SUFFICIENT CONDITIONS FOR UNIQUENESS OF POSITIVE SOLUTIONS AND NON EXISTENCE OF SIGN CHANGING SOLUTIONS FOR ELLIPTIC DIRICHLET PROBLEMS

DISSERTATION

Presented to the Graduate Council of the University of North Texas in Partial Fulfillment of the Requirements For the Degree of

DOCTOR OF PHILOSOPHY

By

Mehran Hassanpour, B.S., M.A.

Denton, Texas

August, 1995
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In this paper we study the uniqueness of positive solutions as well as the non existence of sign changing solutions for Dirichlet problems of the form

$$\Delta u + g(\lambda, u) = 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $\Delta$ is the Laplace operator, $\Omega$ is a region in $\mathbb{R}^N$, and $\lambda > 0$ is a real parameter. For the particular function $g(\lambda, u) = |u|^p u + \lambda$, where $p = \frac{4}{N-2}$, and $\Omega$ is the unit ball in $\mathbb{R}^N$ for $N \geq 3$, we show that there are no sign changing solutions for small $\lambda$ and also we show that there are no large sign changing solutions for $\lambda$ in a compact set. We also prove uniqueness of positive solutions for $\lambda$ large when $g(\lambda, u) = \lambda f(u)$, where $f$ is an increasing, sublinear, concave function with $f(0) < 0$, and the exterior boundary of $\Omega$ is convex. In establishing our results we use a number of methods from non-linear functional analysis such as rescaling arguments, methods of order, estimation near the boundary, and moving plane arguments.
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CHAPTER 1

INTRODUCTION

Boundary value problems arise naturally in mathematical physics. Their study involves investigation of topics such as existence, uniqueness, bifurcation, and approximation of solutions. A vast array of methods such as spectral theory of linear operators, contractions, fixed point theorems, and variational techniques have proven useful. Here we give results concerning the problem of non-existence of sign changing solutions and uniqueness of positive solutions for semilinear elliptic problems. In proving these results, we use rescaling arguments, methods of order, estimation near the boundary, and moving plane arguments.

We consider boundary value problems of the form

\[ \begin{cases} 
\Delta u + g(\lambda, u) = 0 & \text{in } \Omega, \\
\mathbf{1} = 0 & \text{on } \partial \Omega,
\end{cases} \quad (1.1) \]

where \( \Delta \) is the Laplace operator, \( g \) is a given function of \( \lambda \) and \( u \), and \( \Omega \) is a region in \( \mathbb{R}^N \).

For our first problem we consider (1.1), when \( g(\lambda, u) = |u|^p u + \lambda, p = \frac{4}{N-2}, N \geq 3 \), (so that \( g \) has critical growth), \( \Omega \) is the unit ball in \( \mathbb{R}^N \), and \( \lambda > 0 \) is a parameter. Under these conditions we consider radially symmetric solutions to (1.1). Therefore we study the solutions of the ordinary differential equation equivalent to (1.1),

\[ -u'' - \frac{(N-1)}{r} u' = |u|^p u + \lambda \quad r \in (0, 1), \quad (1.2) \]
where (') denotes differentiation with respect to $r$.

Our main result for problem (1.1) under these conditions is the following theorem.

**Theorem 1.1** Let $N \geq 3$, $\Omega$ the unit ball in $\mathbb{R}^N$, $\lambda > 0$ and $g(\lambda, u) = |u|^pu + \lambda$, $p = \frac{4}{N-2}$. Then there exists a constant $\lambda^*$ (depending on $N$) such that

1. there are no sign changing solutions for (1.2) – (1.4), when $\lambda \leq \lambda^*$, and

2. if $\lambda \in [\lambda_1, \lambda_2]$, for $\lambda_1, \lambda_2 > 0$, then there exists $d^* > 0$, such that if $u$ satisfies (1.1) – (1.4) then $\|u\|_\infty \leq d^*$.

In Chapter 4 we consider the problem (1.1) assuming that the exterior boundary of $\Omega$ is convex, $\lambda > 0$, $g(\lambda, u) = \lambda f(u)$, where $f : [0, \infty) \to \mathbb{R}$ is a smooth, increasing, concave function satisfying

$$\lim_{u \to \infty} f(u) = \infty,$$  
(1.5)

$$f(0) < 0,$$  
(1.6)

and

$$\lim_{u \to \infty} \frac{f(u)}{u} = 0.$$  
(1.7)

Our main result for equation (1.1) under these assumptions is the following theorem.
Theorem 1.2 Let $N \geq 2$, $\Omega$ smooth bounded region in $\mathbb{R}^N$ with convex exterior boundary $S$, $\lambda > 0$, $g(\lambda, u) = \lambda f(u)$, and $f$ defined as above. Then there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$, the equation (1.1) has exactly one nonnegative solution. In addition this solution is positive in $\Omega$.

Problems such as (1.1) arise in a variety of situations, in the theory of nonlinear diffusion generated by nonlinear sources, in the theory of thermal ignition of gases (see D.D. Joseph and T.S. Lundgren [16], I.M. Gelfand [12]), in quantum field theory and statistical mechanics (see W. Strauss [21], Coleman, Glazer and A. Martin [10], H. Berestycki and P.L. Lions [3]), and in the theory of gravitational equilibrium of stars (see D.D. Joseph and T.S. Lundgren [16], P.L. Lions [18]). A most striking feature of such problems is that positive solutions of (1.1) need not be unique.

For example Joseph and Lundgren [16] showed that the problem (1.1) with $g(\lambda, u) = \lambda e^u$ has the following uniqueness properties:

There exists a finite positive value $\lambda_*$ (depending on $N$) such that (1.1) has

1. no solution when $\lambda > \lambda_*$ ($N \geq 1$),
2. one solution when $\lambda = \lambda_*$ ($N \geq 1$),
3. two solutions when $0 < \lambda < \lambda_*$ ($N = 1, 2$),
4. an infinite number of solutions when $\lambda = 2$ ($N = 3$),
5. a finite but large number of solutions when $|\lambda - 2| \neq 0$ is small ($N = 3$),
6. an infinite number of solutions when $|\lambda - 2(N - 2)| \neq 0$ is small ($N < 10$) and

7. one solution for each $\lambda < 2(N - 2)$ ($N \geq 10$).

In a recent paper [2], I. Ali and A. Castro established uniqueness for large positive solutions of (1.1), where they considered problem (1.1) with $g(\lambda, u) = |u|^pu + \lambda$ with $p = \frac{4}{N-2}$, and $\lambda \in \mathbb{R}$. We now state their result which will be used later in proving Theorem 1.1.

**Theorem 1.3** (I. Ali and A. Castro) There exists a continuous function $F : (0, \infty) \rightarrow (0, \infty)$ such that $u$ is a positive solution to (1.1) if and only if $\lambda = F(u(0))$. If $u_1$ and $u_2$ are positive solutions to (1.1) with $u_1(0) = u_2(0)$ then $u_1 \equiv u_2$. Moreover $\lim_{d \to 0} F(d) = 0$, $\lim_{d \to \infty} F(d) = 0$, and there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$ then (1.1) has exactly two solutions. Moreover, the set $\{(\lambda, u) \in (0, \lambda_0) \times (0, \infty)\}$, where $u$ satisfies (1.1), is connected.

To prove Theorem 1.1 we will estimate integrals involving solutions of (1.1). Using these estimates and analyzing the energy for these solutions together with some identities of Pohazaev type [19], we obtain the proof of Theorem 1.1.

Results like Theorem 1.2 have been proven (see [7] and [17]) when $f(0) \geq 0$ (the positone case). In this case one can assume, without loss of generality, that $f(u) \geq 0$ for every $u \geq 0$. Thus, by the maximum principle every nonnegative solution is strictly positive in the interior. Further, in the case $f(0) > 0$, using the subsolution $w = 0$, a priori estimates near the boundary for positive solutions are easily obtained. In this
paper we obtain a priori growth estimates for positive solutions to (1.1) by analyzing the integral of the solutions on surfaces parallel to the exterior boundary of \( \Omega \) namely \( S \). In particular we show that near \( S \) nonnegative solutions grow as fast as \( \lambda d(x, \partial \Omega) \), where \( d(x, \partial \Omega) \) is the distance from \( x \) to the boundary of \( \Omega \). Hence we also prove that nonnegative solutions are also positive.

A main ingredient in obtaining the estimates is the use of moving planes device as developed in [13] (see also [8] and [11]).

For the particular case in which \( \Omega \) is a ball, Theorem 1.2 was proven in [4]. The methods used in [4] depend heavily on one-dimensional arguments, which do not extend to arbitrary smooth regions with convex outer boundary. In [5] it was shown that even when \( \Omega \) is a ball there are values of \( \lambda \) for which (1.1) has more than one solution. We do not know if for each smooth bounded region there are values of \( \lambda \) for which (1.1) has more than one positive solution.

For existence results on positive solutions for large \( \lambda > 0 \) see [6]. Semipositone problems occur naturally in applications such as population models with constant harvesting effort. For a survey on semipositone problems see [9].
This chapter contains some preliminary results on the solution to initial value problems for (1.1) and the maximal principle. First we establish an identity which is a type of *Pohozaev Identity* (see [19]). Given \( d \in \mathbb{R} \), and \( \lambda \in \mathbb{R} \), let \( u(r, \lambda, d) \equiv u(r) \) be a solution of (1.2) – (1.3), where \( u(0) = d \). Define

\[
E(r, \lambda, d) = \frac{(u'(r, \lambda, d))^2}{2} + \frac{(u(r, \lambda, d))^{p+2}}{p+2} + \lambda u(r, \lambda, d) \equiv E(r).
\]  

(2.1)

**Lemma 2.1** If \( 0 < r^* < r \), then

\[
r^{N-1}H(r) - (r^*)^{N-1}H(r^*) = \frac{N + 2}{2} \int_{r^*}^r \lambda s^{N-1}u(s)ds,
\]  

(2.2)

where

\[
H(r) = rE(r) + \frac{(N - 2)}{2} u(r, \lambda, d)u'(r, \lambda, d).
\]

**Proof.** Multiplying (1.2) by \( r^N u'(r) \) and integrating over \([r^*, r]\), we obtain

\[
r^N E(r) = \int_{r^*}^r s^{N-1} \left\{ N \left( \frac{u^{p+2}(s)}{p+2} + \lambda u \right) - \left( \frac{N - 2}{2} \right) (u'(s))^2 \right\} ds
\]

\[
+ \ (r^*)^N E(r^*).
\]  

(2.3)

Similarly, multiplying (1.2) by \( r^{N-1}u(r) \) and integrating over \([r^*, r]\) we obtain

\[
\int_{r^*}^r s^{N-1}(u'(s))^2ds = u'(r)u(r)r^{N-1} - u'(r^*)u(r^*)(r^*)^{N-1}
\]

\[
+ \int_{r^*}^r s^{N-1}(u^{p+2}(s) + \lambda u)ds.
\]  

(2.4)
Substituting (2.4) in (2.3), we obtain (2.2), which completes the proof.

Taking \( r^* = 0 \) in (2.2) we get

\[
\frac{N + 2}{2} \int_0^r \lambda s^{N-1} u(s) ds = \frac{r^N (u'(r))^2}{2} + \frac{r^N u^{p+2}(r)}{p + 2} + r^N \lambda u(r) + \frac{N - 2}{2} r^{N-1} u'(r) u(r).
\] (2.5)

Now for a solution \( u(r) \) of (1.1), we define the function

\[
h(r) = -\frac{ru'(r)}{u(r)}, \quad r \in [0, r_0),
\] (2.6)

where \( r_0 \) is the first zero of the function \( u \). It is clear that \( h \) is continuous, and \( h(0) = 0 \). Since \( u(r_0) = 0 \), we see that \( \lim_{r \to r_0^-} h(r) = \infty \). Furthermore, \( h \) is an increasing function. Indeed

\[
h'(r) = \frac{-u'(r)u(r) - ru(r)u''(r) + r(u'(r))^2}{(u(r))^2}.
\] (2.7)

Replacing (1.2) in (2.7) gives

\[
h'(r) = \frac{(N - 2)u'(r)u(r) + ru^{p+2}(r) + r\lambda u(r) + r(u'(r))^2}{(u(r))^2}.
\] (2.8)

Combining (2.5) and (2.8) we infer

\[
h'(r) = \frac{\lambda(N + 2) r^{1-N} \int_0^r s^{N-1} u(s) ds - r\lambda u(r) + (1 - \frac{2}{p+2}) ru^{p+2}(r)}{(u(r))^2}.
\] (2.9)

Since \( u \) is a decreasing function on \((0, r_0)\), it follows from (2.9) that

\[
h'(r) \geq \frac{(1 - \frac{2}{p+2}) ru^{p+2}(r) + \frac{2}{N} \lambda r u(r)}{(u(r))^2}
\]

\[
= (1 - \frac{2}{p+2}) ru^p(r) + \frac{2}{N} \lambda r[u(r)]^{-1}
\]

\[
\geq \frac{2}{N} ru^p(r) > 0.
\] (2.10)
Lemma 2.2 If $u$ is a solution to (1.1), then there exists $M_0 > 0$ and a unique $r^* \in (0, r_0)$ such that $u(r^*) = M_0 (r^*)^{-\frac{2}{p}}$. Moreover, if $0 < M < M_0$ then there exist exactly two numbers $r_1, r_2 \in (0, r_0)$ such that $u(r_i) = M r_i^{-\frac{2}{p}}$, for $i = 1, 2$.

Proof. Let $r^* \in [0, r_0]$ be such that $M_0 = \max \{u(r) r_0^2 : r \in [0, r_0] \} = u(r^*) r_0^2$. Thus, the graph of $u$ is tangent to the graph of $M_0 r^{-\frac{2}{p}}$ at $r^*$, and $u(r) \leq M_0 r^{-\frac{2}{p}}$ for all $r \in [0, r_0]$. Next for $M < M_0$ we show that the graph of $u$ intersects the graph of $M r^{-\frac{2}{p}}$ at exactly two points. Suppose $0 < r_1 < r_2 < r_3 < r_0$ are the first three numbers such that $u(r_i) = M r_i^{-\frac{2}{p}}$, for $i = 1, 2, 3$. Since $u$ is decreasing on $(0, r_0)$, $u(r_1) < u(r_2) < u(r_3)$. Let $Z(r) = M r^{-\frac{2}{p}}$, then we have $Z(r_2) = u(r_2), Z'(r_2) > u'(r_2), Z(r_3) = u(r_3)$, and $Z'(r_3) < u'(r_3)$. Hence,

$$h(r_3) = \frac{-r_3 u'(r_3)}{u(r_3)} < \frac{-r_3 Z'(r_3)}{Z(r_3)} = \frac{2}{p},$$

and

$$h(r_2) = \frac{-r_2 u'(r_2)}{u(r_2)} > \frac{-r_2 Z'(r_2)}{Z(r_2)} = \frac{2}{p}.$$

But then $h(r_3) < h(r_2)$ with $r_2 < r_3$, which contradicts that $h$ is an increasing function on $(0, r_0)$. Assuming that $u(r^*) r^{\frac{2}{p}} = u(r) r^{\frac{2}{p}} = M_0$ for some $r \in [0, r^*]$, we see that $h(r^*) = h(r) = \frac{2}{p}$ which once again contradicts that $h$ is an increasing function. Hence $r^*$ is unique.

From (2.5) and the quadratic formula we obtain

$$r u'(r) = -\frac{N - 2}{2} u(r) \pm \frac{1}{2} A(r),$$

(2.11)
\[ [A(r)]^2 = (N - 2)^2 u^2(r) - \frac{8}{p + 2} r^2 u^{p+2}(r) - 8r^2 \lambda u(r) \]  
\[ + 4(N + 2) \lambda \frac{1}{r^{N-2}} \int_0^r s^{N-1} u(s) ds. \]  

From (2.11) we have

\[ \frac{2}{N - 2} h(r) = \pm \frac{1}{N - 2} \frac{A(r)}{u(r)}. \]  

Since \( h(0) = 0 \) and \( \lim_{r \to r_0^-} h(r) = \infty \) we see from (2.13) that for \( r \) near zero

\[ \frac{2}{N - 2} h(r) = 1 - \frac{1}{N - 2} \frac{A(r)}{u(r)}, \]  

and for \( r \) near \( r_0 \)

\[ \frac{2}{N - 2} h(r) = 1 + \frac{1}{N - 2} \frac{A(r)}{u(r)}. \]

The fact that \( h \) is an increasing function together with (2.14), and (2.15) implies that there exists a unique \( r \) such that \( \frac{2}{N - 2} h(r) = 1 \), that is, \( A(r) = 0 \). Since \( A(r) = 0 \) implies \( h(r) = \frac{2}{p} \) and \( r^* \) is the only element in \([0, r_0]\) for which \( h(r) = \frac{2}{p} \) we see that \( r^* = r \). Using that \( u(r) = M_0 r^{\frac{2}{p}} \) and integrating (2.10) on \([0, r^*]\) we obtain

\[ M_0 \leq \left( \frac{N(N - 2)}{2} \right)^{1/p} . \]  

Thus, from (2.16) and the definition of \( M_0 \) we get

\[ u(r) \leq \left( \frac{N(N - 2)}{2} \right)^{1/p} r^{-\frac{2}{p}} . \]
Computing $E'(r)$ and using the equation (1.2) we have

$$E'(r) = u'(r)[u''(r) + u^{p+1}(r) + \lambda]$$

$$= -\left(\frac{N - 1}{r}\right)(u'(r))^2 \leq 0.$$  

Therefore $E$ is a decreasing function of $r$ on $[0,1]$.

The remaining results in this chapter will be used in chapter 4, where we prove Theorem 1.2. First we discuss the method of order. Consider the boundary value problem

$$\begin{cases}
Lu + f(x,u) = 0 \quad \text{in } D, \\
Bu = g \quad \text{on } \partial D,
\end{cases}$$

where

$$Lu = \sum_{i,j=1}^{N} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i},$$

is a second order uniformly elliptic operator and $B = a(\frac{\partial u}{\partial v}) + bu = g$.

An upper solution to (2.19) is a function $\phi$ satisfying

$$L\phi + f(x,\phi) \leq 0 \quad \text{in } D,$$

$$B\phi \geq g \quad \text{on } \partial D.$$

A lower solution to (2.19) is a function $\psi$ satisfying

$$L\psi + f(x,\psi) \geq 0 \quad \text{in } D,$$

$$B\psi \leq g \quad \text{on } \partial D.$$
In what follows, we assume that $\partial D$, $f$, $g$, and the coefficients of the operator $L$, are smooth.

**Theorem 2.3** Let $\phi$ be an upper solution and $\psi$ a lower solution, with $\psi \leq \phi$ on $D$. Then there exists a solution $u$ to the boundary value problem (2.19) with $\psi \leq u \leq \phi$.

**Proof.** See [20].

Next we state some notations and results concerning maximum principle and the so called "moving plane arguments". Proofs can be found in [8]. The following notations follow that of [13]. Let

$$T_t := \{ x \in \mathbb{R}^N : x \cdot e = t \},$$

$$\Sigma_t := \{ x \in \Omega : x \cdot e > t \},$$

where $e = (1, 0, \ldots, 0)$ and $(\cdot)$ denotes the usual inner product in $\mathbb{R}^N$. Let $\Sigma'_t$ be the reflection of $\Sigma_t$ across $T_t$. For $x \in \mathbb{R}^N$ we will denote by $x^t$ the reflection of $x$ across $T_t$. Also $u_t$ will denote the partial derivative of $u$ with respect to $x_i$, similarly $u_{ij}$.

Further, for $x \in \partial\Omega$ we will denote by $\nu(x) = (\nu_1(x), \ldots, \nu_N(x))$ the outward unit normal at $x$.

**Lemma 2.4** Let $f$ satisfy (1.6) and $u : \overline{\Omega} \to \mathbb{R}$ be a nonnegative solution to (1.1). If for some $t \in (0, 1)$ we have $\Sigma'_t \subset \Omega$,

$$u(x) \leq u(x^t) \quad \forall x \in \Sigma_t,$$

$$u(x) \leq u(x^t) \quad \forall x \in \Sigma_t,$$
and

\[ u(\alpha) < u(\alpha^t) \quad \text{for some } \alpha \in \Sigma_t, \]

then

\[ u(x) < u(x^t) \quad \forall x \in \Sigma_t, \]

and

\[ u_1(x) < 0 \quad \forall x \in T_t \cap \Omega. \]

**Proof.** See proof of Lemma 2.2 in [13] and observe that one requires that only \( u \geq 0. \]

**Lemma 2.5** Let \( f \) satisfy (1.6), \( u : \overline{\Omega} \to \mathbb{R} \) be a solution to (1.1) and \( z \in \partial\Omega. \) If \( v_1(z) > 0 \) and there exists \( \varepsilon > 0 \) such that \( u(x) \geq 0 \) for \( x \in \overline{\Omega} \) with \( \|x - z\| < \varepsilon, \) then there exists \( \varepsilon_1 \in (0, \varepsilon) \) such that \( u_1(x) < 0 \) for \( x \in \Omega \) with \( \|x - z\| < \varepsilon_1. \)

**Proof.** (See [8]).

Next we state the Hopf's Maximum Principle in the form:

**Lemma 2.6** Suppose \( L \) is an elliptic operator, \( u \leq 0 \) is a \( C^2 \) solution of the differential inequality, \( Lu \geq 0. \) Further, suppose there is a ball \( B \) in \( \Omega \) with a point \( P \in \partial\Omega \) on its boundary and suppose \( u \) is continuous in \( \Omega \cup P \) and \( u(P) = 0. \) Then if \( u \neq 0 \) in \( B \) we have for an outward directional derivative at \( P, \)

\[ \frac{\partial u}{\partial v}(P) > 0, \]
in the sense that if $Q$ approaches $P$ in $B$ along a radius then

$$\lim_{Q \to P} \frac{u(P) - u(Q)}{|P - Q|} > 0.$$

Proof. (See [13]).
CHAPTER 3

NON EXISTENCE OF SIGN CHANGING SOLUTIONS

In this chapter we prove Theorem 1.1, by showing that under the assumptions of Theorem 1.1, \( \int_0^r s^{N-1}u(s)ds \leq 0 \) for some \( r \in [0,1] \). This on the other hand contradicts the assumption that a solution \( u \) of (1.1) has a second zero in \([0,1]\), since \( E'(r) \leq 0 \) for \( r \in [0,1] \). Thus showing that there can not be any sign changing solution for the equation (1.1). By (2.17) we have

\[
\int_0^r s^{N-1}u(s)ds \leq \left( \frac{N(N-2)}{2} \right)^{\frac{1}{p}} \int_0^r s^{\frac{N-2}{p}}ds \leq \frac{2}{N+2} \left( \frac{N(N-2)}{2} \right)^{\frac{N-2}{4}} r_0^{\frac{N+2}{2}}.
\]

Let \( r_1 \in (r_0,1) \) be such that \( u'(r_1) = 0 \), and \( u' < 0 \) on \([r_0,r_1]\). Since \( E' \leq 0 \) we have \( E(r_1) \leq E(r_0) \), which implies that

\[
\frac{|u'|^{p+2}(r_1)}{p+2} + \lambda u(r_1) \leq \frac{(u'(r_0))^2}{2}.
\]

By (2.5) and (3.1), for \( r = r_0 \) we obtain

\[
|u'(r_0)|^2 = (N+2)\lambda r_0^{-N} \int_0^{r_0} s^{N-1}u(s)ds \leq 2 \left( \frac{N(N-2)}{2} \right)^{\frac{N-2}{4}} \lambda r_0^{\frac{N-2}{2}}.
\]

On the other hand since \( E' \leq 0 \), and \( u \) has a second zero we have

\[
|u(r_1)| \geq \left[ \left( \frac{2N}{N-2} \right) \lambda \right]^\frac{N-2}{N+2}.
\]
Let \( r \geq r_1 \) be such that \( u \leq \frac{1}{2} u(r_1) \) on \([r_1, r]\). Thus multiplying (1.2) by \( r^{N-1} \) and integrating over \([r_1, t] \), for \( t \leq r \) we obtain

\[
\frac{\partial u}{\partial t} \leq t^{1-N} \int_{r_1}^t s^{N-1} ||u||^{p+1}(r_1) - \lambda] ds. \tag{3.5}
\]

Integrating (3.5) and using the fact that \( u(r) = \frac{1}{2} u(r_1) \) we infer

\[
\frac{1}{2} |u(r_1)| \leq ||u||^{p+1}(r_1) - \lambda] \int_{r_1}^t t^{1-N} \left( \frac{t^N}{N} - \frac{r_1^N}{N} \right) dt \tag{3.6}
\]

\[
\leq \frac{||u||^{p+1}(r_1)}{2N} (r^2 - r_1^2).
\]

Using (3.6) we obtain

\[
r^2 \geq N||u||^{-p}(r_1) + r_1^2. \tag{3.7}
\]

**Lemma 3.1** If \( u \) is any solution of (1.1) and \( r_1, r \) are defined as above, then

\[
\int_{r_1}^r s^{N-1} u(s) ds \leq -\frac{N^{N-2}}{2} |u(r_1)|^{-\left(\frac{N+2}{N-2}\right)}. \tag{3.8}
\]

**Proof.** Since \( u(r) = \frac{1}{2} u(r_1) \), by (3.7) we have

\[
\int_{r_1}^r s^{N-1} u(s) ds \leq \frac{u(r_1)}{2N} (r^N - r_1^N) \leq \frac{u(r_1)}{2N} [N r^{N/2} - r_1^{N/2}] \tag{3.9}
\]

\[
\leq \frac{u(r_1)}{2N} [(N||u(r_1)||^{-p} + r_1^{N/2}) - r_1^{N/2}] \leq \frac{u(r_1)}{2N} (N||u(r_1)||^{-p})^{N/2} \leq -\frac{N^{N-2}}{2} |u(r_1)|^{1-\frac{pN}{2}}
\]

\[
\leq -\frac{N^{N-2}}{2} |u(r_1)|^{-\left(\frac{N+2}{N-2}\right)}.
\]

This completes the proof. \(\blacksquare\)
Suppose \( r_2 \in [0, 1] \) is the second zero of \( u \). Using (2.2), and the fact that \( E(r_2) > 0 \) we obtain

\[
\left(-\frac{N+2}{2}\right) \lambda \int_{r_1}^{r_2} s^{N-1} u(s) \, ds = r_2^N E(r_2) - r_1^N E(r_1) \geq -r_1^N E(r_1). \tag{3.9}
\]

By (3.8) and (3.9) we have

\[
\begin{align*}
    r_1^N E(r_1) &\geq \left(-\frac{N+2}{2}\right) \lambda \int_{r_1}^{r_2} s^{N-1} |u(s)| \, ds \\
&\geq \left(-\frac{N+2}{2}\right) \lambda \int_{r_1}^{r_2} s^{N-1} |u(s)| \, ds \\
&\geq \left(-\frac{N+2}{2}\right) \lambda N^{\frac{N-2}{2}} \left| \frac{2}{N\lambda} \right|^{-\left(\frac{N+2}{2}\right)}.
\end{align*}
\tag{3.10}
\]

Using the definition of \( E \) together with (3.10), and the fact that \( u(r_1) < 0 \) we get

\[
\begin{align*}
    r_1^N \left| \frac{u(r_1)}{p+2} \right| &\geq \left(-\frac{N+2}{2}\right) N^{\frac{N-2}{2}} \lambda |u(r_1)|^{-\left(\frac{N+2}{2}\right)}.
\end{align*}
\tag{3.11}
\]

Since \( p + 1 = \frac{N+2}{N-2} \), and \( \frac{2N}{N-2}, \frac{N+2}{2} \) are both greater than one for \( N \geq 3 \), using (3.11) we infer

\[
\left| u(r_1) \right|^{p+2} \geq \frac{N^{\frac{N-2}{2}}}{2} - \lambda r_1^{-N} |u(r_1)|^{-\left(\frac{N+2}{2}\right)}.
\tag{3.12}
\]

But (3.12) further implies that

\[
\left| u(r_1) \right|^{2p+3} \geq \frac{N^{\frac{N-2}{2}}}{2} - \lambda r_1^{-N}.
\tag{3.13}
\]

Since \( 2p + 3 = \frac{2N+3}{N-2} \), by (3.13) we obtain

\[
\begin{align*}
\left| u(r_1) \right| &\geq 2^{-\left(\frac{2N+3}{2}\right)} N^{\frac{2N+3}{2}} \lambda^{\frac{N-3}{2}} r_1^{-N} \left(\frac{N-2}{2}\right) \\
&\geq 2^{-\left(\frac{2N+3}{2}\right)} N^{\frac{2N+3}{2}} \lambda^{\frac{N-3}{2}} r_1^{\frac{2N-3}{2}}.
\end{align*}
\tag{3.14}
\]
Lemma 3.2 If \( u \) is a solution of (1.1), \( \lambda \leq 1 \), and \( r_0, r_1 \) are as above. Then there exists a \( K := K(N) \) such that

\[
|u(r_1)| \leq K \lambda^{ \frac{(2N+1)(N-2)}{2(N+2)}} |r_0| \lambda^{ \frac{(N-2)^2}{2(N+2)}}.
\]

Proof. Since \( E' \leq 0 \) we have

\[
\frac{|u(r_1)|^{p+2}}{p+2} + \lambda u(r_1) \leq \frac{(u'(r_0))^2}{2}. \tag{3.15}
\]

From (3.15) we get

\[
\frac{|u(r_1)|^{p+2}}{p+2} \leq \frac{(u'(r_0))^2}{2} - \lambda u(r_1) \leq \frac{(u'(r_0))^2}{2} + \lambda |u(r_1)|. \tag{3.16}
\]

Multiplying both side of (3.16) by \( (p + 2) \) and then dividing by \( |u(r_1)| \) we obtain

\[
|u(r_1)|^{p+1} \leq \frac{p + 2}{2} \frac{(u'(r_0))^2}{|u(r_1)|} + (p + 2) \lambda \tag{3.17}
\]

By (3.3), (3.14), and (3.17) we infer

\[
|u(r_1)|^{p+1} \leq \left( \frac{N}{N-2} \right) \left( \frac{N(N-2)}{2} \right)^{\frac{N-2}{4}} r_0^{\frac{2-2N}{4}} \lambda^{\frac{2N}{4}} r_0^{\frac{2N+4}{4}} \lambda^{\frac{2N+4}{4}} \tag{3.18}
\]

\[
+ \frac{2N}{N-2} \lambda
\]

\[
\leq 2^{\frac{N-2}{2}} N^{-\frac{(2N+2)}{4}} (N-2)^{\frac{N-2}{4}} \lambda^{\frac{2N+4}{4}} r_0^{\frac{2-2N}{4}}
\]

\[
+ \frac{2N}{N-2} \lambda.
\]
The fact that \( \frac{2N+4}{3N+2} < 1 \) for \( N \geq 3 \) implies that for \( \lambda < 1, \lambda < \frac{2N+4}{3N+2} \). On the other hand since \( r_0 < 1 \) we have, \( \lambda \frac{2N+4}{3N+2} r_0 \frac{2-N}{2} \). Therefore by (3.18) we obtain

\[
|u(r_1)|^{p+1} \leq \frac{2N}{N-2} \lambda \frac{2N+4}{3N+2} r_0 \frac{2-N}{2}
\]

Now since \( p + 1 = \frac{N+2}{N-2} \), by (3.19) we have

\[
|u(r_1)| \leq K(N) \lambda \left[ \frac{(2N+4)(N-2)}{4} r_0 \right]^{\frac{N-2}{4}}
\]

where

\[
K(N) = \left[ \frac{2N}{N-2} - \frac{(2N+4)(N-2)}{4} \right]^{\frac{N-2}{4}}
\]

and this completes the proof of the lemma. \( \square \)

Next we state and prove a lemma which will also be needed in proving Theorem 1.1.

**Lemma 3.3** If \( u \) is a solution of (1.1), \( r_0, r_1, \) and \( r \) are as in Lemma 3.2. Then there exists \( K_1 := K_1(N) \) such that if \( \lambda \leq K_1 \)

\[
\int_0^r s^{N-1} u(s) ds \leq 0.
\]

**Proof.** By (3.1), (3.8), and (3.20) we infer

\[
\int_0^r s^{N-1} u(s) ds = \int_0^{r_1} s^{N-1} u(s) ds + \int_{r_1}^r s^{N-1} u(s) ds \quad (3.21)
\]
\[ \int_0^{r_0} s^{N-1}u(s)\,ds + \int_{r_1}^r s^{N-1}u(s)\,ds \leq \frac{2}{N+2} \left( \frac{N(N-2)}{2} \right)^\frac{N-2}{2} r_0^\frac{N+2}{2} \]

\[ - \frac{N^\frac{N-2}{2}}{2} |u(r_1)|^{-\left(\frac{N+3}{2}\right)} \]

\[ \leq \frac{2}{N+2} \left( \frac{N(N-2)}{2} \right)^\frac{N-2}{2} r_0^\frac{N+2}{2} \]

\[ - \frac{N^\frac{N-2}{2}}{2} [K(N)]^{-\frac{N+3}{2}} \lambda^{-\left(\frac{2N+4}{3}\right)} r_0^\frac{N-2}{2}. \]

Now to show that \( \int_0^r s^{N-1}u(s)\,ds \leq 0 \), from (3.21) it suffices to show that

\[ \frac{2}{N+2} \left( \frac{N(N-2)}{2} \right)^\frac{N-2}{2} r_0^\frac{N+2}{2} \leq \frac{N^\frac{N-2}{2}}{2} [K(N)]^{-\frac{N+3}{2}} \lambda^{-\left(\frac{2N+4}{3}\right)} r_0^\frac{N-2}{2}. \] (3.22)

Now (3.22) is satisfied if

\[ \frac{2}{N+2} \left( \frac{N(N-2)}{2} \right)^\frac{N-2}{2} r_0^2 \leq \left[ \frac{(N-2) N^\frac{N-2}{2}}{2 \left[ \frac{1}{3}(N-2) \right] N^{-\left(\frac{5N+4}{4}\right)}(N-2) N^{-\left(\frac{5N+4}{4}\right)}} \right] \lambda^{-\left(\frac{2N+4}{3}\right)}. \] (3.23)

On the other hand for (3.23) to hold

\[ r_0^2 \leq \frac{N+2}{2} \left( \frac{2}{N(N-2)} \right)^\frac{N-2}{2} \times \left[ \frac{(N-2) N^\frac{N-2}{2}}{2 \left[ \frac{1}{3}(N-2) \right] N^{-\left(\frac{5N+4}{4}\right)}(N-2) N^{-\left(\frac{5N+4}{4}\right)}} \right] \lambda^{-\left(\frac{2N+4}{3}\right)}. \] (3.24)

Or equivalently \( r_0 \) must satisfy

\[ r_0 \leq \left[ \frac{(N+2)(N-2) N^\frac{N-2}{2}}{2 \left[ \frac{1}{3}(N-2) \right] N^{-\left(\frac{5N+4}{4}\right)}(N-2) N^{-\left(\frac{5N+4}{4}\right)} + 8N} \right]^{1/2} \times \frac{2^\frac{N-2}{6}}{N(N-2)} \lambda^{-\left(\frac{2N+4}{3}\right)\frac{1}{N}}. \] (3.25)

\[ \leq K'(N) \lambda^{-\frac{N+2}{N+2}}. \]
where
\[
K'(N) = \left[ \frac{(N+2)(N-2)2^{\frac{n-1}{2}}N^{\frac{n-2}{2}}}{\left[ 2^{\frac{13(n-10)}{2(n-2)}}N^{-(n+1)}(N-2)^{\frac{n-2}{2}} + 8N^{\frac{n+1}{2}}(N-2)^{\frac{n-2}{2}} \right]} \right]^{1/2}.
\]

Now since \( r_0 < 1 \), (3.25) is satisfied if
\[
1 \leq K'(N)\lambda^{-\left(\frac{n+2}{2}N^2\right)}.
\]

But (3.26) holds if
\[
\lambda^{\frac{n+2}{2N^2}} \geq \frac{1}{K'(N)},
\]
or equivalently if
\[
\lambda \leq \left[ K'(N) \right]^{-\frac{3N^2+2}{2N^2}}.
\]

Thus taking
\[
K_1(N) = \left[ K'(N) \right]^{-\frac{3N^2+2}{2N^2}}
\]
completes the proof of the lemma.\[\blacksquare\]

Proof of part (1) of Theorem 1.1:

In Lemma 3.3 we established that, under the assumption that \( u \) has a second zero in \([0,1]\) namely \( r_2, \int_0^r s^{N-1}u(s)ds \leq 0 \) where \( r \in [r_1, r_2] \) is such that \( u(s) \leq \frac{1}{2}u(r) \) for \( s \in [r_1, r] \). On the other hand since \( u \leq 0 \) on \([r, r_2]\), \( \int_{r_2}^r s^{N-1}u(s)ds \leq \int_{r_2}^r s^{N-1}u(s)ds \leq 0 \). Now from (2.5) we have
\[
\frac{N+2}{2} \lambda \int_0^{r_2} s^{N-1}u(s)ds = \frac{r_2^N(u'(r_2))^2}{2}.
\]

By (3.27) we obtain
\[
\int_0^{r_2} s^{N-1}u(s)ds = \frac{1}{N+2} \lambda r_2^N(u'(r_2))^2.
\]
Hence we have a contradiction since the right hand side of (3.28) is greater than or equal to zero. This completes the proof of part (1) of Theorem 1.1.

To prove part (2) of the Theorem 1.1 we proceed by defining the function \( v \) on \([0,1]\) via

\[
v(r) = \left( \frac{1}{\beta^{2+\frac{2}{p}}} \right) u\left( \frac{r}{\beta}, \lambda, d \right),
\]

\[v(0) = \left( \frac{d}{\beta^{2+\frac{2}{p}}} \right),\]

\[v'(0) = 0,
\]

where \( u \) is a solution of (1.2) — (1.4), and \( \beta > 0 \). Thus

\[
v'' + \left( \frac{N-1}{r} \right) v' + v^{p+1} = 
\]

\[
= \left( \frac{1}{\beta^{2+\frac{2}{p}}} \right) u''(\frac{r}{\beta}) + \left( \frac{1}{\beta^{1+\frac{2}{p}}} \right) \left( \frac{N-1}{r} \right) u'(\frac{r}{\beta}) + 
\]

\[
= \left( \frac{1}{\beta^{2+\frac{2}{p}}} \right) u''(\frac{r}{\beta}) + \left( \frac{1}{\beta^{2+\frac{2}{p}}} \right) \left( \frac{N-1}{\beta} \right) u'(\frac{r}{\beta}) + 
\]

\[
+ \left( \frac{1}{\beta^{2+\frac{2}{p}}} \right) u^{p+1}(\frac{r}{\beta}) = 
\]

\[
= \left( \frac{1}{\beta^{2+\frac{2}{p}}} \right) (-\lambda)
\]

which implies that \( v(r) = u\left( \frac{r}{\beta^{2+\frac{2}{p}}}, \lambda, d \right), \) Thus for all \( \beta > 0 \) we obtain

\[
u\left( \frac{r}{\beta}, \lambda, d \right) = \beta^{\frac{2}{p}} u\left( \frac{r}{\beta^{2+\frac{2}{p}}}, \lambda, d \right),
\]

(3.29)

Next we will prove the following lemma, which estimates the first zero, \( r_0 \), of a solution \( u \) to (1.2) — (1.4).

**Lemma 3.4** If \( r_0 \) is the first zero of a solution \( u \) to (1.2) — (1.4), then \( r_0 \leq O(\lambda^{\frac{2}{N+2}}) \).
Proof. Multiplying both side of equation (1.2) by $r^{N-1}$ and integrating on $[0, r_0]$, we obtain

$$u'(r_0) = -r_0^{1-N} \int_0^{r_0} s^{N-1} |u|^{p+1} + \lambda] ds.$$  \hspace{1cm} (3.30)

On the other hand by (2.5) and (3.3) we have

$$u'(r_0) \geq -2\left(\frac{N(N-2)}{2}\right)^{\frac{N-2}{4}} \lambda^\frac{1}{2} r_0^{\frac{2-N}{4}}.$$ \hspace{1cm} (3.31)

By (3.30) and (3.31) we infer

$$2\left(\frac{N(N-2)}{2}\right)^{\frac{N-2}{4}} \lambda^\frac{1}{2} r_0^{\frac{2-N}{4}} \geq r_0^{1-N} \int_0^{r_0} s^{N-1} |u|^{p+1} + \lambda] ds \hspace{1cm} (3.32)$$

$$\geq r_0^{1-N} \int_0^{r_0} s^{N-1} \lambda ds$$

$$\geq \frac{r_0}{N}.\lambda$$

Now (3.32) implies that

$$r_0 \leq 2^{\frac{N-2}{N+2}} N(N-2)^{\frac{N-2}{4}} \lambda^{-\frac{1}{N+2}}.$$ \hspace{1cm} \text{(3.33)}

This completes the proof of the lemma.

In order to complete the proof of part (2) of Theorem 1.1 we will proceed by proving the next three Lemmas, in which first we establish an estimate for the value of $u$ at the first zero of $u'$, that is at $r_1$, then using this estimate we will obtain a necessary condition for $\int_0^{r_1} s^{N-1}u(s)ds \leq 0$, and finally we establish a relation between $u(0) = d$ and the first zero of $u$ namely $r_0$. 


Lemma 3.5 If \( u \) is a solution of (1.2) – (1.4), \( \lambda > 1 \), and \( r_0, r_1 \) are as above. Then there exists a \( K_2 := K_2(N) \) such that

\[
|u(r_1)| \leq K_2 \lambda^{\frac{N-2}{N+2}} r_0^{-\frac{(2-N)^2}{4(N+2)}}.
\]

Proof. By (3.17), (3.4), and (3.3) we infer

\[
|u(r_1)|^{p+1} \leq \frac{N}{N-2} \frac{|u'(r_0)|^p}{|u(r_1)|} + \frac{2N}{N-2} \lambda
\]

\[
\leq \left( \frac{N}{N-2} \right)^2 \left( \frac{N(N-2)}{2} \right)^{\frac{N-2}{4}} \lambda r_0^{\frac{2-N}{2}} \left( \frac{2N}{N-2} \right)^{-\frac{N-2}{N+2}} + \frac{2N}{N-2} \lambda,
\]

because \( \left( \frac{2N}{N-2} \right)^{-\frac{N-2}{N+2}} \leq 1 \). The fact that \( \frac{4}{N+2} < 1 \) for \( N \geq 3 \) implies that for \( \lambda < 1 \), \( \lambda^{\frac{4}{N+2}} < \lambda \). On the other hand since \( r_0 < 1 \) we have \( \lambda < \lambda r_0^{\frac{2-N}{2}} \). Thus by (3.33) we obtain

\[
|u(r_1)|^{p+1} \leq N^{\frac{N+2}{4}} \left( \frac{N-2}{2} \right)^{\frac{N-2}{4}} \lambda r_0^{\frac{2-N}{2}} + \frac{2N}{N-2} \lambda r_0^{\frac{2-N}{2}}
\]

\[
\leq \left[ \left( \frac{N-2}{N-2} \right)^{\frac{N+2}{4}} \left( \frac{N-2}{2} \right)^{\frac{N-2}{4}} + 2N \right] \lambda r_0^{\frac{2-N}{2}}.
\]

Since \( p + 1 = \frac{N+2}{N-2} \), (3.34) gives

\[
|u(r_1)| \leq \left[ \left( \frac{N-2}{N-2} \right)^{\frac{N+2}{4}} \left( \frac{N-2}{2} \right)^{\frac{N-2}{4}} + 2N \right]^{\frac{N-2}{N+2}} \lambda^{\frac{N-2}{N+2}} r_0^{-\frac{(2-N)^2}{4(N+2)}}.
\]

Hence taking

\[
K_2(N) = \left[ \left( \frac{N-2}{N-2} \right)^{\frac{N+2}{4}} \left( \frac{N-2}{2} \right)^{\frac{N-2}{4}} + 2N \right]^{\frac{N-2}{N+2}}
\]

completes the proof of the lemma.\( \blacksquare \)
Lemma 3.6 Suppose $u$ is a solution of (1.2)–(1.4), $r_0, r_1, r$ are as in Lemma 3.2 and $\lambda > 1$. Then there exists $K_3 := K_3(N)$ such that $\int_0^r s^{N-1}u(s)ds \leq 0$ if $r_0 \leq K_3\lambda^{-\frac{1}{2}}$.

Proof. By (3.1), (3.8), and Lemma 3.5 we have

$$\int_0^r s^{N-1}u(s)ds = \int_0^{r_0} s^{N-1}u(s)ds + \int_{r_0}^{r_1} s^{N-1}u(s)ds$$

$$\leq \int_0^{r_0} s^{N-1}u(s)ds + \int_{r_0}^{r_1} s^{N-1}u(s)ds$$

$$\leq \frac{2}{N+2}\left(\frac{N(N-2)}{2}\right)^{\frac{N+2}{2}} r_0^{\frac{N+2}{2}} \leq K''(N)\lambda^{-1}r_0^{\frac{N-2}{2}},$$

(3.36)

From (3.35), $\int_0^r s^{N-1}u(s)ds \leq 0$ if

$$\frac{2}{N+2}\left(\frac{N(N-2)}{2}\right)^{\frac{N+2}{2}} r_0^{\frac{N+2}{2}} \leq K''(N)\lambda^{-1}r_0^{\frac{N-2}{2}},$$

where $K''(N) = \frac{N^{\frac{N-2}{2}}}{(N-2)N^{\frac{N+2}{2}}(N-2)N^{\frac{N+2}{2}} + 2N}$. By (3.36), $\int_0^r s^{N-1}u(s)ds \leq 0$ if

$$r_0 \leq \left[\frac{2^{N-10} |N(N-2)|^{\frac{N+2}{4}} (N^2 - 4)}{(N - 2)N^{\frac{N+2}{4}} (N-2)^{\frac{N-6}{4}} + 2N}\right]^{\frac{1}{2}} \lambda^{-\frac{1}{2}}.$$ 

Thus taking

$$K_3(N) = \left[\frac{2^{N-10} |N(N-2)|^{\frac{N+2}{4}} (N^2 - 4)}{(N - 2)N^{\frac{N+2}{4}} (N-2)^{\frac{N-6}{4}} + 2N}\right]^{\frac{1}{2}}$$

completes the proof of the lemma.$\blacksquare$

Next we state and prove a lemma, in which we establish that the first zero of a solution $u$ to (1.2)–(1.4) must tend to zero as $u(0) = d$ tends to infinity.
Lemma 3.7 Suppose \( \lambda \in [\lambda_1, \lambda_2] \), where \( \lambda_1, \lambda_2 > 0 \). If \( r_n \) is the first zero of \( u(\cdot, \lambda, d_n) \) and \( (d_n) \to \infty \) as \( n \to \infty \), then \( (r_n) \to 0 \) as \( n \to \infty \).

Proof. Suppose not. By definition of \( r_n \) we have, \( u(r_n, \lambda, d_n) = 0 \), for \( n \in N \). Let \( r_m = \inf\{r_n; u(r_n, \lambda, d_n) = 0\} \). Thus there exists a \( M > 0 \) such that, \( r_m \geq M > 0 \).

For \( n \) large enough let \( \lambda_n := \lambda_n(d_n) \) be such that \( u(\cdot, \lambda_n, d_n) \) is the unique positive solution of \( (1.2) - (1.4) \). Note that the existence of such a unique positive solution for \( d_n \) large enough is given by the Theorem 1.3. Let \( (\beta_n) \) be a sequence of positive real numbers define by

\[
\lambda_n = \beta_n^{2+\frac{2}{p}} \lambda .
\]  (3.37)

For \( j > n \) let

\[
d_j = \frac{d_n}{\beta_n^{\frac{2}{p}}} .
\]  (3.38)

By (3.29), (3.37), and (3.38) we infer

\[
0 = u(r_j, \lambda, d_j) = u(r_j, \lambda, \frac{d_n}{\beta_n^{\frac{2}{p}}})
\]  (3.39)

\[
= u(r_j, \frac{\lambda_n}{\beta_n^{2+\frac{2}{p}}}) = \beta_n^{-\frac{2}{p}} u(\frac{r_j}{\beta_n}, \lambda_n, d_n) .
\]

Since \( u(\cdot, \lambda_n, d_n) \) is the unique positive solution of \( (1.2) - (1.4) \), (3.39) gives, \( r_j = \beta_n \), but this is a contradiction since \( (\beta_n) \to 0 \) as \( n \to \infty \) and \( r_j > r_m \geq M > 0 \). This completes the proof of the lemma. \( \blacksquare \)

Proof of part (2) of Theorem 1.1:

By part (1) of the Theorem 1.1, we need only to consider \( \lambda \in [\lambda_1, \lambda_2] \), for which
Suppose $\lambda \in [\lambda_1, \lambda_2]$. Suppose $r_0, r$ are as in Lemma 3.6. From Lemma 3.6 we have, $\int_0^r s^{N-1} u(s)ds \leq 0$ if $r_0 \leq K_3\lambda^{-\frac{1}{2}}$, where $K_3$ depends only on $N$. Since $\lambda_1 \leq \lambda \leq \lambda_2$, we have $\lambda_2^{-\frac{1}{2}} \leq \lambda^{-\frac{1}{2}} \leq \lambda_1^{-\frac{1}{2}}$. Therefore if $r_0 \leq K_3\lambda_2^{-\frac{1}{2}}$ we obtain $\int_0^r s^{N-1} u(s)ds \leq 0$. By Lemma 3.7 we have, choosing $u(0) = d$ large enough, $r_0 \leq K_3\lambda_2^{-\frac{1}{2}}$, thus $\int_0^r s^{N-1} u(s)ds \leq 0$. But this contradicts the assumption that $u$ has a second zero in $[0, 1]$. This completes the proof of part (2) of Theorem 1.1.
CHAPTER 4

UNIQUENESS OF NONNEGATIVE SOLUTIONS

We began by analyzing nonnegative solutions of (1.1) near $S$. Since $S$ is smooth, by the $\epsilon$-neighbourhood theorem [15], there exists $\epsilon > 0$ such that

$$\Omega^\epsilon := \{ z \in \bar{\Omega} : d(z, S) \leq \epsilon \}$$

(4.1)

$$= \{ x + s\eta(x) : s \in [0, \epsilon], x \in S \},$$

where $\eta(x)$ is the inward unit normal to the $S$ at the point $x$. By further restricting $\epsilon$ we can assume that if $x \in \Omega^\epsilon$ then $d(x, S) = d(x, \partial \Omega)$.

For the remaining of this chapter whenever we refer to solutions $u$ of (1.1) we consider $g(\lambda, u) = \lambda f(u)$, where $f$ is as in Theorem 1.2.

Using Lemma 2.5, Lemma 2.6, and convexity of $S$, by following the argument of ([11], pg 45), there exists a $\delta > 0$ such that if $z = x + t\eta(x) \in \Omega^\epsilon, x \in S$ and $v = \frac{\eta(x) + w}{\|\eta(x) + w\|}$ with $(w, \eta(x)) = 0, \|w\| < \delta$, then

$$\langle \nabla u(z), v \rangle \geq 0$$

(4.2)

for any non-negative solution $u$ of (1.1). Thus as pointed out in [11], for each $z$ such that $d(z, S) \leq \epsilon/2$ we have

$$u(y) \geq u(z) \quad \text{for} \quad y \in z + T_z,$$

(4.3)
where $T_z$ is a cone congruent to

$$\{ (s_1, \cdots, s_N) : \sum_{i=1}^N s_i^2 \leq (\epsilon/2)^2, \sum_{i=2}^N s_i^2 \leq \delta^2 s_1^2, s_1 \geq 0 \}. $$

For $t \in [0, \epsilon]$ let $\Omega_t = \{ x \in \Omega : d(x, S) > t \}$ and $S_t = \{ x \in \Omega : d(x, S) = t \} \subset \partial \Omega_t$. For $u \geq 0$ solution to (1.1) let $w(t) = \int_{\Omega_t} f(u(x)) \, dx$. Hence

$$w'(t) = -\int_{S_t} f(u(x)) \, da_t \quad (4.4)$$

$$= -\int_{S_t} f(u(x + t\eta(x)))g_1(t, x) \, da,$$

where $da_t$ is the differential of volume in $S_t$, $da$ the differential of volume in $\partial \Omega$, and $g_1(t, x)$ the Jacobian of the change of variables $x \to x + t\eta(x)$ for $x \in S$. Taking one more derivative with respect to $t$ we infer

$$w''(t) = -\int_{\partial \Omega} f'(u(x + t\eta(x))) \frac{\partial u}{\partial \eta}(x + t\eta(x))g_1(t, x) \, da \quad (4.5)$$

$$- \int_{S_t} f(u(x + t\eta(x))) \frac{\partial g_1}{\partial t}(t, x) \, da$$

$$= -\int_{S_t} f'(u(y)) \frac{\partial u}{\partial \eta} \, da_t$$

$$- \int_{S_t} f(u(y))g_2(t, y) \, da_t,$$

where $g_2$ is the product of $\frac{\partial g_1}{\partial t}$ with the Jacobian of the transformation $x + t\eta(x) \to x$.

We note that $g_1$ and $g_2$ are bounded functions independent of $u$.

**Theorem 4.1** For nonnegative solutions $u$ to (1.1) we have

$$\lim_{\lambda \to \infty} (\max \{ u(x) : d(x, S) \leq \epsilon/2 \}) = \infty.$$
Proof. We will proceed by contradiction. Suppose \( \{ (\lambda_n, u_n) \} \) is a sequence of solutions to (1.1) with \( \lambda_n \to \infty, u_n \geq 0 \) in \( \Omega \) such that \( \max \{ u(x) : d(x, S) \leq \epsilon/2 \} \leq M \) for some \( M \in \mathbb{R} \). For the sake of simplicity in the notation we write \( \lambda_n = \lambda \) and \( u_n = u \). Since \( f \) is concave, and by (4.2) \( \partial u / \partial n \geq 0 \) in \( \Omega' \), (4.5) yields

\[
w''(t) \geq f'(0) \int_{S_t} \frac{\partial u}{\partial \eta} d\sigma - m = f'(0) \lambda w(t) - m, \tag{4.6}
\]

where \( m = \sup \{ |m(t)| ; t \in [0, \epsilon/2] \} \), \( m(t) = \int_{S_t} f(u(y)) g_2(t, y) d\sigma \). Similarly

\[
w''(t) \leq -\lambda f'(M) w(t) + m. \tag{4.7}
\]

Let \( d = \sqrt{\lambda f'(M)} \). Multiplying (4.7) by \( \sin(dt) \) and integrating by parts on \([0, t]\), \( t \in [0, \pi/d] \), we have

\[
w'(t) \sin(dt) + d(w(0) - w(t) \cos(dt)) \leq \frac{\pi m}{d}.
\]

In particular, for \( t = \frac{\pi}{d} \) and \( t = \frac{\pi}{2d} \), we infer \( w(0) + w(\frac{\pi}{2d}) \leq \frac{\pi m}{\sqrt{\lambda f'(M)}} \) and \( w'(\frac{\pi}{2d}) \leq \frac{\pi m}{d} \).

The later inequality and the mean value theorem imply that there exists \( t_1 \in [0, \frac{\pi}{2d}] \) such that

\[
w''(t_1) = \frac{2d}{\pi} (w'(\frac{\pi}{2d}) - w'(0)) \leq \frac{2d}{\pi} (\frac{\pi m}{d} + f(0)|S|)
\]

\[
\leq \pi m + \sqrt{\lambda f'(M)} f(0)|S| \sqrt{\lambda} \leq -k_1 \sqrt{\lambda}
\]

for \( \lambda \) large enough and \( k_1 = -\frac{f(0)|S|}{\sqrt{\lambda f'(M)}} \), where we used that \( w'(0) = -f(0)|S| \), and \( |S| \) is the measure of \( S \). This and (4.6) yield

\[
w''(t_1) \geq -\lambda f'(0) \lambda w(t_1) \leq -k_1 \sqrt{\lambda} + m.
\]

Thus

\[
w(t_1) \geq \frac{k_1 \sqrt{\lambda} - m}{\lambda f'(0)} \geq \frac{k_1}{2 \sqrt{\lambda} f'(0)} = k_2 / \sqrt{\lambda}, \tag{4.8}
\]
for $\lambda$ large. Multiplying (4.7) by $\sin(d(t - t_1))$ and integrating on $[t_1, t_1 + \frac{\pi}{d}]$ we have

$$w(t_1 + \frac{\pi}{d}) + w(t_1) \leq \frac{m\pi}{\lambda f'(M)}.$$  

(4.9)

By (4.2) we have $\frac{\partial u}{\partial \eta} \geq 0$ in $\Omega^c$. Integrating (1.1) in $\Omega$, we infer

$$\lambda w(t) = \int_{\Omega_t} -\Delta u = \int_{\partial\Omega_t} \frac{\partial u}{\partial \eta} \geq 0,$$  

(4.10)

for all $t \in [0, \epsilon/2]$. Now taking $\lambda$ large enough, $t_1 + \frac{\pi}{d} \leq \epsilon/2$. By (4.8) and (4.9) we have

$$w(t_1 + \frac{\pi}{d}) \leq \frac{m\pi}{\lambda f'(M)} - \frac{k_2}{\sqrt{\lambda}} \leq 0,$$

for $\lambda$ large enough. This contradicts (4.10), thus completing the proof.

Next we establish some global estimates for nonnegative solutions of (1.1). For this we need a result, which we will state as a lemma. For $z \in \Omega$ with $d(z, S) \leq \epsilon/2$ and $T_z$ as in (4.3), let $\sigma_z := \sigma$ denote the solution to $-\Delta \sigma = \chi_z$ in $\Omega$, $\sigma = 0$ on $\partial \Omega$, where

$$\chi_z = \begin{cases} 1 & \text{if } x \in z + T_z \\ 0 & \text{if } x \notin z + T_z. \end{cases}$$

By Hopf's maximum principle (Lemma 2.6), there exists a positive real number $C := C(z)$ such that if $d(z, S) < \epsilon/2$ then

$$\sigma_z(x) \geq C \varphi(x) \text{ for all } x \in \Omega,$$  

(4.11)

where $\varphi$ is the solution to $-\Delta \varphi = 1$ in $\Omega$ and $\varphi = 0$ on $\partial \Omega$. Next we prove that $C$ in (4.11) can be chosen independent of $z$ if in addition we assume $d(z, S) \geq \epsilon/4$ and $x \in \Omega^c$. 
Lemma 4.2 Suppose $z \in \Omega$ such that $\varepsilon/4 \leq d(z, S) \leq \varepsilon/2$, $x \in \Omega^c$, and $C$ as in (4.11) then $C$ can be chosen independent of $z$.

Proof. Suppose not. There exists a sequence $\{z_n\}$ in $\Omega$ with $\varepsilon/4 \leq d(z_n, S) \leq \varepsilon/2$ for $n = 1, 2, \ldots$ such that for some sequence $\{y_n\}$ in $\Omega^c$,

$$\sigma_{z_n}(y_n) \leq \frac{1}{n} \varphi(y_n).$$

(4.12)

Since $\{z_n\}$ is bounded it has a convergent subsequence which we denote again by $\{z_n\}$, let $z = \lim_{n \to \infty} z_n$. Hence $\varepsilon/4 \leq d(z, S) \leq \varepsilon/2$, and $\chi_{z_n} \to \chi_z$ uniformly. Without loss of generality we can assume $\{y_n\} \to y$, with $y \in \Omega^c$. By (4.12) $\sigma_{z_n}(y_n) \to 0$ which implies that $\sigma_z(y) = 0$, so $y \in S$. Therefore for $n$ large enough $d(y_n, S) < \varepsilon$, let $b(y_n)$ be the projection of $y_n$ onto $S$, by the mean value theorem there exists a sequence $\{s_n\}$ in $\Omega^c$ such that for each $n$,

$$-\frac{\partial \sigma_{z_n}}{\partial \eta}(x_n) = \frac{\sigma_{z_n}(y_n) - \sigma_{z_n}(b(y_n))}{\|y_n - b(y_n)\|} \leq \frac{1}{n} \varphi(y_n) \leq \frac{1}{n} \|y_n - b(y_n)\| \leq K \frac{1}{n},$$

for some constant $K > 0$. Since $x_n \to y$, we have $-\frac{\partial \sigma_{z_n}}{\partial \eta}(x_n) \to -\frac{\partial \sigma_z}{\partial \eta}(y)$, which further implies that $-\frac{\partial \sigma_z}{\partial \eta}(y) = 0$, which contradicts $-\frac{\partial \sigma_z}{\partial \eta}(y) > 0$ (Lemma 2.6). Thus there exists $C$, independent of $x, z$ such that

$$\sigma_z(x) \geq C \varphi(x) \text{ for all } x \in \Omega, \varepsilon/4 \leq d(z, \partial \Omega) \leq \varepsilon/2.$$  

(4.13)

This completes the proof of the lemma. ■
The next theorem establishes a global estimate for nonnegative solutions of (1.1).

**Theorem 4.3** There exists $\lambda > 0$, such that if $u$ is a nonnegative solution of (1.1), and $\lambda > \lambda$ then $u(x) \geq \lambda d(x, \partial \Omega)$ for all $x \in \Omega$.

**Proof.** Let $M > 0$ be such that $P := Cf(M) + f(0) > 0$, where $C$ is as in (4.13). By Theorem 4.1, for $\lambda$ large enough, there exists $z_1 \equiv z \in \Omega^{\epsilon/2}$ such that $u(z) \geq M$.

Since $z = x + t\eta(x)$ for some $x \in S$, $t \in [0, \epsilon/2]$, and $u(x + s\eta(x))$ is a nondecreasing function for $s \in [0, \epsilon/2]$, without loss of generality we can assume that $\epsilon/4 < t \leq \epsilon/2$.

Let $u_1, u_2$ satisfy $\Delta u_1 + \lambda f(M)\chi_x = 0$ in $\Omega$, $u_1 = 0$ on $\partial \Omega$ and $\Delta u_2 - \lambda f(0) = 0$ in $\Omega$, $u_2 = 0$ on $\partial \Omega$. By (4.3), $u(y) \geq M$ for all $y \in z + T_z$. On the other hand since $f(u(x)) \geq f(0)$ on $\Omega - (z + T_z)$, we have

$$-\Delta u \geq \lambda f(M)\chi_x + \lambda f(0)$$

$$= -\Delta(u_1 - u_2).$$

Thus by the maximum principle

$$u \geq u_1 - u_2 \geq C\lambda f(M)\varphi + \lambda f(0)\varphi \equiv \lambda P\varphi \text{ for } x \in \Omega. \quad (4.14)$$

To justify the second inequality in (4.14) consider the following,

$$\Delta(\lambda f(0)\varphi) = \lambda f(0)\Delta \varphi = -\lambda f(0).$$

This implies that $\Delta(\lambda f(0)\varphi + u_2) = 0$. Thus by the maximum principle $\lambda f(0)\varphi = -u_2$.

On the other hand from the definition of $u_1$ we have $-\Delta(\frac{1}{\lambda f(M)}u_1) = \chi_x$ in $\Omega$, and
By Lemma 4.2 we have \( \frac{1}{\lambda f(M)} u_1 \geq C \varphi \), which implies that \( u_1 \geq C \lambda f(M) \varphi \). From the fact that \( u_1 \geq C \lambda f(M) \varphi \) and \( -u_2 = \lambda f(0) \varphi \), we have the second inequality in (4.14). To proceed with the proof of the theorem we state and prove the following claim.

Claim: There exists a constant \( L \) independent of \( \lambda \) such that

\[
\varphi(x) \geq L d(x, \partial \Omega) \quad \text{for all } x \in \Omega.
\]

Proof of the claim: By Lemma 2.6 and compactness of \( \partial \Omega \), there exists \( a_1 > 0 \) such that \( \frac{\partial \varphi}{\partial \eta} \geq a_1 \) on \( \partial \Omega \). By continuity of \( \varphi \), there exists \( \epsilon > 0 \) such that \( \frac{\partial \varphi}{\partial \eta} \geq \frac{a_1}{2} \) on \( \Omega^\epsilon \). Suppose \( y \in \Omega^\epsilon \). By the \( \epsilon \)-neighbourhood theorem [15], \( y = x + t \eta(x) \), for some \( x \in S \) and \( t \in [0, \epsilon] \). By the mean value theorem, there exists \( \xi \in \Omega^\epsilon \) such that

\[
\frac{\partial \varphi}{\partial \eta}(x) = \frac{\varphi(y) - \varphi(x)}{\|y - x\|} = \frac{\varphi(y)}{d(y, S)} = \frac{\varphi(y)}{d(y, \partial \Omega)}
\]

for \( \epsilon \) small enough. Thus \( \varphi(y) \geq \frac{a_1}{2} d(y, \partial \Omega) \) for all \( y \in \Omega^\epsilon \). On the other hand on \( \overline{\Omega^\epsilon} \), \( \varphi > 0 \). Because \( \overline{\Omega^\epsilon} \) is compact, there exists \( a_2 > 0 \) such that \( \varphi \geq a_2 \) on \( \overline{\Omega^\epsilon} \). Hence

\[
\varphi(x) \geq a_2 = \frac{d(x, \partial \Omega)}{d(x, \partial \Omega)} a_2 \geq \frac{a_2}{\text{diam} \Omega} d(x, \partial \Omega)
\]

for \( x \in \overline{\Omega^\epsilon} \), where \( \text{diam} \Omega \) is the diameter of \( \Omega \). Taking \( L = \max \{ \frac{a_2}{\text{diam} \Omega}, \frac{a_1}{2} \} \) completes the proof of the claim.

By (4.14) and (4.15) we have

\[
u(x) \geq K \lambda d(x, \partial \Omega) \quad \text{for } x \in \Omega\]
for \( K = PL \). Let \( D = \Omega' - \Omega \). Therefore for \( x \in D \) we have \( u(x) \geq \overline{K}\lambda \) where \( \overline{K} = \frac{L}{2} K \). Let \( u_3 \) satisfy \(-\Delta u_3 = \chi_D \) in \( \Omega \), \( u_3 = 0 \) on \( \partial\Omega \). By (1.6), for \( \lambda \) large enough

\[
f(\overline{K}\lambda)u_3 + f(0)\varphi \geq d(x, \partial\Omega) \text{in } \Omega.
\] (4.16)

Now

\[
-\Delta u \geq \lambda[f(\overline{K}\lambda)\chi_D + f(0)]
\]
\[= \lambda f(\overline{K}\lambda)(-\Delta u_3) + \lambda f(0)(-\Delta \varphi)
\]
\[= -\Delta(\lambda\{f(\overline{K}\lambda)u_3 + f(0)\varphi\}).
\]

Thus by the maximum principle and (4.16) we have

\[
u \geq \lambda\{f(\overline{K}\lambda)u_3 + f(0)\varphi\} \geq \lambda d(x, \partial\Omega) \text{for } x \in \Omega.
\]

This completes the proof of the theorem.■

Next we obtain a priori bound for positive solutions to (1.1).

**Lemma 4.4** For each \( \lambda > 0 \), there exists \( M := M(\lambda) \) such that if \( u \) is a positive solution of (1.1), then \( \|u\|_{L^\infty(\Omega)} \leq M \).

**Proof.** Let \( p > N/2 \). By a priori bound for elliptic boundary value problems (see [9], pg 242), there exists \( C_1 := C_1(p, \Omega) \) such that if \(-\Delta v = g(v) \) in \( \Omega \), \( v = 0 \) on \( \partial\Omega \) then

\[
\|v\|_{H^2(\Omega)} \leq C_1\|g\|_{L^p(\Omega)}.
\]

On the other hand by (1.7) there exists \( K := K(\lambda) \), such that \( |f(t)| \leq \frac{1}{2C_1\lambda}|t| + K \) for all \( t \geq 0 \). Hence

\[
\|u\|_{H^2(\Omega)} \leq C_1\lambda\|f(u)\|_{L^p(\Omega)} \leq \frac{1}{2}\|u\|_{L^p(\Omega)} + C_1\lambda K|\Omega|^{1/p}.
\]
Thus for $\lambda$ large enough we have $\|u\|_{H^{2,p}(\Omega)} \leq 2C_1\lambda K^\prime |\Omega|^{1/p}$. Finally, since $p > N/2$, by Sobolev inequalities [1], there exists $C_2 := C_2(p, \Omega)$ such that $\|u\|_{L^\infty(\Omega)} \leq C_2 \lambda K$, which completes the proof of lemma.$\blacksquare$

To complete the proof of Theorem 1.2 we will use the following two lemmas.

**Lemma 4.5** Suppose $u$ is a nonnegative solution of (1.1). Then given $\alpha > 0$, there exists a $b > 0$ such that

$$G(u) := f(u) - uf'(u) \geq b \text{ whenever } u \geq \alpha.$$

**Proof.** Differentiating $G(u)$, by concavity of $f$ we have

$$G'(u) = -uf''(u) \geq 0.$$

Therefore to establish the existence of $b$, it suffices to show that $G(u) \geq 0$ for some $u \geq 0$. Suppose $\beta > 0$ is such that $f(\beta) = 0$. Let $\alpha > \beta$. We will show that $G(t) \geq 0$ for some $t \geq \alpha$. Suppose not. Therefore

$$f(t) \leq tf'(t) \text{ for all } t \geq \alpha. \quad (4.17)$$

Dividing both side of (4.17) by $tf(t) > 0$, since $t > \beta$, and integrating on $[\alpha, t]$ we infer

$$\frac{t}{\alpha} \leq \frac{f(t)}{f(\alpha)}.$$

Thus $0 < \frac{f(\alpha)}{\alpha} \leq \frac{f(t)}{t}$. But this is a contradiction to (1.7), which completes the proof of the lemma.$\blacksquare$
Lemma 4.6 Suppose \( a > 0, \ p > N, \ \Omega_+ = \{x \in \Omega; d(x, \partial \Omega) \geq \frac{x}{a}\} \), and \( \Omega_- = \overline{\Omega} - \Omega_+ \).

If \( m \) and \( n \) satisfy \(-\Delta m = h_+ \) in \( \Omega \), \( m = 0 \) on \( \partial \Omega \) and \(-\Delta n = h_- \) in \( \Omega \), \( n = 0 \) on \( \partial \Omega \), where \( h_+ \) and \( h_- \) denote the characteristic functions of \( \Omega_+ \) and \( \Omega_- \) respectively.

Then \( m \to \varphi \) and \( n \to 0 \) in \( C^1(\overline{\Omega}) \) as \( \lambda \to \infty \).

Proof. First we show that \( n \to 0 \), once we establish this we can then use it to show that \( m \to \varphi \) in \( C^1(\overline{\Omega}) \) as \( \lambda \to \infty \). Taking the \( L^p \) norm of \( h_- \) we have

\[
\|h_-\|_{L^p} = \left( \int_{\Omega_-} 1 \right)^{\frac{1}{p}} \leq K_1 \lambda^{-\frac{1}{p}},
\]

for some constant \( K_1 \), independent of \( \lambda \). Therefore by the a priori estimate for elliptic boundary value problems [9], there exists \( K_2 = K_2(p, \Omega) \) such that

\[
\|n\|_{H^{2,p}} \leq K_2 \|h_-\|_{L^p} \leq K_1 K_2 \lambda^{-\frac{1}{p}}.
\]

On the other hand since \( p > N \), by Sobolev inequalities [1], we infer

\[
\|n\|_{C^1} \leq K_4 \lambda^{-\frac{1}{p}},
\]

for some constant \( K_4 \) independent of \( \lambda \). Hence \( n \to 0 \) in \( C^1(\overline{\Omega}) \) as \( \lambda \to \infty \). To see that \( m \to \varphi \) in \( C^1(\overline{\Omega}) \) as \( \lambda \to \infty \), note that \(-\Delta (m - \varphi) = -h_-\). So similarly we can show that \( m - \varphi \to 0 \) in \( C^1(\overline{\Omega}) \) as \( \lambda \to \infty \). This completes the proof of lemma. \( \blacksquare \)

Proof of Theorem 1.2:

Given \( \lambda > 0 \) by (1.7), there exists \( J := J(\lambda) > \lambda f(M) \), (see Lemma 4.4) such that \( J \geq \lambda f(J \varphi) \). For any positive solution \( u \) of (1.1), we obtain \(-\Delta (J \varphi - u) = J - \lambda f(u) > \)
\( J - \lambda f(M) > 0 \). Let \( \bar{u} \) be the limit of the sequence defined by

\[
\begin{align*}
    u_0(x) &= J\varphi(x) \\
    u_{n+1}(x) &= \int_{\Omega} G(x, y) \lambda f(u_n(x)) \, dy,
\end{align*}
\]

where \( G \) is the Green function for \(-\Delta\) with Dirichlet boundary condition in \( \Omega \). It is well known that \( \bar{u} \) is a solution to (1.1) and \( u \leq \bar{u} \) for any positive solution \( u \) of (1.1) [20]. To complete the proof of Theorem 1.2 we will show that \( \bar{u} = u \). Since \( \bar{u}, u \) are solutions of (1.1), we obtain

\[
\begin{align*}
    \Delta(\bar{u} - u) + \lambda(f(\bar{u}) - f(u)) &= 0 \quad \text{in } \Omega \\
    \bar{u} - u &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]  

By the mean value theorem there exists \( \xi \), such that \( u \leq \xi \leq \bar{u} \) in \( \Omega \) and

\[
\begin{align*}
    \Delta(\bar{u} - u) + \lambda f'(\xi)(\bar{u} - u) &= 0 \quad \text{in } \Omega \\
    \bar{u} - u &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]  

Multiplying (1.1) by \( \bar{u} - u \), (4.19) by \( u \), integrating we have

\[
\lambda \int_{\Omega} (f(u) - uf'(u))(\bar{u} - u) \leq 0,
\]  

where we have used that \( f'(\xi) \leq f'(u) \) because \( f \) is concave and \( u \leq \xi \). By Lemma 4.5 there exists \( b > 0 \) and \( a > 0 \) such that \( f(u) - uf'(u) \geq b \) whenever \( u \geq a \). From Theorem 4.3 \( u(x) \geq a \) if \( d(x, \partial\Omega) \geq \frac{a}{\lambda} \). Let \( \Omega_+ \) and \( \Omega_- \) be as in Lemma 4.6, by (4.20) we have

\[
\int_{\Omega_-} f(0)(\bar{u} - u) + \int_{\Omega_+} b(u - u) \leq 0,
\]  

(4.21)
where we have used that $f(u) - uf'(u) \geq f(0)$ for all $u \geq 0$. Let $m$ and $n$ be as in Lemma 4.6, multiplying (4.19) by $bm + f(0)n$, and integrating by parts we obtain

$$f(0) \int_{\Omega_-} (\bar{u} - u) + b \int_{\Omega_+} (\bar{u} - u) = \lambda \int_{\Omega} f'(\xi)(\bar{u} - u)(bm + f(0)n).$$

(4.22)

By Lemma 4.6, for $\lambda$ large enough, $bm + f(0)n > 0$. Thus (4.21), (4.22), and the assumption that $f$ is an increasing function imply that $\bar{u} = u$ for every nonnegative solution $u$ of (1.1). This completes the proof of Theorem 1.2.
BIBLIOGRAPHY


