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ERGODIC THEOREMS FOR NONLINEAR CONTRACTION SEMIGROUPS IN A HILBERT SPACE

by

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ABSTRACT

Two ergodic theorems are presented for means of nonlinear contraction semigroups in a Hilbert space. These means are generated by a class of averaging kernels which includes the usual Abel and Cesàro-(C,α) kernels.

INTRODUCTION

In their note [3], Brézis and Browder proved an ergodic theorem for a general averaging process for nonlinear contraction mappings in a Hilbert space. We present similar results here for nonlinear contraction semigroups in a Hilbert space. These results include as special cases the theorems of Baillon [1] and of Baillon and Brézis [2] for Cesàro means and of the authors [6] for Abel means of nonlinear contraction semigroups.

Throughout this report, H denotes a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$; C is a closed bounded convex subset of H. The notations $\to$ and $\rightharpoonup$ refer to convergence in the norm topology and weak topology in H, respectively.

1. The Averaging Process

For each $\lambda \geq 0$, let $k(\lambda,\cdot)$ be a real-valued nonnegative integrable function on $[0,\infty)$ with $\int_0^\infty k(\lambda,t)dt = 1$, such that $\int_0^t k(\lambda,\tau)d\tau \to 0$ as $\lambda \downarrow 0$ for every finite t. Such k can be used to define an averaging process on $L_\infty([0,\infty);H)$ in the following way. For any given $x \in L_\infty([0,\infty);H)$, let

$$(1) \qquad y(\lambda) := \int_0^\infty k(\lambda,t)x(t)dt \qquad \lambda \geq 0 .$$

The integral is well-defined as the strong limit of Riemann sums and defines an element of H for each $\lambda \geq 0$. By analogy with Hardy, [5], Section 3.2, we say that the averaging process (1) is strongly (weakly) regular if $y(\lambda) \to \bar{x}$ $(y(\lambda) \rightharpoonup \bar{x})$ as $\lambda \downarrow 0$, whenever $x(t) \to \bar{x}(x(t) \rightharpoonup \bar{x})$ as $t \to \infty$. Obviously, strong regularity implies weak regularity.

Lemma 0. The averaging process (1) is strongly regular.

Proof. Suppose $x(t) \to \bar{x}$ as $t \to \infty$. Then, given any $\varepsilon > 0$, there exists a $t(\varepsilon)$ such that $|x(t) - \bar{x}| < \frac{1}{2}\varepsilon$ for all $t > t(\varepsilon)$. Let $M = \sup\{|x(t)-\bar{x}|: t \in [0,\infty)\}$. Without loss of generality we may assume $M < \infty$. With $t(\varepsilon)$ fixed, we subsequently choose $\lambda(\varepsilon)$ such that $\int_0^{t(\varepsilon)} k(\lambda,t)dt < \frac{1}{2}M^{-1}\varepsilon$ for all $\lambda < \lambda(\varepsilon)$. Thus,

$$|y(\lambda)-\bar{x}| = \left|\int_0^\infty k(\lambda,t)(x(t)-\bar{x})dt\right|$$

$$\leq \int_0^{t(\varepsilon)} k(\lambda,t)|x(t)-\bar{x}|dt + \int_{t(\varepsilon)}^\infty k(\lambda,t)|x(t)-\bar{x}|dt < \varepsilon$$

for all $\lambda < \lambda(\varepsilon)$, which proves the lemma. $\square$

Let $\{S(t,\cdot): t \geq 0\}$ be a continuous semigroup of nonlinear contraction mappings of C into itself. Consider the real interval [0,R] for any R > 0. For $x \in C$, let $\sigma_R(\lambda,x)$ be defined by the integral

$$\sigma_R(\lambda,x) := \int_0^R k_R(\lambda,t)S(t,x)dt , \qquad \lambda \geq 0 ,$$

where $k_R(\lambda,t) = k(\lambda,t)/\int_0^R k(\lambda,t)dt$. Since C is convex, $\sigma_R(\lambda,x) \in C$ for every R > 0. The sequence $\{\sigma_{R_n}(\lambda,x): n=0,1,\ldots\}$ is Cauchy in the norm topology of H for any increasing sequence $\{R_n: n=0,1,\ldots\}$. Let $\sigma(\lambda,x)$ denote its limit. As C is closed, $\sigma(\lambda,x) \in C$; $\sigma(\lambda,x)$ has the following representation,

$$\sigma(\lambda,x) = \int_0^\infty k(\lambda,t)S(t,x)dt .$$

The object of this note is to study the convergence of $\sigma(\lambda,x)$ as $\lambda \downarrow 0$.

2. Weak Convergence

Let F denote the set of fixed points of the semigroup S in H, which is closed, convex, and nonempty, cf. [4], Remark 2.5. We denote by Proj_F the projection of H on F. In this section we prove the following theorem on the weak convergence of $\sigma(\lambda,x)$.

Theorem 1. Suppose $k(\lambda,\cdot)$ is of bounded variation on $[0,\infty)$ for each $\lambda \geq 0$. Let $v^+(\lambda,t)$ be the positive variation of $k(\lambda,\cdot)$ on [0,t], and let the function $k^+(\lambda,\cdot)$ be defined by $k^+(\lambda,t) = k(\lambda,0+) + v^+(\lambda,t)$ for all $t \in [0,\infty)$. If $\lim_{\lambda\downarrow 0} \int_0^\infty d_t k^+(\lambda,t) = 0$, then $\sigma(\lambda,x)$ converges weakly as $\lambda \downarrow 0$ to a fixed point of S. This fixed point is also the strong limit of $\{\mathrm{Proj}_F S(t,x): t \geq 0\}$ as $t \to \infty$.

Before considering the proof of this theorem we note that, by [4], remark 3.4 and Theorem A2, there exists a unique maximal dissipative set $A \subset H\times H$ such that its minimal section A^0 is the generator of S on the domain D(A) of A, where D(A) is dense in C. The proof of Theorem 1 depends upon the following lemma.

<u>Lemma 2</u>. Let $\{\lambda_n : n=0,1,\ldots\}$ be a sequence of positive real numbers tending to zero. If $x \in D(A)$ and $\sigma(\lambda_n,x) \to \ell$, then $\ell \in F$.

<u>Proof</u>. Take any $[v,w] \in A$, then

$$(w,\sigma(\lambda,x)-v) = \int_0^\infty k(\lambda,t)(w,S(t,x)-v)dt .$$

Now, observe that $(w,S(t,x)-v) = (A^0S(t,x),S(t,x)-v) + (w-A^0S(t,x),S(t,x)-v)$, where the last inner product is positive because of the dissipativity of A, cf. [4], Definition 2.1. Thus,

$$(w,\sigma(\lambda,x)-v) \geq \int_0^\infty k(\lambda,t)(A^0S(t,x),S(t,x)-v)dt .$$

By [4], Corollary 3.1, $A^0S(t,x) = (d/dt)^+S(t,x)$ for $t \geq 0$, where $(d/dt)^+$ denotes the right derivative, so

$$(w,\sigma(\lambda,x)-v) \geq \frac{1}{2}\int_0^\infty k(\lambda,t)(d/dt)^+|S(t,x)-v|^2dt$$
$$= -\frac{1}{2}k(\lambda,0+)|x-v|^2 - \frac{1}{2}\int_0^\infty |S(t,x)-v|^2 d_tk(\lambda,t) .$$

Since $k(\lambda,\cdot)$ is of bounded variation on $[0,\infty)$, it is the difference $k^+(\lambda,\cdot)-k^-(\lambda,\cdot)$ of two nondecreasing functions -- viz., $k^+(\lambda,t) = k(\lambda,0+) + v^+(\lambda,t)$ and $k^-(\lambda,t) = -v^-(\lambda,t)$, where $v^+(\lambda,t)$ and $v^-(\lambda,t)$ are the positive and negative variations, respectively, of $k(\lambda,\cdot)$ on $[0,t]$, cf. [7], Section

42. Thus,

$$(w,\sigma(\lambda,x)-v) \geq -\frac{1}{2}k(\lambda,0+)|x-v|^2 - \frac{1}{2}\int_0^\infty |S(t,x)-v|^2 d_t k^+(\lambda,t) .$$

Now, let $\{\lambda_n: n=0,1,\ldots\}$ be a sequence of positive real numbers tending to zero. C is bounded; hence, $|k(\lambda,0+)||x-v|^2 \leq |k(\lambda,0+)|(\mathrm{diam}\ C)^2 \to 0$ and $\int_0^\infty |S(t,x)-v|^2 d_t k^+(\lambda,t) \leq \int_0^\infty d_t k^+(\lambda,t)(\mathrm{diam}\ C)^2 \to 0$ as $\lambda \downarrow 0$, so if $\sigma(\lambda,x) \rightharpoonup \ell$ as $\lambda \downarrow 0$, then $(w,\ell-v) \geq 0$ for all $[v,w] \in A$. Since A is maximal dissipative, $A = \{[\phi,\psi] \in H\times H: (y-\psi,x-\phi) \leq 0 \text{ for all } [x,y] \in A\}$, cf. [4], Lemma 2.2. It follows that $0 \in A\ell$ and, hence, $\ell \in F$. □

Proof of Theorem 1. Suppose $x \in D(A)$. The set $\{\sigma(\lambda,x): \lambda \geq 0\}$ is sequentially weakly compact. Hence, there exists at least one weakly convergent subsequence of $\{\sigma(\lambda_n,x): n=0,1,\ldots\}$; we assume that this subsequence coincides with the sequence itself. Let ℓ denote its weak limit. Then by Lemma 2, $\ell \in F$. We show next that ℓ is the strong limit of $\mathrm{Proj}_F\, S(t,x)$ as $t \to \infty$. Define $y(t) := \mathrm{Proj}_F\, S(t,x)$ for $t \geq 0$. Then $y(t)$ converges in norm as $t \to \infty$ to an element $y \in F$, cf. [2], Lemma 3. Consequently, $\int_0^\infty k(\lambda,t)y(t)dt$ converges in norm as $\lambda \downarrow 0$ to y, as the averaging process defined by k is strongly regular. In order to prove that $\ell = y$ it suffices to show that $(f-y,\ell-y) \leq 0$ for all $f \in F$. The latter inequality holds true if

$$(*) \qquad \lim_{\lambda\downarrow 0} \int_0^\infty k(\lambda,t)(f-y,S(t,x)-y(t))dt \leq 0$$

for all $f \in F$.

By virtue of the definition of $y(t)$ we have the inequality $(f-y(t), S(t,x)-y(t)) \leq 0$ for any $f \in F$, so

$$\begin{aligned}(f-y,S(t,x)-y(t)) &\leq (y(t)-y,S(t,x)-y(t)) \\ &\leq |y(t)-y||S(t,x)-y(t)| .\end{aligned}$$

Given any $\varepsilon > 0$, there exists a $t(\varepsilon)$ such that $|y(t)-y| < \frac{1}{2}(\text{diam } C)^{-1}\varepsilon$ for all $t \geq t(\varepsilon)$, and a $\lambda(\varepsilon)$ such that $\int_0^{t(\varepsilon)} k(\lambda,t)dt < \frac{1}{2}(\text{diam } C)^{-2}\varepsilon$ for all $\lambda < \lambda(\varepsilon)$. Then

$$\int_0^{\infty} k(\lambda,t)(f-y,S(t,x)-y(t))dt < \varepsilon$$

for all $\lambda < \lambda(\varepsilon)$, which proves (*).

Since any subsequence of $\{\sigma(\lambda_n,x)\colon n=0,1,\ldots\}$ which is weakly convergent, converges to the same limit, it follows that the sequence itself is weakly convergent. This proves the theorem for $x \in D(A)$.

Now, suppose $x \in C$. Since $D(A)$ is dense in C, there exists a sequence $\{x_i\colon i=1,2,\ldots\}$ with $x_i \in D(A)$, such that $x_i \to x$. Then we know that $\sigma(\lambda,x_i)$ is weakly convergent as $\lambda \downarrow 0$ to ℓ_i, say, and $\ell_i \in F$. For any $w \in H$ with $|w| = 1$ we have

$$|(\sigma(\lambda,x)-\sigma(\mu,x),w)| \leq |\sigma(\lambda,x)-\sigma(\lambda,x_i)| + |(\sigma(\lambda,x_i)-\sigma(\mu,x_i),w)| + |\sigma(\mu,x)-\sigma(\mu,x_i)| .$$

Since σ is a contraction, the first and last term of the right member can each be estimated by $|x-x_i|$, so

$$|(\sigma(\lambda,x)-\sigma(\mu,x),w)| < 2|x-x_i| + |(\sigma(\lambda,x_i)-\sigma(\mu,x_i),w)|$$

and, hence,

$$\limsup_{\lambda,\mu \downarrow 0} |(\sigma(\lambda,x)-\sigma(\mu,x),w) < 2|x-x_i| .$$

This implies that $\{\sigma(\lambda,x)\colon \lambda \geq 0\}$ is weakly Cauchy and, hence, weakly convergent as $\lambda \downarrow 0$ to ℓ, say. We now show that $\ell \in F$. We have the inequality

$$(\sigma(\lambda,x)-\ell,w)| \leq 2|x-x_i| + |(\sigma(\lambda,x_i),-\ell_i,w)| ,$$

whence

$$|(\ell_i-\ell,w)| \le 2|(\ell_i-\sigma(\lambda,x_i),w)| + |(\sigma(\lambda,x_i)-\sigma(\lambda,x),w)| + 2|x-x_i|$$

$$\le 2|(\ell_i-\sigma(\lambda,x_i),w)| + 3|x-x_i| \ .$$

Given any $\varepsilon > 0$, there exists a $i(\varepsilon)$ such that $|x-x_i| < \varepsilon/6$ for all $i \ge i(\varepsilon)$. With i thus fixed, we can choose a $\lambda(\varepsilon,i)$ such that $|(\ell_i-\sigma(\lambda,x_i),w)| < \varepsilon/4$ for all $\lambda < \lambda(\varepsilon,i)$. It follows that $\lim_{i\to\infty}(\ell_i-\ell,w) = 0$ for all $w \in H$, so $\ell_i \rightharpoonup \ell$ and, since F is closed, $\ell \in F$.

It remains to be shown that ℓ is the strong limit of $\mathrm{Proj}_F\, S(t,x)$ as $t \to \infty$. The proof is identical to the corresponding proof for $x \in D(A)$ given above and will, therefore, be omitted. □

Let $\sigma(0,x)$ denote the weak limit of $\sigma(\lambda,x)$ as $\lambda \downarrow 0$. We have the following corollary of Theorem 1.

<u>Corollary 3</u>. The operator $\sigma(0,\cdot)$ is a contractive mapping of C into F, which satisfies $S(t,\sigma(0,x)) = \sigma(0,x)$ for all $x \in C$, $t \ge 0$. If k satisfies the additional condition $\lim_{\lambda\downarrow 0}\int_0^\infty |k(\lambda,t+\tau)-k(\lambda,t)|dt = 0$ for each $\tau > 0$, then also $\sigma(0,S(t,x)) = \sigma(0,x)$ for all $x \in C$, $t \ge 0$.

<u>Proof</u>. The first part of the corollary is an immediate consequence of Theorem 1. The second part follows from the identity

$$\sigma(\lambda,S(t,x)) = \sigma(\lambda,x) - \int_0^t k(\lambda,\tau)S(\tau,x)d\tau$$

$$- \int_0^\infty (k(\lambda,\tau+t)-k(\lambda,\tau))S(\tau+t,x)d\tau \ ,$$

where $\sigma(\lambda,x)$ converges weakly to $\sigma(0,x)$ and the remaining terms in the right member converge (strongly) to 0 as $\lambda \downarrow 0$. □

3. Strong Convergence

We assume that the kernel k has the following additional property: for any bounded function q on $[0,\infty)$, the convergence of $\int_0^\infty k(\lambda,t)q(t)dt$ to a limit δ as $\lambda \downarrow 0$ implies the convergence of $\int_0^\infty\int_0^\infty k(\lambda,t)k(\lambda,s)q(|t-s|)dsdt$ to the same limit δ as $\lambda \downarrow 0$. We refer to this property as Property (A) of the kernel k.

In this section we prove the following theorem on the strong convergence of $\sigma(\lambda)$.

Theorem 4. Suppose $0 \in C$, $S(t,0) = 0$, and that, for some $c \geq 0$, S satisfies the inequality

$$|S(t,u)+S(t,v)|^2 \leq |u+v|^2 + c\{|u|^2 - |S(t,u)|^2 + |v|^2 - |S(t,v)|^2\}$$

for all $u,v \in C$. If $\lim_{\lambda\downarrow 0}\int_0^\infty |k(\lambda,t+\tau) - k(\lambda,t)|dt = 0$ for each $\tau > 0$, then $\sigma(\lambda,x)$ converges strongly as $\lambda \downarrow 0$.

The suppositions of the theorem hold true in particular if $C = -C$ and $S(t,-x) = -S(t,x)$ for all $x \in C$.

The proof of this theorem depends upon the following lemma.

Lemma 5. Suppose $(S(t,x),S(t+\tau,x)) \to q(\tau)$ as $t \to \infty$, uniformly for $\tau \in [0,\infty)$. If $\lim_{\lambda\downarrow 0}\int_0^\infty |k(\lambda,t+\tau)-k(\lambda,t)|dt = 0$ for each $\tau > 0$, then $\sigma(\lambda,x)$ converges strongly as $\lambda \downarrow 0$.

Proof. Let ℓ denote the weak limit of $\sigma(\lambda,x)$ as $\lambda \downarrow 0$. We first show that $\lim_{\lambda\downarrow 0}\int_0^\infty k(\lambda,t)q(t)dt = |\ell|^2$.

Consider the difference $(S(t,x),\ell) - \int_0^\infty k(\lambda,\tau)q(\tau)d\tau$, the first term of which converges to $|\ell|^2$, as we will show presently. We have the identity

$$(S(t,x),\ell) - \int_0^\infty k(\lambda,\tau)q(\tau)d\tau$$

$$= (S(t,x),\ell-\sigma(\lambda,x)) + \int_0^\infty k(\lambda,\tau)S(t,x),S(\tau,x))d\tau$$

$$- \int_0^\infty k(\lambda,\tau)(S(t,x),S(t+\tau,x)d\tau$$

$$+ \int_0^\infty k(\lambda,\tau)\{(S(t,x),S(t+\tau,x))-q(\tau)\}d\tau \ .$$

Splitting the integral in the second term of the right member, $\int_0^\infty = \int_0^t + \int_t^\infty$, and introducing a new variable of integration ($\tau' = \tau-t$) in the latter integral, we find the identity

$$(S(t,x),\ell) - \int_0^\infty k(\lambda,\tau)q(\tau)d\tau$$

$$= (S(t,x),\ell-\sigma(\lambda,x)) + \int_0^t k(\lambda,\tau)(S(t,x),S(\tau,x))d\tau$$

$$+ \int_0^\infty (k(\lambda,\tau+t)-k(\lambda,\tau))(S(t,x),S(t+\tau,x))d\tau$$

$$+ \int_0^\infty k(\lambda,\tau)\{(S(t,x),S(t+\tau,x))-q(\tau)\}d\tau \ .$$

Given any $\varepsilon > 0$, there exists a $t(\varepsilon)$ such that $|(S(t,x),S(t+\tau,x))-q(\tau)| < \varepsilon/4$ for all $t > t(\varepsilon)$, uniformly for $\tau \in [0,\infty)$. Hence, the fourth term in the right member of the above identity is less than $\varepsilon/4$ in absolute value if $t > t(\varepsilon)$. Now, for a fixed $t > t(\varepsilon)$, we can choose $\lambda(\varepsilon,t)$ such that each of the remaining terms in the right member is less than $\varepsilon/4$ in absolute value for all $\lambda < \lambda(\varepsilon,t)$: the first term by virtue of the weak convergence $\sigma(\lambda,x) \rightharpoonup \ell$, the second by virtue of the condition $\lim_{\lambda\downarrow 0}\int_0^t k(\lambda,\tau)d\tau = 0$ for each $t \geq 0$, and the third by virtue of the condition on k stated in the lemma. It follows that

(*) $$|(S(t,x),\ell) - \int_0^\infty k(\lambda,\tau)q(\tau)d\tau| < \varepsilon$$

for $t > t(\varepsilon)$ and $\lambda < \lambda(\varepsilon,t)$. Thus, $(S(t,x),\ell)$ is Cauchy and, therefore, has a limit p, say, as $t \to \infty$. As the averaging process defined by k is weakly regular, it follows that $(\sigma(\lambda,x),\ell) \to p$ as $\lambda \downarrow 0$. But $\lim_{\lambda\downarrow 0}(\sigma(\lambda,x),\ell) = |\ell|^2$, so with the inequality (*) it follows that

$$\lim_{\lambda\downarrow 0} \int_0^\infty k(\lambda,\tau)q(\tau)d\tau = |\ell|^2 \quad .$$

By virtue of Property (A) we then have also

$$\lim_{\lambda\downarrow 0} \int_0^\infty\int_0^\infty k(\lambda,t)k(\lambda,s)q(|t-s|)dsdt = |\ell|^2 \quad .$$

Now, consider the quantity $|\sigma(\lambda,x)|^2 = \int_0^\infty\int_0^\infty k(\lambda,t)k(\lambda,s)(S(t,x), S(s,x))dsdt$. By supposition, there exists a bounded measurable function η defined on $[0,\infty)$, such that $\eta(\omega) \to 0$ as $\omega \to \infty$ and $|(S(t,x),S(s,x))-q(|t-s|)| < \eta(\min(t,s))$. Then,

$$|\sigma(\lambda,x)|^2 \leq \int_0^\infty\int_0^\infty k(\lambda,t)k(\lambda,s)q(|t-s|)dsdt$$
$$+ \int_0^\infty\int_0^\infty k(\lambda,t)k(\lambda,s)\eta(\min(t,s))dsdt \quad .$$

The first term in the right member tends to $|\ell|^2$ as $\lambda \downarrow 0$; the second can be estimated by $2\int_0^\infty k(\lambda,t)\eta(t)dt$, which tends to zero as $\lambda \downarrow 0$. Hence, $\limsup_{\lambda\downarrow 0}|\sigma(\lambda,x)|^2 \leq |\ell|^2$. Since $|\sigma(\lambda,x)-\ell|^2 = |\sigma(\lambda,x)|^2 - 2(\sigma(\lambda,x),\ell) + |\ell|^2$ and $\sigma(\lambda,x) \rightharpoonup \ell$, it follows that $\limsup_{\lambda\downarrow 0}|\sigma(\lambda,x)-\ell|^2 < \limsup_{\lambda\downarrow 0}|\sigma(\lambda,x)|^2-|\ell|^2 \leq 0$, so $\lim_{\lambda\downarrow 0}|\sigma(\lambda,x)-\ell|^2$ exists and is equal to zero. []

<u>Proof of Theorem 4</u>. It suffices to prove that the inequality given in the statement of the theorem implies the uniform convergence of $(S(t,x), S(t+\tau,x))$ as $t \to \infty$. We start from the identity

$$(S(t+\sigma,x),S(t+\sigma+\tau,x)) - (S(t,x),S(t+\tau,x))$$
$$= (S(t+\sigma,x),S(t+\sigma+\tau,x) + S(t+\sigma,x)) - |S(t+\sigma,x)|^2$$
$$- (S(t,x),S(t+\tau,x) + S(t,x)) + |S(t,x)|^2 .$$

Using the law of cosines, $2(a-b,a-c) = |a-b|^2 + |a-c|^2 - |b-c|^2$ for any three vectors $a,b,c \in H$, we find

$$2[(S(t+\sigma,x),S(t+\sigma+\tau,x)) - (S(t,x),S(t+\tau,x))]$$
$$= \{|S(t,x)|^2 - |S(t+\sigma,x)|^2\} + \{|S(t+\tau,x)|^2 - |S(t+\tau+\sigma,x)|^2\}$$
$$+ \{|S(t+\tau+\sigma,x) + S(t+\sigma,x)|^2 - |S(t+\tau,x) + S(t,x)|^2\} ,$$

which can be estimated by virtue of the inequality given in the statement of the theorem by the quantity

$$(1+c)[\{|S(t,x)|^2 - |S(t+\sigma,x)|^2\} + \{|S(t+\tau,x)|^2 - |S(t+\tau+\sigma,x)|^2\}] .$$

Since S is a contractive semigroup, $|S(t,x)|^2$ is nonincreasing with limit $q(0)$, say. Hence, for any given $\varepsilon > 0$, we can find $t(\varepsilon)$ such that the expressions in braces above are less than ε in absolute value for all $t > t(\varepsilon)$. □

4. Examples

In this section we show that the results of the preceding sections are applicable to the usual Abel and Cesàro averaging procedures. The kernel for Abel averaging is given by

$$k_A(\lambda,t) := \lambda e^{-\lambda t} , \qquad t \geq 0 ,$$

and the kernel for Cesàro averaging of order α ($\alpha > 0$) is given by

$$k_{C,\alpha}(\lambda,t) := \begin{cases} \alpha\lambda(1-\lambda t)^{\alpha-1} & \text{for } 0 \leq t \leq 1/\lambda \\ 0 & \text{for } t \geq 1/\lambda . \end{cases}$$

It is trivially verified that these kernels are real-valued and nonnegative for $\lambda \geq 0$, that they satisfy the normalization condition $\int_0^\infty k(\lambda,t)dt = 1$, and that they are such that $\lim_{\lambda \downarrow 0} \int_0^t k(\lambda,\tau)d\tau = 0$ for all finite t. The functions $k_A(\lambda,\cdot)$ and $k_{C,\alpha}(\lambda,\cdot)$ are of bounded variation on $[0,\infty)$, with $k_A^+(\lambda,t) = \lambda$, $k_{C,\alpha}^+(\lambda,t) = k_{C,\alpha}(\lambda,t)$ for $0 < \alpha < 1$, and $k_{C,\alpha}^+(\lambda,t) = \alpha\lambda$ for $\alpha \geq 1$. Hence, $\int_0^\infty d_t k^+(\lambda,t) = 0$ for $\lambda \geq 0$ in all cases, except for the (C,α)-kernel with $0 < a < 1$, in which case $\int_0^\infty d_t k_{C,\alpha}^+(\lambda,t) = -\alpha\lambda$. Hence, $\lim_{\lambda \downarrow 0} \int_0^\infty d_t k^+(\lambda,t)dt = 0$ in all cases. Thus, the conditions of Theorem 1 are satisfied for the Abel and Cesàro-(C,α) averaging procedures and we conclude that the Abel and Cesàro-(C,α) means converge weakly to a fixed point of the semigroup.

Next, we turn to the verification of the conditions on the kernel k which are necessary for the application of Theorem 4. First we verify that the Abel and Cesàro-(C,α) kernels have Property (A).

Let q be a bounded function on $[0,\infty)$. Suppose $\int_0^\infty k_A(\lambda,t)q(t)dt$ converges to a limit δ as $\lambda \to \infty$. Then,

$$\int_0^\infty\int_0^\infty k_A(\lambda,t)k_A(\lambda,s)q(|t-s|)dsdt$$

$$= \lambda^2\left[\int_0^\infty\int_0^t e^{-\lambda(t+s)}q(t-s)dsdt + \int_0^\infty\int_t^\infty e^{-\lambda(t+s)}q(s-t)dsdt\right]$$

$$= \lambda^2\left[\int_0^\infty\int_0^t e^{-\lambda(2t-s)}q(s)dsdt + \int_0^\infty\int_0^\infty e^{-\lambda(2t+s)}q(s)dsdt\right]$$

$$= 2\lambda^2 \int_0^\infty\int_0^\infty e^{-\lambda(2t+s)}q(s)dtds$$

$$= \int_0^\infty k_A(\lambda,s)q(s)ds \;,$$

whence Property (A) follows for the Abel kernel.

For the verification of Property (A) for the Cesàro-(C,α) kernel it is more convenient to use the variable $\xi = 1/\lambda$. We put $\hat{k}_{C,\alpha}(\xi,t) :=$ $k_{C,\alpha}(1/\xi,t)$; thus, $C_\alpha(\xi,q) := \int_0^\infty \hat{k}_{C,\alpha}(\xi,t)q(t)dt$ is the Cesàro-(C,α) mean of q. Now, suppose that $C_a(\xi;q)$ converges to a limit δ as $\xi \to \infty$. Then,

$$I := \int_0^\infty\int_0^\infty \hat{k}_{C,\alpha}(\xi,t)\hat{k}_{C,\alpha}(\xi,s)q(|t-s|)dsdt$$

$$= \alpha^2\xi^{-2\alpha}\left[\int_0^\xi\int_0^t (\xi-t)^{\alpha-1}(\xi-s)^{\alpha-1}q(t-s)dsdt + \int_0^\xi\int_t^\xi (\xi-t)^{\alpha-1}(\xi-s)^{\alpha-1}q(s-t)dsdt\right]$$

$$= \alpha^2\xi^{-2\alpha}\left[\int_0^\xi\int_0^t (\xi-t)^{\alpha-1}(\xi-(t-s))^{\alpha-1}q(s)dsdt + \int_0^\xi\int_0^{\xi-t} (\xi-t)^{\alpha-1}(\xi-(t+s))^{\alpha-1}q(s)dsdt\right].$$

Interchanging the order of the integration in both terms we obtain the expression

$$I = \alpha^2\xi^{-2\alpha}\left[\int_0^\xi\int_s^\xi (\xi-t)^{\alpha-1}(\xi-(t-s))^{\alpha-1}q(s)dtds + \int_0^\xi\int_0^{\xi-s} (\xi-t)^{\alpha-1}(\xi-(t+s))^{\alpha-1}q(s)dtds\right].$$

The two terms inside the brackets have the same value, so

$$I = 2\alpha^2\xi^{-2\alpha}\int_0^\xi\int_0^{\xi-s} (\xi-s-t)^{\alpha-1}(\xi-t)^{\alpha-1}q(s)dtds .$$

Again interchanging the order of the integrations we find the expression

$$I = 2\alpha^2\xi^{-2\alpha}\int_0^\xi (\xi-t)^{\alpha-1}\int_0^{\xi-t} (\xi-s-t)^{\alpha-1}q(s)dsdt ,$$

where the inner integral is recognized as a Cesàro-(C,α) mean of q,

$$I = 2\alpha^2\xi^{-2\alpha}\int_0^{\xi}(\xi-t)^{2\alpha-1}C_\alpha(\xi-t;q)dt$$

$$= 2\alpha^2\xi^{-2\alpha}\int_0^{\xi}t^{2\alpha-1}C_\alpha(t;q)dt\ .$$

Thus,

$$\left|\int_0^\infty\int_0^\infty\hat{k}_{C,\alpha}(\xi,t)\hat{k}_{C,\alpha}(\xi,s)q(|t-s|)dsdt - \int_0^\infty\hat{k}_{C,\alpha}(\xi,t)q(t)dt\right|$$

$$\leq 2\alpha\ \xi^{-2\alpha}\int_0^{\xi}t^{2\alpha-1}|C_\alpha(t;q)-\delta|dt + |C_\alpha(\xi;q)-\delta|\ .$$

Given $\varepsilon > 0$, there exists a $t(\varepsilon)$ such that $|C_\alpha(t;q)-\delta| < \varepsilon/3$ for all $t > t(\varepsilon)$ amd, consequently, $2\alpha\xi^{-2\alpha}\int_{t(\varepsilon)}^{\xi}t^{2\alpha-1}|C_\alpha(t;q)-\delta|dt < \varepsilon/3$. Finally, $2\alpha\xi^{-2\alpha}\int_0^{t(\varepsilon)}t^{2\alpha-1}|C_\alpha(t;q)-\delta|dt < \varepsilon/3$ for all $\xi > \xi(\varepsilon)$. Hence,

$$\lim_{\xi\to\infty}\left[\int_0^\infty\int_0^\infty\hat{k}_{C,\alpha}(\xi,t)\hat{k}_{C,\alpha}(\xi,s)q(|t-s|)dsdt - \int_0^\infty\hat{k}_{C,\alpha}(\xi,t)q(t)dt\right] = 0\ ,$$

which proves Property (A) for the Cesàro-(C,α) kernel.

The condition $\lim_{\lambda\downarrow 0}\int_0^\infty|k(\lambda,t+\tau)-k(\lambda,t)|dt = 0$ for each $\tau > 0$ is immediate for the Abel kernel. For the Cesàro-(C,α) kernel one has

$$\int_0^\infty|k_{C,\alpha}(\lambda,t+\tau)-k_{C,\alpha}(\lambda,t)|dt$$

$$= \begin{cases}(1-\lambda\tau)^\alpha + 2(\lambda\tau)^\alpha - 1 & \text{if} \quad 0 < \alpha \leq 1\\ 1 - (1-\lambda\tau)^\alpha & \text{if} \quad \alpha > 1\ ,\end{cases}$$

which tends to zero as $\lambda\downarrow 0$.

Thus, the conditions of Theorem 4 are satisfied for the Abel and Cesàro-(C,α) averaging procedures and we conclude that, under the appropriate conditions on the semigroup, the Abel and Cesàro-(C,α) means converge strongly to a fixed point of the semigroup.

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