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SECONDARY STATES OF VIBRATING PLATES

by

Bernard J. Matkowsky, Leonard J. Putnick,
and Edward L. Reiss

ARGONNE NATIONAL LABORATORY, ARGONNE, ILLINOIS

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ABSTRACT

A previously developed perturbation method is used to obtain a new class of periodic motions for the nonlinear vibrations of rectangular, elastic plates. The dynamic von Kármán plate theory is used in the analysis. The new solutions arise by secondary bifurcation from the periodic solutions that bifurcate from the natural frequencies of free vibrations of the linearized plate theory. The new motions are a linear combination of two modes of the linearized theory.

1. Introduction

The natural frequencies ω_{mn} and the normal modes $\phi_{mn}(x,y)$, $m,n = 1,2,\dots$ of free vibrations of elastic plates are determined from a linear theory of vibrating plates, see e.g. [1]. They are the eigenvalues and eigenfunctions, respectively of this theory. However, a linear theory does not determine the amplitude of the free normal modes. It is necessary to employ a nonlinear plate theory to obtain these amplitudes. Then as we show in section 3 the frequencies ω_{mn} are bifurcation points for periodic solutions of the nonlinear theory. That is, branches of temporally periodic solutions of period $2\pi/\omega(\epsilon)$ of the nonlinear problem bifurcate from the basic solution at each ω_{mn} such that $\omega(0) = \omega_{mn}$. Here ϵ is an amplitude of the periodic solutions, and the basic solution corresponds to the motionless flat plate. Furthermore, the bifurcation is supercritical. This means that the frequency ω increases as the amplitude ϵ increases. Thus, the plate responds like a "stiff spring" when it oscillates. The stiffness results from the stretching of the middle plane of the plate due to the plate's motion normal to this plane. The stretching tends to decrease the amplitude of the normal motion, and hence to effectively increase the plate's stiffness. The Poincaré-Linstedt perturbation method is used in our analysis, see e.g. [2]. In previous studies of the

free nonlinear oscillations of rectangular plates, one term Galerkin methods are usually employed, see [3] for references.

Conversely, if a thrust is applied to the midplane, for example, by an applied end shortening the effective stiffness of the plate is changed. Consequently the natural frequencies vary as the thrust varies. They increase (decrease) if the thrust is tensile (compressive). However as we demonstrate, the thrust can produce a more dramatic change in the nonlinear response of the plate. Specifically, we show that there are critical values of the thrust, which are less than the buckling load of the plate, such that for thrusts near these critical values, there are secondary bifurcations of periodic solutions of the nonlinear plate theory. That is, new periodic solutions bifurcate from the basic solution. A typical response curve that illustrates this is shown in Figure 1 where the thrust parameter is denoted by λ . We refer to the bifurcation points ω_{mn} and the solutions that branch from them as the primary bifurcation points and the primary states, respectively. Similarly, the points at which the new periodic solutions bifurcate from the primary state are called secondary bifurcation points and the new solutions are then called secondary states.

As the thrust λ varies, the positions of the primary and secondary bifurcation points vary. Specifically as λ approaches a critical value, the secondary bifurcation points coalesce with its primary bifurcation point as we demonstrate in this paper. A consequence of this result is described by considering a plate vibrating in a state corresponding to a point on a primary branch with thrust λ and small amplitude $\epsilon_0 < \epsilon_s(\lambda)$, see Figure 1. Then by keeping ϵ_0 fixed and varying λ we conclude that there is a value λ_0 such that $\epsilon_0 = \epsilon_s(\lambda_0)$ and a range of values of λ for which $\epsilon_s(\lambda) < \epsilon_0$. Thus as λ varies through λ_0 , the solution may jump so that the plate vibrates in a state corresponding to a point on the secondary branch.

In our analysis we consider simply supported rectangular plates that are compressed by a uniaxial end-shortening, and we employ the dynamic von Kármán theory of plates. Furthermore we use the ideas of secondary bifurcation and the perturbation methods presented in [4] and which were used in [5,6] to determine the secondary bifurcation points and the secondary states. The method relies on the previously stated observation that secondary bifurcation

points can coalesce with primary bifurcation points as a parameter such as λ varies.

The results are summarized in section 5. The secondary branch, near the secondary bifurcation point, is shown to be a linear combination of the normal mode corresponding to the basic primary branch, and a cross mode. The cross mode is the normal mode of the primary branch whose primary bifurcation point coalesces with the primary and secondary bifurcation points of the basic primary branch as λ approaches a critical value.

2. Formulation

We consider a simply supported rectangular plate of aspect ratio ℓ . The edges of the plate are compressed by rigid, "greased" blocks. Thus a uniform and specified end shortening C_0 is applied in the X direction, see Figure 2. The edge displacement C_1 in the Y direction is adjusted so that the total edge force in the Y direction vanishes. These boundary conditions were considered in [7] for a time independent problem. The equations of motion for the plate in nondimensional form is given by

$$(2.1a) \quad \Delta^2 \bar{w} + \lambda \bar{w}_{xx} + \bar{w}_{tt} = [\bar{f}, \bar{w}] ,$$

$$(2.1b) \quad \Delta^2 \bar{f} = -\frac{1}{2}[\bar{w}, \bar{w}] ,$$

where $\bar{w}$ is the displacement in the z direction and $\bar{f}$ is a stress function. The boundary conditions are

$$(2.1c) \quad \bar{w} = \Delta \bar{w} = 0 ,$$

$$(2.1d) \quad \bar{f}_N = \bar{f}_{NNN} = 0 ,$$

where the subscript N denotes derivatives with respect to the outward normal to the boundary B of the plate. In the dimensionless variables, the plate occupies the region $0 \leq x \leq \ell$, $0 \leq y \leq 1$, where the aspect ratio ℓ is defined by

$$(2.2) \quad \ell \equiv a/b .$$

The boundary conditions (2.1d) are equivalent to

$$(2.3) \quad \bar{f}_N = (\Delta \bar{f})_N = 0 .$$

The nonlinear operator $[\bar{f}, \bar{g}]$ in (2.1) is defined for any two sufficiently differentiable functions $\bar{f}$ and $\bar{g}$ by

$$(2.4) \quad [\bar{f}, \bar{g}] \equiv \bar{f}_{xx} \bar{g}_{yy} + \bar{f}_{yy} \bar{g}_{xx} - 2 \bar{f}_{xy} \bar{g}_{xy} .$$

The "thrust" parameters λ in (2.1a) is defined by

$$(2.5) \quad \lambda \equiv 24(1-\nu^2)(b/h)^2 C_0/a ,$$

where ν is Poisson's ratio and h is the thickness of the plate. It is negative (positive) for tension (compression).

We seek periodic solutions of (2.1) of period $2\pi/\omega$. Thus following the Poincare-Linstedt method, we define new variables t , $w(x,y,t)$, $f(x,y,t)$ by

$$(2.6a) \quad t \equiv \omega \bar{t} ,$$

$$(2.6b) \quad w(x,y,t) \equiv \bar{w}(x,y,t/\omega) ,$$

$$(2.6c) \quad f(x,y,t) \equiv \bar{f}(x,y,t/\omega) .$$

Thus the von Kármán plate theory becomes

$$(2.7a) \quad \Delta^2 w + \lambda w_{xx} + \omega^2 w_{tt} = [f, w] ,$$

$$(2.7b) \quad \Delta^2 f = -\frac{1}{2} [w, w] ,$$

$$(2.7c) \quad w = \Delta w = \bar{N} \cdot \nabla f = \bar{N} \cdot \nabla (\Delta f) = 0 , \quad \text{on } h ,$$

where the initial conditions have been omitted since we seek periodic solutions of period 2π , i.e.

$$(2.7d) \quad w(x, y, t+2\pi) = w(x, y, t) ,$$

$$(2.7e) \quad f(x, y, t+2\pi) = f(x, y, t) ,$$

for all t . We note that the solution of $\Delta^2 f = g(x, y)$ with boundary conditions (2.7c) is unique within an additive constant. We set that constant equal to zero since f is a stream function.

We observe that

$$(2.8) \quad w = f \equiv 0$$

is a solution of (2.7). We refer to (2.8) as the basic solution. By linearizing (2.7) about the basic solution (2.8) we obtain the classical theory

$$(2.9) \quad \begin{aligned} \Delta^2 w + \lambda w_{xx} + \omega^2 w_{tt} &= 0 , \\ w = \Delta w &= 0 , \quad \text{on } B , \end{aligned}$$

for the compressed or stretched vibrating elastic plate. Thus, the natural frequencies ω_{mn} , $m, n=1, 2, \dots$, are given by

$$(2.10) \quad \omega_{mn} = (m\pi/\ell)(\lambda_{mn} - \lambda)^{1/2} ,$$

where the critical thrusts λ_{mn} of the static buckling theory are defined by

$$(2.11) \quad \lambda_{mn} \equiv \pi^2 [(m/\ell)^2 + n^2]^2 / (m/\ell)^2 .$$

The buckling load of the plate, $\lambda_c(\ell)$, is then defined by

$$(2.12) \quad \lambda_c(\ell) \equiv \min_{m, n} \lambda_{mn}(\ell) = \min_m \lambda_{m1}(\ell) .$$

For simplicity of presentation, we restrict the values of λ to the interval

$$(2.13) \quad \lambda < \lambda_c(\ell) .$$

Then, all the frequencies ω_{mn} are real. In addition, since the stability of the basic state is determined by the solutions of (2.9) we conclude that if

$\lambda < \lambda_c(\ell)$, then the basic state (2.8) is neutrally stable for all values of $\omega \geq 0$.

The normal modes of the linearized theory (2.9), corresponding to the frequencies (2.10) are given by

$$(2.14a) \quad w_{mn}(x,y,t) = \{A_{mn} \sin t + B_{mn} \cos t\} \phi_{mn}(x,y) ,$$

where the spatial modes ϕ_{mn} are defined by

$$(2.14b) \quad \phi_{mn}(x,y) = 2\ell^{-1/2} \sin(m\pi x/\ell) \sin n\pi y .$$

The amplitudes A_{mn} and B_{mn} in (2.14a) are not determined by the linear theory. However, for any specific mode w_{mn} , their ratio can be determined by observing that (2.9) is translationally time invariant. Since we are considering only periodic solutions, the time axis can always be shifted so that for the specified mode

$$(2.15) \quad \frac{\partial w_{mn}(x,y,0)}{\partial t} = 0 .$$

Then, (2.14a) and (2.15) imply $A_{mn} = 0$ and hence

$$(2.16) \quad w_{mn} = B_{mn} \cos t \phi_{mn} .$$

3. The Primary States

To demonstrate that solutions of the nonlinear problem (2.7) branch from each natural frequency ω_{mn} of the linear problem, we first define an amplitude ϵ of the solutions by

$$(3.1) \quad \epsilon^2 \equiv \pi^{-1} \int_0^1 \int_0^\ell \int_0^{2\pi} w^2(x,y,t) dt dx dy .$$

Then we seek solutions branching from the generic value ω_{mn} by the Poincaré-Linstedt perturbation method. That is, we assume

$$(3.2a) \quad w = \sum_{j=1}^{\infty} w^{(j)} \epsilon^j ,$$

$$(3.2b) \quad f = \sum_{j=1}^{\infty} f^{(j)} \epsilon^j ,$$

$$(3.2c) \quad \omega^2 = \omega_{mn}^2 + \sum_{j=1}^{\infty} \Omega_j \epsilon^j .$$

We note that the assumed solution (3.2) reduces to the basic solution when $\epsilon = 0$. Substituting (3.2) into the von Karman equations (2.7) and equating coefficients of the same power of ϵ , we obtain a sequence of problems

$$(3.3a) \quad \begin{aligned} \Delta w^{(j)} &= \Delta^2 w^{(j)} + \lambda w_{xx}^{(j)} + \omega_{mn}^2 w_{tt}^{(j)} \\ &= - \sum_{i=1}^{j-1} \lambda_i w_{tt}^{(i)} + \sum_{i=1}^{j-1} [f^{(i)}, w^{(j-i)}] = r_j , \end{aligned}$$

$$(3.3b) \quad \Delta^2 f^{(j)} = - \frac{1}{2} \sum_{i=1}^{j-1} [w^{(i)}, w^{(j-i)}] = p_j ,$$

where

$$(3.3c) \quad w^{(j)} = \Delta w^{(j)} = f_N^{(j)} = (\Delta f^{(j)})_N = 0 \quad \text{on } B ,$$

where $j=1,2,\dots$. The summations in (3.3a) and (3.3b) are assumed to be zero when $j < 2$.

Setting $j = 1$ in (3.3), we find that

$$(3.4a) \quad w^{(1)} = \{A_{mn}^{(1)} \sin t + B_{mn}^{(1)} \cos t\} \phi_{mn}(x,y) ,$$

$$(3.4b) \quad f^{(1)} = 0 .$$

We normalize the amplitude by the condition

$$(3.5) \quad w_t(x,y,0) = 0 ,$$

which is a generalization of (2.15). Condition (3.5) and the definition of ϵ , (3.1), yields

$$(3.6) \quad w^{(1)} = \cos t \phi_{mn}(x, y) .$$

To obtain a more complete description of the primary state, we find additional terms in the expansion. For $j \geq 2$, equations (3.3) are inhomogeneous. For a solution to exist we must satisfy the orthogonality condition

$$(3.7) \quad \int_0^1 \int_0^l \int_0^{2\pi} r_j \cos t \phi_{mn} dt dx dy = 0 \quad j \geq 2 .$$

Applying the orthogonality conditions (3.7) and solving (3.3) we find

$$(3.8a) \quad w = w_p(x, y, t; \epsilon, \lambda, \kappa) \equiv \cos t \phi_{mn} \epsilon + O(\epsilon^3) ,$$

$$(3.8b) \quad f = f_p(x, y, t; \epsilon, \lambda, \kappa) \equiv f_{mn}(x, y, t) \epsilon^2 + O(\epsilon^3) ,$$

$$(3.8c) \quad \omega^2 = \omega_p^2(\epsilon, \lambda, \kappa) \equiv \omega_{mn}^2(\lambda, \kappa) + \kappa \epsilon^2 + O(\epsilon^3) ,$$

where the coefficients f_{mn} and κ are defined by

$$(3.9) \quad \begin{aligned} f_{mn} &= (n^2 \kappa / 8m^2)_{\lambda} \cos(2m\pi x/l) + (m/n\lambda)^4 \cos 2n\pi y \cos^2 t , \\ \kappa &= [3(n\pi)^4 / (8\lambda)]_{\lambda} [1 + (m/n\lambda)^4] . \end{aligned}$$

The subscript, p , in (3.8) denotes the primary bifurcation state that branches from ω_{mn} .

In obtaining (3.8), we have assumed that the bifurcation point ω_{mn} is simple. That is, the linearized problem, with the normalization (2.15) has a unique (within a multiplicative constant) mode corresponding to $\omega = \omega_{mn}$. However, as can easily be deduced from (2.10) for every distinct pair of integers (m, n) and (p, q) , there are critical values of λ that are given by

$$(3.10) \quad \tilde{\lambda} \equiv \lambda_{mnpq} \equiv \frac{p^2 \lambda_{pq} - m^2 \lambda_{mn}}{p^2 - m^2} ,$$

such that $\omega_{mn}(\tilde{\lambda}) = \omega_{pq}(\tilde{\lambda})$. Then the eigenvalue $\omega_{mn} = \omega_{pq}$ is multiple, i.e. the distinct eigenfunctions w_{mn} and w_{pq} correspond to the same eigenvalue.

Since ω given in (3.9) is positive, (3.8c) demonstrates that the solution (3.2) branches supercritically, as we show in the response diagram, $\epsilon = \epsilon(\omega)$ in Figure 1. Thus, the frequency of the free oscillations of the plate increases as the amplitude ϵ increases. We observe from (3.9) that Ω is a monotonically decreasing function of the aspect ratio λ . Thus, the narrower the plate, the slower the frequency increases with the amplitude. As $\lambda \rightarrow 0$, $\omega \rightarrow \infty$, and hence the perturbation expansion (3.8) is not valid for small λ .

To lowest order in ϵ , the transverse motion of the mid-plane of the plate, as given (3.8a), is the same as the motion (2.16) of the linear theory. However, the amplitude B_{mn} of the linear theory is arbitrary, while the amplitude ϵ of the nonlinear theory is obtained from (3.8c)

$$(3.11) \quad \epsilon = \pm [(\omega^2 - \omega_{mn}^2)/\omega^2]^{1/2}.$$

The nonlinear oscillations of plates are frequently studied using a one-term Galerkin expansion. That is, approximate solutions of (2.7) (or other boundary conditions) are sought in the form

$$(3.12) \quad w = A(t)\phi_{mn},$$

where the amplitude equation for $A(t)$ is obtained from the Galerkin method. The result (3.8a) suggests that (3.12) is an accurate assumption for small amplitude motions.

4. Secondary Bifurcation Points

The points of secondary bifurcation of periodic solutions from the primary state (3.8) are determined by solving the linear problem

$$(4.1a) \quad \Delta^2 \psi + \lambda \psi_{xx} + \omega_p^2(\epsilon, \lambda) \psi_{tt} = [f_p(\epsilon, \lambda), \psi] + [g, w_p(\epsilon, \lambda)],$$

$$(4.1b) \quad \Delta^2 g = -[w_p(\epsilon, \lambda), \psi],$$

$$(4.1c) \quad \psi = \Delta \psi = 0, \quad g_N = (\Delta g)_N = 0 \quad \text{on } B,$$

$$(4.1d) \quad \psi(x,y,t+2\pi) = \psi(x,y,t) \quad \text{for all } t ,$$

$$(4.1e) \quad g(x,y,t+2\pi) = g(x,y,t) \quad \text{for all } t .$$

Problem (4.1) is obtained by linearizing (2.7) about the primary state (3.8). We have omitted showing the dependence of the primary state on x , y , t , and ℓ . Since problem (4.1) is linear and homogeneous, we seek solutions that are normalized by the condition

$$(4.2) \quad \langle \psi, \psi \rangle = 1 ,$$

where we have used the notation

$$(4.3) \quad \langle F, G \rangle \equiv \int_0^{2\pi} \int_0^{\ell} \int_0^1 F(x,y,t)G(x,y,t)dydxdt$$

for the inner product of any two functions F and G .

We observe that $\psi = g \equiv 0$ is a solution of (4.1) for all values of ε , and λ . We wish to determine the values of $\varepsilon_s = \varepsilon(\lambda)$ for which (4.1) has non-trivial solutions. Then in the response diagram, such as Figure 1, the coordinates of the secondary bifurcation points of the primary state are given by

$$(4.4) \quad \varepsilon_s = \varepsilon(\lambda) \quad \omega_s(\lambda) = \omega_p[\varepsilon_s(\lambda), \lambda] .$$

In addition, it follows from (3.8) that the primary states at the secondary bifurcation points are given by

$$(4.5) \quad \begin{aligned} w_s(x,y,t;\lambda,\ell) &= w_p[x,y,t;\varepsilon_s(\lambda),\lambda,\ell] \\ f_s(x,y,t;\lambda,\ell) &= f_p[x,y,t;\varepsilon_s(\lambda),\lambda,\ell] . \end{aligned}$$

Since w_p and f_p are functions of x,y,t , the differential equations in (4.1) have variable coefficients, and hence the linear problem (4.1) is difficult to solve exactly.

To solve (4.1) by the perturbation method described in [4] we first observe from (4.4) that the secondary bifurcation points vary with λ . In

fact, as can be shown for a class of problems [5,6], as λ approaches the critical value $\tilde{\lambda}$ given by (3.10), secondary bifurcation points coalesce with the primary bifurcation points ω_{mn} and ω_{pq} to form the multiple primary bifurcation point, $\omega_{mn}(\tilde{\lambda}) = \omega_{pq}(\tilde{\lambda})$. Or, equivalently, as $|\lambda - \tilde{\lambda}|$ increases from zero, the multiple primary bifurcation point splits into its constituent simple primary bifurcation points ω_{mn} and ω_{pq} and into one or more secondary bifurcation points. These secondary bifurcation points vary along one or both of the primary states branching from ω_{mn} , and ω_{pq} as $|\lambda - \tilde{\lambda}|$ increases. This suggests a perturbation method in the small parameter δ that is defined by

$$(4.6) \quad \delta \equiv |\lambda - \tilde{\lambda}|$$

to obtain the secondary bifurcation points on the generic primary state (3.8). Thus we seek solutions as power series in the parameter $|\delta|^{1/2}$ *. Thus we assume

$$(4.7a) \quad \varepsilon = b\delta^{1/2} + \sum_{j=1}^{\infty} b_j \delta^{j/2},$$

$$(4.7b) \quad \psi = \sum_{j=0}^{\infty} \psi^{(j)} \delta^{j/2},$$

$$(4.7c) \quad g = \sum_{j=0}^{\infty} g^{(j)} \delta^{j/2}.$$

We use (4.7a) to express the primary state (3.8) in terms of δ . Substituting the result of this calculation and (4.7b) and (4.7c) into (4.1) and equating coefficients of the power of δ to zero, we obtain a sequence of problems. The first three of these problems are

$$(4.8a) \quad M\psi^{(0)} \equiv \Delta^2 \psi^{(0)} + \tilde{\lambda} \psi_{xx}^{(0)} + \omega_{mn}(\tilde{\lambda}) \psi_{tt}^{(0)} = 0,$$

$$(4.8b) \quad M\psi^{(1)} = b_0 \cos t [g^{(0)}, \phi_{mn}],$$

*For other secondary bifurcation problems, different powers of δ may be needed.

$$\begin{aligned}
 (4.8c) \quad M\psi^{(2)} = & -\text{sgn}(\delta)\psi_{xx}^{(0)} - \left\{ -\text{sgn}(\delta) \frac{m^2\pi^2}{\ell^2} + b_0^2 \Omega \right\} \psi_{tt}^{(0)} \\
 & + b_0^2 \cos^2 t [\phi_{mn}, \eta^{(0)}] + b_0 \cos t [g^{(1)}, \phi_{mn}] \\
 & + b_1 \cos t [g^{(0)}, \phi_{mn}] ,
 \end{aligned}$$

$$(4.8d) \quad \psi^{(j)} = \Delta\psi^{(j)} = 0 \quad \text{on } B, \quad j=1,2,3 ,$$

$$(4.8e) \quad \psi_t^{(j)}(x,y,0) = 0 \quad j=1,2,3 ,$$

$$(4.9a) \quad \Delta^2 g^{(0)} = 0 ,$$

$$(4.9b) \quad \Delta^2 g^{(1)} = -b_0 \cos t [\phi_{mn}, \psi^{(0)}] ,$$

$$(4.9c) \quad \Delta^2 g^{(2)} = -b_0 \cos t [\phi_{mn}, \psi^{(1)}] - b_1 \sin t [\phi_{mn}, \psi^{(0)}] ,$$

$$(4.9d) \quad g_N^{(j)} = (\Delta g^{(j)})_N = 0 \quad \text{on } B, \quad j=1,2,3 ,$$

where $\text{sgn}(\delta)$ is the sign function.

Equations (4.8)-(4.9) are solved by employing the same techniques which yielded the primary state. The principal difference is that there are two orthogonality conditions instead of one because of the multiplicity of the eigenvalue. We confine ourselves to a statement of the results:

$$(4.10a) \quad \psi = \phi_{pq} \cos t + O(\delta^{1/2}) ,$$

$$(4.10b) \quad g = g_1 |\delta|^{1/2} + O(\delta) ,$$

where the constant b and the coefficient g_1 are given by

$$(4.11a) \quad g_1 = b\lambda \cos^2 t \{ (mq+np)^2 [\chi_{m-p, n+q} + \chi_{m+p, n-q}] - (mq-np)^2 [\chi_{m-p, n-q} + \chi_{m+p, n+q}] \} ,$$

$$(4.11b) \quad \lambda_{i,j} = \frac{\cos(i\pi x/\ell) \cos j\pi y}{(i^2 + j^2 \ell^2)^2} ,$$

$$(4.11c) \quad b^2 = (\pi^{3/2})^3 (p^2 - m^2) K_{mnpq} \operatorname{sgn}(\delta) ,$$

where the constant K_{mnpq} is given by

$$(4.12a) \quad K_{mnpq}^{-1} = \frac{3\pi^5}{16\lambda} \begin{cases} n^4 (1 + (m/n\ell)^4) - Z_{m,n,p,q} - Z_{m,n,-p,q} & n \neq q \\ n^4 (1 + (m/n\ell)^4) - (4n^4 + (mp/\ell^2)^2) - Y_{m,n,p} - Y_{m,n,-p} & n = q \end{cases} ,$$

where the constants $Z_{m,n,p,q}$ and $Y_{m,n,p}$ are defined by

$$(4.12b) \quad Z_{mnpq} = (mq+np)^4 \left\{ \frac{1}{[(m-p)^2 + (n+q)^2 \ell^2]^2} + \frac{1}{[(m+p)^2 + (n-q)^2 \ell^2]^2} \right\} > 0 ,$$

$$(4.12c) \quad Y_{mnp} = \frac{n^4 (m+p)^4}{[(m-p)^2 + 4n^2 \ell^2]^2} > 0 .$$

For secondary bifurcation to occur, the right side of (4.11c) must be positive. This implies that for any given quartet of positive integers (m,n,p,q) , $\operatorname{sgn} \delta$ must either be positive or negative. Equivalently, secondary bifurcations occur only for $\lambda > \tilde{\lambda}$ or $\lambda < \tilde{\lambda}$ depending on the sign of $(p^2 - m^2)k_{mnpq}$. By inserting (4.6), (4.7) and (4.11c) into (3.8c), we obtain the secondary bifurcation points as

$$(4.13) \quad \omega_s^{(2)}(\lambda, \delta) = \omega_{mn}^2(\tilde{\lambda}, \ell) + |\delta| \left\{ -\operatorname{sgn}(\delta) \frac{M^2 \pi^2}{\ell^2} + \Omega b^2 \right\} + O(\delta^{3/2}) .$$

5. Secondary Bifurcation States

The secondary bifurcation states are now obtained by a perturbation expansion in the neighborhood of the secondary bifurcation points. Thus, we define a new small amplitude parameter ρ by

$$(5.1) \quad \rho^2 \equiv \frac{1}{\pi} \int_0^1 \int_0^L \int_0^\pi (w - w_p)^2 dt dx dy .$$

Then we seek solutions of (2.7) by expanding w , f and ω^2 in power series in ρ . That is,

$$(5.2a) \quad w(\rho, \delta) = w_p(\delta) + \psi(\delta)\rho + \sum_{j=2}^{\infty} w^j(\delta)\rho^j ,$$

$$(5.2b) \quad f(\rho, \delta) = f_p(\delta) + g(\delta)\rho + \sum_{j=2}^{\infty} f^j(\delta)\rho^j ,$$

$$(5.2c) \quad \omega^2(\rho, \delta) = \omega_s^{(2)}(\delta) + B_1(\delta)\rho + \sum_{j=2}^{\infty} B_j(\delta)\rho^j ,$$

where $\{\psi(\delta), g(\delta)\}$ are the eigenfunctions (4.10) of (4.1).

Series representations for w_p , f_p , ω_p^2 , g , ψ were found in sections 3 and 4. We substitute (5.2) into (2.7) and equate coefficients of the same power of ρ and obtain a sequence of problems. We consider the first three:

$$(5.3a) \quad \begin{aligned} \Delta^2 w_p + \omega_p^2 w_{pxx} + \omega_p^2 w_{ptt} &= [f_p, w_p] \\ w_p &= \Delta w_p = 0 \quad \text{on the boundary} \\ w_t(x, y, 0) &= 0 , \end{aligned}$$

$$(5.3b) \quad \begin{aligned} \Delta^2 f_p &= -\frac{1}{2} [w_p, w_p] \\ f_{pN} = (\Delta f_p)_N &= 0 \quad \text{on } B . \end{aligned}$$

$$(5.4a) \quad \Delta^2 \psi + \lambda \psi_{xx} + \omega_p^2 \psi_{tt} = [f_p, g] + [g, \psi] - B_1(\delta) w_{ptt} ,$$

$$\psi = \Delta \psi = 0 \quad \text{on } B ,$$

$$\psi(x, y, 0) = 0 \quad \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \int_0^L v^2 dx dy dt = 1 ,$$

$$(5.4b) \quad \Delta^2 g = -[w_p, \psi] ,$$

$$g_N = (\Delta g)_N = 0 \quad \text{on } B .$$

$$(5.5a) \quad \Delta^2 w^{(2)} + \lambda w_{xx}^{(2)} + \omega_p^2 w_{tt}^{(2)} = [f_p, w^{(2)}] + [\psi, g] \\ + [f^{(2)}, w^p] - B_1 \psi_{tt} - B_2 w_{ptt} ,$$

$$w^{(2)} = \Delta w^{(2)} = 0 \quad \text{on } B ,$$

$$w_t^{(2)}(x, y, 0) = 0 \quad \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \int_0^L \psi w^{(2)} dx dy dt = 0 ,$$

$$(5.5b) \quad \Delta^2 f^{(2)} = -[w_p, w^{(2)}] - \frac{1}{2} [\psi, \psi] ,$$

$$f_N^{(2)} = (\Delta f^{(2)})_N = 0 \quad \text{on } B .$$

Since $\{w_p, f_p\}$ is a solution of (2.5) when $\omega^2 = \omega_p^2$, (5.3) is satisfied identically. Since $\{\eta, \psi\}$ is a solution of (4.4) when $\omega^2 = \omega_p^2$ we are led to the conclusion that

$$(5.6a) \quad B_1(\delta) w_{ptt} = 0$$

and

$$(5.6b) \quad B_1(\delta) = 0 .$$

To determine $w^{(2)}$, $f^{(2)}$ and $B^{(2)}$, we assume that we can write

$$(5.7a) \quad w^{(2)}(\delta) = \sum_{k=0}^{\infty} w^{(2k)} \delta^{k/2} ,$$

$$(5.7b) \quad f^{(2)}(\delta) = \sum_{k=0}^{\infty} f^{(2k)} \delta^{k/2} ,$$

$$(5.7c) \quad B_2(\delta) = \sum_{k=0}^{\infty} b_{2k} \delta^{k/2} .$$

Substituting (5.7), (3.8), (4.10) into (5.5) and (5.1), we obtain a sequence of problems to determine the coefficients in (5.7). The first two of these are

$$(5.8a) \quad M w^{(20)} = 0 \quad w^{20} = \Delta w^{(20)} = 0 \quad \text{on } b \\ w_t^{20}(x, y, 0) = 0 ,$$

$$(5.8b) \quad \Delta^2 f^{(20)} = -\frac{1}{2} [\psi_0, \psi_0] \quad f_N^{20} = (\Delta f^{20})_N = 0 \quad \text{on } B ,$$

$$(5.8c) \quad \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \int_0^1 \eta_0 w_{20} dy dx dt = 0 .$$

$$(5.9) \quad M w^{(21)} = [g_1, \psi^{(0)}] + [f^{(20)}, b_0 \cos t \phi_{mn}] + B_2 h_0 \cos t \phi_{mn} , \\ w^{(21)} = \Delta w^{(21)} = 0 \quad \text{on the } B , \\ w^{(21)}(x, y, 0) = 0 .$$

The operator M is defined in (4.8a). $\psi^{(0)}$, and g_1 are defined in (4.10a) and (4.10b), respectively.

Problem (5.8a) and the normalization condition (5.8c) imply

$$(5.12) \quad w^{20} = \beta \phi_{mn} \cos t ,$$

where β is to be determined. The solution of (5.8b) is

$$(5.13) \quad f^{(20)} = f_{pq} ,$$

where f_{pq} is defined in (3.9). Since (5.9) is an inhomogeneous eigenvalue problem, and the null space of the corresponding homogeneous problem is spanned by $\phi_{mn} \cos t$ and $\phi_{pq} \cos t$, the inhomogeneous terms must satisfy two compatibility conditions. One of these is satisfied identically because no term of the form $\sin t \phi_{pq}$ appears in the nonhomogeneous term. The other condition yields

$$(5.14) \quad B_2 = \frac{3\pi q}{8\ell} \begin{cases} Z_{m,n,p,q} + Z_{m,n,-p,q} > 0 & q \neq n \\ [4n^4 + (pm/\ell^2)^2] + Y_{mnpq} + Y_{mn-p} > 0 & q = n \end{cases}$$

Thus, we have found the following approximation for w , f and ω :

$$(5.15a) \quad w = [b\phi_{mn}|\delta|^{1/2} + \rho\phi_{pq}]\cos t + \dots,$$

$$(5.15b) \quad f = f_{mn}|\delta| + g_1\rho|\delta|^{1/2} + f_{pq}\rho^2 + \dots,$$

$$(5.15c) \quad \omega^2 = \omega_s^{(2)}(\lambda, \delta) + B_2\rho^2 + \dots$$

Since $B_2 > 0$, (5.15c) shows that the secondary states branch supercritically from the primary states at the secondary bifurcation points.

The shape of the middle plane of the plate as it vibrates in a secondary state is given by (5.15a). It is a standing wave whose spatial structure is a linear combination of the primary mode ϕ_{mn} and the cross mode ϕ_{pq} and it depends on the relative magnitude of $|\delta|$ and ρ . We recall that δ measures the deviation of the thrust from its critical value $\tilde{\lambda}$, while ρ measures the amplitude of the primary state at the secondary bifurcation point. For fixed δ and small ρ , the secondary states are small distortions of the primary state by the cross mode. As ρ increases, the secondary states gradually deform from ϕ_{mn} to ϕ_{pq} . Contour lines of these shapes are shown in Figures 2-4 of [8], where related results were obtained for a problem in magnetohydrodynamics.

We wish to emphasize that the secondary states (5.15) have the property that the secondary bifurcation point approaches a multiple primary bifurcation point as $\delta \rightarrow 0$. However, it has been demonstrated by numerical computations

[9,10] for other problems, that there are secondary bifurcation points which do not arise by the splitting of multiple, primary bifurcation points.

The analysis presented in this paper is valid for small δ . However, it may be possible that the secondary bifurcation points and states persist for larger values of δ . Then a given primary state may have many secondary bifurcation points, because as δ varies, its primary bifurcation point may coalesce with other primary bifurcation points. Each coalescence may produce new secondary points.

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Figure 1

Figure 2