UNIVERSAL BRANCHED COVERINGS

DISSERTATION

Presented to the Graduate Council of the
University of North Texas in Partial
Fulfillment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

By

Débora Tejada, B.A., M.S.

Denton, Texas

May, 1993
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In this paper, the study of $k$-fold branched coverings for which the branch set is a stratified set is considered.

First of all, the existence of universal $k$-fold branched coverings over CW-complexes with stratified branch set is proved using Brown's Representability Theorem. Next, an explicit construction of universal $k$-fold branched coverings over manifolds is given. Finally, some homotopy and homology groups are computed for some specific examples of Universal $k$-fold branched coverings.
PREFACE

I wish to express gratitude to my advisor Professor Neal Brand for his insightful suggestions, comments, and remarks as well as his encouragement that constantly improved this research. I am also grateful to Professors Melvin Hagan, Joseph Kung, Luca Zamboni, and Tom Jacob. Their comments have been appreciated. I am indebted to Professor Stephen Curran for his assistance and help at the beginning of the present work. I owe special thanks to Professor Phillip Griffith for the numerous conversations that we had prior to initiating my work in Algebraic Topology. I am also deeply grateful to Professor Alfonso Castro and his family for their encouragement and support. I would like to express my thanks to the National University of Colombia, the Colombian Institute of Sciences (Colciencias), and the Department of Mathematics of the University of North Texas, for their support while this work was under way.

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Débora Tejada
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>0. PREFACE</td>
<td>iii</td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1. History</td>
<td></td>
</tr>
<tr>
<td>2. Chapter References</td>
<td></td>
</tr>
<tr>
<td>2. EXISTENCE OF CLASSIFYING SPACES</td>
<td>8</td>
</tr>
<tr>
<td>1. Introduction</td>
<td></td>
</tr>
<tr>
<td>2. Brown's Representability Theorem</td>
<td></td>
</tr>
<tr>
<td>3. Branched Coverings over Manifolds</td>
<td></td>
</tr>
<tr>
<td>4. Branched Coverings over Simplicial Complexes</td>
<td></td>
</tr>
<tr>
<td>5. Chapter References</td>
<td></td>
</tr>
<tr>
<td>3. CONSTRUCTION OF CLASSIFYING SPACES</td>
<td>33</td>
</tr>
<tr>
<td>1. Introduction</td>
<td></td>
</tr>
<tr>
<td>2. The Simplest Case</td>
<td></td>
</tr>
<tr>
<td>3. General Case</td>
<td></td>
</tr>
<tr>
<td>4. Commutativity of Diagrams 2.4 and 2.5</td>
<td></td>
</tr>
<tr>
<td>5. Chapter References</td>
<td></td>
</tr>
<tr>
<td>4. EXAMPLES AND CALCULATIONS</td>
<td>59</td>
</tr>
<tr>
<td>1. Preliminaries</td>
<td></td>
</tr>
<tr>
<td>2. Relations among the Homotopy Groups</td>
<td></td>
</tr>
</tbody>
</table>
3. Van Kampen's Theorem and Mayer-Vietoris Sequences

4. Chapter References

REFERENCES ................................................................. 73
CHAPTER 1

INTRODUCTION

1. History

Since the beginning of the present century, the characterization of compact connected 2-manifolds is known (see [10] and [21]). More precisely, a compact connected 2-manifold is homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes. Although considerable work has been done in order to classify 3-manifolds, we do not have a similar theorem for 3-manifolds yet.

Working on this subject, Poincaré stated his famous conjecture: “Every connected compact 3-manifold whose fundamental group is zero should be homeomorphic to $S^3$” (see [10]). We could say that the statement of this conjecture marked the beginning of the modern era in Algebraic Topology. Even more, much of the progress in this area has been obtained trying to solve the mentioned conjecture.

In 1920, Alexander (see [1]) showed that every closed oriented $n$-manifold is a branched covering of $S^n$. In the same paper he also stated, without proof, that for $n = 3$ the branch set could be taken to be a link in $S^3$. In part, because of this result, Alexander started the study of Knots and Links. The importance of his work in this field is now easily recognized.

In 1957, R.H. Fox formulated as a topological concept the idea of a branched
covering space (see [11] and [12]). Only in 1974, Hilden and Montesinos (independently) proved the statement made by Alexander for \( n = 3 \) (see [16] and [24]). We observe that Hilden’s and Montesinos’ methods are very different. While Hilden (see [16] and [17]) used more traditional techniques in Algebraic Topology, the originality of Montesinos’ work was remarkable (see [22], [23], [24], [25], [26], [27], [28], and [29]). The ideas developed by Montesinos, using surgery on Knots and Links, as well as his examples, inaugurated a new style. As a consequence, the hope of giving an answer to Poincaré’s conjecture is still alive.

On the other hand, the study of branched coverings has been considered by many other authors using different points of view. For example; I. Berstein and A.L. Edmonds studied the degree of a branched covering in [2]. They also considered the construction of branched coverings of low-dimensional manifolds in [3]. In [4] and [5] N. Brand found necessary conditions for the existence of branched coverings, he also constructed classifying spaces in [4] and [6]. N. Brand and G. Brumfiel studied concordance classes of branched coverings in [7] and [8]. S.M. Gersten gave an explicit method for constructing branched coverings from \( S^2 \) to \( S^2 \) in [13] and [14]. Recently, in [15] T. Harikae considered 3-fold irregular branched coverings of some spatial graphs. H.M. Hilden and R.D. Little developed a general method of constructing branched covering spaces of spheres and other manifolds in [18]. R.D. Little studied projective spaces as branched coverings in [19] and [20].

This research is a continuation of Brand’s work. In 1978, N. Brand (see
constructed classifying spaces for $k$-fold branched coverings ($k \in \mathbb{N}$) that have branch set a submanifold of codimension 2. In this paper, the $k$-fold branched coverings with stratified branch sets (see Definition 3.1, Chapter 2) are studied. Besides proving the existence of universal $k$-fold branched coverings, an explicit construction for classifying spaces of branched coverings is given here.

In Chapter 2, first of all, only $k$-fold branched coverings over manifolds with branch set a stratified set are considered. Several properties for these $k$-fold branched coverings are proved, among them are generalizations of two well known transversality theorems. In fact, it is shown that any smooth function $f : M \to N$ is homotopic to a smooth function that is transverse to a stratified set $K \subset N$ (see Proposition 3.2, Chapter 2). It is also proved that the preimage of a stratified set under a function that is transverse to it is also a stratified set (see Corollary 3.3, Chapter 2).

Next, the $k$-fold branched coverings over simplicial and CW-complexes are considered. In order to construct a functor from the category of CW-complexes to the category of sets, some basic facts are proved (see for example Propositions 4.2, 4.6 and Lemmas 4.3 and 4.4 in Chapter 2). Finally, using Brown's Representability Theorem (see [9]) as a principal tool, the existence of classifying spaces for $k$-fold branched coverings over CW-complexes for which the branch set is a stratified set is shown (see Theorem 4.13, Chapter 2). Theorem 4.13 is the principal result in Chapter 2.
Unfortunately, the proof of Theorem 4.13 (Chapter 2) is not constructive. So, Chapter 3 is devoted to constructing universal $k$-fold branched coverings over manifolds with a stratified branch set. First of all, the simplest universal $k$-fold branched covering is considered and then the two more general cases are studied. The intrinsic nature of the stratified sets permits a nice inductive construction. Actually, the first step of this induction was done by Brand in [4]. The method consists in taking a tubular neighbourhood around the submanifold that is placed at the deepest level ($l$) of the stratified branch set. Outside of this tubular neighbourhood the induction hypothesis guarantees the existence of the universal space. Inside of the tubular neighbourhood, the universal normal disc bundle gives the universal space. Those two spaces are glued together by taking a double mapping cylinder to get the Universal Space for the $k$-fold branched coverings of type $l$.

In Chapter 4, first of all, we study some sequence of functions among the Homotopy groups of the classifying spaces. Next, using traditional algebraic topology techniques such as Van Kampen’s Theorem and Mayer-Vietoris Sequences we study a specific example. Finally we give information about the homotopy and homology groups of some particular classifying spaces.
2. Chapter References


CHAPTER 2

EXISTENCE OF CLASSIFYING SPACES

1. Introduction

In this chapter, our principal goal is to prove the existence of a classifying space for $k$-fold branched coverings over CW-complexes. To accomplish this purpose, we are going to apply Brown's Representability Theorem as our most important tool.

Section 2 states Brown's theorem which says that if a specific functor satisfies certain conditions then there is a classifying space for it. Section 3 focuses on the basic concepts of $k$-fold branched coverings of manifolds (with or without boundary) and some of their properties. In Section 4 we see that the concepts and properties developed in Section 3 play an important role. The concepts give a natural framework for the definitions of $k$-fold branched coverings over simplicial complexes and over CW-complexes. The properties permit us to construct a functor from the category of CW-complexes to the category of sets, and prove that this functor verifies all the conditions that Brown's Representability Theorem requires.

2. Brown's Representability Theorem

In this section we will state Brown's Representability Theorem which will be applied at the end of Section 4 in order to prove the existence of a classifying space
for \( k \)-fold branched coverings over pointed CW-complexes.

First of all, we are going to give some basic definitions and notations:

**Definition 2.1.**

(a) Let \( X \) and \( Y \) be topological spaces, let \( A \subset X \) and let \( I \) be the unit interval \([0,1]\). We say that the pair \((X, A)\) has the homotopy extension property if any continuous map \( f : A \times I \to Y \) admits a continuous extension \( F : X \times I \to Y \).

(b) Let \( C \) be the category of topological spaces with base point which admit a CW-complex structure and continuous maps preserving base points. A triple \((X, X_1, X_2)\) will be called a proper triad of \( C \) if \( X = X_1 \cup X_2, X_1, X_2, \) and \( X_1 \cap X_2 \) are in \( C \), all have the same base point, and \((X_1, X_1 \cap X_2)\) and \((X_2, X_1 \cap X_2)\) each have the homotopy extension property.

**Notation 2.2.** If \( X \) and \( Y \) belong to \( C \), \([X, Y]\) will denote the set of homotopy classes of maps of \( X \) into \( Y \) with respect to homotopies which leave the base point of \( X \) fixed. \([\bullet, Y]\) will denote the functor from \( C \) to \( S \) (where \( S \) is the category of sets with a distinguished element and maps preserving distinguished elements) which assigns to each \( X \) in \( C \) the set \([X, Y]\) with the class of the constant map as distinguished element, and assigns to each map \( f : X \to X' \) the map \( F : [X', Y] \to [X, Y] \) defined as follows: \( F[g] = [g \circ f] \) where \([g]\) denotes the homotopy class of \( g \).

Now, we are ready to state Brown's Theorem (see [3]) which guarantees the existence of classifying spaces for functors that verify some special conditions.
Theorem 2.3. If $H : C \to S$ is a contravariant functor, and $H$ satisfies the Conditions A, B, C, D listed below, there is a space $Y$ in $C$, unique up to homotopy type, such that the functors $[\bullet, Y]$ and $H$ are naturally equivalent.

A. If $f, g : X \to Y$ are homotopic, $H(f) = H(g)$.

B. (1) If $p$ is a point $H(p)$ contains only one element. (2) Suppose $(X, X_1, X_2)$ is a proper triad, $A = X_1 \cap X_2$ and $j_i : A \to X_i$ and $k_i : X_i \to X$ are the inclusion maps, $i = 1, 2$. If $u_1 \in H(X_1)$ and $u_2 \in H(X_2)$ are such that $H(j_1)u_1 = H(j_2)u_2$, then there is a $v$ in $X$ such that $H(k_1)v = u_1$ and $H(k_2)v = u_2$. Furthermore, if $A$ is a point, $v$ is unique.

C. Suppose $S^n_\alpha$ is a collection of disjoint $n$-spheres whose wedge product $\vee S^n_\alpha$ is in $C$. Let $i_\alpha : S^n_\alpha \to \vee S^n_\alpha$ be the inclusion map. Then $\prod H(i_\alpha) : H(\vee S^n_\alpha) \to \prod (S^n_\alpha)$ is bijective.

D. Suppose $X_1 \subset X_2 \subset \cdots \subset X_n \cdots$ is a collection of subcomplexes of $X = \bigcup X_n$ in $C$ with respect to some CW-complex structure on $X$ such that $X^n_n = X^n$ (where $X^n$ is the $n$-skeleton of $X$). Let $i_n : X_n \to X$ be the inclusion map. Let $\varprojlim H(X_n)$ be the inverse limit of $H(X_n)$ with respect to the maps induced by the inclusions of $X_n$ to $X_m$. Then the function $\varprojlim H(i_n) : H(X) \to \varprojlim H(X_n)$ is an epimorphism.

Sections 3 and 4 are devoted to prove that a specific functor verifies all the hypotheses of Brown's Theorem. In this way we show the existence of a classifying space for $k$-fold branched coverings over CW-complexes.
3. Branched Coverings over Manifolds

According to the plan outlined in Section 1, this section studies \( k \)-fold branched coverings over manifolds and some of their properties. The definitions and propositions exposed in this section are essentially the base for the rest of the paper.

Definitions 3.1.

(a) Let \( N^n \) be a smooth manifold, and let \( l \) be a natural number, we say that \( K \subset N \) is a stratified set in \( N \) of type \( l \) if there is a sequence \( K_l \subset K_{l-1} \subset K_{l-2} \subset \cdots \subset K_3 \subset K_2 = K \) of closed sets in \( N \), such that \( (K_j - K_{j+1}) \) is a smooth manifold without boundary of codimension \( j \) and \( K_j - K_{j+1} = K_j \) for every \( j \), \( j = 2, \ldots, l-1 \).

(b) Given \( M, N \) manifolds, \( Z \) a submanifold of \( N \), and \( f : M \rightarrow N \) a smooth function we say that \( f \) is transverse to \( Z \) if every \( x \in f^{-1}(Z) \) satisfies the following equation:

\[
df_x(T_x(M)) + T_y(Z) = T_y(N),
\]

in other words, for every \( x \in f^{-1}(Z) \), the tangent space of \( N \) at \( y = f(x) \) \( (T_y(N)) \) is spanned by the tangent space of \( Z \) at \( y \) and the image of the tangent space of \( M \) at \( x \) under the derivative \( df_x : T_x(M) \rightarrow T_y(N) \) (see [5] p.28).

(c) Given \( M, N \) manifolds, \( K \) a stratified set in \( N \) of type \( l \), and \( f : M \rightarrow N \) a smooth function, we say that \( f \) is transverse to \( K \) if \( f \) is transverse to each of the following manifolds: \( K_l \) and \( (K_j - K_{j+1}) \) for \( j = 2, \ldots, l-1 \).
Remark. Given $M, N$ manifolds, $Y$ a submanifold of $N$, and $f : M \to N$ a smooth function. A well known theorem (see [5] p.70) says: that $f$ is homotopic to a smooth function $g : M \to N$ which is transverse to $Y$.

The following proposition generalizes this theorem. Similar results are found in [9].

**Proposition 3.2.** Let $M^n, N^n$ be smooth manifolds. If $g_0 : M \to N$ is any smooth function, there exists $g : M \to N$ homotopically equivalent to $g_0$ such that $g$ is transverse to $K$, where $K$ is a stratified set in $N$ of type $l$.

**Proof.** The existence of $g_1 : M \to N$ homotopically equivalent to $g_0$ such that $g_1$ is transverse to $K_1$ is known because $K_1$ is a smooth manifold. Now assume that $g_s : M \to N$ is homotopically equivalent to $g_{s-1}$ and $g_s$ is transverse to the manifolds $K_l, (K_l - K_{l-1}), \ldots, (K_{l-(s-1)} - K_{l-(s-2)})$. Let us show that there is $g_{s+1} : M \to N$ homotopically equivalent to $g_s$ such that $g_{s+1}$ is transverse to the manifolds $K_l, (K_l - K_{l-1}), \ldots, (K_{l-s} - K_{l-(s-1)})$.

Since $g_s$ is transverse to $K_l, (K_l - K_{l-1}), \ldots, (K_{l-(s-1)} - K_{l-(s-2)})$, then for each of these manifolds there is a tubular neighbourhood that contains only regular points of $g_s$.

In fact, since $g_s$ is transverse to the manifold $K_l$, each point $y \in K_l$ is a regular value. Using Brown and Sard's theorem (see [6]) we could construct a regular neighbourhood of $K_l$ by taking around each $y$ in $K_l$ a ball $B_{\epsilon(y)}$ containing only regular points. Moreover, we take the function $\epsilon : K_l \to \mathbb{R}$ as a smooth
function. Let \( K_i^{e(y)} = \bigcup_{y \in K_i} B_{e(y)} \), then \( K_i^{e(y)} \) is a tubular neighbourhood (see [6]) that contains only regular values of \( g_s \). Consider the smaller tubular neighbourhood given by \( K_i^{e(y)_{\frac{1}{2}}} = \bigcup_{y \in K_i} B_{e(y)_{\frac{1}{2}}} \). Therefore \( g_s \) is transverse to \( K_i^{e(y)} \) and also to \( K_i^{e(y)_{\frac{1}{2}}} \).

In a similar way we choose tubular neighbourhoods for \( (K_{i-1} - K_i), \ldots, (K_{i-(s-1)} - K_{i-(s-2)}) \), and let us denote by \( K_i^{e(y)} \) the union of all tubular neighbourhoods of the first type, and by \( K_i^{e(y)_{\frac{1}{2}}} \) the union of the second type of neighbourhoods. Note that they are open sets in \( N \) and, moreover, \( g_s \) is transverse to both of them.

Let \( A \) be the closed set \( g_s^{-1} \left( K_i^{e(y)_{(s-1)}} \right) \), \( A \) is contained in the open set \( g^{-1} \left( K_i^{e(y)_{(s-1)}} \right) \). Since \( (K_{i-s} - K_{i-(s-1)}) \) is a manifold and \( g_s \) restricted to the open neighbourhood \( g^{-1} \left( K_i^{e(y)_{(s-1)}} \right) \) of \( A \) is already transverse to \( (K_{i-s} - K_{i-(s-1)}) \) then by Thom's Transversality Theorem (see [2] p.34), there is a function \( g_{s+1} : M \to N \) homotopically equivalent to \( g_s \) (rel \( A \)) such that \( g_{s+1} \) is transverse to \( (K_{i-s} - K_{i-(s-1)}) \). Thus \( g_{s+1} \) is transverse to \( K_i, (K_i - K_{i-1}), \ldots, (K_{i-s} - K_{i-(s-1)}) \).

Inductively, there exists \( g_{l-1} : M \to N \) homotopically equivalent to \( g_0 \) that is transverse to the stratified set \( K = K_2 \). This function \( g_{l-1} \) is exactly the desired function \( g \).

The next corollary generalizes the statement that the inverse image of a manifold under a continuous function that is transverse to it is still a manifold with the same codimension (see [5] and [9]).
Corollary 3.3. Let $M, N, K, g$ be as in the last proposition. Then $g^{-1}(K)$ is a stratified set in $M$ of type $l$.

Proof. In fact, by continuity of $g$, $g^{-1}(K_1) \subset g^{-1}(K_{l-1}) \subset \cdots \subset g^{-1}(K_2) = g^{-1}(K) \subset M$ is a sequence of closed sets. Since $g$ is transverse to $K$ then $g$ is transverse to the following manifolds: $K_1, (K_{l-1} - K_1), \ldots, (K_3 - K_4), (K_2 - K_3)$. Therefore, their inverse image under $g$ are smooth manifolds of codimension $l, l - 1, \cdots, 2$, respectively. Hence $g^{-1}(K)$ is a stratified set in $M$ of type $l$. $lacksquare$

Definitions 3.4.

(a) Let $\tilde{N}, N$ be smooth manifolds without boundary, let $k, l$ be natural numbers, and let $f : \tilde{N} \to N$ be a smooth function. We say that $f$ is a $k$-fold branched covering over a manifold without boundary (of type $l$) if $f$ verifies:

(i) The branch set $K$ is a stratified set in $N$ of type $l$.

(ii) $f$ is transverse to $K$.

(iii) $f|_{f^{-1}(N - K)}$ is a $k$-fold covering.

(iv) $f|_{f^{-1}(K_1)}, f|_{f^{-1}(K_{l-1} - K_1)}, \ldots, f|_{f^{-1}(K_3 - K_4)}$ are $k_s$-coverings over their respective components, where $k_s$ is less than $k$.

(b) Let $\tilde{N}, N$ be smooth manifolds with boundary. Let $f : \tilde{N} \to N$ be a smooth function. We say that $f$ is a $k$-fold branched covering over a manifold with boundary if:

(i) $f|_{(\tilde{N} - \partial \tilde{N})} : (\tilde{N} - \partial \tilde{N}) \to (N - \partial N)$ and $f|_{\partial \tilde{N}} : \partial \tilde{N} \to \partial N$ are $k$-fold branched coverings over a manifold without boundary.
(ii) The following diagram is commutative:

\[ \partial \tilde{N} \times I \xrightarrow{i} \tilde{N} \]

\[
\begin{array}{c}
\partial N \times I \\
\downarrow f_\ast \times \text{id}
\end{array}
\xrightarrow{f} \begin{array}{c}
\partial N \times I \\
\downarrow j
\end{array} \xrightarrow{f} N
\]

where \( i \) and \( j \) are the embeddings obtained by "collaring" \( \partial \tilde{N} \) and \( \partial N \) in \( \tilde{N} \) and \( N \), respectively (see [4]), and \( I \) is the closed interval \([0,1] \) in \( \mathbb{R} \).

Remarks. 1) If no ambiguity arises, in most cases we are going to omit the words "with boundary" or "without boundary". 2) If \( H \) is a set in \( N \), \( f^{-1}(H) \) will be denoted by \( \tilde{H} \), for example: \( \tilde{K}_j = f^{-1}(K_j) \).

Roughly speaking, the next proposition says that the pullback of a \( k \)-fold branched covering is also a \( k \)-fold branched covering.

**Proposition 3.5.** Let \( M \) be a smooth manifold. Let \( f, \tilde{N}, N, \tilde{K}_j, K, j = 2, \ldots, l \) be defined as before. Let \( g : M \to N \) be any smooth map transverse to \( K \), and let \( \tilde{M} \) be the fiber product in the diagram

\[ \begin{array}{c}
\tilde{M} \\
\downarrow f_1
\end{array} \xrightarrow{f} \begin{array}{c}
\tilde{N} \\
\downarrow f
\end{array} \xrightarrow{f} N
\]

Then \( f_1 : \tilde{M} \to M \) is a \( k \)-fold branched covering.

**Proof.** In fact, since \( g \) is transverse to \( K \), the sequence of closed sets \( g^{-1}(K_1) \subset g^{-1}(K_{l-1}) \subset \cdots \subset g^{-1}(K_2) \subset g^{-1}(K_2) = g^{-1}(K) \) is such that \( g^{-1}(K_i) \) is a smooth
manifold of codimension 1 in \( M \) and \( g^{-1}(K_j) - g^{-1}(K_{j-1}) = g^{-1}(K_j - K_{j-1}) \) is a manifold of codimension \( j \) in \( M \), for every \( j, j = 2, \ldots, l - 1 \).

Moreover, since the restriction of \( f \) to the preimages of \( K_1, (K_{l-1} - K_1), \ldots, (K_2 - K_3), (N - K) \) under \( f \) are finite coverings, the pullbacks of them under \( g \) are also finite coverings.

In order to finish our proof, we have to show that the branch set of \( f_1 \) is just \( g^{-1}(K) \). If \( x \in (M - g^{-1}(K)) \) then \( x \) is not a branched point because the restriction of \( f_1 \) to \( f_1^{-1}(M - g^{-1}(K)) \) is a finite covering.

Let \( x \in g^{-1}(K) \), we want to show that \( x \) is a branched point. We need only to notice that the cardinality of the set \( f_1^{-1}(x) \) is different from the cardinality of the set \( f_1^{-1}(x_0) \) when \( x_0 \) is not a branched point. In fact, if \( z \) is any element in \( M \), the cardinality of \( f_1^{-1}(z) \) is equal to the cardinality of \( f^{-1}(g(z)) \) because of the definition of the fiber product.

**Definition 3.6.**

(a) Let \( M \) be a manifold with \( f_1 : \widetilde{M}_1 \to M \) and \( f_2 : \widetilde{M}_2 \to M \) branched coverings over \( M \). Then \( f_1 \) and \( f_2 \) are equivalent up to homeomorphism if there is a homeomorphism \( h : \widetilde{M}_1 \to \widetilde{M}_2 \) such that \( f_2 \circ h = f_1 \). The map \( h \) is called a branched covering homeomorphism.

(b) Two branched coverings \( f_1 \) and \( f_2 \) are concordant if there is a branched covering \( F : \widetilde{W}^{n+1} \to M^n \times I \) such that \( F|_{F^{-1}((M \times \{0\})} \) is equivalent up to homeomorphism with \( f_1 \) and \( F|_{F^{-1}((M \times \{1\})} \) is equivalent up to homeomorphism with \( f_2 \).
Let us denote by $B_{k,i}(M)$ the set of all concordance classes of $k$-fold branched coverings over $M$ of type $l$.

**Proposition 3.7.** $B_{k,i}(\bullet)$ is a contravariant functor from the category of smooth manifolds to the category of sets.

**Proof.** Let $M, N, \tilde{N}$ be smooth manifolds (with or without boundary). Let $g : M \to N$ be a smooth map and let $f : \tilde{N} \to N$ be a $k$-fold branched covering (see Definition 3.4). By Proposition 3.2, without loss of generality, we can assume that $g$ is transverse to $K$, where $K$ is the branch set of $f$. By Proposition 3.5 $f_1 : \tilde{M} \to M$ is a $k$-fold branched covering of type $l$, where $\tilde{M}$ is the fiber product in diagram (3.2). Let us define $B_{k,i}(f)$ as the concordance class of $f_1$. We need to show that $B_{k,i}$ is well defined.

First of all, let us prove that if $f' : \tilde{N}' \to N$ is a branched covering concordant to $f : \tilde{N} \to N$, then $B_{k,i}(f) = B_{k,i}(f')$. In other words, we need to show that the pullbacks of $f$ and $f'$ under $g$ are concordant. Without loss of generality, we assume that $g$ is also transverse to $K'$, the branch set of $f'$. Let $F : W^{n+1} \to N \times I$ be a branched covering that makes $f$ and $f'$ concordant. Consider the pullback of $F$ under $g \times id$. It is clear that this pullback makes the pullback of $f$ concordant with the pullback of $f'$ under $g$, i.e., $B_{k,i}(f) = B_{k,i}(f')$.

We need also to prove that if $g_1, g_2 : M \to N$ are homotopic functions that are both transverse to $K$, and if $f_1$ and $f_2$ are the pullbacks of $g_1, g_2$, respectively, under $g$ then $f_1$ and $f_2$ are concordant.
Let \( H : M \times I \rightarrow N \) be a homotopy from \( g_1 \) to \( g_2 \). By the same technique used in the proof of Proposition 3.2 we can construct a map \( H_1 : M \times I \rightarrow N \) that agrees with \( H \) in \( M \times \{0,1\} \) and is transverse to \( K \). Taking \( W_1 \) as the fiber product in the next diagram:

\[
\begin{array}{ccc}
W_1 & \longrightarrow & \tilde{N} \\
\downarrow F_1 & & \downarrow f \\
M \times I & \underleftarrow{H_1} & N
\end{array}
\]

we get the branched covering \( F_1 : W_1 \rightarrow M \times I \) which implies the concordance between \( f_1 \) and \( f_2 \).

**Definition 3.8.**

(a) Let \( \tilde{M}, M \) be manifolds. A **pointed branched covering** is a \( k \)-fold branched covering \( f : \tilde{M} \rightarrow M \) together with a base point \( * \in (M - K) \) (where \( K \) is the branch set of \( f \)) and a one-to-one correspondence of \( \{1,2,\ldots,k\} \) with the set \( f^{-1}(*) \). This correspondence is denoted by \( c(f)_i \) where \( 1 \leq i \leq k \).

(b) Two pointed branched coverings \( f_1 \) and \( f_2 \) are **equivalent up to homeomorphism** if there is a branched covering homeomorphism \( h \) which preserves the labeling, that is, \( h(c(f_1)_i) = c(f_2)_i \).

(c) Two pointed \( k \)-fold branched coverings are **concordant** if there is a branched covering concordance \( F : \tilde{W}^{n+1} \rightarrow M \times I \) such that \( * \times I \) does not intersect the branch set of \( F \) and \( c(f_1)_i \) and \( c(f_2)_i \) are in the same components of \( F^{-1}(*) \times I \) when \( F|_{F^{-1}(M \times \{0\})} \) and \( F|_{F^{-1}(M \times \{1\})} \) are equivalent to \( f_1 \) and \( f_2 \), respectively.
Let us denote by $B_{k,\ell}(M, \ast)$ the set of all concordance classes of pointed $k$-fold branched coverings over the manifold $M$.

**Proposition 3.9.** $B_{k,\ell}(\bullet, \ast)$ is a contravariant functor from the category of pointed smooth manifolds to the category of sets with a distinguished element.

**Proof.** The proof is similar to the proof of Proposition 3.7. ■

### 4. Branched Coverings over Simplicial Complexes

In this section we want to prove the existence of a functor that operates over the category of CW-complexes which verifies all the hypotheses of Theorem 2.3. Since every CW-complex is homotopically equivalent to a simplicial complex, our first work will be done for simplicial complexes. At the end of the section we will proceed to generalize the theorems to the CW-complexes.

**Note.** Let $\Delta_m$ be an $m$-simplex. Notice that for every face $\Delta_{m-1}$ of $\Delta_m$ there is a one-to-one function $sl: \Delta_{m-1} \times I \to \Delta_m$ such that $(\Delta_m - sl(\Delta_{m-1} \times [0,1]))$ is homeomorphic to $\Delta_m$. We are going to call this kind of function a “slice”.

Now, let us give some definitions:

**Definition 4.1.**

(a) Let $\Delta_m$ be an $m$-simplex. We say that $f: \Delta_m \to \Delta_m$ (where $\Delta_m$ is not necessarily an $m$-simplex) is a $k$-fold branched covering over the simplex $\Delta_m$ if for every face $\Delta_s$ of $\Delta_m$ the function $f|_{f^{-1}(\text{int} \Delta_s)}: f^{-1}(\text{int} \Delta_s) \to \text{int} \Delta_s$ (where $\text{int}$
means interior) is a \( k \)-fold branched covering over the manifold \( \text{int} \Delta_s \), and for each face \( \Delta_{s-1} \) of \( \Delta_s \), we have the following commutative diagram:

\[
\begin{array}{ccc}
 f^{-1}(\Delta_{s-1}) \times I & \longrightarrow & f^{-1}(\Delta_s) \\
 f \times I & \\ \\
 \Delta_{s-1} \times I & \longrightarrow & \Delta_s \\
 \end{array}
\]

(4.1)

where \( I \) is the closed interval \([0,1]\), and \( f^{-1}(\Delta_{s-1}) \times I \rightarrow f^{-1}(\Delta_s) \) is some embedding.

(b) Let \( \Sigma \) be a simplicial complex. Let \( f : \tilde{\Sigma} \rightarrow \Sigma \) be a continuous function (where \( \tilde{\Sigma} \) is not necessarily a simplicial complex). We say that \( f \) is a \( k \)-fold branched covering over the simplicial complex \( \Sigma \) if for every \( m \)-simplex \( \Delta_m \) contained in any subdivision of \( \Sigma \) the function \( f : \tilde{\Delta}_m \rightarrow \Delta_m \) is a \( k \)-fold branched covering over \( \Delta_m \).

The next proposition preserves the spirit of Proposition 3.5.

**Proposition 4.2.** Let \( \Sigma_1, \Sigma_2 \) be simplicial complexes, \( f : \tilde{\Sigma}_2 \rightarrow \Sigma_2 \) be a \( k \)-fold covering, and let \( g : \Sigma_1 \rightarrow \Sigma_2 \) be a simplicial function. If \( \tilde{\Sigma}_1 \) is the fiber product in the following diagram:

\[
\begin{array}{ccc}
 \tilde{\Sigma}_1 & \longrightarrow & \tilde{\Sigma}_2 \\
 f_1 & \\ \\
 \Sigma_1 & \longrightarrow & \Sigma_2 \\
 \end{array}
\]

(4.2)

then the pullback \( f_1 \) is a \( k \)-fold covering over the simplicial complex \( \Sigma_1 \).

**Proof.** Let \( \Delta_n \) be any \( n \)-simplex contained in \( \Sigma_1 \). Since \( g : \Sigma_1 \rightarrow \Sigma_2 \) is a simplicial function, \( g(\Delta_n) \) is a simplex in \( \Sigma_2 \). Moreover, for every \( \Delta_s \subset \Delta_n \), \( g(\text{int} \Delta_s) \) is
equal to the interior of the simplex $g(\Delta_s)$. The linearity of $g$ in $\Delta_s$ implies that $g$ is transverse to the branch set of $f|_{f^{-1}(\text{int} g(\Delta_s))} : f^{-1}(\text{int} g(\Delta_s)) \to \text{int} g(\Delta_s)$ and by Proposition 3.5 the restriction $f_1|_{f^{-1}(\text{int} \Delta_s)} : f^{-1}(\text{int} \Delta_s) \to \text{int} \Delta_s$ is a $k$-fold branched covering.

Now, consider the slice : $\Delta_{s-1} \times I \subset \Delta_s$, for every $\Delta_{s-1} \subset \Delta_s$. The linearity of $g$ in $\Delta_n$ implies that $g(\Delta_{s-1}) \times I$ is also a slice inside $g(\Delta_s)$, and for some embedding $f_1^{-1}(\Delta_{s-1}) \times I \to f_1^{-1}(\Delta_s)$ the following diagram commutes:

\[
\begin{array}{c}
\Delta_{s-1} \times I \\
\downarrow \text{slice} \\
\Delta_s
\end{array}
\begin{array}{c}
\xrightarrow{f_1^{-1}(\Delta_{s-1}) \times id} \\
\downarrow f_1 \\
\xrightarrow{f_1}
\end{array}
\begin{array}{c}
f_1^{-1}(\Delta_s) \\
\end{array}
\]

Therefore $f_1 : \Sigma_1 \to \Sigma_1$ is a $k$-fold branched covering of the simplex $\Delta_n$.

Since we are working with simplicial complexes we need to modify Definition 3.6 for simplicial complexes.

**Definition 4.3.**

(a) Let $\Sigma$ be a simplicial complex, $\widetilde{\Sigma}_1, \widetilde{\Sigma}_2$ be topological spaces and let $f_1 : \widetilde{\Sigma}_1 \to \Sigma, f_2 : \widetilde{\Sigma}_2 \to \Sigma$ be branched coverings. We say that $f_1$ and $f_2$ are equivalent up to homeomorphism if there is a homeomorphism $h : \widetilde{\Sigma}_1 \to \widetilde{\Sigma}_2$ such that $f_2 \circ h = f_1$.

(b) Two $k$-fold branched coverings $f_1, f_2$ over a simplicial complex $\Sigma$ are concordant if there is a branched covering $F : W \to \Sigma \times I$ over the simplicial
complex $\Sigma \times I$ such that $F|_{F^{-1}(\Sigma \times \{j\})}$ is equivalent up to homeomorphism with $f_{j+1}$, for $j = 0, 1$.

We are now in a position to prove the following lemmas:

**Lemma 4.4.** Let $\Sigma_1, \Sigma_2$ be simplicial complexes. Let $f_2 : \tilde{\Sigma}_2 \rightarrow \Sigma_2$, $f'_2 : \tilde{\Sigma}'_2 \rightarrow \Sigma_2$ be two concordant branched coverings over $\Sigma_2$, and let $g : \Sigma_1 \rightarrow \Sigma_2$ be a simplicial function. Then the pullbacks $g^* f_2$ and $g^* f'_2$ given in the diagrams below are concordant.

\[
\begin{array}{ccc}
\tilde{\Sigma}_1 & \rightarrow & \tilde{\Sigma}_2 \\
\downarrow g^* f_2 & & \downarrow f_2 \\
\Sigma_1 & \rightarrow & \Sigma_2
\end{array}
\]

\[(4.4)\]

\[
\begin{array}{ccc}
\tilde{\Sigma}'_1 & \rightarrow & \tilde{\Sigma}'_2 \\
\downarrow g^* f'_2 & & \downarrow f'_2 \\
\Sigma_1 & \rightarrow & \Sigma_2
\end{array}
\]

\[(4.5)\]

**Proof.** Since $f_2$ and $f'_2$ are concordant, there is a function $F : W \rightarrow \Sigma \times I$ that makes them concordant.

Let $W^*$ be the fiber product in the following diagram:

\[
\begin{array}{ccc}
W^* & \rightarrow & W \\
\downarrow F^* & & \downarrow F \\
\Sigma_1 \times I & \rightarrow & \Sigma_2 \times I
\end{array}
\]

\[(4.6)\]

Hence $g^* f_2$ and $g^* f'_2$ are concordant. \qed
Lemma 4.5. Let $\Sigma_1, \Sigma_2$ be simplicial complexes. Let $g_1 : \Sigma_1 \to \Sigma_2$, $g_2 : \Sigma_1 \to \Sigma_2$ be two homotopic simplicial functions. And let $f_2 : \tilde{\Sigma}_2 \to \Sigma_2$ be a branched covering over the simplicial complex $\Sigma_2$. Then the branched coverings $g_1^*f_2$ and $g_2^*f_2$ are concordant.

**Proof.** Since $g_1$, and $g_2$ are homotopic, there is a continuous function $H : \Sigma_1 \times I \to \Sigma_2$ such that $H|_{\Sigma_1 \times \{0\}} = g_1$ and $H|_{\Sigma_1 \times \{1\}} = g_2$.

Define $W$ as the fiber product in the following diagram:

$$
\begin{CD}
W @>>> \tilde{\Sigma}_2 \\
@V F^* VV @VV f_2 V \\
\Sigma_1 \times I @>>> \Sigma_2
\end{CD}
$$

(4.7)

then $F^*$ produces the desired concordance between $g_1^*f_2$ and $g_2^*f_2$. ■

**Notation 4.6.** Let $\Sigma$ a simplicial complex. Let us denote by $B_k(\Sigma)$ the set of all concordance classes of $k$-fold branched coverings over the simplicial complex $\Sigma$.

**Proposition 4.7.** $B_k(\bullet)$ is a contravariant functor from the category of simplicial complexes to the category of sets.

**Proof.** It comes straightforward from Proposition 4.2 and Lemmas 4.4 and 4.5. ■

The following definition is similar to Definition 3.8.

**Definition 4.8.**

(a) Let $\tilde{\Sigma}$ be a topological space, $\Sigma$ be a simplicial complex. A pointed branched covering over the simplicial complex $\Sigma$ is a $k$-fold branched covering $f : \tilde{\Sigma} \to \Sigma$.
together with a base point \( * \in (\Sigma - K) \) (where \( K \) is the branch set of \( f \)) and a one-to-one correspondence of \( \{1, 2, \ldots, k\} \) with the set \( f^{-1}(\cdot) \). This correspondence is denoted by \( c(f_i) \) where \( 1 \leq i \leq k \).

(b) Two pointed branched coverings \( f_1 : \Sigma_1 \to \Sigma \) and \( f_2 : \Sigma_2 \to \Sigma \) over the simplicial complex \( \Sigma \) are equivalent up to homeomorphism if there exists a homeomorphism \( h : \Sigma_1 \to \Sigma_2 \) such that \( f_2 \circ h = f_1 \) and \( h \) preserves the labeling, that is, \( h(c(f_1)_i) = c(f_2)_i \).

(c) Two pointed \( k \)-fold branched coverings over a simplicial complex \( \Sigma \) are concordant if there is a branched covering concordance \( F : \overset{\cdot}{W} \to \Sigma \times I \) such that \( * \times I \) does not intersect the branch set of \( F \) and \( c(f_1)_i, c(f_2)_i \) are in the same components of \( F^{-1}(\cdot \times I) \) when \( F|_{F^{-1}(\Sigma \times \{0\})} \) and \( F|_{F^{-1}(\Sigma \times \{1\})} \) are equivalent to \( f_1 \) and \( f_2 \), respectively.

As before, \( B_k(\Sigma, \cdot) \) denotes the set of all concordance classes of pointed \( k \)-fold branched coverings over the simplicial complex \( \Sigma \).

It is not difficult to see that Proposition 4.2, Lemmas 4.4, and 4.5 extend for pointed branched coverings over simplicial complexes. Finally, using these extensions the following proposition is straightforward. Really, it is just the extension of Proposition 4.7.

**Proposition 4.9.** \( B_k(\cdot, \cdot) \) is a contravariant functor from the category of pointed simplicial complexes to the category of sets with distinguished element the constant function.
So far, we have proved the existence of the functor $B_k(\bullet,\ast)$. Now, we will proceed to verify Conditions A, B, C, and D, of Theorem 2.3. Actually, the extension of Lemma 4.5 for the case of pointed branched coverings proves that the functor $B_k(\bullet,\ast)$ verifies Condition A. The first part of Condition B is straightforward. The following proposition will state the second part of Condition B.

**Proposition 4.10.** Let $(\Sigma,\Sigma_1,\Sigma_2)$ be a proper triad (see Definition 2.1) in the category of pointed simplicial complexes $C$. Let $A = \Sigma_1 \cap \Sigma_2$ and $j_i : A \to \Sigma_i$, $k_i : \Sigma_i \to \Sigma$ be the canonical inclusions. If $f_1 \in B_k(\Sigma_1,\ast)$ and $f_2 \in B_k(\Sigma_2,\ast)$ are such that the pullbacks $j_i^*f_1$ and $j_i^*f_2$ are concordant, then there is $f \in B_k(\Sigma,\ast)$ such that $k_i^*f$ is concordant to $f_i$, $\forall i, i = 1, 2$. Furthermore, if $A$ is a point then $f$ is unique.

**Proof.** First of all, let us take subdivisions in $A, \Sigma_1, \Sigma_2$ such that every simplex in $A$ is a simplex in $\Sigma_1$ and in $\Sigma_2$.

Since $j_1^*f_1$ and $j_2^*f_2$ are concordant over $A$ then there exists $F : W \to A \times I$ such that $F|_{F^{-1}(A \times \{0\})}$ is equivalent up to homeomorphism to $j_1^*f_1|(j_1^*f_1)^{-1}(A)$ and $F|_{F^{-1}(A \times \{1\})}$ is equivalent up to homeomorphism to $j_2^*f_2|(j_2^*f_2)^{-1}(A)$. Without loss of generality, we can assume that $F|_{F^{-1}(A \times \{0\})}$ and $F|_{F^{-1}(A \times \{1\})}$ are exactly the functions $j_1^*f_1|(j_1^*f_1)^{-1}(A)$ and $j_2^*f_2|(j_2^*f_2)^{-1}(A)$, respectively.

Consider the space $((A \times I) \cup \Sigma_1)/\sim$ (where $a \sim (a,0), \forall a \in A$). Call this space $X_1$. Let $id_{X_1} : X_1 \to X_1$ be the identity function. Since $(\Sigma_1,A)$ has the homotopy extension property, there exists a homotopy $H : \Sigma_1 \times I \to X_1$ such that
$H|_{X_1} = id_{X_1}$.

If $id_i : \Sigma_i \to \Sigma_i$ is the identity function, then $id_i^* f_i$ coincides with $j_i^* f_i$ in $(j_i^* f_i)^{-1}(A)$, for $i = 1, 2$. For $i = 1, 2$, let us denote $(j_i^* f_i)^{-1}(A)$ and $(id_i^* f_i)^{-1}(\Sigma_i)$ by $\tilde{A}_i$ and $\tilde{\Sigma}_i$, respectively.

Now, consider the function $G : (W \cup \tilde{\Sigma}_1) \to X_1$ defined by $G(x) = F(x)$ if $x \in W$ and $G(x) = (id_1^* f_1)(x)$ if $x \in \tilde{\Sigma}_1$. It is well defined because $F$ coincides with $id_1^* f_1$ in $\tilde{A}_1$.

Let $H^* G$ be the pullback in the following diagram:

$$(H^* G)^{-1}(\Sigma_1 \times I) \rightarrow W \cup \tilde{\Sigma}_1$$

$H^* G \downarrow$

$\Sigma_1 \times I \rightarrow X_1$ $G$ $H$

Then, $H^* G|_{(H^* G)^{-1}(A \times \{0\})}$ is equivalent up to homeomorphism to $j_1^* f_1$ and $H^* G|_{(H^* G)^{-1}(A \times \{1\})}$ is equivalent up to homeomorphism to $j_2^* f_2$.

Construct the following space: $X = ((H^* G)^{-1}(\Sigma_1 \times \{1\}) \cup \tilde{\Sigma}_2)/\sim$ where we use the homeomorphism that makes $H^* G|_{(H^* G)^{-1}(A \times \{1\})}$ equivalent to $j_2^* f_2$ to identify the elements in $f_2^{-1}(A)$ with the elements in $(H^* G)^{-1}(A \times \{1\})$. Now let us define $f : X \to ((\Sigma_1 \times \{1\}) \cup \Sigma_2)/\sim$ (where $a \sim (a, 1)$ for every $a \in A$) by $f(x) = H^* G(x)$ if $x \in (H^* G)^{-1}(\Sigma_1 \times \{1\})$ and $f(x) = id_2^* f_2(x)$ if $x \in \tilde{\Sigma}_2$. Clearly $f$ is a well defined branched covering over $((\Sigma_1 \times \{1\}) \cup \Sigma_2)/\sim$.

Since $((\Sigma_1 \times \{1\}) \cup \Sigma_2)/\sim$ is isomorphic to $\Sigma = \Sigma_1 \cup \Sigma_2$ then we can say
that \( f \in B_k(\Sigma, \ast) \). Moreover, the construction of \( f \) implies that \( k^*_i f \) is concordant to \( f_i \), \( (i = 1, 2) \). Hence \( f \) is the desired function.

It is also straightforward that if \( A \) reduces to a single point the construction of \( f \) is unique up to concordance.

The next proposition will verify Condition D for the functor \( B_k(\ast, \ast) \). Condition C will be verified in Proposition 4.12.

**Proposition 4.11.** Suppose \( \Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_n \subset \cdots \) is a collection of simplicial subcomplexes of \( \Sigma = \bigcup_n \Sigma_n \in \mathcal{C} \) with respect to some simplicial-complex structure on \( \Sigma \) such that \( \Sigma^n_n = \Sigma^n \) (where \( X^n \) is the \( n \)-skeleton of \( X \)). Let \( i_n : \Sigma_n \to \Sigma \) be the inclusion map. Let \( \lim B_k(\Sigma_n, \ast) \) be the inverse limit of \( B_k(\Sigma_n, \ast) \) with respect to the maps induced by the inclusions \( \psi^*_m \) of \( \Sigma_n \) into \( \Sigma_m \).

Then \( \lim(i^*_n) : B(\Sigma, \ast) \longrightarrow \lim B(\Sigma_n, \ast) \) is an epimorphism.

**Proof.** First of all, recall that the inverse limit of \( B_k(\Sigma_n, \ast) \) is:

\[
\lim B(\Sigma_n, \ast) = \{ (f_n) \in \prod_n B_k(\Sigma_n, \ast) \mid f_n = (\psi^*_m)^* f_m, m \leq n \}
\]

(see [8]). Let \( (f_n) \in \lim B_k(\Sigma_n, \ast) \); we want to find \( f \in B_k(\Sigma, \ast) \) such that \( \lim(i^*_n)(f) = (f_n) \). Let us construct \( f \) inductively.

Notice that \( f_1 = (\psi^*_1)^* f_2 \), in other words, \( f_1 \) and the restriction \( f_2|_{\Sigma_1} \) are concordant, i.e., we have a function \( F : W \to \Sigma_1 \times I \) such that \( F|_{F^{-1}(\Sigma_1 \times \{0\})} \) is equivalent up to homeomorphism to \( f_2|_{\Sigma_1} \) and \( F|_{F^{-1}(\Sigma_1 \times \{1\})} \) is equivalent up to homeomorphism to \( f_1 \).
Using the same techniques developed in the proof of Proposition 4.10, we construct a function \( f'_2 \in B_k(\Sigma_2, \ast) \) such that \( f'_2|_{(f'_2)^{-1}(\Sigma_1)} = f_1 \).

Assume we have \( f'_{n-1} \in B_k(\Sigma_{n-1}, \ast) \) such that \( f'_{n-1}|_{(f'_{n-1})^{-1}(\Sigma_{n-2})} = f'_{n-2} \).

A similar construction will provide \( f'_n \in B_k(\Sigma_n, \ast) \) such that \( f'_n|_{(f'_n)^{-1}(\Sigma_{n-1})} = f'_{n-1} \).

Hence we have a collection of functions \( f_1 = f'_1, f'_2, f'_3, \ldots \) such that \( f'_n \) is an extension of \( f'_{n-1}, \forall n \in \mathbb{N} \).

Let \( \tilde{\Sigma} = \bigcup_n \tilde{\Sigma}_n \) where \( \tilde{\Sigma}_n = (f'_n)^{-1}(\Sigma_n) \) and define \( f : \tilde{\Sigma} \to \Sigma \) by \( f(x) = f'_n(x) \) if \( x \in (f'_n)^{-1}(\Sigma_n) \). It is obvious that \( f \) is well defined. Hence \( f \in B_k(\Sigma, \ast) \).

Because of the construction of \( f'_n \), we have that \( f'_n \) is concordant to \( f_n \). Therefore, \( \lim(i^*_n)(f) = (f_n) \) which implies that \( \lim(i^*_n) : B(\Sigma, \ast) \to \lim B(\Sigma_n, \ast) \) is an epimorphism. \( \blacksquare \)

In the following proposition we regard the boundary of an \((n + 1)\)-simplex as an \( n \)-sphere.

**Proposition 4.12.** Suppose \( \{S^n_\alpha\} \) is a collection of disjoint \( n \)-spheres whose wedge product \( \bigvee S^n_\alpha \) is in the category of simplicial complexes \( C \). Let \( i_\beta : S^n_\beta \to \bigvee S^n_\alpha \) be the inclusion map.

Then the function \( \prod i^*_\beta : B_k(\bigvee S^n_\alpha, \ast) \to \prod B_k(S^n_\alpha, \ast) \) is bijective.

**Proof.** Let \( (f_\alpha) \in \prod B_k(S^n_\alpha, \ast) \), where \( f_\alpha \in B(S^n_\alpha, \ast) \). Recall, that for every \( f_\alpha, \ast \) is not a branched point and there is also a one-to-one correspondence of \( \{1, 2, \ldots, k\} \) with \( f^{-1}_\alpha(*) \). This correspondence is denoted by \( c(f_\alpha)_i \), for every \( \alpha \).

Let \( X = (\bigcup_\alpha f^{-1}_\alpha(S^n_\alpha))_{/\sim} \), where \( x_\alpha \sim x_\beta \) if \( f_\alpha(x_\alpha) = f_\beta(x_\beta) = \ast \), and
the corresponding $i \in \{1, 2, \ldots, k\}$ under $c(\alpha)_i$ and $c(\beta)_i$ is the same for both elements.

Now, let us define $f : X \to \vee S^a_n$ by $f(x) = f_\alpha(x)$ if $x \in f_\alpha^{-1}(S^a_n)$. It is clear that $f$ is a well defined branched covering in $B_k(\vee S^a_n, \ast)$. Moreover, $\prod i^*_\beta(f) = (f_\alpha)$. Therefore the function $\prod i^*_\beta$ is surjective.

Let us prove the injectivity. Let $f, g \in B_k(\vee S^a_n, \ast)$ and assume that $\prod i^*_\beta(f) = (f_\alpha)$ and $\prod i^*_\beta(g) = (g_\alpha)$, in other words, $i^*_\alpha f = f_\alpha$ and $i^*_\alpha g = g_\alpha$, for every $\alpha$. $(f_\alpha) = (g_\alpha)$ means that $f_\alpha$ is concordant to $g_\alpha$ for each $\alpha$. So, there exists a branched covering $F_\alpha : W_\alpha \to S^a_n \times I$ such that $F_\alpha|_{F_\alpha^{-1}(S^a_n \times \{1\})}$ is equivalent up to homeomorphism to $f_\alpha$ and $F_\alpha|_{F_\alpha^{-1}(S^a_n \times \{0\})}$ is equivalent up to homeomorphism to $g_\alpha$. Recall that $\ast \times I$ does not intersect the branch set. Therefore $F_\alpha^{-1}(\ast \times I)$ contains exactly $k$ copies of $I$ and $c(f_\alpha)_i$ belongs to the same component of $F_\alpha^{-1}(\ast \times I)$ that contains $c(g_\alpha)_i$. Let us denote by $F^{-1}(\ast \times I)_i$ the copy of $I$ that contains $c(f_\alpha)_i$.

Without loss of generality, we could assume that if $\alpha \neq \beta$ then $W_\alpha \neq W_\beta$.
Let $\tilde{X} = (\cup_\alpha W_\alpha)/\sim$, where $\sim$ is defined in the following way: $x \sim y$ if there are $\alpha, \beta$, and $i$ ($i = 1, \ldots, k$) such that $x \in F_\alpha^{-1}(\ast \times I)_i$, $y \in F_\beta^{-1}(\ast \times I)_i$, and $F_\alpha(x) = F_\beta(y)$. Consider $F : \tilde{X} \to \vee S^a \times I$ defined by $F(x) = F_\alpha(x)$ if $x \in W_\alpha$. Clearly $F$ is a well defined function such that $F|_{F^{-1}(\vee S^a_n \times \{1\})}$ is equivalent up to homeomorphism to $f$ and $F|_{F^{-1}(\vee S^a_n \times \{0\})}$ is equivalent up to homeomorphism to $g$. Hence $f$ is concordant to $g$, i.e., $\prod i^*_\beta$ is injective. ■
Recall that every CW-complex is homotopically equivalent to a simplicial complex, i.e., if $X$ is a CW-complex there exists a simplicial complex $\Sigma$ and maps $h : X \to \Sigma$, $h' : \Sigma \to X$ such that $h \circ h'$ is homotopic to the identity of $\Sigma$ and $h' \circ h$ is homotopic to the identity of $X$.

We shall say that a function $f : \tilde{X} \to X$ is a $k$-fold branched covering over the CW-complex $X$, if the function $(h')^*f : \tilde{\Sigma} \to \Sigma$ is a $k$-fold branched covering over the simplicial complex $\Sigma$. Keeping the same spirit, let us extend the Definition 4.8 for CW-complexes. In the same way, let $B_k(X, \ast)$ be the set of all concordance classes of $k$-fold branched coverings over the CW-complex $X$. Similarly, $B_k(\bullet, \ast)$ represents a contravariant functor from the category of based CW-complexes to the category of sets with a distinguished element. Then Propositions 4.2, 4.7, 4.9, 4.10, 4.11, 4.12, and Lemmas 4.4, 4.5 also hold for the functor $B_k(\bullet, \ast)$ in the category of CW-complexes.

So far, we have proved that if $C$ is the category of CW-complexes (not necessarily finite) the functor $B_k(\bullet, \ast)$ satisfies Conditions A, B, C, D of Brown's Representability Theorem (Theorem 2.3). Hence the following theorem is true.

**Theorem 4.13.** Let $C$ be the category of spaces with base point $\ast$ which has as objects all topological spaces admitting a CW-complex structure. Let $S$ be the category of sets with a distinguished element. Let $B_k(\bullet, \ast) : C \to S$ be the functor defined above. Then there is a space $Y$ in $C$, unique up to homotopy type such that the functors $[\bullet, Y]$ and $B_k(\bullet, \ast)$ are naturally equivalent.
Proof. In fact, Proposition 4.9 says that the functor $B_k(\bullet, \ast)$ is a contravariant functor. Moreover, the functor $B_k(\bullet, \ast)$ verifies all the conditions of Theorem 2.3. Hence we obtain the existence of a classifying space $Y$ for $k$-fold branched coverings over CW-complexes. ■
5. Chapter References


CHAPTER 3

CONSTRUCTION OF CLASSIFYING SPACES

1. Introduction

A branched covering with branch set a stratified set of type $l$ is called a branched covering of type $l$ ($l \in \mathbb{N}$) (see Definition 3.4, Chapter 2). The construction of classifying spaces for branched coverings of type 2 was done by Brand in [1] and [2]. If $k \in \mathbb{N}$, let us denote by $(E(2), BR_k(2), \gamma(2))$ the universal $k$-fold branched covering of type 2, i.e., if $\tilde{M}$, and $M$ are manifolds, and $f : \tilde{M} \to M$ is a $k$-fold branched covering of type 2, then there exists a continuous function $c : M \to BR_k(2)$ such that the pullback $c^*\gamma(2)$ is concordant to $f$. Furthermore, Brand showed (see [1] p.235) that we can find $c : M \to BR_k(2)$ such that $c^*\gamma(2)$ is equivalent to $f$ up to homeomorphism.

Let $\tilde{M}$, and $M$ be smooth manifolds, let $k$ be a natural number, and let $f : \tilde{M} \to M$ be a $k$-fold branched covering of type $l$ with branch set $K$. Recall that $K$ is a stratified set of type $l$ for which the different pieces or strata associated to it are the closed sets $K_2, K_3, \ldots, K_{l-1}, K_l$ (see Definitions 3.1 and 3.4 of Chapter 2).

Let us assume that $f^{-1}(K)$ is also a stratified set of type $l$ such that $f^{-1}(K_l) \subset f^{-1}(K_{l-1}) \subset \cdots \subset f^{-1}(K_3) \subset f^{-1}(K_2) = f^{-1}(K)$ where $f^{-1}(K_l)$ is a submanifold in $\tilde{M}$ of codimension $l$, $(f^{-1}(K_{j-1}) - f^{-1}(K_j))$ is a submanifold in $\tilde{M}$ of codimension $j - 1$, and $f^{-1}(K_{j-1}) - f^{-1}(K_j) = f^{-1}(K_{j-1})$.  

33
Notation. Let us denote \( f^{-1}(K_l) \) by \( \tilde{K}_l \).

In this chapter we give an inductive construction for \( (E(l), BR_k(l), \gamma(l)) \) the universal \( k \)-fold branched covering of type \( l \). We consider three cases. The level of difficulty depends on the local condition that is different for each case. In Section 2, we consider the simplest case. Section 3 is devoted to the study of the more complicated cases. For simplicity, we give only the complete proof for the case considered in Section 2. In Section 4, we prove the commutativity of some diagrams that are used in Section 2.

2. The Simplest Case

Here, we are going to consider functions \( f \) that are \( k \)-fold branched coverings of type \( l \) \((l, k \in \mathbb{N})\) which satisfy the following conditions:

a1) \( f|_{\tilde{K}_l} : \tilde{K}_l \to K_l \) is an \( r \)-fold covering, where \( r \in \mathbb{N} \).

b1) Local condition: Let \( \tilde{\eta}_l \) and \( \eta_l \) be the disk normal bundles of \( \tilde{K}_l \) and \( K_l \), respectively. Hence, their fiber is the closed disk \( D_l \) with radius 1. If \( T\tilde{K}_l, TK_l \) are the closure of the open tubular neighbourhoods of \( \tilde{K}_l \) and \( K_l \), respectively, we know (see [4] p.76 and [5] page 115) that the total space \( E_{\tilde{\eta}_l} \) of \( \tilde{\eta}_l \) is diffeomorphic to \( T\tilde{K}_l \). In the same way: \( E_{\eta_l} \cong TK_l \). Furthermore, \( T\tilde{K}_l = \bigcup_{y \in \tilde{K}_l} D^l_{\epsilon(y)}(y) \) where \( D^l_{\epsilon(y)}(y) \) is a closed disk of radius \( \epsilon(y) \) centered at \( y \), where \( \epsilon \) is a smooth positive function on \( \tilde{K}_l \). Similarly \( TK_l = \bigcup_{x \in K_l} D^l_{\epsilon(x)}(x) \).

For any \( x \in K_l \) and any \( y \in f^{-1}(x) \subset \tilde{K}_l \) we are going to assume that
\[ f(D_{e(y)}^l(y)) = D_{e(x)}^l(x), \]
and that there are coordinate systems \( \phi_x : D_{e(x)}^l(x) \to D^l \), \( \phi_y : D_{e(y)}^l(y) \to D^l \), where \( D^l \) is the closed disk of radius 1 centered at the origin in \( \mathbb{R}^l \) and \( \phi_x(x) = 0, \phi_y(y) = 0 \). Moreover we assume that the following diagram is commutative:

\[
\begin{array}{ccc}
D_{e(y)}^l(y) & \overset{f}{\longrightarrow} & D_{e(x)}^l(x) \\
\phi_y \downarrow & & \downarrow \phi_x \\
D^l & \overset{f_0}{\longrightarrow} & D^l
\end{array}
\]

where the function \( f_0 : D^l \to D^l \) is defined in the following way:

Let \( S^{l-1} \) be the \((l-1)\)-sphere of radius 1. In order to define \( f_0 \), we choose \( g : S^{l-1} \to S^{l-1} \) a \( k \)-fold branched covering of type \( l-1 \), and for any \( z \in D^l \) let

\[
f_0(z) = |z|^p g\left( \frac{z}{|z|} \right), \quad \text{for some } p \geq 2.
\]

In this way, \( f_0 : D^l \to D^l \) is a \( k \)-fold branched covering of type \( l \) for which the branch submanifold of codimension \( l \) reduces to the center 0 of the disk \( D^l \).

\textbf{Remark.} Roughly speaking, we want the function \( f \) to be "equal to" \( f_0 \) for each fiber.

Thanks to S.M. Gersten (see [3]) it is well known how to construct \( k \)-fold branched coverings of type 2 from \( S^2 \) to \( S^2 \). Unfortunately, however, we do not know a simple way for constructing \( k \)-fold branched coverings from \( S^{l-1} \) to \( S^{l-1} \) of type \( l-1 \) when \( l \) is greater than 4.
c1) Let $\tilde{G}$, and $G$ be the groups of coordinate transformations of the normal bundles $\tilde{\eta}_i, \eta_i$, respectively. There exists a homomorphism $\mu : \tilde{G} \to G$ of groups such that the diagram:

$$
\begin{array}{ccc}
D^l & \xrightarrow{\tilde{g}} & D^l \\
\downarrow f_o & & \downarrow f_e \\
D^l & \xrightarrow{\mu(\tilde{g})} & D^l
\end{array}
$$

(2.2)

commutes, for every $\tilde{g} \in \tilde{G}$.

Condition a1 implies the existence of a classifying function $r : K_i \to B_{S_r}$ such that $f|\tilde{K}_i$ is the pullback in the next diagram:

$$
\begin{array}{ccc}
\tilde{K}_i & \xrightarrow{\tilde{r}} & E_{S_r} \\
f|\tilde{K}_i & \downarrow & \downarrow \gamma_{S_r} \\
K_i & \xrightarrow{r} & B_{S_r}
\end{array}
$$

(2.3)

where $S_r$ is the symmetric group on $r$ elements and $(E_{S_r}, B_{S_r}, \gamma_{S_r})$ is the universal $r$-fold covering.

Let $(E_G, B_G, p_G), (E_{\tilde{G}}, B_{\tilde{G}}, p_{\tilde{G}})$ be the universal $G$, and $\tilde{G}$-bundles, respectively (see [5] p.52). Let us denote $f|\tilde{K}_i$ and $f|E_{\eta_i}$ by $F, \tilde{F}$, respectively. If $\tilde{e} : \tilde{K}_i \to B_{\tilde{G}}$ and $e : K_i \to B_G$ are the classifying functions of the bundles $\tilde{\eta}_i, \eta_i$, then the following diagram commutes:
where $BF(< \tilde{g}, t > G) = < \mu(\tilde{g}), t > G$ (see (** Section 4), and $B\tilde{F}(< \tilde{g}, t >) = < \mu(\tilde{g}), t >$, for every $< \tilde{g}, t > G \in E_{\tilde{G}}$ and $< \tilde{g}, t > \in B_{\tilde{G}}$ (see (*) in Section 4).

The commutativity of this diagram is proved in Section 4 (see Diagram 4.10).

Because of the action of $\tilde{G}$ and $G$ over $D^l$ we can change the fiber of $(E_G, B_G, p_G), (E_{\tilde{G}}, B_{\tilde{G}}, p_{\tilde{G}})$ to $D^l$. This change produces the following commutative diagram (see Diagram (4.9)):
where \((E_G)_{D^l}\) and \((E_{\overline{G}})_{D^l}\) denote the total spaces with fiber \(D^l\), and we have:

\[
(B\tilde{F})_{D^l}([\tilde{e}, y]) = [B\tilde{F}(\tilde{e}), f_0(y)], \quad \forall \tilde{e} \in E_{\overline{G}}, \forall y \in D^l \quad \text{(see (***) in Section 4)}.
\]

(for more details about the Diagrams 2.4 and 2.5, and the definitions of functions \(BF, B\tilde{F}, (B\tilde{F})_{D^l}, h_{D^l}\), and \(\tilde{h}_{D^l}\) see Section 4). The definition of the function \((B\tilde{F})_{D^l}\) implies that if we restrict it to the fiber \(D^l\) we obtain exactly the function \(f_0 : D^l \rightarrow D^l\) that was fixed in the local Condition b1.

Combining Diagrams 2.3 and 2.5, we obtain a new commutative diagram:

\[
\begin{array}{ccc}
E_{\eta_l} & \xrightarrow{E_{\eta_l}} & E_{S_{\tau}} \times (E_{\overline{G}})_{D^l} \\
\downarrow \gamma_{S_{\tau} \times (B\tilde{F})_{D^l}} & & \downarrow \text{id} \times (p_{G_{\overline{G}}})_{D^l} \\
K_l & \xrightarrow{\gamma_{S_{\tau} \times BF}} & E_{S_{\tau}} \times B_{\overline{G}} \\
\end{array}
\]

where the function \(E_{\eta_l} \rightarrow E_{S_{\tau}} \times (E_{\overline{G}})_{D^l}\) is defined by:

\[(x, y) \mapsto (\tau(x), \tilde{h}_{D^l}(x, y)), \quad \forall x \in K_l, y \in D^l.\]

Recall that we are taking \(D^l\) as fiber in \(\eta_l\). It is routine to show that the previous diagram is commutative. Notice that \((B\tilde{F})_{D^l}\) restricted to the \((l-1)\)-sphere \(S^{l-1}\)
is a \( k \)-fold branched covering of type \( l - 1 \) thus:

\[
\gamma_{S_r} \times (B\hat{F})_{S^{l-1}} : E_{S_r} \times (E_G)_{S^{l-1}} \to B_{S_r} \times (E_G)_{S^{l-1}}
\]

is also a \( k \)-fold branched covering of type \( l - 1 \).

If \(((BR_k(l - 1), E(l - 1), \gamma(l - 1))\) is the universal \( k \)-fold branched covering of type \( l - 1 \), the function \( \gamma_{S_r} \times (B\hat{F})_{S^{l-1}} \) is the pullback in the following diagram:

\[
\begin{array}{ccc}
E_{S_r} \times (E_G)_{S^{l-1}} & \xrightarrow{\bar{c}} & E(l - 1) \\
\downarrow\gamma_{S_r} \times (B\hat{F})_{S^{l-1}} & & \downarrow\gamma(l - 1) \\
B_{S_r} \times (E_G)_{S^{l-1}} & \xrightarrow{c} & BR_k(l - 1)
\end{array}
\]

Let us define:

\[
BR_k(l) = (B_{S_r} \times (E_G)_{D^{l'}}) \cup_c BR_k(l - 1),
\]

\[
E(l) = (E_{S_r} \times (E_G)_{D^{l'}}) \cup_{\bar{c}} E(l - 1), \quad \text{and}
\]

\[
\gamma(l) = (\gamma_{S_r} \times (B\hat{F})_{D^{l'}}) \cup_{\bar{c}} \gamma(l - 1),
\]

where \( BR_k(l) = (B_{S_r} \times (E_G)_{D^{l'}}) \cup_c BR_k(l - 1) = ((B_{S_r} \times (E_G)_{D^{l'}}) \cup BR_k(l - 1)) \, /\sim \) and \( x \sim c(x) \) for every \( x \in (E_G)_{S^{l-1}} \). Similarly, using identifications with the function \( \bar{c} \) we define \( E(l) \) and \( \gamma(l) \).

**Theorem 2.1.** The triple \((E(l), BR_k(l), \gamma(l))\) (defined above) is the universal \( k \)-fold branched covering for functions that verify the Restrictions a1, b1, and c1.
Proof. In fact, let $M$ be a manifold, and let $g : M \to BR_k(l)$ be a smooth function. Deforming $g$ slightly we can make $g$ transverse to the branch set of the next restriction function:

$$\gamma(l)|_{(E_s \times (E_G)_{D^l})} = (\gamma_s \times (B \hat{F})_{D^l}) : (E_s \times (E_G)_{D^l}) \to (B_s \times (E_G)_{D^l})$$

Since $(B \hat{F})_{D^l}$ restricted to the fiber $D^l$ is just $f_0$ a $k$-fold branched covering of type $l$, then it is clear that $(\gamma_s \times (B \hat{F})_{D^l})$ is also a $k$-fold branched covering of type $l$, let us say that $R$ is its branch set. Thus $R$ is a stratified set of type $l$ in $(B_s \times (E_G)_{D^l})$. Moreover if $R_l$ is the submanifold of codimension $l$ related to the stratified set $R$, it is clear that $R_l = (B_s \times B_G)$ where $B_G$ is the space of zero vectors of $(E_G)_{D^l}$.

Pulling back, we see that $g^{-1}(R_l) = g^{-1}(B_s \times B_G)$ is a submanifold of $M$ of codimension $l$ (call it $K_l$), and the function:

$$h_{K_l} = (g^*(\gamma_s \times (B \hat{F})_{D^l}))|_{(g^*(\gamma_s \times (B \hat{F})_{D^l}))^{-1}(K_l)}$$

is thus an $r$-covering. Furthermore, $g^{-1}(B_s \times (E_G)_{D^l})$ gives us a tubular neighbourhood of $K_l$, let us call it $TK_l$, and by Proposition 3.5 of Chapter 2, the function

$$h_{TK_l} = (g^*(\gamma_s \times (B \hat{F})_{D^l}))|_{(g^*(\gamma_s \times (B \hat{F})_{D^l}))^{-1}(TK_l)}$$

is a $k$-fold branched covering over $TK_l$ of type 1.

On the other hand, since $g^{-1}(BR_k(l - 1)) = (M - g^{-1}(B_s \times (E_G)_{D^l})) = (M - TK_l)$ we obtain a $k$-fold branched covering of type $l - 1$ over $(M - TK_l)$, let
us call it: \( h_{(M - TK,)} \). The definition of \( BR_k(l) \) implies that \( h_{TK,} \) coincides with \( h_{(M - TK,)} \) on the inverse image \( g^{-1}(B_{S_r} \times (E_G)_{S_t - 1}) \). Hence, the pullback \( g^* \gamma(l) \) in the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{M} & \longrightarrow & E(l) \\
g^* \gamma(l) \downarrow & & \downarrow \gamma(l) \\
M & \longrightarrow & B(l)
\end{array}
\]

is a \( k \)-fold branched covering of type \( l \) over \( M \).

Now, let \( f : \tilde{M} \rightarrow M \) be a \( k \)-fold branched covering of type \( l \) that verifies the Conditions \( a_1, b_1, \) and \( c_1 \). We claim that there exists \( \overline{c}^l : M \rightarrow BR_k(l) \) that classifies \( f \). In fact, since \( f|_{(\tilde{M} - TK,)} \) is a \( k \)-fold branched covering of type \( l - 1 \) there is \( c_0 : (M - TK,) \rightarrow BR_k(l - 1) \) that classifies \( f|_{(\tilde{M} - TK,)} \). Let us define \( \overline{c}^l \) in the following way:

\[
\begin{align*}
\overline{c}^l(z) &= c_0(z) & \text{if } z \in (\tilde{M} - TK,), \\
\overline{c}^l(z) &= (r(x), h_{D^l(z)}) & \text{if } z \in D^l_{\epsilon(x)}(x), \text{ and } x \in K_l.
\end{align*}
\]

Where \( r \) is the function defined in (2.3). Recall that \( TK_l = \bigcup_{x \in K_l} D^l_{\epsilon(x)}(x) \).

Because of the definition of \( BR_k(l) \), \( \overline{c}^l \) is well defined. Furthermore, \( \overline{c}^l \) classifies \( f \), i.e., \( f \) is the pullback in the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{M} & \longrightarrow & E(l) \\
\overline{c}^* \gamma(l) = f \downarrow & & \downarrow \gamma(l) \\
M & \longrightarrow & BR_k(l)
\end{array}
\]
Hence, \((E(l), BR_k(l), \gamma(l))\) is the universal \(k\)-fold branched covering of type \(l\). 

**Remark.** Notice that \(f\) is not only concordant with the pullback but equals the pullback \(\overline{e^*\gamma(l)}\) up to homeomorphism. In fact, since the result is true for \(l = 2\) (see [1] p. 235), induction over \(l\) will prove the statement.

### 3. General Case

Here we are going to generalize the previous case. First of all, let us put the following restrictions over the \(k\)-fold branched covering \(f\) of type \(l\).

**a2) \(f|_{\tilde{K}_l} : \tilde{K}_l \to K_l\) is an \(r\)-fold covering for some \(r \in \mathbb{N}\).** Moreover, we assume that there is a partition \(\{\tilde{K}_{l,b}\}_{b \in B}\) of \(\tilde{K}_l\) such that for each \(b \in B\)

**b2) Local condition:** For each \(b \in B\), let \(\tilde{\eta}_{l,b}\) and \(\eta_l\) be the disk normal bundles of \(\tilde{K}_{l,b}\) and \(K_l\), respectively. If \(T\tilde{K}_{l,b}, TK_l\) are the closure of the open tubular neighbourhoods of \(\tilde{K}_{l,b}\) and \(K_l\), respectively, then the total space \(E_{\tilde{\eta}_{l,b}}\) of \(\tilde{\eta}_{l,b}\) is diffeomorphic to \(T\tilde{K}_{l,b} = \bigcup_{y \in \tilde{K}_l} D^i_{\epsilon(y)}(y)\). In the same way: \(E_{\eta_l} \cong TK_l = \bigcup_{x \in K_l} D^i_{\epsilon(x)}(x)\). Now, for any \(b \in B\), \(x \in K_l\) and \(y_b \in f^{-1}(x) \subset \tilde{K}_{l,b}\) we are going to assume that \(f(D^i_{\epsilon(y_b)}(y_b)) = D^i_{\epsilon(x)}(x)\), and that there are coordinate systems \(\phi_x : D^i_{\epsilon(x)}(x) \to D^i, \phi(y) : D^i_{\epsilon(y_b)}(y_b) \to D^i\) (where \(D^i\) is the closed disk of radius 1 centered at the origin in \(\mathbb{R}^l\), and \(\phi_x(x) = 0, \phi(y_b(y_b)) = 0\)) such that \(\phi_x \circ f|_{D^i_{\epsilon(y_b)}(y_b)} = f_{0,b} \circ \phi_y\) where \(f_{0,b} : D^i \to D^i\) is a fixed \(k_b\)-fold branched covering.
of type $l$ such that $f_{0,b}(z) = |z|^p g_b(z^p)$ where $p \geq 2$ and $g_b : S^{l-1} \to S^{l-1}$ is some $k_b$-fold branched covering of type $l - 1$. Let us fix the partition $\{k_b\}_{b \in B}$ of $k$.

c2) Let $\tilde{G}_b$, and $G$ be the groups of coordinate transformations of the normal bundles $\tilde{\eta}_{l,b}$, and $\eta_l$, respectively. For each $b \in B$ there exists a homomorphism $\mu_b : \tilde{G}_b \to G$ of groups such that the diagram:

\[
\begin{array}{ccc}
D^l & \xrightarrow{\tilde{g}_b} & D^l \\
\downarrow f_{0,b} & & \downarrow f_{0,b} \\
D^l & \xrightarrow{\mu(\tilde{g}_b)} & D^l
\end{array}
\]

(3.1) commutes, for every $\tilde{g}_b \in \tilde{G}_b$.

Similarly to the case studied in Section 2, for each $b \in B$ the function

\[
(\gamma_{S_{r_b}} \times (B\tilde{F}_b)_{D^l}) : (E_{S_{r_b}} \times (E_{\tilde{G}_b})_{D^l}) \to (B_{S_{r_b}} \times (E_{G})_{D^l}).
\]

is a $k_b$-fold branched covering of type $l$.

Now, let $X_{D^l} = \bigcup_{b \in B}(X_{r_b} \times (E_{\tilde{G}_b})_{D^l})$ where $X_{r_b} = \prod_{c \in B} Y_{r_c}$ is such that $Y_{r_c} = B_{S_{r_c}}$, for all $c$ different from $b$, and $Y_{r_b} = E_{S_{r_b}}$. Here, $\bigcup$ means disjoint union.

Let us define

\[
\Phi_{D^l} : X_{D^l} \to \left( \prod_{b \in B} B_{S_{r_b}} \times (E_{G_b})_{D^l} \right)
\]

in the following way: let $x \in X_{D^l}$, then there is $b \in B$ such that $x \in (X_{r_b} \times (E_{\tilde{G}_b})_{D^l})$, in other words, $x = ((x_c)_{c \in B}, e)$ where $(x_c)_{c \in B} \in X_{r_b}$ and $e \in (E_{\tilde{G}_b})_{D^l}$. Let
\( \Phi_{D'}(x) = ((y_c)_{c \in B}, (B_{\Phi_D})_{D'}(c)) \) with \( y_c = x_c \) if \( c \) is different from \( b \) and \( y_b = \gamma_{S_{r_b}}(x_b) \). Clearly \( \Phi_{D'} \) is a well defined function. Since \( (B_{\Phi_D})_{D'} \) is a \( k_b \)-fold branched covering of type \( l \), and \( \{k_b\}_{b \in B} \) is a partition of \( k \), then \( \Phi_{D'} \) is a \( k \)-fold branched covering of type \( l \). Furthermore, if \( X_{S^{l-1}} = \bigsqcup_{b \in B} (X_{r_b} \times (E_{G_b})_{S^{l-1}}) \) then \( \Phi_{S^{l-1}} = \Phi_{D'}|_{X_{S^{l-1}}} \) is a \( k \)-fold branched covering of type \( l - 1 \). Hence, there is a classifying function \( c \) such that \( c^* \gamma(l - 1) = \Phi_{S^{l-1}} \), i.e., the next diagram commutes:

\[
\begin{array}{ccc}
X_{S^{l-1}} & \xrightarrow{\overline{c}} & E(l - 1) \\
\Phi_{S^{l-1}} = c^* \gamma(l - 1) & & \gamma(l - 1) \\
\prod_{b \in B} B_{S_{r_b}} \times (E_{G_b})_{S^{l-1}} & \xrightarrow{c} & BR_k(l - 1)
\end{array}
\]

Similarly to the previous case we define:

\[
BR_k(l) = \left( \prod_{b \in B} B_{S_{r_b}} \times (E_{G_b})_{D'} \right) \cup c BR_k(l - 1),
\]

\[
E(l) = (X_{D'}) \cup c E(l - 1) \quad \text{and}
\]

\[
\gamma(l) = (\Phi_{D'}) \cup c \gamma(l - 1).
\]

**Theorem 3.1.** \((E(l), BR_k(l), \gamma(l))\) is the universal \( k \)-fold branched covering of type \( l \) for functions that verify \( a2, b2, \) and \( c2. \)

**Proof.** In fact, we need only to notice that the pullback of \( \gamma(l) \) under any smooth function \( g : M \rightarrow BR_k(l) \) is just a \( k \)-fold branched covering verifying Conditions \( a2, b2, \) and \( c2. \) Similarly to the last case we can construct a classifying function for any \( k \)-fold branched covering that verifies \( a2, b2, \) and \( c2. \)
Now, let us consider the general case. Let us have the following list of restrictions:

**a3)** Let \( \{K_{i,a}\}_{a \in A} \) and \( \{\tilde{K}_{i,b_a}\}_{b_a \in B_a} \) be partitions of \( K_i \) and \( \tilde{K}_i \), respectively, such that \( f(\tilde{K}_{i,b_a}) = K_{i,a} \) for every \( b_a \in B_a \). We assume that \( f|_{\bigcup_{b_a \in B_a} K_{i,b_a}} : \bigcup_{b_a \in B_a} K_{i,b_a} \to K_{i,a} \) is an \( r_{b_a} \)-fold covering, and each \( f|_{\tilde{K}_{i,b_a}} : \tilde{K}_{i,b_a} \to K_{i,a} \) is an \( r_{b_a} \)-fold covering, where \( \{b_a\}_{b_a \in B_a} \) is a fixed partition of \( r_a \).

**b3)** Local condition: We assume that for any \( b_a \) the restriction of \( f \) to each fiber of the disk normal bundle of \( \tilde{K}_{i,b_a} \) is a fixed function \( f_{0,b_a} : D^l \to D^l \) (defined as we did before with \( f_{0,b} \)) which is a \( k_{b_a} \)-fold branched covering of type \( l \) where \( \{k_{b_a}\}_{b_a \in B_a} \) is a fixed partition of \( k \).

**c3)** For each \( b_a \) there is a homomorphism of groups \( \mu_{b_a} : G_{b_a} \to G_a \) such that \( f_{0,b_a} \circ \tilde{g}_{b_a} = \mu_{b_a}(\tilde{g}_{b_a}) \circ f_{0,b_a} \) for any \( \tilde{g}_{b_a} \in G_{b_a} \), where \( G_{b_a} \) and \( G_a \) are the groups of coordinate transformations of the disk normal bundles of \( \tilde{K}_{i,b_a} \) and \( K_{i,a} \), respectively.

In this case, for each \( a \in A \) we get

\[
(\Phi_a)_{D^l} : (X_a)_{D^l} \to (\prod_{b_a \in B_a} B_{S_{b_a}} \times (E_{G_{b_a}})_{D^l})
\]

a \( k \)-fold branched covering of type \( l \). Let

\[
\Theta_{D^l} = \bigcup_{a \in A} (\Phi_a)_{D^l} : \bigcup_{a \in A} (X_a)_{D^l} \to \bigcup_{a \in A} \bigcup_{b_a \in B_a} \left( \prod_{b_a \in B_a} B_{S_{b_a}} \times (E_{G_{b_a}})_{D^l} \right)
\]

such that \( \Theta_{D^l}(x) = (\Phi_a)_{D^l}(x) \) if \( x \in (X_a)_{D^l} \). Here \( \bigcup \) means disjoint union. Hence, \( \Theta_{D^l} \) is a \( k \)-fold branched covering of type \( l \), and \( \Theta_{S^l-1} \) is a \( k \)-fold branched covering
of type \( l - 1 \). Therefore, there is a classifying function

\[
c : \bigcup ( \prod_{a \in A \ b_s \in B_s} B_{a \ b_s} \times (E_{G_{b_s}})_{S^{l-1}}) \to BR_k(l - 1)
\]

such that \( c^* \gamma(l - 1) = \Theta_{S^{l-1}} \). 

As in the last case, it is not difficult to prove the following theorem.

**Theorem 3.2.** The triple \((E(l), BR_k(l), \gamma(l))\) defined by:

\[
BR_k(l) = \left( \bigcup ( \prod_{a \in A \ b_s \in B_s} B_{a \ b_s} \times (E_{G_{b_s}})_{D^l}) \right) \cup_c BR_k(l - 1),
\]

\[
E(l) = \left( \bigcup (\mathfrak{X}_a)_{D^l} \right) \cup_{\mathfrak{X}} E(l - 1) \quad \text{and}
\]

\[
\gamma(l) = \Theta_{D^l} \cup_{\mathfrak{X}} \gamma(l - 1).
\]

is a universal \( k \)-fold branched covering for functions that satisfies the Conditions a3, b3, and c3.

4. Commutativity of Diagrams 2.4 and 2.5

Since the commutativity of Diagrams (2.4) and (2.5) is not trivial, this Section is devoted to give most of the details for these proofs. The definitions and notations used in this section have been taken from the book *Fibre bundles* by Dale Husemoller (see [5]).

Let \( \tilde{K}_l \) and \( K_l \) be manifolds, and let \( \tilde{\eta}_l = (E_{\tilde{\eta}_l}, p_{\tilde{\eta}_l}, \tilde{K}_l) \), \( \eta_l = (E_{\eta_l}, p_{\eta_l}, K_l) \) be two \( D^l \)-bundles with structure topological groups \( \tilde{G} \) and \( G \), respectively. Let us
assume the existence of functions $F : \tilde{K}_i \to K_i$ and $\tilde{F} : E_{\eta_i} \to E_{\eta_1}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
E_{\eta_1} & \xrightarrow{\tilde{F}} & E_{\eta_1} \\
\downarrow{\bar{p}_{\eta_i}} & & \downarrow{p_{\eta_1}} \\
\tilde{K}_i & \xrightarrow{F} & K_i
\end{array}
$$

(4.1)

Let us also assume that if $\{U_j\}_{j \in J}$ is a system of coordinate neighbourhoods of $\eta_i$, (i.e., $\eta_i| U_j$ is trivial for all $j \in J$) then $\{F^{-1}(U_j)\}_{j \in J}$ is a system of coordinate neighbourhoods of $\tilde{\eta}_i$. We are going to denote $F^{-1}(U_j)$ by $\tilde{U}_j$, for all $j \in J$.

Since $K_i$ is a manifold, there is a countable partition of unity (see [5]) $\{u_j : K_i \to [0,1]\}_{j \in \mathbb{N}}$ such that $\eta_i| u_j^{-1}(0,1)$ is trivial for each natural number $j$. Let us denote the set $u_j^{-1}(0,1)$ by $U_j$. So $\{U_j\}_{j \in \mathbb{N}}$ is a system of coordinate neighbourhoods of the bundle $\eta_i$. By our initial assumption over $F$ the family $\{\tilde{U}_j\}_{j \in \mathbb{N}}$ is a system of coordinate neighbourhoods of the bundle $\tilde{\eta}_i$. Moreover, the family $\{u_j \circ F\}_{j \in \mathbb{N}}$ is a partition of unity for this system.

Now, let us assume that there is a continuous homomorphism of groups $\mu : \tilde{G} \to G$ such that the following diagram commutes for all $i, j$ natural numbers:

$$
\begin{array}{ccc}
\tilde{U}_i \cap \tilde{U}_j & \xrightarrow{\tilde{g}_{ij}} & \tilde{G} \\
\downarrow{F} & & \downarrow{\mu} \\
U_i \cap U_j & \xrightarrow{g_{ij}} & G
\end{array}
$$

(4.2)

where $\{\tilde{g}_{ij}\}$ and $\{g_{ij}\}$ are the systems of coordinate transformations of $\tilde{\eta}_i$ and $\eta_i$, respectively. Let $X$ be constructed as in 3.2 (p.62) of Husemoller [5], more precisely, let $X$ be defined in the following way:
Let $Z$ be the disjoint union of the family $\{U_j \times G\}_{j \in \mathbb{N}}$. In this way $Z$ is a topological space. An element of $Z$ is of the form $(k, g, j)$ where $k \in U_j \subset K_l, g \in G$ and $j \in \mathbb{N}$. On $Z$ we define an equivalence relation by: $(k, g, j) \sim (k', g', i)$ if $k = k'$ and $g' = g_{ij}(k)g$. $X$ is the set of equivalence classes of this relation in $Z$. Let $q : Z \to X$ such that $q(k, g, j) = < k, g, j >$, where $< k, g, j >$ is the equivalence class of the element $(k, g, j)$ of $Z$. We say that a set $V$ in $X$ is an open set if $q^{-1}(V)$ is open in $Z$. Thus $X$ is a topological space. If we define the projection function $p_X : X \to K_l$ by $p_X(< k, g, j >) = k$, it is not difficult to see that the triple $\xi = (X, p_X, K_l)$ is a principal $G$-bundle. This triple is known as the principal $G$-bundle associated with the $G$-bundle $\eta$. Similarly, we get $\tilde{\xi} = (\tilde{X}, p_{\tilde{X}}, \tilde{K}_l)$ the principal $\tilde{G}$-bundle associated with the $\tilde{G}$-bundle $\tilde{\eta}$ (see [7] p.41).

Since $\{u_j : K_l \to [0, 1]\}_{j \in \mathbb{N}}$ (resp. $\{u_j \circ F : \tilde{K}_l \to [0, 1]\}_{j \in \mathbb{N}}$) is a countable partition of unity, then (see [5] p.48) $\xi$ (resp. $\tilde{\xi}$) is a numerable principal $G$-bundle (resp. $\tilde{G}$-bundle). Using Theorem 12.2 (p.55) from Husemoller [5], we guarantee the existence of a $G$-morphism $(\lambda, \varepsilon) : \xi \to \omega_G$ ( resp. a $\tilde{G}$-morphism $(\tilde{\lambda}, \tilde{\varepsilon}) : \tilde{\xi} \to \omega_G$) where $\omega_G = (E_G, p_G, B_G)$ is the $G$-universal bundle (resp. $\omega_G$ is the $\tilde{G}$-universal bundle). In other words, $\varepsilon$ and $\tilde{\varepsilon}$ classify the bundles $\xi$ and $\tilde{\xi}$, i.e., $p_G \circ \lambda = \varepsilon \circ p_X$ and $p_{\tilde{G}} \circ \tilde{\lambda} = \tilde{\varepsilon} \circ p_{\tilde{X}}$.

Define $\hat{F} : \tilde{X} \to X$ by $\hat{F}(< \tilde{k}, \tilde{g}, i >) = < F(\tilde{k}), \mu(\tilde{g}), i >$ where $< \tilde{k}, \tilde{g}, i > \in \tilde{X}, \tilde{k} \in \tilde{K}_l, \tilde{g} \in \tilde{G}, i \in \mathbb{N}$.

Note that $\hat{F}$ is well defined. In fact, assume that $< \tilde{k}, \tilde{g}, i > = < \tilde{k}', \tilde{g}', j >$ in
\( \bar{X} \), then \( \bar{k} = \bar{k}' \) and \( \bar{g}' = \bar{g}_{ij}(\bar{k}) \). So

\[
\bar{F} < \bar{k}', \bar{g}', j > = \bar{F} < \bar{k}, \bar{g}_{ij}(\bar{k}) \bar{g}, j > = \langle F(\bar{k}), \mu(\bar{g}_{ij}(\bar{k})) \bar{g} \rangle, j > = \langle F(\bar{k}), \mu(\bar{g}) \rangle, j > = \langle F(\bar{k}), \mu(\bar{g}), j > = \bar{F} < \bar{k}, \bar{g}, i >
\]

therefore, \( \bar{F} \) is well defined.

Clearly \( F \circ p_{\bar{X}} = p_X \circ \bar{F} \) because \( p_{\bar{X}} < \bar{k}, \bar{g}, i > = \bar{k} \) and \( p_X < k, g, i > = k \).

Now, let \( BF : E_G \to E_G \) be defined by \( BF < \bar{g}, t > = \langle \mu(\bar{g}), t > \) (see construction of \( E_G \) in [5] p.52). Let us see that \( BF \) is well defined. In fact, recall that \( < \bar{g}, t > = (t_0 \bar{g}_0, t_1 \bar{g}_1, \ldots) \) where each \( \bar{g}_i \in \bar{G}, t_i \in [0, 1] \), only a finite number of \( t_i \) are different from 0, and \( \sum_{i=1}^{\infty} t_i = 1 \). Then:

\[
(\ast) \quad BF < \bar{g}, t > = \langle \mu(\bar{g}), t > = (t_0 \mu(\bar{g}_0), t_1 \mu(\bar{g}_1), \ldots)
\]

and if \( < \bar{g}, t > = < \bar{g}', t' > \) then \( t_i = t'_i \) for each natural number \( i \), and \( \bar{g}_i = \bar{g}'_i \) for all \( i \) with \( t_i = t'_i > 0 \), so \( \mu(\bar{g}_i) = \mu(\bar{g}'_i) \) for all \( i \) such that \( t_i = t'_i > 0 \). Hence, \( t_0 \mu(\bar{g}_0), t_1 \mu(\bar{g}_1), \ldots) = (t_0 \mu(\bar{g}'_0), t_1 \mu(\bar{g}'_1), \ldots) \) in \( E_G \), i.e., \( < \mu(\bar{g}), t' > = < \mu(\bar{g}'), t' > \) and therefore \( BF < \bar{g}, t > = BF < \bar{g}', t' > \). Hence, \( BF \) is well defined. Moreover, \( BF \)
induces a well defined function $BF : B\tilde{G} \to B\sigma$ such that if $< \tilde{g}, t > \tilde{G} \in B\tilde{G}$ then

\[
BF(< \tilde{g}, t > \tilde{G}) = (B\tilde{F} < \tilde{g}, t >)\sigma = < \mu(\tilde{g}), t > \sigma.
\]

In fact, let us see that $BF$ is a well defined function. Let $< \tilde{g}, t > \tilde{G}, < \tilde{g}', t' > \tilde{G}$ elements of $B\tilde{G}$ such that $< \tilde{g}, t > \tilde{G} =< \tilde{g}', t' > \tilde{G}$ then there is $\tilde{s} \in \tilde{G}$ such that

$< \tilde{g}', t' > =< \tilde{g}, t > \tilde{s} =< \tilde{g}s, t >$ and

\[
B\tilde{F} < \tilde{g}', t' > = B\tilde{F} < \tilde{g}s, t >
\]

$= < \mu(\tilde{g}s), t >$

$= < \mu(\tilde{g})\mu(s), t >$

$= < \mu(\tilde{g}), t > \mu(s) = B\tilde{F} < \tilde{g}, t > \mu(s).
\]

Since $\mu(s) \in G = \mu(\tilde{G})$ then $BF(< \tilde{g}, t > \tilde{G}) = BF(< \tilde{g}', t' > \tilde{G})$, i.e., $BF$ is well defined. Furthermore, the following diagram commutes:

$$
\begin{array}{ccc}
E\tilde{G} & \xrightarrow{B\tilde{F}} & E\sigma \\
\downarrow p_{\tilde{G}} & & \downarrow p_{\sigma} \\
B\tilde{G} & \xrightarrow{BF} & B\sigma
\end{array}
$$

(4.3)
In the following diagram:

\[
\begin{array}{c}
\xymatrix{
X \ar[r]^{p_X} \ar[d]_{p_X} & E_G \ar[d]_{p_G} & B_{\widetilde{G}} \ar[d]_{p_{\widetilde{G}}} \\
X \ar[u]_{F} & E'_{\widetilde{G}} & B_{\widetilde{G}} \ar[u]_{p_{\widetilde{G}}} \\
K_i \ar[u]_{\epsilon} & B_{\widetilde{G}} \ar[u]_{\lambda} 
} \end{array}
\]

(4.4)

We already know that \( F \circ p_X = p_X \circ \widetilde{F} \), \( \epsilon \circ p_X = p_G \circ \lambda \), \( \widetilde{F} \circ p_X = p_G \circ \widetilde{\lambda} \)

and \( BF \circ p_G = p_G \circ B\widetilde{F} \). To get commutativity, we need to show that \( \lambda \circ \widetilde{F} = B\widetilde{F} \circ \widetilde{\lambda} \). Let \( < \tilde{k}, \tilde{g}, i > \in X \). By definition of \( \widetilde{\lambda} \) (see [5] p.55) \( \widetilde{\lambda}(< \tilde{k}, \tilde{g}, i >) = (u_0 \circ F(\tilde{k})\tilde{g}, u_1 \circ F(\tilde{k})\tilde{g}, \ldots) \) where \( \{ u_i \circ F \}_{i \in \mathbb{N}} \) is a partition of unity. Therefore:

\[
B\widetilde{F} \circ \widetilde{\lambda}(< \tilde{k}, \tilde{g}, i >) = B\widetilde{F}(u_0 \circ F(\tilde{k})\tilde{g}, u_1 \circ F(\tilde{k})\tilde{g}, \ldots)
\]

\[
= (u_0(F(\tilde{k}))\mu(\tilde{g}), u_1(F(\tilde{k}))\mu(\tilde{g}), \ldots)
\]

and also:

\[
\lambda \circ \widetilde{F}(< \tilde{k}, \tilde{g}, i >) = \lambda(< F(\tilde{k}), \mu(\tilde{g}), i >)
\]

\[
= (u_0(F(\tilde{k})))\mu(\tilde{g}), u_1(F(\tilde{k})))\mu(\tilde{g}), \ldots)
\]
So, \( \lambda \circ \tilde{F} = BF \circ \tilde{\lambda} \).

On the other hand, \( \epsilon \circ F = BF \circ \tilde{\epsilon} \) because all the other squares in diagram (4.4) commute. Hence, the total diagram (4.4) commutes.

Let

\[
\tilde{\xi}[D^l] = (\tilde{X}_{D^l}, (p_{\tilde{X}})_{D^l}, \tilde{K}_l),
\]

\[
\xi[D^l] = (X_{D^l}, (p_X)_{D^l}, K_l),
\]

\[
\omega_G[D^l] = ((E_G)_{D^l}, (p_G)_{D^l}, B_G)
\]

and

\[
\omega_G[D^l] = ((E_G)_{D^l}, (p_G)_{D^l}, B_G)
\]

be the fiber bundles with fiber \( D^l \) (see [5] p.43).

First of all notice that \( \tilde{\eta}_l \cong \tilde{\xi}[D^l] \) and \( \eta_l \cong \xi[D^l] \) (where \( \cong \) means diffeomorphism) because the \( D^l \)-bundle is unique (see [7] Number 3.2). Let us define \( \lambda_{D^l} : X_{D^l} \to (E_G)_{D^l} \) by \( \lambda_{D^l}([x, y]) = [\lambda(x), y] \) where \([x, y] \) denotes an element in \( X_{D^l} = X \times D^l / \sim \) (see [5] p.44). Let us show that \( \lambda_{D^l} \) is well defined. Let \([x, y] = [x', y'] \in X_{D^l} \) then there is \( s \in G \) such that \((x', y') = (x, y)s = (xs, s^{-1}y) \).

Then:

\[
\lambda_{D^l}[x', y'] = \lambda_{D^l}[xs, s^{-1}y]
\]

\[
= [\lambda(xs), s^{-1}y]
\]

\[
= [\lambda(x)s, s^{-1}y]
\]

\[
= [\lambda(x), y]
\]

\[
= \lambda_{D^l}[x, y].
\]
This shows that \( \lambda_{D^i} \) is well defined. Similarly, we define

\[
\tilde{\lambda}_{D^i} : \tilde{X}_{D^i} \to (E_G)_{D^i} \text{ by } \tilde{\lambda}_{D^i}([\tilde{x}, y]) = [\tilde{X}(\tilde{x}), y].
\]

Since \((p_X)_{D^i}([x, y]) = (p_X)(x)\) and \((p_G)_{D^i}([e, y]) = p_G(e)\) for all \([x, y] \in X_{D^i}\) and \([e, y] \in (E_G)_{D^i}\), it is easy to check that \((\lambda_{D^i}, \epsilon) : \xi[D^i] \to \omega_G[D^i] \) is a \(G\)-morphism. Similarly \((\tilde{\lambda}_{D^i}, \tilde{\epsilon}) : \tilde{\xi}[D^i] \to \tilde{\omega}_G[D^i] \) is a \(\tilde{G}\)-morphism.

Recall that \(\eta_l\) and \(\eta_i\) are \(D^i\)-bundles. Since \(\tilde{F} : E_{\eta_l} \to E_{\eta_l}\) preserves fibers (see Diagram (4.1)), if \(\tilde{k} \in \tilde{K}_l\) then the function \(\tilde{F}|_{P_{\eta_l}^{-1}(\tilde{k})} : P_{\eta_l}^{-1}(\tilde{k}) \to p_{\eta_l}^{-1}(F(\tilde{k}))\) can be considered as a function from \(D^i\) to \(D^i\). Roughly speaking, let us assume that for every fiber the restriction of \(\tilde{F}\) to each fiber is a fixed function \(f_0 : D^i \to D^i\) such that \(f(\tilde{s}y) = \mu(\tilde{s})f(y)\) for every \(\tilde{s} \in \tilde{G}\) and for every \(y \in D^i\).

Let us define \(F_{D^i} : \tilde{X}_{D^i} \to X_{D^i}\) by:

\[
F_{D^i}([\tilde{x}, y]) = [\tilde{F}(\tilde{x}), f(y)], \quad \forall \tilde{x} \in \tilde{X}, y \in D^i.
\]

It is a well defined function. In fact, let \([\tilde{x}, y] = [\tilde{x}', y']\) then there is \(\tilde{s} \in \tilde{G}\) such that \((\tilde{x}', y') = (\tilde{x}, y)\tilde{s} = (\tilde{x}\tilde{s}, \tilde{s}^{-1}y).\) So:

\[
F_{D^i}([\tilde{x}', y']) = F_{D^i}([\tilde{x}\tilde{s}, \tilde{s}^{-1}y])
= [\tilde{F}(\tilde{x}\tilde{s}), f_0(\tilde{s}^{-1}y)]
= [\tilde{F}(\tilde{x})\mu(\tilde{s}), \mu(\tilde{s})^{-1}f_0(y)]
= [\tilde{F}(\tilde{x}), f_0(y)]
= F_{D^i}([\tilde{x}, y]).
\]
Thus $F_{D^l}$ is well defined. Now, since $(p_{\bar{X}})_{D^l}[\bar{x}, y] = p_{\bar{X}}(\bar{x})$ and $(p_X)_{D^l}[x, y] = p_X(x)$ it is clear that the next diagram commutes:

$$
\begin{array}{ccc}
\tilde{X}_{D^l} & \xrightarrow{F_{D^l}} & X_{D^l} \\
(p_{\bar{X}})_{D^l} & \downarrow & \downarrow (p_X)_{D^l} \\
\tilde{K}_l & \xrightarrow{F} & K_l
\end{array}
$$

(4.5)

Similarly, we could define the function $(BF)_{D^l} : (E_G)_{D^l} \to (E_G)_{D^l}$ by:

$$(***) \quad (BF)_{D^l}([\bar{e}, y]) = [BF(\bar{e}), f_0(y)] \quad \text{where } \bar{e} \in E_G, \text{ and } y \in D^l.$$

Also we get the commutativity in the following diagram:

$$
\begin{array}{ccc}
(E_G)_{D^l} & \xrightarrow{(BF)_{D^l}} & (E_G)_{D^l} \\
(p_{\bar{G}})_{D^l} & \downarrow & \downarrow (p_{\bar{G}})_{D^l} \\
B_{\bar{G}} & \xrightarrow{BF} & B_G
\end{array}
$$

(4.6)

Now, let us consider the following diagram:

$$
\begin{array}{ccc}
\tilde{X}_{D^l} & \xrightarrow{\lambda_{D^l}} & (E_G)_{D^l} \\
\downarrow \lambda_{D^l} & & \downarrow \lambda_{D^l} \\
X_{D^l} & \xrightarrow{(p_X)_{D^l}} & (E_G)_{D^l} \\
\downarrow (p_X)_{D^l} & & \downarrow (p_X)_{D^l} \\
\tilde{K}_l & \xrightarrow{\epsilon} & B_G
\end{array}
$$

(4.7)
Let us see that it commutes. We already know:

\[ F \circ (p_X)_{D'} = (p_X)_{D'} \circ F_{D'} \]

\[ BF \circ (p_G)_{D'} = (p_G)_{D'} \circ (BF)_{D'} \]

\[ \epsilon \circ (p_X)_{D'} = (p_G)_{D'} \circ \lambda_{D'} \]

\[ \tilde{\epsilon} \circ (p_X)_{D'} = (p_G)_{D'} \circ \tilde{\lambda}_{D'} \]

\[ \epsilon \circ F = BF \circ \tilde{\epsilon} \]

In order to finish the proof of the commutativity of (4.7), we have to show that \( \lambda_{D'} \circ F_{D'} = (BF)_{D'} \circ \tilde{\lambda}_{D'} \). In fact, let \([\bar{x}, \bar{y}] \in X_{D'}\).

\[
\lambda_{D'}(F_{D'}[\bar{x}, y]) = \lambda_{D'}([\bar{F}(\bar{x}), f_0(y)])
\]
\[
= [\lambda(\bar{F}(\bar{x})), f_0(y)]
\]
\[
= [B \bar{F}(\bar{\lambda}(\bar{x})), f_0(y)]
\]
\[
= (BF)_{D'}[\bar{\lambda}(\bar{x}), y]
\]
\[
= (BF)_{D'}(\tilde{\lambda}_{D'}[\bar{x}, y])
\]

therefore, the diagram (4.7) commutes.

**Remark.** Notice that the bundles \( \eta_l = (E_{\eta l}, p_{\eta l}, \bar{K}_l) \) and \( \tilde{\eta}[D'] = (\tilde{X}_{D'}, (p_{\tilde{X}})_{D'}, \bar{K}_l) \) are equivalent bundles. Similarly \( \eta_l \cong \xi[D'] \). Let \( \tilde{h} : E_{\eta l} \to \tilde{X}_{D'} \) and \( h : E_{\eta l} \to X_{D'} \) be the homeomorphisms that give the mentioned equivalences. Then, the following diagram commutes:
Now, if $\tilde{h}_{Dl} = \lambda_{Dl} \circ \tilde{h}$ and $h_{Dl} = \lambda_{Dl} \circ h$ then the commutativity of the next diagram is clear.
Similar arguments transform diagram (4.4) into the diagram:

(4.10)

Therefore, both Diagrams (2.4) and (2.5) are commutative. In fact, Diagrams (4.9) and (4.10) are just Diagrams (2.5), and (2.4), respectively.
5. Chapter References


CHAPTER 4

EXAMPLES AND CALCULATIONS

1. Preliminaries

Section 2 is devoted to the study of some homomorphisms among the Homotopy groups of the classifying spaces of $k$-fold branched coverings of type $l$. In Section 3 we show how Van Kampen's Theorem and the Mayer-Vietoris sequence can be used directly on the classifying spaces to obtain the results in Section 2. We also compute some homotopy and homology groups for some specific examples.

First of all, let us give some preliminaries. Let $M, N$ be smooth manifolds, let $f : M \to N$ be a $k$-fold branched covering of type 2, i.e., its branch set $K_2$ is a submanifold of $N$ of codimension 2. For each component of $K_2$ we can associate a partition $\sum_{i=1}^{m} k_i s_i = k$ of $k$. Let $P$ be a set of partitions of $k$. Let us denote by $BR_k^P(2)$ the classifying space of $k$-fold branched coverings of type 2 for which the associated partitions belong to $P$. On the other hand, each partition $\sum_{i=1}^{m} k_i s_i = k$ of $k$ corresponds to a conjugacy class of elements of $S_k$ (the symmetric group of $k$ elements) which in cycle notation contains $s_i$ disjoint $k_i$-cycles, $1 \leq i \leq m$.

In [2] Brand and Brumfiel showed that the fundamental group of $BR_k^P(2)$ is just the quotient group $S_k / < P >$ where $< P >$ is the normal subgroup of $S_k$ generated by the conjugate classes associated to all partitions in $P$. In Section 2 we are going to see that we get a similar result for any $l$. 
In [1] (p. 235) Brand gave an explicit construction of the classifying space $BR_k^P(2)$. Inductively (using the same methods that we used in Section 3, Chapter 3), we extend this definition for any natural number $l$ greater than 1. In other words, we define:

$$BR_k^P(l) = (\bigcup_{v \in A} (B_{G_{r_{v_a}}} \times (E_{G_{r_{v_a}}})_{D_v})) \cup_c BR_k^P(l - 1).$$

Notations. The total space associated to $BR_k^P(l)$ is denoted by $E_k^P(l)$, and the projection from $E_k^P(l)$ to $BR_k^P(l)$ is denoted by $\gamma^P$.

2. Relations among the Homotopy Groups

As a consequence of the construction of the classifying spaces given in Chapter 3, it is clear that for every natural number $l$ greater than 1, $BR_k^P(l) \subset BR_k^P(l + 1)$. So, it is clear that for every $i \in \mathbb{N}$ and $l \geq 2$ we get a homomorphism of homotopy groups:

$$\psi_{i,l} : \Pi_i(BR_k^P(l), *) \to \Pi_i(BR_k^P(l + 1), *),$$

where $*$ is a base point in $BR_k^P(2)$. Notice that if $f : M \to N$ is a $k$-fold branched covering of type $l + 1$ classified by $g : N \to BR_k^P(l + 1)$ then $f$ is concordant to a $k$-fold branched covering of type $l$ if and only if there is map $g' : N \to BR_k^P(l)$ so that the composition $N \xrightarrow{g'} BR_k^P(l) \xleftarrow{} BR_k^P(l + 1)$ is homotopic to $g$. 

Proposition 2.1. For every \( l \) greater than or equal to \( i \) the homomorphism \( v_{i,l} \) is surjective.

Proof. In fact, since \( l \geq i \) then every pointed \( k \)-fold branched covering of type \( l + 1 \) over \( S^i \) is just a pointed \( k \)-fold branched covering of type \( l \) because we can not have a codimension \( l + 1 \) subset in \( S^i \).

Proposition 2.2. For every \( l \) greater than \( i \) the homomorphism \( v_{i,l} \) is injective.

Proof. Let \( \phi : (S^i, *) \to (BR_k^P(l + 1), *) \) and \( \psi : (S^i, *) \to (BR_k^P(l + 1), *) \) be homotopic pointed continuous functions. Therefore, the pullbacks \( \phi^* \gamma^P(l + 1) \) and \( \psi^* \gamma^P(l + 1) \) belong to the same concordance class of pointed \( k \)-fold branched coverings of type \( l + 1 \) over \( S^i \). It means that there is a pointed \( k \)-fold branched covering \( F : W \to (S^i \times I, * \times I) \) of type \( l + 1 \) such that \( F|_{F^{-1}(S^i \times \{0\})} \) is equivalent up to homeomorphism to \( \phi^* \gamma^P(l + 1) \) and \( F|_{F^{-1}(S^i \times \{1\})} \) is equivalent up to homeomorphism to \( \psi^* \gamma^P(l + 1) \). Now, since \( l \geq 1 \), every pointed \( k \)-fold branched covering of type \( l + 1 \) over \( S^i \times I \) shall be of type \( l \) because in dimension \( i + 1 \leq l + 1 \) there is no codimension \( l + 1 \) subsets. Therefore, the pullbacks \( \phi^* \gamma^P(l + 1) \) and \( \psi^* \gamma^P(l + 1) \) are also concordant under pointed \( k \)-fold branched coverings of type \( l \). Hence, \( v_{i,l} \) is injective.

As a consequence of Propositions 2.1 and 2.2, we have the following sequences of functions:

a. \( \Pi_1(BR_k^P(2), *) \xrightarrow{\sim} \Pi_1(BR_k^P(3), *) \xrightarrow{\sim} \Pi_1(BR_k^P(4), *) \xrightarrow{\sim} \cdots \)
b. $\Pi_2(\mathcal{B} \mathcal{R}_k^P(2), \star) \to \Pi_2(\mathcal{B} \mathcal{R}_k^P(3), \star) \to \Pi_2(\mathcal{B} \mathcal{R}_k^P(4), \star) \to \cdots$

c. $\Pi_3(\mathcal{B} \mathcal{R}_k^P(3), \star) \to \Pi_3(\mathcal{B} \mathcal{R}_k^P(4), \star) \to \Pi_3(\mathcal{B} \mathcal{R}_k^P(5), \star) \to \cdots$

The following proposition implies that for a specific partition of $k$ any $k$-fold branched covering of type 3 is just of type 2.

First of all, let us recall the classical Riemann-Hurwitz formula (see [3]). Let $f : M^2 \to N^2$ be a $k$-fold branched covering of type 2, with branch set a finite set $K_2 = \{x_1, \cdots, x_r\}$, and such that for each point $y \in \tilde{K}_2 = f^{-1}(K_2)$ there is a neighbourhood $U$ of $y$ in $M^2$ on which the restricted map $f|_U : U \to f(U)$ is equivalent up to homeomorphism to the map $z \to z^n$ of the complex numbers for some integer $n \geq 1$. The number $n$ is called the ramification number. We may associate to the $k$-fold branched covering $f$ a branch data $D = \{A_1, A_2, \cdots, A_r\}$, where $A_i = [n_{i1}, n_{i2}, \cdots, n_{ij_i}]$ is a partition of $k$, and the numbers $n_{ij}$ are the ramification numbers of the points in the fiber $f^{-1}(x_i)$. Let $v(D) = \sum_{ij}(n_{ij} - 1)$ then the Riemann-Hurwitz formula is:

$$\chi(M) = k \chi(N) - v(D).$$

Where $\chi$ is the Euler characteristic. For example if $M = N = S^2$ then $v(D) = 2k - 2$.

**Proposition 2.3.** Let $f : M \to N$ be a $k$-fold branched covering of type 3. If $K_3$ is not the empty set then $f|_{(M - \tilde{K}_3)}$ is a $k$-fold branched covering of type 2 for which the associated set $P$ of partitions of $k$ is different from $\{\{k\}\}$.

**Proof.** Let $x \in K_3$. Let us consider the tubular neighbourhoods of $K_3$ and $\tilde{K}_3$,
respectively. Now, if we consider the function $f$ locally around some $y \in f^{-1}(x)$, it is possible to find coordinate systems in such a way that the restriction of the function $f$ to the fiber of $y$ in the normal bundle of $\tilde{K}_3$ is equivalent up to homeomorphism to a $k$-fold branched covering $f_0 : D^3 \to D^3$ of type 3. Moreover, without loss of generality, we choose the “disk” around $y$ in such a way its boundary intersects all the branches of the branched set that connect to $y$. So, $f_0|_{S^2} : S^2 \to S^2$ is a $k$-fold branched covering of type 2.

Now, assuming that the ramification number associated to every point in the branch set of $f_0$ is $k$, then using the Riemann-Hurwitz formula we must have only two branched points for $f_0|_{S^2}$. If it happens, it is easy to see that $x$ does not belong to $K_3$ but to $(K_2 - K_3)$ (see Definition 3.1, Chapter 2) which contradicts our initial assumption. Hence, the set $P$ of partitions associated to $f|_{(M - \tilde{K}_3)}$ is different from $\{\{k\}\}$. 

**Corollary 2.4.** If $P = \{\{k\}\}$ then

$$\Pi_2(BR_k^P(2)) \cong \Pi_2(BR_k^P(3)) \cong \Pi_2(BR_k^P(4)) \cong \Pi_2(BR_k^P(5)) \cong \cdots .$$

**Proof.** In fact, the first congruence in the sequence comes from the previous proposition. Proposition 2.1, and 2.2 prove the rest of the congruences. 

**Corollary 2.5.** If $P = \{\{k\}\}$ then for every $i$, $\Pi_i(BR_k^P(2)) \cong \Pi_i(BR_k^P(3))$.

**Proof.** The proof is obvious. 

The next proposition shows that $\Pi_2(BR^P_k(2))$ is not always isomorphic with $\Pi_2(BR^P_k(3))$.

**Proposition 2.6.** If $P = \{\{4\}, \{2,1,1\}\}$ then the homomorphism:

$$v_{2,2} : \Pi_2(BR^P_k(2), \#) \to \Pi_2(BR^P_k(3), \#)$$

is not injective.

**Proof.** In fact, let us consider the following example: Using the methods developed by S.M. Gersten in [3], we can see that there exists a 4-fold branched covering $g : S^2 \to S^2$ of type 2 such that the branch data of $g$ is $\{[4], [2,1,1], [2,1,1], [2,1,1]\}$. It means that the branch set of $g$, $K_2 = \{x,y,z,w\}$ is such that $g^{-1}(x)$ has only one element with ramification number 4, and each of the sets $g^{-1}(y)$, $g^{-1}(z)$, and $g^{-1}(w)$ has only one element with ramification number 2 and two with ramification number 1.

Let us extend the function $g : S^2 \to S^2$ to a function $f_0 : D^3 \to D^3$ as we did before in Section 2, Chapter 3. In other words:

$$f_0(z) = |z|^2 g\left(\frac{z}{|z|}\right), \quad \forall z \in D^3.$$ 

So, $f_0$ is a 4-fold branched covering of type 3 from $D^3$ to $D^3$. Let us see $f_0$ in the following picture:
where the lines represent the branching set, and the notation \( x[4] \) means that 
\( f_0(x[4]) = x \), and the ramification number of \( x[4] \) is 4. Now let \( a \in \text{int}(D^3) \) such that \( a \) is not a branched point (see Figure 2.2), and let us cut out a small open ball 
\( B_\epsilon(a) \) around the point \( a \). Since \( a \) is not a branched point, we can choose the ball 
\( B_\epsilon(a) \) such that it does not intersect the branch set of \( f_0 \). Therefore, \( f_0^{-1}(B_\epsilon(a)) \) consists of 4 disjoint sets each one homeomorphic to \( B_\epsilon(a) \).
where \( f_0(B_1 \cup B_2 \cup B_3 \cup B_4) = B_4(a) \).

Hence, it is clear that the function \( f_0|_{(D^3 - f_0^{-1}(B_4(a)))} \) is equivalent up to homeomorphism to a \( k \)-fold branched covering \( F : W \to S^2 \times I \) of type 3, such that \( F|_{F^{-1}(S^2 \times \{1\})} \) is equivalent up to homeomorphism to \( g : S^2 \to S^2 \), and \( F|_{F^{-1}(S^2 \times \{0\})} \) is a 4-fold covering (it has empty branch set). Let us denote \( F|_{F^{-1}(S^2 \times \{0\})} \) by \( h \). So, \( g \) is concordant to \( h \) under a 4-fold branched covering of type 3.

Now, let us prove that \( g \) is not concordant to \( h \) under a 4-fold branched covering of type 2. In fact, using branched type 2, we see that the branched component that contains the point \( x[4] \) has to connect to another point with ramification number 4 belonging either to the domain of \( g \) or to the domain of \( h \). Nevertheless, the only point with ramification number 4 inside these domains is \( x[4] \). Moreover, if we assume that the points \( y[2], z[2] \) are in the same component, then the component that contains the point \( w[2] \) can not connect to any other point (see Figure 2.3).

\[ \text{Figure 2.3} \]
In particular the last example also says that for $P = \{\{4\}, \{2, 1, 1\}\}$ then $\Pi_2(BR^P_k(2))$ is different from zero.

The next corollary shows that all the $k$-fold branched coverings of type $n$ from $S^n$ to $S^n$ are concordant under type $n + 1$.

**Corollary 2.7.** If $g : S^n \to S^n$ and $g' : S^n \to S^n$ are two $k$-fold branched coverings of type $n$, then they are concordant under a $k$-fold branched covering of type $n + 1$.

**Proof.** In fact, using the same kind of arguments given in the previous proof we can show that both $k$-fold branched coverings $g$ and $h$ are concordant to a $k$-fold covering (with empty branch set). □

Actually, the last corollary is also a corollary of the following proposition because given any function $g : S^n \to S^n$, always there is an extension $g' : D^{n+1} \to D^{n+1}$ of $g$. Moreover, if $g$ is a $k$-fold branched covering of type $n$ there is an extension $g'$ of $g$ which is a $k$-fold branched covering of type $(n + 1)$ (see the construction of $f_0$ in Section 2, Chapter 1).

**Proposition 2.8.** Let $M^n$ be a smooth $n$-manifold. Given a $k$-fold branched covering $f : M^n \to S^n$ of type $n$, then $f$ is concordant (under type $(n + 1)$) to a $k$-fold covering (with empty branch set) $g : \bigcup_{i=1}^k S^n(i) \to S^n$ if and only if there is a $k$-fold branched covering $f' : W' \to D^{n+1}$ of type $(n + 1)$ that is an extension of $f$, where $W'$ is an $(n + 1)$-manifold with boundary $M^n$. Here each $S^n(i)$ is a copy of $S^n$.

**Proof.** Suppose $f$ and $g$ are concordant (under type $(n + 1)$). So, there is a $k$-fold
branched covering \( F : W \to S^n \times I \) of type \((n + 1)\) (where \( W \) is a \((n + 1)\)-manifold with boundary \( M^n \cup (\bigcup_{i=1}^k S^n(i)) \) such that \( F|_{F^{-1}(S^n \times \{1\})} \) is equivalent up to homeomorphism to \( f \), and \( F|_{F^{-1}(S^n \times \{0\})} \) is equivalent up to homeomorphism to \( g \).

Let \( g' : \bigcup_{i=1}^k D^{n+1}(i) \to D^{n+1} \) (each \( D^{n+1}(i) \) is a copy of \( D^{n+1} \)) be an extension of \( g \). Now, let \( W' = (W \cup (\bigcup_{i=1}^k D^{n+1}(i)))/\sim \) where we identify the copy \( S^n(i) \) of \( S^n \) to the boundary of the respective copy \( D^{n+1}(i) \) of \( D^{n+1} \). So the boundary of \( W' \) is just \( M^n \). Also, let \( D = (S^n \times I \cup D^{n+1})/\sim \), where we identify \( S^n \times \{0\} \) to the boundary of \( D^{n+1} \). Clearly \( D \cong D^{n+1} \). Now, let us define \( f' : W' \to D \) in the following way: \( f'(x) = F(x) \) if \( x \in W \), and \( f'(x) = g'(x) \) if \( x \in \bigcup_{i=1}^k D^{n+1}(i) \). Hence, it is clear that \( f' \) is an extension of \( f \).

Now, let \( W' \) be an \((n + 1)\)-manifold with boundary \( M^n \). Let us assume there is a \( k \)-fold branched covering \( f' : W' \to D^{n+1} \) of type \((n + 1)\) such that \( f = f'|_{\partial W'} : \partial W' \to S^n \) is a \( k \)-fold branched covering of type \( n \). Let us choose \( x \in \text{int}(W') \) such that \( x \) is not a branch point. So, there is a small open ball \( B_\epsilon(x) \) that is disjoint with the branch set, and such that \( f^{-1}_\epsilon(B_\epsilon(x)) = \bigcup_{i=1}^k B_i \), where each \( B_i \cong (D^{n+1} - S^n) \). Consider \( F : (W' - \bigcup_{i=1}^k B_i) \to (D^{n+1} - B_\epsilon(x)) \). Then, it is clear that \( F \) is equivalent up to homeomorphism to a \( k \)-fold branched covering \( F' : (W' - \bigcup_{i=1}^k B_i) \to S^n \times I \) of type \((n + 1)\) that makes \( f \) and \( g \) concordant.

Remark. We conjecture that if \( M^2 \) is the torus \( S^1 \times S^1 \) and \( f : M^2 \to S^2 \) is a \( k \)-fold branched covering of type \( 2 \) with some associated set \( P \) of partitions of \( k \) then \( f \) is not concordant to a \( k \)-fold covering \( g : \bigcup_{i=1}^k S^2(i) \to S^2 \). If our conjecture were
true we would get $\Pi_2(BR_k^p(3)) \neq 0$ and then $\Pi_2(BR_k^p(l)) \neq 0$ for every $l \geq 2$.

3. Van Kampen's Theorem and Mayer-Vietoris Sequences

In the last section we showed that $\Pi_1(BR_k^p(l)) \cong \Pi_1(BR_k^p(l + 1))$ for every $l \geq 2$. Using the Hurewicz Theorem (see [4]) we see that the reduced homology group $\tilde{H}_1(BR_k^p(l)) \cong \Pi_1(BR_k^p(l))$ for every $l \geq 2$. Moreover, since $\Pi_1(BR_k^p(2)) \cong S_k \langle P \rangle$ (see [2]) then for every $l \geq 2$ we have:

$$\Pi_1(BR_k^p(l)) \cong \tilde{H}_1(BR_k^p(l)) \cong S_k \langle P \rangle.$$

In contrast with the last section, in this section, we use the construction of classifying spaces. Actually, we use the Van Kampen's Theorem and the Mayer Vietoris sequence to reproduce the results of Section 2. We think that exploiting the construction of classifying spaces we should be able to do more calculations. This section shall be considered as the beginning of such an investigation.

Let $f : M \to N$ be a $k$-fold branched covering of type $l$ that verifies the Conditions a1, and b1, given in Section 2 of the previous chapter. For this special case we assume $r = 1$, i.e., the restriction $f|_{\tilde{K}_l} : \tilde{K}_l \to K_l$ is a diffeomorphism. Furthermore, we suppose the following restriction:

$c1')$ The groups $\tilde{G}$ and $G$ of coordinate transformations of the normal bundles $\tilde{\eta}_l$ and $\eta_l$ are trivial groups. In this case we have:

$$BS_r \times (EG)_{D^l} \cong \{0\} \times D^l \cong D^l.$$
because \( r = 1 \) and \( G \) is a trivial group. Hence \( BR_k(l) = D^l \cup_c BR_k(l - 1) \) where \( c : S^{l-1} \to BR_k(l - 1) \) is the classifying function of \( f_0 : S^{l-1} \to S^{l-1} \) (see Section 2, Chapter 3). Now, if \( P \) is a set of partitions of \( k \), then it is clear that:

\[
BR_k^P(l) = D^l \cup_c BR_k^P(l - 1).
\]

Applying Van Kampen’s Theorem \((l - 2)\) times we get:

\[
\Pi_1(BR_k^P(l)) \cong \Pi_1(D^l) \ast_{\Pi_1(S^{l-1})} \Pi_1(BR_k^P(l - 1)) \cong \Pi_1(BR_k^P(l - 1)) \cong \cdots \\
\cdots \cong \Pi_1(BR_k^P(4)) \cong \Pi_1(D^4) \ast_{\Pi_1(S^3)} \Pi_1(BR_k^P(3)) \cong \Pi_1(BR_k^P(3)) \\
\cong \Pi_1(D^3) \ast_{\Pi_1(S^2)} \Pi_1(BR_k^P(2)) \cong \Pi_1(BR_k^P(2)) \cong S_k \langle < P > \rangle
\]

because \( \Pi_1(D^3) \cong \Pi_1(S^2) = \{0\} \). So, we have gotten again the expected result using Van Kampen’s Theorem.

Now, let us consider the reduced Mayer-Vietoris sequence for \( \tilde{H}_*(BR_k^P(l)) \):

\[
\cdots \to \tilde{H}_2(BR_k^P(l)) \to \tilde{H}_1(S^{l-1}) \to \tilde{H}_1(D^l) \ast_{\Pi_1(S^{l-1})} \tilde{H}_1(BR_k^P(l - 1)) \to \tilde{H}_1(BR_k^P(l)) \to 0
\]

where \( i : S^{l-1} \to D^l \) is the natural inclusion and \( j = c : S^{l-1} \to BR_k^P(l - 1) \) is the classifying function of \( f_0 : S^{l-1} \to S^{l-1} \).

Since \( \tilde{H}_1(S^{l-1}) = \{0\} \) then \( \tilde{H}_1(BR_k^P(l - 1)) = \tilde{H}_1(BR_k^P(l - 1)) \) and by Hurewicz Theorem (see [4]):

\[
\tilde{H}_1(BR_k^P(l)) \cong \Pi_1(BR_k^P(l)) \cong (S_k \langle < P > \rangle)
\]
obtaining again the same result as before.

Examples.

i) If \( k = 4 \) and \( P = \{\{2,2\}\} \) then \( < P > = A_4 \). So:

\[
\Pi_1(BR_4^P(2)) \cong (S_4 / A_4) \cong \mathbb{Z}_2 \cong \widetilde{H}_1(BR_4^P(2)).
\]

ii) If \( k = 4 \) and \( P = \{\{3,1\}\} \) then \( < P > = A_4 \). Hence:

\[
\Pi_1(BR_4^P(2)) \cong (S_4 / A_4) \cong \mathbb{Z}_2 \cong \widetilde{H}_1(BR_4^P(2)).
\]

As in the example studied in Proposition 2.6, we get for this case that:

\[
\Pi_2(BR_4^P(2)) \neq 0.
\]

iii) If \( k = 4 \) and \( P = \{\{4\},\{2,2\},\{3,1\}\} \) then \( < P > = S_4 \). So:

\[
\Pi_1(BR_4^P(2)) \cong (S_4 / S_4) \cong \{0\} \cong \widetilde{H}_1(BR_4^P(2)).
\]

Similarly to the last example we get for this case:

\[
\Pi_2(BR_4^P(2)) \neq 0.
\]

iv) If \( k = 5 \) and \( P = \{\{2,2,1\}\} \) then \( < P > = A_5 \). Therefore:

\[
\Pi_1(BR_5^P(2)) \cong (S_5 / A_5) \cong \mathbb{Z}_2 \cong \widetilde{H}_1(BR_5^P(2)).
\]

v) If \( k = 5 \) and \( P = \{\{4,1\},\{2,1,1,1\}\} \) then \( < P > = S_5 \). So:

\[
\Pi_1(BR_5^P(2)) \cong (S_5 / S_5) \cong \{0\} \cong \widetilde{H}_1(BR_5^P(2)).
\]
4. Chapter References


REFERENCES


