EXPERIMENTAL SYNCHRONIZATION OF CHAOTIC ATTRACTORS USING CONTROL

DISSERTATION

Presented to the Graduate Council of the University of North Texas in Partial Fulfillment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

By

Timothy C. Newell, B.S. M.S.
Denton, Texas
December 1994
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The focus of this thesis is to theoretically and experimentally investigate two new schemes of synchronizing chaotic attractors using chaotically operating diode resonators. The first method, called synchronization using control, is shown for the first time to experimentally synchronize dynamical systems. This method is an economical scheme which can be viably applied to low dimensional dynamical systems. The other, unidirectional coupling, is a straightforward means of synchronization which can be implemented in fast dynamical systems where timing is critical. Techniques developed in this work are of fundamental importance for future problems regarding high dimensional chaotic dynamical systems or arrays of mutually linked chaotically operating elements.

In the control technique, synchronization is achieved when required feedback amplitude modulates the drive signal of only one of the resonators for a fraction of each period. This feedback is the difference in observed signals from each resonator multiplied by a factor that is a predetermined function of the chaotic attractor. A numerical analysis of the process shows how the synchronizing feedback forces both the local and global Lyapunov multipliers, the indicators of stability, of the composite dynamical system to be less than one. While the theoretically prescribed feedback factor presents the optimum choice of feedback, simpler more experimentally accessible factors may also be incorporated to achieve synchronization.

In the unidirectional coupling method, feedback is created by amplifying the difference between the observed signals with a constant factor then continuously adding it to the slave resonator driving wave. The synchronizing feedback can be divided into a dissipation and driving term. The former term, if acting alone, will force the
slave resonator into a periodic orbit. The latter drives the slave resonator in concert with the other. The result is that the original slave attractor is replaced by an attractor which is synchronous to the driver. In contrast to the control method, the synchronizing amplification factor can only be determined during the synchronization experiment.
ACKNOWLEDGMENTS

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Commencing with the insightful work of Poincaré at the end of the 19th century the study of nonlinear and chaotic dynamics has consisted of understanding the nature of nonlinear phenomena by developing and refining a formalism for its study. Since 1980, the computer has had a tremendous influence in the growth of the field as it allows for vast amounts of information to be created and analyzed in a short amount of time. In 1990, a new paradigm of post modern chaos was created with the publication of two revolutionary papers. The first paper, *Synchronization in Chaotic Systems*, by Louis Pecora and Thomas Carroll [1] at the Naval Research Laboratory describes how a system operating chaotically can be synchronized with an identical system also operating chaotically. This was originally surprising since at first thought the concept of synchronization and chaos appears to be an oxymoron. The second paper entitled *Controlling Chaos*, by Edward Ott, Celso Grebogi, and James Yorke [2] at the University of Maryland points out that since an infinite number of unstable periodic orbits are embedded in a chaotic attractor, any one of them can be stabilized by making small changes to a parameter of the system. This too was astonishing since chaotic motion was often thought to be unpredictable [3]. The significance of these two papers is that by demonstrating applications of chaos, the view point of the chaos community has been changed. Before the publication of these papers, applied science and engineering had usually considered chaos like noise as an undesirable feature to be avoided at all costs. But now the windows of application have been opened to the extent that an entire chaotic circuit has been placed into a single integrated circuit
The Pecora and Carroll method of synchronization has been demonstrated in a variety of chaotic electrical circuits and is being used to test the validity of data encryption. And the Ott, Grebogi, and Yorke (OGY) control technique has been illustrated in chaotic lasers, biological systems, and chemical reactions, as well as chaotic electronic devices. In the short time since the publication of these two papers other techniques have been developed for the synchronization of chaos and control of the unstable periodic orbits. Coupling chaotic objects with a connection that mutually affects each element has been experimentally demonstrated in chaotic lasers and electrical circuits. A control method known as time-delayed feedback is proving fruitful in the stabilization of low periodic orbits in electrical circuits. This method, described below, can potentially be implemented all-optically so as to function for dynamical systems that operate in the GHz regime. While the above schemes have been well documented experimentally, two alternative methods of synchronization have remained untested. The first synchronization method, named synchronization using control, is the extension of the OGY control theory and has been proposed by Ying-Cheng Lai and Celso Grebogi [5] in April of 1993. The second method, to be called unidirectional coupling synchronization, was investigated independently by the Phillips Laboratory chaos group [6, 7] and the group of Rul’kov, Volkovskii, Rodriguez-Lozano, Del Rio, and M.G. Velarde [8] is to couple chaotic elements with a unidirectional term so that one chaotic element becomes the master of the other slave chaotic element which receives the steering signal.

The intent of this thesis is to examine and establish the validity and feasibility of synchronizing chaotic electrical circuits, diode resonators, by the method of synchronization using control and by the unidirectional coupling technique. Once it is shown that the schemes are successful, it is important to test the tolerances of synchronization not only to variations between the individual circuits but also to the range of factors which are required to produce a synchronizing feedback signal. Continuing
the study, one must determine how simply the necessary feedback can be computed so as to minimize the complexity of the experimental apparatus. Furthermore we wish to examine numerically the Lyapunov multipliers of the two resonator systems in order to determine the correlations between orbits for different types of coupling. The goal is to demonstrate that these are valid methods which can be used either as an efficient technique (synchronization by control) or as a method capable of operating at very high frequencies (unidirectional coupling). The experimental nonlinear system to be used in the study is the diode resonator which is composed of a $p-n$-junction diode in series with an inductor and a resistor and is driven by an externally supplied sine wave. Though the resonator is physically a simple circuit, it exhibits an extraordinary wealth of nonlinear phenomena and could be established as a paradigm for non-autonomous chaotic circuits.

This thesis is organized as follows. In this chapter, we provide an exposition of landmark developments of the experimental work performed in the control and synchronization of chaos since 1990. From this review, it is seen that interest lies in the two alternative synchronization topics to be studied by this dissertation. In chapter 2, the published work on the diode resonator is reviewed. We complement this work by presenting our numerical and experimental work on characterizing the diode resonator. Numerical work consists of deriving the dynamical equations of the resonator starting from Kirchhoff’s laws and incorporating established models of the nonlinear diode. From the model, the Lyapunov spectrum is computed as a function of the drive wave amplitude and the Kaplan-Yorke dimensionality conjecture is calculated. From the Kaplan-Yorke conjecture, we see that the resonator has a nearly 1-dimensional attractor. Subsequently we determine the unstable direction of the resonator and show that its orientation is constant. Experimental work is composed of recording and examining bifurcations which occur due to changes in either the amplitude, frequency, or DC bias of the driving sine wave. Other work consists of plotting first return maps and using the time series to compute the mutual
information in order to optimize the time delay for reconstruction of the attractor by
delay coordinates.

In chapter 3 the theme of control is introduced. First the Ott, Grebogi, and
Yorke chaos control theory is presented. This is followed by the derivation of the
synchronization control scheme of Lai and Grebogi. As an introductory exercise,
both algorithms are applied to control the logistic map and then to synchronize one
logistic map with another.

Synchronization between two diode resonators is then analyzed. As an initial
test, it is proposed that the synchronization scheme be adapted to one that is easily
implemented experimentally. This is referred to as synchronization by occasional
proportional feedback (SOPF). Subsequently the Lai and Grebogi prescribed steering
perturbation is constructed from first return maps. This factor is expressed as a
function of the peaks of the circuit current and can be implemented experimentally.
We also propose that synchronization will occur if the steering perturbation is only
applied while the resonator is operating in unstable regions of the attractor. In the
stable regions, we apply no feedback. This proposal bridges the gap between the
SOPF technique and the scheme prescribed by Lai and Grebogi. Numerical studies
are then performed which show how the synchronizing feedback and the simplified
constant affects the ability, quantified by the Lyapunov multipliers, of two resonators
to synchronize.

In chapter 4, the experiments are described which show the various types of feed-
back that can synchronize two nearly identical resonators using control. The first
experiment is an implementation of SOPF in order to achieve synchronization. In a
second experiment, the true factor as prescribed by the Lai and Grebogi algorithm is
generated and applied in real time to synchronize the resonators. The third experi-
ment is a demonstration that synchronization indeed occurs when feedback is applied
when the resonator is operating in unstable regions of the attractor.

Chapter 5 describes the experimental arrangement of the unidirectional coupling
technique and the results of experiments performed. It is shown how the unidirectional coupling technique can be considered as a quenching of one chaotic orbit and being driven in resonance with the chaotic orbit desired for synchronization.

Finally in chapter 7, the main points of this dissertation are summarized and conclusions drawn. Comparisons are made between the two studied methods of synchronization. Possibilities for future studies are presented along with open problems which currently exist.

The organization of the remainder of this chapter is dedicated to a historical review of synchronization and control theory and corresponding experimental work. The topics of synchronization and control are introduced in section II. In section III, the Pecora and Carroll synchronization method is outlined and the pertinent experiments and applications demonstrating the technique are reviewed. Subsequently, the other synchronization techniques of mutual coupling and unidirectional coupling are presented along with experimental studies performed. In section IV, the Ott, Grebogi, and Yorke control method is summarized and original experiments presented. Finally the time-delayed feedback scheme of chaos control is described and the verifying experimental work is reviewed. The chapter is summarized in section V.

What is Control and Synchronization?

If two perfectly identical but chaotic systems are both started from points in a phase space which are only infinitesimally different, then as the trajectories follow their respective orbits the distance between the two will increase exponentially on average for short time segments. Separation is occurring despite the fact that the volume occupied by the orbit is shrinking. This can be seen in Fig. 1.1 which is a series of graphs showing the evolution at different times of points in a chaotic Lorentz attractor [9]. Initially, a cluster of 10,000 points are tightly packed into a small region of the attractor. As time progresses these points are rapidly being stretched into an infinitesimally thin filament which spreads over the entire attractor until the points
could be located almost anywhere. These points seem to wander through the butterfly shaped attractor with no obvious discernible pattern of motion. But if one looks at short time segments, one realizes that the trajectory of a point may at times look like a cyclic orbit traced out by a simple harmonic oscillator. Other times, it is still periodic but with a number of loops. These are the unstable periodic orbits (UPO) which are densely packed in the attractor. An infinite number of these UPOs exist but they are traced out only for a short time as the extreme sensitivity to conditions causes a drift between one trajectory and another. If there was some manner in which one could steer the phase space trajectory along a defined UPO, then the composite jumble would resolve into the simple periodic cycle. This is a technique for controlling the unstable periodic orbits.

Now say that the 10,000 points represented 10,000 Lorentz systems. To expect that the orbit point of each system follow each other in step as they wander throughout the attractor seems unlikely at best - especially when realizing the impossibility of perfectly identical physical systems. Yet this mirroring is the desired synchronization. Therefore, some influence will have to be exerted on one or both of the systems in order for synchronization to occur.

Synchronization

The Pecora and Carroll Synchronization Method

The method of synchronization developed by Pecora and Carroll [1, 10] entails the division of the system into two subsystems designated as the driver and response. This kind of division, graphically pictured in Fig. 1.2, is as follows. Given an n-dimensional dynamical system, \( F(\vec{x}) \), where \( \vec{x} = (x_1, x_2, ..., x_n) \), divide it into a subsystem \( G(\vec{y}, \vec{z}) \) with dimension \( m \) and a subsystem \( H(\vec{y}, \vec{z}) \) with dimension \( k \) where \( \vec{y} = (y_1, y_2, ..., y_m) \)
and \( \bar{z} = (z_1, z_2, \ldots, z_k) \) so that \( n = m + k \). Then

\[
\frac{dy}{dt} = G(\bar{y}, \bar{z})
\]

(1.1)

and

\[
\frac{d\bar{z}}{dt} = H(\bar{y}, \bar{z}).
\]

(1.2)

Next observe a trajectory, \( \bar{z}(t) \), of the \( H(\bar{y}, \bar{z}) \) subsystem. Can the original \( F(\bar{x}) \) system be divided so that any small perturbation to \( \bar{z}(t) \), i.e. \( \bar{z}(t) + \delta \bar{z} \), will converge back onto the original orbit? Equivalently one asks, can a subsystem \( H(\bar{y}, \bar{z}) \) be constructed so that the error signal defined by \( \Delta \bar{z} = |\bar{z}(t) - \bar{z}(t)| \) converges to zero. If the answer is yes, then synchronization will occur. The \( H(\bar{y}, \bar{z}) \) subsystem is named the response subsystem and \( G(\bar{y}, \bar{z}) \) the driver. One then constructs a duplicate subsystem, \( H'(\bar{y}, \bar{z}') \), so that

\[
\frac{d\bar{z}'}{dt} = H'(\bar{y}, \bar{z}').
\]

(1.3)

The two response subsystems are linked to the \( G(\bar{y}, \bar{z}) \) driver subsystem by the \( \bar{y} \) variable. Then the trajectories, \( \bar{z}(t) \) and \( \bar{z}'(t) \) should synchronize onto each other even when they are driven chaotically by the driver. It remains to answer the question as to the condition in which \( \Delta \bar{z}(t) \to 0 \). Consider the linearization of the difference in the two dynamical response systems \( H(\bar{y}, \bar{z}) \) and \( H'(\bar{y}, \bar{z}') \) about \( \bar{z}(t) \)

\[
\frac{d\Delta \bar{z}}{dt} = D_\bar{z} H(\bar{y}, \bar{z}) \Delta \bar{z}.
\]

(1.4)

One needs to determine the eigenvalues of the Jacobian, \( D_\bar{z} H(\bar{y}, \bar{z}) \), as \( t \to \infty \). These eigenvalues have been named conditional Lyapunov exponents by Pecora and Carroll [10, 11]. These are referred to as conditional because they are not a subset of the Lyapunov exponents for the entire system and they are dependent on the chaotic drive signal \( \bar{y}(t) \). We can calculate the conditional Lyapunov exponents by the following procedure. Since \( \bar{z} \) is a response variable being driven by a chaotic
signal, calculation of the eigenvalues is not trivial since one must not only integrate Eq. 1.4 but also the system \((\bar{y}, \bar{z})\). Alternatively, one uses a matrix \(A(t)\) in place of \(\bar{z}\) where \(A(0)\) is the identity matrix. The linearized equation in \(A(t)\) is

\[
\frac{d\delta A}{dt} = D_zH(y, z)A
\] (1.5)

where \(\delta A\) is the linearized distortion of \(A(t)\) due to the Jacobian matrix \(D_zH(y, z)\). Physically this process of propagating \(A(t)\) forward in time causes the deformation of some initially orthogonal coordinate system, \(A(0)\). By evaluating the time evolution of the matrix \(A(t)\), the rotating of the coordinates along with the contraction of the phase space can be determined and the conditional Lyapunov exponents of the subsystem can be obtained [12]. These exponents must all be negative in order for synchronization to occur. While this is a necessary condition for synchronization, the fundamental linear stability theorem shows that it is also sufficient [10]. As a result, if one can determine an appropriate division of the dynamical equations and construct the ensuing response system the synchronization will occur. An example is the Lorentz system which is given by

\[
\frac{dx}{dt} = \sigma(y - x), \quad \frac{dy}{dt} = r x - y - xz, \quad \frac{dz}{dt} = xy - bz.
\] (1.6, 1.7, 1.8)

These equations can be divided in either of two ways. The first way relies on \(y\) as a drive variable. The response subsystem to be created and driven by \(y\) is

\[
\frac{dx_1}{dt} = \sigma(y - x_1) \quad \frac{dz_1}{dt} = x_1 y - bz_1.
\] (1.9, 1.10)
Alternatively, use \( z \) as the drive variable and create the subsystem defined by the equations

\[
\begin{align*}
\frac{dy_2}{dt} &= rz_2 - y_2 - xz_2 \\
\frac{dz_2}{dt} &= xy_2 - bz_2.
\end{align*}
\] (1.11, 1.12)

In either case, the conditional Lyapunov exponents for the subsystem are negative and hence either system will synchronize with the original.

This method of synchronization has been tested on electronic circuits whose dynamical equations are usually known \([13, 14, 15, 16, 17]\). In chaotic lasers, chemical, or biological systems it is not clear that one can create the appropriate subsystems. As pointed out by Carroll and Pecora \([13]\), even circuits exist which cannot be divided so that the conditional Lyapunov exponents are all negative.

Experimental observation of this form of synchronization was published in the original paper of Pecora and Carroll \([1]\) and in a more complete form in Ref.\([13]\). Their experimental system was a chaotic electrical circuit which consisted of an unstable oscillator with an added hysteretic element. The dynamical equations governing the voltages and currents induced in the circuit are

\[
\begin{align*}
\frac{dx}{dt} &= y + \gamma x + c(\alpha z - \beta x) \\
\frac{dy}{dt} &= -\omega x + \delta_2 y \quad (1.13)
\end{align*}
\]

and

\[
\frac{dz}{dt} = \left( \frac{1}{\alpha} \right) [(1 - (\alpha z - \beta x)^2)(S z - D + \alpha x)/\epsilon - \delta_3(\alpha z - \beta x)/\epsilon - \beta y - \beta\gamma x - c\beta(\alpha z - \beta x)].
\] (1.15)

A suitable driver subsystem consisting of a portion of the hysteretic element and the unstable oscillator was determined from the dynamical equations. The drive signal
is identified by the dynamical variable: \( z \). The response subsystem was identified to be governed by the dynamical equations in \( x \) and \( y \). A duplicate subsystem was then constructed. The single driver drove both response systems into synchronization. The quality of synchronization was extremely dependent on the identical character of each response system. When one of the response systems was slightly different, degradation of the degree of synchronization was observed.

Cuomo and Oppenheim at M.I.T. have utilized the Pecora method to synchronize an electrical circuit of the Lorentz system \([14, 18]\). As seen above, the Lorentz equations can be divided into either of two drive-response subsystems satisfying the requirement of the conditional Lyapunov exponents. Furthermore, these two distinct response systems can be combined into a single system having a three-dimensional state space. This produces a synchronizable response system which is structurally similar to the original Lorentz system. What this division means is that not only can the response system be driven into synchronization with the driver by means of a suitable drive variable, say \( x(t) \), but also the response system generates its own version of the drive variable \( x'(t) \).

The M.I.T. group's interest lies in applications regarding communications which are buried in the chaos of the signal so as to present a data encrypted signal. Transmittal of the hidden information is accomplished by comparing the drive signal \( x(t) \) issued by a transmitter circuit to the regenerated drive signal \( x'(t) \) created by a receiver circuit. One way to do this is to modulate a parameter of the transmitter circuit with minute digital pulses. The receiver circuit under conditions of no modulation will synchronize \( x(t) \) with \( x'(t) \). Whenever a pulse modulates the parameter of the transmitter circuit, the transmitted drive signal will lose synchronization with the regenerated one and from this loss of synchronization the bit is decoded. Since the signal is chaotic, the bit stream is effectively buried in the chaos. Encoding schemes such as this one have also been demonstrated using Chua's circuit \([17]\). Another manner of synchronization is to add a signal to the \( x(t) \) and then extract the signal by
evaluating the difference between \( x(t) \) and \( x'(t) \). This method requires that both circuits be identical and the impressed signal be small. As long as a third party cannot duplicate the response system, the signals will be masked in the chaotic signal.

In the electrical engineering community, the paradigm of chaotic circuits is the Chua circuit [19, 20]. In essence, this circuit is a RLC resonator using a nonlinear resistor with two break points. It is capable of being divided into an acceptable response system in two different manners. Synchronization was achieved between a duplicated subsystem and the suitable subsystem when both were driven by the drive subsystem [15]. In the experiment, the drive signal from one complete circuit was buffered from the response subsystem in order to isolate the drive from feedback generated by the response systems. If feedback were to occur, then the dynamical equations of the original drive-response system would be altered by the presence of the response variables from the duplicated subsystem. In this case, it is unclear if synchronization could be achieved.

Beyond Pecora and Carroll

If two chaotic systems are linked together so as to share common signals, synchronization can occur if the coupling strength exceeds a certain level. This is known as mutual entrainment. In linear systems, the idea is not new. Huygens observed in 1667 that clocks would synchronize when they were mounted on the same wall. However, the extension of synchronization in this manner to chaotic systems was not anticipated until recently. The coupling can be through either one variable or several. In the event of one variable it is known as scalar coupling and physically implies one link. For several variables coupling can be made through several variables which is known as vector coupling and naturally implies several links. The effect on the governing dynamical equations is to add an equivalent term to both systems. For the
case of two linked systems, this coupling is seen by

$$\frac{dx}{dt} = F(x) + \alpha(y_i - x_i), \quad (1.16)$$

$$\frac{dy}{dt} = F(y) - \alpha(x_i - y_i) \quad (1.17)$$

where $F$ refers to the underlying dynamical equations, $\alpha$ is the coupling strength, and $x_i$ refers to the $i$th component of the system. The nomenclature of dissipative synchronization is often used to describe Eqs 1.16 and 1.17 because of the damping manner in which the coupling enters into the dynamical equations. Note that in Eq. 1.16 one has the term $-\alpha x_i$ and in Eq. 1.17 the term $-\alpha y_i$ appears. In electrical circuits, the physical meaning of the coupling is via the current flow from one system to another. In the governing equations, the coupling parameter is inversely proportional to the resistance used while the difference term links the voltages at the nodes on either side of the resistor. This is logical since as the resistance goes to infinity, the two circuits become independent. This method is perhaps a more feasible approach to synchronization to the method proposed by Carroll and Pecora. It is not necessary to understand how to divide the system into a drive and response subsystem, and it is applicable to systems such as lasers which probably cannot be divided in such a manner in any case. However, one must know the system well enough so that coupling links are placed appropriately.

Waller and Kapral observed mutual synchronization in 1984 [21] when studying the dynamical behavior of two coupled Rossler oscillators. They noticed that the oscillators synchronized if the coupling term was subtracted from the dynamical equations. However, since their interest was to explore routes into chaos, their work studied the effect of the coupling when it was added to the dynamical equations. Afraimovich, Verichev, and Rabinovich [22] first studied mutual stochastic synchronization in dissipatively coupled self-excited noise oscillators. Rul'kov, Volkovski, and Rodriguez-Lozano, [23] keying on this work observed synchronization using two iden-
tical chaotic electric circuits which were coupled using a resistor. The current flow between one circuit and the other created the dissipation term which affected both systems equally.

As in the work of Pecora and Carroll, an understanding of synchronization can be obtained through the use of Lyapunov exponents. By linking two n-dimensional systems, a 2n-dimensional phase space is created. There exists a synchronization manifold in this phase space which is defined by setting the n variables of one system equal to the equivalent variables of the other. Thus any tangent motion entirely within this manifold is completely synchronous while motion perpendicular to the manifold is not. The question is then to determine if the coupling factor between the two systems can be set so that motion perpendicular to the synchronization manifold is compressed onto it. To answer this consider the effect of the coupling parameter, α, on the Lyapunov exponents of the composite system. These can be divided into exponents characterizing stretching tangent to the manifold and those characterizing stretching perpendicular to the manifold. For any motion solely along the manifold, these tangent exponents do not depend on α since as seen in Eq. 1.16 and Eq. 1.17 the coupling term disappears. Thus these tangent exponents are just those for a single uncoupled system. Motion perpendicular to the manifold is quantified by perpendicular Lyapunov exponents. In order to calculate the perpendicular Lyapunov exponents, one must introduce error variables defined by

$$X = \bar{x} - \bar{y}. \quad (1.18)$$

By inserting the error variables into the dynamical system, one obtains

$$\frac{d\tilde{X}}{dt} = F(\bar{x}) - F(\bar{y}) - 2\alpha X_i \quad (1.19)$$

where $X_i$ refers to the $i$th coordinate coupling. Linearization gives a set of linear
equations

\[
\frac{d\vec{X}}{dt} = D_x F \vec{X} - 2\alpha X_i
\]  

(1.20)

where \( D_x F \) is the Jacobian matrix. The coupling has added the term \(-2\alpha\) into the \(i\)th element of the \(n\)-dimensional matrix. If this term can drive all the eigenvalues of the error system's Jacobian matrix negative then synchronization will occur.

As an aside, one potential way to create a two positive Lyapunov exponent, hyperchaotic, dynamical system is to combine two chaotic systems by introduction of a weak coupling term. If the coupling can be made, then in the \(2n\)-dimensional system there will be two positive exponents, two zero exponents, and the rest are negative. However, as the coupling term is strengthened, the lower of the positive exponents becomes negative.

Numerical laser synchronization by means of mutual coupling of the evanescent fields was performed by Winful and Rahman [24]. Their model system was a three element array of semiconductor lasers coupled by means of overlapping their evanescent fields. For certain coupling strengths, synchronization is obtained between two of the three lasers while the third remains chaotic and unsynchronized to the other two. This is the result of the spatial symmetry of the array.

Two numerical examples with themes of mutual coupling have been demonstrated with arrays. An early study of networks of synchronized states has been performed by Kowalski, Albert, and Gross [25] in 1990. Each element of the network was coupled in the case of a map to the system mean or in a dynamical system to an influence function. This influence function is identical for all interacting components of the network and is analogous to a mean-field coupling. With this scheme, the group was able to synchronize four Lorentz units regardless of the initial conditions for each unit. The intention of their work was to model potential neurophysiological activity in signal propagation between axons. Their coupling function was designed to emulate the process of receiving input, processing it, and then outputting some influence on the other units.
A mathematical study of an array of coupled systems has been performed by Heagy et al. [26]. The work explores possibilities of the synchronization of arrays via a universal shift invariant coupling function. The notation of shift invariance is due to a coupling symmetry whereby any element of the array is interchangeable with another without altering the coupling factor. This includes diffusive coupling and global coupling. The authors present a formalism for linearizing, diagonalizing, and obtaining the transverse (equivalent to the perpendicular or conditional) Lyapunov exponents for the system.

An example of experimental dissipative mutual coupling has been demonstrated using two identical Chua's circuits [19, 27]. The circuits, described above in the Pecora synchronization section, were coupled by means of a variable resistor which could be placed at various points in the circuit so as to link any of the equivalent variables of the two systems. Of these three variables (implying three different ways of scalar coupling), synchronization was obtained experimentally with two and numerically with the third. Interestingly, an analysis of the perpendicular Lyapunov exponents indicated that synchronization could not be achieved using two of the three permutations.

R. Roy and S. Thornburg Jr. [28] at Georgia Tech have succeeded in the mutual coupling of the intracavity fields between two lasers. The experiment consisted of propagating both beams in the same crystal. The amount of coupling was determined by the overlap of one beam with the other.

Unidirectional Coupling Synchronization

The concept of unidirectional synchronization is mathematically similar to that of mutual coupling. While it is the subject of Chapter 5, for this historical review it will be mentioned for completeness. The idea is identical to that of mutual coupling with the exception that the coupling term is added only to one of the equations leaving the other untouched. A potential benefit of this type of coupling is the proposition
that an array of systems could be synchronized to a single master system.

Newell, Alsing, Gavrielides, and Kovanis [7] demonstrated synchronization via unidirectional coupling using two chaotic diode resonators (See chapter 5). Independently Rul’kov, Volkovskii, Rodriguez-Lozano, Del Rio, and Velarde [8] also obtained unidirectional synchronization in an chaotic circuit. In both groups, an operational amplifier integrated circuit was used to sum the signal from the driving circuit into the slave.

Unidirectional coupling was recently realized in a pair of CO₂ lasers [29]. The result was a synchronization of passive Q-switched pulsations (PQS) for each system. PQS occurs when a saturable absorber inside the laser cavity induces a self-sustained pulsation in the single mode oscillations of the laser. When the PQS-chaotic beam of the master laser, made chaotic by an equivalent absorber in the master cavity was propagated through the absorber of the slave cavity, the slave pulsations synchronized with those of the master. The coupling strength was varied by an iris in the master beam’s path.

Control of Chaos

The Ott, Grebogi, and Yorke Control Experiments

In the late 1980’s several papers appeared which indicated that chaos could be guided into a stable periodic orbit. [30, 31, 32, 33, 34]. However, the pre-eminent demonstration of the control of chaos is that of Ott, Grebogi, and Yorke [2]. They realized that because there are an infinite number of unstable periodic orbits embedded within the chaotic attractor, one can use a method similar to standard linear control theory in which one modulates an observable signal of the system in order to correct for noise and other unwanted effects [35]. The OGY recipe is identical to the control technique presented in the Introduction. One determines the desired periodic orbits buried in the chaos, then waits for the chaotic wandering of the orbit trajectory to drift into the neighborhood of the designated periodic orbit. When it does, it is evaluated and
a correcting perturbation is applied to the chaotic system so that its orbit is shifted onto the periodic orbit. An important point to be stressed is that whatever controlling influence is exerted, it is minimal and in principle does not alter the attractor. One modification pointed out by Dressler and Nitsche [36, 37], is that if using a time series to reconstruct the attractor, one must modify the OGY formula so as to take into account previous perturbations. This is because the scalar time series is influenced by the previous perturbations. The OGY method has been shown to be a particular case of the pole placement technique [38, 39]. This technique is a method whereby one makes alterations to the system in order to set unstable eigenvalues, regulator poles, of the system to zero in order to stabilize an orbit [35].

The OGY idea was first tested experimentally by Ditto, Rauseo, and Spano at the Naval Surface Warfare Center to control the chaotic behavior of a magnetoelastic ribbon [40]. Since the fundamental period of vibration was 1.18 seconds, there was ample time to calculate by computer the required feedback factor. Motivated by this success the field blossomed. Investigators applied the OGY algorithm to control chaos in chemical reactions [41], the chaotic flow of water [42], chaotic electronic circuits [43], lasers [44, 45, 46, 47, 48], and even in biological systems such as the tissue from a rabbit heart [49].

Experimentally in electronic circuits, the time required to calculate OGY perturbations necessary to stabilize the unstable periodic orbits limits the range of implementation in dynamical systems. This is due to the inherent timing delays in the electronics devices required to implement the algorithm. As a result, the feedback perturbations of the first experiments were applied at a frequency of only a few hundred Hertz or less. Earl Hunt at Ohio University [43] successfully adapted the OGY algorithm for the $53k\text{Hz}$ sine wave driven diode resonator and has stabilized orbits from period 1 up to period 23. As implied by OGY, Hunt's scheme measures the difference between the actual current peaks in the resonator and the desired current peaks of the periodic orbit. This voltage converted value is stored in a sample and
hold amplifier, i.e. an analog memory. If the measured peak is within the neighborhood of the desired peak, the feedback process continues. Otherwise, the control electronics wait until the next peak occurs. In the favorable case the stored difference is amplified by a constant factor instead of the calculated OGY factor. The resulting perturbation amplitude modulates, for some fraction of the period, the driving sine wave. This technique is known as occasional proportional feedback (OPF). By using the constant amplification factor regardless of the period of the orbit desired to stabilize the frequency range of control is extended and the ease of implementation is enhanced. However, advantages gained by this simplification are offset by a difficulty in predicting which orbit will be controlled.

Using Hunt’s OPF technique, Roy, Murphy, Maier, Gills, and Hunt at Georgia Tech [44] controlled chaos in a diode laser pumped Nd:YAG laser which contains a potassium titanyl phosphate (KTP) crystal in the cavity. For certain rotational orientations of the KTP and YAG crystals the system becomes chaotic. The controlling feedback was obtained by sampling the laser’s output intensity at the relaxation oscillation frequency of 118kHz. The amplified difference between this intensity and the desired intensity was then used to modulate the control signal to the diode laser pump. In this manner, orbits up to period 9 were stabilized for a duration of up to several minutes.

At the University of North Texas [45], original experiments were performed in which a CO₂ laser, driven into a chaotic state by injecting a feedback beam modulated by an electro-optic modulator, was controlled by a variation of the OPF scheme. The feedback influenced the electro-optic modulator.

A direct implementation of the OGY algorithm has also successfully controlled periods 1 through 6 in a chaotic NMR laser [46]. This experiment was complicated by the need of calculating the OGY perturbation in a 6 dimensional embedding space. The calculated feedback modulated the quality factor of the magnetic field cavity at a frequency of 120Hz to stabilize the orbits. Bielawski, Derozier, and Glorieux at the
University of Lille [47] stabilized the unstable periodic orbits in a Nd-doped optical fiber laser that was operating at 15kHz. They did not measure the difference between the laser's output intensity of the desired orbit and the observed intensity. Instead they time delayed the output intensity by $1/15\mu s$ and measured the difference between this intensity and the observed intensity. This difference was sampled and held at some fixed point in the cycle. It was then constantly amplified and applied to every other period for a small fraction of the period.

OGY since 1990

Following the publication of the original OGY paper, a number of modifications and enhancements to the OGY technique have appeared in the press. For a critical review of these works see Alsing et al. [39] and Gavrielides et al. [50]. The explicit OGY procedure has been successfully implemented in dynamical systems which are operating at frequencies below a few hundred Hertz. One variation is Hunt's OPF technique as applied to control a period 1 orbit. In this particular case, the constant feedback factor corresponds to the OGY factor. A part of the problem lies in the speed of evaluation necessary to apply the perturbation in real time. Alsing et al. [39] has addressed this problem by projecting the OGY formalism one or more iterates in advance. By looking at the $n$-th iterate, one calculates the necessary perturbation to be applied at the $n$th iterate so as to stabilize the UPO at the $n+1$th iterate. In casting OGY in this manner, one has gained time to evaluate the necessary feedback. The technique has proved numerically successful in controlling low period orbits.

Time-Delayed Feedback Control of Chaos

The above work in controlling chaos has been applied to relatively slow systems in which analog electronics are capable of evaluating and implementing any required perturbations. However, for fast dynamical systems, any computer or analog estimate of the required feedback is likely to fail since there is simply not enough time for
its implementation. An avenue for controlling such systems has been theoretically explored by Pyragas [51] at the University of Tubingen and subsequently by Rui’k’ov, Tsimring, and Abarbanel [52]. Pyragas showed that the stable steady state that was made unsteady by the chaos in the system could be stabilized by continuously applying feedback to the dynamical system. The correcting signal is obtained by subtracting a dynamical variable of the system from the same output signal but time-delayed by the period of the desired unstable periodic orbit. The amplified feedback is then added back into the system. Explicitly,

$$\frac{d\bar{x}}{dt} = F(\bar{x}) + \alpha[x_i(t) - x_i(t - T_{upo})]$$

(1.21)

where $F(\bar{x})$ represents the governing equations, $\alpha$ is a constant amplification factor, $x_i$ is an observable variable, and $T_{upo}$ is the period of the unstable periodic orbit. The strength of this method is that the feedback term can be readily calculated since the primary operation is simply the subtraction of the two signals. When control is successful, the time delayed signal is equal to the actual one thus the sustaining feedback goes to zero. Experimental demonstration of the time-delayed feedback algorithm was subsequently performed by Pyragas and Tamasevicius [53] to stabilize periods 1 through 4 in a 4MHz driven nonlinear oscillator composed of a tunnel diode, inductor and capacitor. The time delayed signal was obtained by propagating the output signal through a long delay line. The output signal was subtracted from the time delayed one and the amplified difference was then added back into the oscillator. This scheme has also been applied by Bielawski, Derozier, and Glorieux [54] in order to control unstable periodic orbits in a carbon dioxide laser that typically operates at $400kHz$. This was accomplished by modulating an intracavity electro-optic modulator with the constantly amplified difference between the output intensity and the one period delayed intensity. In order to obtain the time delayed signal, they propagated the laser light through a $600m$ long fiber. An ultra fast diode resonator operating at $11MHz$ (almost thirty times faster than the laser of the Glorieux and
three times faster than the Tamasevicius circuit) was stabilized by Gauthier, Socolar, Concannon [55] in a similar manner. The delay signal was created by propagating the signal through a long cable.

Summary

In summary, the 1990 synchronization work of Pecora and Carroll along the OGY control theory has opened the possibilities for a wide range of experimental exploration testing the use of nonlinear dynamical systems. Among the opportunities for practical applications presented by these two schemes reside cardiac controlling devices, secure communication electronics, and control devices for chaotically operating lasers. The development of synchronization by mutual coupling has also been well studied numerically and experimentally. The control scheme of time-delayed feedback is a technique which is proving viable for fast dynamical systems. Additionally, it may be adapted to an all optical method of feedback so as to control semiconductor lasers driven into a chaotic state by external feedback. The concept of synchronization by unidirectional coupling is a recent development which presents a viable method of synchronization for arrays of chaotic elements.
Figure 1.1: Chaos in the Lorentz attractor

The time evolution of a tight cluster of 10,000 points along the Lorentz attractor commences in (a). The bead rapidly stretches, (b) and (c), so that points are soon scattered across the entire attractor (d).
Figure 1.2: The Pecora and Carroll synchronization technique

A chaotic system $F$ is divided into a drive system $G$ and a response system $H$. An identical system to $H$, $H'$, is constructed. If the conditional Lyapunov exponents, derived from $H - H'$, are negative then $H'$ will synchronize to $H$ when both are driven by $G$. 
CHAPTER 2

THE DIODE RESONATOR

Varactor diodes when coupled with inductors were often used for such devices as television remote controls. These diodes are designed so that the capacitance which arises due to the depletion region of the pn-junction is related to the applied voltage. By applying a bias voltage to the diode, a tunable resonator could be created and the television controlled from a comfortable position. In this chapter, the diode resonator as a chaotic dynamical system is studied. This system exhibits such an extraordinary range of nonlinear phenomena, such as period doubling bifurcations [56], crisis [57], intermittency [58], hysteresis [59], and period bubbling [59, 60], that one can consider it as a paradigm for experimental nonlinear studies. Phenomena such as hysteresis and period bubbling which can be observed in the resonator can also be observed in doped fiber lasers driven chaotic by pump modulation [61] and semiconductor lasers [62]. Hence the richness of the physics of the circuit as well as its ease of experimental implementation presents an excellent choice for study. The organization of this chapter is as follows. In section II, a review is presented of the historical work performed on this system. In section III, an early formulation of the diode resonator is presented. The dynamical equations of the resonator used in the numerical work are derived and explained in section IV. In section V experimental results are then displayed to enhance the previously performed work and set up the synchronization experiments described in the subsequent chapters. Problems arising in the experimental implementation of the resonator are discussed in section VI. And a summary is presented in section VII.
The Diode Resonator

Our system of interest is the diode resonator which is composed of a 1N4004 silicon rectifier diode, a 33\,mH inductor (DC resistance 243\,\Omega), and a 90.5\,\Omega resistor in series. The circuit is sine wave driven at a frequency that can range from 20kHz to 300kHz. The diode and inductor give rise to the nonlinear resonator while the resistor influences the quality factor, \(Q\), of the resonator and provides a convenient method of measuring the chaotic current.

Though only composed of three elements, this physically simple circuit has a rich history of characterization. It was originally studied as a chaotic system by Linsay [56] in 1981 who put together a varactor diode, inductor and resistor. The varactor diode was chosen since it is designed to have a nonlinear voltage dependent capacitance. The diode resonator was designed to be a nonlinear damped driven oscillator. Linsay observed, in the frequency domain, the period doubling route into chaos. Recording the threshold voltages at which the period doublings occurred, Linsay was able to experimentally test the Feigenbaum universality hypothesis by measuring the number \(\delta\) at 4.5 ± 0.6. (The predicted value is 4.67.)

Subsequent studies of the circuit was performed by Testa, Perez, and Jeffries [63] who first obtained bifurcation diagrams of the current with respect to the drive voltage. In addition, this work featured a calculation of the Feigenbaum number \(\delta\) by directly measuring the width and location of the bifurcation created branches. Perez and Jeffries observed tangent bifurcations [58] and that the system can also undergo the intermittent route to chaos [64]. Rollins and Hunt [65] followed by Hilborn [57] studied the onset of crisis in the resonator as the chaotic attractor collided with the stable period-3 orbit. Extensive characterization of the resonator and resistively coupled resonators was performed by Buskirk and Jeffries [59] who modeled the circuit as a damped driven oscillator. They obtained and compared numerical and experimental Poincaré sections, bifurcation diagrams and return maps. Reverse bifurcations out of chaos as the drive frequency was increased were also observed. In addition to the
single resonator circuit, oscillators coupled through resistors have been characterized
[59, 66, 67]. However these investigations are outside the scope of this thesis.

The resonator can undergo bifurcations as a function of each of the drive parameters: amplitude, frequency, and wave bias. Shifting the DC offset of the drive wave upwards can cause bifurcations and reverse bifurcations [68]. Since the thermal drift of the charge carriers is, naturally, a function of the thermal voltage, $e/kT$, temperature dependent bifurcations have been shown [69].

An Early Formalism of the Resonator

To accompany the experimental work being performed on the diode resonator, Rollins and Hunt [70, 71] proposed a model of the resonator in which the system is modeled by exactly solvable equations when forward and reverse biased. They considered the nonlinear nature of the diode to arise from the fact that it continues to conduct for some time after the applied voltage is switched from positive (forward biased) to negative (reverse biased). This amount of time is known as the reverse recovery time for a diode. Depending on the diode, it can range from a few nanoseconds to tens of microseconds. Since their model has an analytical solution for the forward biased case and one for the reverse biased case, the nonlinear nature is introduced when incorporating reverse recovery current to match boundary solutions for the two regions.

The reverse recovery time, $\tau_r$, is exponentially dependent on the maximum forward current for previous cycles.

$$\tau_r = \tau_m [1 - \exp(-|I_m| I_c)]$$  \hspace{1cm} (2.1)

where $I_m$ is the maximum current of the previous cycle, $\tau_m$ and $I_c$ are parameters which describe the diode.
The equation governing the motion while the diode is forward biased is then

\[ I(t; A) = (V_0 Z_a) \cos(\omega t - \theta_a) + Ae^{-RL} + V_f R \]  

(2.2)

and

\[ V_d = -V_f \]  

(2.3)

where \( I(t; A) \) is the current, \( V_d \) is the voltage drop across the diode, \( Z_a^2 = R^2 + \omega^2 L^2 \), \( \theta_a = \arctan(\omega LR) \), \( R \) is the resistor, \( L \) is the inductor, \( \omega \) is the driving frequency, and \( A \) is a constant to be determined by the boundary conditions. In the reversed biased region

\[ I(t; B, \Phi) = (V_0 Z_b) \cos(\omega t - \theta_b) + Be^{-2RL} \cos(\omega_b t - \theta) \]  

(2.4)

and

\[ V_d(t; B, \Phi) = V_o \cos(\omega t) - I(t; B, \Phi) R - L \dot{I}(t; B, \Phi). \]  

(2.5)

Here \( B \) and \( \Phi \) are constants determined by the boundary conditions. In this equation, \( Z_b^2 = R^2 + (L\omega)^2(\omega^2 - \omega_b^2) \), \( \theta_b = \arctan[L(\omega^2 - \omega_b^2)R] \), \( \omega^2 = 1/LC \), and \( \omega_b^2 = \omega_0^2 - (R/2L)^2 \).

This formulation is successful in producing first return maps and bifurcation diagrams. One advantage of this model is that it successfully predicts the hysteresis seen in the diode. Rollins and Hunt proposed this model to counter the notion of chaos due to the nonlinearity of the capacitance. Instead, chaos is due solely to the reverse recovery time of the diode. Corroborating their hypothesis is the fact that one can't observe chaos in resonators incorporating fast switching diodes. It has been suggested [72] that the Rollins and Hunt model is more successful at modeling a varactor diode resonator instead of a rectifier diode resonator. The difference in the two is that the varactor diode is designed to function with only a space-charge capacitance while as pointed out below, the rectifier diode is dominated by the diffusion capacitance at
forward biases. This would mean that Rollins and Hunt underestimate the influence of capacitance in rectifier diodes. However, the gist of their proposal is valid. The time it takes for the chaos is a result of the diffusion and recombination time of the injected carriers. This time though is mediated through the resonance of the cavity.

In this thesis, though the validity of this formulation is not challenged, we instead prefer to construct the model of the resonator starting from Kirchoff's laws and utilizing the large amount of work which has been performed to model the diode [73]. The latter approach also predicts the observed nonlinear phenomena of the resonator and does not require discontinuous functions and their associated boundary matching problems.

Dynamical Equations of Motion

The dynamical system can be numerically modeled by an application of Kirchoff's laws. The inductor and resistor combination is straightforward while the rectifier diode can be represented as an ohmic resistance in series with a parallel nonlinear resistor and nonlinear capacitors [73, 74, 75]. Figure 2.1 shows the circuit along with the modeled version of the circuit. The voltage drop of the model circuit is given by:

\[ V_0 \sin \omega t = V_d + L \frac{dI}{dt} + I(R + R_s) \]  

(2.6)

where \( V_0 \) is the amplitude of the drive wave, \( V_d \) is the voltage drop across the diode, \( L \) is the inductance, \( I \) is the total current through the circuit, \( R \) is the inline resistance, and \( R_s \) a small added resistance to take into account an ohmic element to the diode. From the conservation of current,

\[ I = I_d + I_{cd} + I_{ct} \]  

(2.7)

where \( I_d \) is the current through the nonlinear resistor branch, \( I_{cd} \) and \( I_{ct} \) are currents in nonlinear capacitances described below. The capacitor currents can be expressed
The diode is represented by a nonlinear resistance in parallel with nonlinear space-charge and diffusion capacitances. The nonlinear resistance describes the familiar $IV$ curve of the diode. The capacitive terms arise when considering the effect of an AC signal on the injected charge carriers.

Figure 2.1: The diode resonator and its model
in terms of a differential equation for the voltage by:

\[
(C_d + C_t) \frac{dV_d}{dt} = I_{cd} + I_{ct}. \tag{2.8}
\]

Here \(C_d\) and \(C_t\) are nonlinear capacitances.

The nonlinear resistance describes the well known current-voltage characteristics of the diode and is given by the Shockley equation

\[
I_d = I_s[\exp(eV/nkT) - 1] \tag{2.9}
\]

where \(I_s\) is the reverse bias saturation current, \(e/kT\) is the thermal voltage, and \(n\) is an emission coefficient implemented to take into account carrier recombination in the depletion zone.

The capacitance terms are modeled from a consideration of the applied AC signal. Charge carrier recombination near the p-n junction creates a depletion region populated primarily by immobile charges. The fixed charges create a junction potential which, under no applied signal, induces a carrier drift that exactly balances their thermal diffusion. When a forward voltage is applied, charge is injected into the diode, builds up nearby the depletion region, and is moved into the depletion region thus shrinking its width. This variance of the depletion region due to the applied voltage causes a depletion (or space-charge) capacitance which can be thought of as a parallel plate capacitance with a voltage dependent width. While the diode is reversed biased, this term is given by:

\[
C_t = C_b(1 - \frac{V_d}{V_J})^{-m} \tag{2.10}
\]

where \(V_J\) is the junction potential, \(C_b\) is the zero voltage bias capacitance and \(m\), a grading coefficient, refers to the variation of the doping concentration across the p-n junction. Since this term would become infinite when \(V = V_J\), it is used whenever
$V_d < \frac{V_J}{2}$. A forward bias modification of the capacitance applied whenever $V_d \geq \frac{V_J}{2}$ is

$$C_t = C_0 \left( \frac{b_1 + \frac{nV_d}{V_J}}{b_2} \right)$$

(2.11)

where $b_1$ and $b_2$ are parameters to ensure continuity of the capacitance, $b_1 = \frac{1}{2}(1 - m)$ and $b_2 = 0.5^{1+m}$. The latter formula is derived from a curve fit to experimental data and is valid as long as the forward applied voltage is not excessively high.

In addition to the space-charge capacitance, a second capacitive effect arises when the finite response time of the mobile charge to the changing field is considered. Since the diode cannot respond immediately to injected charge, a charge build up occurs in the vicinity of the depletion region. The charge build up, $Q$, is proportional to the injected current, $I$

$$Q = \tau I$$

(2.12)

where the proportionality factor, $\tau$, describes the amount of time it takes the majority carriers to cross the p-type or n-type material. The variance of this charge with respect to the voltage defines this capacitance, viz.

$$C_d = C_0 e^{\frac{V_d}{n k T}}.$$ 

(2.13)

With the following transformations: $I = \frac{\omega d I}{V_j}, \nu = \frac{V_d}{V_j}, V_1 = \frac{V_d}{V_j}, \beta = \frac{\omega L}{R + R_s}, \alpha = \frac{\varepsilon V_J}{n k T}$, $\gamma = \frac{\beta I_d (R + R_s)}{V_j}$, $c_1 = (R + R_s) \beta \omega C_0$, $c_2 = \frac{C_b}{C_0}$, and $\tau = \omega t$, the governing differential equations are transformed into the dynamical equations:

$$\frac{d\nu}{d\tau} + \frac{\nu}{\beta} - \nu = V_1 \sin(\tau)$$

(2.14)

$$\frac{d\nu}{d\tau} = G(I, \nu)$$

(2.15)
where

\[ G(I, V) = \frac{I - \gamma(e^{\alpha V} - 1)}{c_1[e^{\alpha V} + c_2 (1 - V)^{-m}]} \quad \text{if} \ V < \frac{1}{2} \]  

(2.16)

\[ G(I, V) = \frac{I - \gamma(e^{\alpha V} - 1)}{c_1[e^{\alpha V} + c_2 (b_2 + mV)^{-b_1}]} \quad \text{if} \ V \geq \frac{1}{2}. \]  

(2.17)

For our particular system, the constants are: \( V_J = 0.34V, I_s = 5.86 \times 10^{-6}A, \) \( n = 1.7, R_s = 0.0422\Omega, C_b = 52.1pF, m = 0.38, C_o = 668pF, \beta = 43.51, \alpha = 7.692, \) \( \gamma = 0.2574, b_1 = 0.38422, b_2 = 0.31, c_1 = 4.26, \) and \( c_2 = 0.078. \) The parameters \( \alpha, \beta, \gamma, \) and \( c_1 \) are dependent on the resistor and inductor chosen for the resonator as well as the driving frequency. The other parameters characterize the 1N4004 diode. An interesting feature to point out is the relationship between \( C_o \) and \( C_b. \) For forward biases, \( C_o, \) the zero bias diffusion capacitance dominates over the space-charge capacitance. Since it takes a relatively large amount of time for the injected charges to diffuse across the diode, quite a large amount of charge can build up. In the reversed biased region, the diffusion capacitance is damped out exponentially and the space-charge capacitance becomes the dominant term.

An intuitive look at the system can be gained by considering the processes involved. Majority charge carriers injected into the forward biased diode drift and diffuse across their respective p-type or n-type regions in a transit time characteristic of the particular diode before subsequent recombination in the depletion region. When the diode is switched from forward to reversed bias, this current persists for a time which is quantified by the reverse recovery time. This response time for carrier buildup, motion, and recombination, similar to the nonlinearly damped oscillator, tends to oppose the natural frequency of the driven resonator which itself is nonlinear due to the capacitances. As long as the driving voltage is low, an equilibrium situation concerning the carriers can be reached for each cycle. But as the voltage is increased and more charges injected, the equilibrium is upset and the carriers respond chaotically to the driving field. If instead of varying the drive voltage, the frequency of oscillation is increased, a similar situation occurs. At low frequencies, there is
ample time for equilibrium to be re-established for each cycle. As the frequency is increased the reduction in time destabilizes the equilibrium since the carrier motion does not respond sufficiently rapidly. The result is a bifurcation into chaos with respect to the frequency. If the frequency is increased high enough, the detuning between the drive frequency and the natural resonance frequency essentially damp any net drift of the carriers and the charges essentially vibrate in place. An equilibrium is re-established since only a negligible amount of charge is being injected into the diode. Hence, one can observe reverse bifurcations occurring in the resonator as the frequency is increased [59]. For our particular system, bifurcations no longer occur when the frequency is neither below $20\,kHz$ nor above $280\,kHz$.

The resonator also exhibits hysteresis. The location of bifurcation points is dependent on the direction of change of the varying parameter. The diode seems to possess a memory which in fact is due to the finite response time of the charge carriers.

The model shows qualitatively equivalent agreement to the experimental circuit. However, the behavior of the physical diodes varies widely from one to another in spite of all being the 1N4004 type. This is due to the extreme sensitivity of the diode to variations in the doping concentrations across the $p$-type and $n$-type substrate. For this reason, quantitative agreement between numerically predicted values (e.g. bifurcation points) and that observed are not likely.

Measurements of Chaos in the Diode Resonator

In this section, experimental observations are recorded so as to characterize the circuit. Figures 2.2, 2.4, and 2.6 plot experimentally obtained bifurcation diagrams as a function of various parameters. These were obtained in the following manner. The diode resonator was constructed and driven using an HP3325A programmable waveform generator which was buffered from the circuit by a LH0002 driver. This particular driver is a high current low impedance (6Ω) device and was used to prevent feedback into the higher impedance waveform generator. The voltage across the
resistor was measured and amplified by a Stanford Research Systems preamplifier. The output of the preamplifier was recorded by a Tektronix RTD710A 10 bit vertical resolution digitizer whose sampling interval was set at 100ns. (See Appendix A.1 for details.)

Figure 2.2(a) is the familiar voltage bifurcation diagram. The peaks of the resonator signal versus the amplitude of the sinusoidal drive wave are plotted. Figure 2.2(b) is a close up of the period doubling cascades into chaos. The numerical equivalent to Fig. 2.2 is shown in Fig. 2.3.

Figure 2.4 is a bifurcation diagram where the drive wave amplitude is constant while the drive frequency is varied. As before, the vertical axis plots the resonator signal peaks. The resonator exhibits both bifurcations and reverse bifurcations as the detuning between the natural resonator frequency and the driving frequency shifts from largely positive to largely negative. Figure 2.6 shows bifurcations as a result of varying the DC offset of the drive signal while keeping both the frequency and the amplitude constant. If the drive signal is biased too low, an equilibrium is established since the diode is mostly reversed biased and being injected by the carriers for only a small amount of time. On the other hand, when the signal is too high, the diode is mostly forward biased and the current is mostly continuous. This figure shows that the forward and reverse bias switching of the diode is crucial to the observation of chaos.

Figure 2.5 shows a first return map obtained by recording a time series of the voltage drop across the master resistor then extracting the values of the peak voltages. The graphs plot the \( n \)th peak versus the \( n+1 \)th peak for a 4.6V amplitude of the drive wave.

The Lyapunov exponents were calculated for experimentally obtained data and using the numerical model. For an experimental estimation, two different codes were used. A description of these codes are found in Appendix A.2 and A.3. One code, developed by the Institute of Nonlinear Science at the University of California San
Figure 2.2: Bifurcation diagram as a function of the drive wave amplitude

(a) An experimental bifurcation diagram obtained by recording the peaks of the voltage drop across the resonator resistor element versus the amplitude of the driving sine wave. (b) A close up of (a) from period 2 into chaos shows the creation of periods 8 and 16.
Figure 2.3: Numerical bifurcation diagram vs. the drive wave amplitude

This diagram is in good agreement with that observed experimentally. However, at the region of the T3-T6 bifurcation, transients remain which give the false impression of complex structure.
By keeping the drive amplitude constant at 9V and increasing the frequency period doubling bifurcations and reverse bifurcations occur in the physical circuit. At too low a frequency carriers have time to diffuse across the diode and no chaos is observed. At too high a frequency the carriers essentially oscillate in place.
Figure 2.5: First return map for the resonator

The graph plots the resistor component peak $n$ (abscissa) vs. peak $n+1$ (ordinate) when the resonator is operating chaotically below the period-3 window. The thinness of the trace indicates a 1-dimensional structure.
Diego [76], computes the Lyapunov spectrum. Using the same time series that produced Fig. 2.5, typical values obtained are $\lambda_1 = 0.16$, $\lambda_2 = -0.1$, and $\lambda_3 = -4.0$. This code, unfortunately, is somewhat unstable (see Appendix A.2). A code produced by Wolf [77] estimates only the largest Lyapunov exponent by observing divergences in nearby trajectories of a reconstructed attractor from the time series. Using the same time series as the above, $\lambda_1 = 0.03$. The largest and least exponent (the third being 0), obtained with the model, are shown in Fig. 2.7(a) for a range of drive voltages. These were calculated using an algorithm which can be found in Ref.[78]. In correspondence with the bifurcation diagram Fig. 2.3, they show the ascent into chaos along with various windows of stability. An estimate for the fractal dimension of the attractor can be estimated using the Kaplan-Yorke conjecture [79]

$$d = j + \frac{\sum_{i=1}^{j} \lambda_i}{-\lambda_{i+1}}$$  \hspace{1cm} (2.18)

where $j$ is the number of nonegative Lyapunov exponents. Since $\lambda_2 = 0$ for nonautonomous systems,

$$d = 2 + \frac{\lambda_1}{|\lambda_3|}.$$  \hspace{1cm} (2.19)

This estimate is shown in Fig. 2.7(b). By this measure the attractor dimensionality ranges up to 2.18 in the chaotic regions. However the dimensionless time, $\tau$, does not contribute to the expansion or contraction of the phase space volume. Then in the subspace spanned by the circuit current and the diode voltage drop the dimensionality is reduced by one to a maximum of 1.18.

Furthermore the diode resonator due to the component resistances can be regarded as a highly damped oscillator and can be approximated by a one-dimensional map [59, 72].

Since it is often desired to reconstruct the attractor from a time series by means of the delay coordinate embedding technique proposed by Takens [80], one needs a
The DC offset, $\Delta$, of the wave ($6V \sin \omega t + \Delta$) is varied from -1V to 0.4V. Period bubbling shows the critical dependence of the chaos on the wave bias.
Figure 2.7: Lyapunov exponents and Kaplan-Yorke dimensionality

(a) The largest and least Lyapunov exponents are numerically calculated as a function of the wave amplitude. The largest exponent shows the ascent into chaos as well as the stable windows. (b) The Kaplan-Yorke conjecture for the dimensionality of the attractor. The calculation is made using the Lyapunov exponents displayed in (a).
proper choice for the time interval between measurements of the system. It has been suggested that the mutual information function can be used to forecast a proper time delay [81]. In short the mutual information function is an indication of information that one knows about point \( x(t + T) \) given point \( x(t) \) (see Appendix A.6). The idea of the delay coordinate reconstruction is that one wants to choose \( T \) so that the relationship between components of the delay vector are minimal without being random. This value is given by the first minimum of the mutual information function. Figure 2.8 shows the mutual information function for delays up to 60\( \mu s \) [82]. It is seen that the first minimum occurs when \( T = 4.5\mu s \).

In order to successfully apply the synchronization scheme of Lai and Grebogi we desire to determine the orientation of the unstable direction of the attractor. In the phase space of the resonator, focus on a two dimensional subspace, i.e. a Poincaré surface of section, by fixing the drive wave phase at \( \frac{\pi}{2} \) which is the drive peaks. In this plane the diode voltage drop is almost constant at about 0.7V. This is due to the fact that at the drive peaks, the diode is forward biased and in a conducting state. On the other hand, the values of the current lie scattered since one is observing the chaotic peaks of the voltage drop across the resistor. From this intuitive examination, it is straightforward that in this particular plane the unstable direction always lies along the circuit current axis.

To determine the unstable direction numerically one can proceed in a manner similar to that defined in [5, 83]. The idea is to observe how an \( n \)-dimensional ball of radius \( \epsilon \) centered on the trajectory deforms as it is propagated along the orbit. One can determine the unstable direction by propagating the ball forward in time and determining the major axis as the ball deforms. Alternatively the stable direction can be obtained by propagating the ball backwards in time and again determining the major axis of the ellipsoid. In our case, it is sufficient to define two dimensional vectors \( \hat{e}(I, V) \) where the first component is along the circuit current (or voltage across the resistor) axis and the second component is along the diode voltage drop.
Figure 2.8: The mutual information

Mutual information plotted as a function of the time delay. The first minimum occurs when $T = 4.5\mu s$. This time is proposed to be the optimum choice for construction of a delay-coordinate embedding of the attractor.
axis. Define unit vectors \( \hat{e}_1 = (1,0) \) and \( \hat{e}_2 = (0,1) \). By propagating these vectors forward in time using Eqtns. 2.14, 2.15, 2.16, 2.17 from one peak to the next, they will rotate into the unstable direction due to compression along the stable direction and expansion along the unstable one. Figure 2.9 shows the evolution of the first component of \( \hat{e}_1 \) versus the peaks of the circuit current. While the direction switches signs, its magnitude remains almost as defined initially signifying that the unstable direction is along the circuit current axis. The second component of \( \hat{e}_1 \), not shown, remains at zero.

Using the Diode Resonator

In a synchronization experiment, the components should usually be identically matched [84]. In practice, this becomes unlikely due to variations of their properties. The primary culprits are the diodes whose characteristics vary due to different doping concentrations in the p-type and n-type material. While it is possible to purchase special matched diodes, these diodes are always fast switching diodes and do not exhibit chaos under the circumstances we desire. Thus a technique must be devised so as to speedily match diodes. Using an inductor and a resistor as a standard, diodes can be reasonably well matched by observing the variances in bifurcation points. The resonator displays a well defined bifurcation from period 2 to period 4 and a chaos to a period 3 window crisis as the drive voltage is increased. We use these two points to match the diodes since the transitions will occur at different voltage levels for diodes with different properties. It is important to keep the temperature constant in order to characterize diodes in this manner since the diode capacitances are a function of the thermal voltage, \( e/kT \). From characterizing diodes in this manner, one finds that the bifurcation points can range from \( 1V \) to \( 4V \) for a period 1 to period 2 bifurcation and from \( 4V \) to over \( 10V \) for a chaos to period 3 transition. Naturally we chose the two diodes which proved to be the most similar in characteristics from the ones tested. Even then, daily temperature fluctuations in the laboratory induced changes in the
The unstable direction can be determined by propagating a unit vector forward in time and observing its rotation. This unit vector is originally defined as \((1,0)\) where the first component lies along the circuit current axis and the second is along the diode voltage drop axis. The plot is of the first component vs. the current peaks. While the sign switches, the magnitude remains fixed at one indicating that the unstable direction at the current peaks always remains along the current axis.
resonators. The solution is to thermally couple the diodes to a conductor and hold the temperature constant by means of a bath.

Summary

In this chapter the diode resonator was introduced as an experimentally accessible chaotic system that demonstrates a wide range of nonlinear phenomena. In addition to the period doubling route into chaos via the drive amplitude, this system also demonstrates bifurcations and reverse bifurcations as the drive frequency is increased. It also displays period bubbling as the DC offset of the drive wave is varied. An important point to be stressed is that the resonator can be considered to have a 1-dimensional attractor. This can be shown by observing the Kaplan-Yorke conjecture for the dimensionality as well as a correlation dimension calculation. Also note that the features of the resonator, bifurcation points, peak current, etc., are strongly dependent on the doping process used to create the pn-junction.
CHAPTER 3

SYNCHRONIZATION USING CONTROL

Introduction

One of the most influential papers to be published in the discipline of nonlinear dynamics is the Controlling Chaos paper of Ott, Grebogi, and Yorke (OGY) [2]. This work successfully introduced the process involved in steering an existing trajectory onto a desired unstable periodic orbit by means of minute perturbations to an existing parameter of the system. The natural extension of the technique is apply perturbations to a system parameter so as to steer an existing trajectory onto a second chaotic orbit. This process of synchronization by control was proposed by Lai and Grebogi [5]. In section II, the OGY theory is presented. To obtain a rudimentary sense of the technique, the process is applied to control the period-1 orbit of the logistic map. From OGY, the synchronization algorithm of Lai and Grebogi, presented in section III, can be rapidly understood. Again as an exercise, two logistic maps are synchronized. The review of the schemes serves as an introduction to the problem of synchronization of the diode resonator. In section IV, following the prescription for the synchronization factor, it is shown how this factor can be derived using an experimentally obtained time series from the diode resonator. An important observation which is made is that stabilizing feedback need only to be made when the diode resonator is operating in an unstable region of the attractor. In the stable region of the attractor orbits do not diverge exponentially and the synchronizing factor is inconsequential. Furthermore, in the stable region of the attractor the prescribed feedback factor is a constant value which amplifies the difference between the master and slave orbits. Therefore the
experimental implementation of a constant multiplication factor which was originally considered as an experimental simplification of the peak dependent synchronizing factor is in fact the prescribed factor while the resonator is operating in the unstable regime of the attractor. A quantitative measure of the ability of synchronization can be found in the local Lyapunov multipliers. These are calculated using the numerical model and show how different feedback factors affect synchronization.

Ott, Grebogi, Yorke Revisited

The idea of OGY is that because there are an infinite number of unstable periodic orbits (UPO) buried densely in the attractor any one of these can be controlled by means of small steering perturbations. The basic idea of the OGY method, seen graphically in Fig. 3.1 is as follows. First, of the experimentally feasible UPOs embedded in the phase space, one chooses an UPO which yields the desired system performance. Second, define a small region around the desired periodic orbit. Due to the ergodic nature of the attractor, eventually the trajectory falls into this small region. When this occurs, small judiciously chosen temporal parameter perturbations are applied to force the trajectory to approach the desired UPO. In principle, detailed knowledge of the dynamical system is not required. However, a precisely measured time series of an observable scalar or vector quantity is necessary so as to calculate the Lyapunov exponents, estimate the dimensionality of the attractor, extract the unstable periodic orbits and determine other structural information about the chaotic system.

The OGY formalism commences by considering a chaotic map

$$\vec{x}_{n+1}(p) = F(\vec{x}_n(p),p)$$

with fixed point

$$\vec{x}_f = F(\vec{x}_f,p_0).$$

Here $p$ represents an existing parameter of the system to which the controlling per-
Figure 3.1: The Ott, Grebogi, and Yorke technique

(a) A point on a Poincaré surface of section, $y_n$, may approach the fixed point, $y_f(p_0)$, but will be driven off in the unstable direction. (b) The application of a minute perturbation causes a shifting of the fixed point along with the stable and unstable directions. (c) With the correct shift, the iterate, $y_{n+1}$, has no component along the unstable direction and moves toward the fixed point along the stable direction.
turbation is to be applied. Note that the fixed point depends on \( p \) and \( \bar{x}_f = \bar{x}_f(p) \). To determine the necessary steering influence on this system, one requires a sufficiently long time series. From the obtained time series first reconstruct the attractor using the delay coordinate technique proposed by Takens [80] (See Appendix A.6). From the reconstructed attractor, extract a Poincaré surface of section. This surface of section is pierced by the crossings of the orbit trajectory. Designate these points by: \( \bar{y}_1, \bar{y}_2, \ldots \bar{y}_n \), and \( \bar{y}_f \) where \( \bar{y}_f \) indicates the desired fixed point. Additionally from the attractor obtain an experimental estimate of the Jacobian matrix. Once this matrix is obtained, the unstable eigenvalues of the system as well as the unstable directions can be determined. Ultimately we are interested in applying the algorithm to the diode resonator. Therefore consider there to be single unstable direction, designated as \( \hat{i}_u \), and a single stable direction, \( \hat{i}_s \). Look at the difference

\[
\bar{x}_{n+1} - \bar{x}_f = F(\bar{x}_n(p), p) - F(\bar{x}_f, p_0).
\]  

(3.3)

Linearizing

\[
\bar{x}_{n+1} - \bar{x}_f = D_x F(\bar{x}_n(p), p) \cdot (\bar{x}_n - \bar{x}_f) + D_p F(\bar{x}_n(p), p) \cdot \Delta p.
\]  

(3.4)

Here \( D_x F \) is the Jacobian matrix and \( D_p F \) is the response of the system to a variation of the perturbed parameter \( p \). The derivatives are evaluated at the desired fixed points. One of the key points to control is to determine how a variation in the perturbing parameter shifts the fixed point. The shift of the fixed point is

\[
\frac{\partial \bar{x}_f}{\partial p} = \frac{\partial}{\partial p} F(\bar{x}_f(p), p) = D_x F \frac{\partial \bar{x}_f}{\partial p} + D_p F
\]  

(3.5)

Rearranging

\[
D_p F = (1 - D_x F) \frac{\partial \bar{x}_f}{\partial p}.
\]  

(3.6)
If Eq. 3.6 is inserted into Eq. 3.4

\[ \tilde{x}_{n+1} - \tilde{x}_f = D_xF(\tilde{x}_n(p), p) \cdot (\tilde{x}_n - \tilde{x}_f) + (1 - D_xF) \frac{\partial \tilde{x}_f}{\partial p} \cdot \Delta p \]  

(3.7)

\[ \bar{x}_{n+1} - \bar{x}_f = D_xF \cdot [(\bar{x}_n - \bar{x}_f) - \frac{\partial \bar{x}_f}{\partial p} \cdot \Delta p] + \frac{\partial \bar{x}_f}{\partial p} \Delta p. \]  

(3.8)

The Jacobian matrix can be expressed as a diagonal matrix so as to isolate the stable and unstable eigenvalues. Then designate \( \hat{i}_u \) and \( \hat{i}_s \) as the eigenvectors pointing along the unstable and stable directions respectively. Define \( \bar{f}_u \) as a contravariant basis vector which points along the unstable direction and \( \bar{f}_s \) as a second contravariant basis vector which points along the stable direction. In this space, \( \bar{f}_k \cdot \hat{i}_j = \delta_{jk} \) where \( \delta_{jk} \) is the Kronecker delta. The diagonal Jacobian matrix can be expressed as

\[ D_xF = \lambda_u \hat{i}_u \bar{f}_u + \lambda_s \hat{i}_s \bar{f}_s \]  

(3.9)

where \( \lambda_u \) and \( \lambda_s \) are eigenvalues with \( |\lambda_u| > 1 > |\lambda_s| \). The perturbation must be made so that the orbit is nudged onto the fixed point. This implies that the difference between the fixed point and the orbit point along the unstable direction be zero. Thus require that

\[ \bar{f}_u \cdot (\bar{x}_{n+1} - \bar{x}_f) = 0. \]  

(3.10)

Then using the orthogonality relations

\[ 0 = \lambda_u \bar{f}_u \cdot (\bar{x}_n - \bar{x}_f) + (1 - \lambda_u) \bar{f}_u \cdot \frac{\partial \bar{x}_f}{\partial p} \Delta p. \]  

(3.11)

Finally, solving for the perturbation gives

\[ \Delta p = \frac{\lambda_u \bar{f}_u \cdot (\bar{x}_n - \bar{x}_f)}{\lambda_u - 1} \frac{\bar{f}_u \cdot \partial \bar{x}_f / \partial p}{\bar{f}_u \cdot \partial \bar{x}_f / \partial p}. \]  

(3.12)

This perturbation is applied only when \( x_n \) is within a reasonably small distance
from $x_f$ Experimentally, one must determine $\lambda_u, \tilde{f}_x$, and the shift of the fixed point with respect to the perturbation. This is accomplished from the delay coordinate reconstruction of the attractor.

Stabilizing the Period 1 Orbit in the Logistic Map

A simple but instructional example of both the Ott, Grebogi, and Yorke control algorithm for a 1-dimensional system can be observed from controlling the period 1 orbit in the logistic map. This map is given by

$$x_{n+1} = F(x_n, \alpha) = \alpha x_n(1 - x_n). \quad (3.13)$$

The factor, $\alpha$, is taken to be the parameter of the system to perturb. Desiring to control a period one orbit, solve for the fixed point viz.

$$x_f = \alpha x_f(1 - x_f) \quad (3.14)$$

which leads to

$$x_f = 1 - \frac{1}{\alpha}. \quad (3.15)$$

We need to determine the unstable eigenvalue of the linear approximation of the map by expanding the map about the fixed point. This is given by

$$D_x F(x, \alpha) = \alpha(1 - 2x_n) \quad (3.16)$$

where the Jacobian is evaluated at the fixed point. At the fixed point, the eigenvalue is simply $\lambda_u = \alpha(1 - 2x_f) = 2 - \alpha$. We next determine how the mapping shifts with respect to $\alpha$. From Eq. 3.6 we can express this as

$$\frac{\partial x_f(\alpha)}{\partial \alpha} = \frac{D_\alpha F}{1 - D_x F}, \quad (3.17)$$
For our case,

$$D_\alpha F = \frac{\partial}{\partial \alpha} F(x_n, \alpha)|_{x=x_f} = x_f(1-x_f)$$

(3.18)

and $D_\alpha F$ is given by Eq. 3.16. Since this is a 1-dimensional mapping, all the above terms are scalars and the eigenvector, $\tilde{f}_u$, never enters into the calculation. Therefore the required perturbation that must be made to $\alpha$ is

$$\Delta p_n = \left( \frac{\lambda_u}{\lambda_u - 1} \right) \left( \frac{1 - \alpha + 2\alpha x_f}{x_f(1-x_f)} \right) (x_n - x_f).$$

(3.19)

If we take $\alpha = 4$, then $x_f = 3/4$ and $\lambda_u = -2$. The required perturbation, Eq. 3.19, becomes $\Delta p_n = 32(x_n - x_f)/3$. One of the points to stress about the OGY technique is that the perturbations to the parameter should be small so as to truly extract the UPOs that are buried in the attractor. Therefore limit the size of the perturbation to only a small percentage of the target parameter. If limited to being less than 10% of $\alpha$, i.e. $\Delta p_n < 0.0375$, the period-1 orbit can be rapidly stabilized within 20 or so iterations.

Lai and Grebogi Synchronization Scheme

The natural extension of chaos control theory is to replace the periodic orbit’s fixed point by a chaotic orbit and determine whether an effective perturbation could be applied to an identical chaotic system in order to stabilize that orbit about the first. This algorithm was proposed by Lai and Grebogi [5] in April 1993 and tested numerically in the Henon map.

To apply the Lai and Grebogi algorithm, one must characterize the behavior of the two systems that we desire to synchronize about their shared attractor. Our particular interest is to merely observe one of the systems’ phase space orbit while to the second slave orbit both observe it and apply some stabilizing feedback to an existing parameter of the slave system in a manner similar to that of the OGY theory. Figure 3.2 is a graphical description of the procedure. To determine the necessary
feedback into the slave system, examine the master orbit and the contraction or divergence of slave orbits which are nearby the identical master orbit. Along each point in the trajectory of the chaotic master orbit identify the stable and unstable manifolds. Adjacent points to the orbit will be attracted to the orbit in the stable directions while being repelled along the unstable direction. As the orbit wanders along the attractor, both stable and unstable directions rotate in some manner governed by the underlying dynamics of the system. Our goal is to make a small perturbation to the slave orbit at point $n$ so that at point $n+1$ there is no difference between the two orbits along the unstable direction. The primary difference between synchronization and control is found in the following point. For OGY, all that was necessary was to determine the unstable directions at the fixed point of the return map. However, for synchronization we must know the unstable direction at each point along the attractor. Once this has been ascertained, the required feedback can be computed by evaluating the effect that small changes to the designated parameter of the slave system have on the slave orbit. With the correcting perturbation, the slave orbit converges to the master along the stable directions. The feedback is effectively applied as long as the it modulates the chosen parameter by only a few percent. Beyond this range, no feedback is applied.

Begin with a master map

$$\tilde{x}_{n+1} = F(\tilde{x}_n, p_o) \tag{3.20}$$

where $p_o$ is a constant parameter. The slave map is

$$\tilde{y}_{n+1} = F(\tilde{y}_n, p) \tag{3.21}$$

where now $p$ is variable. We desire to observe the error signal given by

$$\tilde{y}_{n+1} - \tilde{x}_{n+1} = F(\tilde{y}_n, p) - F(\tilde{x}_n, p_o). \tag{3.22}$$
Linearizing this difference as in OGY gives a similar result

\[
\ddot{y}_{n+1} - \ddot{x}_{n+1} = D_{\dot{y}}F(\ddot{y}_n, p) \cdot (\ddot{y}_n - \ddot{x}_n) + D_xF(\bar{y}_n, p) \cdot \Delta p_n. \quad (3.23)
\]

Here \( D_{\dot{y}}F \) is the Jacobian matrix evaluated at \( \ddot{y} = \ddot{x} \) and \( p = p_0 \); \( D_xF \) is the shift of the mapping with respect to the parameter to be perturbed and is also to be evaluated at \( \ddot{y} = \ddot{x} \) and \( p = p_0 \). As in OGY, define the unstable and stable directions by \( \dot{i}_{u(n)} \) and \( \dot{i}_{s(n)} \). And define basis vectors by \( \bar{f}_u(n) \) and \( \bar{f}_s(n) \) where \( \bar{f}_j(n) \cdot \dot{i}_{k(n)} = \delta_{jk} \).

The requirement of the difference at \( n + 1 \) being 0 along the unstable direction leads to:

\[
(\ddot{y}_{n+1} - \ddot{x}_{n+1}) \cdot \bar{f}_{u(n+1)} = 0. \quad (3.24)
\]

Substitution and algebra produce the perturbation factor [5],

\[
\Delta p_n = \frac{D_{\dot{y}}F(\ddot{y}_n, p) \cdot (\ddot{y}_n - \ddot{x}_n) \cdot \bar{f}_u(n+1)}{-D_xF(\bar{y}_n, p) \cdot \bar{f}_u(n+1)}. \quad (3.25)
\]

The physical interpretation of the equation is as follows. The perturbation factor is naturally related to the proximity of the slave orbit to the master. To this separation, the strength of the amplification factor is directly dependent on the Jacobian matrix. This matrix essentially tells us just how strong the unstable direction is. Hence the greater the unstable eigenvalues, the greater the feedback perturbation must be. The denominator is the measure of how sensitive the mapping is to minute changes in the parameter to be perturbed. In the event that the mapping is very sensitive this shift is large and so the resulting perturbation must be relatively small. On the other hand, if the mapping is insensitive to the perturbation, then the denominator is small and so the perturbation must be substantial. In principle the perturbation is to be applied whenever it is a small fraction of the unperturbed parameter. Hence, one should not apply the feedback if the separation between the two orbits is large or the denominator of Eq. 3.25 is small. The former case is bound to happen due to the
ergodicity of chaotic trajectories. The latter case is curious in that it states that there may be parts of the attractor in which one is unable to activate the control scheme since the perturbation factor could become infinite.

Synchronizing a Pair of Logistic Maps

Apply this control method to synchronize a pair of logistic maps. Consider Eq. 3.14, leaving $\alpha$ as a constant, as the master equation. Define a slave equation

$$y_{n+1} = F(y_n, p) = py_n(1 - y_n)$$

(3.26)

where $p$ is the parameter we wish to perturb so as to synchronize $y_n$ about $x_n$. The Jacobian of the slave map evaluated on the master is

$$D_y F(y, p) |_{y=x, p=\alpha} = \alpha(1 - 2x_n)$$

(3.27)

while the shift, identically as in the control scheme is

$$D_p F(y, p) |_{y=x, p=\alpha} = x_n(1 - x_n).$$

(3.28)

Hence the perturbation, $\Delta p_n$, that must be made to the parameter $p$ in the slave equation is given by:

$$\Delta p_n = \frac{\alpha(1 - 2x_n)}{x_n(1 - x_n)}(y_n - x_n).$$

(3.29)

The perturbation is applied only if it is small enough so as not to knock the map out of its domain. For this to occur, either $x_n$ is close to $y_n$, and $x_n$ is neither close to zero nor close to 1. The above algorithm is easily coded and the rapidity of synchronization depends on the size of $\Delta p$ allowed to perturb the slave system.
Synchronizing the Diode Resonator

The goal is to apply the synchronization algorithm to the case of two diode resonators which are operating chaotically and independently (excluding the required feedback). We do require that both resonators be driven in phase and with the same amplitude and DC bias. In order to achieve synchronization, the perturbation algorithm must be adapted from the above mapping to the dynamical system. From the two chaotic dynamical variables, current ($I$) and voltage drop across the diode ($V_d$), select the current through the resonator as the observable signal. This we physically measure as the voltage drop across the resistor element of the resonator. Ohm’s law provides the linear relationship: $I^M(t) = V^M(t)/R$ where the superscript $M$ is introduced to distinguish the master diode resonator from the slave (superscript $S$). From a recorded time series of $V^M(t)$ we generate the delay space reconstruction of the attractor, determine the Jacobian matrix elements, and compute the shift of the attractor due to variations in the parameter to be perturbed. The iterative feedback factor, Eq. 3.25, derived for a map is translated to the case of a dynamical system by the following recipe. At some point in the drive cycle, designated as $t_p$, measure both the master, $V^M(t_p)$ and slave $V^S(t_p)$ signals. From $V^M(t_p)$ calculate the prescribed amplification factor as well as the difference $V^S(t_p) - V^M(t_p)$. Form the product of the two above factors and modulate the chosen system parameter for all or part of the drive period. The process is to be repeated for each cycle. Since the amplitude chaotic resonator is driven with a defined frequency, it is logical to evaluate the feedback at the peaks of the current and modulate the chosen parameter for some fraction of the known period. Thus we rename the iterative dependent factor as a peak dependent factor so that

$$\Delta p_n \rightarrow \Delta p[t_p(n)] \quad (3.30)$$

where $t_p(n)$ refers to the time of occurrence of the $n$th peak. Of the possible slave resonator parameters to apply the necessary perturbation, the drive wave amplitude
presents the easiest parameter to modulate as a voltage controlled amplitude (VCA) input is an often available feature of waveform generators. Therefore, when discussing the perturbation, \( \Delta p \), we are referring to a shift of the amplitude voltage. The result is that the driving voltage for slave resonator which appears in Eq. 2.10 is replaced by

\[
V_0 \sin \omega t \rightarrow V_0 + \Delta p[t_p(n)] \sin \omega t \quad \text{for} \quad t_p(n) < t < t_d
\]

\[
\rightarrow V_0 \sin \omega t \quad \text{for} \quad t_d < t < t_p(n + 1). \tag{3.31}
\]

The feedback is to be applied from \( t_p(n) \) up to the point, \( t_d \), which can vary from \( t_p(n) \) to the next drive wave peak \( t_p(n + 1) \). The differential equation concerning the voltage drop across the resonator, Eq. 3.1, becomes

\[
\{V_0 + \Delta p[t_p(n)]\} \sin \omega t = \frac{dI}{dt} + V_d + I(R + R_s). \tag{3.32}
\]

where \( V_0 \) refers to the amplitude of the drive wave, \( V_d \) is the voltage drop across the diode, \( R \) is the inline resistor and \( R_s \) is the ohmic resistance of the diode.

In terms of the physical variables of the diode \( \bar{\mathbf{x}} = (V^M, V^M_d), \bar{\mathbf{y}} = (V^S, V^S_d), F(\bar{\mathbf{y}}, p) \rightarrow F(V^S, V^S_d, p) \) where \( F \) refers to the governing dynamical equations described in chapter 2, and \( \tilde{f}_{u(n)} \) remains as the notation for unstable eigenvector.

The Peak Dependent Feedback Factor

The next step is to determine the desired feedback \( \Delta p[t_p(n)] \). From the Kaplan-Yorke dimensionality conjecture (see chapter 2), the resonator can be considered as nearly one dimensional. Furthermore, at the current peaks the unstable direction always lies along the current axis (See Fig. 2.9). Thus the unstable eigenvector, \( \tilde{f}_{u(n)} \), is constantly oriented along the current axis. Then the peak dependent formula reduces to a 1 dimensional calculation. Next the voltage drop across the diode, \( V_d \), for both master and slave resonator is virtually identical at 0.7V on the current peaks since
the diodes are forward biased. Hence $V_d^M - V_d^S$ is considered negligible with respect to the current. We therefore need only the difference in the voltage drops across the resistors, $V^M(t) - V^S(t)$ to calculate the perturbation. Applying these conclusions to the feedback formula yields a much simpler factor to calculate. The unstable eigenvector, $f_u$, cancels in the numerator and denominator. The Jacobian matrix becomes a scalar which physically tells us how the 1 dimensional system changes with respect to a variation in the current. The shift of the map with respect to a variation in the drive amplitude is also one dimensional. By combining the above, the feedback factor becomes

$$\Delta p = -\left. \frac{\partial F_1}{\partial V^M} \frac{\partial F_1}{\partial p} \right| [V^M(t_p) - V^S(t_p)].$$ (3.33)

In the above, $F_1$ refers to the 1 dimensional Jacobian term to be calculated below.

We have also taken advantage of Ohm's law to recast the dynamical current $I^{M(S)}$ as $V^{M(S)}$.

Synchronization by Occasional Proportional Feedback

Considering the experimental difficulty involved in the calculation of these terms, the first attempt to obtain synchronization in this manner is to adapt the occasional proportional synchronization technique of Hunt [43] to our needs. The iterate dependent factor, Eq. 3.25, translated to our implementation as Eq. 3.33 is replaced by a constant to be designated as $\alpha$ multiplied by the difference in the two measured voltages. In this case the feedback perturbation, Eq. 3.33 simplifies to

$$\Delta p = \alpha[V^M(t_p) - V^S(t_p)].$$ (3.34)

And Eq. 3.31 becomes
\[ V_o \sin \omega t \rightarrow (V_o + \alpha (V^M(t_p) - V^S(t_p))) \sin \omega t \quad \text{for} \quad t_p(n) < t < t_d \]
\[ \rightarrow V_o \sin \omega t \quad \text{for} \quad t_d < t < t_p(n + 1). \quad (3.35) \]

This is to be called synchronization by occasional proportional feedback (SOPF). The experimental implementation of the scheme is described in chapter 6. In Eq. 3.35 the feedback is to be evaluated on the peaks of the drive wave. In practice, the difference can be measured at any point of the orbit. Hence the definition of \( t_p \) is relaxed to be some point in the cycle. As to be seen below, this simplification successfully synchronized the chaotic resonator.

Synchronization Using the Peak Dependent Feedback Factor

The natural experimental progression beyond SOPF is to actually calculate and apply the peak dependent feedback factor in order to obtain synchronization and compare if this more complex method is superior to usage of the constant term. The resonator is still considered as 1-dimensional and we use Eq. 3.33 for the perturbation. Thus we need to calculate and apply in real time the ratio of the Jacobian matrix to the shift. The evaluation of this term is determined at the peaks of the drive wave when the orientation of the unstable direction is assured. We measure the difference in the voltage drops across the slave and master resistors, \( V^S(t) - V^M(t) \), as the error signal and need only to calculate the ratio of the Jacobian, \( \partial F_1/\partial V^M \), to the parameter induced shift, \( \partial F_1/\partial p \). This can be accomplished using the first return maps. Figure 3.3 displays two return maps taken at a drive amplitude of 4.6V and 5.1V respectively. These maps were used to experimentally determine the feedback factor. Since each return map is composed of approximately 300 points and are contaminated by noise a curve fit is interpolated so as to average the fluctuations in nearby points. Both return maps are divided into four sections so as to create the most accurate piecewise
continuous curve. A low order polynomial curve fit is determined for each section. The fitted curve is then used in the calculations. The result is an estimation for the perturbation as a function of the voltage drop across the master resistor, $V^M(t)$.

For the 1-dimensional consideration of this system, the slope of the return map at each point is the Jacobian evaluated at that point. This can be shown by differentiation of the map

$$v_{n+1} = F(v_n)$$

(3.36)

with respect to $v_n$ which leads to the 1-dimensional Jacobian

$$\frac{\delta v_{n+1}}{\delta v_n} = DF.$$  (3.37)

The term $D_pF$ is obtained from the first return map of the resonator by considering how the return map reacts to a perturbation applied to the amplitude of the drive voltage. For a driving voltage $V_0$, we look at how the point $x_n(V_0)$ is mapped into $x_{n+1}$. Then at a slightly higher voltage, $V_0 + \delta$, we look at how the same point, $x_n(V_0)$ is mapped into the new point $x_{n+1}(V_0 + \delta)$. The shifting is seen in Fig. 3.3; the 5.1V return map has slid along the upper branch of the 4.6V map. The experimental approximation for the shift is then

$$D_pF \approx \frac{[x_{n+1}(V_0 + \delta) - x_{n+1}(V_0)]}{\delta}. \quad (3.38)$$

Since we are using a shift large enough to prevent the return maps from overlapping, we see that the highest points along the abscissa of the 5.1V map have no correspondence on the 4.6V map, and the lowest abscissa points of the 4.6V map have no corresponding points on 5.1V map. A meaningful approximation for $D_pF$ is made only for points which occur on both maps. For the periphery points, we extrapolate $D_pF$.

The 1-dimensional iterate dependent amplification factor is the negative of the
quotient of the two above terms. Figure 3.4 plots the result of the calculation versus the peaks of $V^M(t)$. While both branches of the lower folded region produce a relatively constant factor, in the upper region the factor drops off rapidly. The diode resonator shifts along an axis almost parallel to the upper branch and the shift, $D_p F$, approaches zero. Hence the feedback factor becomes quite large. It is not possible to apply the exact feedback perturbation since the large amplification of the unavoidable noise along with any differences in $V^S(t)$ and $V^M(t)$ would unacceptably over modulate the drive wave. A numerical calculation of the feedback factor, shown in Fig. 3.5, displays a similar shape, but does not drop off as drastically. The numerically calculated return map shifts upward more than experimentally observed so as to prevent $D_p F$ from approaching zero. Problems caused by the shift of the return map are not particular to this dynamical system. In the case of the Lorentz system, the shift of the return map produces an intersection with the unshifted map [85]. When these pathologies occur, one either chooses to not apply feedback or approximate the feedback from the nearby regions. Finally it must be pointed out that this type of analysis is applicable because of the simple shape of the return map for the diode resonator.

On inspection of the return maps, one remarks that the slope of the lower branches is greater than unity while the slope of the upper section is less than one. It follows from the linearized dynamics of the one dimensional system that the lower section is unstable; nearby orbits in this region are divergent. On the other hand, the upper branch is stable and adjacent orbits here should remain temporarily adjacent until the phase space trajectory wanders into the unstable regime. The salient feature to capitalize is that a feedback perturbation need only be applied when the system is in an unstable region of the attractor. In the stable regions, the magnitude of the Jacobian is less than one and the feedback factor should become unnecessary. Therefore, we propose that synchronization is possible if feedback is only applied when the Jacobian is greater than 1. For the diode resonator, this means that if feedback is
applied only when the voltage peaks are less than 48mV synchronization can occur. Refer to the iterative feedback factor seen in Fig. 3.4. In the critical regions of the attractor below 48mV, this factor is constant. Therefore an experiment is proposed which would apply the SOPF constant factor but only if the voltage peaks are below 48mV. Outside this region nothing is to modulate the drive wave. A successful experiment would indicate a natural bridge between the simplified scheme of SOPF and the actual implementation of the peak dependent factor. It would indicate that it is unnecessary to exactly duplicate the large negative sloped region of the peak dependent feedback factor and would explain in part the reason why SOPF is so successful. When the constant factor of SOPF was adjusted so as to achieve synchronization, then it was essentially the prescribed peak dependent feedback factor.

Lyapunov Exponents and the Peak Dependent Feedback

An understanding of the ability for systems to synchronize can be gained by computing the Lyapunov exponents locally as well as globally. The often used global Lyapunov exponents quantify the average instability of the entire phase space to small perturbations. Local Lyapunov exponents, on the other hand, characterize instabilities of the attractor along small sections of the orbit. They are dependent on the location of the evaluated point and provide detailed information about the regional stability of the attractor [86].

Lyapunov exponents are determined by considering the evolution of an infinitesimal displacement, $\tilde{y}_n$, from some initial point, $\tilde{x}_n$. A linearization of the mapping yields the resulting displacement, $\tilde{y}_{n+1}$

$$y_{n+1} = DF(\tilde{x}_n) \cdot \tilde{y}_n$$

(3.39)

where DF is the Jacobian matrix of the mapping function. The direction of the
displacement is given by $\frac{y_n}{|y_n|}$ while the magnitude is given by

$$m_n = \frac{|y_{n+1}|}{|y_n|}. \quad (3.40)$$

The value $m_n$ is referred to as the local Lyapunov multiplier (LLM). It expresses the amount of growth ($m_n > 1$) or contraction ($m_n < 1$) of the deviation in a localized region of the attractor surrounding $x_n$. Note that in general, the value obtained for $m_n$ is dependent on the direction chosen for the displacement and that for each dimension of the dynamical system there will be a corresponding Lyapunov exponent.

For the diode resonator, we start $y_n$ as a displacement of the current at a current peak $n$. The differing trajectory is propagated to the $n+1$th peak where it is compared to the trajectory which would have occurred had no initial displacement occurred. The ratio of the final difference, $y_{n+1}$, to the initial displacement, $y_n$, is taken to be the local Lyapunov multiplier for the peak $n$. At the $n+1$th peak the process is repeated with a new infinitesimal displacement in order to determine $m_{n+1}$. Global Lyapunov multipliers are then defined by stepping the deviation through the linearized mapping indefinitely and calculating the geometrical mean of the local multipliers. In time the initial perturbation will have grown enough so that the linear approximation is rendered invalid. At these points, the perturbation is renormalized. The global Lyapunov multiplier is then

$$m_g = \prod_{n=0}^{\infty} \left( \frac{|y_{n+1}|}{|y_n|} \right)^{\frac{1}{n+1}}. \quad (3.41)$$

The global Lyapunov exponent is then defined as

$$\lambda = \ln(m_g). \quad (3.42)$$

At this point we must be careful to point out that this process is used to determine the largest Lyapunov multiplier. This is because any initial deviation, $\tilde{y}_n$, will rotate
in the direction of the rate of the largest expansion as it is stepped through the linearized mapping. However, since our system of interest is the driven diode resonator which has only 1 positive Lyapunov exponent, we do not consider rotations of the displacement as detrimental to the calculation. For high dimensional systems the reader is referred to [78, 87].

Lyapunov multipliers can be used to predict the occurrence of synchronization. Consider the above mentioned infinitesimal displacement, $\vec{y}_n$, as a vector pointing from one orbit to a second orbit. However, instead of the linearizing the mapping of the single orbit, one observes the linearized relationship between the difference of the master and slave orbits. Explicitly, the evolution of the displacement is measured by evaluating the propagation of a point along the primary trajectory and an equivalent point along the secondary orbit. The secondary orbit is perturbed by the effect of some feedback; otherwise, this calculation would simply duplicate the above estimate of the Lyapunov exponent. The evolution of the displacement defines whether synchronization of the second orbit to the first is to be achieved or not. For synchronization to occur, the corresponding global Lyapunov multiplier must be less than one.

The significance of the algorithm proposed by Lai and Grebogi for maps is that it provides a technique to calculate and apply perturbations so as to set the local Lyapunov multipliers along the unstable directions to zero at every iteration. Since the global Lyapunov multiplier will be minimized as well, this technique represents the ideal situation and presents the most rapid scheme possible for synchronization. However, the requirement for convergence is only that the global Lyapunov multipliers have a magnitude less than unity. Since this is less stringent than the demand of the original theory, it is possible to adapt the Lai and Grebogi perturbation formula to the experimentally feasible calculation of SOPF. The implementation of the constant factor can be successful if it satisfies the criteria of the global Lyapunov multipliers being less than unity. Note that setting each of the LLMs to be less than one does
not in general guarantee synchronization. As pointed out by So and Ott [88, 89], for high dimensional systems the product of the eigenvalues of the Jacobian matrices is not necessarily equal to the eigenvalue of the product of the Jacobian matrices. In such systems, the feasibility of the Lai and Grebogi scheme is uncertain. One would perhaps have to replace the scalar feedback perturbation with a vector feedback so as to stabilize all unstable directions.

A numerical calculation of the LLM's versus the voltage peaks of the resistor component of the resonator (designated as $V^M(t)$) were obtained by first synchronizing a slave system to the master and then applying a small randomly generated kick to the slave system at each peak. Both systems were propagated forward in time until the next peak at which point the resulting difference between the two systems was evaluated. The ratio between the final trajectory difference and the initial kick was considered to be the LLM for the initial peak. Global Lyapunov multipliers were then obtained by the geometrical averaging method described above. Figure 3.6 shows the effect of the feedback perturbation on the LLMs. The LLMs are plotted as a function of the chaotic peaks. In the event of no applied feedback to the second orbit, Fig. 3.6(a) the LLM's are predominantly greater than one and no synchronization is achieved. Note that even with no feedback, the LLMs along the upper branch are less than one. Though globally unstable, the attractor has regions of stability.

We next look at amplitude modulating the drive wave of the second system by the term $\alpha(V^S(t) - V^M(t))$ where $\alpha$ is a simplifying constant amplification factor described above. The dimensionless factor, ($\alpha = 0.3$), applied in Fig. 3.6(b) is just sufficient to shift enough of the LLMs downward so that synchronization is realized. Even though some of the LLMs are greater than one, the global product remains below unity. Figure 3.6(c) shows the optimum constant value ($\alpha = 0.5$) for synchronization. The iterative dependent term prescribed by the Lai and Grebogi algorithm, Fig. 3.6(d), presents the optimum case. However, since the algorithm was obtained for maps, the LLM will not necessarily be zero.
Synchronization can be achieved for the range of constant amplification values which produce a negative global Lyapunov exponent. Figure 3.7(a) is a graph of the largest and least global Lyapunov exponents versus constant amplification factors. Figure 3.7(b) is plot of the standard deviation of the difference between peaks of the voltage drops across the master and slave resistor. The graph shows the range of constants for which synchronization can be achieved. Only for a band of values is the global Lyapunov exponent negative. The corresponding synchronization of the master and slave is striking as the standard deviation drops to zero for the band. These plots are a numerical illustration of that which is observed experimentally and reported below.

Summary

In summary, the minute modulation of some available system parameter can cause the control of unstable periodic orbits (OGY) or the synchronization of chaotic trajectories (Lai and Grebogi). The perturbation is chosen in order to shift the attractor so that the slave orbit trajectory is entrained along stable direction and convergent to the desired fixed point or master orbit point.

In the nearly 1-dimensional diode resonator, the amplitude of the driving sine wave is the most convenient parameter to modulate. Subsequently, two time series taken at slightly different values of the amplitude are sufficient to calculate the iterate dependent synchronization factor. From these time series of the resistor voltage drop a first return map can be constructed. The 1 dimensional Jacobian as a function of the voltage peaks is simply the slope of this map. And the shift, $D_pF$, can be evaluated by observing changes in return maps taken at two different drive amplitudes. Then the negated ratio of the Jacobian to the shift gives the peak dependent amplification factor. This factor is constant for low voltage peaks and drops off steeply when the peaks are above $48mV$. Therefore, one must duplicate this curve in order to successfully implement the true synchronization scheme.
By observing the return map for this 1-dimensional system, it is realized that there are regions of the attractor (defined by the peaks being above 48 mV) which are stable. In these stable regions, we suggest synchronization can be maintained with no applied feedback. Furthermore, if feedback is applied in these regions, their effect on the synchronization is marginal. When feedback must be applied, the peak dependent factor is merely a constant value which amplifies the difference between the master and slave signals. Therefore on the basis of these observations, we suggest that using the SOPF constant amplification value will achieve synchronization when this factor is adjusted so as to match the prescribed one. Also experimental synchronization can be achieved when perturbations are applied only if the resistor voltage peaks are less than 48 mV.

The possibility of synchronization can be quantified by computing the local and global Lyapunov multipliers. We see that the effect of the perturbation is to reduce the local multipliers. Then when the global multiplier is reduced to be less than one, synchronization can occur. From this analysis it is seen that the Lai and Grebogi peak dependent feedback factor presents the most efficient means of synchronization. It is also observed that the global multiplier can be reduced under the unity barrier by implementation of a of constant amplification factor (SOPF). Thus experimental techniques can be developed to synchronize chaotic diode resonators without the need for real time computation of elaborate feedback factors.
Figure 3.2: Synchronization using control

(a) While one orbit, \( y \), may approach a second, \( x \), eventually it will move off along the \textit{iterate dependent} unstable direction, \( f_u \). (b) By applying the correct perturbations to the \( y \) system, we direct its orbit so that it has no component along the unstable direction, \( f_u \), of the \( x \) orbit. The \( y \) orbit then approaches the \( x \) orbit along the stable direction \( f_s \), synchronizing the two.
Figure 3.3: Shifting of the first return maps

Two return maps recorded at 4.6V and 5.1V (upper) superimposed show the effect of a perturbation to the drive wave amplitude. The shift of the map is obtained by measuring the change in peak \( (x_n(5.1V) - x_n(4.6V))/0.5V \). The approximation of the 1-dimensional Jacobian is the slope of the 4.6V map. Note that above 48mV the slope is less than one. The attractor is locally stable in this region.
Figure 3.4: Peak dependent feedback factor from experimental data

The peak dependent feedback amplification factor is plotted as a function of the chaotic resistor voltage peaks. This is obtained using data obtained from Fig. 3.3. Since the return map shifts along the upper branch, the denominator is quite small and the factor drops off rapidly. Below 48mV the factor is approximately constant regardless of the two lower branches of the return map.
Figure 3.5: Numerical calculation of the peak dependent feedback factor

The amplification factor is plotted as a function of the chaotic peaks. As seen experimentally, the lower section is almost constant at 0.5 (dimensionless units). However, the numerical return map shifts upwards more than experimentally observed and the drop off of the upper section is not as severe.
A calculation of the local Lyapunov multipliers plotted as a function of the peaks of the chaotic current. These describe the separation (> 1) or contraction (< 1) of the master and slave orbits as they are propagated from one peak to the next. (a) The master and slave resonator are operating independently. Note that for the high peaks, the multipliers are less than one. (b) With a constant feedback amplification of 0.3 some local multipliers remain greater than unity. However synchronization occurs because the global Lyapunov multiplier is reduced below 1. (c) An optimum constant feedback factor of 0.5. The perturbation reduces substantially the multipliers in the unstable region of the attractor but does little to influence those in the stable regions. (d) The peak dependent feedback factor according to the Lai and Grebogi algorithm influences the local Lyapunov multipliers strongly. This represents the optimum feedback factor possible to obtain synchronization using the control technique.
Figure 3.7: Lyapunov exponents and standard deviation

(a) The largest and least Lyapunov exponents of the linked master slave system versus the applied constant feedback factor. Only for a narrow band of values is synchronization obtainable. (b) The plot shows the standard deviation of the difference between the master and slave orbits as the constant feedback synchronizing factor is increased. The results are in concert with (a). The plots are in excellent agreement with that observed experimentally.
CHAPTER 4

EXPERIMENTAL SYNCHRONIZATION OF DIODE RESONATORS USING CONTROL

Introduction

The algorithm proposed by Lai and Grebogi has been successful in numerical synchronization of the logistic, Hénon, and Ikeda maps. However, it is uncertain whether an actual implementation of the procedure would be successful due to the inherent noise in the system and the impossibility of constructing identical chaotic systems. Hence the goal is to experimentally implement, using the diode resonator, not only the actual algorithm, but also variations on it in order to observe whether synchronous signals can be obtained from the two resonators. The resonator presents an ideal system to implement. Since it is a non-autonomous system driven by an external sine wave, the Poincaré surface of section can be easily created by strobing on the peaks of the drive wave and any feedback can be applied for the period of the drive signal. Experimentally, waveform generators usually incorporate strobe signal outputs and voltage controlled amplitude inputs so that the required electronics are somewhat simplified. This makes the experimental arrangement one in which most of the design time can be focused on the resonator itself and the creation of a feedback signal.

As an initial experiment, we speculate that the peak dependent factor prescribed by the formula can be replaced by a constant amplification factor which is to multiply the difference between the two observed signals in order to create the feedback perturbation. This we designate as synchronization by occasional proportional feedback (SOPF). Motivation for using the constant factor is found in the occasional propor-
tional feedback method of control originally developed by Hunt [43] to control the unstable periodic orbits of the chaotic diode resonator. In addition, as described in Chapter 3, it was seen that the theoretically proposed feedback factor is constant in the critical unstable regions of the attractor. Furthermore, the calculation of the Lyapunov multipliers indicate that synchronization can be achieved for a range of constant values of the amplification factor. Finally, this method is the simplest to experimentally implement since no effort is spent constructing the feedback factor.

A more ambitious test will be to experimentally generate the prescribed peak dependent feedback factor so as to duplicate the Lai and Grebogi algorithm as closely as possible. That this can be accomplished is largely dependent on the fact that the resonator is approximately a 1-dimensional system whose unstable direction is constantly oriented. As seen above, this simplifies the peak dependent factor without violating the theoretical premise of synchronization.

Finally test the hypothesis that feedback need only be applied when the resonator is in an unstable region of the attractor. When applied, the feedback is the constantly amplified difference between the two signals. And it is identical to the peak dependent factor. When the resonator is in a stable region of the attractor apply nothing for nearby orbits should not diverge exponentially. This particular scheme would prove to be the most efficient method of synchronization yet devised. The small correcting signal would only have to be administered for only a fraction of the time.

This chapter is organized as follows. In section II, the general experimental arrangement is described in detail. With exception of the determination of the feedback factor, this circuit is identical for all the experiments performed The achievement of synchronization by SOPF is detailed in section III. Results are presented showing the tolerance of synchronization to variations in the properties of the resonators, to changes in the amplification factor, and to phase differences in the drive wave. The amount of time required for the resonators to synchronize is also displayed. In section IV, we describe the implementation of the theoretically prescribed feedback factor.
The resulting synchronization is described and compared to that obtained by the SOPF technique. The third experiment of synchronization when necessary is presented in section V. It is seen that synchronization can be achieved when feedback is applied for only 52° of the peaks. The results are summarized in section VI.

Experimental Setup

Each of the experiments performed are based on the following circuitry described in detail here. While the feedback is proportional to the difference in the voltage drops across the master and slave resistor, the multiplying factor varies between each experiment. The explicit details of the proportional amplifying factor are described in the subsections concerning each experiment. Figure 4.1 is a block diagram of the circuit. Figure 4.2 is a schematic of the first constant amplification feedback experiment. Figure 4.3 is a schematic of the second experiment and Fig. 4.4 is a schematic of the third experiment.

The voltage drop across the resistor of one circuit, designated as the master signal $V^M(t)$, presents a convenient chaotic signal to represent the attractor of the circuit. From a second diode resonator circuit operating under similar conditions we select the corresponding chaotic signal, designated as the slave $V^S(t)$, which is to be synchronized to $V^M(t)$. The difference in the two voltage signals, $V^S(t)-V^M(t)$, is obtained and amplified by a factor $\alpha$ by Analog Devices AD521 instrumentation amplifier. Hence from Ohm's law, we are observing the chaotic current in each resonator.

In all experiments, the two waveform generators which drove each resonator, an HP3325A and a WaveTek 166, were phase locked at the same frequency by linking the sync output of the HP3325A waveform to the trigger input of the WaveTek 166. However, when the two were directly connected, a 90° phase shift is created since the sync pulse is issued at the peaks of the HP3325A and the triggered wave began at the 0V crossing. It is then necessary to construct a delay generator. This is accomplished with a single 74LS123 dual monostable multivibrator. The rising
Synchronization of chaotic trajectories by control can be summarized by the block diagram. One measures a scalar signal from both the master ($V_n^M$) and slave ($V_n^S$) chaotic systems. Using the two signals a feedback perturbation is calculated and applied to modulate an existing dynamic parameter of the slave system.
Figure 4.2: Schematic diagram of the SOPF experiment

A block diagram of the experimental apparatus. The setup consists of two diode resonators each composed of a 1N4004 diode, 33mH inductor and a 90.5Ω resistor. The amplified difference between the chaotic voltages dropped by the resistor of each circuit, $V^S(t) - V^M(t)$, is measured by an instrumentation amplifier. On the hold strobe (which occurs at the signal peaks), this difference is held by a sample and hold device and gated into the amplitude modulation input of the slave sine wave generator.
Figure 4.3: Schematic diagram of Lai and Grebogi experiment

Schematic diagram of the experiment in which the Lai and Grebogi peak dependent factor is produced in real time. $V^S(t)-V^M(t)$ is obtained and amplified by the instrumentation amplifier. The amplified and DC biased $V^M(t)$ drives a diode-resistor rectifier element which produces a curve similar to Fig. 3.10. The product of the feedback components is input into a sample and hold device. On the drive wave peaks, the input is frozen and gated into the amplitude modulation input of the slave drive wave generator.
The block diagram for the application of feedback only in the unstable attractor region is similar to the diagrams for the previous two experiments. In this case, the amplified signal $V^M(t)$ is input into a comparator. If $V^M(t)$ is greater than a threshold voltage the feedback procedure is impeded. Otherwise, the difference, $V^S(t_p) - V^M(t_p)$, is amplified by a constant factor and resulting term modulates the drive wave exactly like the other experiments.
edge of the sync pulse triggered the first one-shot which issued a TTL pulse with a
duration of approximately the RC time constant of the discharging external resistor
and capacitor which complete the circuit. The falling edge of this created pulse
triggered the second one shot whose output pulse is used to trigger the WaveTek. By
adjusting the resistor component, the duration of the first pulse could be varied and
subsequently, a variable phase difference between the two waves could be created.
This allowed an observation of the persistance of synchronization as a function of
phase difference.

Though the output impedances of both waveform generators were rated at 50Ω in
practice there is a slight variation in the drive properties of the two generators. The
natural feedback of the resonators, due to their impedance, altered the drive signal of
each waveform generator differently. Though mild, this difference contributed to the
error in the degree of synchronization which could be achieved. A partial solution to
the problem is to buffer both waveform generators with low impedance high current
LH0002 line drivers. These high current buffers minimized the feedback and aided in
maintaining an equality of drive signal.

Both diode resonators were driven with the output of the LH0002 devices. Each
resonator is insulated from the loading effects of the oscilloscope by using two of the
four operational amplifiers of a quad LF347. The voltage drop across each resistor is
input into the noninverting input of an op amp which is routed in a voltage follower
mode. The voltage drop across the resistor is then measured with the oscilloscope or
waveform digitizer by observing the outputs of the op amp.

The difference in the resistor voltage drop of the two resonators is obtained using
an Analog Devices AD521 instrumentation amplifier. These differential amplifiers are
ideal for this purpose since their high common-mode-rejection-ratio and precision are
exactly what is needed for measuring this small difference. Furthermore, by adjusting
the external scale resistor a gain ranging from 1 to 1000 can be realized.

The sample and hold amplifier used is a Harris HA-5320. This device follows the
input wave as long as a control input is held TTL logic low. When the control goes high, the signal is frozen at that point for the duration of the logic high signal. When the control line is released the device resumes following the signal. However, there is a finite time from when the control line is released to when the device accurately follows the input signal. This acquisition time limits the speed of the device. For the HA-5320 this time is \(1\mu s\). This meant that feedback could not be applied for the entire cycle but rather any time to within \(1\mu s\) of the next drive wave peak. Therefore, the feedback perturbation is evaluated on the rising edge of the TTL signal from the feedback timing circuit and is a constant as long as this signal remains high.

The output of the sample and hold amplifier is then input into an DG303 analog switch, which is routed to act as a two channel multiplexer with the second channel being grounded. Like the sample and hold device, the switch is controlled by TTL logic. While the control line is held low logic, the output of the switch is the ground input, i.e. ground. On logic high, the switch output is the feedback perturbation which is being held by the sample and hold amplifier. The output of the switch is input into the amplitude modulation input of the driving waveform generator which drove the slave resonator. Therefore when the feedback timing circuit held issued a TTL logic high signal, the feedback perturbation is gated into the waveform generator. Otherwise nothing is applied to the waveform generator. It is necessary to rout the switch as a multiplexer in order to eliminate the effect of charge injection. This effect occurs because of a capacitance associated with the switch. Injected charge from the signal charges the capacitor to a voltage given by \(qC\). This voltage is then present when the switch is disabled and creates an undesirable feedback into the amplitude modulation input of the waveform generator. By routing the switch as a multiplexer with the other input being ground, charge injection can be avoided.

The feedback timing circuit is created using a single 74LS123 dual monostable multivibrator. The idea is identical to the circuit used to phase lock the two drive waves. The rising edge of the sync pulse generated by the WaveTek waveform generator trig-
gered the first one-shot which issued a TTL pulse with a duration of approximately the RC time constant of the accompanying resistor and capacitor. The falling edge of this created pulse triggered the second one shot. As with the first one-shot, the duration of the pulse could be varied by adjusting the RC time constant of its associated resistor and capacitor. The second pulse is used to control both the sample and hold amplifier and the analog switch. By varying the duration first pulse, the controlling signal could be issued at any point in the drive cycle. By varying the second pulse, the duration of the feedback could be varied.

In the algorithm proposed by Lai and Grebogi feedback is not applied whenever its magnitude is larger than some predefined value. Hence, the observer must wait until the master and slave trajectory is sufficiently close enough to satisfy this criteria. Furthermore, if the influence of noise is large enough, synchronization should be lost until the two orbits drifted close to each other. In the experimental arrangement, no circuitry is included to limit this feedback. When the feedback signal is applied, its initial magnitude is at most 25% of the peak-to-peak voltage of the driving sine wave. Thus while initial deformation of the attractor is not negligible, nor is it catastrophic. The short duration of these large kicks restricts the energy imparted to the driving wave.

Synchronization by Occasional Proportional Feedback

Figure 4.2 shows the circuitry used to realize the feedback. Note the Analog Devices AD521 Instrumentation Amplifier. Since this device has an adjustable gain, the amplification, determined by the ratio of the 100kΩ resistor to the variable gain resistor, is used to create a constant factor to approximate the peak dependent feedback factor. The output of the device is then directly input into the sample and hold integrated circuit.

The amplitudes and DC offset of each waveform generator are originally set so that each resonator is operating chaotically just below the period 3 window. There
is nothing sacrosanct about this choice of amplitude. It is chosen because, it is easier to place both resonators in the particular region. In the course of experimental work, the drive amplitude is varied over the chaotic range of the resonator. The initial goal is to drive each resonator identically so that they share the same attractor. It is rather difficult to construct identical resonators. (See chapter 2 on how diodes can be matched.) As a result, the attractor of each resonator can never be identical. Furthermore, using waveform generators built by different manufacturers negatively influences the process. Minor problems include any phase differences between the two generators and noticeable differences in the power spectra of the two. An unexpected source of error lies in the DC bias of each sine wave. Small DC shifts in the drive signal changed in a large manner the bifurcation points of the resonator. Thus any difference in this bias produced a large signal difference. Figure 2.6, a bifurcation plot with respect to the DC offset shows the scope of the problem. The most fruitull method to place the resonator is similar regions of the attractor is to insure that the two waveforms were in phase and subsequently observe the root-mean-square voltages of each resistor. Then adjust the drive amplitude and DC offset of the HP3325A so as to equalize the two measured voltages.

Figure 4.5 is a plot of the voltage across the slave resistor versus that of the master while unsynchronized. As the system is nonautonomous and in phase, the area is more hexagonal in shape than square. Feedback pulses are then applied commencing on the peaks of the driving wave and having a duration of 8\(\mu s\) (56% of the 14.3\(\mu s\) period). The gain is slowly increased until synchronization is achieved (Fig. 4.6). This rather impure synchronization is primarily due to the diodes not being ideally matched. and also to the driving differences described above. Note the slight bulge on the lower side of the figure. This occurs on those occasions when the resistor voltages are traversing from one of the lowest troughs to the highest peaks. There is a slight phase difference between the two signals and, combined with the relatively large slope, this produces the well-above-average difference. In spite of these variations, the feedback pulses
are only 5% of the peak-to-peak voltage of the driving wave. The important point to observe is that this method of synchronization is tolerant to a wide range of variances.

The next step is to try and achieve a superior synchronization. From the assortment of two dozen diodes available, the optimum pair is chosen using the criteria established in chapter 2. Figure 4.7 is the resulting plot of the master versus the slave for this case. The quality of synchronization is clearly superior to that obtained priorhand. The synchronization process has reduced the root-mean-square difference in the master and slave signals by a factor of 10. Additionally, the average heights of feedback pulses necessary to maintain synchronization have been reduced to less than 1% of the peak-to-peak voltage of the driving wave.

Figure 4.8(a) is a time-domain plot of the slave resistor voltage along with the master. At time $t = 0$, feedback is applied and within a few cycles, synchronization is achieved, as seen from the difference of the two signals in Fig. 4.8(b). Compare this figure to Fig. 4.9, in which the diodes are not as well matched. The time to achieve synchronization is noticeably longer. Note that the time it takes to converge to synchronization depends on the location of the two systems in their respective attractors when the feedback signal is initially applied. On average, closely matched diodes converge to synchronization more rapidly than poorly matched ones. Figure 4.10 shows the the driving slave sine wave along with feedback signal necessary to maintain synchronization. In this is the optimized case almost no feedback is being applied. The driving wave is barely being perturbed.

Synchronization can be achieved for a band of amplification factors ranging from a low of 50 to a high of 150. An optimum value is approximately 80. These values take into account the gain of the instrumentation amplifier, attenuation across the circuit, and the gain of the amplitude modulation input to the slave waveform generator. While outside this band it is rapidly lost, inside it is relatively stable. Synchronization is also sign dependent. It is not possible to obtain valid results when the sign of the
Figure 4.5: Unsynchronized diodes

The voltage across the slave resistor, $V^S(t)$ (abscissa) versus that of the master, $V^M(t)$ (ordinate) while signals are unsynchronized. Since this is a driven system with both drive waves operating in phase, a complete square is not observed.
Figure 4.6: Synchronization by occasional proportional feedback: unmatched diodes $V^S(t)$ vs. $V^M(t)$. Synchronization using occasional proportional feedback as originally achieved using unmatched diodes. The imperfect synchronization is primarily due to differences between the properties of diodes. The bulge arises from a slight phase difference existing between the chaotic signals as they traverse from a low trough to a high peak.
amplification is inverted. These results are in complete agreement with that obtained numerically and shown in Fig. 3.7.

By adjusting the phase difference between the driving waves, good synchronization can be maintained with up to a 12° phase difference and a crude synchronization maintained for up to 20°. The value of this measurement is to demonstrate the possibility of relaxing the zero phase difference requirement between the drive signals. Since the feedback is a constant evaluated at one point of the cycle, it is not necessary to have the entire time series of the master system, merely the value of the master evaluated at the holding point. This implies that synchronization can be achieved for an arbitrary constant phase difference between the two driving waveform generators. One would measure the master signal at its peak level, wait for the peak of the slave, then evaluate and apply the difference. In this latter scenario lie potential applications for fast dynamical systems. If the computational time of the amplification factor were lengthy in comparison to the period of the driving signal, then measure the master at a time sufficiently before the slave peak so that computations could be performed. Furthermore, since only the peak of the master is needed, it is possible for a computer with a digital-to-analog converter to supply a master signal, updating it at the driving frequency, in order to synchronize the slave resonator.

With the feedback pulses commencing at the peaks, we can apply the feedback for up to 12μs (84% of the period) after which synchronization can no longer be maintained. Thus, when feedback occurs as the driving wave is approaching its peak, synchronization is lost. This observation is a characteristic of the diode resonators and is due to the polarization of the diodes. Along the positive half of the drive wave, they are switching forward-biased. As the wave approaches its peak, any change in the drive amplitude alters the current through the diode accordingly and tends to overcompensate the perturbation needed to maintain synchronization. As an aside, note that the acquisition time for the HA-5320 sample and hold amplifier is 1μs. Thus technically feedback cannot be reliably applied whenever the duration is greater that
Occasional proportional feedback in the best case as seen by recording $V^S(t)$ vs. $V^M(t)$. The optimum synchronization is achieved when not only are the diodes are matched but there is no temperature difference between the two.
Figure 4.8: Time to achieve synchronization for matched diodes

(a) A time series of $V^M(t)$ (dotted line) and $V^S(t)$ (solid line) for closely matched diodes. Feedback starts at $t = 0$. (b) $\Delta V(t) = V^S(t) - V^M(t)$ for the closely matched diodes shows the length of time necessary for synchronization to occur for this particular run.
Figure 4.9: Time to achieve synchronization for unmatched diodes

(a) $V^M(t)$ (dotted line) and $V^S(t)$ (solid line) for unmatched diodes. Feedback commences at $t = 0$. Though visual differences remain, the two are well synchronized.

(b) $\Delta V(t) = V^S(t) - V^M(t)$ for the unmatched diodes. On average, synchronization takes longer to be achieved than in Fig. 4.8.
Figure 4.10: Feedback signal and drive wave

The feedback signal, $\Delta p$, and the perturbed slave driving wave corresponding to Fig. 4.7. Note that the troughs of the driving wave are imperceptibly perturbed according to the prescription $(V_0 + \Delta p)\sin\omega t$. 
Continuing this line of observation, we find that the feedback only needs to alter the troughs of the driving wave. While the original trigger point of feedback is on each peak of the sine wave, it can be adjusted until the phase difference between it and the peak is as great as 145°, i.e., just above the trough of the wave. When triggered at this point in the cycle, the duration of the feedback pulse can be reduced to less than 3µs (20% of the period). This is again due to the asymmetry of the diodes. In this reversed-bias region, the current does not appreciably alter with changes in the amplitude. Hence, by tweaking the minimum voltage level seen by the diode, the necessary minor shifts in the attractor can be obtained.

Synchronization Using Peak Dependent Feedback

The goal of this experiment is to reproduce in an analog circuit the exact synchronization scheme of Lai and Grebogi [5]. This calls for an amplification factor to \( V_M(t) - V_S(t) \) which is a function of the orbit trajectory in the attractor. As seen in chapter 3, this term is an experimental approximation of the negative quotient of the Jacobian to the map shift,

\[
\alpha = -\frac{\delta F \delta F^{-1}}{\delta v \delta p}, \quad (4.1)
\]

and is obtained from the first return maps. In the above equation, the terms are evaluated at each peak of \( V_M(t) \). Experimentally, these peaks are used to determine the required factor. The calculated feedback curve portrayed in Figs. 3(b) and (c) is reminiscent of the well known current-voltage curve of rectifier diodes. Hence a rectifier composed of a diode in series with a resistor is used to create the desired factor.

A block diagram of this synchronization experiment is shown in Fig. 4.3. The circuit generates the peak dependent amplification factor and implements this calculated perturbation in real time as prescribed by the synchronization algorithm.
$V^M(t)$ is amplified with a variable gain and in addition is DC biased. This signal is then used to drive the diode-resistor rectifier. While the voltage drop across the rectifier diode is such as to keep it reversed biased and relatively non-conducting, the slight voltage drop across the resistor can be used as the constant portion of the factor. As the diode voltage drop crosses the forward bias threshold, the increasing conductance of the diode is reflected by the increase in the voltage drop across the resistor. In the forward biased region, the rectifier diode, content with dropping only about 0.7V, is essentially transparent. The voltage drop across the resistor is then that applied to the rectifier less 0.7V and the sloped line portion of the alpha factor is obtained. By varying the DC offset and the gain, the location of the bend and the slope of the line can be varied with respect to the applied voltage peaks of $V^M(t)$. The voltage drop of the rectifier resistor is then inverted, amplified and also DC offset with a LF355 operational amplifier. The slope of the factor could be adjusted by changing the gain of the amplifier. Additionally the entire factor could be shifted upwards or down by varying the DC offset.

By using all four of the parameters available, the shape of the feedback factor could be varied from a constant line of slope zero to the desired peak dependent feedback factor. The composite feedback perturbation is created by multiplying $V^M(t) - V^S(t)$ by the rectifier amplification term. This is done using an Analog Devices AD534 multiplier. This device is routed so as to operate with a unity scale factor. The product is input into the sample and hold amplifier described above.

Figure 4.11 shows the experimentally implemented amplification factor $\alpha$ versus the peaks of $V^M(t)$.

It does not fall off as rapidly as the factor shown in Fig. 3.4 since this would produce an unacceptably large modulation of the slave drive signal. The product of this factor and $V^S(t) - V^M(t)$ amplitude modulates the slave waveform generator for 12$\mu$s of the 14.285$\mu$s period. Figure 4.12(a) shows $V^M(t)$ and $V^S(t)$ while Fig. 4.12(b) is plots the difference between the two signals. At time $t = 0$ the feedback is applied. In
Figure 4.11: Peak dependent amplifying factor

The experimentally applied feedback factor is seen by recording the amplified output of the rectifier element on the drive wave peaks. The shape and location of the form are adjustable parameters.
contrast to the numerical algorithm of Lai and Grebogi, feedback is applied irrespectively to the location of the master and slave trajectories in their respective phase space orbits. Even while unsynchronized, perturbations applied to the slave drive wave are relatively small. After 200μs, a time which is representative for this particular system, the two signals are virtually identical. The length of time to synchronize is rather long considering that in the case of SOPF synchronization could be achieved in as little as 30μs. However, diodes used in this experiment were different from those used in previous work. The time needed to stabilize one system about the other is dependent on not only the matching of the attractors, but also on the shape and dimension of the attractor itself.

While synchronization can be achieved when the slave resonator is merely nearby the master in their respective bifurcation diagrams, a superior synchronization occurs when both systems can be driven so that their attractors are identical. In this latter case, the sustaining feedback as well as the difference between \( V^M(t) \) and \( V^S(t) \) is minimized. In the optimum case, the ratio of the feedback pulse heights to the amplitude of the drive wave is less than 2%.

When the peaks of \( V^M(t) \) reach above 48mV the degree of synchronization is only slightly effected by the magnitude of the feedback factor. The slope of this feedback factor curve (Fig. 4.11) is varied from zero to a quite large value yet synchronization persisted with degeneration increasing moderately for the higher slopes. This tolerance is to be expected since, as observed in Section III, the attractor is stable in this regime. A more critical parameter is the upwards or downwards shifting of the feedback curve. Synchronization occurs only for a small range of shifting since it is the constant portion of the factor which critically effects the unstable region of the attractor. If feedback is applied longer than 12μs of the 14.285μs period, synchronization could not be maintained. (The acquisition time of the sample and hold amplifier is 1μs.) This breakdown also occurred in the SOPF experiment when a constant application factor is utilized. While the diode is forward biased, a small
Figure 4.12: Synchronization using the peak dependent feedback factor

(a) Voltage drop across the master, $V^M(t)$, (solid line) and slave, $V^S(t)$, resistors. The feedback perturbations commence at $t = 0$. (b) $\Delta V(t) = V^S(t) - V^M(t)$ plotted in the time domain. Synchronization is achieved in about 175\,\mu s. A comparison with Fig. 4.8 is tenuous since the pair of diodes used in this experiment were different.
change in the applied voltage as the drive wave approaches its peak is leading to a large difference in the current through the resonator. The perturbation is too much in this quadrant. The minimum amount of time feedback could be applied before synchronization is lost, when initially commencing at the voltage peaks, is $8\mu s$.

Synchronization by Feedback
Only When Necessary

From the local Lyapunov exponents and equivalently the slope of the return map, it is seen that the resonator is stable in parts of the attractor. One would expect that in these regions, it is not necessary to apply feedback perturbations. This experiment is designed to test this hypothesis. The goal is to apply a perturbation only when the voltage peaks of the master resistor, $V^M(t)$, are below a cutoff threshold. In the case of a $4.6V$ drive voltage the perturbation factor is constant whenever the peaks of $V^M(t)$ are less than $48mV$. Above this level, the resonator is in a stable regime. Peaks less than $48mV$ occur $64\%$ of the time. Thus feedback should be inhibited for more than a third of the sampling periods. The block diagram of the experiment is shown in Fig. 4.4. The implementation of the constant amplification term is accomplished by using the gain of the instrumentation amplifier as described in the first experiment. Additionally, one must discriminate as to when not to apply the steering perturbations. Using a LF355 operational amplifier, $V^M(t)$ is amplified by a factor $\beta$ (producing $\beta V^M(t)$) then input into a LM319 comparator for comparison with an arbitrary voltage level designated as $V_L$. The operational amplifier with gain $\beta$ is necessary for two reasons. First, comparators are essentially operational amplifiers designed to operate either shut off (when the noninverting input is less than the inverting input) or completely saturated (when the noninverting input is greater than the inverting input). When the comparator goes saturated the device draws a fare amount of current. This current draw would distort the signal if no op amp is in place. Secondly, when the noninverting input is equal to the inverting input
the comparator is unstable and can generate a lot of noise. Therefore, to minimize this noise, one desires an input signal with a high slew rate. Thus $\beta$ should be large. A pull-up resistor tied to a 5V source on the output is used to create TTL logic for this particular comparator.

The output of the comparator is input into the active low input of the 74LS123 used for the feedback timing circuit. As long as $\beta V^M(t) < \beta V_t$ with $V_t = 48mV$ the comparator holds the TTL output low and the feedback timing circuit is enabled. However, when above this threshold voltage, the TTL high signal disables the 74LS123. A nuance of this design is that because the one-shot is triggered on the rising (active high input) or falling (active low input) edge of the input pulse, the feedback circuit could be triggered at undesirable points in the cycle. The solution to the problem is to have the duration of the sync output pulse from the waveform generator be as short as possible so as not to overlap edges of the two input signals. Fortunately the WaveTek 166 features a variable width pulse and potential problems were avoided. In cases where the timing circuit is disabled, the analog switch is never closed and no feedback is applied.

The driving signal for the slave resonator is then

$$V_0 \sin \omega t \rightarrow (V_0 + \Delta p) \sin \omega t \quad \text{for} \quad V^M(t_p) \leq 48mV \quad \text{and} \quad t_p(n) < t < t_d$$

$$\rightarrow V_0 \sin \omega t \quad \text{for} \quad V^M(t_p) > 48mV \quad \text{or} \quad t_d < t < t_p(n + 1)$$

(4.2)

Here $t_p(n)$ is the time at which peak $n$ occurs and

$$\Delta p = \alpha [V^S[t_p(n)] - V^M[t_p(n)]]$$

(4.3)

With the threshold set so as to disable the feedback when $V^M(t)$ is above 48mV, this scheme successfully synchronized the two resonators. The degree of synchronization is equivalent to the plot shown in Fig. 4.7 obtained for the first experiment. Figure 4.13 shows the applied feedback along with $V^M(t)$. From the graph, no mod-
ulation is applied for the higher peaks and when applied only slightly perturbed the driving signal. Note that after a period with no applied feedback, the following kick remains relatively small and often smaller than those applied for several periods. This indicates that though the two systems were occasionally drifting apart, they were in a stable region of the attractor. It is possible for synchronization to occur by simply applying the feedback at intervals on the order of the inverse of the Lyapunov exponent since it takes a finite time for the trajectories to diverge. However if this were the case the applied feedback after a non-modulating period would be large since it is directly proportional to the difference in the two orbits.

By adjusting the threshold level of the comparator, the process could be impeded for any voltage level of $V_M(t)$. Synchronization, as expected, occurs when the cutoff threshold is any value above 48mV. Synchronization persisted when the cutoff level is reduced to 43mV. At this level feedback is applied on only 52% of the peaks. Below this level, the feedback is not sufficient to maintain a viable synchronization.
Figure 4.13: Chaotic signal and feedback pulses

Voltage drop across the master resistor, $V^M(t)$, along with the feedback perturbation. Note that feedback is only applied on the lower peaks, i.e. in the unstable regions of the attractor. In the stable regions, indicated by the upper peaks, no feedback is applied.
Figure 4.14: Chaotic signal and minimum feedback pulses

Synchronizing feedback can be withheld for any peaks above 43mV and synchronization maintained. Plotted is the voltage drop across the master resistor, $V^M(t)$, along with the feedback perturbation. Note that no feedback is applied whenever the peaks exceed 43mV.
Summary

In summary, synchronization of the chaotic resonators has been successfully obtained in each of the three experiments performed. In the first experiment, we evaluated the difference in the current (measured as the voltage drop across the resistor element of the resonator) between the two resonators on the peaks of the current. This difference was multiplied by a constant factor which represented a simplification of the Lai and Grebogi synchronization algorithm. The resulting product amplitude modulated for a fraction of the drive period the sine wave which drove only the slave resonator. In this manner a superb synchronization was achieved (See Fig. 4.7).

A second experiment replaced the constant amplification factor with the peak dependent factor as prescribed by the theoretical formalism. While this experiment was successful as well, synchronization was not noticeably superior to that obtained in the first experiment. This is due to the fact that in the unstable regime of the attractor the constant factor is identical to the prescribed factor. In the stable regime of the attractor, the form of the feedback factor is reasonably inconsequential to the synchronization. This was witnessed by varying the shape of the feedback factor. Synchronization was quite dependent on the DC bias of the entire factor. However, the slope of the upper portion of the factor (See Fig 4.11) had little consequence to the synchronization.

This conjecture of feedback application only when necessary was demonstrated by the third experiment. In this test, feedback was applied only in the unstable regime of the attractor. This area is defined by a peak of the resistor voltage being less than 48mV. In the stable regimes (peaks above 48mV) no feedback is applied. The synchronization observed in this experiment is identical to the previous two. Synchronization can be maintained when feedback applied only if the peaks were less than 43mV. Since peaks this high or lower represent only 52% of the possible peak heights, this technique offers a highly efficient means of synchronization.

The range of synchronization was tested versus variations in certain available
parameters. If the diodes are not well matched, then the synchronization that occurs is not as exact as when using matched diodes. This is to be expected since in the former case, there are differences between each attractor. On the other hand, synchronization does occur despite differences not only in the diode but also in the driving signal supplied to each resonator and the day to day temperature of the laboratory. There are a range of values of the SOPF amplification factor for which synchronization is obtained. This result is in complete agreement with the numerical predictions when calculating the global Lyapunov multiplier. The phase difference in the two waves which drove each resonator could be varied by as much as 12° and the two signals remained synchronous. For these reasons we consider synchronization by control to be a viable technique.
Andy Warholl made a film, *Empire*, 1964, in which the camera, focused on the Empire State Building, was simply left running for eight hours and five minutes. The point of the opus was to emphasize a minimalist point of view. Translated into synchronization, our motivation for a minimal technique is speed. The more efficient and simple the technique, the more likely the approach is to synchronize and control fast dynamical systems. This brings us to synchronization by unidirectional coupling.

Starting with two identical dynamical systems, we continuously measure the difference between observable signals from the two elements. This difference is amplified by a constant factor and the resulting product is added into one of the elements, designated as the slave, leaving the other, designated as the master, to operate untouched. Consider the $x$ system to be the master and $y$ as the slave. The coupling is then

$$\frac{d\bar{x}}{dt} = F(\bar{x}),$$

$$\frac{d\bar{y}}{dt} = F(\bar{y}) + \alpha(x_i - y_i)$$

where $F$ represents the dynamical systems and the coupling is through the $i$th component of the state vector. From observing Eq. 5.1, the coupling signal can be considered as an error signal since it is a measurement of the error in the synchronization. It approaches zero as the two systems synchronize. Unidirectional coupling presents a fast method of achieving synchronization and is limited electronically only by prop-
agation delays inherent in the integrated circuits necessary for implementation. If implemented optically in chaotic lasers such as semiconductor lasers subjected to destabilizing optical injection, the technique could potentially synchronize relaxation oscillations in the GHz range.

Further motivation for this technique lies in the interest of synchronizing arrays of slaves to a single master. A synchronous array could be constructed by linking the elements in either of two ways. The first would be to unidirectionally couple each slave in parallel with a single master. By synchronizing to the master, each slave element would be made synchronous with each other even though there would be no communication between individual slaves. The other manner of linkage is to chain each element so that the slave of element $n$ is the master of element $n+1$. By using this type of coupling, influences applied to any one element will have consequences to all subsequent elements. In either case, any desired controlling influences will only have to be applied to a single element, the master of the former method and element 1 of the latter.

In this chapter the experimental results of this technique are presented. In section II, the experimental apparatus is detailed. Experimental results of the synchronization obtained are then presented. It is shown that this method is tolerant to noise inherent in the arrangement as well as variances in the properties of the resonator components. In section III the effect of the coupling term on the resonator dynamical equations is examined. Further experiments demonstrate how the synchronizing coupling signal can be decomposed into a damping term and a driving term. The damping term when applied without the driving term will force the slave resonator into a periodic orbit. The driving term which is due to the master resonator drives the slave resonator in concert with the chaotic master. The chapter is summarized in section IV.
Unidirectional Synchronization Experiments

Setup

A description of how the coupling signal is obtained and applied is an apt beginning. From the chaotic attractor of a diode resonator circuit select out an arbitrary chaotic signal designated as the master signal $V^M(t)$ which, as in the previous work, is the voltage drop across the resistor. From another identical diode resonator circuit operating under the same conditions we wish to select another chaotic signal, designated as the slave $V^S(t)$, and synchronize it to $V^M(t)$. This is accomplished by measuring the difference between the two signals and feeding back a time varying, proportional amount $\alpha [V^M(t) - V^S(t)]$ to the sine wave driving the slave resonator. From Ohm’s law, the current through each resonator is then $V^M(t)/R$ and $V^S(t)/R$ where $R$ is the resistor component. The difference in the two currents is amplified and added to the driving wave. Hence the coupling constant in terms of the current is $\alpha R(I^M(t) - I^S(t))$. Then in the governing equations for the resonator, the master resonator is unchanged. However, for the slave equation, Kirchoff’s law concerning the voltage drop of the circuit, Eq. 2.6, is modified to being

$$V_o \sin \omega t + \alpha R(I^M - I^S) = V^S + L \frac{dI^S}{dt} + I^S(R + R_s).$$  \hspace{1cm} (5.2)

Here the superscripts, $M$ and $S$ refer to the master and slave resonator respectively. The quantity $\alpha$ is to be determined experimentally by adjusting the gain of an amplifier which acts on the difference voltage, until the two chaotic signals lock together. Note that this choice of sign for the feedback means that in absence of a master signal, $V^M(t)$, a dissipative term, $-\alpha V^S(t)$ is added into the slave dynamical system.

Experimental Arrangement

A block diagram of the experimental setup is shown in Fig. 5.1. Each diode resonator consists of a 1N4004 diode, a 33$mH$ inductor and a 90$\Omega$ resistor in series. Both
circuits were driven by the same Wave Tek 166 waveform generator at 80.5 kHz. The drive signal in the initial experiments was 2.92 Vrms which placed the master resonator in a regime just below its period-3 window. In order to insure that no signal from one of the resonators is being inadvertently fed back into the other resonator via the mutual drive wave, each resonator is buffered from the other by means of LF355 operational amplifiers which are operating in a noninverting mode. The master resonator is driven only with a sinusoidal wave. To the slave resonator, the feedback signal is added to the drive wave as described by Eq. 5.2. Both resonators are isolated from the loading effects of the observing oscilloscope by means of LF355 operational amplifiers operating in a voltage follower mode. These JFET op amps have high input impedances and relatively low load capacitances so as not to perturb the resonator. The voltage drop across each resistor is input into these op amps. Using another LF355, the difference between the resistors of each resonator, \( V_M(t) \) and \( V_S(t) \) is obtained and amplified by a constant factor \( \alpha \).

The amplified difference is input into a LM319 comparator and a DG303 analog switch. The comparator is routed to create a window comparison between \( |\alpha(V_M(t) - V_S(t))| \) and \( V_t \) where \( V_t \) is an arbitrary threshold voltage. As long as the difference is less than an arbitrary value, \( \delta V \), the comparator is held at TTL logic high (> 2.5 V) and the analog switch is held closed so that the amplified difference is used as the feedback perturbation \( \delta p \). However, outside this range, the switch is opened and no feedback is applied. Under conditions of no applied feedback, both master and slave signal are approximately 40 mVrms.

Observations

Under conditions of zero feedback, the driving voltage for the master and slave circuits are on average identical. Slight variations in the slave driving voltage do exist due the unavoidable noise involved in passing it through the summing amplifier and LH0002 driver. In Fig. 5.2(a) is an oscilloscope trace of \( V_M(t) \) versus \( V_S(t) \). The resulting
A schematic illustration of the strategy for synchronizing two almost identical chaotic circuits. The difference in the voltage drop of the master and the slave resistor is amplified and input into a comparator. As long as the perturbation is within an adjustable window, the switch remains closed and the feedback is continuously summed with the driving wave.
wide parallelogram region of phase space demonstrates that each signal was executing a different chaotic trajectory. The 45° slope of the figure is due to the chaotic peaks being approximately in phase as each signal followed the driving voltage. In Fig. 5.3(a) the un-amplified signal difference $V^M(t) - V^S(t)$ is plotted.

Initially, the drive signal to both resonators was set at 2.92Vrms. The threshold window was closed so that no feedback was applied and the gain factor of the operational amplifier was decreased to a minimal level. Then the window was completely opened and the amplification gain factor was increased. When the slave signal synchronized to the master, the slave driving voltage dropped to 2.83Vrms. This implied a feedback of approximately 3.5% relative to the master driving voltage of 2.92Vrms. The oscilloscope trace of $V^M(t)$ versus $V^S(t)$ in Fig. 5.2(b) clearly demonstrates that synchronization has been achieved. Below this feedback level the signals would not synchronize completely but rather would become partially synchronized.

During synchronization the absolute value of the signal difference $|V^M(t) - V^S(t)|$, fluctuated in time, never exceeding 2.5mV as shown in Fig. 5.3(b). In this optimized case, the gain, $\alpha$, was approximately 40. With this value the feedback signal $\delta V_f(t)$ never exceeded $\pm 100\, \text{mV}$. Once the signals were locked together, the comparator window became superfluous since the amplified signal difference was always within the window comparator implying that the analog switch then remained continuously closed. The purpose of the comparator window was to ensure that arbitrarily large perturbations are not being summed to the slave driving voltage. However, the use of a window comparator was not essential for achieving synchronization. The signals would still synchronize in an experimental arrangement in which the window comparator and the analog switch were removed and the amplified signal difference was fed directly into the slave driving voltage. Again the relative feedback level was approximately 3.5%.

Synchronization could not be maintained once the gain of the amplifier was greater than approximately 85. At this point the amplification of the inevitable difference
in the two resonator signals caused too much distortion of the driving signal. At the maximum gain for which synchronization could be achieved, the feedback exceeded $250mV$. This corresponded to a relative feedback level of 8.5%.

Once synchronization had been achieved for a given drive amplitude, $V_0$, along with a set amplification factor $\alpha$, $V_0$ could be varied across the full range of the resonator bifurcation diagram (Fig. 2.2), without varying the gain, and the synchronization would continue to persist. Note that since both resonators are driven by the same waveform generator, we are not radically driving one system away from its natural attractor as would be the case if one resonator was driven at one voltage and the other driven at a much different voltage. Thus synchronization could be maintained as the signals were changed freely between chaotic and periodic orbits. This observation follows from the fact that the chaotic attractors of both resonators are similar regardless of the drive amplitude and, more importantly, synchronization is feasible for a wide enough range of amplification factors that the same gain factor was neither too strong nor too weak in all parts of the bifurcation diagram.

When the resonators were initially placed in the naturally occurring period 3 orbit, synchronization was problematic. This seems to indicate that the amount of feedback necessary to influence a stable orbit is greater than that needed to influence the trajectory of a chaotic orbit. However, if the resonators were synchronized in a chaotic regime, then the drive wave was increased to the point that the master resonator went into a period-3 orbit, the slave usually remained synchronized to the master and would follow the master into the period-3 orbit. On the other hand, if the master was too near the period-3 window, the feedback would occasionally push the slave into a crisis and it would go into a period-3 orbit. Naturally synchronization was lost. Synchronization could not be re-established until the drive signal was reduced enough so that the slave dropped back into a chaotic orbit. Since the resonator exhibits hysteresis, this meant that a substantial reduction in the amplitude had to be made.
Figure 5.2: Unidirectional synchronization

(a) Oscilloscope trace phase portrait $V^M(t)$ versus $V^S(t)$ of the chaotic circuits when no feedback is applied. (b) The phase portrait when feedback is applied. The thinness of the trace indicates almost perfect synchronization.
Figure 5.3: Recorded difference of the two chaotic signals

The master-slave signal difference $V^M(t) - V^S(t)$: (a) no feedback applied, (b) feedback applied. The maximum signal difference is reduced by over a factor of ten when the signals are synchronized.
As mentioned above, once synchronization was achieved the amplified signal difference always remained within the window comparator. This implied that the analog switch remained closed so that feedback was applied continuously. In another experiment, the feedback signal was chopped in time once synchronization was achieved so that the feedback was turned off periodically. The experiment tested the persistence of synchronization. Additionally, this test provides a crude estimate of the Lyapunov exponent of the system. The largest interruption interval in which the feedback signal could be turned off and synchronization still maintained was 10\(\mu s\) which was 80.5\% of the period. The quality of synchronization degraded rather slowly as the chopping time was increased from 0\(\mu s\) to 10\(\mu s\) at which point synchronization was rather degraded. Beyond this time, synchronization was rapidly and completely lost. While the time of persistence can be related to the inverse of the Lyapunov exponent, it is presently unclear as to the exact relationship. This synchronization time can be used as forecast a crude estimate of the Lyapunov exponent. Since

\[ |\Delta V| = |\Delta V_0| \exp(\lambda t_h) \tag{5.3} \]

where \(t_h\) is the so-called horizon time, the Lyapunov exponent is

\[ \lambda = \frac{1}{t_h} \ln \frac{|\Delta V|}{|\Delta V_0|}. \tag{5.4} \]

In synchronization \(|\Delta V_0| \approx 1.5 mV\). 10\(\mu s\) later, the two are coming out of synchronization and \(|\Delta V| \approx 5 mV\). With these numbers, \(\lambda = 120 kHz\). To make a correspondence between these units and the dimensionless factor calculated numerically, divide \(\lambda\) by the driving frequency of the resonator (80.5 kHz) and by 2\(\pi\). This gives \(\lambda = 0.24\), a number about twice as large as the numerical calculation indicates.

Finally, care was taken to ensure that the two diode resonators were constructed as identically as possible and operated under similar driving voltages. (See Chapter 2 for details.) It is noteworthy to point out that synchronization could still be achieved
when the resonator circuits differed slightly from each other due to differences in the various circuit components. In addition synchronization could also be achieved when there existed noticeable differences between the master and slave driving voltages. In these cases the synchronization was somewhat degraded in the sense that the waveform of the slave signal showed noticeable differences from that of the master signal. A comparison of \(V^M(t)\) versus \(V^S(t)\) similar to Fig. 5.2(b) revealed a straight line which grew thicker towards the higher voltage end. For slightly dissimilar resonator circuits or nearly identical operating conditions, the chaotic trajectory content of the attractors are different for the two circuits. However, the synchronization employed here allows for successful synchronization of similar chaotic orbits.

**Theoretical Aspects of Unidirectional Synchronization**

Having performed the experimental verification of this method, we study the technique of unidirectional coupling scheme in order to determine underlying aspects concerning the reasons of synchronization. For convenience write Eq. 5.1 again,

\[
\frac{d\vec{x}}{dt} = F(\vec{x}),
\]

where \(\vec{x}\) is the master and the slave dynamical system which is

\[
\frac{d\vec{y}}{dt} = F(\vec{y}) + \alpha(x_i - y_i).
\]

As before \(\alpha\) is the coupling parameter and \(x_i\) refers to the \(i\)th component. The feedback error signal is based on the difference between one or more equivalent observable signals of the two chaotic systems is generated and continuously added back into only the slave system leaving the other untouched. This particular coupling scheme is scalar in that only one component is being measured. As an aside, other schemes have recently appeared which involve vector coupling [90]. In this case, the feedback is \(K(\vec{x} - \vec{y})\) where \(K\) is a matrix whose dimensionality is the same as that of the dy-
namical equations. Note that this vector coupling method has not yet been performed experimentally.

Instead of looking at the error signal as a single entity, recast Eq. 5.6 as

$$\frac{d\vec{y}}{dt} = G(\vec{y}) + \alpha x_i$$  \hspace{1cm} (5.7)

where

$$G(\vec{y}) = F(\vec{y}) - \alpha y_i$$  \hspace{1cm} (5.8)

Cast in this light the function $G(\vec{y})$ includes a damping term, $-\alpha y_i$. Then in Eq. 5.7 $\alpha x_i$ plays the role of an external drive signal. We desire to look at the effect of only the dissipation term on the undriven system, i.e. remove the $\alpha x_i$ term in Eq. 5.7. Then observe the effect of the damping as the $\alpha$ factor is increased from some negligible value to a point where the synchronization of the two resonators occurs. This is to be done by recording the peaks of $V^S(t)$ as a function of $\alpha$ so as to generate an $\alpha$ bifurcation diagram. At each value of $\alpha$ re-establish the synchronizing feedback, $\alpha[V^M(t) - V^S(t)]$, and record the standard deviation of the difference between the master and slave system so as to determine at what point in the bifurcation diagram synchronization commences. Experimentally a DPDT switch is placed on the non-inverting input of the op amp which subtracts the slave signal (inverting input) from the master (non-inverting input). One pole of the switch is connected to the master resonator while the other is grounded. In this manner, either the synchronizing error signal can be fed back into the slave or just the damping $-\alpha y$ term.

The experimental result is shown in Fig. 5.4(a) and (b). It is important to point out that this bifurcation experiment was performed on a pair of resonators which had different diodes than those used in the original experiment described above. Substantial variations in the diodes exist. In the original work, synchronization was lost when $\alpha > 85$. In this experiment, synchronization was not lost until $\alpha > 130$.

Figure 5.4(a) shows the standard deviation, in units of $mV$, between the master
and slave signals versus $\alpha$. Figure 5.4(b) is a corresponding bifurcation diagram of the slave resonator as a function of $\alpha$ when the master input is shorted. At negligible levels of $\alpha$, the slave resonator operates chaotically as expected and the master and slave are out of synchronization. However as $\alpha$ is increased the damping causes reverse bifurcations out of chaos into a period-4 orbit and finally into a period-1 orbit. Not shown on the graph is the fact that on further increase of $\alpha$ ($\alpha > 130$) the drive wave is over modulated and synchronization is lost. This collapse is potentially a topic of future studies. Note that $\alpha$ can be quite large. Thus the damping term, $\alpha V^S(t)$, can be a significant fraction (up to 50%) of the input sine wave. The distortion of the wave by this dissipation term can be crudely summarized. The effect of the term is to decrease the amplitude of the sine wave and to shifting of the wave downward. Figure 5.4(a) can be considered a rough composite of the bifurcation diagram obtained as a function of the drive wave amplitude, Fig. 2.2 and the DC bias bifurcation diagram, Fig. 2.6.

The period-1 orbit produced by the distorted drive wave is not the same as the natural period-1 orbit due to a low amplitude sinusoidal drive wave. The distorted wave gives rise to an attractor which differs from the usual attractor. Thus the period-1 orbit in the new phase space follows a different trajectory than for pure sinusoidal driving.

Referring to Fig. 5.4(b), as the $\alpha$ factor is being increased, the two resonators, when the complete synchronization signal is applied to the slave resonator, slip into synchronization as evidenced by the dropping of the standard deviation of the error signal. The graph displays a remarkable result: synchronization can only be achieved when the slave is driven into a periodic orbit. Furthermore, a good synchronization can only be achieved when the slave is in a period-1 orbit. The effect of the feedback is to place the slave resonator into a periodic orbit and simultaneously drive the periodic slave in resonance with the master. Taken individually the two components constitute a sizable fraction of the drive wave, heavily damping the system on one
Figure 5.4: Bifurcations as a function of the gain

(a) The standard deviation of the difference between the master and slave orbits as the amplification factor $\alpha$ is increased. For a large range of $\alpha$, synchronization can be achieved. Not shown is the deviation when $\alpha > 120$. For these values, synchronization is lost. (b) A bifurcation diagram of the slave resonator with the damping term $-\alpha V^s(t)$ only applied. As $\alpha$ is increased, the resonator reverse bifurcates out of chaos into a period-1 orbit. Correspondingly, synchronization only occurs when $\alpha$ is large enough so as to place the slave into the period-1 orbit.
hand while heavily driving it on the other. However, when considered as a composite, the two influences negate and the sum coupling signal is quite small.

A handicap of this method of synchronization is that there is no a priori means of calculating $\alpha$ experimentally before the synchronization is attempted. A single time series of a dynamical system will not provide enough information to derive this value. Instead, one would have to generate the bifurcation diagram in $\alpha$ (Fig. 5.4(a)). Even then one can only anticipate that the value necessary to damp the system into a period-1 orbit is sufficient to synchronize the two systems.

Summary

Two chaotic signals generated by separate but identical diode resonators could be synchronized by a simple proportional feedback algorithm. To achieve synchronization, the feedback had to be applied for a minimum duration of $1\mu s$. The synchronization was maintained by applying relative feedback levels between 3.5% and 8.5%. Once synchronization was established, it could be maintained as the driving voltage was altered while the feedback gain was held constant. We found this synchronizing proportional feedback scheme to be robust and easier to implement than other traditional synchronization methods. The combination of this synchronization scheme with current chaotic controlling algorithms could potentially find use in many applications including communications and chaotic lasers.

If the coupling term is broken into its damping and driving parts, synchronization is seen in a different light. In order for synchronization to occur, the damping term must be sufficient enough so as to place the slave resonator in a stable period-1 orbit. The drive term then causes the replacement of the slave periodic orbit with the chaotic master orbit so that two operate in concert.

Finally, the synchronizing values of $\alpha$ are determined by either trial and error as in the initial experiments performed or by generating a bifurcation diagram as a function of $\alpha$. There is no established means of calculating $\alpha$ from time series.
CHAPTER 6

SUMMARY AND CONCLUSIONS

In the five years since the publication of the Pecora and Carroll synchronization paper and the Ott, Grebogi, and Yorke control work, new experimental opportunities for the exploration of nonlinear phenomena have been created. Their work along with the enormous reservoir of preceding research has stimulated the studies of applied nonlinear dynamics. The results have been the development of time-delayed feedback chaos control, synchronization of chaotic elements by the mutual coupling of the elements, by a unidirectional coupling method and by the control formula of Lai and Grebogi. Among the practical applications which are being developed because of this research are cardiac control devices, secure communications equipment, and synchronized arrays of stabilized lasers.

Because it is experimentally time consuming to test new ideas on sophisticated electronics or sensitive laser systems, it is important to have an experimentally accessible nonlinear system to use as a proving ground. Though it is a physically simple device composed of only a diode, inductor and resistor, the diode resonator exhibits nonlinear phenomena that are also found in other dynamical systems such as externally modulated fiber lasers [54] and semiconductor lasers subjected to optical injection [62]. For a range of drive parameters, the resonator has a nearly 1-dimensional attractor and displays chaos in the amplitude of its current. For these reasons, the diode resonator can serve as a paradigm for nonlinear experimental studies for driven dynamical systems.
Summary

A goal of this thesis was to perform, for the first time, the synchronization using the control scheme originally proposed by Lai and Grebogi [5]. In the case of the diode resonator, the amplitude of the driving wave was chosen to be the dynamical parameter to which the required minute feedback perturbations are applied. The necessary feedback is composed of the difference between the voltage drops across the two resonator resistors multiplied by an amplification factor which is a function of the peaks of the voltage drop across the master resistor, \( V_M(t) \). As an initial hypothesis, this amplification factor be approximated by a constant value. This is designated as synchronization by occasional proportional feedback (SOPF). The next step forward was to determine, using only experimentally obtained data, the actual peak dependent perturbation. Its physical significance can be seen by observing the components of the feedback perturbation. We first need to know how far apart one orbit is from another. Next we need to estimate the strength of the instability, i.e. the Jacobian matrix. Finally we must recognize how sensitive the dynamical system is to changes in its dynamical parameters, i.e. the shift. When a dynamical system can be considered to be 1-dimensional, these factors are scalars and can be extracted using first return maps. The feedback factor is then directly proportional to the product of the Jacobian and the orbit difference and inversely proportional to the shift. We measure each of these factors on the peaks of the current, compose the appropriate product, and modulate the amplitude of the driving wave for some fraction of the drive period. An estimation for the Jacobian, expressed as a function of the current peaks, is the slope of the return map. Next we measure the upward shifting of the return map which is caused by a small variation in the amplitude of the drive wave. The negative of the ratio of the Jacobian to the shift is the peak dependent amplification factor. If the peaks of \( V_M(t) \) are low, then this factor is a constant. After the peaks reach a height of 48 mV, this factor drops off rapidly. Thus the SOPF factor is identical to the peak dependent factor whenever the peaks are
For the resonator, the Jacobian is less than one whenever the peaks of the diode resonator are higher than $48mV$. Hence the resonator is locally stable in these regions. This natural stability implies that it is not necessary to administer feedback here since synchronization will be maintained. Thus the amplification factor of the feedback, as long as it is small, plays little role in determining synchronization. This observation was verified experimentally. Below $48mV$ the Jacobian is greater than one and the resonator is locally unstable. In this unstable region, the peak dependent factor is constant. Therefore in the unstable region, SOPF is identical to the Lai and Grebogi prescribed algorithm yet represents a technique experimentally simple to implement.

The possibility of synchronization can be quantified by computing the local and global Lyapunov multipliers. From the computation of these, one finds that the effect of the perturbation is to force the local multipliers to be less than one. When the global multiplier, i.e. the geometrical average of the local multipliers, is reduced to less than one, synchronization can occur. Because it pushes the local multipliers to the zero floor, the Lai and Grebogi peak dependent feedback factor presents the most efficient means of synchronization. Yet the global multipliers are also less than unity for a band of constant factors as suggested by SOPF. By calculating Lyapunov multipliers for other dynamical systems, it may be possible to ascertain appropriate simplifications to the feedback formalism.

As a first experiment, the SOPF scheme was implemented. Synchronization was obtained using a constant amplification factor multiplied by the voltage drops across each resonator component. This factor was applied starting on the peaks of the circuit current and having a duration of between 8 to $12\mu s$ (56% to 84% of the drive period). Below this period of time, the feedback did not significantly affect the amplitude and the synchronization was lost. For times greater than $12\mu s$, the feedback over modulated the drive wave again destroying the synchronization. Synchronization is reasonably tolerant of variances in the resonator properties, the manner in which each
resonator is driven, and laboratory temperature fluctuations. Naturally the superior synchronization occurs for ideally matched conditions.

In the second experiment the peak dependent factor as prescribed by the theoretical formalism was implemented successfully. This factor was approximated by the inverted $I - V$ curve of a rectifier diode in series with a resistor. Important observations concerned the feedback factor. The DC bias of the factor (the shifting upwards or downwards of the entire factor) played a large role in whether synchronization could be maintained or not. On the other hand, the slope of the upper section was relatively inconsequential.

As a third experiment, feedback was applied only in the unstable regime of the attractor. This area is defined by a peak of the resistor voltage being less than 48mV. In the stable regimes (peaks above 48mV) no feedback is applied. The synchronization observed in this experiment is identical to the previous two experiments. Synchronization could be maintained when feedback was applied only if the peaks were less than 43mV. Since peaks this high or lower represent only 52% of the possible peak heights, this technique offers a highly efficient means of synchronization.

An alternative method of synchronization is the technique of unidirectional coupling. In this method, like the synchronization by control scheme, we amplify by a constant factor the measured difference in the voltage drops across the two resistors. In contrast to SOPF, the resulting term is continuously added to the drive wave of one of the resonators leaving the other to operate independently. Hence there are two differences between unidirectional coupling and the synchronization by control scheme. In the former, feedback is added to the governing equations. While for synchronization using control, a parameter is modulated by the feedback. Secondly feedback is continuously applied in the unidirectional coupling scheme. For the control method, we must evaluate the feedback occasionally and apply it for some fraction of the drive period. Together the two methods can be used for a variety of dynamical systems. Synchronization by control presents an economical method to be used for slow sys-
tems such as biological ones. Unidirectional coupling, due to its basic simplicity, is suited for high speed dynamical systems.

Experimentally synchronization by the unidirectional coupling technique using two diode resonators was successfully demonstrated. The synchronization was maintained by applying relative feedback levels between 3.5% and 8.5% of the drive wave. Since one waveform generator drove both resonators, we could first synchronize the two and then vary the amplitude of the drive wave so as to observe the persistence of synchronization across the bifurcation diagram. Once synchronization was established, it could be maintained as the driving voltage was altered while the feedback gain was held constant. Hence synchronization is tolerant to the amplification factor.

If the coupling term is broken into its damping and driving parts, synchronization is seen in a different light. To investigate this, the influence of the master resonator was removed from the feedback term leaving only the damping term as the applied feedback to the slave resonator. As the intensity of the damping feedback was increased, the slave resonator reverse bifurcated out of chaos and into a period one orbit. At each level of amplification, the influence of the drive term was restored to the feedback in order to determine if synchronization would occur. Synchronization would not occur until the feedback amplification, \( \alpha \), is large enough to place the damped slave resonator into the period one orbit. At this point, the drive term will force the slave resonator in concert with the chaotic master. Essentially, the slave attractor is replaced by an attractor which is synchronous to the master.

The unidirectional coupling technique, for the diode resonator, is stable in operation and easier to implement than either the method of Pecora and Carroll or that of Lai and Grebogi. However, the necessary feedback is determined experimentally only through trial and error. Synchronization using control, on the other hand provides, a procedure for the determination of the correct feedback. Consider the following scenario. Enclosed in a black box is a low dimensional chaotic dynamical system which issues a scalar time series. An observer, privileged to only the time series, desires to
synchronize two of these boxes. Then by following the synchronization using control procedure outlined in chapter 3, the observer will be able to determine the correct feedback necessary to synchronize two black boxes. However, if the observer wanted to synchronize the two by unidirectionally coupling, then they will only be able to guess the correct amplifying factor.

This unidirectional coupling scheme produces a synchronization equal and often superior to the synchronization by control experiments described above. This is because the feedback signal is a continuously applied error signal. As the resonators are pulled together the error drops to zero and, in principle, a perfect synchronization would occur. Contrast this method with the synchronization by control scheme which is a technique derived for maps but implemented in a dynamical system. When applied to flows, feedback is calculated on a peak, applied for the entire period, then re-evaluated on the next peak. However we found that by applying feedback too long, the resonator was over modulated and synchronization lost. This was due to an over modulation of the drive wave and not to electronic limitations such as the acquisition time of the sample and hold integrated circuit. Thus, though synchronization by control is also based on the difference between the two orbits, feedback consists of a series of finite time pulses. Due to the nature of chaos, during the interval when feedback is not applied, the dynamical systems will diverge. Hence a feedback pulse is almost guaranteed. While this does not negate the viability of the scheme, the more accessible approach to synchronization is via unidirectional coupling - as long as the necessary amplification factor can be evaluated.

Future Possibilities

Several possibilities exist for future consideration as extensions of this work. Controlling arrays of synchronized chaotic dynamical systems presents one option. Current work with arrays has been for the most part limited to numerical studies [24, 25]. A possibility of an experimental implementation is to create an array of unidirectional-
ally coupled elements whereby element \( n \) is the master of element \( n+1 \). Though this is not an array in the traditional sense of the word (because some type of mutual coupling between the elements does not exist), it still presents possibilities for identical synchronization between each element. Furthermore, the first element becomes the master of the chain. By controlling the single master, all the elements of the array could be controlled as well as synchronized. Thus only one control device is required to stabilize the chaos in the master. Potential problems are found in the differences between the individual elements. As seen in the performed experiments, synchronization depends on effectively matching the elements. It would be expected that the quality of synchronization would diminish as more and more elements were added to the chain. However, differences could be mitigated by proper adjustment of the coupling strengths between each element.

Fiber lasers are increasingly being used in the communication industry. These lasers, whose fiber core is lightly doped (\( \approx 200\text{ppm} \)) with either Erbium or Neodymium, present a possibility of immediately implementing this work in optical systems. Fibers can be driven chaotic by driving the intensity of the semiconductor laser pump. Because relaxation oscillations are less than 20kHz, the electronic circuits, constructed as part of this research, could be used to modulate the current of the diode laser. Conceivably synchronization could then be obtained. A potential problem with the scenario is that the relaxation oscillations of the two lasers are not likely to be either in phase or even at the same frequency. These problems would have to be solved before the desired result is obtained.

An important area of future research will focus on dynamical systems with high dimensional chaotic attractors and on spatiotemporal chaos. Arrays of evanescently coupled fiber lasers can be used to test newly developed ideas concerning control of spatiotemporal chaos [91, 92]. While synchronizing these arrays presents a completely new problems, the work performed here will serve as a fundamental introduction for future research in these fields.
A third option to consider concerns fast dynamical systems which operate in the high MHz or even GHz range. At speeds this fast, control schemes using electrical devices become difficult to implement due to the propagation delays inherent in the devices. For the diode laser, chaotic instability occurs when reflections from an external mirror are introduced into the laser cavity. The goal is to control the unstable steady state operation of this laser whose relaxation oscillations are approximately $4GHz$. To do this, a new all optical technique of implementing control will have to be developed using the electrical methods as guidelines. The time delayed feedback control scheme described in chapter 1 is a method which can be successfully adapted for these speeds. We direct the chaotic output radiation into a Michelson interferometer in order to construct the required feedback. This interferometer splits the incident radiation and directs it to separate mirrors some distance from the beam splitter. Reflected light from both paths returning to the splitter are recombined with a phase difference proportional to the optical path difference. Thus this device creates the necessary time delayed feedback which, in principle, will stabilize the chaotic oscillations.

Finally, a goal of this performed research was to demonstrate that applications of chaos are experimentally feasible. Because of the successful results, future uses of nonlinear dynamical systems including and beyond the possibilities described above becomes a more tangible prospect.
APPENDIX A

TOOLS OF CHAOS
Introduction

If experimental chaotic data is to be processed, then there must be a means of recording the signal followed by appropriate data analysis. In this appendix, the data acquisition apparatus is described and the process used to generate bifurcation diagrams is explained. Subsequently the question always arises as to whether the system is operating chaotically. The most definitive indication of chaos from a time series is via a Lyapunov exponent calculation. Several research groups have written code which purports to calculate the exponents from a scalar time series. A few of these programs are described below. The lesson to be learned is clear: proceed with caution. It is not just noise which can influence the data. The resolution of the data can be influential as well. Finally required input parameters of the programs can play a major role.

In section (A.2) data acquisition is discussed. In section (A.3) the Lyapunov exponent algorithm of the INLS (Institute of Nonlinear Studies, Univ. of California, San Diego) is analyzed. The Wolf method [77] is presented in section (A.4). In section (A.5), other methods are briefly mentioned. Finally in section (A.6) the delay coordinate reconstruction of the attractor is introduced using the mutual information as an indication of the proper delay time to choose.

Data Aquisition Programming

Traditionally most graphical experimental results are unimpressive photographs of an oscilloscope screen. However, the computer in conjunction with digitizing units is being increasingly used to obtain, analyze, and display information. The requirements for this latter process are: a digitizing unit capable of being driven by external commands, a method of communicating with the host computer, and efficient software to drive the process and analyze the data. Because both the HP3325A waveform generator and the RTD710A digitizer feature GPIB (General Purpose Interface Bus) IO ports, both data and instructional commands could be received from and sent
to an Hewlett-Packard Apollo 400S workstation. Using the HP VEE-Test program (an iconic programming language for engineering problem solving) developed specifically for HP workstations, a program was written to program the instruments and to receive and analyze data. For example consider the flow of the program which can obtain bifurcation diagrams. First set the waveform generator to the appropriate amplitude, frequency, and DC offset values. Next instruct the digitizer to record a time series of the resonator and retrieve this information. Then extract the peaks of these time series, graph and save these peaks. The appropriate waveform generator parameter is then incremented by a small amount and the process repeated. In this manner, a bifurcation diagram could be generated in a few minutes. Furthermore, one can zoom in on a particular region of the bifurcation diagram by simply adjusting the range of parameters. In contrast, a decent bifurcation diagram created numerically using a Sun SPARC 10 workstation took a several hours.

One of the drawbacks of recording and displaying data in this manner as opposed to the traditional method of photographing an oscilloscope screen is disproportionate weighing given to the distribution of points. To illustrate the problem consider the period 2 region of the bifurcation diagram shown in Fig. 2.2(b). Each branch of the period two region is composed of approximately 150 peaks per recording interval. Say that 148 of these lie one on top of the other while the last two, due to noise, create the thickness of the lines. These show up visually as being equivalent to the majority of points. In contrast, on the oscilloscope screen, one merely turns down the display intensity and the noisy points fade away eclipsed in brightness by the majority. For this reason, sometimes simple oscilloscope patterns can appear superior to plots obtained in the above manner. In particular, in the synchronization literature, the authors often plot the synchronized signals on a X-Y axis. Using a photo of the oscilloscope, these lines can appear quite thin. However, a digitization process of the same experiment would shown a much thicker line. With time, this problem could be solved, but one must balance aesthetics with deadlines.
The INLS Lyapunov Exponent Algorithm

There are several methods of determining the Lyapunov exponents from a time series. One of the most recent to appear in the literature is that proposed by the INLS (Institute for Nonlinear Science, University of California, San Diego) group of Abarbanel, Brown, Bryant, and Kennel [76, 93, 94, 95, 96]. Since an algorithm which can predict these exponents is highly desirable, this appendix steps through the procedure as proposed by this group. In short, the method involves two key steps. The first is to accurately estimate the Jacobian matrix of the dynamical system. Once this matrix is found, attempt to determine the linear evolution of the mapping so as to obtain the Lyapunov exponents.

To determine the Jacobian matrix from a time series, use the delay-coordinate reconstruction of the attractor (See below). Defining a vector in the delay coordinate space as $\mathbf{y}$, take a sample, $N_b$, vectors all lying close to $\mathbf{y}(k)$ and ordered by

$$|\mathbf{y}(k) - \mathbf{y}^1(k)| < |\mathbf{y}(k) - \mathbf{y}^2(k)| < \ldots |\mathbf{y}(k) - \mathbf{y}^{N_b}(k)|.$$

These points will map as

$$\mathbf{y}(k+1) = F(\mathbf{y}(k)), \mathbf{y}^1(k+1) = F(\mathbf{y}^1(k)), \ldots \mathbf{y}^{N_b}(k+1) = F(\mathbf{y}^{N_b}(k)).$$

Look at small deviations from the center of the neighborhood $\mathbf{y}(k)$.

$$\mathbf{z}(k) = \mathbf{y}(k) - \mathbf{y}(k).$$

These deviations map as

$$\mathbf{z}(k+1) = F(\mathbf{y}(k)) - F(\mathbf{y}(k))\mathbf{z}(k) + 1 = F(\mathbf{y}(k) + \mathbf{z}(k)) - F(\mathbf{y}(k)).$$
An expansion gives

\[ \bar{z}(k + 1) = H_1 \bar{z}(k) + H_2 \bar{z}(k) \bar{z}(k) + \text{higher orders}. \quad (A.5) \]

From this, \( H_1 \) is the desired Jacobian matrix. By using a least squares procedure, determine these Taylor expansion coefficients. At first glance there is no need to keep the higher orders since one is only interested in the first order term. However, by keeping these in the expansion, pressure is taken off the linear term and the estimation of the Jacobian matrix will be more accurate.

Once the Jacobian matrix has been obtained, one can then determine the eigenvalues, and stable and unstable directions. Additionally one can obtain from the Jacobian, i.e. the determinant of the Jacobian matrix, the rate of contraction of the phase space. The matrix allows a calculation of the evolution of a region about an arbitrary initial condition. Hence by observing how a ball of points centered on this orbit point deforms, the stable and unstable directions are obtained.

The second step is to use the obtained Jacobian matrix in order to evaluate the Lyapunov exponents. Given some mapping

\[ \bar{y}_{k+1} = F(\bar{y}_k). \quad (A.6) \]

Then making a small perturbation to \( \bar{y}_k \), i.e. \( \bar{y}_k \rightarrow \bar{y}_k + \delta \bar{y}_k \) the linearized mapping becomes

\[ \delta \bar{y}_{k+1} = DF(\bar{y}_k) \delta \bar{y}_k. \quad (A.7) \]

If this process is repeated \( L \) steps

\[ \delta \bar{y}_{k+L} = DF(\bar{y}_{k+L-1})...DF(\bar{y}_{k+1})DF(\bar{y}_k)\delta \bar{y}_k\delta \bar{y}_{k+1} = DF^L(\bar{y}_k)\delta \bar{y}_k. \quad (A.8) \]
The Oseledec matrix is defined as

\[
O_{SL}(x, L) = [DF^L(\tilde{y}_k)]^T [DF^L(\tilde{y}_k)]^{-2L}.
\]  

(A.9)

In the limit that \( L \to \infty \)

\[
O_{SL}(x) = \lim_{L \to \infty} O_{SL}(x, L).
\]

(A.10)

This matrix is well defined and its eigenvalues are independent of the initial conditions that lead to the orbit of attraction. The eigenvalues are real, positive, and given by \( e^{\lambda_c} \). The \( \lambda_c \) are the global Lyapunov exponents of the mapping. The INLS method of finding Lyapunov exponents from an experimental time series is to determine the Jacobian matrix from the data points then calculate the eigenvalues from the Oseledec Matrix. We want to find the eigenvalues when \( L \) is large. However, this matrix is likely to be ill-conditioned and the round-off errors will degrade the results significantly. A method to avoid this problem is by use of the so called QR decomposition. The QR decomposition consists of decomposing some matrix, \( M \), into \( M = Q \cdot R \) where \( Q \) is an orthogonal matrix and \( R \) is a right (upper triangular) matrix. The useful property of triangular matrices is that the product of triangular matrices is triangular and their eigenvalues are just the diagonal elements.

To determine the eigenvalues, first consider some matrix \( A \) such that

\[
A = A(N)A(N - 1)...A(1).
\]

(A.11)

Now perform a QR decomposition on the matrix \( A(i)Q(i - 1) \) This gives

\[
A(i)Q(i - 1) = Q(i)R(i)
\]

(A.12)

and so

\[
A(i) = Q(i)R(i)Q^T(i - 1)
\]

(A.13)
where $Q(0) = I$. The first component of $A$ is then

$$A(1)Q(0) = A(1) = Q(1)R(1).$$ \hfill (A.14)$$

This is a recursive procedure from which one builds up to the entire $A$ matrix. And so the entire matrix $A$ can be decomposed into

$$A = Q(N)R(N)Q^T(N-1)\cdot Q(N-1)R(N-1)Q^T(N-2)\cdot ...R(1)$$

$$= Q(N)R(N-1)R(N-2)...R(1).$$ \hfill (A.15)$$

By equating the Oseledec matrix with $A$, we can write

$$(DF^L)^TDF^L = Q_1(2L)R_1(2L)R_1(2L-1)...R_1(1) = M_1$$ \hfill (A.16)$$

where the subscripts are for notational purposes. Now define

$$M_2 = R_1(2L)R_1(2L-1)...R_1(1)Q_1(2L).$$ \hfill (A.17)$$

Then

$$M_2 = Q_1^T(2L)M_1Q_1(2L).$$ \hfill (A.18)$$

Now perform a QR decomposition on $M_2$

$$M_2 = Q_2(2L)R_2(2L)R_2(2L-1)...R_2(1).$$ \hfill (A.19)$$

If the process is repeated $k$ times, one has

$$M_k = Q_k(2L)R_k(2L)R_k(2L-1)...R_k(1).$$ \hfill (A.20)$$

The point of repeating this for $k$ steps is that $Q_k \rightarrow I$ for $k$ being only 2 or 3. This means that $M_k$ is triangular and the eigenvalues are just the diagonal elements.
Therefore the Lyapunov exponents are

\[ \lambda_\alpha = \frac{1}{2L} \sum_{j=1}^{2L} \ln| R_k(j)_{\alpha\alpha} |. \]  \hspace{1cm} (A.21)

Figure A.1 shows a calculation of the Lyapunov exponents for data generated numerically for the Lorentz system shows how convergence on the correct exponent is a function of the number of Jacobians matrices, \( L \), used in the calculation of the Oseledec matrix. INLS has noted that if one looks at the exponents when \( L \) is a small number, then the obtained exponents measure the diversion for only a small region of the attractor. Hence we can call these local Lyapunov exponents. They are a function of the position of the orbit in the attractor. For small values of \( L \) the variation in the exponent can be substantial.

A problem with this code is that it was developed in double precision code using numerical experimental data. In reality, the precision of the data is a function of the speed of the dynamical system. Currently most digitizing recorders are 8 to 10 bit vertical resolution. This means that at best one must work with 4 significant digits of information. This can be enhanced by going to very slow dynamical systems where the data can be recorded using high precision recording voltmeters. As an aside, the ANALOGIC corporation makes a 16 bit vertical resolution digitizer capable of sampling up to 1 MHz. Hence if sampling at the Nyquist frequency, one could record a dynamical system whose maximum frequency is 500 kHz. A second problem with the code is that it is unstable when the data is periodic. For example, if one uses a simple sine wave as the data, the code will predict a large positive Lyapunov exponent. This renders the code rather useless. The code also demands that one specify the number \( L \) of Jacobian matrix to calculate, the number of neighbors, \( Nb \), the time delay \( T \), the dimensionality of the system, \( n \), and the order of expansion \( H_i \). To one extent or another, the Lyapunov exponents are dependent on the values used.

Figure A.2 shows a calculation of the Lyapunov exponent as a function of the number
Figure A.1: Largest Lyapunov exponent of the Lorentz system

The plot shows how the Lyapunov exponent decreases as the number of Jacobians used to estimate the Oseledec matrix increases. The two traces represent calculations for floating point precision data and two decimal point precision data.
of neighbors used for estimating the Jacobian. All the other parameters were kept constant. If the data, generated from the Lorentz model, is double precision (lower trace), there is no problem. On the other hand, for limited precision data certain values of $Nb$ generate largely erroneous results. Furthermore, exponents generated for the double precision data are lower than the limited precision data.

Figure A.3 shows an even more dangerous case of using the the wrong number of vectors in the construction of the delay space. These exponents were calculated using experimentally obtained data from the diode resonator. It is uncertain as to the particular problems which have caused this abnormality.

In summary, in order to implement this code successfully, one must have a time series which is precisely measured as well as accurate.
Figure A.2: Lorentz system Lyapunov exponents

The graph shows how the largest Lyapunov exponent is a function of the number of vectors used to determine the Jacobian matrix. In the case of floating point precision, no computation truncation occurs. However, for the case of 2 decimal point precision, pathologies arise.
Figure A.3: Lyapunov exponents of the Lorentz system

The largest and second Lyapunov exponent as a function of the number of data points in the file. For certain regions unknown computational limitations cause anomalous readings. Note that the second exponent should be zero.
Wolf Algorithm to Calculate the Largest Lyapunov Exponent

One of the earlier (1984) techniques to determine the largest Lyapunov exponent was published by Wolf et al. [77]. In the original work, a simple code was shown which carries out the process. Though a modern version is available with source code in both FORTRAN and C (The latter is a direct translation from the FORTRAN. Ugly!) [97], the improvements lie not in the technique of the evaluation but rather in a program which sorts the data into a form more rapidly digested by the exponent code. The code then runs up to 1000 times faster [97].

As with the INLS code, it is useful to step through the process. Given a time series \( x(t) \), an \( n \)-dimensional phase space reconstruction is made in delay coordinates. Thus as with other calculations, one must be able to estimate the dimensionality of the attractor and the delay time. Since this algorithm was produced before Fraser's suggestion about the first minimum of the mutual information function being used as the proper delay time, Wolf leaves the choice of \( T \) arbitrary. Wolf suggests that \( T \) be chosen as about \( 1/3 \) the period of the orbit which in fact is quite close to the first minimum of the delay coordinate for not only the diode resonator but also the Lorentz system. A delay vector, designated as the fiducial, pointing to some arbitrary point in the orbit is chosen and then the nearest neighbor to this point is determined. From these two points, the Euclidean distance between them, designated as \( L(t_0) \), is calculated. The evolution of these two vectors is followed for a short time, \( t_1 \). The distance between the two will have evolved into \( L(t_1) \) as the second vector will have diverged from the fiducial orbit. If the proximity of the two vectors remains close, the evolution is continued. On the hand, if the separation is larger than a user set value, the program then records the initial separation, the final separation, and the evolution time. Having recorded this information, it searches for the nearest vector to the evolved fiducial. The criteria of this search is that the distance between the fiducial and the newly found vector is small and the angular separation is also small. Here the angle of separation is defined by the fiducial as its vertex and points along
the rays by the old evolved vector and the newly found one. What this means is that in a multidimensional phase space, we don’t accidentally choose a vector which will evolve along a separate branch of the attractor. Both the initial separation and the final separation are parameters input by the user. This is necessary since the scale of the data and the noise level are known only to the user. The process is continued until the fiducial has traversed entire data series. The estimate for the largest Lyapunov exponent is then

\[
\lambda_1 = \frac{1}{t_M - t_o} \sum_{k=1}^{M} \log_2 \frac{L(t_k)}{L(t_{k-1})}
\]  

(A.22)

where \( M \) refers to the total number of replacement steps.

Note that Wolf evaluates the Lyapunov exponent using \( \log_2 \) while in INLS Lyapunov exponent calculation incorporates the natural logarithm, \( \ln \). For this reason, there are apparent discrepancies in the published works of each group [76, 77]. If one calculates the Lyapunov multiplier for each calculation then approximately the same values are obtained. For example, the principle Lyapunov exponent for the Lorentz system is 1.51 and 2.16 for the INLS and Wolf codes respectively. Calculating \( LM_{INLS} = e^{1.51} = 4.53 \) and \( LM_{Wolf} = 2^{2.16} = 4.47 \). This is a 13% difference.

This code appears to be quite insensitive to the number of significant digits of data. Using numerical data obtained from the Lorentz system, the calculated Lyapunov exponent varied only a few percent as the number of significant digits was reduced from 8 to 2. This is really the crucial test. A noted above, the accuracy of experimental data is limited by the measuring apparatus.

A digitized sine wave was used as a test to check if the program could evaluate a periodic orbit. Using a two-dimensional reconstructed phase space, the Wolf program successfully calculated the principal Lyapunov exponent to be zero.
Other Lyapunov Exponent Codes

In the course of examining Lyapunov exponent codes, a code by Lathrop and Kostelich [98] and one written by K. Briggs [99] were examined. In short, these codes were deficient. The code of Lathrop and Kostelich was unwieldy and for the effort spent, I was not able to successfully compute the exponents for the Lorentz system.

The Briggs's code crashed repeatedly. To give credit to Briggs's, he is aware of its limitations [100]. The point he was trying to make can be found within the code of Abarbanel. Briggs realized that when trying to determine the Jacobian of a dynamical system from a time series, one should do a least squares fit using higher order polynomials rather than a linear fitting. This is because a linear interpolation will not be able to handle the natural curving of the points along the attractor.

The Mutual Information

The idea of the mutual information is drawn from the information theory work of Shannon [101] who showed that the entropy of a system is a useful measure of its information content. The mutual information can be determined by the following procedure. Consider a source $S$ which issues events $s_i$, where we don't know $s_i$ before it is emitted, but we do know the probability of occurrence: $p_1, p_2, p_3, \ldots$ Is there a measure, to be called $H$, of the uncertainty in the outcome? If so, $H$ must (a) be continuous and (b) be maximum if all $p_i$ are equally likely. Furthermore, (c) if the choice can be broken down into separate choices then $H$ will be the weighted sum of the individual choices so that $H$ of separate choices equals $H$ of the single choice, viz $H(a,b,c) = H(a,b) + aH(b,c)$. The measure, $H$, which satisfies the above is the entropy of the system.

$$H(S) = - \sum_i P_s(s_i) \log P_s(s_i).$$  \hspace{1cm} (A.23)

Here we are summing over the possible messages. If the possible outcomes are equivalent (like dice), then the entropy is $H(S) = \log P_s(s_i)$, i.e. this is the maximum
possible value for the entropy and it means that we don’t know anything. On the other hand, if we know everything, then $H(S) = 0$.

Consider a coupled system $(S, Q)$ where $s$ refers to a measurement of $S$ and $q$ refers to a measurement of $Q$. If we measure system $S$ and find it to be in state $s_i$, then what uncertainty is there in a measurement of $q$? To answer this, we calculate the entropy given $s_i$:

$$H(Q | s_i) = - \sum_j P_{q|s}(q_j | s_i) \log P_{q|s}(q_j | s_i).$$  \hspace{1cm} (A.24)

The notation of $P_{q|s}(q_j | s_i)$ means that we are first given $s_i$. Since we know $s_i$ we have to sum over all the possible outcomes of $q$.

We can write

$$P_{q|s}(q_j | s_i) = \frac{P_{sq}(s_i, q_j)}{P_s(s_i)}$$  \hspace{1cm} (A.25)

where now we are saying that $P_{sq}(s_i, q_j)$ is the probability that $s_i$ occurs first followed by $q_j$.

Thus we can write:

$$H(Q, s_i) = - \sum_j \frac{P_{sq}(s_i, q_j)}{P_s(s_i)} \log \frac{P_{sq}(s_i, q_j)}{P_s(s_i)}.$$  \hspace{1cm} (A.26)

Now given that we measure $S$, what is the uncertainty in a measurement of $Q$. This time we don’t know that the particular state $s_i$ is going to occur. Therefore we must consider all $s_i$ events and we must weigh these events with the probability that they occur. The result is an averaging over $s_i$. 
\[ H(Q|S) = \sum_i P_s(s_i) H(Q|s_i) \]
\[ = -\sum_i P_s(s_i) \sum_j \frac{P_{sq}(s_i, q_j)}{P_s(s_i)} \log \frac{P_{sq}(s_i, q_j)}{P_s(s_i)} \]
\[ = -\sum_{i,j} P_{sq}(s_i, q_j) \log \frac{P_{sq}(s_i, q_j)}{P_s(s_i)} \]
\[ = -\sum_{i,j} P_{sq}(s_i, q_j) \log P_{sq}(s_i, q_j) + \sum_i P_{sq}(s_i, q_j) \log P_s(s_i). \quad (A.27) \]

In the second term, we can sum over the \( q_j \) which just gives one. Thus:

\[ H(Q|S) = -\sum_{i,j} P_{sq}(s_i, q_j) \log P_{sq}(s_i, q_j) + \sum_i P_s(s_i) \log P_s(s_i) \quad (A.28) \]

Thus the uncertainty in a measurement in \( Q \) is

\[ H(Q|S) = H(S, Q) - H(S). \quad (A.29) \]

The mutual information can now be defined. \( H(Q) \) is the uncertainty we have in \( Q \) taken by itself. From this uncertainty, subtract off the information gained by measuring \( S \). Then

\[ I(Q, S) = H(Q) - H(Q|S) = H(Q) + H(S) - H(S, Q). \quad (A.30) \]

This answers the question, given a measurement of \( s \), how much information can be predicted about \( q \).

Takens [80] has proposed that the attractor can be reconstructed from a scalar time series. This is because (a) a time series will be some scalar function of the state vector of the system and (b) since this is deterministic chaos, there is a relationship between points in the time series and points along the attractor. One needs a sufficiently long time series of a scalar observable \( x(t) \). The delay coordinate reconstruction of the
attractor where a point in this phase space is defined by

\[ \tilde{y}(t) = [x(t), x(t - T), \ldots, x(t - (D - 1)T)]. \]

(A.31)

Here \( D \) is the lowest integer larger than the expected fractal dimensionality of the system and \( T \) is a time delay chosen so as to maximize the information contained in each axis of the space. An acceptable choice for the delay time \( T \) is the first minimum of the mutual information [81]. One chooses the first minimum of the mutual information because it minimizes the twistings in the attractor which could lead to erroneous values of the Lyapunov exponent. In addition to determining the optimum time for the delayed feedback, the mutual information can be used to calculate correlations between time series. See [24, 90] for examples.
APPENDIX B

SCHEMATIC DIAGRAM OF SYNCHRONIZATION CIRCUIT


[11] In Ref. [1] these were referred to as sub-Lyapunov exponents. The nomenclature was changed in Ref. [10].


[37] Private conversation with Uta Dressler, North Island Beach San Diego CA July 1993.


[82] I am indebted to Thomas-Martin Kruel at the University of Wuerzburg Germany for allowing his computer program to be accessible via anonymous ftp.


[84] This may not always be true. M. Ding and E. Ott have considered the synchronization of subsystems when driven by chaotic drivers as originally proposed by L. Pecora and T. Carroll. They show that nonidentical subsystems can often


[97] The code along with pertinent information may be obtained by sending $5 to A. Wolf, Dept. of Physics, The Cooper Union, 51 Astor Place, NY NY 10003.


[100] Email correspondence with Keith Briggs. He said that the available code was a prototype and likely needed modifications.