ANL-79-53

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ON THE EQUIVALENCE BETWEEN
RESOLVENT CONSISTENCY AND CONVERGENCE FOR
NONLINEAR QUASI-CONTRACTIVE ALGORITHMS

by

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June 1979

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TABLE OF CONTENTS

	Page
ABSTRACT.	3
1. STATEMENT OF THE PROBLEM.	3
2. PROOF OF THE THEOREM.	4
3. APPLICATIONS.	.12
REFERENCES.	.14

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ABSTRACT

Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, D a closed convex subset of E , and C a nonexpansive retract of D . Let $F(t): D \rightarrow C$, $0 \leq t < \infty$, be a continuous family of mappings with $F(0) = I$ on C , such that $|F(t)x - F(t)y| \leq M(t)|x - y|$ for x and y in D and $t \geq 0$ with $M(t) = 1 + \omega t + o(t)$ as $t \rightarrow 0+$ for some $\omega \geq 0$. If $\lim_{n \rightarrow \infty} F\left(\frac{t}{n}\right)^n x = S(t)x$ exists for each x in C uniformly on compact t intervals, then $\lim_{t \rightarrow 0+} \left(I + \frac{\lambda}{t}(I - F(t))\right)^{-1} x = J_\lambda x$ exists for each x in D and $0 < \lambda < 1/\omega$.

1. Statement of the Problem

Let D be a closed convex subset of a Banach space E , and let C be a closed subset of D . C is said to be a nonexpansive retract of D if there exists a retraction $P: D \rightarrow C$ which is a nonexpansive mapping (that is, $|Px - Py| \leq |x - y|$ for all x and y in D). The norm of E is said to be uniformly Gâteaux differentiable if for each y in the unit sphere of E , the limit $\lim_{t \rightarrow 0} (|x + ty| - |x|)/t$ is approached uniformly for x in the unit sphere of E . We denote the identity mapping by I . The purpose of this report is to prove the following result.

Theorem. Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, D a closed convex subset of E , and C a nonexpansive retract of D . Let $F(t): D \rightarrow C$, $0 \leq t < \infty$, be a family of mappings such that

- (1) $|F(t)x - F(t)y| \leq M(t)|x - y|$ for x and y in D and $t \geq 0$ with
 $M(t) = 1 + \omega t + o(t)$ as $t \rightarrow 0+$ for some $\omega \geq 0$;
- (2) $F(t)x$ is continuous in $t \geq 0$ for $x \in C$ and in $t > 0$ for $x \in D$;
- (3) $F(0) = I$ on C .
 If
- (a) $\lim_{n \rightarrow \infty} F\left(\frac{t}{n}\right)^n x = S(t)x$
 exists for each $x \in C$ uniformly on compact t intervals,
 then
- (b) $\lim_{t \rightarrow 0+} \left(I + \frac{\lambda}{t} (I - F(t))\right)^{-1} x = J_{\lambda} x$
 exists for each $x \in D$ with $0 < \lambda < 1/\omega$.

2. Proof of the Theorem

The proof of the theorem will be preceded by several remarks.

A smoothness assumption on E is necessary for the conclusion of the theorem to hold [6, p. 272]; but we do not know if uniform Gâteaux differentiability can be replaced by Gâteaux differentiability. However, a somewhat different proof shows that continuity in t of $F(t)x$ is required only for $x \in C$. Combining the theorem with a known converse [2, Theorem 3.2], we do indeed obtain the equivalence between resolvent consistency and convergence (see Corollary 1). The linear case is due to Chernoff [3, Theorem 1.1] (see also [7, Theorem IX.3.6] and [4, Theorem 3.1]). In the proof of the theorem we use ideas due to Crandall-Liggett [5] and Baillon [1] (see also [8]). Corollary 2 is an extension of Baillon's result. There are other applications of the Theorem in addition to the corollaries mentioned after its proof.

These applications deal with extensions of accretive operators, the geometry

of Banach spaces, and properties of nonlinear semigroups. They will be presented elsewhere.

Proof of Theorem. We first note that for $x \in D$ and $0 < \lambda < 1/\omega$.

$\left(I + \frac{\lambda}{t} (I - F(t))\right)^{-1} x$ does exist for all $0 < t$ that are small enough. To see this, define $T: D \rightarrow D$ by

$$Ty = \frac{t}{\lambda+t} x + \frac{\lambda}{\lambda+t} F(t)y, \quad y \in D.$$

If t is small enough, T has a unique fixed point $y_{\lambda,t} = \left(I + \frac{\lambda}{t} (I - F(t))\right)^{-1} x$. For a fixed λ , we denote $y_{\lambda,t}$ by y_t .

We also observe that we may assume that D is separable. To see this, define $K_0 = \{x\}$, $K_{n+1} = \text{co}(K_n \cup \{F(t)y: y \in K_n \text{ and } t > 0\})$, and $K = \text{cl}(\cup \{K_n: n \geq 0\})$. (Here $\text{co}(B)$ and $\text{cl}(B)$ denote the convex hull and the closure of a subset B of E .) $K \subset D$ is closed, convex, separable, and invariant under T . Therefore $\{y_t\} \subset K$ and we may replace D with K and C with $K \cap C$.

Next we show, by induction on k , that for each fixed $x \in C$ and k , $F(t)^k x$ is continuous in t . Suppose this is true for k . To prove continuity for $k+1$, note that $|F(t)^{k+1} x - F(t_0)^{k+1} x| \leq |F(t)F(t)^k x - F(t)F(t_0)^k x| + |F(t)F(t_0)^k x - F(t_0)F(t_0)^k x| \leq M(t)|F(t)^k x - F(t_0)^k x| + |F(t)F(t_0)^k x - F(t_0)F(t_0)^k x|$.

For $T \geq 0$ define

$$\alpha(T, x) = \sup\{|F(t)^k x - x|: 0 \leq kt \leq T\}.$$

We claim that $\lim_{T \rightarrow 0+} \alpha(T, x) = 0$ for $x \in C$. If not, there is a positive ϵ and sequences $\{k_n\}, \{t_n\}$ such that $k_n t_n \rightarrow 0$ and $|F(t_n)^{k_n} x - x| > \epsilon$. Clearly $t_n \rightarrow 0$. If $\{k_n\}$ remains bounded, then we immediately reach a contradiction. Therefore

we may assume that $k_n \rightarrow \infty$. But then

$$|F(t_n)^k x - x| \leq |F(k_n t_n / k_n)^k x - S(k_n t_n) x| + |S(k_n t_n) x - x| < \varepsilon$$

for n large enough, by (a) and the continuity of S in t .

Now we wish to prove that $\{y_t\}$ remains bounded as $t \rightarrow 0+$. Since $F(t)y_t = (1 + \frac{t}{\lambda})y_t - \frac{t}{\lambda}x$, we have for $k \geq 0$,

$$\begin{aligned} M(t)|y_t - F(t)^k x| &\geq |F(t)y_t - F(t)^{k+1} x| \\ &= |(1 + \frac{t}{\lambda})(y_t - F(t)^{k+1} x) + \frac{t}{\lambda}(F(t)^{k+1} x - x)| \\ &\geq |y_t - F(t)^{k+1} x| + \frac{t}{\lambda}|y_t - F(t)^{k+1} x| - \frac{t}{\lambda}|F(t)^{k+1} x - x| \\ &\geq |y_t - F(t)^{k+1} x| + \frac{t}{\lambda}|y_t - x| - \frac{2t}{\lambda}|F(t)^{k+1} x - x|. \end{aligned}$$

Fix a positive T , define n and α_t by $T = nt + \alpha_t$, $0 \leq \alpha_t < t$, and let $P: D \rightarrow C$ be a nonexpansive retraction. For $0 \leq k \leq n$ we have

$$\begin{aligned} |F(t)^k x - x| &\leq |F(t)^k x - F(t)^k Px| \\ &\quad + |F(t)^k Px - Px| + |Px - x| \leq (1 + M(t)^k)|x - Px| + \alpha(T, Px), \text{ so that} \\ |y_t - F(t)^k x| &\geq M(t)^{-1}|y_t - F(t)^{k+1} x| - \frac{2t}{\lambda}(M(t)^{-1} + M(t)^k)|x - Px| \\ &\quad - \frac{2t}{\lambda}M(t)^{-1}\alpha(T, Px) + \frac{t}{\lambda}M(t)^{-1}|y_t - x|, \text{ and} \\ M(t)^{-k}|y_t - F(t)^k x| &\geq M(t)^{-k-1}|y_t - F(t)^{k+1} x| - \frac{2t}{\lambda}(M(t)^{-k-1} + 1)|x - Px| \\ &\quad - \frac{2t}{\lambda}M(t)^{-k-1}\alpha(T, Px) + \frac{t}{\lambda}M(t)^{-k-1}|y_t - x|. \end{aligned}$$

Summing from $k = 0$ to $k = n-1$, we obtain

$$\begin{aligned}
|y_t - x| &\geq M(t)^{-n} |y_t - F(t)^n x| - \frac{2t}{\lambda} \left(n + \sum_{k=1}^n M(t)^{-k} \right) |x - Px| \\
&\quad - \frac{2t}{\lambda} \left(\sum_{k=1}^n M(t)^{-k} \right) \alpha(T, Px) + \frac{t}{\lambda} \left(\sum_{k=1}^n M(t)^{-k} \right) |y_t - x| \geq \\
&\quad M(t)^{-n} |y_t - x| - M(t)^{-n} (|x - Px| + \alpha(T, Px) + M(t)^n |x - Px|) \\
&\quad - \frac{2t}{\lambda} \left(n + \sum_{k=1}^n M(t)^{-k} \right) |x - Px| - \frac{2t}{\lambda} \left(\sum_{k=1}^n M(t)^{-k} \right) \alpha(T, Px) \\
&\quad + \frac{t}{\lambda} \left(\sum_{k=1}^n M(t)^{-k} \right) |y_t - x|.
\end{aligned}$$

Thus

$$\begin{aligned}
|y_t - x| &\leq \left\{ \left(1 + \frac{2t}{\lambda} \frac{M(t)^n - 1}{M(t) - 1} \right) \alpha(T, Px) \right. \\
&\quad \left. + \left(1 + M(t)^n + \frac{2nt}{\lambda} M(t)^n + \frac{2t}{\lambda} \frac{M(t)^n - 1}{M(t) - 1} \right) |x - Px| \right\} / \\
&\quad \left(1 - M(t)^n + \frac{t}{\lambda} \frac{M(t)^n - 1}{M(t) - 1} \right),
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{t \rightarrow 0+} |y_t - x| &\leq \left\{ \left(2 + \frac{\lambda \omega}{e^{\omega T} - 1} \right) \alpha(T, Px) + \right. \\
&\quad \left. \left(2 + \frac{2T\omega e^{\omega T}}{e^{\omega T} - 1} + \lambda \omega \frac{e^{\omega T} + 1}{e^{\omega T} - 1} \right) |x - Px| \right\} / (1 - \lambda \omega).
\end{aligned}$$

We proceed with a lemma.

Lemma 1. Let $\{y_m\}$ be a bounded sequence contained in a separable subset D of a Banach space E . Then there is a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ such that $\lim_{k \rightarrow \infty} |y_{m_k} - z|$ exists for all $z \in D$.

Proof. Let $\{z_n\}$ be dense in D . By a diagonal argument we obtain a subsequence $\{m_k\}$ such that $f(z_n) = \lim_{k \rightarrow \infty} |y_{m_k} - z_n|$ exists for all n . Suppose $z \in D$ and let $z = \lim_{i \rightarrow \infty} z_{n_i}$. To show that $\lim_{k \rightarrow \infty} |y_{m_k} - z|$ exists, we first note that $||y_{m_k} - z_{n_i}| - |y_{m_k} - z_{n_j}|| \leq |z_{n_i} - z_{n_j}|$, so that $|f(z_{n_i}) - f(z_{n_j})| \leq |z_{n_i} - z_{n_j}|$ and

$a = \lim_{i \rightarrow \infty} f(z_{n_i})$ exists. Since $-|z - z_{n_i}| \leq |y_{m_k} - z| - |y_{m_k} - z_{n_i}| \leq |z - z_{n_i}|$, we have

$$\limsup_{k \rightarrow \infty} |y_{m_k} - z| \leq |z - z_{n_i}| + f(z_{n_i}) \xrightarrow{i \rightarrow \infty} a,$$

$$\liminf_{k \rightarrow \infty} |y_{m_k} - z| \geq f(z_{n_i}) - |z - z_{n_i}| \xrightarrow{i \rightarrow \infty} a,$$

and the result follows. $\square$

Let $\{y_{t_m}\}$ be a subsequence of $\{y_t\}$ with $t_m \rightarrow 0$ such that $f(z) = \lim_{m \rightarrow \infty} |y_{t_m} - z|$ exists for all $z \in D$. Then f attains its minimum over D at a point in C . To see this, note that since $|f(z_1) - f(z_2)| \leq |z_1 - z_2|$, f is continuous. It is also convex and $f(z) \rightarrow \infty$ as $|z| \rightarrow \infty$. Therefore f attains its minimum at $u \in D$. We also have $y_t - Py_t = F(t)y_t - Py_t - \frac{t}{\lambda}(y_t - x)$, and $|F(t)y_t - Py_t| \leq |F(t)y_t - y_t| = \frac{t}{\lambda}|y_t - x|$. Thus $|y_t - Py_t| \leq \frac{2t}{\lambda}|y_t - x| \xrightarrow{t \rightarrow 0+} 0$. Consequently, $|y_{t_m} - Pu| \leq |y_{t_m} - Py_{t_m}| + |y_{t_m} - u|$, and $f(Pu) \leq f(u)$.

Lemma 2. Let D be a closed convex subset of a real Banach space E with a uniformly Gâteaux differentiable norm, and let $\{y_m\}$ be a sequence in D such that $f(z) = \lim_{m \rightarrow \infty} |y_m - z|$ exists for all $z \in D$. Let $J: E \rightarrow E^*$ denote the duality map. If f attains its minimum over D at u , then $\limsup_{m \rightarrow \infty} (z - u, J(y_m - u)) \leq 0$ for all $z \in D$.

Proof. For z in D and $0 \leq t \leq 1$, we have

$$\begin{aligned} |y_m - u|^2 &= |y_m - tu - (1-t)z + (1-t)(z - u)|^2 \\ &\geq |y_m - tu - (1-t)z|^2 + 2(1-t)(z - u, J(y_m - tu - (1-t)z)). \end{aligned}$$

Let $\epsilon > 0$ be given. Then, since the norm of E is uniformly Gâteaux differentiable, $|(z - u, J(y_m - tu - (1-t)z) - J(y_m - u))| < \epsilon$ if t is near enough 1. Thus

$$\begin{aligned} (z - u, J(y_m - u)) &< \epsilon + (z - u, J(y_m - tu - (1-t)z)) \\ &\leq \epsilon + \frac{1}{2(1-t)} \{|y_m - u|^2 - |y_m - tu - (1-t)z|^2\}, \end{aligned}$$

and

$$\limsup_{m \rightarrow \infty} (z-u, J(y_m - u)) \leq \varepsilon + \frac{1}{2(1-t)} \{f^2(u) - f^2(tu + (1-t)z)\} \leq \varepsilon. \quad \square$$

To continue the proof of the Theorem, let v belong to C . For $0 \leq k \leq n-1$ we have $M(t)^2 |y_t - F(t)^k v|^2 \geq |F(t)y_t - F(t)^{k+1} v|^2 = |y_t - F(t)^{k+1} v + \frac{t}{\lambda}(y_t - x)|^2 \geq |y_t - F(t)^{k+1} v|^2 + \frac{2t}{\lambda}(y_t - x, J(y_t - F(t)^{k+1} v))$. Consequently,

$$\begin{aligned} \frac{2t}{\lambda} \sum_{k=1}^n M(t)^{-2k} (y_t - x, J(y_t - F(t)^k v)) \\ \leq |y_t - v|^2 - M(t)^{-2n} |y_t - F(t)^n v|^2. \end{aligned}$$

We will show now that if $t_m \rightarrow 0$, then there is a subsequence of $\{y_{t_m}\}$ that is strongly convergent.

There is a subsequence of $\{y_{t_m}\}$, which we denote by $\{y_m\}$, such that $f(z) = \lim_{m \rightarrow \infty} |y_m - z|$ exists for all $z \in D$. Let f attain its minimum over D at $u \in C$. Let T be positive and define n_m by $T = n_m t_m + \alpha_m$, where n_m is an integer and $0 \leq \alpha_m < t_m$. Now we note that for $v \in C$,

$$\begin{aligned} (y_t - x, J(y_t - v)) - (y_t - x, J(y_t - F(t)^k v)) = \\ |y_t - v|^2 + (v - x, J(y_t - v)) - |y_t - F(t)^k v|^2 \\ - (F(t)^k v - x, J(y_t - F(t)^k v)) = \\ \{|y_t - v| - |y_t - F(t)^k v|\} \{|y_t - v| + |y_t - F(t)^k v|\} \\ + (v - x, J(y_t - v) - J(y_t - F(t)^k v)) \\ - (F(t)^k v - v, J(y_t - F(t)^k v)). \end{aligned}$$

Thus

$$\begin{aligned}
& |(y_t - x, J(y_t - v)) - M(t)^{-2k} (y_t - x, J(y_t - F(t)^k v))| \\
& \leq |(y_t - x, J(y_t - v)) - (y_t - x, J(y_t - F(t)^k v))| \\
& + (1 - M(t)^{-2k}) |(y_t - x, J(y_t - F(t)^k v))| \\
& \leq \alpha(T, v) M_1 + |(v - x, J(y_t - v) - J(y_t - F(t)^k v))| \\
& + \alpha(T, v) M_2 + (1 - M(t)^{-2k}) M_3 < \varepsilon
\end{aligned}$$

if T is small enough.

We apply the above to $v = u$ to obtain for small enough T ,

$$\begin{aligned}
\frac{2(T - \alpha_m)}{\lambda} (y_m - x, J(y_m - u)) &= \frac{2t_m n_m}{\lambda} \frac{1}{n_m} \sum_{k=1}^{n_m} (y_m - x, J(y_m - u)) \leq \\
&\frac{2t_m n_m}{\lambda} \frac{1}{n_m} \sum_{k=1}^{n_m} M(t_m)^{-2k} (y_m - x, J(y_m - F(t_m)^k u)) \\
&+ \frac{2t_m n_m}{\lambda} \varepsilon \leq |y_m - u|^2 - M(t_m)^{-2n_m} |y_m - F(t_m)^{n_m} u|^2 + \frac{2T}{\lambda} \varepsilon .
\end{aligned}$$

We also have, by (a), $\lim_{m \rightarrow \infty} F(t_m)^{n_m} u = S(T)u$. Thus

$$\frac{2T}{\lambda} \limsup_{m \rightarrow \infty} (y_m - x, J(y_m - u)) \leq f^2(u) - e^{-2\omega T} f^2(S(T)u) + \frac{2T}{\lambda} \varepsilon ,$$

$$\limsup_{m \rightarrow \infty} (y_m - x, J(y_m - u)) \leq \frac{\lambda}{2T} (1 - e^{-2\omega T}) f^2(S(T)u) + \varepsilon , \text{ and}$$

$$\limsup_{m \rightarrow \infty} (y_m - x, J(y_m - u)) \leq \lambda \omega f^2(u) .$$

Since $x \in D$, we also have

$$\limsup_{m \rightarrow \infty} (x - u, J(y_m - u)) \leq 0 .$$

Therefore

$$f^2(u) \leq \lambda \omega f^2(u) , \text{ and } f(u) = 0 .$$

For $x \in D$ and small enough positive s , define $z_s = (I + \frac{\lambda}{s} (I - S(s)P))^{-1}x$. By what we have proved so far (with $F(t)$ replaced by $S(t)P$), there is a sequence $\{s_p\}$ convergent to zero such that $v = \lim_{p \rightarrow \infty} z_{s_p}$ exists. ($S(t)P$ satisfies (a) because it can be shown that S is, in fact, a semigroup.)

Now suppose that $\lim_{m \rightarrow \infty} y_{t_m} = u$. We complete the proof of the Theorem by showing that $u = v$.

To this end, let $\epsilon > 0$ be given and, as before, define integers $\{n_m\}$ by $0 < T = n_m t_m + \alpha_m$, $0 \leq \alpha_m < t_m$. We have for $1 \leq k \leq n_m$,

$$\begin{aligned} (y_{t_m} - x, J(y_{t_m} - F(t_m)^k P z_{s_p})) - (u - x, J(u - v)) = \\ (y_{t_m} - u, J(y_{t_m} - F(t_m)^k P z_{s_p})) + \\ (u - x, J(y_{t_m} - F(t_m)^k P z_{s_p})) - (u - x, J(u - v)), \end{aligned}$$

and since $v \in C$,

$$\begin{aligned} |F(t_m)^k P z_{s_p} - v| &\leq |F(t_m)^k P z_{s_p} - F(t_m)^k P v| \\ &+ |F(t_m)^k P v - v| \leq M(t_m)^k |z_{s_p} - v| + \alpha(T, v). \end{aligned}$$

Therefore

$$|(y_{t_m} - x, J(y_{t_m} - F(t_m)^k P z_{s_p})) - (u - x, J(u - v))| < \epsilon_1$$

if t_m , s_p and T are small enough. It follows that

$$\begin{aligned} \frac{2(T - \alpha_m)}{\lambda} (u - x, J(u - v)) &= \frac{2n_m t_m}{\lambda} \frac{1}{n_m} \sum_{k=1}^{n_m} (u - x, J(u - v)) \\ &\leq \frac{2n_m t_m}{\lambda} \frac{1}{n_m} \sum_{k=1}^{n_m} M(t_m)^{-2k} (y_{t_m} - x, J(y_{t_m} - F(t_m)^k P z_{s_p})) \\ &+ \frac{2T}{\lambda} \epsilon \leq |y_{t_m} - P z_{s_p}|^2 - M(t_m)^{-2n_m} |y_{t_m} - F(t_m)^{n_m} P z_{s_p}|^2 + \frac{2T}{\lambda} \epsilon, \end{aligned}$$

and

$$\begin{aligned} \frac{2T}{\lambda} (u-x, J(u-v)) &\leq |u-Pz_{s_p}|^2 - e^{-2\omega T} |u-S(T)Pz_{s_p}|^2 + \frac{2T}{\lambda} \varepsilon \leq \\ &|u-z_{s_p}|^2 - e^{-2\omega T} |u-S(T)Pz_{s_p}|^2 + \frac{2T}{\lambda} \varepsilon . \end{aligned}$$

This last inequality holds for all small enough T and s_p . Choosing $T = s_p$, we obtain

$$\begin{aligned} \frac{2s_p}{\lambda} (u-x, J(u-v)) &\leq |u-z_{s_p}|^2 - e^{-2\omega s_p} |u-z_{s_p} - \frac{s_p}{\lambda} (z_{s_p} - x)|^2 + \frac{2s_p}{\lambda} \varepsilon \\ &\leq (1-e^{-2\omega s_p}) |u-z_{s_p}|^2 + \frac{2s_p}{\lambda} e^{-2\omega s_p} (z_{s_p} - x, J(u-z_{s_p})) + \frac{2s_p}{\lambda} \varepsilon , \end{aligned}$$

so that

$$(u-x, J(u-v)) \leq \lambda\omega |u-v|^2 + (v-x, J(u-v)) + \varepsilon ,$$

and

$$(1-\lambda\omega) |u-v|^2 \leq \varepsilon .$$

This completes the proof of the Theorem. Note that the proof that $u = v$ is valid in any smooth E .

Combining the Theorem with [2, Theorem 3.2], we see that convergence ((a)) is indeed equivalent to resolvent consistency ((b)) for quasi-contractive nonlinear algorithms $F(t)$.

3. Applications

Corollary 1. Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, and let C be a closed convex subset of E . Let $F(t): C \rightarrow C$, $0 \leq t < \infty$, be a continuous family of mappings with $F(0) = I$ such that $|F(t)x - F(t)y| \leq M(t)|x-y|$ for x and y in C and $t \geq 0$ with $M(t) = 1 + \omega t + o(t)$ as $t \rightarrow 0+$ for some $\omega \geq 0$. Then the following are equivalent.

$$(a) \quad \lim_{n \rightarrow \infty} F\left(\frac{t}{n}\right)^n x = S(t)x$$

exists for each $x \in C$ uniformly on compact t intervals;

$$(b) \quad \lim_{t \rightarrow 0+} \left(I + \frac{\lambda}{t} (I - F(t)) \right)^{-1} x = J_{\lambda} x$$

exists for each $x \in C$ and $0 < \lambda < 1/\omega$.

J_{λ} is the resolvent (on C) of a set $A \subset E \times E$ such that $A + \omega I$ is accretive. S is a quasi-contractive semigroup of type ω that is generated by $-A$ via the exponential formula.

Recall that the strong generator A_0 of a semigroup S is defined by $A_0 x = \lim_{t \rightarrow 0+} (x - S(t)x)/t$. The next corollary extends [1, Théorème 3] because a uniformly smooth Banach space has a uniformly Gâteaux differentiable norm (the converse is false).

Corollary 2. Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, and C a closed convex subset of E . If S is a quasi-contractive semigroup on C , then the domain of its strong generator is dense in C .

Our last corollary is known in Hilbert space, where every closed convex subset is a nonexpansive retract of the space. Note that the assumption that C is a nonexpansive retract of E is necessary in general [9, Theorem 2.3].

Corollary 3. Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, and C a closed convex nonexpansive retract of E . If S is a quasi-contractive semigroup of type ω on C , then there is a unique set $A \subset E \times E$ such that $A + \omega I$ is accretive, $\text{cl}(D(A)) = C$, $R(I + \lambda A) = E$ for $0 < \lambda < 1/\omega$, and $-A$ generates S via the exponential formula.

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