TOPICS IN FRACTAL GEOMETRY

DISSERTATION

Presented to the Graduate Council of the
University of North Texas in Partial
Fulfillment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

By

JingLing Wang, B.A., M.A.
Denton, Texas
August, 1994

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In this dissertation, we study fractal sets and their properties, especially the open set condition, Hausdorff dimensions and Hausdorff measures for certain fractal constructions. We begin by introducing some known results of self-similar sets and give a short summary of each chapter in Chapter 1. In Chapter 2, we discuss the existence and uniqueness of a graph directed self-similar measure list and extend Schief's result concerning the open set condition for self-similar sets to graph directed self-similar sets.

In Chapter 3, we introduce certain ratio self-similar fractals with overlaps and show that these kinds of fractals have positive Hausdorff measures with respect to the corresponding similarity dimensions. We prove in Chapter 4 that the strong open set condition is equivalent to the conformal measure being zero on the boundary of a set U satisfying the open set condition.

Statistically self-similar fractals and their properties are studied in Chapter 5. We extend Graf's δ-condition to a weak δ-condition. Furthermore, we show that for certain statistically self-similar fractals in $\mathbb{R}^d$, if Graf's δ-condition is not satisfied, their Hausdorff measures will be zero almost surely.

Finally in Chapter 6, we study linear cellular automata. We show that linear cellular automata can be generated by graph-directed constructions. Therefore their Hausdorff dimensions and measures can be calculated by applying the results for graph directed constructions.

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CHAPTER I

INTRODUCTION

This dissertation is divided into five parts. Although each part is self-contained and can be read without referring to the others, they do share a basic framework and are studies of fractals using Hausdorff measure, Hausdorff dimension and Hausdorff metric. We gather these basics together in this chapter.

In 1919, Felix Hausdorff [Ha] published his theory of measure and dimension. He paid tribute to Caratheodory who in 1914 gave a new treatment of Lebesgue measure. Based on Caratheodory's theory, Hausdorff established the entire theory of measure and 'fractional' dimension, which is known today as the Hausdorff dimension.

Hausdorff Measure and Hausdorff Dimension.

Let (X, d) be a metric space and $s \geq 0$ be a fixed real number. For every $\delta > 0$ and $E \subset X$, we define

$$\mathcal{H}^s_\delta(E) = \inf\{\sum_{i=1}^{\infty}(\operatorname{diam} U_i)^s : E \subset \cup_{i=1}^{\infty} U_i, \operatorname{diam} U_i \leq \delta\},$$

and

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta \geq 0} \mathcal{H}^s_\delta(E), \tag{1.1}$$

where $\mathcal{H}^s(E)$ is called the s-dimensional Hausdorff measure of E. It is clear that $\mathcal{H}^s(E)$ relates to the local geometric structure of E. The set function $\mathcal{H}^s$ is a Borel regular measure [Ro], but is normally not finite on bounded sets. If $f : X \to X$ is Lipschitz, then $\mathcal{H}^s(f(A)) \leq (\operatorname{Lip} f)^s \mathcal{H}^s(A)$. If $S : X \to X$ is a similarity map, *i.e.* there is a constant r such that $|S(x) - S(y)| = r|x - y|$ for all x, y in X, then $\mathcal{H}^s(S(A)) = r^s \mathcal{H}^s(A)$.

For each $E \subset X$, there is a unique real number s, called the Hausdorff dimension of E, denoted by $\dim_H E$, such that

$$\mathcal{H}^t(E) = \begin{cases} \infty & \text{if } t < s; \\ 0 & \text{if } s < t. \end{cases}$$

The δ-parallel body of E is the set of points within distance δ of E, that is,

$$E_\delta = \{x \in X : \inf_{y \in E} d(x, y) \leq \delta\}.$$

The Hausdorff metric d_H is defined on the collection of all nonempty compact subsets of X by $d_H(E, F) = \inf\{\delta : E \subset F_\delta \quad \text{and} \quad F \subset E_\delta\}$. A simple check shows that d_H is a metric.

The term "fractal" was introduced by Mandelbrot (1975) [Ma] for sets with a highly irregular structure to which the methods of classical calculus can not be applied. Fractal geometry provides a general tool for the study of irregular sets. One reason for studying them arises from the fact that irregular sets provide a much better representation of many natural phenomena. For instance, they can be used to model the Brownian motion of particles, turbulence in fluids, the growth of plants, geographical coastlines and surfaces, the dynamics of discrete variables in discrete space and time, (see [Fa1] [Fe] [Ma] [Schr]).

In general, we can characterize a fractal set F as follows [Fa1]:

(1) F has a fine structure, *i.e.*, detail on arbitrarily small scales.

(2) F is too irregular to be described in traditional geometrical language, both locally and globally.

(3) Often F has some form of self-similarity, perhaps approximate or statistical.

(4) Usually the Hausdorff dimension of F is greater than its topological dimension.

(5) In most cases of interest, F is defined in some very simple recursive manner.

Among the fractal sets, those with the additional property of being self-similar are particularly interesting. A theory of strictly self-similar compact sets has been developed by Moran [Mo] in 1946, and later extended by Hutchinson [Hu] in 1981. The main results of Moran and Hutchinson can be described as follows:

(a) Let $X = (X, d)$ be a complete metric space and $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a finite set of contraction maps on X. Then there exists a compact set K such that $K = \cup_{i=1}^n S_i(K)$. Furthermore, if $\Psi(A) = \cup_{i=1}^n S_i(A)$, and $\Psi^k(A) = \Psi(\Psi^{k-1}(A))$, then for any nonempty closed and bounded set A,

$$\Psi^k(A) \to K \quad \text{in Hausdorff metric.}$$

The compact set K is called the invariant set w.r.t. $\mathcal{S}$ or the limit set of the iterated function system of $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$.

(b) In addition to the hypothesis of (a), suppose there is a probability vector $(\rho_1, \ldots, \rho_n)$ such that $\sum_{i=1}^n \rho_i = 1$. Then there exists a unique Borel probability measure μ on X such that $\mu = \sum_{i=1}^n \rho_i \mu \circ S_i^{-1}$, and the support of μ is K. The measure μ is called the invariant measure w.r.t. $(\mathcal{S}, (\rho_i))$.

(c) As a special case of (a) and (b), if $X = \mathbb{R}^d$ and $S_i \in \mathcal{S}$ are similarity maps with similarity ratios $r_i < 1$, then the invariant set $K = \cup_{i=1}^n S_i(K)$ is called the self-similar set w.r.t. $\mathcal{S}$. Moreover Moran and Hutchinson gave a criterion that guarantees that the sets $S_i(K)$, $i = 1, \cdots, n$, do not overlap too much, namely the open set condition. The open set condition (OSC) says that there is an open set $U \neq \emptyset$, such that

(i) $\bigcup_{i=1}^n S_i(U) \subset U$,

(ii) $S_i(U) \bigcap S_j(U) \neq \emptyset$ if $i \neq j$.

If $\mathcal{S}$ satisfies the OSC, then the Hausdorff dimension of K is s, where s satisfies $\sum_{i=1}^{n} r_i^s = 1$, and is called the similarity dimension of $\mathcal{S}$. Also the Hausdorff measure of K is positive and finite, *i.e.* $0 < \mathcal{H}^s(K) < \infty$. In addition, if we let $\rho_i = r_i^s$, $i = 1, 2, \ldots, n$, be the probability vector, then the invariant measure μ w.r.t. $(\mathcal{S}, (r_i^s))$ is called the self-similar measure, and it is, up to a constant, the same as the restricted Hausdorff measure $\mathcal{H}^s \lfloor K$, *i.e.*

$$\mu = \frac{1}{\mathcal{H}^s(K)} \mathcal{H}^s \lfloor K.$$

As an illustration, let K be the Sierpinski gasket (see Figure 1.1) generated by $S_1, S_2, S_3 : \mathbb{R}^2 \to \mathbb{R}^2$ where $S_1(x, y) = \frac{1}{2}(x, y)$, $S_2(x, y) = \frac{1}{2}(x, y) + (\frac{1}{2}, 0)$ and $S_3(x, y) = \frac{1}{2}(x, y) + (\frac{1}{4}, \frac{\sqrt{3}}{4})$. Then $S_1(K)$, $S_2(K)$ and $S_3(K)$ are just the left, right and up 'halves' of K, so that $K = S_1(K) \cup S_2(K) \cup S_3(K)$. Thus K is invariant for mappings S_1, S_2 and S_3, which represent the fundamental self-similarities of the Sierpinski gasket. The Hausdorff dimension of K is $\frac{\log 3}{\log 2}$.

Since 1991, the theory of self-similar set has been further developed. For instance, given a self-similar set K generated by similarity maps $\mathcal{S} = (S_1, S_2, \cdots, S_n)$, Schief [Sch] proved that the following conditions are equivalent.

(i) $\mathcal{H}^s(K) > 0$, where s satisfies $\sum_{i=1}^{n} r_i^s = 1$.

(ii) $\mathcal{S}$ satisfies the open set condition (OSC).

(iii) $\mathcal{S}$ satisfies the strong open set condition (SOSC), *i.e.* there is an open set U satisfying the OSC and $U \cap K \neq \emptyset$.

Remark. *We are more interested in the set U satisfying the SOSC, since $U \cap K \neq \emptyset$. There are examples where an open set U satisfies the OSC, but $U \cap K = \emptyset$. However,*

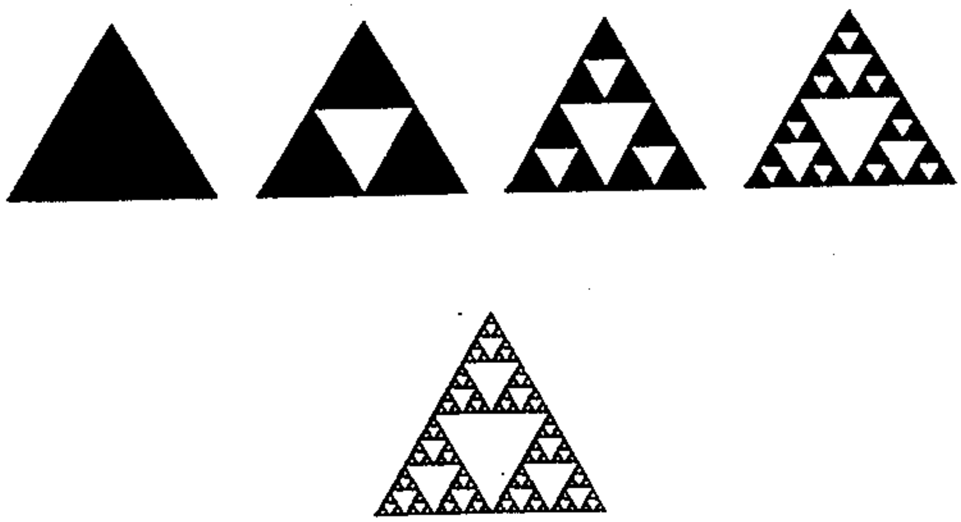

FIGURE 1.1 CONSTRUCTION OF THE SIERPINSKI GASKET

using Schief's result, if $\mathcal{H}^s(K) > 0$, then we know that there exists an open set U satisfying the SOSC.

During the 1980s, Moran and Hutchinson's results were extended to variety of general cases. Graph directed self-similar sets were introduced and studied by Mauldin & Williams [MW1]. Random self-similar constructions were investigated by Falconer [Fa2], Mauldin and williams [MW2], and Graf [Gr]. Recently, infinite conformal iterated function systems are undergoing investigation by Mauldin and Urbanski [MU]. This dissertation is a collection of studies of these fractal sets and their geometric properties.

In Chapter 2, we will give a basic definition of graph directed self-similar sets, then extend Moran and Hutchinson's result to the graph directed case, *i.e.*, we will prove that there exists a unique invariant measure list $(\mu_u)_{u\in V}$ w.r.t. probabilities $(\rho_i)_{i\in E}$. We will also extend Schief's results to strongly connected graph directed constructions.

In Chapter 3, we will attempt to answer a question asked by Mauldin (in a private communication) whether a fractal set K with overlapping construction can still have positive Hausdorff measure w.r.t. its Hausdorff dimension. Such a fractal set F can be constructed by a certain ratio self-similar construction. Furthermore, we will show that the fractal set F constructed is also a fractal set in the sense of Taylor's definition, *i.e.* the Hausdorff dimension and the packing dimension of F are equal. The packing dimension and packing measure are defined as follows:

Let $s \geq 0$ be a fixed real number, for any $\delta > 0$ and $E \subset \mathbb{R}^d$, define

$$\mathcal{P}^s_\delta(E) = \sup\{\sum_{i=1}^{\infty}(2r_i)^s : x_i \in E, r_i < \delta, B(x_i, r_i) \,\text{disjoint}\},$$

and

$$\mathcal{P}^s_0(E) = \lim_{\delta \to 0} \mathcal{P}^s_\delta(E),$$

where $B(x, r)$ denotes the closed ball with center x and radius r.

The set function $\mathcal{P}^s_0$ fails to be countable subadditive (cf. [TT]), so a further stage of the definition is needed,

$$\mathcal{P}^s(E) = \inf\{\sum_{i=1}^{\infty} \mathcal{P}^s_0(E_i) : \; E \in \cup_{i=1}^{\infty} E_i\}. \tag{1.2}$$

The set function $\mathcal{P}^s$ is a Borel measure introduced by Taylor & Tricot [TT] and is called s-dimensional packing measure. Recall the definition of Hausdorff measure by (1.1). We have (cf. [RT]) for all $s \geq 0$, and $E \subset \mathbb{R}^d$,

$$0 \leq \mathcal{H}^s(E) \leq \mathcal{P}^s(E) \leq \infty. \tag{1.3}$$

Both Hausdorff measure and packing measure relate to the local geometric structure of $E \subset \mathbb{R}^d$: $\mathcal{H}^s$ uses economical covers by sets of small diameter; and $\mathcal{P}^s$ uses efficient packings by small balls centered in E.

The packing dimension s of E denoted by $\dim_P(E)$ is defined by

$$\mathcal{P}^t(E) = \begin{cases} \infty & \text{if } t < s; \\ 0 & \text{if } s < t. \end{cases}$$

Taylor defines a set E to be a fractal set if

$$\dim_H(E) = \dim_P(E). \tag{1.4}$$

In Chapter 4, the notion of conformal iterated function system (c.i.f.s) is introduced. We will extend the result in the self-similar case and show that, for a c.i.f.s, the SOSC is equivalent to the conformal measure being zero on the boundary of an open set U satisfying the OSC.

In Chapter 5, we will study the geometric properties of random statistically self-similar sets. In particular, we are interested in the so called δ-condition introduced by Graf, which gives a sufficient condition for the Hausdorff measure of a statistically self-similar fractal set K to be positive almost surely. We will show that for certain random fractals K, if the δ-condition is not satisfied, then the Hausdorff measure of K is zero almost surely. An example in $\mathbb{R}^2$ is provided. Moreover, in Section 5.3, we will give a weak δ-condition and show that the δ-condition and the weak δ-condition are equivalent.

Finally in Chapter 6, fractal sets generated by linear cellular automata are studied. We will associate a fractal set generated by a p-state cellular automaton with a graph directed construction, where p is a prime number. Therefore, by applying the results for graph directed construction, we can calculate the Hausdorff dimension and Hausdorff measure for a fractal set generated by a p-state cellular automaton.

CHAPTER II

THE OPEN SET CONDITION FOR GRAPH DIRECTED SELF-SIMILAR SETS

As we mentioned in Chapter 1, Moran and Hutchinson have studied self-similar sets and developed some basic theorems. The concept and theory of self-similar sets were then extended to graph directed self-similar sets by Mauldin and Williams [MW1] between 1985 and 1988. This extension provides a way to study a larger class of sets. For example, in Chapter 6, we will apply this theory to cellular automata. In this chapter, we will give the basic definitions of graph directed self-similar sets and establish some propositions and theorems, which extend the theory of self-similar sets. In particular, we will show that the OSC and the SOSC are equivalent for certain graph directed self-similar sets.

2.1 Definitions and Notations of Graph Directed Self-Similar Sets

Graph directed self-similar sets are defined and constructed as follows (cf. also [Edg]):

Let (V, E) be a directed graph, where V is the set of vertices and E is the set of edges, such that for each $u \in V$, there are some edges $e \in E$ coming from u. If u, $v \in V$ are vertices and $e \in E$ is an edge, then we denote the set of edges with initial vertex u by E_u, the set of edges from u to v by E_{uv}, the initial vertex of e by $i(e)$, and the terminal vertex of e by $t(e)$.

A list $((V, E), (X_u)_{u \in V}, (S_e)_{e \in E}, (r_e)_{e \in E})$ where:

(1) (V, E) is a directed graph.

(2) X_u is a compact metric space.

(3) $S_e : X_v \to X_u$ is a similarity map, where $e \in E_{uv}$.

(4) r_e is the similarity ratio of S_e such that, for each cycle $\alpha = [e_1\, e_2\, \ldots\, e_q : t(e_q) = i(e_1)]$, $r_\alpha = r_{e_1} r_{e_2} \ldots r_{e_q} < 1$.

is called a Mauldin-Williams graph (MW-graph).

Remark. *By using rescaling (cf. [Edg] p 116), we can, without loss of generality, assume $r_e < 1$ for all $e \in E$.*

If $G = ((V, E), (X_u)_{u\in V}, (S_e)_{e\in E}, (r_e)_{e\in E})$ is a MW-graph, there exists a unique invariant list $(K_u)_{u\in V}$, where each K_u is a nonempty compact subset of X_u, satisfying

$$K_u = \bigcup_{e\in E_u} S_e(K_{t(e)}) \quad \text{for all } u \in V.$$

Moreover, according to [MW1], the sets $(K_u)_{u\in V}$ can be constructed by the following recursive process:

For each $u \in V$, choose a nonempty closed subset J_u of X_u and construct recursively for each $u \in V$ a sequence $(K_{u,n})_{n\in N}$ of nonempty compact subsets of X_u as follows:

(1) Let

$$K_{u,1} = \bigcup_{e_1\in E_u} S_{e_1}(J_{t(e_1)}).$$

(2) Let

$$\begin{aligned} K_{u,2} &= \bigcup_{e_1\in E_u} S_{e_1}(K_{t(e_1),1}) \\ &= \bigcup_{e_1\in E_u} \bigcup_{e_2\in E_{t(e_1)}} (S_{e_1} \circ S_{e_2})(J_{t(e_2)}). \end{aligned}$$

(3) Let

$$K_{u,3} = \bigcup_{e_1\in E_u} S_{e_1}(K_{t(e_1),2})$$

$$= \bigcup_{e_1 \in E_u} \bigcup_{e_2 \in E_{t(e_1)}} (S_{e_1} \circ S_{e_2})(K_{t(e_2),1})$$
$$= \bigcup_{e_1 \in E_u} \bigcup_{e_2 \in E_{t(e_1)}} \bigcup_{e_3 \in E_{t(e_2)}} (S_{e_1} \circ S_{e_2} \circ S_{e_3})(J_{t(e_3)}).$$

Continuing this process, we obtain a sequence of nonempty compact sets

$$\left(K_{u,n} := \bigcup_{e_1 \in E_u} \bigcup_{e_2 \in E_{t(e_1)}} \cdots \bigcup_{e_n \in E_{t(e_{n-1})}} (S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_n})(J_{t(e_n)}) \right)_{n \in N}.$$

It is known that $K_{u,n}$ converges to K_u w.r.t. the Hausdorff metric as $n \to \infty$ (cf. Mauldin and Williams [MW1]). The sets $(K_u)_{u \in V}$ are called graph directed self-similar sets (or a invariant list) associated with G, and the set $K = \cup_{u \in V} K_u$ is called the graph directed construction object. In the original case studied by Moran and Hutchinson, the graph (V, E) has only one vertex, u, and the set $K = K_u$ is called a self-similar set.

Definition 2.1.1. *Let $G = ((V, E), (X_u)_{u \in V}, (S_e)_{e \in E}, (r_e)_{e \in E})$ be a MW-graph such that, for each $u \in V$, X_u is a compact subset of $\mathbb{R}^d$. We say G satisfies the open set condition (OSC) if and only if there exists a list $(U_u)_{u \in V}$ of sets, where U_u is a nonempty, open and bounded subset of X_u, satisfying*

$$\bigcup_{e \in E_u} S_e(U_{t(e)}) \subset U_u \quad \text{for all } u \in V,$$
$$S_e(U_{t(e)}) \bigcap S_{e'}(U_{t(e')}) = \emptyset \quad \text{for all } u \in V \text{ and } e, e' \in E_u \text{ with } e \neq e'.$$

Furthermore, if $U_u \cap K_u \neq \emptyset$ for all $u \in V$, then we say G satisfies the strong OSC (SOSC).

Recall in Chapter 1, for a self-similar set K generated by similarity maps $\mathcal{S} = (S_1, S_2, \cdots, S_n)$, if $\mathcal{S}$ satisfies the OSC, then the Hausdorff dimension of K is s where s satisfies $\sum_{i=1}^{n} r_i^s = 1$, and the $\mathcal{H}^s$ measure is positive *i.e.* $\mathcal{H}^s(K) > 0$.

Moreover, both OSC and SOSC are equivalent to $\mathcal{H}^s(K) > 0$. However, the situation for a graph directed construction G is more complicated. Mauldin and Williams calculated the Hausdorff dimension when G satisfies the OSC. The Hausdorff dimension s is determined by $\Phi(s) = 1$, where $\Phi(s)$ is the spectral radius of the matrix $A_s = \left(\sum_{e \in E_{uv}} r_e^s\right)_{u,v \in V}$. If in addition, the graph G is strongly connected, *i.e.* for each pair of vertices u and v, there is a directed path from u to v, then the $\mathcal{H}^s$ measure of K is positive and finite *i.e.* $0 < \mathcal{H}^s(K) < \infty$. However, do we still have the equivalencies among the OSC, the SOSC and the $\mathcal{H}^s$ measure of K being positive? We will investigate it in Section 2.2.

We provide here some notations which will be used in Sections 2.2. Given a MW-graph G, we define E^*_{uv} for the set of all finite paths α with initial vertex u and terminal vertex v. We will also say that such a path goes from u to v, or it connects u and v. The number of edges in a path is its length, written by $|\alpha|$. We will write $E^{(n)}_{uv}$ for the set of all paths from u to v of length n; and $E^{(n)}_u$ for the set of all paths of length n with initial vertex u; and $E^{(n)}$ for the set of all paths of length n. The empty set convention will work out best if we say (by convention) that for each $u \in V$, the set $E^{(0)}_{uu}$ has only one element, which is the empty path from u to itself. Of course we may identify E with $E^{(1)}$ and E_{uv} with $E^{(1)}_{uv}$. We define E^* to be the set of all finite paths, and E^*_u the set of all finite paths in the graph starting from vertex u. Note that E^* is a disjoint union of E^*_u for all $u \in V$. If $\alpha \in E^*$, then we denote the initial vertex of α by $i(\alpha)$ and the terminal vertex by $t(\alpha)$. Also if the strings α, β represent paths, and the terminal vertex of α is equal to the initial vertex of β, *i.e.* $t(\alpha) = i(\beta)$, then the concatenated string $\alpha\beta$ represents a path as well. A path α with $i(\alpha) = t(\alpha)$ is called a cycle.

We will consider infinite paths as well. An infinite string ω corresponds to an infinite path if the terminal vertex for each edge matches the initial vertex for the next edge. We write $E^{(\omega)}$ for the set of all infinite paths for the MW-graph G, and call it the string space. If $u \in V$ is a vertex, then we write $E_u^{(\omega)}$ for the set of all infinite paths starting at u. If $\alpha \in E^*$, then we write $[\alpha] = \{\sigma \in E^{(\omega)} : \quad \alpha \preceq \sigma\}$ and call it the cylinder set generated by α. Thus, $[\alpha]$ is the set of all infinite paths that begin with the path α.

The following abbreviations are important.

If $\alpha = (e_1, e_2, \ldots, e_n) \in E^*$, we write

$$S_\alpha = S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_n}$$

$$K_\alpha = S_\alpha \left(K_{t(\alpha)}\right)$$

$$r_\alpha = r_{e_1} r_{e_2} \ldots r_{e_n}.$$

Furthermore, set

$$r_{\max} = \max\{r_e : e \in E\}$$

$$r_{\min} = \min\{r_e : e \in E\}.$$

Definition 2.1.2. *Let G be a MW-graph, A projection map π from $E^{(\omega)}$ to $X = \cup_{u \in V} X_u$ is defined by*

$$\pi : E^{(\omega)} \to X$$

$$\sigma = (e_1 e_2 \ldots) \to \cap_{k=1}^{\infty} S_{\sigma|_k}(X_{t(\sigma|_k)}),$$

where $\sigma|_k = (e_1 e_2 \ldots e_k)$.

Definition 2.1.3. *Let G be a MW-graph, we define the construction matrix by*

$$A_s = \left(\sum_{e \in E_{uv}} r_e^s \right)_{u,v \in V},$$

where $s \geq 0$ satisfies $\Phi(s) = 1$ and $\Phi(s)$ is the spectral radius of A_s. The number s is call the dimension of graph G.

Remark. *If G is strongly connected, then A_s is irreducible.*

2.2 Theorems and Proofs

This section contains five propositions and a theorem. Proposition 2.2.1 shows that there exists a "natural" probability measure μ^* on the string space $E^{(\omega)}$, such that the left-shift map T on $E^{(\omega)}$ is ergodic w.r.t. μ^*. By using the projection map $\pi : E^{(\omega)} \to X = \bigcup_{u \in V} X_u$, we get a pull down measure $\hat{\mu} = \mu^* \circ \pi^{-1}$ on X. The measure μ^* is natural in that $\hat{\mu}$ is equivalent to the Hausdorff measure $\mathcal{H}^s\lfloor K$. Proposition 2.2.2 shows that the Hausdorff measure of the intersection $K_e \bigcap K_{e'}$ is zero for $e \neq e'$. This proposition is very useful in the proofs of the rest of propositions. Next, we introduce the notion of a graph directed self-similar measure list $(\mu_u)_{u \in V}$ on $(X_u)_{u \in V}$ for a given MW-graph G. Then we provide the existence and uniqueness of the graph directed self-similar measure list $(\mu_u)_{u \in V}$ in Proposition 2.2.3. In Proposition 2.2.4, we show that, up to a constant, the restricted Hausdorff measure $\mathcal{H}^s\lfloor K_u$ is the same as μ_u. Proposition 2.2.5 presents two properties of the self-similar measure list $(\mu_u)_{u \in V}$, which are used in the proof of Theorem 2.2.6. Finally, we prove Theorem 2.2.6 that both the OSC and the SOSC are equivalent to $\mathcal{H}^s(K) > 0$ for strongly connected MW-graphs.

Proposition 2.2.1. *Let G be a strongly connected MW-graph satisfying the OSC. Then there exists a unique ergodic T-invariant probability measure μ^* on the string space $E^{(\omega)}$ such that the image measure $\mu^* \circ \pi^{-1}$ and the restricted Hausdorff measure $\mathcal{H}^s\lfloor K$ are equivalent.*

Proof of Proposition 2.2.1. We first define a measure μ^* on the string space $E^{(\omega)}$, then we show that it is ergodic w.r.t. the left-shift map T.

Let $A_s = \left(\sum_{e \in E_{uv}} r_e^s\right)_{u,v \in V}$ be the construction matrix of G. Since the graph

G is strongly connected, A_s is irreducible. The Perron-Frobenius Theorem [Wal] tells us that there exists a left eigenvector $p = (p_u^s)_{u\in V}$ and a right eigenvector $q = (q_v^s)_{v\in V}$ such that

$$pA = p \quad i.e. \quad \sum_{u\in V}\sum_{e\in E_{uv}} p_u^s r_e^s = p_v^s \quad \text{and} \quad p_u > 0 \quad \text{for all } u \in V; \tag{2.1}$$

and

$$Aq = q \quad i.e. \quad \sum_{v\in V}\sum_{e\in E_{uv}} r_e^s q_v^s = q_u^s \quad \text{and} \quad q_u > 0 \quad \text{for all } u \in V. \tag{2.2}$$

We can normalize p, q such that

$$\sum_{u\in V} p_u^s q_u^s = 1. \tag{2.3}$$

In view of (2.1) and (2.2), one can define a Borel probability measure μ_n^* on C_n, by putting $\mu_n^*([\alpha]) = p_{i(\alpha)}^s r_\alpha^s q_{t(\alpha)}^s$, where C_n is the algebra generated by the cylinder sets of the form $[\alpha]$, where $\alpha \in E^{(n)}$. Using (2.1) and (2.2), we have the following equalities:

$$\begin{aligned}\sum_{u,v\in V}\sum_{e\in E_{uv}} \mu_1^*[e] &= \sum_{u,v\in V}\sum_{e\in E_{uv}} (p_u r_e q_v)^s \\ &= \sum_{u\in V} p_u^s \sum_{v\in V}\sum_{e\in E_{uv}} (r_e q_v)^s \\ &= \sum_{u\in V} (p_u q_u)^s \\ &= 1.\end{aligned}$$

Since $[\alpha] = \cup_{e\in E_v}[\alpha e]$, where $\alpha \in E_{uv}^{(n)}$, we have

$$\begin{aligned}\mu_{n+1}^*(\cup_{e\in E_v}[\alpha e]) &= \sum_{v'\in V}\sum_{e\in E_{vv'}} \mu_{n+1}^*[\alpha e] \\ &= \sum_{v'\in V}\sum_{e\in E_{vv'}} (p_u r_\alpha r_e q_{v'})^s\end{aligned}$$

$$
\begin{aligned}
&= (p_u r_\alpha)^s \sum_{v' \in V} \sum_{e \in E_{vv'}} (r_e q_{v'})^s \\
&= (p_u r_\alpha q_v)^s \\
&= \mu_n^*([\alpha]).
\end{aligned}
$$

Using Kolmogorov's Existence Theorem, there exists a unique probability measure μ^* on $E^{(\omega)}$ such that $\mu^*([\alpha]) = \mu^*_{|\alpha|}([\alpha])$ for all $\alpha \in E^*$. Using (2.1) again, we get

$$
\begin{aligned}
\mu^*_{n+1}\left(\cup_{v' \in V} \cup_{e \in E_{v'u}} [e\alpha]\right) &= \sum_{v' \in V} \sum_{e \in E_{v'u}} \mu_n^*([e\alpha]) \\
&= \sum_{v' \in V} \sum_{e \in E_{v'u}} (p_{v'} r_e r_\alpha q_{t(\alpha)})^s \\
&= \left(\sum_{v' \in V} \sum_{e \in E_{v'u}} (p_{v'} r_e)^s \right) (r_\alpha q_{t(\alpha)})^s \\
&= (p_{i(\alpha)} r_\alpha q_{t(\alpha)})^s \\
&= \mu_n^*([\alpha]).
\end{aligned}
$$

This tells us that the left-shift map $T : E^{(\omega)} \to E^{(\omega)}$ by $T\big((e_1 e_2 \dots)\big) = (e_2 e_3 \dots)$ is μ^*-measure preserving.

Now we will show that μ^* is ergodic. Let A be a Borel set in $E^{(\omega)}$ with $\mu^*(A) > 0$, there exists $u \in V$ such that $\mu^*(A_u) > 0$ where $A_u = A \cap E_u^{(\omega)}$. Since the nested family of sets $\{[\alpha] : \ \alpha \in E_u^*\}$ generates the Borel σ-algebra on $E_u^{(\omega)}$, for every $n \geq 0$ and every $\alpha \in \cup_{v \in V} E_{vu}^{(n)}$, we can find a subfamily B of E_u^* consisting of mutually incomparable finite strings and such that $A_u \subset \{[\beta] : \beta \in B\}$ and $\sum_{\beta \in B} \mu^*([\alpha\beta]) = \mu^*(\cup\{[\alpha\beta] : \beta \in B\}) \leq \lambda \mu^*(\alpha A_u)$, where $\alpha A_u = \{\alpha\omega : \omega \in A_u\}$, and $\lambda > 1$. Then

$$
\begin{aligned}
\mu^*(T^{-n}(A) \cap [\alpha]) &= \mu^*(\alpha A) \\
&\geq \frac{1}{\lambda} \sum_{\beta \in B} \mu^*([\alpha\beta]) \\
&= \frac{1}{\lambda} \sum_{\beta \in B} p^s_{i(\alpha)} r^s_\alpha r^s_\beta q^s_{t(\beta)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\lambda} p_{i(\alpha)}^{s} r_{\alpha}^{s} \sum_{\beta \in B} r_{\beta}^{s} q_{t(\beta)}^{s} \\
&\geq \frac{1}{\lambda} p_{i(\alpha)}^{s} r_{\alpha} \frac{q_{t(\alpha)}^{s}}{q_{\max}^{s}} \sum_{\beta \in B} \frac{p_{i(\beta)}^{s}}{p_{\max}^{s}} r_{\beta}^{s} q_{t(\beta)}^{s} \\
&= \frac{1}{\lambda} q_{\max}^{-s} p_{\max}^{-s} \mu^{*}([\alpha]) \sum_{\beta \in B} \mu^{*}([\beta]) \\
&\geq \frac{1}{\lambda} q_{\max}^{-s} p_{\max}^{-s} \mu^{*}([\alpha]) \mu^{*}\big(\cup \{[\beta] : \beta \in B\}\big) \\
&\geq \frac{1}{\lambda} a \mu^{*}([\alpha]) \mu^{*}(A_u),
\end{aligned}
$$

where $a = q_{\max}^{-s} p_{\max}^{-s}$.

Therefore, we have

$$
\begin{aligned}
\mu^{*}\big(T^{-n}(E_u^{(\omega)} \backslash A_u) \cap [\alpha]\big) &= \mu^{*}\left([\alpha] \backslash \big(T^{-n}(A_u) \cap [\alpha]\big)\right) \\
&= \mu^{*}([\alpha]) - \mu^{*}\big(T^{-n}(A_u) \cap [\alpha]\big) \\
&\leq \big(1 - \frac{1}{\lambda} a \mu^{*}(A_u)\big) \mu^{*}([\alpha]).
\end{aligned}
$$

Hence, for every Borel set $A \subset E^{(\omega)}$ with $\mu^{*}(A) < 1$, for every $n \geq 0$ and for every $\alpha \in E^{(n)}$, we get

$$
\begin{aligned}
\mu^{*}\big(T^{-n}(A) \cap [\alpha]\big) &= \mu^{*}(T^{-n}(A_u) \cap [\alpha]), \qquad \text{where } u = t(\alpha) \\
&\leq \big(1 - \frac{1}{\lambda} a \mu^{*}(E_u^{(\omega)} \backslash A_u)\big) \mu^{*}([\alpha]). \qquad (2.4)
\end{aligned}
$$

In order to complete the proof of ergodicity of T, suppose that $T^{-1}A = A$ and $0 < \mu^{*}(A) < 1$, this implies $0 < \mu^{*}(A_u) < \mu^{*}(E_u^{(\omega)})$ for each $u \in V$. Let $d(A) = \min\{\mu^{*}(E_u^{(\omega)} \backslash A_u) : u \in V\}$, then $0 < d(A) < 1$. We can set $\gamma = 1 - \frac{1}{\lambda} a d(A)$. Note that $0 < \gamma < 1$. In view of (2.4), for every $\alpha \in E^{*}$, we get

$$
\mu^{*}(A \cap [\alpha]) = \mu^{*}\big(T^{-|\alpha|}(A) \cap [\alpha]\big) \leq \gamma \mu^{*}([\alpha]).
$$

Now, take $\eta > 1$ such that $\gamma\eta < 1$, and choose a subfamily R of E^{*} consisting of mutually incomparable finite strings and such that $A \subset \cup\{[\alpha] : \alpha \in R\}$ and

$\mu^*(\cup\{[\alpha] : \alpha \in R\}) \leq \eta\mu^*(A)$. Then $\mu^*(A) = \sum_{\alpha\in R} \mu^*(A\cap[\alpha]) \leq \sum_{\alpha\in R} \gamma\mu^*([\alpha]) = \gamma\mu^*(\{[\alpha] : \alpha \in R\}) \leq \gamma\eta\mu^*(A)$. This contradiction completes the proof of ergodicity of T.

Now let's look at the measure μ^* on the subspace $E_u^{(\omega)}$ for each $u \in V$.

Since

$$\begin{aligned}
\mu^*(E_u^{(\omega)}) &= \sum_{v\in V} \sum_{e\in E_{uv}} \mu^*([e]) \\
&= \sum_{v\in V} \sum_{e\in E_{uv}} (p_u r_e q_v)^s \\
&= p_u^s \sum_{v\in V} \sum_{e\in E_{uv}} (r_e q_v)^s \\
&= p_u^s q_u^s,
\end{aligned}$$

we have $\mu^*(E_u^{(\omega)}) = p_u^s q_u^s$, and $\mu^*(E^{(\omega)}) = \sum_{u\in V} \mu^*(E_u^{(\omega)}) = \sum_{u\in V} p_u^s q_u^s = 1$ by (2.3)

By using the projection map $\pi : E^{(\omega)} \to X = \cup_{u\in V} X_u$, we get a pull down measure $\mu^* \circ \pi^{-1}$ on X such that if $A \subset X$

$$\mu^* \circ \pi^{-1}(A) = \mu^*\big(\pi^{-1}(A)\big). \tag{2.5}$$

Since G satisfies the OSC, it follows that the pull down measure $\mu^* \circ \pi^{-1}$ is, up to a constant, the same as the restricted Hausdorff measure $\mathcal{H}^s\lfloor K$. This can be verified by a similar argument by Mauldin and Williams [MW1] (cf. also [Edg] p 172). □

Remark. *In Proposition 2.2.1, if we do not assume G satisfying the OSC, we can still get*

$$\mathcal{H}^s\lfloor K \ll \mu^* \circ \pi^{-1},$$

by a similar argument as in [Edg].

Proposition 2.2.2. *Let G be a strongly connected MW-graph and $K = \cup_{u \in V} K_u$ be the graph directed self-similar object of G, if $\mathcal{H}^s(K) > 0$, then*

$$\mathcal{H}^s(S_e(K_v) \cap S_{e'}(K_{v'})) = 0$$

for all $e \in E_{uv},\ e' \in E_{uv'}$ with $e \neq e'$.

Proof of Proposition 2.2.2. We will prove the proposition by contradiction.

Let $D = S_e(K_v) \cap S_{e'}(K_{v'})$, suppose $\mathcal{H}^s(D) > 0$, then $\mu^*(\pi^{-1}(D)) > 0$ since $\mathcal{H}^s\lfloor K \ll \mu^* \circ \pi^{-1}$.

Let $B_e = \{\sigma \in E_u^{(\omega)} : \sigma(1) = e,\ \pi(\sigma) \in D\}$, then $\mu^*(B_e) > 0$ for at least one $e \in E$ say e_i. Clearly

$$\pi(E_u^{(\omega)} \setminus B_{e_i}) = K_u, \tag{2.6}$$

since if $x \in D$, then x has at least two preimages σ, σ' such that $\sigma(1) \neq \sigma'(1)$. We can choose one, say σ, such that $\sigma(1) \neq e_i$, then $\sigma \in E_u^{(\omega)} \setminus B_{e_i}$ and $\pi(\sigma) = x$. Clearly $\mu^*(E_u^{(\omega)} \setminus B_{e_i}) < p_u^s q_u^s = \mu^*(E_u^{(\omega)})$.

Let $F_u = \{\sigma \in E^{(\omega)} : \sigma = (e_1 e_2 \cdots)$ with infinitely many e_j such that $t(e_j) = u\}$.

Claim: $\mu^*(F_u) = 1$

Proof of Claim:

Let $F_u^0 = \{\sigma \in E^{(\omega)} : \sigma = (e_1 e_2 \cdots) \quad t(e_j) \neq u \quad \text{for all } j\}$, and let $e \in E$ such that $t(e) = u$, then $[e] \cap F_u^0 = \emptyset$. Hence $\mu^*(F_u^0) < 1$.

Since

$$T^{-1}(E^{(\omega)} \setminus F_u^0) \subset E^{(\omega)} \setminus F_u^0,$$

ergodicity of T implies $\mu^*(F_u^0) = 0$.

Since

$$E^{(\omega)} \setminus F_u = \cup_{n \geq 0} T^{-n}(F_u^0),$$

we obtain $\mu^*(E^{(\omega)} \setminus F_u) = 0$ *i.e.* $\mu^* F_u = 1$.

Let $\mathcal{B}_u = \{[\alpha] : \alpha \in E^*_{uu}\}$. Clearly $\mathcal{B}_u$ induces the same σ-algebra on $E_u^{(\omega)} \cap F_u$ as $\{[\alpha] : \alpha \in E^*_u\}$ does. Since $(E_u^{(\omega)} \setminus B_{e_i}) \cap F_u$ is open in $E_u^{(\omega)} \cap F_u$, there exist pairwise disjoint cylinder sets $[\alpha_k] \in \mathcal{B}_u$ such that $(E_u^{(\omega)} \setminus B_{e_i}) \cap F_u = \cup_k [\alpha_k]$. Hence

$$\begin{aligned}
\mu^*(E_u^{(\omega)} \setminus B_{e_i}) &= \mu^*\left((E_u^{(\omega)} \setminus B_{e_i}) \cap F_u\right) \\
&= \sum_k \mu^*([\alpha_k]) \\
&= \sum_k p_u^s r_{\alpha_k}^s q_u^s \\
&= p_u^s q_u^s \sum_k r_{\alpha_k}^s \\
&< p_u^s q_u^s.
\end{aligned}$$

The last inequality is because $\mu^*(E_u^{(\omega)} \setminus B_{e_i}) < \mu^* E_u^{(\omega)} = p_u^s q_u^s$. So there is some $0 < \lambda < 1$ such that $\sum_k r_{\alpha_k}^s = \lambda < 1$.

Let $K'_u = \pi((E_u^{(\omega)} \setminus B_{e_i}) \cap F_u)$. Then $\mathcal{H}^s(K_u \setminus K'_u) = 0$ by (2.6) and the Claim. Since $(E_u^{(\omega)} \setminus B_{e_i}) \cap F_u = \cup_k [\alpha_k]$, we can cover K'_u by the sets $S_{\alpha_k}(K'_u)$, $k = 1, 2, \ldots$ and also by the smaller sets $S_{\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_m}}$ where $k_1, k_2, \ldots, k_m \in \{1, 2, \ldots\}$ for each m. Hence, $\mathcal{H}^s(K'_u) = 0$ follows from

$$\begin{aligned}
\sum_{k_1, k_2, \ldots, k_m} |S_{\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_m}}(K'_u)|^s &= \sum_{k_1, k_2, \ldots, k_m} r^s_{\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_m}} |K'_u|^s \\
&= |K'_u|^s \lambda^m \\
&\to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Consequently, we have $\mathcal{H}^s(K_u) = 0$. This contradicts $\mathcal{H}^s(K_u) > 0$. $\square$

Recall in Chapter 1, for a self-similar set K w.r.t. similarity maps $\mathcal{S} = (S_1, S_2, \cdots, S_n)$, there exists a unique self-similar measure μ w.r.t. $(\mathcal{S}, (r_i))$ such that

$$\mu = \frac{1}{\mathcal{H}^s(K)} \mathcal{H}^s \lfloor K.$$

Here we extent this result to graph directed self-similar constructions.

Definition 2.2.1. *A MW-graph with probabilities is a list*

$$\mathcal{S} = ((V,E),\, (X_u)_{u\in V},\, (S_e)_{e\in E},\, (r_e)_{e\in V},\, (\rho_e)_{e\in E})$$

where

(1) $G = ((V,E),\, (X_u)_{u\in V},\, (S_e)_{e\in E},\, (r_e)_{e\in E})$ *is a MW-graph,*

(2) $\sum_{e\in E_u} \rho_e = 1$ *for each* $u \in V$ *and* $0 \le \rho_e \le 1$.

Definition 2.2.2. *A graph directed self-similar measure list* $(\mu_u)_{u\in V}$ *associated with a MW-graph* G *with probabilities is a list of Borel probability measures* $(\mu_u)_{u\in V}$ *on* $(X_u)_{v\in V}$*, which satisfies, for each* $u \in V$

$$\mu_u = \sum_{e\in E_u} \rho_e \mu_{t(e)} \circ S_e^{-1}. \tag{2.7}$$

In order to investigate the existence and uniqueness of such a measure list, we need the following definitions and notations.

We define $\mathbf{M}(X_u)$ to be the set of probability measures on X_u, and $\mathbf{M}(X) = \prod_{u\in V} \mathbf{M}(X_u)$ to be the product space of $\mathbf{M}(X_u)$.

Let $C(X_u) = \{f : X_u \to \mathbb{R} : f \text{ is continuous}\}$ equipped with the ∞-norm. Recall that X_u is a compact metric space for each $u \in V$, so f is continuous implies that f is uniformly continuous.

For $\mu \in \mathbf{M}(X_u)$, $f \in C(X_u)$, we define:

$$\mu(f) = \int f d\mu.$$

Using the Riesz Representation Theorem, $\mathbf{M}(X_u)$ can be identified with a convex subset of the unit ball in $C(X_u)^*$. This allows us to get a topology on $\mathbf{M}(X_u)$ from the weak* topology on $C(X_u)^*$.

Definition 2.2.3. *The weak topology on* $\mathrm{M}(X_u)$ *is the smallest topology making each of the maps:* $\mu \to \int_{X_u} f d\mu \quad (f \in C(X_u))$ *continuous.*

Remark. $\mathrm{M}(X_u)$ *is compact and metrizable in the weak topology [Wal. Theorem 6.5]. In fact, a compatible metric is given by [Wal]. However, for our purpose, a new compatible metric has to be defined.*

Definition 2.2.4. *For each pair* $\mu, \nu \in \mathrm{M}(X_u)$*, let*

$$L_u(\mu, \nu) = \sup\left\{ \left| \int f d\mu - \int f d\nu \right| : \ f \in C(X_u) \quad \text{and} \quad \mathrm{Lip}\, f \leq 1 \right\}.$$

It is clear that L_u is a metric on $\mathrm{M}(X_u)$. Moreover, the L_u metric topology and the weak topology coincide on $\mathrm{M}(X_u)$, as indicated in Lemma 2.2.3.

Lemma 2.2.3. *The* L_u *metric topology and the weak topology coincide on* $\mathrm{M}(X_u)$.

Proof of Lemma 2.2.3.

(i) $\{\mu_n\}$ converging to μ in L_u metric topology implies $\{\mu_n\}$ converging to μ in weak topology, because the set of Lipschitz functions on X_u is dense in $C(X_u)$.

(ii) Let $\{\mu_n\}$ converges to μ in the weak topology, then Theorem 6.8 in [Par] implies that

$$\lim_{n\to\infty} \sup_{f\in\mathcal{A}} \left| \int f d\mu_n - \int f d\mu \right| = 0$$

for every family $\mathcal{A} \subset C(X_u)$ which is equicontinuous at all the points $x \in X_u$ and uniformly bounded.

Pick any $x_0 \in X$, since the set

$$\mathcal{A}_0 = \{f \in C(X_u) : f(x_0) = 0 \quad \text{and} \quad \mathrm{Lip}\, f \leq 1\}$$

is equicontinuous at all the points of X_u and uniformly bounded, so we have

$$\lim_{n\to\infty} \sup_{f\in\mathcal{A}_0} \left| \int f d\mu_n - \int f d\mu \right| = 0.$$

Now using the fact that, for any $f \in C(X_u)$ with $\operatorname{Lip} f \leq 1$,

$$\int f d\mu_n - \int f d\mu = \int (f - f(x_0)) d\mu_n - \int (f - f(x_0)) d\mu,$$

we have $\mu_n \to \mu$ in L_u metric topology. $\square$

Hence the compactness of $\mathbf{M}(X_u)$ in the weak topology implies that L_u is also complete.

Definition 2.2.5. *For $(\mu_u)_{u \in V},\ (\nu_u)_{u \in V} \in \mathbf{M}(X) = \prod_{u \in V} \mathbf{M}(X_u)$, let*

$$L((\mu_u)_{u \in V},\ (\nu_u)_{u \in V}) = \sup_{u \in V} L_u(\mu_u,\ \nu_u).$$

It is clear that L is a metric on $\mathbf{M}(X)$ and induces the product topology of the weak topologies, by Lemma 2.2.3. Therefore the compactness of $\mathbf{M}(X)$ allows us to use the Banach Contraction Mapping Principle.

Definition 2.2.6. *Let $\Psi : \mathbf{M}(X) \to \mathbf{M}(X)$ be the map, defined by:*

$$\Psi((\mu_u)_{u \in V}) = \left(\sum_{e \in E_u} \rho_e \mu_{t(e)} \circ S_e^{-1} \right)_{u \in V} .$$

Remark. *Comparing Definitions 2.2.6 and 2.2.2, we conclude: a graph directed self-similar measure list $(\mu_u)_{u \in V}$ is a fixed point of Ψ. Therefore, we can also call a graph directed self-similar measure list an invariant measure list.*

Proposition 2.2.3. *Let*

$$\mathcal{S} = ((V, E),\ (X_u)_{u \in V},\ (S_e)_{e \in E},\ (r_e)_{e \in V},\ (\rho_e)_{e \in E})$$

be a MW-graph with probabilities. Then we have the following:

(i) *$\Psi : \mathbf{M}(X) \to \mathbf{M}(X)$ is a contraction map in the L metric.*

(ii) *There exists a unique* $(\mu_u)_{u\in V} \in \mathbf{M}(X)$, *such that*

$$\Psi((\mu_u)_{u\in V}) = (\mu_u)_{u\in V} \quad i.e.\ \mu_u = \sum_{e\in E_u} \rho_e \mu_{t(e)} \circ S_e^{-1} \quad \text{for each } u \in V.$$

Moreover, if $(\nu_u)_{u\in V} \in \mathbf{M}(X)$, *then* $\{\Psi^k((\nu_u)_{u\in V})\}_k$ *converges to* $(\mu_u)_{u\in V}$ *in the* L *metric, and therefore in the product topology of weak topologies.*

Proof of Proposition 2.2.3. Assertion (ii) follows immediately from (i) since $(\mathbf{M}(X)\,,\,L)$ is a complete metric space.

To establish (i), suppose $f \in C(X_u)$ with $\operatorname{Lip} f \le 1$ and let $r = \max_{e\in E} r_e < 1$. Then for $(\mu_u)_{u\in V},\ (\nu_u)_{u\in V} \in \mathbf{M}(X)$, and for each $u \in V$,

$$\begin{aligned}
&\left| \sum_{e\in E_u} \rho_e \mu_{t(e)} \circ S_e^{-1}(f) - \sum_{e\in E_u} \rho_e \nu_{t(e)} \circ S_e^{-1}(f) \right| \\
=&\left| \sum_{e\in E_u} \rho_e \left(\mu_{t(e)} \circ S_e^{-1}(f) - \nu_{t(e)} \circ S_e^{-1}(f)\right) \right| \\
=&\left| \sum_{e\in E_u} \rho_e \left(\int f \circ S_e d\mu_{t(e)} - \int f \circ S_e d\nu_{t(e)}\right) \right| \\
\le& \sum_{e\in E_u} \rho_e r \left| \int r^{-1} f \circ S_e d\mu_{t(e)} - \int r^{-1} f \circ S_e d\nu_{t(e)} \right| \\
\le& \sum_{e\in E_u} \rho_e r L_{t(e)}(\mu_{t(e)}, \nu_{t(e)}) \\
\le&\ rL((\mu_u)_{u\in V},\ (\nu_u)_{u\in V}).
\end{aligned}$$

The last two inequalities follow from $\operatorname{Lip}(r^{-1} f \circ S_e) \le r^{-1}\cdot 1 \cdot r_e \le 1$, $\sum_{e\in E_u} \rho_e = 1$ and the definition of L.

Hence

$$L\left(\Psi((\mu_u)_{u\in V}),\ \Psi((\nu_u)_{u\in V})\right) \le rL((\mu_u)_{u\in V},\ (\nu_u)_{u\in V}) \quad \square$$

Remark. *The graph* G *in Proposition 2.2.3 is not required to be strongly connected.*

Corollary of Proposition 2.2.3. *Let G be a strongly connected MW-graph and $\rho_e = q_{i(e)}^{-s} r_e^s q_{t(e)}^s$, where $(q_u)_{u \in V}$ is the same as in (2.2). Then there exists a unique graph directed invariant measure list $(\mu_u)_{u \in V}$ such that for each $u \in V$,*

$$\mu_u = \sum_{e \in E_u} q_u^{-s} r_e^s q_{t(e)}^s \mu_{t(e)} \circ S_e^{-1}.$$

Proof of the Corollary. In order to apply Proposition 2.2.3, We only need to check that

$$\sum_{e \in E_u} q_u^{-s} r_e^s q_{t(e)}^s = 1 \quad \text{for each } u \in V.$$

However, this follows immediately from (2.2) □

The following proposition tells us the relation between the invariant measure list $(\mu_u)_{u \in V}$ and the restricted Hausdorff measure list $(\mathcal{H}^s \lfloor K_u)_{u \in V}$.

Proposition 2.2.4. *Let G be a strongly connected MW-graph, $(\mu_u)_{u \in V}$ be the self-similar measure list associated with $\rho_e = q_{i(e)}^{-s} r_e^s q_{t(e)}^s$ and $\mathcal{H}^s(K) > 0$. Then for each $u \in V$, we have*

$$\mu_u = (\mathcal{H}^s(K_u))^{-1} \mathcal{H}^s \lfloor K_u.$$

Proof of Proposition 2.2.4. By uniqueness of the graph directed self-similar measure list $(\mu_u)_{u \in V}$, we only need to show that the measure list $\left((\mathcal{H}^s(K_u))^{-1} \mathcal{H}^s \lfloor K_u\right)_{u \in V}$ is also a graph directed self-similar measure list associated with $\rho_e = q_{i(e)}^{-s} r_e^s q_{t(e)}^s$. We will need the following two equalities.

$$\mathcal{H}^s(K_u) = \mathcal{H}^s(\cup_{e \in E_u} S_e(K_{t(e)})) = \sum_{v \in V} \sum_{e \in E_{uv}} r_e^s \mathcal{H}^s(K_v), \tag{2.8}$$

$$\mathcal{H}^s \lfloor S_e(K_v) = r_e^s (\mathcal{H}^s \lfloor K_v) \circ S_e^{-1}. \tag{2.9}$$

The first equality follows from Proposition 2.2.2; the second equality can be shown as follows:

$$\mathcal{H}^s \lfloor S_e(K_v)(E) = \mathcal{H}^s(S_e(K_v) \cap E)$$

$$= \mathcal{H}^s\left(S_e\big((K_v) \cap S_e^{-1}(E)\big)\right)$$

$$= r_e^s \mathcal{H}^s(K_v \cap S_e^{-1}(E))$$

$$= r_e^s \big(\mathcal{H}^s \lfloor K_v\big) \circ S_e^{-1}(E).$$

We divide the proof of Proposition 2.2.4 into two claims.

Claim 1. There is a constant $\lambda > 0$ such that $\mathcal{H}^s(K_u) = \lambda q_u^s$ for each $u \in V$.

Proof of Claim 1. By (2.8), we have

$$\sum_{v \in V} \sum_{e \in E_{uv}} r_e^s \mathcal{H}^s(K_v) = \mathcal{H}^s(K_u).$$

Therefore, the vector $(\mathcal{H}^s(K_u))_{u \in V}$ is a right eigenvector of the matrix

$$A = \Big(\sum_{e \in E_{uv}} r_e^s\Big)_{u,v \in V}$$

with eigenvalue 1. Since A is irreducible with spectral radius 1, and the vector $(q_u^s)_{u \in V}$ is also a right eigenvector with eigenvalue 1, by (2.2). The Perron-Frobenius Theorem implies that there is a constant $\lambda > 0$ such that

$$\mathcal{H}^s(K_u) = \lambda q_u^s \quad \text{for each} \quad u \in V.$$

Hence $(\mathcal{H}^s(K_u))^{-1}\mathcal{H}^s\lfloor K_u = \lambda^{-1} q_u^{-s} \mathcal{H}^s \lfloor K_u$.

Claim 2. $\big((\mathcal{H}^s(K_u))^{-1}\mathcal{H}^s\lfloor K_u\big)_{u \in V}$ is an invariant measure list associated with $\rho_e = q_{i(e)}^{-s} r_e^s q_{t(e)}^s$. That is

$$(\mathcal{H}^s(K_u))^{-1}\mathcal{H}^s\lfloor K_u = \sum_{v \in V} \sum_{e \in E_{uv}} q_u^{-s} r_e^s q_v^s \big((\mathcal{H}^s(K_v))^{-1}\mathcal{H}^s\lfloor K_v\big) S_e^{-1}. \tag{2.10}$$

Proof of Claim 2.

$$\text{The right hand side of (2.10)} = \sum_{v \in V} \sum_{e \in E_{uv}} q_u^{-s} r_e^s q_v^s \lambda^{-1} q_v^{-s} \big(\mathcal{H}^s \lfloor K_v\big) \circ S_e^{-1}$$

$$
\begin{aligned}
&= \sum_{v\in V}\sum_{e\in E_{uv}} q_u^{-s} r_e^s \lambda^{-1} \big(\mathcal{H}^s\lfloor K_v\big) \circ S_e^{-1} \\
&= \lambda^{-1} q_u^{-s} \sum_{v\in V}\sum_{e\in E_{uv}} \mathcal{H}^s\lfloor S_s(K_v) \\
&= \lambda^{-1} q_u^{-s} \mathcal{H}^s\lfloor K_u \\
&= \text{the left hand side of (2.10)},
\end{aligned}
$$

where the third equality follows from (2.9) and the fourth equality follows from Proposition 2.2.2 and (2.8), using the fact that $K_u = \cup_{e\in E_u} S_e(K_{t(e)})$. This completes the proof of Claim 2.

Hence by uniqueness of the invariant measure list, we get $\mu_u = (\mathcal{H}^s(K_u))^{-1}\mathcal{H}^s\lfloor K_u$ for each $u \in V$. $\square$

Proposition 2.2.5. *Let G be a strongly connected MW-graph and $\mathcal{H}^s(K) > 0$, then the graph directed self-similar measure list $(\mu_u)_{u\in V}$ associated with $\rho_e = q_{i(e)}^{-s} r_e^s q_{t(e)}^s$ has the following properties:*

(i) *For $e \in E_{uv}$, $e' \in E_{uv'}$, with $e \neq e'$,*

$$\mu_u(S_e(K_v) \cap S_{e'}(K_{v'})) = 0. \tag{2.11}$$

(ii) *For $e \in E_u$, we have*

$$\mu_u(S_e(A)) = q_u^{-s} r_e^s q_{t(e)}^s \mu_{t(e)}(A). \tag{2.12}$$

Proof of Proposition 2.2.5.

(i) Follows from Propositions 2.2.2 and 2.2.4.

To prove (ii), we first consider $A = K_v$. Since for $e \in E_{uv}$,

$$\mu_u(S_e(K_v)) = \sum_{e'\in E_u} q_u^{-s} r_{e'}^s q_{t(e')}^s \mu_{t(e')} \circ S_{e'}^{-1}(S_e(K_v)) \geq q_u^{-s} r_e^s q_{t(e)}^s,$$

therefore $K_u = \cup_{e\in E_u} S_e(K_{t(e)})$ together with (2.11) implies that

$$1 = \mu_u(K_u) = \sum_{e\in E_u} \mu_u(S_e(K_{t(e)})) \geq \sum_{e\in E_u} q_u^{-s} r_e^s q_{t(e)}^s = 1.$$

Hence

$$\mu_u(S_e(K_v)) = q_u^{-s} r_e^s q_{t(e)}^s.$$

Now for any $A \subset K_v$, $e \in E_{uv}$, since

$$\mu_u(S_e(A)) = \sum_{e'\in E_u} q_u^{-s} r_{e'}^s q_{t(e')}^s \mu_{t(e')} \circ S_{e'}^{-1}(S_e(A)) \geq q_u^{-s} r_e^s q_{t(e)}^s \mu_v(A), \qquad (2.13)$$

and

$$\begin{aligned} q_u^{-s} r_e^s q_{t(e)}^s = \mu_u(S_e(K_v)) &= \mu_u(S_e(A)) + \mu_u(S_e(K_v \setminus A)) \\ &\geq q_u^{-s} r_e^s q_{t(e)}^s \mu_v(A) + q_u^{-s} r_e^s q_{t(e)}^s \mu_v(K_v \setminus A) \\ &= q_u^{-s} r_e^s q_{t(e)}^s, \end{aligned}$$

where the inequality follows from (2.13), we obtain $\mu_u(S_e(A)) = q_u^{-s} r_e^s q_{t(e)}^s \mu_v(A)$. □

Theorem 2.2.6. *Let G be a strongly connected MW-graph, then the following conditions are equivalent.*

(i) *The OSC*

(ii) *The SOSC*

(iii) $\mathcal{H}^s(K) > 0$.

Proof of Theorem 2.2.6. It is clear that: the SOSC $\Rightarrow$ the OSC $\Rightarrow$ $\mathcal{H}^s(K) > 0$. Therefore, we need only to prove that $\mathcal{H}^s(K) > 0$ implies the SOSC. We will divide the proof into four steps.

Recall that d_H denotes the Hausdorff metric defined on the collection of all nonempty compact subsets of $\mathbb{R}^d$. We will denote by *dist* the Euclidean metric on

$\mathbb{R}^d$ and $N(E,\epsilon)$ the set of points y such that $dist(y,E) < \epsilon$, *i.e.* $N(E,\epsilon)$ is the ϵ Euclidean neighborhood of E.

Step 1. Let $x > 0$, we will show that there exists $\delta_u > 0$ for each $u \in V$ such that for all $\alpha, \beta \in E^*_{uv}$ with

$$r_\beta > x r_\alpha, \tag{2.14}$$

we have

$$d_H(S_\alpha(K_v), S_\beta(K_v)) \geq \delta_v r_\alpha.$$

Proof. For each $u \in V$, choose open sets $U_1^u, U_2^u, \ldots, U_{n(u)}^u$ of X_u such that $K_u \subset U_u := \cup_i U_i^u$ and $\sum_i \mu_u(U_i^u) \leq (1+x^s)\mu_u(K_u)$, where $(\mu_u)_{u\in V}$ is the same as in Proposition 2.2.5. Let $\delta_v = dist(K_v, \complement U_v)$, suppose $d_H(S_\alpha(K_v), S_\beta(K_v)) < \delta_v r_\alpha$, since

$$\begin{aligned} dist(S_\alpha(K_v), \complement S_\alpha(U_v)) &= dist(S_\alpha(K_v), S_\alpha(\complement U_v)) \\ &= r_\alpha dist(K_v, \complement U_v) \\ &= r_\alpha \delta_v, \end{aligned}$$

so $S_\beta(K_v) \subset S_\alpha(U_v)$. This implies

$$\begin{aligned} q_u^{-s} r_\alpha^s q_v^s (1+x^s) &= q_u^{-s} r_\alpha^s q_v^s + q_u^{-s} r_\alpha^s q_v^s x^s \\ &< q_u^{-s} r_\alpha^s q_v^s + q_u^{-s} r_\beta^s q_v^s \quad \text{by (2.14)} \\ &= \mu_u(S_\alpha(K_v)) + \mu_u(S_\beta(K_v)) \quad \text{by Proposition 2.2.5 (ii)} \\ &= \mu_u(S_\alpha(K_v) \cup S_\beta(K_v)) \quad \text{by Proposition 2.2.5 (i)} \\ &\leq \sum_i \mu_u(S_\alpha(U_i^v)) \quad \text{since} \quad S_\alpha(K_v) \cup S_\beta(K_v) \subset \cup_i S_\alpha(U_i^v) \\ &= \sum_i q_u^{-s} r_\alpha^s q_v^s \mu_v(U_i^v) \\ &\leq q_u^{-s} r_\alpha^s q_v^s (1+x^s) \mu_v(K_v) \end{aligned}$$

$$= q_u^{-s} r_\alpha^s q_v^s (1 + x^s).$$

This is a contradiction, since $q_u^{-s} r_\alpha^s q_v^s (1 + x^s) \not< q_u^{-s} r_\alpha^s q_v^s (1 + x^s)$.

Step 2. Given $0 < d \leq 1$, we set

$$I_d^u = \{\alpha \in E_u^* : r_\alpha \leq d < r_{\alpha\big|_{|\alpha|-1}}\} \quad \text{for all } u \in V,$$

$$I_d^{uv} = \{\alpha \in E_{uv}^* : r_\alpha \leq d < r_{\alpha\big|_{|\alpha|-1}}\} \quad \text{for all } u, v, \in V.$$

Clearly $I_d^u = \cup_{v \in V} I_d^{uv}$, and $K_u = \cup_{\alpha \in I_d^u} K_\alpha$. Recall $K_\alpha = S_\alpha(K_{t(\alpha)})$. Let $0 < \epsilon < \frac{1}{3}$, fix $u \in V$ and $\xi \in E_u^*$. Let $G_\xi = N(K_\xi, \epsilon r_\xi)$ be the ϵr_ξ Euclidean neighborhood of $K_\xi = S_\xi(K_{t(\xi)})$. Set

$$I^{uv}(\xi) = \{\alpha \in I_{r_\xi}^{uv} : K_\alpha \cap G_\xi \neq \emptyset\},$$

$$I^u(\xi) = \{\alpha \in I_{r_\xi}^u : K_\alpha \cap G_\xi \neq \emptyset\}.$$

Clearly $I^u(\xi) = \cup_{v \in V} I^{uv}(\xi)$.

Claim 2. $\gamma_u = \sup_{\xi \in E_u^*}\{\# I^u(\xi)\} < \infty$.

Proof of Claim 2. Clearly it is enough to show $\gamma_{uv} = \sup_{\xi \in E_u^*}\{\# I^{uv}(\xi)\} < \infty$ for each $v \in V$. Let $z \in K_u$ denote $B_u = B(z,\, 3|K_u|)$ the closed ball center at z with the Euclidean radius $3|K_u|$. Fix $\xi \in E_u^*$, denote $d = r_\xi$. For $\alpha, \beta \in I^{uv}(\xi)$, we have

$$r_\alpha \leq d < r_{\alpha\big|_{|\alpha|-1}} \quad \Rightarrow \quad r_\alpha r_{\min} < d r_{\min},$$

$$r_\beta \leq d < r_{\beta\big|_{|\beta|-1}} \quad \Rightarrow \quad d r_{\min} \leq r_\beta.$$

So $r_\alpha r_{\min} < r_\beta$. Now we can apply Step 1 for $x = r_{\min}$ to get δ_v and

$$d_H(S_\alpha(K_v),\, S_\beta(K_v)) \geq \delta_v r_\alpha \geq \delta_v d r_{\min} = \delta_v r_{\min} r_\xi,$$

for arbitrary $\alpha, \beta \in I^{uv}(\xi)$. Hence for each pair of $\alpha, \beta \in I^{uv}(\xi)$, the preimages of $S_\alpha(K_v)$ and $S_\beta(K_v)$ under S_ξ are compact subsets of B_u and

$$d_H(S_\xi^{-1}(S_\alpha(K_v)), S_\xi^{-1}(S_\beta(K_v))) \geq \delta_v r_{\min}.$$

From the Blaschke Selection Theorem and the total boundedness of the compact set, we known that there is a bound γ on the possible cardinality of a collection of closed subsets of B_u, which are at least $\delta_v r_{\min}$ apart in the Hausdorff metric. So

$$\gamma_{uv} = \sup_{\xi \in E_u^*} \{\# I^{uv}(\xi)\} \le \gamma < \infty.$$

Step 3. For each $u \in V$, choose $\xi^u \in E_u^*$ such that $\# I^u(\xi^u) = \gamma_u$.

Claim 3. For each $u \in V$ and $\alpha \in E_{uu}^*$, we have

$$I^u(\alpha\xi^u) = \{\alpha\beta : \beta \in I^u(\xi^u)\}.$$

Proof of Claim 3.

(1) "$\supseteq$". Let $\alpha\beta \in$ RHS, then $r_\beta \le r_{\xi^u} < r_{\beta\big|_{|\beta|-1}}$ and $K_\beta \cap G_{\xi^u} \ne \emptyset$. Hence $\emptyset \ne S_\alpha(K_\beta \cap G_{\xi^u}) = K_{\alpha\beta} \cap G_{\alpha\xi^u}$, and $r_{\alpha\beta} \le r_{\alpha\xi^u} < r_{\alpha\beta\big|_{|\alpha\beta|-1}}$. Thus $\alpha\beta \in I^u(\alpha\xi^u)$.

(2) "$\subseteq$ ". Since $\# I^u(\alpha\xi^u) \le \# I^u(\xi^u)$, therefore $I^u(\alpha\xi^u) \subseteq \{\alpha\beta : \beta \in I^u(\xi^u)\}$.

Step 4. For each $u \in V$, define $W_u = \cup_{\alpha \in E_u^*} G^*_{\alpha\xi^{t(\alpha)}}$, where $G^*_\sigma = N(K_\sigma, \frac{1}{2}\epsilon r_\sigma)$.

Claim 4. The open set list $(W_u)_{u \in V}$ satisfies the SOSC.

Proof of Claim 4.

i. Since $K_{\xi^u} \subset G^*_{\xi^u} \subset W_u$, so $K_u \cap W_u \ne \emptyset$, for each $u \in V$.

ii. For each $e \in E_{vu}$,

$$S_e(W_u) = S_e(\cup_{\alpha \in E_u^*} G^*_{\alpha\xi^{t(\alpha)}}) = \cup_{\alpha \in E_u^*} G^*_{e\alpha\xi^{t(\alpha)}} \subset W_v.$$

iii. For each $e \in E_{vu}$, $e' \in E_{vu'}$ with $e \ne e'$,

$$S_e(W_u) \cap S_{e'}(W_{u'}) = \emptyset.$$

Since if not, there exist $\alpha \in E_u^*$, $\beta \in E_{u'}^*$, such that

$$G^*_{e\alpha\xi^{t(\alpha)}} \cap G^*_{e'\beta\xi^{t(\beta)}} \neq \emptyset, \quad \text{and} \quad r_{e\alpha\xi^{t(\alpha)}} \geq r_{e'\beta\xi^{t(\beta)}}.$$

If y is an element of this intersection, there exist $y_1 \in K_{e\alpha\xi^{t(\alpha)}}$, and $y_2 \in K_{e'\beta\xi^{t(\beta)}}$ such that

$$\begin{aligned} dist(y, y_1) &< \frac{1}{2}\epsilon r_{e\alpha\xi^{t(\alpha)}} \\ dist(y, y_2) &< \frac{1}{2}\epsilon r_{e'\beta\xi^{t(\beta)}}, \end{aligned}$$

so $dist(y_1, y_2) \leq dist(y_1, y) + dist(y, y_2) < \epsilon r_{e\alpha\xi^{t(\alpha)}}$, hence

$$e'\beta\xi^{t(\beta)} \in I^v(e\alpha\xi^{t(\alpha)}).$$

If $v = t(\alpha)$, Step 3 implies that $e'\beta\xi^{t(\beta)} = e\alpha\sigma$ for some $\sigma \in I^v(\xi^v)$. We get contradiction, since $e \neq e'$. If $v \neq t(\alpha)$, we can find $\omega \in E_{t(\alpha)v}$, since the directed graph G is strongly connected. Clearly, we have $\omega e'\beta\xi^{t(\beta)} \in I^{t(\alpha)}(\omega e\alpha\xi^{t(\alpha)})$. Step 3 implies $\omega e'\beta\xi^{t(\beta)} = \omega e\alpha\sigma$, for some $\sigma \in I^{t(\alpha)}(\xi^{t(\alpha)})$. This is a contradiction as well. $\square$

2.3 A Counter Example for a General MW-Graph G

In this section we give an example of a MW-gragh G which satisfies the OSC but not the SOSC.

Let $G = ((V, E), (X_u)_{u\in E}, (S_e)_{e\in E}, (r_e)_{e\in E})$ be the MW-graph, where $V = \{A, B, C, D, F\}$ and $E = \{AB, BA, AA, BB, CD, DC, CC, DD, FA, FB, FC, FD\}$ (see Figure 2.1).

The compact metric spaces are defined as follows (see Figure 2.2): for a fixed $0 < \epsilon < \frac{1}{3}$,

$$X_A = [0, \frac{1}{3}] \times [-\epsilon, \frac{1}{3}], \qquad X_B = [\frac{2}{3}, 1] \times [-\epsilon, \frac{1}{3}],$$
$$X_C = [0, \frac{1}{3}] \times [\frac{2}{3}, 1+\epsilon], \qquad X_D = [\frac{2}{3}, 1] \times [\frac{2}{3}, 1+\epsilon],$$

$$X_F = \left([0, \frac{1}{3}] \times [-\frac{1}{3}, \frac{1}{3}]\right) \bigcup \left([\frac{2}{3}, 1] \times [-\frac{1}{3}, \frac{1}{3}]\right).$$

For each $u \in V$, X_u is equipped with Euclidean metric. We denote by $B_{X_u}(z, \delta)$ the open ball in X_u with center at z and radius δ.

The similarity maps are defined as follows:

$$\begin{aligned}
&S_{AB} : X_B \to X_A && (x,y) \to \frac{1}{3}(x,y),\\
&S_{AA} : X_A \to X_A && (x,y) \to \frac{1}{3}(x,y),\\
&S_{BA} : X_A \to X_B && (x,y) \to \frac{1}{3}(x,y) + (\frac{2}{3}, 0),\\
&S_{BB} : X_B \to X_B && (x,y) \to \frac{1}{3}(x,y) + (\frac{2}{3}, 0),\\
&S_{CD} : X_D \to X_C && (x,y) \to \frac{1}{3}(x,y) + (0, \frac{2}{3}),\\
&S_{CC} : X_C \to X_C && (x,y) \to \frac{1}{3}(x,y) + (0, \frac{2}{3}),\\
&S_{DC} : X_C \to X_D && (x,y) \to \frac{1}{3}(x,y) + (\frac{2}{3}, \frac{2}{3}),\\
&S_{DD} : X_D \to X_D && (x,y) \to \frac{1}{3}(x,y) + (\frac{2}{3}, \frac{2}{3}),\\
&S_{FA} : X_A \to X_F && (x,y) \to (x,y),\\
&S_{FB} : X_B \to X_F && (x,y) \to (x,y),\\
&S_{FC} : X_C \to X_F && (x,y) \to (x,y) + (0, -1),\\
&S_{FD} : X_D \to X_F && (x,y) \to (x,y) + (0, -1).
\end{aligned}$$

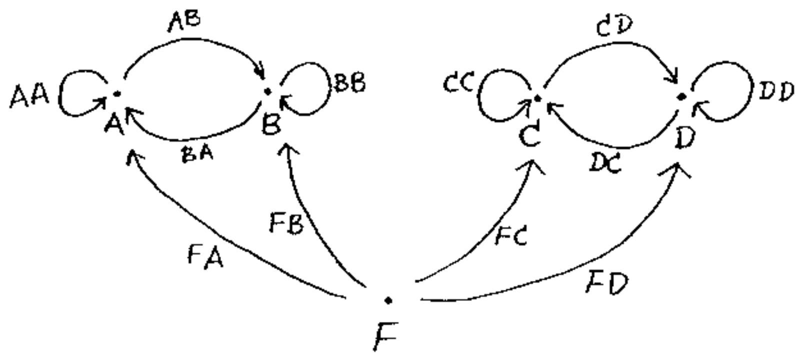

FIGURE 2.1 GRAPH OF THE EXAMPLE

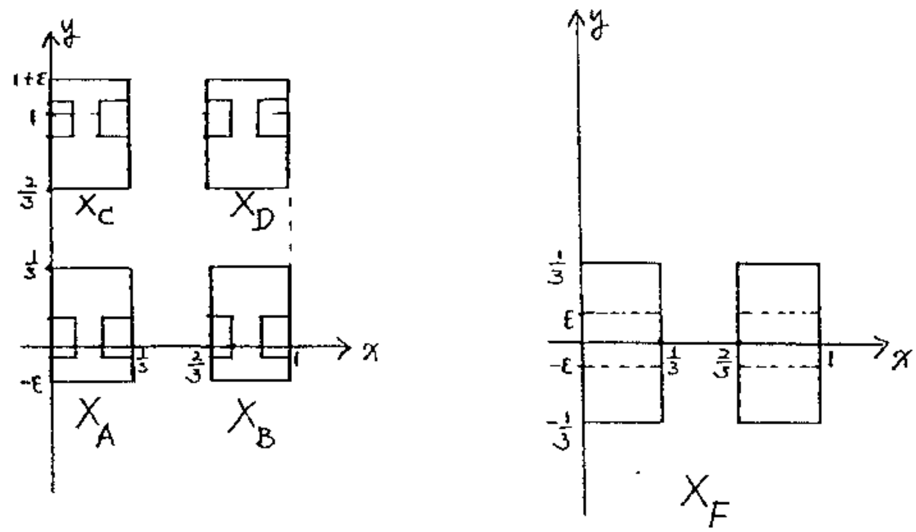

FIGURE 2.2 STRUCTURE OF THE FRACTAL

Note that the last four maps are not contractions, however, by using rescaling (see [Edg] p.116), we can make the maps become contractions. Moreover, G is not strongly connected. If we let $K = \mathcal{C} \times \{0\}$ where $\mathcal{C}$ is the standard Cantor set, then the limit sets $(K_u)_{u\in V}$ are:

$$K_A = \frac{1}{3}K, \qquad K_B = \frac{1}{3}K + (\frac{2}{3}, 0),$$
$$K_C = \frac{1}{3}K + (0,1), \qquad K_D = \frac{1}{3}K + (\frac{2}{3}, 1),$$
$$K_F = K.$$

Note that:

$$S_{FA}(K_A) = S_{FC}(K_C), \quad S_{FB}(K_B) = S_{FD}(K_D).$$

Theorem.

(i) *G satisfies the OSC.*

(ii) *G does not satisfy the SOSC.*

Proof.

(i) Choose the list of open sets $(U_A, U_B, U_C, U_D, U_F)$ as follows:

$$U_A = (0, \frac{1}{3}) \times (0, \frac{1}{3}), \qquad U_B = (\frac{2}{3}, 1) \times (0, \frac{1}{3}),$$
$$U_C = (0, \frac{1}{3}) \times (\frac{2}{3}, 1), \qquad U_D = (\frac{2}{3}, 1) \times (\frac{2}{3}, 1),$$

$$U_F = S_{FA}(U_A) \cup S_{FB}(U_B) \cup S_{FC}(U_C) \cup S_{FD}(U_D).$$

It is easily seen that the list $(U_A, U_B, U_C, U_D, U_F)$ satisfies the OSC.

(ii) We prove the assertion by contradiction. Suppose the list $(O_A, O_B, O_C, O_D, O_F)$ satisfies the SOSC. Let $z \in O_A \cap K_A$, then

$$S_{FA}(z) \in S_{FA}(K_A) = S_{FC}(K_C) \subseteq S_{FC}(\overline{O}_C). \tag{2.15}$$

On the other hand, the SOSC

Since $z \in O_A$, there exists $0 < \delta < \epsilon$, such that $B_{X_A}(z, \delta) \subseteq O_A$. Hence, using the fact that the map S_{FA} is the identity map, we have

$$B_{X_F}(S_{FA}(z), \delta) = S_{FA}(B_{X_A}(z, \delta)) \subseteq S_{FA}(O_A).$$

Also, since $S_{FA}(z) \in S_{FC}(\overline{O}_C)$ by (2.15), there exists a sequence $\{z_n\} \subset O_C$, such that

$$S_{FC}(z_n) \to S_{FA}(z) \text{ as } n \to \infty.$$

Hence, for n big enough, we have $S_{FC}(z_n) \in B_{X_F}(S_{FA}(z), \delta)$. This implies

$$S_{FC}(O_C) \cap S_{FA}(O_A) \neq \emptyset.$$

In view of (2.16), we get a contradiction. $\square$

CHAPTER III

THE HAUSDORFF MEASURES OF SOME RATIO SELF-SIMILAR SETS WITH OVERLAPS

For self-similar sets and strongly connected graph directed self-similar sets, we know that the OSC is equivalent to the Hausdorff measure being positive on the limit set (cf. [Sch] and Chapter 2). This means if the limit set has positive Hausdorff measure, then the pieces $\{S_\alpha(K_{t(\alpha)})\}_{\alpha\in E_u^{(n)}}$ which make up the limit set cannot overlap too much. However, what will happen if we allow some overlaps? Certainly the nonempty compact limit set exists even if the OSC is not satisfied (cf. [Hu] and [Mo]). It is reasonable to assume that the dimension of this limit set might be smaller than otherwise would be expected, since the Hausdorff measure would be zero if the OSC is not satisfied. Falconer[Fa3] studied the Hausdorff dimension of some self-similar sets with overlaps. He proved that even if the OSC does not hold for any set U, the Hausdorff dimension of the limit set is "usually" the same as its similarity dimension (cf. Theorem 1 of [Fa3]).

There is another interesting question: are there any fractal sets defined by overlapping constructions, having positive Hausdorff measure w.r.t. its Hausdorff dimension? From Schief's result and our result in Chapter 2, we conclude that if there is such a fractal set with overlaps, it can not be a self-similar set or strongly connected graph directed self-similar set. Nevertheless, we will give an example of a fractal set in this chapter, defined by an overlapping construction and yet having positive and finite Hausdorff measure w.r.t. its Hausdorff dimension. The example is based on a ratio self-similar construction. In Section 3.1, we will introduce the notion of a ratio self-similar construction; and prove the main theorem and give an example of overlapping construction having positive and finite Hausdorff dimension in Section 3.2.

3.1 A Ratio Self-Similar Construction

In this section we give the definition of a ratio self-similar construction and its properties.

Let $J \neq \emptyset$ be a compact subset of $\mathbb{R}^d$ with $J = \overline{\text{int} J}$, and let $0 < r_i < 1$, $i \in I = \{1, 2, \cdots n\}$ where $n > 1$. A ratio self-similar construction based on a seed set J and a set of similarity ratios $S = \{r_i : \ i \in I\}$ is a family,

$$\mathcal{J} = \big(J(\alpha)\big)_{\alpha \in D^*}$$

where $D^* = \cup_{k=0}^{\infty} I^k$, $I^k = \{1, 2, \ldots, n\}^k$ and $I^0 = \emptyset$, such that

(1) $J(\emptyset) = J$

(2) For each $\alpha \in D^*$ and $i \in \{1, 2, \ldots, n\}$, the set $J(\alpha i)$ is a subset of $J(\alpha)$

(3) For each $\alpha \in D^*$ and $i \in \{1, 2, \ldots, n\}$, the set $J(\alpha i)$ is similar to $J(\alpha)$ with $|J(\alpha i)|/|J(\alpha)| = r_i$.

The ratio self-similar set F constructed by $\mathcal{J}$ is

$$F = \bigcap_{k=0}^{\infty} \bigcup_{|\alpha|=k} J(\alpha).$$

Remark. *The term "ratio self-similar set" is introduced by Moran [Mo]. However the definition here is a little bit different from his, since we don't require $\{J(\alpha i)\}_{i=1}^{n}$ to be non-overlapping. Also note that the self-similar set is a particular type of a ratio self-similar set, in which the similarity maps are specified and are the same at each level.*

Let $D = \{1, 2, \ldots, n\}^{\mathbb{N}}$, and define $\pi : D \to F$ by

$$\{\pi(\omega)\} = \bigcap_{k=1}^{\infty} J(\omega|_k) \quad \text{where} \quad \omega|_k = (\omega_1, \omega_2, \ldots, \omega_k) \in D^*.$$

Since $0 < r_i < 1$, π maps D continuously onto F. Moreover, if the sets $\{J(\alpha) : |\alpha| = k\}$ are disjoint for each k, then π is a homeomorphism [Mo].

Definition 3.1.1. *The similarity dimension s of a ratio self-similar construction is the unique number which satisfies*

$$\sum_{i=1}^{n} r_i^s = 1. \tag{3.1}$$

The equation (3.1) is called the redistribution of mass equation. Let $\hat{\mu}$ be the infinite product measure on D determined by the probability vector: $(r_1^s, r_2^s, \cdots, r_n^s)$; *i.e.*, the measure of a cylinder set is given by

$$\hat{\mu}\big([\alpha]\big) = r_\alpha^s \quad \big(\text{recall that} \quad r_\alpha = r_{\alpha_1} r_{\alpha_2} \ldots r_{\alpha_{|\alpha|}}\big).$$

In view of (3.1), there is an unique probability measure $\hat{\mu}$ defined on the Borel σ-algebra in D which extends $\hat{\mu}$.

Definition 3.1.2. *Let μ be the image measure of $\hat{\mu}$ under π, i.e.*

$$\mu(A) = \hat{\mu}(\pi^{-1}(A)).$$

Then μ is a probability measure and $\mu(F) = 1$. $\hat{\mu}$ is the restriction of mass measure on F.

For a ratio self-similar set F, we are interested in its Hausdorff dimension and Hausdorff measure.

Remark. *If for each* $\alpha \in D^*$, *the sets* $\{J(\alpha i)\}_{i=1}^n$ *are non-overlapping subsets of* $J(\alpha)$, *then we have* $0 < \mathcal{H}^s(F) < \infty$, *where the* s *is the similarity dimension. Hence* $\dim_H(F) = s$ *[Mo]. If* $\{J(\alpha i)\}_{i=1}^n$ *overlap too much, the Hausdorff dimension may be less than the similarity dimension (cf. [Ban]).*

In the next section, we will give a condition on the amount of overlap guaranteeing that the Hausdorff dimension of F equals the similarity dimension. Furthermore, the Hausdorff measure of F is positive and finite w.r.t. the similarity dimension. We will use the following properties.

Mass Distribution Principle. *Let* μ *be a mass distribution on* F, *and suppose that for some* s *there are numbers* $c > 0$ *and* $\delta > 0$ *such that*

$$\mu(U) \leq c|U|^s$$

for all Borel sets U *with* $|U| \leq \delta$. *Then* $\mathcal{H}^s(F) \geq \mu(F)/c$.

Remark. *Our version of the Mass Distribution Principle differs from [Fa1, Theorem 4.2], since we require the sets* U *to be Borel sets. However, the result still holds, since Hausdorff measure can be computed by using open covers (cf. [Ro]).*

Definition 3.1.3. *Let* U *be a subset of* $\mathbb{R}^d$. *We define* $\mathcal{C}(U)$ *as follows*

$$\mathcal{C}(U) = \{\alpha \in D^* : |J(\alpha)| \leq |U| < |J(\alpha|_{|\alpha|-1})|,\ U \cap F \neq \emptyset\}. \tag{3.2}$$

Clearly if $\alpha, \beta \in \mathcal{C}(U)$, *then* $\alpha \not\prec \beta$ *and* $\beta \not\prec \alpha$.

We denote by $\mathcal{L}^d$ the Lebesgue measure in $\mathbb{R}^d$ and $B(0,1)$ the unit ball in $\mathbb{R}^d$. For any set U, the Isodiametric Inequality (cf. [EG] p 69) implies that

$$\mathcal{L}^d(U) \leq \left(\frac{|U|}{2}\right)^d \mathcal{L}^d(B(0,1)). \tag{3.3}$$

Also, since $J(\alpha)$ is similar to J,

$$\mathcal{L}^d(\mathrm{int}(J(\alpha))) = \left(\frac{|J(\alpha)|}{|J|}\right)^d \mathcal{L}^d(\mathrm{int}(J)). \tag{3.4}$$

3.2 The Theorem and An Example

In this section, we will first state the main theorem and give an example. Finally we will provide the proof.

Theorem 3.2.1. *Let F be a ratio self-similar set based on a seed set J and a set of similarity ratios $S = \{r_i : i \in I = \{1, 2, \cdots, n\}\}$. Choose $0 < r < r_{\min}$, where $r_{\min} = min\{r_1, r_2, \ldots, r_n\}$, and $l > 0$ such that*

$$l^d \mathcal{L}^d(B(0,1)) \leq \mathcal{L}^d(int(J)). \tag{3.5}$$

Suppose that

(i) *There is a constant K such that for any $\alpha \in I^k$ with $k \geq K$,*

$$\left| \, int(J(\alpha i) \bigcap J(\alpha j)) \, \right| \leq l r^{k+1} |J| \quad for \quad i \neq j \tag{3.6}$$

(ii) *If $\alpha, \alpha' \in I^k$ and $\alpha \neq \alpha'$ then*

$$int \left(J(\alpha i) \bigcap J(\alpha' j) \right) = \emptyset \quad for \quad i, j \in \{1, 2, \ldots, n\}. \tag{3.7}$$

Then we have $0 < \mathcal{H}^s(F) \leq \mathcal{P}^s(F) < \infty$ where s is the similarity dimension of F, and consequently $dim_H(F) = dim_P(F) = s$.

Remark 1. *The conditions (i) and (ii) are the so called controlled overlapping conditions (see Figure 3.1).*

Remark 2. *Since our construction is a ratio self-similar construction, unlike self-similar sets, the set F can not in general be constructed as a limit set of an iterated function system. At each level k, we usually choose similarity maps $S_1^{(k)}, S_2^{(k)}, \ldots, S_n^{(k)}$ according to the fixed ratio list $r_1, r_2, \ldots, r_n$ and conditions (i) and (ii). Therefore it is not appropriate to discuss the OSC. However, from our definition, it is clear that the construction allows certain degree of overlapping.*

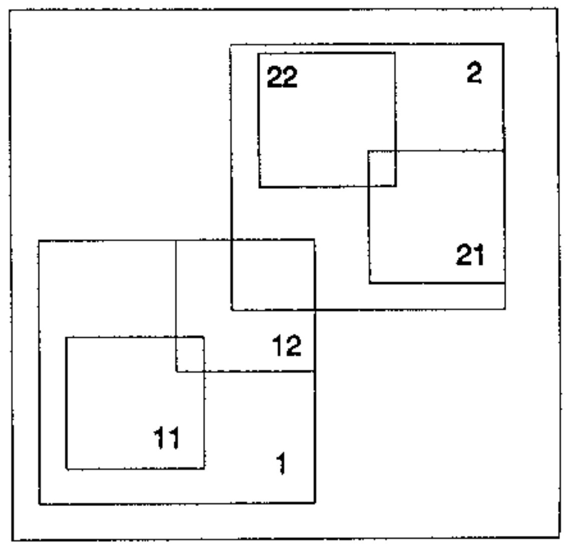

FIGURE 3.1 RATIO CONSTRUCTION WITH CONTROLLED OVERLAPS.

Example. Let $J = [0, 1]$ be the unit interval in $\mathbb{R}^1$, and $r_1 = r_2 = 1/3$ be the ratio list. Let $l = 1/2$, then $l\mathcal{L}^1(B(0,1)) = \mathcal{L}^1(J)$. Let $r = 1/4$, then $r < r_{\min} = 1/3$. We will define a binary ratio self-similar construction by the following recursion (see Figure 3.2).

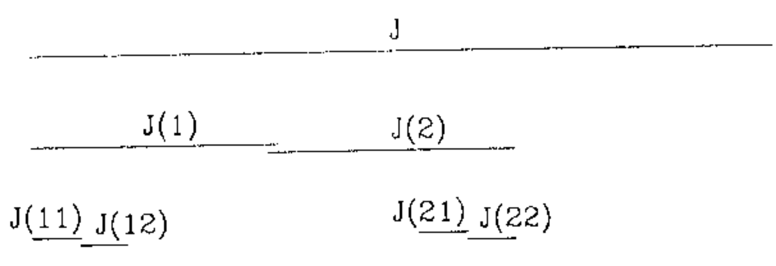

FIGURE 3.2 A BINARY RATIO SELF-SIMILAR CONSTRUCTION.

1. The start of the induction:

   Put $J(\emptyset) = J$, $J(1) = [0, 1/3]$, and $J(2) = [1/3 - 1/2(1/4)^1,\ 2/3 - 1/2(1/4)^1]$;

2. The inductive step:

   If $\alpha \in D^*$ and $J(\alpha) = [a\,,\,b]$ has been constructed, then define $J(\alpha i)$ as follows,

   a) if $\alpha(|\alpha|) = 1$, then

$$J(\alpha 1) = [a\,,\ a + \frac{1}{3}(b-a)]$$
$$J(\alpha 2) = [a + \frac{1}{3}(b-a) - \frac{1}{2}(\frac{1}{4})^{|\alpha|+1}\,,\ a + \frac{2}{3}(b-a) - \frac{1}{2}(\frac{1}{4})^{|\alpha|+1}]$$

   b) if $\alpha(|\alpha|) = 2$ then

$$J(\alpha 1) = [b - \frac{2}{3}(b-a) + \frac{1}{2}(\frac{1}{4})^{|\alpha|+1}\,,\ b - \frac{1}{3}(b-a) + \frac{1}{2}(\frac{1}{4})^{|\alpha|+1}]$$
$$J(\alpha 2) = [b - \frac{1}{3}(b-a),\ \ b].$$

This construction satisfies

(1) $|J(\alpha)| = (\frac{1}{3})^{|\alpha|} = r_\alpha |J|$

(2) $|J(\alpha 1) \cap J(\alpha 2)| = \frac{1}{2}(\frac{1}{4})^{|\alpha|+1} = lr^{|\alpha|+1}$

In order to show the construction also satisfies the condition (ii) in the Theorem 3.2.1 (see the Figure 3.3), we only need to verify that

$$(J(\alpha 11) \cup J(\alpha 12)) \bigcap (J(\alpha 21) \cup J(\alpha 22)) = \emptyset.$$

Since

$$\begin{aligned}
&\Big|\ J(\alpha 11) \cup J(\alpha 12)\ \Big| + \Big|\ J(\alpha 21) \cup J(\alpha 22)\ \Big| \\
&= (\frac{1}{3})^{|\alpha|+2} - \frac{1}{2}(\frac{1}{4})^{|\alpha|+2} + (\frac{1}{3})^{|\alpha|+2} - \frac{1}{2}(\frac{1}{4})^{|\alpha|+2} \\
&= 2(\frac{1}{3})^{|\alpha|+2} - (\frac{1}{4})^{|\alpha|+2},
\end{aligned}$$

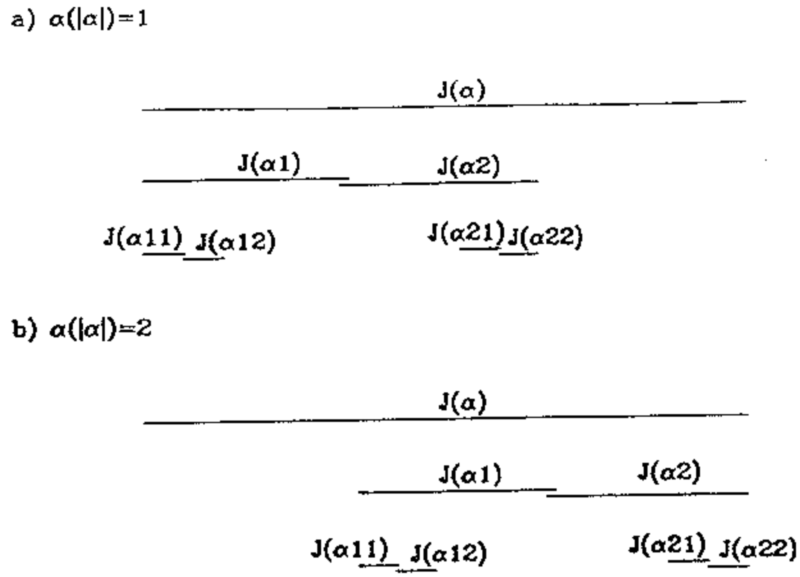

FIGURE 3.3 THE RECURSIVE CONSTRUCTION PROCESS.

and

$$|J(\alpha 1) \cup J(\alpha 2)| = (\frac{1}{3})^{|\alpha|+1} - \frac{1}{2}(\frac{1}{4})^{|\alpha|+1} = 3(\frac{1}{3})^{|\alpha|+2} - 2(\frac{1}{4})^{|\alpha|+2},$$

then

$$|J(\alpha 1) \cup J(\alpha 2)| > |J(\alpha 11) \cup J(\alpha 12)| + |J(\alpha 21) \cup J(\alpha 22)|.$$

Hence

$$(J(\alpha 11) \cup J(\alpha 12)) \bigcap (J(\alpha 21) \cup J(\alpha 22)) = \emptyset.$$

Since the construction satisfies all the conditions in Theorem 3.2.1, the ratio self-similar set $F = \cup_{k=0}^{\infty} \cap_{|\alpha|=k} J(\alpha)$ has the following property

$$0 < \mathcal{H}^s(F) \leq \mathcal{P}^s(F) < \infty, \quad \text{where} \quad s = \frac{\log 2}{\log 3} \text{ is the similarity dimension}$$

and $\dim_H(F) = \dim_P(F) = \frac{\log 2}{\log 3}$.

To prove the theorem, we need the following lemmas.

Lemma 3.2.2. *There is a positive integer N such that if $k \geq N$ and $\alpha \in I^k$, then*

$$\left(\frac{r^{|\alpha|}}{r_\alpha}\right)^d < \frac{1}{2(n-1)(\frac{1}{2}|J|)^d}. \tag{3.8}$$

Proof of Lemma 3.2.2. Since $0 < r < r_{\min}$, for any $\alpha \in I^k$, we have

$$\frac{r^k}{r_\alpha} \le \frac{r^k}{r_{\min}^k} = \left(\frac{r}{r_{\min}}\right)^k \to 0 \quad \text{as } k \to \infty.$$

This implies

$$\left(\frac{r^k}{r_\alpha}\right)^d \to 0 \quad \text{as } k \to \infty.$$

Therefore we can choose N large enough so that (3.8) holds. $\square$

Lemma 3.2.3. *There exists a constant $0 < \lambda < \infty$ and $\delta > 0$ such that for any set U with $U \cap F \neq \emptyset$, if $|U| \le \delta$ then $\#\mathcal{C}(U) \le \lambda$.*

Proof of Lemma 3.2.3. Let $K_1 = \max(K, N)$, where K is the same as in (3.6) and N is the same as in Lemma 3.2.2. Let $\delta = |J|\min\{r_\alpha :\ \alpha \in I^{K_1}\}$. Clearly $\delta > 0$.

Let U be a set with $U \cap F \neq \emptyset$ and $|U| \le \delta$. Note that this implies that for any $\alpha \in \mathcal{C}(U)$, we have $|\alpha| \ge K_1$.

Let $x \in U \cap F$, and $B = B(x, 2|U|)$, we have

$$\{J(\alpha) :\ \alpha \in \mathcal{C}(U)\} \subset B.$$

This implies

$$\begin{aligned}
\mathcal{L}^d(B) &\ge \sum_{\alpha \in \mathcal{C}(U)} \left(\mathcal{L}^d(\mathrm{int}(J(\alpha))) - \sum_{i \neq \alpha(|\alpha|)} \mathcal{L}^d(\mathrm{int}(J(\alpha) \cap J(\alpha|_{|\alpha|-1} i))) \right) \\
&\ge \sum_{\alpha \in \mathcal{C}(U)} \left((\frac{|J(\alpha)|}{|J|})^d \mathcal{L}^d(\mathrm{int}(J)) - \sum_{i \neq \alpha(|\alpha|)} (\frac{1}{2} l r^{|\alpha|} |J|)^d \mathcal{L}^d(B(0,1)) \right) \\
&\quad \text{by (3.4) and (3.3)} \\
&\ge \sum_{\alpha \in \mathcal{C}(U)} \left(r_\alpha^d \mathcal{L}^d(\mathrm{int}(J)) - (n-1)(\frac{1}{2} r^{|\alpha|} |J|)^d \mathcal{L}^d(\mathrm{int} J) \right) \quad \text{by (3.5)} \\
&= \mathcal{L}^d(\mathrm{int} J) \sum_{\alpha \in \mathcal{C}(U)} (r_\alpha^d - (n-1)(\frac{1}{2} r^{|\alpha|} |J|)^d)
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{L}^d(\mathrm{int}J) \sum_{\alpha \in \mathcal{C}(U)} r_\alpha^d (1 - (n-1)(\frac{1}{2}|J|)^d (\frac{r^{|\alpha|}}{r_\alpha})^d) \\
&\geq \frac{1}{2}\mathcal{L}^d(\mathrm{int}J) \sum_{\alpha \in \mathcal{C}(U)} r_\alpha^d \quad \text{by (3.8)} \\
&\geq \frac{1}{2}\mathcal{L}^d(\mathrm{int}J) \sum_{\alpha \in \mathcal{C}(U)} \left(\frac{|U| r_{\min}}{|J|}\right)^d \quad \text{by (3.2).}
\end{aligned}$$

Since $\mathcal{L}^d(B) = \mathcal{L}^d(B(x, 2|U|)) = (2|U|)^d \mathcal{L}^d(B(0,1))$, we have

$$\begin{aligned}
2^d |U|^d \mathcal{L}^d(B(0,1)) &\geq \frac{1}{2}\mathcal{L}^d(\mathrm{int}J)|U|^d \left(\frac{r_{\min}}{|J|}\right)^d \#\mathcal{C}(U) \\
\#\mathcal{C}(U) &\leq \frac{2(2^d)\mathcal{L}^d(B(0,1))|J|^d}{r_{\min}^d \mathcal{L}^d(\mathrm{int}J)}. \quad \square
\end{aligned}$$

Lemma 3.2.4. *For any set U with $U \cap F \neq \emptyset$, we have*

$$\pi^{-1}(U \cap F) \subset \bigcup \{[\alpha] : \alpha \in \mathcal{C}(U)\}.$$

Proof of Lemma 3.2.4. Suppose $\sigma \in \pi^{-1}(U \cap F)$, then $\pi(\sigma) = x \in U \cap F$. Since $|J(\sigma|_p)| \to 0$ as $p \to \infty$, there is a smallest positive integer k such that $|J(\sigma|_p)| \leq |U|$. Then $\alpha = \sigma|_k \in \mathcal{C}(U)$ and $\sigma \in [\alpha]$. $\square$

Proof of Theorem 3.2.1. By using the fact $\mathcal{H}^s(F) \leq \mathcal{P}^s(F)$, see (1.3) in Chapter 1, we only need to show $0 < \mathcal{H}^s(F)$ and $\mathcal{P}^s(F) < \infty$.

Claim 1. $\mathcal{H}^s(F) > 0$.

Proof of Claim 1. We will use the Mass Distribution Principle. Let λ and δ be the same as in Lemma 3.2.3, and $c = \frac{|J|^s}{\lambda}$.

For any Borel set U with $|U| \leq \delta$, the following holds,

1) if $U \cap F = \emptyset$, $\mu(U) = 0$, since μ is supported on F. Consequently

$$\mu(U) \leq c|U|^s. \tag{3.9}$$

2) if $U \cap F \neq \emptyset$, then

$$\mu(U) = \mu(U \cap F) = \hat{\mu}(\pi^{-1}(U \cap F)).$$

Lemma 3.2.4 therefore implies that

$$\mu(U) \leq \sum_{\alpha \in \mathcal{C}(U)} \hat{\mu}([\alpha]).$$

Since

$$\hat{\mu}([\alpha]) = r_\alpha = \frac{|J(\alpha)|^s}{|J|^s} \leq \frac{|U|^s}{|J|^s},$$

then

$$\mu(U) \leq \#\mathcal{C}(U)\frac{|U|^s}{|J|^s} \leq \lambda\frac{|U|^s}{|J|^s} = c|U|^s. \tag{3.10}$$

Now (3.9), (3.10) and the Mass Distribution Principle yield

$$\mathcal{H}^s(F) > \frac{1}{c}\mu(F) > 0. \quad \square$$

Claim 2. $\mathcal{P}^s(F) < \infty$.

Proof of Claim 2. Let $x = \pi(\omega) \in F$, $\omega \in D$ and $0 < \delta < |J|$. Also let $k \geq 0$ be the smallest k such that $J_{\omega|_k} \subset B(x, \delta)$, then

$$\pi^{-1}B(x,\delta) \supset \pi^{-1}J_{\omega|_k} \supset [\omega|_k].$$

This implies

$$\mu(B(x,\delta)) = \hat{\mu} \circ \pi^{-1}(B(x,\delta)) \geq \hat{\mu}[\omega|_k] = r^s_{\omega|_k}. \tag{3.11}$$

By the minimality of k, we conclude that $J_{\omega|_{k-1}}$ is not contained in $B(x, \delta)$. Thus

$$\delta < |J_{\omega|_{k-1}}| = r_{\omega|_{k-1}}|J| \leq \frac{r_{\omega|_k}}{r_{\min}}|J|.$$

This implies

$$r_{\omega|_k} > \frac{r_{\min}}{J}\delta. \tag{3.12}$$

Hence (3.11) and (3.12) imply that

$$\mu B(x,\delta) \geq \left(\frac{r_{\min}}{J}\right)^s \delta^s.$$

Therefore, by applying Theorem 1.1 (b) of [RT] or Theorem 6.11 of [Mat], we get $\mathcal{P}^s(F) < \infty$.

From above Claim 1 and Claim 2, we conclude

$$0 < \mathcal{H}^s(F) \leq \mathcal{P}^s(F) < \infty.$$

Consequently, we obtain $\dim_H(F) = \dim_P(F) = s$. $\square$

CHAPTER IV

THE OPEN SET CONDITION FOR INFINITE CONFORMAL ITERATED FUNCTION SYSTEMS

Finite iterated function systems, their limit sets and open set conditions (OSC) have been carefully studied by many people in the past. Given a self-similar set K generated by a finite number of contracting similarity maps $\mathcal{S} = (S_1, \cdots, S_n)$ on $\mathbb{R}^d$, if s is the similarity dimension of $\mathcal{S}$ and μ is the corresponding self-similar measure, Schief [Sch], Lau and Wang [LW] proved that the following four conditions:

(i) OSC

(ii) $\mathcal{H}^s(K) > 0$

(iii) SOSC

(iv) $\mu(\partial U) = 0$ for certain set U satisfies the OSC,

are equivalent.

However, it is unknown whether these equivalencies still hold for a self-conformal set J generated by countable families of conformal contractions. In this chapter, we will show that for an iterated function system with an infinite set of generators consisting of conformal maps, if m is the unique conformal measure, then the SOSC is equivalent to $m(\partial U) = 0$ for certain set U satisfying the OSC; but whether the OSC and SOSC are equivalent is still unknown. We will first introduce the notions of infinite conformal iterated function systems (c.i.f.s.), self-conformal sets and conformal measures. Then we will study the open set conditions (OSC) for the self-conformal sets.

4.1 Definitions and Notations

The definitions and notations of conformal iterated function systems can be found in [MU]. In general, our notation is as in that paper.

Let (X, ρ) be a nonempty compact metric space, and I be a countable set with at least two elements. Also let $S = \{\phi_i : X \to X \ : i \in I\}$ be a collection of injective contractions from X to X for which there exists $0 < \lambda < 1$ such that

$$\rho(\phi_i(x), \phi_i(y)) \le \lambda \rho(x, y), \tag{4.1}$$

for every $i \in I$ and every pair of points $x, y \in X$. Such a collection S is called an iterated function system, frequently abbreviated as i.f.s.

Let $I^* = \cup_{n \ge 1} I^n$, for $\omega \in I^n$ where $n \ge 1$, define

$$\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}.$$

If $\omega \in I^* \cup I^\infty$, and $k \ge 1$ does not exceed the length of ω, we denote by $\omega|_k$ the string $\omega_1 \omega_2 \ldots \omega_k$. One can see that, given $\omega \in I^\infty$, the compact sets $\phi_{\omega|_k}(X),\ k \ge 1$, are decreasing and their diameters converge to zero. In fact, by (4.1), we have,

$$\operatorname{diam}(\phi_{\omega|_k}(X)) \le \lambda^k \operatorname{diam}(X). \tag{4.2}$$

This implies that the set

$$\pi(\omega) = \bigcap_{k=1}^{\infty} \phi_{\omega|_k}(X) \tag{4.3}$$

is a singleton. Therefore (4.3) defines a map $\pi : I^\infty \to X$ which, in view of (4.2), is continuous.

Let $\sigma : I^\infty \to I^\infty$ denote the left shift map on I^∞, that is $\sigma(\omega) = \omega_2 \omega_3 \cdots$. We will frequently use the following obvious relation

$$\pi \circ \sigma(\omega) = \phi_{\omega_1}^{-1} \circ \pi(\omega). \tag{4.4}$$

The main object of our study is the set $J = \pi(I^\infty)$, called the limit set associated to the system $S = \{\phi_i : X \to X,\ i \in I\}$. Since for every $i \in I$, we have $\phi_i(\pi(\omega)) = \pi(i\omega)$, rewriting (4.4) in the form $\pi(\omega) = \phi_{\omega_1}(\pi(\sigma(\omega)))$, we obtain

$$J = \bigcup_{i \in I} \phi_i(J). \tag{4.5}$$

Note that if I is finite, then J is compact. If the system $S = \{\phi_i : X \to X \ :\ i \in I\}$ is pointwise finite (meaning that each element of X belongs to a finite number of elements of $\phi_i(X)$), then the family $\{\phi_\omega(X) : \omega \in I^n\}$ is pointwise finite for every $n \geq 1$, and therefore

$$J = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in I^n} \phi_\omega(X). \tag{4.6}$$

Thus J is a $F_{\sigma\delta}$ subset of X.

An iterated function system $S = \{\phi_i : X \to X \ :\ i \in I\}$, is said to satisfy the Open Set Condition (OSC), if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$, and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$. It satisfies the strong open set condition (SOSC) if in addition $J \cap U \neq \emptyset$.

An iterated function system S satisfying OSC, is said to be conformal (c.i.f.s.) if the following conditions are satisfied:

(1) X is a regular connected subset of an Euclidean space $\mathbb{R}^d$, that is $\overline{\mathrm{Int}_{\mathbb{R}^d}(X)} = X$. We could and will assume that $U = \mathrm{Int}_{\mathbb{R}^d}(X)$.

(2) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset \mathbb{R}^d$, there exists an open cone $\mathrm{Con}(x, \alpha, l) \subset \mathrm{Int}(X)$ with vertex x, angle α (*i.e.* the Lebesgue measure on the unit sphere S^{d-1}), and altitude l.

(3) There exists an open connected set $X \subset V \subset \mathbb{R}^d$ such that all maps ϕ_i, $i \in I$, extend to $C^{1+\epsilon}$ diffeomorphisms on V, and are conformal on V.

(4) Bounded Distortion Property (BDP). There exists $K \geq 1$ such that $|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$ for every $\omega \in I^*$ and every pair of points $x, y \in V$.

Remark. *A self-similar i.f.s. is a special case of a c.i.f.s., where all the maps are similarity maps.*

Definition. *The limit set $J = \pi(I^\infty)$, associated to a c.i.f.s.S, is called the self-conformal set.*

Definition. *Given $t \geq 0$, a Borel probability measure m is said to be t-conformal provided $m(J) = 1$, and for every Borel set $A \subset X$,*

$$m(\phi_i(A)) = \int_A |\phi'_i|^t dm \quad \text{for every} \quad i \in I, \tag{4.7}$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0 \quad \text{for every pair} \quad i, j \in I,\ i \neq j. \tag{4.8}$$

It is easy to show that for every Borel set $A \subset X$ and every pair $\omega, \tau \in I^*$ such that $\omega \not\prec \tau, \tau \not\prec \omega$, we have,

$$\sum_{\omega \in I^n} \int |\phi'_\omega|^t dm = 1 \tag{4.9}$$

$$m(\phi_\omega(A)) = \int_A |\phi'_\omega|^t dm \tag{4.10}$$

$$m(\phi_\omega(X) \cap \phi_\tau(X)) = 0. \tag{4.11}$$

Remark. *In the self-similar case, the conformal measure m becomes the self-similar measure $\mu = \sum_{i \in I} r_i^t \mu \circ S_i^{-1}$, where S_i is similarity map and r_i is the similarity ratio of S_i.*

Mauldin and Urbanski gave a sufficient and necessary condition for the existence of an unique δ-conformal measure associated with a c.i.f.s. (cf. [MU], Chapter 3), where δ is the unique number such that

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in I^n} \|\phi'_\omega\|^\delta = 0.$$

They defined a c.i.f.s. S to be regular, if S admits a δ-conformal measure. Furthermore, they showed that for a regular c.i.f.s. S, there exists an unique ergodic σ-invariant probability measure μ^* on I^∞. Its image measure $m^* = \mu^* \circ \pi^{-1}$ is equivalent to the conformal measure m (cf. [MU] Theorem 3.8 and Remark 3.12).

4.2 Theorems and Proofs

We will study the geometric properties of the regular c.i.f.s.S in this section. In particular, we will show that the SOSC is equivalent to $m(\partial U) = 0$ for certain set U satisfying the OSC.

Theorem 4.2.1. *Let S be a regular c.i.f.s., and m be the unique δ-conformal measure, then the SOSC is equivalent to $m(\partial U) = 0$ for certain set U satisfying the OSC.*

In order to prove Theorem 4.2.1, we need the following lemmas.

Lemma 4.2.2. *Let U be a set satisfying the OSC, then $m(U) = m(\cup_{i\in I}\phi_i(U))$.*

Remark. *Lemma 4.2.2 becomes trivial in the self-similar case, since in this case we have $\sum_{i\in I} r_i^\delta = 1$, and this implies*

$$\mu(\cup_{i\in I} S_i(U)) = \sum_{i\in I} \mu S_i(U) = \sum_{i\in I} r_i^\delta \mu(U) = \mu(U).$$

Proof of Lemma 4.2.2. Let $m^* = \mu^* \circ \pi^{-1}$, where μ^* is the unique ergodic σ-invariant probability measure on I^∞ (cf. [MU] p.16). Let J be the self-conformal set generated by S, we define

$$J^* = \{x \in J : \pi^{-1}(x) \text{ is unique}\},$$

and define $T: J^* \to J^*$ by

$$T(x) = \phi_{\omega_1}^{-1}(x) \quad \text{where} \quad x = \pi(\omega).$$

Then the following diagram commutes, and the projection map is one to one.

$$\begin{array}{ccc} I^\infty & \xrightarrow{\sigma} & I^\infty \\ \pi \downarrow & & \downarrow \pi \\ J^* & \xrightarrow{T} & J^* \end{array} \tag{4.12}$$

By Theorem 3.8 and Remark 3.12 in [MU], we have $m^*(J \setminus J^*) = 0$ and m^* is equivalent to m. These imply that $m^*(J^*) = 1$.

Therefore, we have,

$$\begin{aligned} m^*(U) = m^*(U \cap J^*) &= \mu^* \pi^{-1}(U \cap J^*) \\ &= \mu^* \sigma^{-1}\big(\pi^{-1}(U \cap J^*)\big) \quad \text{since } \sigma \text{ is } \mu^* \text{ invariant} \\ &= \mu^* \pi^{-1} T^{-1}(U \cap J^*) \quad \text{since the diagram (4.12) commutes} \\ &= \mu^* \pi^{-1}\big(\cup_{i \in I} \phi_i(U \cap J^*)\big) \\ &= m^*\big(\cup_{i \in I} \phi_i(U) \bigcap \cup_{i \in I} \phi_i(J^*)\big) \\ &= m^*(\cup_{i \in I} \phi_i(U)) \quad \text{since} \quad m^*(\cup_{i \in I} \phi_i(J^*)) = 1. \end{aligned}$$

Hence, we get

$$m^*(U) = m^*(\cup_{i \in I} \phi_i(U)).$$

The equivalency between m^* and m implies

$$m(U) = m(\cup_{i \in I} \phi_i(U)). \quad \square$$

Lemma 4.2.3. *Let* $F = \partial U \bigcup \big(\cup_{\omega \in I^*} \phi_\omega(\partial U) \big)$, *then* $m(F) = 0$ or 1.

Proof of Lemma 4.2.3. Since T is ergodic w.r.t. m^*, we only need to show

$$T^{-1}(F) \subset F.$$

Since

$$\begin{aligned} T^{-1}(F) = \cup_{i \in I} \phi_i(F) &= \cup_{i \in I} \phi_i \big(\partial U \bigcup \cup_{\omega \in I^*} \phi_\omega(\partial U)\big) \\ &= \big(\cup_{i \in I} \phi_i(\partial U)\big) \bigcup \big(\cup_{i \in I} \cup_{\omega \in I^*} \phi_{i\omega}(\partial U)\big) \\ &\subset F, \end{aligned}$$

by the ergodicity of T, we have $m^*(F) = 0$ or 1. The equivalency between m^* and m implies $m(F) = 0$ or 1. $\square$

Lemma 4.2.4. $m(\partial U) = 0$ or 1.

Proof of Lemma 4.2.4. In view of Lemma 4.2.3, we only need to show

$$m(\partial U) = m(F).$$

Since

$$\begin{aligned} m(\partial U) + m(U) &= m(\overline{U}) \\ &= 1 \\ &= \sum_{\omega \in I^n} \int |\phi_\omega|^\delta dm \quad \text{by (4.9)} \\ &= \sum_{\omega \in I^n} m(\phi_\omega(\overline{U})) \quad \text{by (4.10)} \\ &= \sum_{\omega \in I^n} \big(m(\phi_\omega(\partial U)) + m(\phi_\omega(U)) \big) \\ &= m(\cup_\omega \phi_{\omega \in I^n}(\partial U)) + m(\cup_{\omega \in I^n} \phi_\omega(U)) \quad \text{by (4.8)} \end{aligned}$$

$$= m(\cup_\omega \phi_{\omega \in I^n}(\partial U)) + m(U) \quad \text{by Lemma 4.2.2,}$$

we get

$$m(\partial U) = m(\cup_{\omega \in I^n} \phi_\omega(\partial U)) \quad \text{for all } n. \tag{4.13}$$

Also, since

$$\begin{aligned} m(\partial U) &= m(\partial U \cap J) \\ &= m\big(\partial U \cap (\cup_{\omega \in I^n} \phi_\omega(\overline{U}))\big) \\ &= m\big(\partial U \cap (\cup_{\omega \in I^n} \phi_\omega(\partial U))\big), \end{aligned} \tag{4.14}$$

in view of (4.13) and (4.14), we get

$$m\left[(\cup_{\omega \in I^n} \phi_\omega(\partial U)) \setminus \partial U\right] = 0 \quad \text{for all } n \geq 1. \tag{4.15}$$

Furthermore, because

$$F = \partial U \bigcup \big(\cup_{\omega \in I^*} \phi_\omega(\partial U)\big) = \partial U \bigcup \big(\cup_{n=1}^\infty \cup_{\omega \in I^n} \phi_\omega(\partial U)\big), \tag{4.16}$$

combining (4.15) and (4.16), we get

$$\begin{aligned} m(F \setminus \partial U) &= m\left([\cup_{n=1}^\infty \cup_{\omega \in I^n} \phi_\omega(\partial U)] \setminus \partial U\right) \\ &= m\left(\cup_{n=1}^\infty \big[(\cup_{\omega \in I^n} \phi_\omega(\partial U)) \setminus \partial U\big]\right) \\ &\leq \sum_{n=1}^\infty m\big[(\cup_{\omega \in I^n} \phi_\omega(\partial U)) \setminus \partial U\big] \\ &= 0, \end{aligned}$$

therefore,

$$m(F) = m(\partial U). \quad \square$$

Now that we proved required Lemmas, we can proceed to prove the Theorem 4.2.1.

Proof of theorem 4.2.1.

"$\Leftarrow$" Since $m(\partial U) = 0$, we have

$$1 = m(J) = m(\overline{U}) = m(\partial U) + m(U) = m(U).$$

Hence $m(U) = 1$. Because m is supported on J, we get

$$U \cap J \neq \emptyset.$$

Therefore the set U also satisfies the SOSC.

"$\Rightarrow$" Suppose SOSC holds for certain open set U, then $U \cap J \neq \emptyset$.

Since $|\phi_\omega(J)| \to 0$ as $|\omega| \to \infty$, there exists $\omega \in I^*$, such that $\phi_\omega(J) \subset U$.

Therefore

$$m(U) \geq m(\phi_\omega(J)) = \int |\phi'_\omega|^\delta dm > 0.$$

Hence

$$m(U) > 0. \tag{4.17}$$

Since

$$m(\partial U) + m(U) = 1,$$

(4.17) implies $m(\partial U) < 1$.

Using Lemma 4.2.4, we get $m(\partial U) = 0$. $\square$

CHAPTER V

THE δ-CONDITION FOR STATISTICALLY SELF-SIMILAR FRACTALS

So far, we have studied fractals which are deterministic. In this chapter, we will study random fractals, in particular, statistically self-similar fractals.

The general concepts of random recursive constructions have been introduced and investigated by Falconer [Fa4], Mauldin and Williams [MW2] and Graf [Gr]. They showed that Moran and Hutchinson's result, as discussed in Chapter 1, has a probabilistic counterpart in the random case. In this chapter, we are interested in a δ-condition introduced by Graf, which gives a sufficient condition for the Hausdorff measure of a statistically self-similar fractal set K to be positive almost surely. It is not known whether the δ-condition is necessary. This chapter is a study of this condition. In particular, we generalize Example 6.8 given by Graf, Mauldin and Williams [GMW], and show that for certain statistically fractal sets in $\mathbb{R}^d$, if the δ-condition is not satisfied, then their Hausdorff measures are zero almost surely.

5.1 Definitions and Notations

This section contains the basic definitions, notations and properties of statistically self-similar constructions, which will be used in the rest of the chapter.

We fix an Euclidean space $\mathbb{R}^d$ and a nonempty compact subset J of $\mathbb{R}^d$ such that $J = \overline{\mathrm{Int}(J)}$. By $\mathrm{Sim}(J)$ we denote the set of all similarities $S : J \to J$, such that the similarity ratio r of S is less than 1. The space $\mathrm{Sim}(J)$ is equipped with the topology of pointwise convergence. Since $\mathrm{Sim}(J)$ can be written as the countable union of completely metrizable subsets, it is a Souslin space (cf. [Sc] or [Ku]). We denote the Borel field of Sim(J) by $\mathcal{F}_0$. Let $\mathcal{C}(J)$ be the space of all nonempty compact subsets of J with the Hausdorff metric d_H, then $(\mathcal{C}(J),\, d_H)$ is a complete separable metric space.

We denote by $\mathbb{N}$ the positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the nonnegative integers. For $n \in \mathbb{N}$, let

$$D^* = \bigcup_{k=0}^{\infty} \{1, \ldots, n\}^k \quad \text{where} \quad \{1, \ldots, n\}^0 = \{\emptyset\}.$$

i.e. D^* is the set of all finite sequences in $\{1, \ldots, n\}$. Clearly D^* is countable.

If we let

$$D_k = \{1, \ldots, n\}^k,$$

then

$$D^* = \bigcup_{k=0} D_k.$$

Moreover, if $\sigma = (\sigma_1, \ldots, \sigma_q)$, $\tau = (\tau_1, \ldots, \tau_p) \in D^*$, then we denote $|\sigma| = q$ and $\sigma\tau = (\sigma_1, \ldots, \sigma_q, \tau_1, \ldots, \tau_p)$.

We denote by $D = \{1, \ldots, n\}^{\mathbb{N}}$ the set of infinite strings equipped with the product topology of the discrete topology on $\{1, \ldots, n\}$.

For $\sigma \in D^* \cup D$ and $k \in \mathbb{N}_0$, where $k \leq |\sigma|$, if $\sigma \in D^*$, let $\sigma|_k = (\sigma_1, \ldots, \sigma_k)$. We define a partial order in $D^* \cup D$ by

$$\sigma \prec \tau \quad \Longleftrightarrow \quad \tau\big|_{|\sigma|} = \sigma.$$

A subset $\mathcal{A} \subset D^*$ is called an antichain if, for each pair of $\sigma, \tau \in \mathcal{A}$, we have $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$, *i.e.* σ and τ are incomparable. A subset $\mathcal{A} \subset D^*$ is called a maximal antichain if $\mathcal{A}$ is an antichain and covers D, *i.e.* for each $\eta \in D$, there is $k \in \mathbb{N}_0$ such that $\eta|_k \in \mathcal{A}$.

For a fixed $n \in \mathbb{N}$, we define

$$\Omega = \left((\mathrm{Sim}(J))^n\right)^{D^*}$$

to be the product space equipped with the product topology denoted by $\mathcal{F}$. Since D^* is countable and $(\mathrm{Sim}(J))^n$ is a Souslin space, therefore, Ω is a metrizable Souslin space. It follows from the properties of Souslin space (cf. [Sc] or [Ku]), that the product of the Borel field of $\mathrm{Sim}(J)^n$ is equal to the Borel field of Ω, and for a given Borel probability measure μ on $(\mathrm{Sim}(J))^n$, there exists a corresponding product measure on Ω denoted by $P = \mu^{D^*}$. Hence $\left(\Omega, \mathcal{F}, P\right)$ becomes a probability space. The element of $\Omega = \left((\mathrm{Sim}(J))^n\right)^{D^*}$ will be denoted by $\omega = (\omega_\sigma)_{\sigma \in D^*}$ where

$$\omega_\sigma = \left(S_{\sigma 1}(\omega), S_{\sigma 2}(\omega), \ldots, S_{\sigma n}(\omega)\right) \in (\mathrm{Sim}(J))^n .$$

Note that for each element $\omega \in \Omega$, we construct an n-ary tree as follows: The nodes of the tree are identified with the finite strings $\sigma \in D^*$, and each node σ has n branches labeled by $(S_{\sigma 1}(\omega), S_{\sigma 2}(\omega), \cdots, S_{\sigma n}(\omega)) \in (\mathrm{Sim}(J))^n$ (see Figure 5.1).

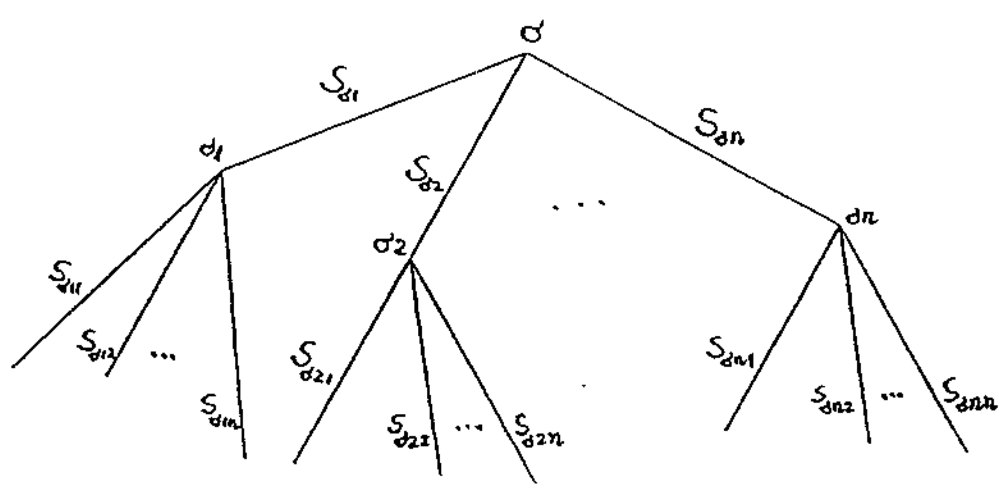

FIGURE 5.1 A RANDOM n-ARY TREE.

Definition 5.1.1. *A statistically self-similar construction modeled on J is the probability space $(\Omega, \mathcal{F}, P)$ together with a family of random subsets of $\mathbb{R}^d$*

$$\mathcal{J} = \{J_\sigma : \sigma \in D^*\}$$

having the following properties:

(1) *$J_\emptyset(\omega) = J$ for almost all $\omega \in \Omega$. For every $\sigma \in D^*$ and for almost all $\omega \in \Omega$, if $J_\sigma(\omega)$ is nonempty, then*

$$J_\sigma(\omega) = S_{\sigma|_1}(\omega) \circ S_{\sigma|_2}(\omega) \circ \cdots \circ S_{\sigma|_{|\sigma|}}(\omega)(J).$$

(2) *For almost every ω and for every $\sigma \in D^*$,*

$$\mathrm{Int} J_{\sigma i} \bigcap \mathrm{Int} J_{\sigma j} = \emptyset \quad \text{for} \quad i, j = 1, 2, \ldots, n \text{ with } i \neq j.$$

(3) *The random vectors $t_\sigma = (r_{\sigma 1}, r_{\sigma 2}, \ldots, r_{\sigma n})$ $\sigma \in D^*$ are i.i.d., where $r_{\sigma i}$ is the similarity ratio of $S_{\sigma i}(\omega) \in Sim(J)$. (For convenience, let $r_\emptyset = \operatorname{diam} J$.)*

We call such a system $\mathcal{J}$ an n-ary statistically self-similar construction. We define the random set K by

$$K(\omega) = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in D_k} J_\sigma(\omega) = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in D_k} S_{\sigma|_1}(\omega) \circ S_{\sigma|_2}(\omega) \circ \cdots \circ S_{\sigma|_{|\sigma|}}(\omega)(J),$$

and we call such a set $K(\omega)$ a statistically self-similar fractal.

Remark. *Property (2) is regarded as the OSC in the random case.*

For an n-ary statistically self-similar construction $\mathcal{J}$ based on a probability space $(\Omega, \mathcal{F}, P)$, we have:

(i) **Theorem 5.1.1.** (cf. Theorem 7.6 in [Gr]) The Hausdorff dimension of $K(\omega)$ is s almost surely, where $s \geq 0$ satisfies

$$E(\sum_1^n r_i^s) = 1.$$

Remark. This theorem is a special case of Mauldin & Williams [MW2] Theorem 1.1. The existence of s is due to the fact that the map : $\beta \to E(\sum_1^n r_i^\beta)$ is continuous and decreasing, where

$$E(\sum_1^n r_i^0) > 1 \quad \text{and} \quad \lim_{\beta\to\infty} E(\sum_1^n r_i^\beta) = 0.$$

(ii) **Theorem 5.1.2.** (cf. Theorem 7.8 in [Gr]) Assume that the following conditions (a) and (b) are satisfied:

(a) $\sum_1^n r_i^s = 1$ for μ-a.s. $(S_1, S_2, \ldots, S_n) \in (\mathrm{Sim}(J))^n$.

(b) There exists a $\delta > 0$, such that $r_i \geq \delta$ for $i = 1, 2, \ldots, n$ and μ-a.s. $(S_1, S_2, \ldots, S_n) \in (\mathrm{Sim}(J))^n$.

Then, we have $0 < \mathcal{H}^s(K(\omega)) < \infty \quad P$-a.s..

Remark. We call condition (b) the δ-condition.

(iii) **Theorem 5.1.3.** (cf. Theorem 7.7 in [Gr]) If condition (a) in Theorem 5.1.2 is not satisfied, *i.e.* if (a') $P(\sum_1^n r_i^s \neq 1) > 0$, then $\mathcal{H}^s(K(\omega)) = 0 \quad P$-a.s..

(iv) In addition, Graf, Mauldin and Williams provided an example in $\mathbb{R}^1$ (see [GMW] Example 6.8), which satisfies condition (a) in Theorem 5.1.2 but not the δ-condition. Yet $\mathcal{H}^s(K(\omega)) = 0 \quad P$-a.s..

We will show in this chapter that for certain statistically self-similar fractal sets K in $\mathbb{R}^d$, if the δ-conditions are not satisfied, then the Hausdorff measures of K are zero almost surely. In addition, an example in $\mathbb{R}^2$ is provided. Furthermore, we will extend the δ-condition to a weak δ-condition, and show that they are equivalent.

In order to investigate the Hausdorff dimension and Hausdorff measure of the random set $K(\omega)$, Mauldin and Williams ([MW2], p 334) introduced the random construction measure ν_ω on the Borel sets of $\mathbb{R}^d$ associated with $\mathcal{J}$. In [GMW], a random measure μ_ω was defined on the Borel sets of D, and the close relation between the two measures μ_ω and ν_ω was discussed. Moreover, a probability measure

Q was given on the Borel sets of the product space $D \times \Omega$. The basic definitions, notations and properties of these three measures ν_ω, μ_ω and Q are provided here. For detailed discussion, we refer to [MW2] and [GMW]. Furthermore, since what we study here is a special case of the random recursive construction in [MW2] and [GMW], all three measures can be applied.

Definitions 5.1.2. For each $\omega \in \Omega$ and $\sigma \in D^*$, we denote

$$l_\sigma(\omega) = \operatorname{diam} J_\sigma(\omega) = \prod_{k=1}^{|\sigma|} r_{\sigma|_k}(\omega), \tag{5.1}$$

where $r_{\sigma|_k}(\omega)$ is the similarity ratio of $S_{\sigma|_k}(\omega)$ and

$$l_\emptyset(\omega) = \operatorname{diam} J. \tag{5.2}$$

It follows from [MW2], that

$$\limsup_{k\to\infty}\{l_\sigma : \sigma \in D_k\} = 0 \;\; P\text{-a.s. } \omega. \tag{5.3}$$

Define the random variables

$$S_{s,k}(\omega) = \sum_{\sigma \in D_k} l_\sigma^s(\omega).$$

Recall that $s \geq 0$ satisfies $E(\sum_1^n r_i^s) = 1$. The Martingale Convergence Theorem yields that the sequence $(S_{s,k})_{k\in\mathbb{N}}$ converges P-a.s. to a random variable $X(\omega)$ with $E\left(X(\omega)\right) = (\operatorname{diam} J)^s$. (see [MW2], Theorem 2.1 for details).

For each $\sigma \in D^*$, we define the random variable $X_\sigma(\omega)$ by

$$X_\sigma(\omega) = \lim_{k\to\infty} \sum_{\tau \in D_k} \prod_{k=1}^{|\tau|} r^s_{\sigma(\tau|_k)}(\omega).$$

By i.i.d. (see Definition 5.1.1 (3)), each X_σ is distributed as $(\operatorname{diam} J)^{-s} X(\omega)$.

Remark. *If the condition*

$$\sum_{1}^{n} r_i^s = 1 \quad for \quad \mu\text{-a.s. } (S_1, S_2, \ldots, S_n) \in Sim(J)^n \tag{5.3}$$

is satisfied, then we have:

$$X(\omega) = (\operatorname{diam} J)^s \quad P\text{-a.s.}, \tag{5.4}$$

and for each $\sigma \in D^*$,

$$X_\sigma(\omega) = 1 \quad P\text{-a.s.}. \tag{5.5}$$

For each $\omega \in \Omega$, define a Borel measure μ_ω on D (see [GMW] p. 4) such that μ_ω satisfies

$$\mu_\omega([\sigma]) = l_\sigma^s(\omega) X_\sigma(\omega), \tag{5.6}$$

where $[\sigma] = \{\eta \in D : \sigma \prec \eta\}$. The map $\omega \to \mu_\omega(A)$ is measurable for every clopen set $A \subset D$.

For each $\omega \in \Omega$, we define a bounded countable additive measure ν_ω on the Borel sets of $\mathbb{R}^d$ satisfying (see [MW2] p. 334 for details):

(1) ν_ω has total mass $X(\omega)$

(2) $\nu_\omega(K(\omega)) = X(\omega)$,

(3) If A is a compact subset of $\mathbb{R}^d$, then ([MW2] Theorem 3.2)

$$\nu_\omega(A) = \lim_{k \to \infty} \sum_{\sigma \in D_k,\, J_\sigma \cap A \neq \emptyset} l_\sigma^s(\omega) X_\sigma(\omega). \tag{5.7}$$

The two measures μ_ω and ν_ω are related, which can be seen later.

We denote by $\mathcal{B}$ the Borel field of D. Recall that $\mathcal{F}$ is the Borel field of Ω. For $B \in \mathcal{B} \otimes \mathcal{F}$ and $\omega \in \Omega$, let $B_\omega = \{\eta \in D : (\eta, \omega) \in B\}$. We define a probability measure on $\mathcal{B} \otimes \mathcal{F}$ by (cf. [GMW] p. 5)

$$Q(B) = (\operatorname{diam} J)^{-s} \int_\Omega \mu_\omega(B_\omega) dP(\omega). \tag{5.8}$$

The expected value of random variables with respect to Q will be denoted by E_Q. Note that for Q-a.s. (η,ω), there is a unique point in $\cap_{k=1}^{\infty} J_{\eta|_k}(\omega)$, denoted by $\hat{\eta}(\omega)$. One of the useful properties of measure Q is that for Q-a.s. (η,ω), $\hat{\eta}(\omega)$ does exist and satisfies

$$Q(D \times \{\omega : K(\omega) \neq \emptyset\}) = 1.$$

Therefore, a natural random map f_ω exists with random domain of definition D_ω such that $f_\omega(\eta) = \hat{\eta}(\omega)$ and

$$\nu_\omega = \mu_\omega \circ f_\omega^{-1}. \tag{5.9}$$

5.2 Theorems and Proofs

Theorem 5.2.1. *Let $\mathcal{J}$ be an n-ary statistically self-similar construction based on a probability space $(\Omega, \mathcal{F}, P)$ and a seed set J of $\mathbb{R}^d$. Suppose that*

(i) $E(\sum_1^n r_i^0) > 1$

(ii) $\sum_1^n r_i^s = 1$ *for μ-a.s. $(S_1, \ldots, S_n) \in (Sim(J))^n$.*

(iii) *For each $\epsilon > 0$, there exists a maximal antichain $\mathcal{D} = \mathcal{D}(\epsilon)$ such that*

$$P(\operatorname{diam} \cup_{\sigma\in\mathcal{D}} J_\sigma < \epsilon \operatorname{diam} J) > 0.$$

Then $\mathcal{H}^s(K(\omega)) = 0$ for P-a.s. ω.

Remark. *The method used to prove Theorem 5.2.1 is similar to that in Example 6.8 of [GMW]. However, since we are dealing with a more general case, other theorems such as Vitali Covering Theorem are also used. We provide here an example in $\mathbb{R}^2$, in which Theorem 5.2.1 can be applied.*

Example. Choose $0 < s < 1$, let $\triangle = \{(t_1, t_2) \in [0, (\frac{1}{3})^{\frac{1}{s}}] \times [(\frac{1}{3})^{\frac{1}{s}}, 1] : 2t_1^s + t_2^s = 1\}$ be a subset of $\mathbb{R}^2$ and λ be the normalized Lebesgue measure on $\triangle$. Let J be an initial unit equilateral triangle with left vertex at origin, define a random ternary construction $\mathcal{J}$ modeled on J as the following recursion: If J_σ is an equilateral triangle with up vertex (a_1, b_1), left vertex (a_2, b_2) and right vertex (a_3, b_3), choose (t_1, t_2) from $\triangle$ at random, and set (see Figure 5.2):

$$J_{\sigma 1} = t_1(J_\sigma - (a_2, b_2)) + (a_2, b_2),$$

$$J_{\sigma 2} = t_1(J_\sigma - (a_2, b_2)) + (a_2 + t_1(a_1 - a_2), b_2 + t_1(b_1 - b_2)),$$

$$J_{\sigma 3} = t_2(J_\sigma - (a_2, b_2)) + (a_2 + t_1(a_3 - a_2), b_2 + t_1(b_3 - b_2)).$$

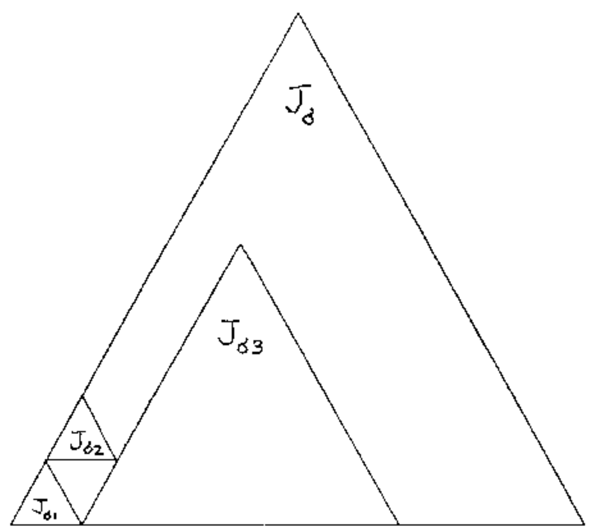

FIGURE 5.2 A RANDOM TERNARY CONSTRUCTION.

Then the corresponding random set is $K = \cap_{k=1}^{\infty} \cup_{\sigma \in \{1,2,3\}^k} J_\sigma$. To see whether it satisfies the three conditions in Theorem 5.2.1, let

$$S_1^{(t_1,t_2)}, S_2^{(t_1,t_2)}, S_3^{(t_1,t_2)} : J \to J$$

be defined by:

$$S_1^{(t_1,t_2)}(x, y) = t_1(x, y),$$

$$S_2^{(t_1,t_2)}(x,y) = t_1(x,y) + (\frac{1}{2}t_1, \frac{\sqrt{3}}{2}t_1),$$
$$S_3^{(t_1,t_2)}(x,y) = t_2(x,y) + (t_2, 0).$$

Then the similarity ratios are $r_1 = t_1$, $r_2 = t_1$ and $r_3 = t_2$. Let μ be the image of normalized Lebesgue measure on $\triangle$ w.r.t. the map

$$\triangle \to \mathrm{Sim}(J)^3 : (t_1, t_2) \to (S_1^{(t_1,t_2)}, S_2^{(t_1,t_2)}, S_3^{(t_1,t_2)}).$$

Hence we have $r_1^s + r_2^s + r_3^s = 2t_1^s + t_2^s = 1$ for μ-a.s. $(S_1, S_2, S_3) \in \mathrm{Sim}(J)^3$, and $||r_3||_\infty = 1$. Therefore the conditions (i) and (ii) in Theorem 5.2.1 are satisfied. According to Theorem 5.1.1, the Hausdorff dimension of K is s P-a.s..

To see this example also satisfies condition (iii) in Theorem 5.2.1, for each j, let $\mathcal{D}_j$ denote the maximal antichain in $\{1,2,3\}^*$ consisting of $2j+1$ sequences:

$$\mathcal{D}_j = \{(1),(2),(31),(32),\cdots,(3\cdots 31),(3\cdots 32),(3\cdots 33)\}.$$

Clearly, $\cup_{\sigma\in\mathcal{D}_j} J_\sigma$ is a subset of the equilateral triangle with left vertex $(0,0)$ and right vertex $(\sum_{\sigma\in\mathcal{D}_j,\sigma(|\sigma|)\neq 2}\prod_{i=1}^{|\sigma|} r_{\sigma|_i}, 0)$, and the diameter of $\cup_{\sigma\in\mathcal{D}_j} J_\sigma$ is

$$\mathrm{diam}\cup_{\sigma\in\mathcal{D}_j} J_\sigma = \sum_{\sigma\in\mathcal{D}_j,\sigma(|\sigma|)\neq 2}\prod_{i=1}^{|\sigma|} r_{\sigma|_i}.$$

Now, by a similar argument as in ([GMW] p. 100), we have, for each ϵ,

$$P(\mathrm{diam}\cup_{\sigma\in\mathcal{D}_j} J_\sigma < \epsilon) > 0.$$

Applying Theorem 5.2.1, we get $\mathcal{H}^s(K) = 0$ P-a.s..

Now we will give two lemmas used in the proof of Theorem 5.2.1. The hypothesis of the lemmas are the same as in Theorem 5.2.1. and their proofs will be provided at the end of Section 5.2.

Lemma 5.2.2. *Let $\mathcal{D}$ be a maximal antichain, then for each $\sigma \in D^*$ and $\epsilon > 0$, we have*

$$P(\mathrm{diam}\cup_{\tau\in\mathcal{D}} J_{\sigma\tau} \geq \epsilon l_\sigma) = P(\mathrm{diam}\cup_{\tau\in\mathcal{D}} J_\tau \geq \epsilon\,\mathrm{diam}\, J).$$

Lemma 5.2.3. *For P-a.s. ω, $\mathcal{H}^s\lfloor K(\omega) \ll \nu_\omega$.*

Proof of Theorem 5.2.1. Given $\epsilon > 0$, let $\mathcal{D} = \mathcal{D}(\epsilon)$ be a maximal antichain with

$$P(\operatorname{diam} \cup_{\sigma\in\mathcal{D}} J_\sigma < \epsilon \operatorname{diam} J) > 0. \tag{5.10}$$

Define a sequence $\{\mathcal{C}_n\}_{n=0}^{\infty}$ of maximal antichains by

$$\mathcal{C}_0 = \{\emptyset\},\ \mathcal{C}_1 = \mathcal{D},\ \mathcal{C}_2 = \mathcal{D} * \mathcal{D},$$

and in general, for each k,

$$\mathcal{C}_{k+1} = \mathcal{C}_k * \mathcal{D} = \{\sigma_1\sigma_2 :\ \sigma_1 \in \mathcal{C}_k,\ \sigma_2 \in \mathcal{D}\}.$$

For each $k \geq 1$, set

$$Z_{\epsilon,k}(\omega) = \sum_{\sigma\in\mathcal{C}_k} l_\sigma^s(\omega) 1_{A(\omega)}(\sigma),$$

where

$$A(\omega) = \{\sigma \in D^* :\ \forall j, \forall \eta \in \mathcal{C}_j\ [\text{ if } \eta \precneqq \sigma \text{ then } \operatorname{diam}(\cup_{\tau\in\mathcal{D}} J_{\eta\tau}) \geq \epsilon l_\eta\ ]\ \}.$$

Claim 1. $\lim_{k\to\infty} E(Z_{\epsilon,k}) = 0$.

Proof of Claim 1.

$$\begin{aligned}
Z_{\epsilon,k+1}(\omega) &= \sum_{\sigma\in\mathcal{C}_{k+1}} l_\sigma^s(\omega) 1_{A(\omega)}(\sigma) \\
&= \sum_{\sigma\in\mathcal{C}_k} \sum_{\tau\in\mathcal{D}} l_{\sigma\tau}^s(\omega) 1_{A(\omega)}(\sigma\tau) \\
&= \sum_{\sigma\in\mathcal{C}_k} \sum_{\tau\in\mathcal{D}} l_\sigma^s \prod_{i=1}^{|\tau|} r_{\sigma(\tau|_i)}^s 1_{A(\omega)}(\sigma) 1_{A(\omega)}(\sigma\tau) \\
&\qquad \text{since} \quad 1_{A(\omega)}(\sigma\tau) = 1_{A(\omega)}(\sigma) 1_{A(\omega)}(\sigma\tau) \\
&= \sum_{\sigma\in\mathcal{C}_k} l_\sigma^s 1_{A(\omega)}(\sigma) \sum_{\tau\in\mathcal{D}} \prod_{i=1}^{|\tau|} r_{\sigma(\tau|_i)}^s 1_{A(\omega)}(\sigma\tau)
\end{aligned}$$

$$= \sum_{\sigma \in \mathcal{C}_k} l_\sigma^s 1_{A(\omega)}(\sigma) 1_{(\operatorname{diam} \cup_{\tau \in \mathcal{D}} J_{\sigma\tau} \geq \epsilon l_\sigma)} \sum_{\tau \in \mathcal{D}} \prod_{i=1}^{|\tau|} r_{\sigma(\tau|_i)}^s$$
$$= \sum_{\sigma \in \mathcal{C}_k} l_\sigma^s 1_{A(\omega)}(\sigma) 1_{(\operatorname{diam} \cup_{\tau \in \mathcal{D}} J_{\sigma\tau} \geq \epsilon l_\sigma)}.$$

The last two equalities follow from $1_{A(\omega)}(\sigma\tau) = 1_{A(\omega)}(\sigma\tau')$ for every $\tau, \tau' \in \mathcal{D}$, $\sum_{i=1}^n r_i^s = 1$ μ-a.s. and $\mathcal{D}$ is a maximal antichain.

Hence, by independence,

$$E(Z_{\epsilon,k+1}) = P(\operatorname{diam} \cup_{\tau \in \mathcal{D}} J_{\sigma\tau} \geq \epsilon l_\sigma) E(Z_{\epsilon,k})$$
$$= P(\operatorname{diam} \cup_{\tau \in \mathcal{D}} J_\tau \geq \epsilon \operatorname{diam} J) E(Z_{\epsilon,k}) \quad \text{by lemma 5.2.2.}$$

By induction and $l_\emptyset = \operatorname{diam} J$, we have

$$E(Z_{\epsilon,k+1}) = P(\operatorname{diam} \cup_{\tau \in \mathcal{D}} J_\sigma \geq \epsilon \operatorname{diam} J)^{k+1} (\operatorname{diam} J)^s.$$

Hypothesis (ii) of Theorem implies that

$$P(\operatorname{diam} \cup_{\tau \in \mathcal{D}} J_\sigma \geq \epsilon \operatorname{diam} J) < 1.$$

Consequently, we obtain

$$E(Z_{\epsilon,k+1}) \to 0 \quad \text{as} \quad k \to \infty.$$

Claim 2. For each $M > 0$, for Q-a.s. (η, ω), there are infinitely many k, such that

$$\nu_\omega \left(\cup_{\sigma \in \mathcal{D}} J_{(\eta|_k)\sigma}(\omega) \right) \geq M^s \left(\operatorname{diam} \cup_{\sigma \in \mathcal{D}} J_{(\eta|_k)\sigma}(\omega) \right)^s, \tag{5.11}$$

where $\mathcal{D} = \mathcal{D}(\frac{1}{M})$.

Proof of Claim 2. In view of (5.7) and (5.5), we have

$$\nu_\omega \left(\cup_{\sigma \in \mathcal{D}} J_{(\eta|_k)\sigma}(\omega) \right) \geq l_{\eta|_k}^s X_{\eta|_k}(\omega) = l_{\eta|_k}^s(\omega).$$

Hence to show (5.11), it is enough to show that for Q-a.s. (η, ω)

$$l_{\eta|_k}(\omega) \geq M \operatorname{diam}\left(\cup_{\sigma \in \mathcal{D}} J_{(\eta|_k)\sigma}(\omega)\right) \quad \text{for infinitely many } k. \tag{5.12}$$

Let $B \subset D \times \Omega$ be the set such that (5.12) is not satisfied. We want to show that $Q(B) = 0$.

Since $B = \cup_{m=1}^{\infty} B_m$ where

$$B_m = \left\{(\eta, \omega) : \text{ for every } \; k \geq m, l_{\eta|_k} < M \operatorname{diam} \cup_{\sigma \in \mathcal{D}} J_{(\eta|_k)\sigma}\right\},$$

clearly $B_1 \subset B_2 \subset \ldots$ is an increasing sequence, hence

$$Q(B) = \lim_{m \to \infty} Q(B_m). \tag{5.13}$$

Set

$$a = E\left(\mu_\omega(\{\eta : \forall k \; l_{\eta|_k} < M \operatorname{diam}(\cup_{\sigma \in \mathcal{D}} J_{\eta|_k \sigma})\})\right). \tag{5.14}$$

For each k, we have

$$a \leq E(\mu_\omega(\{\eta : \text{ if } \sigma \in \mathcal{C}_k \text{ and } \sigma \prec \eta \text{ then } \sigma \in A(\omega)\})).$$

Hence,

$$a \leq E\left(\sum_{\sigma \in \mathcal{C}_k} l_\sigma^s 1_{A(\omega)}(\sigma) X_\sigma\right) = E\left(\sum_{\sigma \in \mathcal{C}_k} l_\sigma^s 1_{A(\omega)}(\sigma)\right).$$

This follows from the definition of μ_ω and $X_\sigma(\omega) = 1$ P-a.s..

Therefore, we have $a \leq E(Z_{\frac{1}{M},k})$ for every k. Claim 1 implies

$$a = 0. \tag{5.15}$$

Since for each $m \geq 1$, we have

$$Q(B_m) = E\left(\mu_\omega(\{\eta : \forall k \geq m \;\; l_{\eta|_k} < M \operatorname{diam}(\cup_{\sigma \in \mathcal{D}} J_{\eta|_k \sigma})\})\right)$$

$$= E\left(\sum_{\gamma\in D_m} \mu_\omega(\{\gamma\eta : \forall k \quad l_{\gamma\eta|_k} < M\,\mathrm{diam}(\cup_{\sigma\in\mathcal{D}} J_{\gamma\eta|_k\sigma})\})\right)$$

$$= E\left(\sum_{\gamma\in D_m} E\left(\mu_\omega(\{\gamma\eta : \forall k \quad l_{\gamma(\eta|_k)} < M\,\mathrm{diam}(\cup_{\sigma\in\mathcal{D}} J_{\gamma(\eta|_k)\sigma})\})|\mathcal{F}_m\right)\right),$$

where $\mathcal{F}_m$ is the σ-field generated by the maps

$$\omega \to (r_{(\sigma|_i)1}, \ldots, r_{(\sigma|_i)n}) \; i = 1, 2, \ldots, m-1, \; \sigma \in D_m,$$

hence, by scaling of μ_ω,

$$Q(B_m) = E\left(\sum_{\gamma\in D_m} E\left(l_\gamma^s \mu_\omega(\{\eta : \forall k \quad l_{\gamma\eta|_k} < M\,\mathrm{diam}(\cup_{\sigma\in\mathcal{D}} J_{\gamma\eta|_k\sigma})\})|\mathcal{F}_m\right)\right)$$

$\Downarrow$ since: l_γ^s is $\mathcal{F}_m$ measurable

$$= E\left(\sum_{\gamma\in D_m} l_\gamma^s E\left(\mu_\omega(\{\eta : \forall k \quad l_{\gamma\eta|_k} < M\,\mathrm{diam}(\cup_{\sigma\in\mathcal{D}} J_{\gamma\eta|_k\sigma})\})|\mathcal{F}_m\right)\right)$$

$\Downarrow$ by independence

$$= E\left(\sum_{\gamma\in D_m} l_\gamma^s E\left(\mu_\omega(\{\eta : \forall k \quad l_{\gamma\eta|_k} < M\,\mathrm{diam}(\cup_{\sigma\in\mathcal{D}} J_{\gamma\eta|_k\sigma})\})\right)\right)$$

$\Downarrow$ by independence

$$= E\left(\sum_{\gamma\in D_m} l_\gamma^s E\left(\mu_\omega(\{\eta : \forall k \quad l_{\eta|_k} < M\,\mathrm{diam}(\cup_{\sigma\in\mathcal{D}} J_{\eta|_k\sigma})\})\right)\right)$$

$$= E\left(\sum_{\gamma\in D_m} l_\gamma^s a\right) \quad \text{by (5.14)}$$

$$= 0. \quad \text{by (5.15)}$$

Therefore $Q(B_m) = 0$ for every $m \geq 1$. In view of (5.13), we have

$$Q(B) = \lim_{m\to\infty} Q(B_m) = 0.$$

This implies that (5.12) holds for Q-a.s. (η, ω).

Claim 3. For each $M > 0$ and P-a.s. ω, there is a set $E_M(\omega) \subset K(\omega)$ such that

$$\nu_\omega(E_M(\omega)) = X(\omega) \quad \text{and} \quad \mathcal{H}^s(E_M(\omega)) \leq \frac{X(\omega)}{M^s}.$$

Proof of Claim 3. For each ω, let

$$F_M(\omega) = \{\eta \in D : \text{ such that (5.11) holds for infinitely many } k \}.$$

Let

$$G_M = \{\omega : \ \mu_\omega(F_M(\omega)) = X(\omega) \text{ and } \limsup_{k\to\infty}\{l_\sigma(\omega) : \ |\sigma| = k\} = 0\}.$$

It follows from Claim 2 and (5.3) that $P(G_M) = 1$. Recall the random map $f_\omega : D_\omega \to \mathbb{R}^d$ given by

$$f_\omega(\eta) = \hat{\eta}(\omega) = \bigcap_{k=1}^{\infty} J_{\eta|_k}(\omega).$$

Let

$$E_M(\omega) = f_\omega(F_M(\omega)).$$

Suppose $\omega \in G_M$, then (5.9) implies that

$$\nu_\omega(E_M(\omega)) = \mu_\omega(f_\omega^{-1}(E_M(\omega))) = \mu_\omega(F_M(\omega)) = X(\omega).$$

Now, to complete the proof of Claim 3, we only need to show that $\mathcal{H}^s\left(E_M(\omega)\right) \leq \frac{1}{M^s}X(\omega)$ by using the Vitali Covering Theorem.

First, note that $E_M(\omega)$ is $\mathcal{H}^s$-measurable, because the complement of $E_M(\omega)$ has ν_ω measure zero, and Lemma 5.2.3 implies $K(\omega) \setminus E_M(\omega)$ has $\mathcal{H}^s$-measure zero. Hence $E_M(\omega)$ is $\mathcal{H}^s$-measurable.

Now we form a Vitali cover for $E_M(\omega)$. For any $\delta > 0$, choose $k_\delta > 0$ such that if $|\sigma| \geq k_\delta$, then $l_\sigma(\omega) < \delta$. For each $\eta \in F_M(\omega)$, there exists the smallest integer $k_\delta(\eta) \geq k_\delta$ such that (5.11) holds.

Define

$$I_\delta(\eta) = \bigcup_{\sigma \in \mathcal{D}} J_{(\eta|_{k_\delta(\eta)})\sigma}.$$

The collection of sets $\mathcal{W}_\omega = \{I_\delta(\eta) : \ \delta > 0, \ \eta \in F_M(\omega)\}$ is a Vitali class for $E_M(\omega)$. The Vitali Covering Theorem (cf. [Fa] p.11) implies that for any given $\epsilon > 0$, we may select a (finite or countable) disjoint sequence $\{U_i\}$ from $\mathcal{W}$ such that

$$\mathcal{H}^s(E_M(\omega)) \le \sum_i |U_i|^s + \epsilon. \tag{5.16}$$

Since each set U_i has the form $I_\delta(\eta)$ for some $\delta > 0$ and $\eta \in F_M(\omega)$, (5.11) and (5.16) imply that

$$\begin{aligned}
\mathcal{H}^s(E_M(\omega)) &\le \sum_i |U_i|^s + \epsilon \\
&\le \sum_i \frac{1}{M^s}\nu_\omega(U_i) + \epsilon \\
&\le \frac{1}{M^s}\nu_\omega(\cup_i U_i) + \epsilon \quad \text{by disjointness} \\
&\le \frac{1}{M^s}X(\omega) + \epsilon \quad \text{since } \nu_\omega \text{ has total mass } X(\omega).
\end{aligned}$$

By letting $\epsilon \to 0$, we get

$$\mathcal{H}^s(E_M(\omega)) \le \frac{1}{M^s}X(\omega). \ \square$$

Claim 4. $\mathcal{H}^s(K(\omega)) = 0$ for P-a.s. ω.

Proof of claim 4. Let

$$A_M = \left\{ \omega : \ \exists \ E_M(\omega) \subset K(\omega) \text{ such that } \nu_\omega(E_M(\omega)) = X(\omega) \right.$$
$$\left. \text{and } \mathcal{H}^s(E_M(\omega)) \le \frac{X(\omega)}{M^s} \right\}.$$

Claim 3 implies $P(A_M) = 1$ for every $M > 0$.

Let

$$A = \{\omega : \exists\, E(\omega) \subset K(\omega) \text{ such that } \nu_\omega(E(\omega)) = X(\omega) \text{ and } \mathcal{H}^s(E(\omega)) = 0\}.$$

Then $A = \cap_{M=1}^{\infty} A_M$.

Since $A_1 \supset A_2 \supset \dots$, we have

$$P(A) = \lim_{M \to \infty} P(A_M) = 1.$$

Hence for P-a.s. ω, there exists a set $E(\omega)$ such that

$$\nu_\omega(E(\omega)) = X(\omega) \quad \text{and} \quad \mathcal{H}^s(E(\omega)) = 0. \tag{5.17}$$

Applying $\mathcal{H}^s \lfloor K(\omega) \ll \nu_\omega$ again, we get

$$\begin{aligned} \mathcal{H}^s(K(\omega) \setminus E(\omega)) &\le \nu_\omega(K(\omega) \setminus E(\omega)) \\ &= \nu_\omega(K(\omega)) - \nu_\omega(E(\omega)) \\ &= X(\omega) - X(\omega) \quad \text{by (5.17)} \\ &= 0. \end{aligned}$$

So

$$\mathcal{H}^s(K(\omega)) = \mathcal{H}^s(E(\omega)) = 0.$$

This completes the proof of Theorem 5.2.1. □

Now we provide the proofs of Lemmas 5.2.2 and 5.2.3.

Proof of Lemma 5.2.2. Since

$$J_{\sigma\tau}(\omega) = S_{\sigma|_1}(\omega) \circ \cdots \circ S_{\sigma|_{|\sigma|}}(\omega) \circ S_{\sigma(\tau|_1)} \circ \cdots \circ S_{\sigma(\tau|_{|\tau|})}(\omega)(J),$$

then

$$\bigcup_{\tau\in\mathcal{D}} J_{\sigma\tau}(\omega) = S_{\sigma|_1}(\omega)\circ\cdots\circ S_{\sigma|_{|\sigma|}}(\omega)\left(\cup_{\tau\in\mathcal{D}} S_{\sigma(\tau|_1)}\circ\cdots\circ S_{\sigma(\tau|_{|\tau|})}(\omega)(J)\right),$$

and

$$\operatorname{diam}\bigcup_{\tau\in\mathcal{D}} J_{\sigma\tau}(\omega) = r_{\sigma|_1}(\omega)\ldots r_{\sigma|_{|\sigma|}}\operatorname{diam}\left(\cup_{\tau\in\mathcal{D}} S_{\sigma(\tau|_1)}\circ\cdots\circ S_{\sigma(\tau|_{|\tau|})}(\omega)(J)\right). \tag{5.18}$$

Moreover, since

$$J_\sigma(\omega) = S_{\sigma|_1}(\omega)\circ\cdots\circ S_{\sigma|_{|\sigma|}}(\omega)(J),$$

and

$$\operatorname{diam} J_\sigma(\omega) = r_{\sigma|_1}(\omega)\ldots r_{\sigma|_{|\sigma|}}(\omega)\operatorname{diam}(J), \tag{5.19}$$

these imply that

$$\begin{aligned}&\{\omega\ :\ \operatorname{diam} J_{\sigma\tau}(\omega)\geq\epsilon\operatorname{diam} J_\sigma\ \}\\ &=\left\{\omega\ :\ \operatorname{diam}\left(\cup_{\tau\in\mathcal{D}} S_{\sigma(\tau|_1)}(\omega)\circ\cdots\circ S_{\sigma(\tau|_{|\tau|})}(\omega)(J)\right)\geq\epsilon\operatorname{diam} J\ \right\}.\end{aligned} \tag{5.20}$$

By i.i.d. (see Definition 5.1 (3)), and that $\mathcal{D}$ is an maximal antichain,

$$\cup_{\tau\in\mathcal{D}} S_{\sigma(\tau|_1)}\circ\cdots\circ S_{\sigma(\tau|_{|\tau|})}(\omega)(J)\quad\text{and}\quad\cup_{\tau\in\mathcal{D}}\ S_{\tau|_1}\circ\cdots\circ S_{\tau|_{|\tau|}}(\omega)(J)$$

are identically distributed for each $\sigma\in\mathcal{D}$. Therefore (5.20) implies that

$$P(\ \operatorname{diam} J_{\sigma\tau}(\omega)\geq\epsilon\operatorname{diam} J_\sigma\ ) = P(\ \operatorname{diam} J_\tau(\omega)\geq\epsilon\operatorname{diam} J\ ).\quad\square$$

Proof of lemma 5.2.3. Let $\Omega_0\subset\Omega$ be such that

$$\Omega_0 = \{\omega : \limsup_{k\to\infty}\{l_\sigma : \sigma\in D_k\} = 0\}.$$

(5.3) implies that $P(\Omega_0) = 1$.

Let $\omega \in \Omega_0$ and $E \subset K(\omega)$ be a Borel set. In view of (5.7) and (5.5), we have

$$\nu_\omega(E) = \lim_{k\to\infty} \sum_{\sigma\in D_k, J_\sigma\cap E\neq\emptyset} l_\sigma^s(\omega)X(\omega) = \lim_{k\to\infty} \sum_{\sigma\in D_k, J_\sigma\cap E\neq\emptyset} l_\sigma^s(\omega).$$

Therefore, for any $\epsilon > 0$, there exists a $K > 0$ such that if $k \geq K$

$$\nu_\omega(E) > \sum_{\sigma\in D_k, J_\sigma\cap E\neq\emptyset} l_\sigma^s(\omega) - \epsilon. \tag{5.21}$$

Since $\omega \in \Omega_0$, by definition of Ω_0 we have, for every $\delta > 0$, there exists a $k_\delta \geq K$ such that

$$\sup_{\sigma\in D_{k_\delta}} l_\sigma \leq \delta.$$

Moreover, since

$$\bigcup_{\sigma\in D_{k_\delta}, J_\sigma\cap E\neq\emptyset} J_\sigma$$

is a δ-cover of E, (5.21) implies that

$$\mathcal{H}_\delta^s(E) \leq \sum_{\sigma D_{k_\delta}, J_\sigma\cap E\neq\emptyset} l_\sigma^s(\omega) < \nu_\omega(E) + \epsilon.$$

By letting $\delta \to 0$, we get

$$\mathcal{H}^s(E) < \nu_\omega(E) + \epsilon.$$

By letting $\epsilon \to 0$, we get

$$\mathcal{H}^s(E) \leq \nu_\omega(E).$$

Finally, the regularity of $\mathcal{H}^s$ leads to

$$\mathcal{H}^s\lfloor K(\omega) \ll \nu_\omega. \quad \square$$

5.3 A WEAK δ-CONDITION

In this section, we will modify the δ-condition to a weak δ-condition and show that they are equivalent. Therefore Graf's result (see Theorem 5.1.2 in Section 5.1) is true under the weak δ-condition.

1. Definitions.

(1) Recall the δ-condition :

There exists a $\delta > 0$ such that $r_i \geq \delta$ for $i = 1, \ldots, n$ and μ-a.s. $(S_1, \ldots, S_n) \in (\mathrm{Sim}(J))^n$, where r_i is the ratio of S_i.

(2) The weak δ-condition :

For P-a.s. $\omega \in \Omega = (\mathrm{Sim}(J)^n)^{D^*}$, there exists $\delta(\omega) > 0$ such that $\inf_{\sigma \in D^*} r_\sigma(\omega) \geq \delta(\omega)$, where $r_\sigma(\omega)$ is the similarity ratio of $S_\sigma(\omega) \in \mathrm{Sim}(J)$.

Remark. *If we define*

$$\phi : \Omega \to [0,1] \quad \textit{such that} \quad \phi(\omega) = \inf_{\sigma \in D^*} r_\sigma(\omega),$$

then the δ-condition means that there exists a $\delta > 0$ such that

$$P(\omega : \phi(\omega) \geq \delta) = 1, \tag{5.22}$$

whereas the weak δ-condition asserts that

$$P(\omega : \phi(\omega) > 0) = 1. \tag{5.23}$$

2. Theorem 5.3.1. *The δ-condition and the weak δ-condition are equivalent.*

Proof. In view of (5.22) and (5.23), the δ-condition implies the weak δ-condition. Therefore all we need to show is that the weak δ-condition implies δ-condition. We will show this by contradiction.

Suppose that the weak δ-condition is satisfied, and the δ-condition is not. For each $m \in \mathbb{N}$, let $A_m = \{(S_1, \ldots, S_n) : \min_i r_i < \frac{1}{m}\}$, then $\mu(A_m) > 0$.

Let $C = \{\omega \in \Omega : \inf_{\sigma \in D^*} r_\sigma(\omega) = 0\}$, then

$$\begin{aligned} C &= \cap_{m=1}^{\infty} \{\omega \in \Omega : \inf_{\sigma \in D^*} r_\sigma(\omega) < \frac{1}{m}\} \\ &= \cap_{m=1}^{\infty} C_m, \end{aligned}$$

where $C_m = \{\omega \in \Omega : \inf_{\sigma \in D^*} r_\sigma(\omega) < \frac{1}{m}\}$.

Clearly

$$C_1 \supset C_2 \supset \ldots .$$

Therefore

$$P(C) = \lim_{m \to \infty} P(C_m).$$

Since $C_m = \{\omega \in \Omega : \inf_{\sigma \in D^*} r_\sigma(\omega) < \frac{1}{m}\}$, we have

$$\begin{aligned} \complement C_m &= \{\omega \in \Omega : \forall \sigma \in D^* \quad r_\sigma \geq \frac{1}{m}\} \\ &= \cap_{\sigma \in D^*} \{\omega \in \Omega : (S_{\sigma 1}, \ldots, S_{\sigma n}) \in \complement A_m\}. \end{aligned}$$

By independence

$$\begin{aligned} P\left(\complement C_m\right) &= \prod_{\sigma \in D^*} P\left(\{\omega \in \Omega : (S_{\sigma 1}, \ldots, S_{\sigma n}) \in \complement A_m\}\right) \\ &= \prod_{\sigma \in D^*} \mu\left(\complement A_m\right) \\ &= 0, \end{aligned}$$

where the last equality is due to the fact that $\mu(A_m) > 0$, i.e. $\mu\left(\complement A_m\right) < 1$.

Hence

$$P(C_m) = 1,$$

so

$$P(C) = \lim_{m \to \infty} P(C_m) = 1$$

i.e.

$$P\left(\omega \in \Omega : \inf_{\sigma \in D^*} r_\sigma(\omega) = 0\right) = 1.$$

This implies that

$$P\left(\omega \in \Omega : \inf_{\sigma \in D^*} r_\sigma(\omega) > 0\right) = 0$$

i.e.

$$P\left(\omega \in \Omega : \phi(\omega) > 0\right) = 0.$$

This contradicts the weak δ-condition. $\square$

CHAPTER VI

THE STUDIES OF LINEAR CELLULAR AUTOMATA USING MW-GRAPHS

The phenomena of Cellular Automata were discovered very early in the history of science. Pascal (1623 -1662)'s triangle, shown in Figure 6.1, was once considered the first example of Cellular Automata [PJS]. However, long before that, a similar Chinese arithmetic triangle had appeared in an ancient science journal around 1303 [PJS], as shown in Figure 6.2. Nevertheless, it was not until 1940s that considerable developments were achieved by Konrad Zuse, Stanislaw Ulam and John von Neumann [TM] to simulate the behavior of complex and spatially extended structures. During the 1970s and 80s, cellular automata received a great revival through the works of Stephen Wolfram [Wo1], who edited an anthology surveying the current research work of cellular automata. Since cellular automata have discrete structures, which allow exact computation, and show considerable richness of behavior, they can be used to model chaotic phenomena. Today cellular automata have become common mathematical models of dynamics of discrete variables in discrete space and time, with applications in physics, chemistry, population dynamics and parallel computing.

General cellular automata can simulate universal structure, yet their long term behavior can be very hard to characterize at same time. By contrast, a special class of cellular automata, linear cellular automata, shows additional structures and permits a much more detailed theoretical analysis. These special linear cellular automata and their properties will be discussed in this chapter.

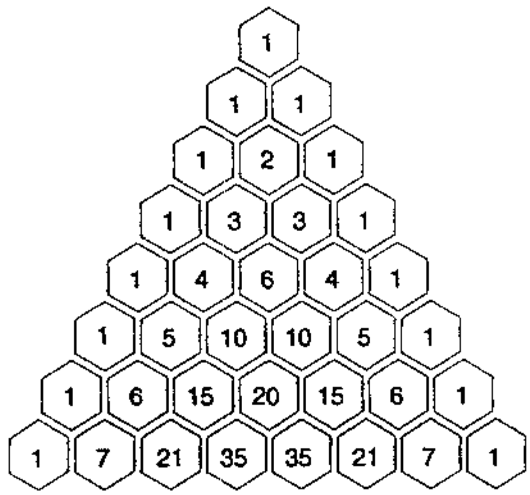

FIGURE 6.1 PASCAL'S TRIANGLE.

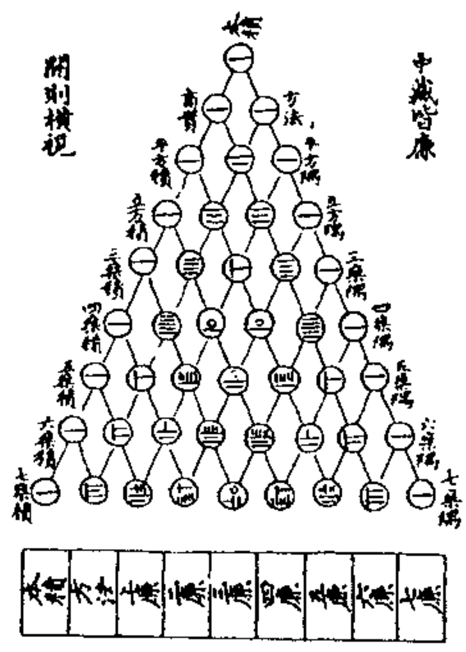

FIGURE 6.2 A CHINESE ARITHMETRIC TRIANGLE.

6.1 INTRODUCTION AND EXAMPLE

Cellular automata can be generally described in terms of two concepts: configuration, and transition rule, defined as follows:

(1) A p-state configuration is a pattern in which each cell of an n-dimensional lattice contains one of the integers: $0, 1, \cdots, p-1$. We use a symbol ω to denote a configuration. Thus, $\omega : \mathbb{Z}^n \to \{0, 1, \cdots, p-1\}$.

(2) A transition rule is a map F which transfers a configuration ω to a new configuration $F(\omega)$.

Cellular automata come in one, two, or many dimensions. The following is an example of one-dimensional cellular automata.

A one-dimensional cellular automaton consists of a row of cells, each containing an initial number, and the transition rule specifying how these numbers change at each time unit. Assuming in the initial state of the automation, all cells are filled with 0's except a single one with a number 1, such as:

$$\cdots 010000000 \cdots .$$

The transition rule F is that the number in each cell is to be replaced by the sum of itself and its left neighbor. Therefore, after one time unit (meaning one application of F), the state of the automation will become as the following:

$$\cdots 011000000 \cdots .$$

Another time unit later, the state will be:

$$\cdots 012100000 \ldots ,$$

and followed by:

$$\cdots 013310000 \cdots ,$$

$$\cdots 014641000 \ldots ,$$

and so on.

In this example, the cellular automaton is in fact a computer which calculates the coefficients of the powers of binomials, such as:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

6.2 Definitions of Linear Cellular Automata (LCA) and Notations

Linear cellular automata (LCA) have been studied by Wolfram [Wo2,Wo3], Willson [Wi1] and Haeseler et al. [HPS1]. Willson has given the definitions and terminology for LCA in case of $p = 2$. We will expand it to the general cases where p is any positive integer greater than or equal to 2.

1. Definition of LCA.

Let $p \geq 2$ be an integer. By $\mathbb{Z}_p[x]$, we denote the ring of polynomials with coefficients in the field $\mathbb{Z}_p$.

In this chapter, we only deal with two-dimensional p-state configurations, *i.e.* each cell can be occupied only by a number in $\mathbb{Z}_p$. A convenient way to describe a two-dimensional configuration ω is given as the following:

A configuration ω is written as a Laurent series in 2 variables, where the first variable s corresponds to space and the second t to time, respectively. The Laurent series expression for ω contains one term $a_{ij}s^it^j$ for each cell (i,j) occupied by a positive integer a_{ij}. The configuration ω can be written then as a sum

$$\sum_{a_{ij}>0} a_{ij}s^it^j. \tag{6.1}$$

For example, the configuration ω with $a_{i,j} = 0$ except $a_{-1,1} = 2$, $a_{0,0} = 1$, and $a_{2,-1} = 5$ can be written as:

$$\omega = 2s^{-1}t^1 + s^0t^0 + 5s^2t^{-1} = 2s^{-1}t^1 + 1 + 5s^2t^{-1}.$$

Suppose a polynomial $r(s) \in \mathbb{Z}_p[s]$ is given, one can define an additive transition rule L by giving the Laurent series of $L(\omega)$:

$$L(\omega) = r(s)\omega \mod p, \tag{6.2}$$

where the multiplication is performed in the usual manner except coefficients are obtained with modulo p.

For example, if $\omega = 1 + 2s^2$ and $r(s) = 2 + s + s^3 \in \mathbb{Z}_3[s]$, then

$$\begin{aligned} L(\omega) &= (1 + 2s^2)(2 + s + s^3) \\ &= 2 + s + s^2 + 2s^5 \mod 3. \end{aligned}$$

We call L additive because it satisfies

$$L(\omega + \tau) = L(\omega) + L(\tau) \mod p,$$

although modulo p prevents it from having ordinary linearity.

The graph construction F induced by r and L shows the evolution of $L^k(\omega)$ in space-time. The process consists of placing a copy of $L(\omega)$ above a copy of ω, a copy of $L^2(\omega)$ above of $L(\omega)$, etc. We can write F as:

$$F(\omega) = (1 + tr(s))\omega. \tag{6.3}$$

Thus, if ω is a one-dimensional configuration in variable s, its global construction will be:

$$\begin{aligned} F(\omega) &= \omega + tL(\omega) \\ F^2(\omega) &= \omega + tL(\omega) + t^2 L^2(\omega) \\ &\vdots \end{aligned}$$

$$F^k(\omega) = \omega + tL(\omega) + t^2L^2(\omega) + \cdots + t^kL^k(\omega)$$
$$\vdots \tag{6.4}$$

One can see that $F^k(\omega)$ demonstrates the evolution $\omega, L(\omega), \ldots, L^k(\omega)$ with respect to time k. The coefficient of t^k is $L^k(\omega)$, which permits us to study the pattern at time k. Figure 6.3 shows a graph construction F induced by $r(s) = 1 + s + s^2$ mod 2 with initial configuration $\omega = 1$, where $k = 13$.

0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0
0	1	1	0	1	0	1	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	1	1	0	0	0	0
0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	0
0	1	1	1	0	0	0	0	0	1	1	1	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	1	1	0	1	1	0	1	1	1	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	0	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 6.3 A Graph Construction F Induced by $r(s) = 1 + s + s^2$ mod 2.

In conclusion, it is clear that, given a polynomial $r(s) \in \mathbb{Z}_p[s]$, one can induce a cellular automaton using (6.2) and (6.3). We call this cellular automaton a linear cellular automaton (LCA) and denote it by $F(r)$.

2. m-Blocks and Induced p×p Matrices.

Here we intend to establish the concepts of m-blocks and induced $p \times p$ matrices which will allow us to relate a LCA with a MW-graph.

From now on, we assume $p \geq 2$ and p is a prime number. Let $F(r)$ be a LCA induced by $r(s) \in \mathbb{Z}_p[s]$, and m be the degree of $r(s)$. A m-block is a sequence $x_{m-1}, \cdots, x_0$ of length m, where $x_i \in \{0, 1, \cdots, p-1\}$. We denote by b_0 the block of zeros, *i.e.* $b_0 = 0, \cdots, 0$. A m-block is nontrivial if it is not the block of zeros.

There are $p^m - 1$ nontrivial m-blocks, and we denote them by $b_1, b_2, \cdots, b_{p^m-1}$. The i-th block b_i is the m-tuple, which is the p-adic expansion of i with initial zeroes to make it into a sequence of length m. Some nontrivial m-blocks are listed here:

$$b_1 = 0, \cdots, 0, 0, 1$$

$$b_{p-1} = 0, \cdots, 0, 0, p-1$$

$$b_p = 0, \cdots, 0, 1, 0$$

$$b_{p^2} = 0, \cdots, 1, 0, 0$$

$$b_{p^m-1} = p-1, p-1, \cdots, p-1, p-1, p-1.$$

In order to define the induced $p \times p$ matrix, we need to study the properties of coefficients for some specific configurations.

Definition. *For a series $\sum_i a_{i,n}s^i$, we call the finite string*

$$a_{i,n},\ a_{i+1,n},\ \cdots,\ a_{i+m,n}$$

a part (or a portion) of the coefficients of $\sum_i a_{i,n}s^i$, where $i \in \mathbb{Z}$ and $m \in \mathbb{N}$ are arbitrary.

Moreover, for a finite string

$$a_{i,n},\ a_{i+1,n},\ \cdots,\ a_{i+m,n},$$

we say it forms a part (or a portion) of the coefficients of a series $\sum_i b_{i,n}s^i$ if

$$(b_{i,n},\ b_{i+1,n},\ \cdots,\ b_{i+m,n}) = (a_{i,n},\ a_{i+1,n},\ \cdots,\ a_{i+m,n}).$$

Suppose ω is an one-dimensional configuration with Laurent series $\sum \omega_i s^i$ and $r(s) = \sum_{i=0}^m a_i s^i$. By (6.2), we have

$$L(\omega) = r(s)\omega(s) = \sum \tau_i s^i, \tag{6.5}$$

where the i-th coefficient of $L(\omega)$ is given by

$$\tau_i = a_0\omega_i + a_1\omega_{i-1} + a_2\omega_{i-2} + \cdots + a_m\omega_{i-m} \mod p. \tag{6.6}$$

Note from (6.5) and (6.6) that the entry in $L(\omega)$ at the site i depends on the coefficients of ω at the site i and at the m sites to the left of i.

Moreover, if we start with an initial (or "seed") configuration $\omega \equiv 1$, using (6.2), we have

$$L(\omega) = r(s), \quad L^2(\omega) = r^2(s), \quad \cdots, \quad L^n(\omega) = r^n(s).$$

Therefore, in order to study the pattern of the LCA, we should consider the orbit $\{r^n(s)\}_{n\in\mathbb{N}}$ under the iteration of r.

We write $r^n(s) = \sum_{l=0}^{\infty} a_{l,n}s^l$, where $a_{l,n} = 0$ for $l > nm$. Using Fermat's Theorem $r(s)^p = r(s^p) \mod p$ (cf.[HW]), we have $r^{pn}(s) = \sum_{l=0}^{\infty} a_{l,n}s^{pl}$. Hence the coefficients of $r^{pn}(s)$ are defined by the coefficients of $r^n(s)$ as follows:

For each $l \geq 0$,

$$a_{pl,pn} = a_{l,n} \quad \text{and} \quad a_{pl+i,pn} = 0 \quad \text{for} \quad i = 1, \cdots, p-1. \tag{6.7}$$

Similarly, if $r^{n+1}(s) = \sum_{l=0}^{\infty} a_{l,n+1}s^l$, then the coefficients of $r^{p(n+1)}(s)$ are defined in the same manner, and we have the following scheme:

$$0\cdots 0a_{p(l-1),p(n+1)}0\cdots 0a_{pl,p(n+1)}0\cdots 0a_{p(l+1),p(n+1)}\ 0\cdots 0 \tag{6.8}$$

$$0\cdots 0a_{p(l-1),pn} \quad 0\cdots 0a_{pl,pn} \quad 0\cdots 0a_{p(l+1),pn} \quad 0\cdots 0, \tag{6.9}$$

where (6.8) and (6.9) are portions of the coefficients of $r^n(s)$ and $r^{n+1}(s)$, respectively.

What remains is to determine the coefficients of $r^{pn+j}(s)$ for $j \in \{1, 2, \cdots, p-1\}$. In view of (6.5) and (6.6), since $r^{pn+j}(s) = r(s)r^{pn+j-1}(s)$, for each pair of positive integers, the coefficient $a_{pl+i,pn+j}$ is determined by $a_{pl+i,pn+(j-1)}$ and the coefficients at the m sites to the left of $pl+i$ as indicated in the following:

$$a_{pl+i-m,pn+(j-1)},\ a_{pl+i-m+1,pn+(j-1)},\cdots,a_{pl+i-1,pn+(j-1)},\ a_{pl+i,pn+(j-1)}.$$

Claim. *All coefficients $a_{pl+i,pn+j}$ where $i,j\in\{0,\cdots,p-1\}$ together with their $m-1$ left neighbors shown in the following scheme,*

$$\begin{array}{ccccc} a_{pl-(m-1),pn+(p-1)} & \cdots & a_{pl,pn+(p-1)} & \cdots & a_{pl+(p-1),pn+(p-1)} \\ a_{pl-(m-1),pn+(p-2)} & \cdots & a_{pl,pn+(p-2)} & \cdots & a_{pl+(p-1),pn+(p-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{pl-(m-1),pn} & \cdots & a_{pl,pn} & \cdots & a_{pl+(p-1),pn} \end{array} \tag{6.10}$$

are determined by the m-block

$$a_{l-m+1,n},\ a_{l-m+2,n},\cdots,a_{l,n}. \tag{6.11}$$

Proof. Since (6.11) is a part of the coefficients of $r^n(s)$, then the following

$$0\ \cdots\ 0\ a_{p(l-m+1),pn}\ 0\ \cdots\ 0\ a_{p(l-m+2),pn}\ 0\ \cdots\ 0\ \cdots\ a_{p(l-1),pn}\ 0\ \cdots\ 0,\ a_{pl,pn}\ 0,\ \cdots,\ 0$$

forms a part of the coefficients for r^{pn}, where

$$a_{p(l-m+i),pn}=a_{l-m+i,n}\quad\text{for}\quad i=1,\cdots,m\quad\text{using (6.9)}.$$

Using (6.5) and (6.6), we obtain part of coefficients for $r^{pn+1}(s)$, as indicated in the following:

$$a_{p(l-m+1)-(p-1)-m,pn+1}\ \cdots\ a_{p(l-1),pn+1}\ \cdots\ a_{pl,pn+1}\ \cdots\ a_{pl+(p-1),pn+1}.$$

Repeating the process, we get a part of coefficients for $r^{pn+2}(s)$. Continuing this procedure, we get a part of coefficients for $r^{pn+(p-1)}(s)$ as follows

$$a_{pl-(m-1),pn+(p-1)}\ \cdots\ a_{pl,pn+(p-1)}\ \cdots\ a_{pl+(p-1),pn+(p-1)}.$$

Thus all coefficients are obtained. □

In terms of polynomials, our observation can be formulated in the following way: For the m-block $b = a_{l-m+1,n}, a_{l-m+2,n}, \cdots, a_{l,n}$, we can consider the corresponding polynomial

$$\tau_b(s) = a_{l-m+1,n} + a_{l-m+2,n}s + \cdots + a_{l,n}s^{m-1}.$$

Then the coefficient $a_{pl+i,pn+j}$ is given by:

$$\left[\tau_b(s^p)r(s)^j\right]_{p(m-1)+i},$$

which is the $\big(p(m-1)+i\big)$-th coefficient of the polynomial $\tau_b(s^p)r(s)^j$.

We can now associate each nontrivial m-block $b_i = x_{m-1}, \cdots, x_0$ with a $p \times p$ matrix:

$$\sigma(b_i) = \left(\omega_{\alpha\beta}\right)_{\alpha,\beta \in \{0,1,\cdots,p-1\}} \tag{6.12}$$

where $\omega_{\alpha\beta}$ is a m-block defined by:

$$\omega_{\alpha\beta} = y_{m-1}, \cdots, y_0 \quad \text{where} \quad y_l = \left[\tau_{b_i}(s^p)r(s)^\beta\right]_{p(m-1)+\alpha-l}. \tag{6.13}$$

Remark. *The set of matrices $\{\sigma(b_i) : i = 1, \cdots, p^m - 1\}$ is regarded as matrix substitution system induced by r (cf. [HPS1]).*

6.3 Linear Cellular Automata and MW-Graphs

As we mentioned before, many people have studied the evolution of LCA since 1980s. Wilson [Wi2] showed that LCA can be generated by fractal sets; Haeseler *et al.* [HPS1] associated LCA with matrix substitution systems. Their research works provide effective tools to explore many features of the pattern formation of LCA.

In this section however, we will provide a different approach which associates each LCA with a MW-graph directed system to study its evolution.

First of all, we will give geometric representations for LCA and introduce the notion of rescaled evolution sets of LCA. Then we will associate the rescaled evolution sets with MW-graph directed fractal sets. Finally we will calculate the Hausdorff dimensions and measures for the rescaled evolution sets of LCA.

1. Geometric Representations and Rescaling Procedures.

We denote by $\big(\mathcal{C}(\mathbb{R}^2), d_H\big)$ the space of nonempty compact subsets of $\mathbb{R}^2$ equipped with the Hausdorff metric.

Let $\omega = \sum_{a_{ij}>0} a_{ij} s^i t^j$ be a configuration as defined in (6.1). We associate ω with a subset A of $\mathbb{R}^2$ such that

$$A = \bigcup_{a_{ij}>0} I_{ij} \quad \text{where} \quad I_{ij} = [0,1]^2 + (i,j).$$

We call the set A a geometric representation of ω.

Let $F(r)$ be a LCA induced by $r \in \mathbb{Z}_p[s]$, and ω be a one dimensional configuration in variable s. In view of (6.2), (6.3) and (6.4), we have

$$\begin{aligned}
F^0(\omega) &= \omega \\
F^1(\omega) &= \omega + tL(\omega) \\
F^2(\omega) &= \omega + tL(\omega) + t^2L^2(\omega) \\
&\vdots \\
F^{n-1}(\omega) &= \omega + tL(\omega) + t^2L^2(\omega) + \cdots + t^{n-1}L^{n-1}(\omega) \\
&\vdots
\end{aligned}$$

Therefore, for each n we associate $F^{n-1}(\omega)$ with its geometric representation $Y_n = \cup_{j=0}^{n-1} \cup_{a_{ij}>0} I_{ij}$. As we monitor the evolution for $n = 1, 2, \cdots$, we see a pattern developing:

$$Y_1 = \bigcup_{a_{i,0}>0} I_{i,0}$$

$$Y_2 = \bigcup_{j=0}^{1} \bigcup_{a_{ij}>0} I_{ij}$$

$$\vdots$$

$$Y_n = \bigcup_{j=0}^{n-1} \bigcup_{a_{ij}>0} I_{ij}$$

$$\vdots$$

As $n \to \infty$, we denote the infinite evolution pattern by

$$Y = \bigcup_{n=1}^{\infty} Y_n = \bigcup_{j=0}^{\infty} \bigcup_{a_{ij}>0} I_{ij}.$$

In addition, a limit set Z can be associated with Y by introducing sequences of rescaled finite parts of Y as follows:

$$Z(n) = \frac{1}{n} Y_n = \bigcup_{j=0}^{n-1} \bigcup_{a_{ij}>0} \frac{1}{n} I_{ij},$$

for any n and any one dimensional configuration ω.

The following theorem is proved by Willson [Wi2].

Theorem 6.3.1. *Suppose $F(r)$ is a LCA induced by $r \in \mathbb{Z}_p[x]$, and ω is any one dimensional configuration in variable s with a finite number of positive terms. Then the sequence $\{Z(p^k)\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}(\mathbb{R}^2, d_H)$. Its limit is independent of ω and is denote by Z:*

$$Z = \lim_{k \to \infty} Z(p^k). \tag{6.14}$$

We call the set Z in (6.14) the rescaled evolution set induced by $F(r)$.

2. The Association of the Rescaled Evolution Set Z with MW-Graph Directed Fractal Sets.

Let $F(r)$ be a LCA induced by $r \in \mathbb{Z}_p[s]$ with degree of m. We define a m-block b to be an accessible m-block, if (i) $b \neq b_0$, and (ii) b is a portion of the coefficients of $r^j(s)$ for some $j \geq 0$.

We then induce a MW-graph as follows:

Let V be the vertex set of all accessible m-blocks. For each $u \in V$, we define a complete metric space $X_u = I = [0,1]^2$. Recall that each m-block b_i is associated with a $p \times p$ matrix $\sigma(b_i)$ as in (6.12) and (6.13). Using the matrix $\sigma(b_i)$, we can define the set of edges of V.

For each pair $(u, v) \in V \times V$, the set of edges from u to v is:

$$E_{uv} = \{(\alpha, \beta) : \ \sigma(u)_{\alpha,\beta} = v\} \quad \text{where} \quad \sigma(u)_{\alpha,\beta} \text{ is the } \alpha\beta\text{-th entry of } \sigma(u).$$

Furthermore, $E = \cup_{uv \in V} E_{uv}$ is the set of edges associate with the vertex set V.

For each $e \in E_{uv}$, $e = (\alpha, \beta)$, define a similarity map $f_e^{uv} : \ X_v \to X_u$ such that

$$f_e^{uv}(x, y) = \left(\frac{x + \alpha}{p}, \frac{y + \beta}{p} \right).$$

Clearly f_e^{uv} has similarity ratio $\delta_e = \frac{1}{p}$ for each $e \in E$.

Hence $G = \left((V, E), (X_u)_{u \in V}, (f_e^{uv})_{e \in E}, r_e = \frac{1}{p} \right)$ forms a MW-graph and satisfies the OSC, since $\text{int} f_e^{uv}(X_v) \cap \text{int} f_{e'}^{uv'}(X_{v'}) = \emptyset$, for $e \neq e'$.

Definition. *The system $(f_e^{uv})_{e \in E}$ corresponding to the MW-graph*

$$G = \left((V, E), (X_u)_{u \in V}, (f_e^{uv})_{e \in E}, r_e = \frac{1}{p} \right)$$

is called a p-adic hierarchical iterated function system.

As indicated in Theorem 1 of [MW1], there exists a unique invariant list $(A_u)_{u\in V}$ of nonempty compact sets $A_u \subset X_u$ such that:

$$A_u = \bigcup_{v\in V} \bigcup_{e\in E_{uv}} f_e^{uv}(A_v).$$

If we denote by $\mathcal{C}(X_u)$ the set of all compact subsets of X_u equipped with the Hausdorff metric d_H, and let $\mathcal{K} = \left(\prod_{u\in V} \mathcal{C}(X_u),\ d_\infty\right)$, where d_∞ is the maximum metric, defined by

$$d_\infty(B, C) = \max_{u\in V} d_H(B_u, C_u) \quad \text{where} \quad B = (B_u)_{u\in V} \ \ C = (C_{u\in V}),$$

also define $\Phi : \ \mathcal{K} \ \to \ \mathcal{K}$ such that

$$\Phi(B)_u = \bigcup_{v\in V} \bigcup_{e\in E_{uv}} f_e^{uv}(B_v).$$

Then Φ is a contraction on $\left(\prod_{u\in V} \mathcal{C}(X_u),\ d_\infty\right)$ and its fixed point is the invariant list $(A_u)_{u\in V}$.

The following theorem is obtained by combining Theorem 4.4 and Proposition 3.2 in [HPS1].

Theorem 6.3.2. *Let $F(r)$ be a LCA induced by r of degree m, and Z be the rescaled evolution set of the LCA. Also let $(A_u)_{u\in V}$ be the invariant list of the MW-graph induced by $F(r)$, then*

$$Z = \bigcup_{j=0}^{m-1} \left(A_{e_j} + (j, 0)\right), \qquad \text{where } e_j = (0 \cdots 0 \overset{j\text{-th}}{1} 0 \cdots 0) \in V.$$

3. Dimensions and Measures of Rescaled Evolution Sets.

Let $F(r)$ be a LCA induced by r of degree m and Z be the rescaled evolution set of the LCA. In view of Theorem 6.3.2, in order to find the Hausdorff dimension of

Z, we only need to find the Hausdorff dimension of the corresponding MW-graph directed fractal sets $(A_u)_{u\in V}$.

Let $G = ((V, E), (X_u)_{u\in V}, (f_e^{uv})_{e\in E}, r_e = \frac{1}{p})$ be the MW-graph induced by $F(r)$. We call the matrix

$$B = (t_{uv})_{u,v\in V}, \quad \text{where} \quad t_{uv} = \#\{e : e \in E_{uv}\}$$

the accessible transition matrix of G. Since p is a prime number, it follows from Corollary 2 of Theorem 4 in [HPS2], that G is strongly connected, which is equivalent to the accessible transition matrix B being irreducible (cf. [BR]).

Using Theorem 3 of [MW1], the Hausdorff dimension α of A_u satisfies $\Phi(\alpha) = 1$, where $\Phi(\alpha)$ is the spectral radius of the construction matrix B_α (see definition 2.1.3),

$$B_\alpha = \left(\sum_{e\in E_{uv}} (\frac{1}{p})^\alpha \right)_{u,v\in V} = \frac{1}{p^\alpha} B. \tag{6.5}$$

Moreover, if λ is the maximum eigenvalue of B, then $\frac{1}{p^\alpha}\lambda$ is the maximum eigenvalue of B_α, and $\frac{1}{p^\alpha}\lambda = \Phi(\alpha)$.

For $\Phi(\alpha) = 1$, we get

$$\alpha = \frac{\log \lambda}{\log p}.$$

Hence the Hausdorff dimension of A_u is

$$\dim_H(A_u) = \frac{\log \lambda}{\log p}.$$

By applying Theorem 6.3.2, we obtain:

$$\dim_H(Z) = \frac{\log \lambda}{\log p}.$$

Also from Theorem 3 of [MW1], we have

$$0 < \mathcal{H}^\alpha(A_u) < \infty \quad \text{for all} \quad u \in V.$$

This implies

$$0 < \mathcal{H}^{\alpha}(Z) < \infty.$$

Finally, we will provide an example to calculate the Hausdorff dimension and measure of a LCA using the techniques discussed in this chapter.

Let $F(r)$ be a LCA induced by $r(s) = 1 + s + s^2 \in \mathbb{Z}_2[s]$ with degree of 2, and Z be the rescaled evolution set. In order to find the associated MW-graph G, we have to find the corresponding vertex set V and the set of edges E induced by $F(r)$. Note that there are four 2-blocks

$$b_0 = 00 \quad b_1 = 01 \quad b_2 = 10 \quad b_3 = 11,$$

and all nontrivial 2-blocks are accessible. Therefore we have the vertex set $V = (b_1,\ b_2,\ b_3)$. Using (6.12) and (6.13), we get

$$\sigma(b_1) = \begin{pmatrix} b_1 & b_3 \\ b_1 & b_2 \end{pmatrix}$$

$$\sigma(b_2) = \begin{pmatrix} b_3 & b_2 \\ b_0 & b_0 \end{pmatrix}$$

$$\sigma(b_3) = \begin{pmatrix} b_2 & b_1 \\ b_1 & b_2 \end{pmatrix}.$$

Hence the sets of edges are as follows:

$$\begin{array}{lll} E_{11} = \{(0,0),(0,1)\} & E_{12} = \{(1,0)\} & E_{13} = \{(1,1)\} \\ E_{21} = \emptyset & E_{22} = \{(1,1)\} & E_{23} = \{(0,1)\} \\ E_{31} = \{(0,0),(1,1)\} & E_{32} = \{(1,0),(0,1)\} & E_{33} = \emptyset. \end{array}$$

The Hierarchical iterated function system $(f_e^{uv})_e$ is then as follows:

$$
\begin{aligned}
f^{11}_{(0,0)} &: X_1 \to X_1, \quad (x,y) \to (\frac{x}{2}, \frac{y}{2}) \\
f^{11}_{(0,1)} &: X_1 \to X_1, \quad (x,y) \to (\frac{x}{2}, \frac{y+1}{2}) \\
f^{12}_{(1,0)} &: X_2 \to X_1, \quad (x,y) \to (\frac{x+1}{2}, \frac{y}{2}) \\
f^{13}_{(1,1)} &: X_3 \to X_1, \quad (x,y) \to (\frac{x+1}{2}, \frac{y+1}{2}) \\
f^{22}_{(1,1)} &: X_2 \to X_2, \quad (x,y) \to (\frac{x+1}{2}, \frac{y+1}{2}) \\
f^{23}_{(0,1)} &: X_3 \to X_2, \quad (x,y) \to (\frac{x}{2}, \frac{y+1}{2}) \\
f^{31}_{(0,0)} &: X_1 \to X_3, \quad (x,y) \to (\frac{x}{2}, \frac{y}{2}) \\
f^{31}_{(1,1)} &: X_1 \to X_3, \quad (x,y) \to (\frac{x+1}{2}, \frac{y+1}{2}) \\
f^{32}_{(1,0)} &: X_3 \to X_3, \quad (x,y) \to (\frac{x+1}{2}, \frac{y}{2}) \\
f^{32}_{(0,1)} &: X_2 \to X_3, \quad (x,y) \to (\frac{x}{2}, \frac{y+1}{2}).
\end{aligned}
$$

As the result, we obtain the accessible transition matrix

$$
B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix},
$$

whose maximum eigenvalue is $\lambda = 1 + \sqrt{5}$.

Hence the Hausdorff dimension α of Z is

$$
\alpha = \frac{\log \lambda}{\log p} = \frac{\log(1+\sqrt{5})}{\log 2}.
$$

Moreover, since matrix B is irreducible, we have

$$
0 < \mathcal{H}^{\alpha}(Z) < \infty.
$$

CHAPTER VII

QUESTIONS

The following is a list of some of the questions that have arisen in my study.

(1) For a conformal iterated function system $\mathcal{S}$, are the OSC and the SOSC equivalent to each other?

(2) For an n-ary random self-similar construction $\mathcal{J}$, suppose $\sum_{i=1}^{n} r_i^s = 1$ μ-a.s.

(i) Is then the δ condition a necessary condition for $\mathcal{H}^s(K(\omega)) > 0$ P-a.s. ?

(ii) If $\mathcal{H}^s(K(\omega)) = 0$ P-a.s., does there exist a Hausdorff gauge function of the form $h(t) = t^s L(t)$, where $L(t)$ is a slowly varying function such that

$$0 < \mathcal{H}^{h(t)}(K(\omega)) < \infty \quad P\text{-a.s. ?}$$

(3) Can we associate a fractal set generated by a M-state linear cellular automaton with a graph directed construction, where $M \geq 2$ is an arbitrary positive integer?

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