}_{k,L} \) for \( \mathfrak{E}_{\pi_{k,L}}, \mathcal{I}_{\pi_{k,L}}, \mathcal{A}_{\pi_{k,L}} \), and \( \mathcal{D}_{\pi_{k,L}} \), respectively.
Since $\mathfrak{g}_N^J$ and $\hat{\mathfrak{g}}_N^J$ are conjugate within $\mathfrak{g}_N^J$, Theorem 3.1 shows that $D(G_N^J \times \hat{\mathfrak{g}}_N^J \mathbb{C}_{k,L}) \cong \mathcal{D}_{k,L}$. Berndt and Schmidt [1] considered the IDO algebra $D(G_1^J \times \hat{\mathfrak{g}}_1^J \mathbb{C}_{k,L})$, the case $N = 1$. They proved that it is not commutative, and its associated graded commutative algebra is generated by two order 2 elements and two order 3 elements, the relation ideal for these generators being generated by one order 6 relation. The complete list of relations in the full non-commutative IDO algebra was determined by Bringmann, Conley, and Richter [3].

We have mentioned that the image of $\mathfrak{z}(\mathfrak{g}_N^J)$ in $\mathcal{D}_{k,L}$ is central. It is clear that the central elements $Z_{ij}$ of $\mathfrak{g}_N^J$ map to the scalars $-L_{ij}$ in $\mathcal{D}_{k,L}$. Let us write $\mathcal{C}_N$ for the image of the Casimir element $\Omega_N$ in $\mathcal{D}_{k,L}$:

$$\mathcal{C}_N := \Omega_N + I_{k,L}.$$ 

One can see from the formula for $\Omega_N$ that $\mathcal{C}_N$ is of order 4 over the center of $\mathfrak{g}_N^J$ for $N > 1$, and of order 3 for $N = 1$. It follows that the image of $\Omega_N$ in the center of $\mathfrak{D}(G_N^J \times \hat{\mathfrak{g}}_N^J \mathbb{C}_{k,L})$ is a differential operator of order 4 for $N > 1$ and of order 3 for $N = 1$. The following theorem is proven in [3].

**Theorem 4.3.** In the case $N = 1$,

1. The center $Z\mathcal{D}_{k,L}$ is the polynomial algebra $\mathbb{C}[C_1]$.
2. The center $ZD(G_1^J \times \hat{\mathfrak{g}}_1^J \mathbb{C}_{k,L})$ is the polynomial algebra generated by the action of the Casimir element $\Omega_1$.

Then the question arises: what is the center of the IDO algebra in the case $N > 1$? In this paper we will answer this question in the case that $N = 2$ and $L$ is invertible. Our main result is as follows:

**Theorem 4.4.** For $N = 2$ and $L$ invertible,

1. The center $Z\mathcal{D}_{k,L}$ is the polynomial algebra $\mathbb{C}[C_2]$.
2. The center $ZD(G_2^J \times \hat{\mathfrak{g}}_2^J \mathbb{C}_{k,L})$ is the polynomial algebra generated by the action of the Casimir element $\Omega_2$. 

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CHAPTER 5

THE CENTERS OF THE ALGEBRAS $\mathcal{D}_{k,L}$

5.1. The Center

We first prove our result assuming that the matrix $L$ defined in Chapter 3 is $\frac{1}{2}I_2$. In this case $\mathcal{I}_{k,L}$ is the ideal in $\mathcal{E}_{k,L}$ generated by the relations $H - k$ and $Z_{ij} + \frac{1}{2}\delta_{ij}$. Therefore in $\mathcal{D}_{k,L}$, $H \equiv k$ and $Z_{ij} \equiv -\frac{1}{2}\delta_{ij}$. For the time being we continue to allow $N$ to be arbitrary.

**Definition.** Let $r_N^J$ be the radical subalgebra of $g_N^J$:

$$r_N^J = \text{Span}\{e_i, f_i, Z_{ij} : 1 \leq i \leq j \leq N\}.$$ 

Note that $\mathcal{I}_{k,\frac{1}{2}I_N} \cap \mathfrak{U}(r_N^J)$ is the ideal generated by the elements $Z_{ij} + \frac{1}{2}\delta_{ij}$. Let $\mathcal{W}$ be the quotient algebra

$$\mathcal{W} = \mathfrak{U}(r_N^J)/\left(\mathcal{I}_{k,\frac{1}{2}I_N} \cap \mathfrak{U}(r_N^J)\right).$$

Clearly $\mathcal{W}$ has a basis

$$\left\{f_{i_1}^{i_2} f_{i_2}^{i_3} \cdots f_{i_N}^{i_1} e_{j_1}^{j_2} \cdots e_{j_N}^{j_N} : i_1, i_2, \ldots, i_N, j_1, j_2, \ldots, j_N \in \mathbb{N}\right\},$$

with relations $[e_i, e_j] = 0$, $[f_i, f_j] = 0$, $[f_i, e_j] = -\delta_{ij}$.

We will use the multinomial notation

$$f^{I} e^{J} = f_{i_1}^{i_2} f_{i_2}^{i_3} \cdots f_{i_N}^{i_1} e_{j_1}^{j_2} \cdots e_{j_N}^{j_N}, \quad |I| = i_1 + i_2 + \cdots + i_N, \quad |J| = j_1 + j_2 + \cdots + j_N.$$ 

Note that the adjoint action of $\mathfrak{sl}_2$ preserves $r_N^J$ and $\mathcal{I}_{k,L}$, so it descends to $\mathcal{W}$. The monomial $f^I e^J$ is of weight $|J| - |I|$. Since $\mathfrak{sl}_2$ acts by derivations, the weight zero subspace

$$\mathcal{W}_0 = \text{Span}\left\{f^I e^J : |J| = |I|\right\}$$

is a subalgebra.

Consider the subspace

$$a_N = \text{Span}\{f_i e_j : 1 \leq i, j \leq N\}$$

of $\mathcal{W}_0$. The reader may check the following lemma.
Lemma 5.1.  

(1) $a_N$ is a Lie subalgebra of $(W_0)_{\text{Lie}}$.

(2) The map $f_i e_j \mapsto \epsilon_{ij}$ is a Lie algebra isomorphism $a_N \rightarrow \mathfrak{gl}_N$.

(3) The center of $a_N$ is $\mathbb{C}\Delta$, where $\Delta = f_1 e_1 + \cdots + f_N e_N$.

(4) $a_N$ generates $W_0$ as an associative algebra.

Our next lemma gives certain commutation relations in $W$.

Lemma 5.2. For $1 \leq i \leq N$ and any positive integer $n$,

(1) $[f_i, e_i^n] = -n e_i^{n-1}$ and $[e_i, f_i^n] = n f_i^{n-1}$.

(2) $f_i^n e_i^m = (f_i e_i - n)(f_i e_i - n + 2) \cdots (f_i e_i - 2)(f_i e_i - 1)f_i e_i$.

Proof. We use induction on $n$. We have $[f_i, e_i] = -\delta_{ii} = -1$, so our claim is true for $n = 1$. Suppose it is true for $n = m$, i.e., $[f_i, e_i^m] = -m e_i^{m-1}$. Then for $n = m + 1$,

$[f_i, e_i^{m+1}] = [f_i, e_i^m] e_i + e_i^m [f_i, e_i] = -m e_i^m - e_i = -(m + 1)e_i^m$,

proving the first part. The proof of second statement is similar.

For (2), we again induct on $n$. By (1),

$f_i^{m+1} e_i^m = f_i f_i^m e_i e_i^m = f_i (e_i f_i^m - m f_i^{m-1})e_i^m = f_i e_i f_i^m e_i^m - m f_i^m e_i^m = (f_i e_i - m) f_i^m e_i^m$.

The result follows. 

□

Definition. Let $A$ be an algebra and let $S$ be any subset of $A$. Then we denote the commutant of $S$ in $A$ by $A^S$.

Lemma 5.3.  

(1) For any algebra $A$ containing $W_0$, $A^{W_0} = A^{a_N}$.

(2) The adjoint action of $\Delta$ is $\text{ad}(\Delta)(f^I e^J) = (|I| - |J|)(f^I e^J)$, so $W^\Delta = W_0$.

Proof. Part (1) follows from Lemma 5.1 (4). Using Part (1) of Lemma 5.2 it is easy to see that $[f_r e_r, f^I e^J] = (i_r - j_r)f^I e^J$. Part (2) follows. 

□

Proposition 5.4. For $N = 2$, $W^{a_2} = \mathbb{C}[\Delta]$.

Proof. Define

$S^n_e := \text{Span}\{e^J : |J| = n\}, \quad S^n_f := \text{Span}\{f^I : |I| = n\}$.
It is easy to see using Lemma 5.2 that
\[
[f_1 e_1, e_1^{j_1} e_2^{j_2}] = -j_1 e_1^{j_1} e_2^{j_2}, \quad [f_1 e_2, e_1^{j_1} e_2^{j_2}] = -j_1 e_1^{j_1-1} e_2^{j_2+1},
\]
\[
[f_2 e_1, e_1^{j_1} e_2^{j_2}] = -j_2 e_1^{j_1+1} e_2^{j_2-1}, \quad [f_2 e_2, e_1^{j_1} e_2^{j_2}] = -j_2 e_1^{j_1} e_2^{j_2}.
\]

Recall that \( a_2 \cong \mathfrak{gl}_2 \) contains a copy of \( \mathfrak{sl}_2 \) via \( H \mapsto f_1 e_1 - f_2 e_2, E \mapsto f_1 e_2, F \mapsto f_2 e_1 \). By the above formulas, \( S^n_e \) has weights \( n, n-2, \cdots, -n \) under this copy of \( \mathfrak{sl}_2 \), and each weight space has multiplicity 1. Therefore \( S^n_e \) is equivalent to the irreducible representation \( L_n \) of \( \mathfrak{sl}_2 \). Similarly one finds \( S^n_f \cong L_n \).

We have seen that \( W_0 = \bigoplus_{n=0}^\infty S^n_f S^n_e \). Under \( \mathfrak{sl}_2 \),
\[ S^n_f S^n_e \cong L_n \otimes L_n \cong L_{2n} \oplus \cdots \oplus L_2 \oplus L_0, \]
so it contains up to a scalar a single \( \mathfrak{sl}_2 \)-invariant element. Since \( \Delta^n \) is \( \mathfrak{sl}_2 \)-invariant, it must be this element modulo terms of lower degree. Thus \( W^{a_2} = \mathbb{C}[\Delta] \). \( \square \)

5.2. A New Basis of \( \mathcal{D}_{k, \frac{1}{2} I_2} \)

The following elements of \( \mathfrak{U}(g_2^J)[\det(Z)^{-1}] \) are defined in [5]:
\[
\tilde{H} = H - \frac{1}{4}(e^T Z^{-1} f + f^T Z^{-1} e), \quad \tilde{E} = E - \frac{1}{4}(e^T Z^{-1} f), \quad \tilde{F} = F + \frac{1}{4} f^T Z^{-1} f.
\]

It is easy to see that
\[
[\tilde{H}, \tilde{E}] = 2\tilde{E}, \quad [\tilde{H}, \tilde{F}] = -2\tilde{F}, \quad [\tilde{E}, \tilde{F}] = \tilde{H}.
\]
Thus \( \tilde{sl}_2 = \text{Span}\{\tilde{H}, \tilde{E}, \tilde{F}\} \) is a copy of \( sl_2 \). It was proven in [5] that in terms of these elements, the Casimir element defined in Chapter 3 is given as follows.

\textsc{Proposition 5.5.} \( \Omega_N = \det(Z)(\tilde{H}^2 + 2\tilde{H} + 4\tilde{F}\tilde{E}) \).

The following theorem is crucial to our argument. Its proof may be found in [5], and is based on ideas of Borho [2], Quesne [9], and Campoamor-Stursburg and Low [4].

\textsc{Theorem 5.6.} \( \mathfrak{U}(g_2^J)[\det(Z)^{-1}] = \mathfrak{U}(\tilde{sl}_2) \otimes_{\text{alg}} \mathfrak{U}(r_N^J)[\det(Z)^{-1}] \).
The main point is that \( \tilde{\mathcal{L}}_2 \) commutes with \( r_N^J \), so the tensor product is in the category of algebras. For \( L = \frac{1}{2} I_2 \), short computations show that modulo \( \mathcal{L}_{k, \frac{1}{2} I_2}, \tilde{H}, \tilde{E}, \) and \( \tilde{F} \) are given by

\[
\tilde{H} \equiv (k + \Delta + 1), \quad \tilde{E} \equiv E + \frac{1}{2}(e_1^2 + e_2^2), \quad \tilde{F} \equiv F - \frac{1}{2}(f_1^2 + f_2^2).
\]

**Theorem 5.7.** The following sets are bases of \( D_{k, \frac{1}{2} I_2} \):

\[
\mathcal{B}_0 := \{ F^i E^j f^I e^J : 2j + |J| - 2i - |I| = 0 \}, \\
\tilde{\mathcal{B}}_0 := \{ \tilde{F}^i \tilde{E}^j f^I e^J : 2j + |J| - 2i - |I| = 0 \}.
\]

**Proof.** By Corollary 3.5, \( \mathcal{E}_{k, \frac{1}{2} I_2} / \mathcal{L}_{k, \frac{1}{2} I_2} \) has a basis \( \{ F^i E^j f^I e^J : i, j, |I|, |J| \in \mathbb{N} \} \). Note that both \( F^i E^j f^I e^J \) and \( \tilde{F}^i \tilde{E}^j f^I e^J \) are of weight \( 2(j - i) + |J| - |I| \). It is now immediate from (2) that \( \mathcal{B}_0 \) is a basis of \( D_{k, \frac{1}{2} I_2} \).

To prove that \( \tilde{\mathcal{B}}_0 \) is also a basis of \( D_{k, \frac{1}{2} I_2} \), define

\[
\mathcal{B}^n_0 = \{ F^i E^j f^I e^J : 2i + |I| \leq n, \ 2(i - j) = |J| - |I| \},
\]

and define \( \tilde{\mathcal{B}}^n_0 \) similarly. By an induction argument based on a symbol calculation, \( \mathcal{B}^n_0 \) and \( \tilde{\mathcal{B}}^n_0 \) have the same span. Since they also have the same cardinality, the result follows. \( \square \)

**Proposition 5.8.**

1. \( \mathcal{W}_0 \subset D_{k, \frac{1}{2} I_2} \).
2. \( D_{k, \frac{1}{2} I_2}^\Delta = \text{Span}_{\mathbb{C}[\mathcal{W}_0]} \{ \tilde{F}^i \tilde{E}^i : i \in \mathbb{N} \} \).
3. \( D_{k, \frac{1}{2} I_2}^{\mathcal{W}_0} = \text{Span}_{\mathbb{C}[\Delta]} \{ \tilde{F}^i \tilde{E}^i : i \in \mathbb{N} \} \).

**Proof.** Part (1) is clear. For Part (2), note that by Lemma 5.3 (2) and Theorem 5.6,

\[
\text{ad}(\Delta)(\tilde{F}^i \tilde{E}^j f^I e^J) = (|I| - |J|) \tilde{F}^i \tilde{E}^j f^I e^J,
\]

so the commutant of \( \Delta \) is spanned by those elements of \( \tilde{\mathcal{B}}_0 \) with \( |I| = |J| \) and hence also \( i = j \).

For Part (3), suppose that \( T \in D_{k, \frac{1}{2} I_2}^{\mathcal{W}_0} \). Since \( \Delta \in \mathcal{W}_0 \), Part (2) shows that

\[
T = \sum_i \tilde{F}^i \tilde{E}^i T_i
\]
for unique elements $T_i \in \mathcal{W}_0$. For any $D \in \mathcal{W}_0$ we have
\[
0 = [D, T] = \sum_i \tilde{F}^i \tilde{E}^i [D, T_i].
\]
Since $[D, T_i]$ is in $\mathcal{W}$, it must be zero because $\mathcal{B}_0$ is a basis of $\mathcal{D}_{k, \frac{1}{2}I_2}$. Therefore by Proposition 5.4, $T_i \in \mathbb{C}[\Delta]$ for all $i$. \hfill \Box

5.3. Proof of Theorem 4.4 for $L = \frac{1}{2}I_2$

The following lemma is elementary.

**Lemma 5.9.** For all $n \in \mathbb{N}$,
\[
\tilde{F}^n \tilde{E}^n = (\tilde{F} \tilde{E} - (n - 1)(\tilde{H} + n))(\tilde{F} \tilde{E} - (n - 2)(\tilde{H} + n - 1)) \cdots (\tilde{F} \tilde{E} - (\tilde{H} + 2))(\tilde{F} \tilde{E} - (\tilde{H} + 2)).
\]

**Proposition 5.10.**

1. $\mathcal{C}_2 = 4\tilde{F} \tilde{E} + (\Delta + k + 2)^2 - 1 + \mathcal{I}_{k, \frac{1}{2}I_2}$.

2. $\mathcal{D}_{\mathcal{V}_0}^{k, \frac{1}{2}I_2} = \mathbb{C}[\Delta, \mathcal{C}_2]$, and $\Delta$ and $\mathcal{C}_2$ are algebraically independent.

**Proof.** Since $\tilde{H} \equiv k + \Delta + 1$ modulo $\mathcal{I}_{k, \frac{1}{2}I_2}$, a simple calculation gives us
\[
\mathcal{C}_2 = \Omega_2 + \mathcal{I}_{k, \frac{1}{2}I_2} = \tilde{H}^2 + 2\tilde{H} + 4\tilde{F} \tilde{E} + \mathcal{I}_{k, \frac{1}{2}I_2} = 4\tilde{F} \tilde{E} + (\Delta + k + 2)^2 - 1 + \mathcal{I}_{k, \frac{1}{2}I_2},
\]
proving Part (1).

By Part (1) and Lemma 5.9, $\tilde{F}^i \tilde{E}^i \in \mathbb{C}[\mathcal{C}_2, \Delta]$ for all $i \in \mathbb{N}$. Then by Proposition 5.8 (3), $\mathcal{D}_{\mathcal{V}_0}^{k, \frac{1}{2}I_2} = \mathbb{C}[\Delta, \mathcal{C}_2]$. By Lemma 5.9, for any positive integer $n$
\[
\mathcal{C}_2^n = (4\tilde{F} \tilde{E} + (\Delta + k + 2)^2 - 1)^n = 4^n \tilde{F}^n \tilde{E}^n + \text{terms with lower degree in } \tilde{F}, \tilde{E}.
\]

By Lemma 5.2, for any positive integer $m$
\[
\Delta^m = (f_1 e_1 + f_2 e_2)^m = f_1^m e_1^m + \text{terms with lower degree in } f_1, e_1.
\]

Now suppose
\[
p(X, Y) = \sum_{0 \leq i \leq n, 0 \leq j \leq m} a_{ij} X^i Y^j
\]
is a polynomial in two variables such that $p(\mathcal{C}_2, \Delta) = 0$. Then
\[
\sum_{0 \leq j \leq m} a_{nj} \mathcal{C}_2^n \Delta^j + \sum_{0 \leq i \leq n-1, 0 \leq j \leq m} a_{ij} \mathcal{C}_2^i \Delta^j = 0,
\]
and so
$$
\sum_{0 \leq j \leq m} a_{nj}4^n \tilde{F}^n \tilde{E}^n f_j^i e_1^j + \text{terms with lower } \tilde{F}, \tilde{E}, f_1, e_1 \text{ degree } = 0.
$$

Since $\tilde{B}_0$ is a basis, $a_{nj} = 0$ for all $j$. It follows that $p(X, Y) \equiv 0$, so $C_2$ and $\Delta$ are algebraically independent.

\[ \square \]

**Lemma 5.11.** For any positive integer $n$, $[\tilde{F} e_1^2, \Delta^n] = \sum_{m=0}^{n-1} \binom{n}{m} 2^{n-m} \Delta^m \tilde{F} e_1^2$.

**Proof.** Since $\tilde{F}$ commutes with $f_i, e_j$, and $\Delta$ acts on $f^I e^J$ by $|I| - |J|$, the proof is an easy induction.

\[ \square \]

**Proof of Theorem 4.4 for $L = \frac{1}{2} I_2$.** We know that $C_2 \in \mathbb{Z}D_{k, \frac{1}{2} I_2} \subset D_{W_0}^{k, \frac{1}{2} I_2} = \mathbb{C}[\Delta, C_2]$. Suppose $T \in \mathbb{C}[\Delta, C_2]$ commutes with $\tilde{F} e_1^2$. Since $C_2$ is central we can write $T = \sum_{i=0}^n p_i \Delta^i$, where $p_i \in \mathbb{C}[C_2]$. Then

$$
0 = [\tilde{F} e_1^2, T] = \sum_{i=0}^n p_i [\tilde{F} e_1^2 \Delta^i].
$$

Using Lemma 5.11 we obtain

$$
\sum_{i=1}^n p_i \sum_{j=0}^{i-1} \binom{i}{j} 2^{i-j} \Delta^j = 0.
$$

Therefore the polynomial $P(X, Y) = \sum_{i=1}^n p_i(X) \sum_{j=0}^{i-1} \binom{i}{j} 2^{i-j} Y^j$ satisfies $P(C_2, \Delta) = 0$ and so is identically zero. Thus $p_i = 0$ for all $i > 0$, so $T = p_0$. This proves that $\mathbb{Z}D_{k, \frac{1}{2} I_2} = \mathbb{C}[C_2]$.

**5.4. The Proof of the Main Result**

We have completed the proof in the case that the matrix $L$ is equal to $\frac{1}{2} I_2$. We now complete the proof in the case that $L$ is any $2 \times 2$ symmetric invertible matrix.

**Proof of Theorem 4.4.** Since $L$ is symmetric and we are assuming that it is also invertible, with the help of Takagi’s factorization [7] we can find a $2 \times 2$ invertible matrix $M$ such that $MLM^T = \frac{1}{2} I_2$. Define

$$
e' := Me, \quad f' := Mf, \quad Z' := MZM^T.$$

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Then
\[ [e'_r, f'_s] = \sum_{j,q} [M_{rj} e_j, M_{sq} f_q] = \sum_{j,q} M_{rj} [e_j, f_q] M_{sq} = \sum_{j,q} -2M_{rj} Z_{jq} M_{sq} = -2Z'_{rs}. \]

Moreover it is easy to check that
\[
\begin{align*}
[H, e'_1] &= e'_1, & [H, f'_1] &= f'_1, & [E, e'_i] &= 0, \\
[F, f'_1] &= 0, & [E, f'_i] &= -e'_i, & [F, e'_i] &= -f'_i, \\
[X, Z'_{ij}] &= 0 \text{ for all } X \in \mathfrak{g}_2^j.
\end{align*}
\]

Hence \( M : \mathfrak{g}_2^j \to \mathfrak{g}_2^j \) defined by
\[
\begin{align*}
H &\mapsto H, \quad E \mapsto E, \quad F \mapsto F, \\
e_i &\mapsto e'_i, \quad f_i \mapsto f'_i, \quad \text{and } Z_{ij} \mapsto Z'_{ij}
\end{align*}
\]
is a Lie algebra automorphism, and
\[
\{H, E, F, e'_1, e'_2, f'_1, f'_2, Z'_{11}, Z'_{12}, Z'_{22}\}
\]
is a new basis of \( \mathfrak{g}_2^j \). Note that the automorphism \( M \) maps \( \mathfrak{k}_2^j \) to itself, and if we have a representation \( \pi_{k,L} \) of \( \mathfrak{k}_2^j \) on \( \mathbb{C}_k \), such that \( \pi_{k,L}(Z'_{ij}) = L_{ij} \), then \( \pi_{k,L}(Z'_{ij}) = \frac{1}{2} \delta_{ij} \).

Therefore by the case \( L = \frac{1}{2} I_2 \), the center of \( \mathcal{D}_{k,L} \) is generated by the image \( C'_2 \) of the Casimir element
\[
\Omega'_2 := \det(Z') \left[ H^2 - 4H + 4EF \right] - \left( H - \frac{5}{2} \right) e'^T Z'^{-1} f' + E f'^T Z'^{-1} f' - e'^T Z'^{-1} e'F \\
+ \frac{1}{4} e'^T (e'^T Z'^{-1} f') Z'^{-1} f' - \frac{1}{4} (e'^T Z'^{-1} e')(f'^T Z'^{-1} f')
\]
in \( \mathcal{D}_{k,L} \). Since \( \det(Z') = \det(M^2) \det(Z) \), \( \Omega'_2 - \det(M^2) \Omega_2 \) is of order \(< 2\) in \( H, E, F \). By Theorem 4.2, any such element of \( \mathfrak{z}(\mathfrak{g}_N^j) \) is of order 0 in \( H, E, F \) and so is polynomial in \( Z_{ij} \). Hence \( C'_2 - \det(M^2)C_2 \) is a scalar, so \( \mathbb{C}[C_2] = \mathbb{C}[C'_2] \).


