A COLLAPSING RESULT USING THE AXIOM OF DETERMINACY AND THE THEORY OF POSSIBLE COFINALITIES

Russell J. May, B.S., M.S.

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APPROVED:

Steven C. Jackson, Major Professor
Daniel Mauldin, Committee Member
Mariusz Urbanski, Committee Member
Neal Brand, Chair of the Department of Mathematics
C. Neal Tate, Dean of the Robert B. Toulouse School of
Graduate Studies

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Assuming the axiom of determinacy, we give a new proof of the strong partition relation on ω_1. Further, we present a streamlined proof that $J_{<\lambda^+}(a)$ (the ideal of sets which force $\operatorname{cof} \prod a \leq \lambda$) is generated from $J_{<\lambda}(a)$ by adding a singleton. Combining these results with a polarized partition relation on $\aleph_{\omega_1}$ and a covering result of Woodin, we obtain our main result that if the nonstationary ideal on ω_1 is ω_2 saturated and there are at least ω many Woodin cardinals plus a measurable cardinal above them all, then some regular cardinal $< \aleph_{\omega_2}$ in $L[\mathbb{R}]$ must collapse in V.

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CONTENTS

CHAPTER 1

Introduction

1.1 Overview

In 1989 Sarahon Shelah proved one of the most astounding results in set theory:

$$\text{if } 2^{\aleph_n} < \aleph_\omega \text{ for all } n < \omega, \text{ then } 2^{\aleph_\omega} < \aleph_{\omega_4}.$$

The main idea of this thesis was to "relativize" this proof to $L[\mathbb{R}]$. Assuming a covering result of Woodin which itself assumes the nonstationary ideal on ω_1 is ω_2 saturated and a large cardinal hypothesis, we show here that some regular cardinal $< \aleph_{\omega_2}$ in $L[\mathbb{R}]$ collapses in V. This gives a new relationship between $L[\mathbb{R}]$ and V. The methods for this proof come mainly from descriptive set theory and Shelah's theory of possible cofinalities of reduced products of regular cardinals (often abbreviated "pcf"). A highly recommended reference for descriptive set theory is [4]. Likewise, a good reference for pcf is [2]. The introduction includes a few well-known results in these fields, not so much as a review, but more to introduce the reader in a relatively gentle setting to methods which will be used later in more complicated contexts. These results also provide enough information to make the proofs in this thesis reasonably self-contained.

Chapter 2 gives a streamlined proof of a result in Shelah's pcf theory. For any set a of regular cardinals let $J_{<\lambda}(a) = \{b \subseteq a : \text{for all ultrafilters } D \text{ on } a,\ b \in D \text{ implies } \mathrm{cof}(\prod a/D) < \lambda\}$. First of all, $J_{<\lambda}(a)$ is an ideal on a and consists of the sets which "force" the cofinality of $\prod a$ to be $< \lambda$. We prove that $J_{<\lambda^+}(a)$ is generated from $J_{<\lambda}(a)$ by adding a singleton. It is significant to note that the only assumption in this proof is $|a|^{+3} < \min a$. There are quicker proofs of this result assuming $2^{|a|} < \min a$ (see [5], for instance). However, if we assume the continuum is small, then our main collapsing result in chapter 4 becomes trivial.

In chapter 3 we assume the axiom of determinacy and prove the strong partition relation Kechris [6] , and S. C. Jackson [3]. The current proof is particularly straightforward and uses only basic set theory. The main new idea is a coding of subsets of ω_1, which is facilitated by inverting pressing-down functions.

In chapter 4 we prove the main result of the thesis. The first step is to show a polarized partition relation on ω_1 length sequences of reasonable cardinals and then show that if a product of cardinals has the finite polarized partition relation, then the product (modulo any nonprincipal measure) is a regular cardinal. Next, we show the existence of a "c.u.b. generator" in the pcf theory. Assuming Woodin's covering result, this is enough to show some regular cardinal $< \aleph_{\omega_2}$ in $L[\mathbb{R}]$ collapses in V.

1.2 Basics of pcf

In this section we present some of the basic notions of the pcf theory, most important of which is a partial ordering of reduced products of regular cardinals. Let a be a set of regular cardinals and J be an ideal on a. Then we can compare any two functions in the product $\prod a = \{h : a \to \bigcup a : \forall \delta \in a\, h(\delta) \in \delta\}$. For any $f, g \in \prod a$ define

$$\begin{aligned} f \leq_J g &\iff \{\delta \in a : f(\delta) > g(\delta)\} \in J \\ f \lneqq_J g &\iff f \leq_J g \wedge \{\delta \in a : f(\delta) < g(\delta)\} \notin J \\ f <_J g &\iff \{\delta \in a : f(\delta) \geq g(\delta)\} \in J. \end{aligned}$$

Clearly, $f <_J g \implies f \lneqq_J g \implies f \leq_J g$. Each $f \in \prod a$ gives rise to an equivalence class $[f]_J = \{f' \in \prod a : f' \leq_J f \wedge f \leq_J f'\}$. We will not distinguish between a function and its equivalence class. The reduced product $\prod a/J = \{[f]_J : f \in \prod a\}$ is also partially ordered by the partial order given above.

There are several terms dealing with sequences in reduced products. We say $\langle f_\alpha : \alpha < \lambda \rangle$ in $\prod a/J$ is *increasing* if it is increasing in the weak sense, i.e. if $\alpha' < \alpha < \lambda$, then $f_{\alpha'} \leq_J f_\alpha$. (Note: in general, when we say a sequence is increasing in any partial order, e.g. $\subseteq$, we mean it is weakly increasing.) The sequence $\langle f_\alpha : \alpha < \lambda \rangle$ is *positively increasing* if $\alpha' < \alpha$ implies $f_{\alpha'} \lneqq_J f_\alpha$. The same sequence is *strictly increasing* if $\alpha' < \alpha$ implies $f_{\alpha'} <_J f_\alpha$. Say $\langle f_\alpha : \alpha < \lambda \rangle$ is *cofinal* in $\prod a/J$ if for all $g \in \prod a$ there exists $\alpha < \lambda$ such that $g <_J f_\alpha$. If λ is a regular cardinal and $\langle f_\alpha : \alpha < \lambda \rangle$ is both strictly increasing and cofinal, we say $\langle f_\alpha : \alpha < \lambda \rangle$ is a λ *scale*. If there is a λ scale in $\prod a/J$, we say $\prod a/J$ has true cofinality λ and write $\mathrm{tcf}(\prod a/J) = \lambda$. In this case $\mathrm{cof}(\prod a/J) = \lambda$ for every ultrafilter D on a which is

We will use two different notions of upper bounds for subsets of $\prod a/J$. Say g (not necessarily in $\prod a$) is a *least upper bound* of $\langle f_\alpha : \alpha < \lambda \rangle$ if for all $\alpha < \lambda$, $f_\alpha \leq_J g$ and if $h \lneqq_J g$, then h is not an upper bound of $\langle f_\alpha : \alpha < \lambda \rangle$, i.e. there exists $\alpha < \lambda$ such that $\{\delta \in a : f_\alpha(\delta) > h(\delta)\} \notin J$. Say g is an *exact upper bound* of $\langle f_\alpha : \alpha < \lambda \rangle$ if g is an upper bound of $\langle f_\alpha : \alpha < \lambda \rangle$ and for any $h <_J g$ there exists $\alpha < \lambda$ such that $h <_J f_\alpha$. Clearly, g

being an exact upper bound implies that g is a least upper bound. Conditions under which the converse is true will be given later (see lemma 2.1.4).

If J is an ideal on a, then $J[b]$ is the ideal *generated* by adding the singleton b to J, namely $\{a' \cup b' : b' \subseteq b,\ a' \in J\}$. Conversely, $J \restriction b$ is the *restriction* of the ideal J to b, i.e. $J \restriction b = J[a - b]$. If $\{f_\alpha : \alpha \in S\}$ is a collection of functions in $\prod a$, then $\operatorname{ptsup}_{\alpha \in S} f_\alpha$ is the function whose value is $\sup_{\alpha \in S} f_\alpha(\delta)$ for each $\delta \in a$. The ideal which we will consider most often is the collection of subsets of a which force $\operatorname{cof}(a) < \lambda$. Specifically, we define $J_{<\lambda}(a) = \{b \subseteq a : \text{for every ultrafilter } D \text{ on } a, \text{ if } b \in D, \text{ then } \operatorname{cof}(\prod a/D) < \lambda\}$. The ideal of nonstationary subsets $\mathcal{I}_{\text{NS}}$ of a regular cardinal will also be used repeatedly.

If $\alpha \in \mathbf{ON}$, $C \subseteq \mathbf{ON}$, and there exists $\beta \in C$ with $\alpha < \beta$, then α^{+C} is the least ordinal in C greater than α. Let $\wp(x)$ denote the power set of x and $\wp_{<\kappa}(x) = \{y \subseteq x : |y| < \kappa\}$. Lastly, let $S^\lambda_\kappa = \{\alpha < \lambda : \operatorname{cof} \alpha = \kappa\}$.

The rest of the lemmas in this section come directly from [2] and are provided for the convenience of the reader. Lemmas 1.2.1 and 1.2.5 in particular will be used repeatedly. Assume throughout that $|a|^+ < \min a$.

Lemma 1.2.1. *$\prod a/J_{<\lambda}$ is λ directed, i.e. if $A \subseteq \prod a$ and $|A| < \lambda$, then there exists $g \in \prod a$ such that for all $f \in A$, $f \leq_{J_{<\lambda}} g$.*

Proof. By induction on cardinals $\mu < \lambda$ we will show that if $A \subseteq \prod a/J_{<\lambda}$ and $|A| = \mu$, then A is bounded. Let $\kappa = |a|^+$.

If $\mu \leq \kappa$, then $g = \operatorname{ptsup}_{f \in A} f$ is an upper bound of A in $\prod a$. If $\mu > \kappa$, then by induction we may assume A is an increasing sequence $\langle f_\alpha : \alpha < \mu \rangle$. If μ is singular, easily $\langle f_\alpha : \alpha < \mu \rangle$ is bounded. So assume μ is a regular cardinal with $\kappa < \mu < \lambda$. Towards a contradiction suppose $\langle f_\alpha : \alpha < \mu \rangle$ is not bounded. We will inductively define a pointwise increasing sequence $\langle g_\beta : \beta < \kappa \rangle$ in $\prod a$. Define $b^\beta_\alpha = \{\delta \in a : f_\alpha(\delta) > g_\beta(\delta)\}$ for all $\alpha < \mu$ and $\beta < \kappa$. Let $g_0 = f_0$. If β is a limit, let $g_\beta = \operatorname{ptsup}_{\beta' < \beta} g_{\beta'}$. For the successor case, since g_β is not an upper bound of $\langle f_\alpha : \alpha < \mu \rangle$, there exists $\alpha_\beta < \mu$ such that $b^\beta_{\alpha_\beta} \notin J_{<\lambda}$. By definition there exists an ultrafilter D_β on $b^\beta_{\alpha_\beta}$ such that $\operatorname{cof} \prod a/D_\beta \geq \lambda$. Further, $D_\beta \cap J_{<\lambda} = \varnothing$. Hence, $\langle f_\alpha : \alpha < \mu \rangle$ is bounded mod D_β, say by h. Let $g_{\beta+1} = \operatorname{ptmax}(g_\beta, h)$. This completes the inductive definition of $\langle g_\beta : \beta < \kappa \rangle$.

Let $\alpha_\kappa = \sup_{\beta < \kappa} \alpha_\beta < \mu$ by the regularity of μ. Fix $\beta < \kappa$. Since $b^\beta_{\alpha_\beta} \subseteq_{J_{<\lambda}} b^\beta_{\alpha_\kappa}$ and $D_\beta \cap J_{<\lambda} = \varnothing$, then $b^\beta_{\alpha_\kappa} \in D_\beta$. On the other hand, $b^{\beta+1}_{\alpha_\kappa} \notin D_\beta$, so $b^{\beta+1}_{\alpha_\kappa} \subsetneq b^\beta_{\alpha_\kappa}$. Thus, $\langle b^\beta_{\alpha_\kappa} : \beta < \kappa = |a|^+ \rangle$ is a strictly decreasing sequence of subsets of a, contradicting the assumption that $\langle f_\alpha : \alpha < \mu \rangle$ was not bounded. $\square$

By definition if D is an ultrafilter on $\mathfrak{a}$ and $D \cap J_{<\lambda} \neq \varnothing$, then $\operatorname{cof}(\prod \mathfrak{a}/D) < \lambda$. The converse is also true, otherwise any cofinal sequence in $\prod \mathfrak{a}/D$ of length $< \lambda$ would be bounded mod $J_{<\lambda}$, hence mod D. Clearly, the map $\lambda \mapsto J_{<\lambda}(\mathfrak{a})$ is increasing and continuous. Define $\operatorname{pcf}(\mathfrak{a}) = \{\lambda : \exists \text{ ideal } J \text{ on } \mathfrak{a} \text{ such that } \lambda = \operatorname{tcf}(\prod \mathfrak{a}/J)\}$. It is easy to see $\lambda \in \operatorname{pcf}(\mathfrak{a})$ iff $J_{<\lambda}(\mathfrak{a}) \subsetneqq J_{<\lambda^+}(\mathfrak{a})$. Hence, $\operatorname{pcf}(\mathfrak{a}) = \{\lambda : \exists \text{ ultrafilter } D \text{ on } \mathfrak{a} \text{ such that } \lambda = \operatorname{cof}(\prod \mathfrak{a}/D)\}$. Also, it is easy to see that $\operatorname{pcf}(\mathfrak{a})$ has a maximal element, which is denoted $\operatorname{maxpcf}(\mathfrak{a})$.

The next lemma is essentially the equivalent of lemma 1.2.1 for an arbitrary ideal.

Lemma 1.2.2. *Suppose J is an ideal on $\mathfrak{a}$, λ is a regular cardinal, and $\langle f_\alpha : \alpha < \lambda \rangle$ in $\prod \mathfrak{a}$ is increasing and unbounded mod J. Then there exist a sequence $\langle b_\alpha : \alpha < \lambda \rangle$ of subsets of $\mathfrak{a}$ and a function $g \in \prod \mathfrak{a}$ such that*

1. *$b_0 \notin J$*
2. *$\forall \alpha', \alpha < \lambda \;\; \alpha' < \alpha \implies b_{\alpha'} \subseteq_J b_\alpha$*
3. *$\forall \alpha' < \lambda \; \langle f_\alpha : \alpha < \lambda \rangle$ is a scale mod $J \restriction b_{\alpha'}$*
4. *g is an upper bound for $\langle f_\alpha : \alpha < \lambda \rangle$ mod the ideal generated by $J \cup \{b_\alpha : \alpha < \lambda\}$.*

Proof. Let $\kappa = |\mathfrak{a}|^+$. We will inductively define a pointwise increasing sequence of functions $\langle g_\beta : \beta < \beta_0 \rangle$ in $\prod \mathfrak{a}$ where $\beta_0 \leq \kappa$. For each $\beta < \beta_0$ and $\alpha < \lambda$ define $b^\beta_\alpha = \{\delta \in \mathfrak{a} : g_\beta(\delta) < f_\alpha(\delta)\}$. The induction will continue until $\langle b^{\beta_0}_\alpha : \alpha < \lambda \rangle$ and g_{β_0} satisfy properties 1-4 or until $\beta_0 = \kappa$. To prove the theorem we will show the latter alternative is impossible.

Let g_0 be an arbitrary element of $\prod \mathfrak{a}$. For limit $\beta < \kappa$, let $g_\beta = \operatorname{ptsup}_{\beta' < \beta} g_{\beta'} \in \prod \mathfrak{a}$, since $\beta < \min \mathfrak{a}$. Now if g_β has been defined, since $\langle f_\alpha : \alpha < \lambda \rangle$ is unbounded, there exists $\alpha_0 < \lambda$ such that $b^\beta_{\alpha_0} \notin J$, i.e. property 1 holds with $b_0 = b^\beta_{\alpha_0}$. Properties 2 and 4 follow immediately (for $b_\alpha = b^\beta_{\alpha_0 + \alpha}$ and $g = g_\beta$). If property 3 holds as well, the theorem is proved. Otherwise, there exist $\alpha_\beta < \lambda$ and $h \in \prod \mathfrak{a}$ such that for each $\alpha < \lambda$, $c^\beta_\alpha = \{\delta \in \mathfrak{a} : f_\alpha(\delta) < h(\alpha)\} \notin J \restriction b^\beta_{\alpha_\beta}$. Thus, $c^\beta_\alpha \cap b^\beta_{\alpha_\beta} \notin J$. Let $g_{\beta+1} = \operatorname{ptmax}(g_\beta, h)$. Note that for any $\alpha < \lambda$, $c^\beta_\alpha \cap b^{\beta+1}_\alpha = \varnothing$. This completes the induction.

This induction cannot continue κ times. To see this, let $\alpha_\kappa = \sup_{\beta < \kappa} \alpha_\beta < \lambda$ since λ contradiction we will show $\langle b^\beta_{\alpha_\kappa} : \beta < \kappa \rangle$ is strictly decreasing. Fix $\beta < \kappa$. One on hand, $b^\beta_{\alpha_\kappa} \notin J$ by properties 1 and 2. On the other hand,

$$b^\beta_{\alpha_\kappa} - b^{\beta+1}_{\alpha_\kappa} \supseteq b^\beta_{\alpha_\kappa} \cap c^\beta_{\alpha_\kappa} \supseteq_J b^\beta_{\alpha_\beta} \cap c^\beta_{\alpha_\kappa} \notin J.$$

So, $\langle g_\beta : \beta < \kappa \rangle$ is strictly decreasing, an impossibility. $\square$

Lemma 1.2.3. *If J is an ideal on a, $\prod a/J$ is λ directed, D is an ultrafilter on a with $D \cap J = \varnothing$, and* $\operatorname{cof} \prod a/D = \lambda$, *then there exists* $b \in D$ *such that* $\operatorname{tcf}(\prod b/J) = \lambda$.

Proof. Let $\langle f_\alpha : \alpha < \lambda \rangle$ be a scale in $\prod a/D$. Since J is λ directed and D respects J, we can take $\langle f_\alpha : \alpha < \lambda \rangle$ to be strictly increasing mod J. Also, $\langle f_\alpha : \alpha < \lambda \rangle$ must be unbounded mod J, or else it would be bounded mod D. Fix $\langle b_\alpha : \alpha < \lambda \rangle$ and g as in lemma 1.2.2. It cannot be the case $b_\alpha \notin D$ for all $\alpha < \lambda$, for otherwise g would be a bound mod D for $\langle f_\alpha : \alpha < \lambda \rangle$. Consequently, some $b_\alpha = b$ is in D, and by property 3 of lemma 1.2.2 $\langle f_\alpha : \alpha < \lambda \rangle$ is a scale in $\prod b/J$. □

Lemma 1.2.4. *If J is an ideal on a and every ultrafilter D on a respecting J has* $\operatorname{cof}(\prod a/D) = \lambda$, *then* $\operatorname{tcf}(\prod a/J) = \lambda$.

Proof. First, observe that $J_{<\lambda} \subseteq J$. Then consider the (possibly improper) ideal $J^* = \{b_1 \cup b_2 : b_1 \in J \vee \operatorname{tcf}(\prod b_2/J) = \lambda\}$. If J^* were proper, then there would be an ultrafilter D on a with $D \cap J^* = \varnothing$. By assumption, $\operatorname{cof}(\prod a/D) = \lambda$. So, by lemma 1.2.3 $D \cap J^* \neq \varnothing$. Thus, J^* could not be proper, showing $\operatorname{tcf}(\prod a/J) = \lambda$. □

Lemma 1.2.5. *If* $b \in J_{<\lambda^+}(a) - J_{<\lambda}(a)$, *then* $\operatorname{tcf}(\prod b/J_{<\lambda}(a)) = \lambda$.

Proof. If D is an ultrafilter on b respecting $J_{<\lambda}$, then $\operatorname{cof}(\prod a/D) \geq \lambda$ and $\operatorname{cof}(\prod a/D) \leq \lambda$. So, by lemma 1.2.4, $\operatorname{tcf}(\prod a/J_{<\lambda} \restriction b) = \lambda = \operatorname{tcf}(\prod b/J_{<\lambda})$. □

1.3 Basics of Descriptive Set Theory

Descriptive set theory is often said to be the study of definable relations in Polish spaces (completely metrizable, separable topological spaces). This section describes some of the basic terminology and results of descriptive set theory. The Polish space which we will use most often is the Baire space ω^ω, the countable product of the integers with the discreet topology. One common tool in descriptive set theory is trees. A *tree* on a set X is a collection of finite sequences $T \subseteq X^{<\omega}$ closed under subsequence. A *branch* of tree T is an infinite sequence $x \in X^\omega$ such that for all $n < \omega$, $x \restriction n \in T$. The collection of all branches of tree T denotes the *rank* of s in T. If $X \subseteq \mathbf{ON}$ and $\beta \in \mathbf{ON}$, $T \restriction \beta$ is the *restriction* of T to β, namely $\{s \in T : \forall n < \operatorname{len}(s)\ s_n < \beta\}$. If T is a tree on a product $X \times Y$, then we identify T with the set $\{(s, t) : \operatorname{len}(s) = \operatorname{len}(t) \wedge ((s_0, t_0), \ldots, (s_{\operatorname{len}(s)-1}, t_{\operatorname{len}(s)-1})) \in T\}$. For any tree T on $X \times Y$ and $x \in X^\omega$ the *section* T_x of the tree T at x is the tree $\{t \in Y : (x \restriction \operatorname{len}(t), t) \in T\}$. The *projection* $p[T]$ of T is $\{x \in X^\omega : \exists y \in Y^\omega \forall n < \omega\ (x \restriction n, y \restriction n) \in T\}$. So, T_x is wellfounded iff $x \notin p[T]$. Let $\mathrm{WO} = \{x \in \omega^\omega : x$ codes a wellorder of $\omega\}$ and $\mathrm{WF} = \{x \in \omega^\omega : x$ codes a wellfounded relation on $\omega\}$. For any $\gamma < \omega_1$ let WO_γ be $\{x \in \mathrm{WO} : |x| < \gamma\}$ and WO'_γ be $\{x \in \mathrm{WO} : |x| = \gamma\}$. Likewise, let WF_γ be $\{x \in \mathrm{WF} : |x| < \gamma\}$ and WF'_γ be $\{x \in \mathrm{WF} : |x| = \gamma\}$.The *Shoenfield tree* T^{WF} on $\omega \times \omega_1$ is

$$\{(\mathrm{s}, \alpha) \ : \ \mathrm{len}(\mathrm{s}) = \mathrm{len}(\alpha) \wedge \forall i, j < \mathrm{len}(\mathrm{s})\ (i <_s j \implies \alpha_i < \alpha_j) \wedge \\ \forall i, j, k < \mathrm{len}(\mathrm{s})(i <_s j \wedge j <_s k \implies i <_s k)\}$$

where $<_s$ is a partial order on a subset of ω coded in the standard manner by s. Then $\mathrm{WF} = p[T^{\mathrm{WF}}]$, and if $x \in \mathrm{WF}$ with $|x| < \beta$, then $x \in p[T^{\mathrm{WF}} \lceil \beta]$.

We briefly discuss the axiom of determinacy, AD, and some of its consequences which we will use. For any $A \subseteq \omega^\omega$ the game G_A is played as follows: players I and II alternately play integers

$$\begin{array}{llllll} \mathrm{I}: & x_0 & & x_2 & & \dots \\ \mathrm{II}: & & x_1 & & x_3 & \end{array}$$

forming $x \in \omega^\omega$. I wins the run of the game iff $x \in A$, otherwise II wins. The *axiom of determinacy* (AD) states that for each $A \subseteq \omega^\omega$ one of the players has a winning strategy in G_A. AD was introduced by Mycielski and Steinhaus in the 1960's. Its consequences have been an active area of research ever since and comprises a good deal of what goes under the heading of descriptive set theory. We will take for granted just a couple consequences of AD. Define Θ to be the supremum of lengths of prewellorders of ω^ω. In [6] Kechris showed the finite partition relation on ω_1, which in turn implies that the c.u.b. filter on ω_1 is a normal measure, denoted W_1^1. In fact, it is the unique normal measure on ω_1. AD implies that every filter on $\kappa < \Theta$ can be extended to an ultrafilter and that every ultrafilter is countably complete. If μ is a *measure* on κ, i.e. a countably complete ultrafilter on κ, and $f : \kappa \to \lambda$, then the *image measure* $f(\mu)$ on λ is $\{B \subseteq \lambda : \mu(f^{-1}(B)) = 1\}$. So B has $f(\mu)$ measure one iff for μ almost all $\alpha \in \kappa$, $f(\alpha) \in B$.

For the sake of completeness we present a construction of the *Kunen tree*. Similar methods will be used later.

Lemma 1.3.1 (AD). *There is a tree T^K on $\omega \times \omega_1$ such that for all $f : \omega_1 \to \omega_1$ there exists $\tau \in \omega^\omega$ such that T^K_τ is wellfounded and for all infinite $\gamma < \omega_1$ we have $f(\gamma) < |T^K_\tau \lceil \gamma|$.*

Proof. Let $A \subseteq \omega^\omega$ be any $\mathbf{\Sigma}^1_1$ complete set and a T^A be a tree on $\omega \times \omega$ which projects to

$\boldsymbol{A}$, i.e. $\boldsymbol{A} = p[T^A]$. Define tree $\tilde{T}$ on $\omega \times \omega \times \omega_1 \times \omega \times \omega$ by

$$(t, a, \alpha, b, c) \in \tilde{T} \iff (a, \alpha) \in T^{\mathrm{WF}} \wedge ta = b \wedge (b, c) \in T^A$$

where $ta = b$ means that in some reasonable manner t codes a partial strategy in a game applied to a partial run of the game a, and that b does not contradict t applied to a. We will show a slight variant of $\tilde{T}$ satisfies the conditions of the lemma. Let $f \colon \omega_1 \to \omega_1$. Consider the game where I plays x, II plays y, and II wins iff

$$x \in \mathrm{WF} \implies T^A_y \text{ is wellfounded } \wedge |T^A_y| > \sup_{\beta \leq |x|} f(\beta).$$

I cannot win this game. Otherwise, if I had a winning strategy σ, then $\sigma(\omega^\omega) \subseteq \mathrm{WF}$ would be $\boldsymbol{\Sigma}^1_1$, and the prewellorder on ω^ω defined by

$$x \leq x' \iff x, x' \in \sigma(\omega^\omega) \wedge |x| < |x'|$$

would also be $\boldsymbol{\Sigma}^1_1$. But by the Kunen-Martin theorem $\boldsymbol{\Sigma}^1_1$ prewellorders of ω^ω are bounded below ω_1. Hence, $\{f(|x|) : x \in \sigma(\omega^\omega)\}$ is also bounded below ω_1, say by α_0. Since $\neg \boldsymbol{A}$ is $\boldsymbol{\Pi}^1_1$ complete, then $\{|T^A_y| : y \notin \boldsymbol{A}\}$ is unbounded in ω_1. So, II can choose $y \in \neg \boldsymbol{A}$ such that $|T^A_y| > \alpha_0$ and beat I's strategy. Hence by AD, II wins. (In later proofs this paragraph is abbreviated "by boundedness".)

Fix a winning strategy τ for II. $\tilde{T}_\tau$ is wellfounded. Otherwise, if $(x, \alpha, y, z) \in [\tilde{T}_\tau]$, then $(x, \alpha) \in [T^{\mathrm{WF}}]$, $\tau(x) = y$, and $(y, z) \in [T^A]$. So, $x \in \mathrm{WF}$ and $y \in \boldsymbol{A}$, contradicting the fact that τ is a winning strategy for II.

For any $x \in \mathrm{WF}$ we can embed $T^A_{\tau(x)}$ into $\tilde{T}_\tau \restriction |x|$. To see this choose $\alpha \in |x|^\omega$ such that $(x, \alpha) \in [T^{\mathrm{WF}} \restriction |x|]$, then map $c \in T^A_{\tau(x)}$ of length n to $(x \restriction n, \alpha \restriction n, \tau(x) \restriction n, c) \in \tilde{T}_\tau \restriction |x|$. So, $f(|x|) < |T^A_{\tau(x)}| \leq |\tilde{T}_\tau \restriction |x||$. Finally, weave the last four coordinates of $\tilde{T}$ on $\omega \times \omega_1 \times \omega \times \omega$ into one coordinate on ω_1, and call this the Kunen tree T^K. Clearly, T^K_τ is wellfounded iff $\tilde{T}_\tau$ is wellfounded, in which case $|\tilde{T}_\tau \restriction \gamma| \leq |T^K_\tau \restriction \gamma|$. Hence, we have shown $f(\gamma) < |T^K_\tau \restriction \gamma|$. □

subset of ω_1. Later it will be convenient for us to use a version of the of the previous lemma for linear orders, as opposed to trees. For this purpose we quote a result from [4].

Lemma 1.3.2. *There is a function* $s \mapsto T^K(s)$ *which assign to each* $s \in \omega^{<\omega}$ *a wellordering of a subset of* ω_1 *with the following properties. If* t *extends* s, *then* $T^K(t) \supseteq T^K(s)$. *For* $x \in$

ω^ω, let $T^K_x = \bigcup_n T^K(x \restriction n)$. So T^K_x is a linear order. Then for any $f\colon \omega_1 \to \omega_1$, there is an $x \in \omega^\omega$ such that T^K_x is a wellordering and for all $\alpha \geq \omega$, $f(\alpha) < |T^K_x \restriction \alpha|$. Furthermore, the map $\mathrm{s} \mapsto T^K(\mathrm{s})$ is Δ^1_1 in the codes, i.e. there are Σ^1_1, Π^1_1 relations $S(n, a, x, y)$, $R(n, a, x, y)$ such that for all $x, y \in \mathrm{WO}$ we have

$$S(n, a, x, y) \iff R(n, a, x, y) \iff [(|x|, |y|) \in T^K(a_0, \ldots, a_{n-1})].$$

Now we briefly discuss norms and scales. A *norm* on a set $\boldsymbol{A} \subseteq \omega^\omega$ is a map from $\boldsymbol{A}$ into $\mathbf{ON}$. Any norm on $\boldsymbol{A}$ induces a prewellorder on $\boldsymbol{A}$, and vice versa. If ϕ is a norm on $\boldsymbol{A}$ and $\boldsymbol{\Gamma}$ is a pointclass, ϕ is called a $\boldsymbol{\Gamma}$ *norm* if there are relations $\leq^{\boldsymbol{\Gamma}}_\phi, \leq^{\check{\boldsymbol{\Gamma}}}_\phi$ such that for all $y \in \boldsymbol{A}$,

$$\forall x \in \omega^\omega\, [x \in \boldsymbol{A} \wedge \phi(x) \leq \phi(y) \iff x \leq^{\boldsymbol{\Gamma}}_\phi y \iff x \leq^{\check{\boldsymbol{\Gamma}}}_\phi y].$$

As long as $\boldsymbol{\Gamma}$ is closed under finite intersections and unions, $\phi\colon \boldsymbol{A} \to \mathbf{ON}$ is a $\boldsymbol{\Gamma}$ norm iff the following "starred" relations are also in $\boldsymbol{\Gamma}$:

$$\begin{aligned} x \leq^*_\phi y &\iff x \in \boldsymbol{A} \wedge (y \notin \boldsymbol{A} \vee \phi(x) \leq \phi(y)) \\ x <^*_\phi y &\iff x \in \boldsymbol{A} \wedge (y \notin \boldsymbol{A} \vee \phi(x) < \phi(y)). \end{aligned}$$

Say $\boldsymbol{\Gamma}$ has the prewellordering property, denoted $\mathrm{pwo}(\boldsymbol{\Gamma})$, if every $\boldsymbol{A} \in \boldsymbol{\Gamma}$ has a $\boldsymbol{\Gamma}$ norm. It is a classical result that $\mathrm{pwo}(\boldsymbol{\Pi}^1_1)$. It is straightforward to show that $\mathrm{pwo}(\boldsymbol{\Gamma})$ implies $\mathrm{pwo}(\exists^{\omega^\omega}\boldsymbol{\Gamma})$. The first periodicity theorem says that if $\boldsymbol{\Gamma}$ is a pointclass closed under $\exists^{\omega^\omega}$ with $\mathrm{pwo}(\boldsymbol{\Gamma})$ and $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ determinacy holds, then $\mathrm{pwo}(\forall^{\omega^\omega}\boldsymbol{\Gamma})$. A *scale* on a set $\boldsymbol{A} \subseteq \omega^\omega$ is a sequence $\langle \phi_n : n < \omega \rangle$ of norms such that if $\{x_m : m < \omega\} \subseteq \boldsymbol{A}$, $\lim_{m\to\infty} x_m = x$, and for each n, $\phi_n(x_m)$ is eventually constant (call this value λ_n), then $x \in \boldsymbol{A}$ and for all n, $\phi_n(x) \leq \lambda_n$. $\boldsymbol{\Gamma}$ has the scale property, denoted $\mathrm{scale}(\boldsymbol{\Gamma})$, if every $\boldsymbol{A} \in \boldsymbol{\Gamma}$ has a $\boldsymbol{\Gamma}$ scale. It is straightforward to show that $\mathrm{scale}(\boldsymbol{\Gamma})$ implies $\mathrm{scale}(\exists^{\omega^\omega}\boldsymbol{\Gamma})$. The second periodicity theorem says that if $\boldsymbol{\Gamma}$ is a pointclass closed under $\exists^{\omega^\omega}$ and $\wedge, \vee$ with $\mathrm{scale}(\boldsymbol{\Gamma})$ and $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ determinacy holds, then $\mathrm{scale}(\forall^{\omega^\omega}\boldsymbol{\Gamma})$. Say $\boldsymbol{A} \subseteq \omega^\omega$ is κ *Suslin* if there exists a tree on $\omega \times \kappa$ such that $\boldsymbol{A} = p[T]$. It is a standard fact that there exists a κ Suslin set which is not κ' Suslin for any cardinal $\kappa' < \kappa$.

A classical theorem of Ramsey states that for any coloring of pairs of integers into two colors there is an infinite set of integers so that all pairs with both entries coming from this set have the same color. We generalize this result. Let $\alpha, \beta, \gamma, \delta \in \mathbf{ON}$. $[\alpha]^\beta$ is the collection of subsets of α of order type β. In general, $\alpha \to \beta^{\gamma}_{\delta}$ means for every partition $p\colon [\alpha]^{\gamma} \to \delta$ there exists $a \in [\alpha]^{\beta}$ such that $p \restriction [a]^{\gamma}$ is constant. If $\delta = 2$, the subscript is usually omitted. The *strong partition relation on a cardinal* κ is the statement $\kappa \to \kappa^{\kappa}$. There is an equivalent formulation of the strong partition relation which in practice is more convenient to use. Say $f\colon \kappa \to \mathbf{ON}$ has *uniform cofinality* ω if for some increasing $f'\colon \omega \cdot \kappa \to \mathbf{ON}$, $f(\alpha) = \sup_{\alpha' < \omega\alpha+\omega} f'(\alpha')$ for all $\alpha < \kappa$. Define $f\colon \kappa \to \mathbf{ON}$ to be of *correct type* if f is strictly increasing, discontinuous, i.e. $f(\alpha) > \sup_{\alpha' < \alpha} f(\alpha')$ for all limit $\alpha < \kappa$, and has uniform cofinality ω. The *c.u.b. version of the strong partition relation*, $\kappa \xrightarrow{\text{c.u.b.}} \kappa^{\lambda}$, means for every partition of the functions of correct type there exists a c.u.b. $C \subseteq \kappa$ which is homogeneous for the functions of correct type from λ to C. The next two lemmas are standard exercises and show the equivalence of these two notions of partition relation.

Lemma 1.3.3. *If* $\kappa \to \kappa^{\omega\lambda}$*, then* $\kappa \xrightarrow{\text{c.u.b.}} \kappa^{\lambda}$.

Proof. Let P be a partition of κ^{λ} of the correct type. This induces a partition P' on $\kappa^{\omega\lambda}$: if $f'\colon \omega\lambda \to \kappa$ and $f(\beta) = \sup_{\beta' < \omega(\beta+1)} f'(\beta')$, then $P'(f') = P(f)$. Let $\boldsymbol{A}$ be homogeneous for P', and define C to be the limit points of $\boldsymbol{A}$. Clearly, $C \subseteq \kappa$ is c.u.b. Now for any $f\colon \lambda \to C$ of correct type (witnesses by $f'\colon \omega\lambda \to \kappa$) there is another witness $f''\colon \lambda \to \boldsymbol{A}$, for instance $f''(\beta') = (\max(f''(\beta'-1), f'(\beta'))^{+A}$. So $P(f) = P'(f')$, and thus C is homogeneous for P. □

Lemma 1.3.4. *If* $\kappa \xrightarrow{\text{c.u.b.}} \kappa^{\lambda}$*, then* $\kappa \to \kappa^{\lambda}$.

Proof. Partition all functions from λ to κ. This partitions the functions of correct type, so let c.u.b. C be homogeneous for the functions of correct type. Define the βth element of $\boldsymbol{A}$ to be the $\omega(\beta+1)$th element of C. Hence, every $f\colon \lambda \to \boldsymbol{A}$ is of the correct type, and $\boldsymbol{A}$ is homogeneous for the partition. □

The notion of a partition relation can be extended to products of ordinals . Suppose k, ℓ, m are ordinal valued sequences of length μ. Then $k \to m^{\ell}$ means for every partition $p\colon \prod_{\alpha<\mu}[k(\alpha)]^{\ell(\alpha)} \to 2$ there exists $b \in \prod_{\alpha<\mu}[k(\alpha)]^{m(\alpha)}$ such that $p \restriction \prod_{\alpha<\mu}[b(\alpha)]^{\ell(\alpha)}$ is increasing and discontinuous, i.e. $\forall\alpha < \mu\ \sup_{\alpha'<\alpha} k(\alpha') < k(\alpha)$, there is an equivalent formulation of the partition relation using block functions. Let $\lambda = \sup_{\alpha<\mu} k(\alpha)$ and for each $\alpha < \mu$, let $b_{\alpha} = [\sup_{\alpha'<\alpha} k(\alpha'), k(\alpha))$ denote the αth block of k. A function $f\colon \lambda \to \lambda$ is a *block function* if for all $\alpha < \mu$, $f \restriction b_{\alpha} \subseteq b_{\alpha}$. There is a one-to-one correspondence between block functions and k^k. The *polarized strong partition relation*, $k \xrightarrow{\text{polar}} k^k$, holds if for every partition $p\colon \{f\colon \lambda \to \lambda : f \text{ is a block function }\} \to 2$ there exists $S \subseteq \lambda$ such that $\forall \alpha < \mu \; |S \cap b_\alpha| = k(\alpha)$ and $p\restriction\{f\colon \lambda \to S : f \text{ is a block function }\}$ is constant. Clearly, $k \to k^k$ iff $k \xrightarrow{\text{polar}} k^k$. As before, there is an equivalent c.u.b. version of the polarized strong partition relation. First, say $C \subseteq \lambda$ is *block c.u.b.* if for all $\alpha < \mu$, $C \cap k(\alpha)$ is c.u.b. in $k(\alpha)$. Then, the *c.u.b. version of the polarized strong partition relation*, $k \xrightarrow{\text{c.u.b.}} k^k$, holds if for every partition $p\colon \{f\colon \lambda \to \lambda : f \text{ is a block function }\} \to 2$ there is a block c.u.b. $C \subseteq \lambda$ such that $p\restriction\{f\colon \lambda \to C : f \text{ is a block function of the correct type}\}$ is constant. As in lemmas 1.3.3 and 1.3.4, $k \xrightarrow{\text{polar}} k^k$ iff $k \xrightarrow{\text{c.u.b.}} k^k$.

CHAPTER 2

Simple Generation

2.1 Preliminaries

In this section we define and show the existence of Shelah sequences (sometimes called c.u.b. guessing sequences) and then show that sequences of functions which respect a Shelah sequence have exact upper bounds mod $J_{<\lambda}$. These lemmas will be used in the next section to prove that $J_{<\lambda^+}$ is generated from $J_{<\lambda}$ by adding a singleton.

Lemma 2.1.1. *Let κ, λ be regular cardinals with $\omega_1 \leq \kappa$ and $\kappa^+ < \lambda$. Then there exists a sequence $\langle A_\alpha : \alpha \in S^\lambda_\kappa \rangle$ such that*

1. *for all $\alpha \in S^\lambda_\kappa$, A_α is a c.u.b. subset of α and $|A_\alpha| = \kappa$;*
2. *for all c.u.b. $C \subseteq \lambda$ there is a stationary $S \subseteq S^\lambda_\kappa$ such that for all $\alpha \in S$ we have $A_\alpha \subseteq C$.*

Proof. Note that it suffices to replace "for all" in (1) by "for almost all" with respect to the c.u.b. filter on λ, since the requirement in (2) only depends on a stationary set. Let (1') denote the "for almost all" version. For each $\alpha \in S^\lambda_\kappa$ fix a c.u.b. subset A_α of α of cardinality κ. We will inductively define a decreasing sequence $\langle C^\beta : \beta < \beta_0 \rangle$ of c.u.b. subsets of λ where $\beta_0 \leq \kappa^+$. For each $\beta < \beta_0$ define $A^\beta_\alpha = A_\alpha \cap C^\beta$. The induction will terminate if $\langle A^\beta_\alpha : \alpha \in S^\lambda_\kappa \rangle$ satisfies (1') and (2) or if the κ^+ step is reached. We will show the induction cannot continue κ^+ steps, thus proving the lemma. For any c.u.b. subset C of λ the sequence $\langle A_\alpha \cap C : \alpha \in S^\lambda_\kappa \rangle$ satisfies (1') since $C \cap \alpha$ is c.u.b. in α for every $\alpha \in S^\lambda_\kappa \cap \lim C$ and the intersection of c.u.b. sets is c.u.b. Hence, the induction will continue only when (2) is not satisfied.

$\beta = 0$: Let $C^0 = \lambda$.

β is a successor: Since $\langle A^{\beta-1}_\alpha : \alpha \in S^\lambda_\kappa \rangle$ did not satisfy (2) there exist c.u.b. subsets in particular $A^\beta_\alpha \subsetneq A^{\beta-1}_\alpha$ for $\alpha \in S^\lambda_\kappa \cap D^{\beta-1}$.

β is a limit: Let $C^\beta = \bigcap_{\beta' < \beta} C^{\beta'}$.

This induction cannot continue κ^+ steps. Otherwise, for each $\alpha \in S^\lambda_\kappa$ the sequence $\langle A^\beta_\alpha : \beta < \kappa^+ \rangle$ is decreasing, and since $|A^0_\alpha| = \kappa$, there is $\beta_\alpha < \kappa^+$ such that $A^\beta_\alpha = A^{\beta_\alpha}_\alpha$ for all $\beta \geq \beta_\alpha$. Since $\kappa^+ < \lambda$, by Fodor's theorem there is a stationary $S \subseteq S^\lambda_\kappa$ and $\beta^* < \kappa^+$ such that $\beta_\alpha = \beta^*$ for all $\alpha \in S$. On the other hand, for any $\alpha \in S \cap D^{\beta^*}$ by the successor case of the induction $A_\alpha^{\beta^*+1} \subsetneq A_\alpha^{\beta^*}$, contradicting the definition of β^*. □

Lemma 2.1.2. *If κ and λ are regular cardinals with $\omega_1 \geq \kappa$ and $\kappa^{++} < \lambda$, then there exists an increasing sequence $\langle \mathcal{S}_\alpha : \alpha < \lambda \rangle$ such that*

1. *for all $\alpha < \lambda$, $\mathcal{S}_\alpha \subseteq \wp_{<\kappa}(\alpha)$ and $|\mathcal{S}_\alpha| < \lambda$;*
2. *if $C \subseteq \lambda$ is c.u.b., then there exists $\alpha \in C \cap S_\kappa^\lambda$ and $\langle \alpha_\beta : \beta < \kappa \rangle$ in C such that $\sup_{\beta<\kappa} \alpha_\beta = \alpha$ and for all $\beta < \kappa$ $\{\alpha_{\beta'} : \beta' < \beta\} \in \mathcal{S}_{\alpha_{\beta+1}}$.*

Notes: Such a sequence $\langle \mathcal{S}_\alpha : \alpha < \lambda \rangle$ is called a *Shelah sequence* or a c.u.b. guessing sequence. Also, this lemma is true assuming only $\kappa^+ < \lambda$, but we will not need this.

Proof. Fix a sequence $\langle C_\delta : \delta \in S_\kappa^{\kappa^{++}} \rangle$ as in lemma 2.1.1. For each $\theta \in S_{\kappa^{++}}^\lambda$ let g_θ enumerate an increasing, cofinal sequence in θ of length κ^{++}. For each $\alpha < \lambda$ define

$$\mathcal{S}_\alpha = \{\operatorname{range} g_\theta \restriction (\beta \cap C_\delta) : \theta \in S_{\kappa^{++}}^\lambda \cap \alpha,\ \delta \in S_\kappa^{\kappa^{++}},\ \beta < \delta\}.$$

Clearly, $\langle \mathcal{S}_\alpha : \alpha < \lambda \rangle$ is an increasing sequence satisfying property 1. To show property 2, let $C \subseteq \lambda$ be c.u.b. Define a normal $f : \kappa^{++} \to C$ by induction. Suppose β is a successor ordinal $< \kappa^{++}$ and $f \restriction \beta$ is given. For each $\delta \in S_\kappa^{\kappa^{++}}$ let $\theta_\delta \in S_{\kappa^{++}}^\lambda$ be least ordinal (if one exists, 0 otherwise) such that

$$g_{\theta_\delta} \restriction (\beta \cap C_\delta) = f \restriction (\beta \cap C_\delta). \tag{2.1}$$

Define $f(\beta) = \left(\max(\sup_{\delta \in S_\kappa^{\kappa^{++}}} \theta_\delta, \sup \operatorname{range} f \restriction \beta)\right)^{+C \cap S_\kappa^\lambda}$. Let $\theta = \sup \operatorname{range} f$. Since both f and g_θ are normal functions unbounded in θ, there is a c.u.b. $D \subseteq \kappa^{++}$ such that $g_\theta \restriction D = f \restriction D$. By lemma 2.1.1 there is $\delta \in S_\kappa^{\kappa^{++}}$ with $C_\delta \subseteq D$. Let $\alpha = f(\delta) \in C \cap S_\kappa^\lambda$ and $\alpha_\beta = f(\beta\text{th element of } C_\delta)$. We claim this choice of α and $\langle \alpha_\beta : \beta < \kappa \rangle$ satisfies property 2. Fix $\beta < \kappa$ and consider $\{\alpha_{\beta'} : \beta' < \beta\}$. Let $\theta' < f(\beta+1)$ be such that $g_{\theta'} \restriction ((\beta + 1) \cap C_\delta) = f \restriction ((\beta + 1) \cap C_\delta)$. Note that $g_\theta \restriction C_\delta = f \restriction C_\delta$ by definition of δ, so for any β at θ'. Then $\{\alpha_{\beta'} : \beta' < \beta\} = \operatorname{range} g_{\theta'} \restriction (\beta \cap C_\delta) \in \mathcal{S}_{\alpha_{\beta+1}}$. □

Lemma 2.1.3. *Suppose $|a| < \kappa$, $\kappa^{+++} < \lambda$, and $\langle f_\alpha : \alpha < \lambda \rangle$ is an increasing sequence in $\prod a / J_{<\lambda}$ which respects a Shelah sequence $\langle \mathcal{S}_\alpha : \alpha < \lambda \rangle$, i.e. $\forall \alpha < \lambda \forall S \in \mathcal{S}_\alpha$ $\operatorname{ptsup}_{\alpha' \in S} f_{\alpha'} \leq_{J_{<\lambda}} f_\alpha$. Then $\langle f_\alpha : \alpha < \lambda \rangle$ has a least upper bound.*

Proof. If not, we will define a positively decreasing (mod $J_{<\lambda}$) sequence $\langle g_\beta : \beta \leq \kappa \rangle$ of upper bounds for $\langle f_\alpha : \alpha < \lambda \rangle$ and derive a contradiction.

$\beta = 0$: Let $g_0 = \text{id}$.

β is a successor: Since $g_{\beta-1}$ is a not a least upper bound of $\langle f_\alpha : \alpha < \lambda \rangle$, then there exists an upper bound $g_\beta \lneqq_{J_{<\lambda}} g_{\beta-1}$.

β is a limit $< \kappa$: For each δ in a define $G^\beta(\delta) = \{g_{\beta'}(\delta) : \beta' < \beta\}$ and for each $\alpha < \lambda$ define $f'_\alpha(\delta) =$ least element of $G^\beta(\delta) \geq f_\alpha(\delta)$. For all $\delta \in a$, $\alpha' < \alpha < \lambda$, and $\beta' < \beta$, it is immediate that $|G^\beta(\delta)| < \kappa$ and $f_{\alpha'} \leq_{\text{ev}} f'_{\alpha'} \leq_{J_{<\lambda}} f'_\alpha \leq_{J_{<\lambda}} g_{\beta'}$. We claim there exists $\alpha^* < \lambda$ such that for all $\alpha \geq \alpha^*$ we have $f'_\alpha =_{J_{<\lambda}} f'_{\alpha^*}$, in which case $g_\beta = f'_{\alpha^*}$ is an upper bound for $\langle f_\alpha : \alpha < \lambda \rangle$, and $g_\beta \lneqq_{J_{<\lambda}} g_{\beta'}$ for all $\beta' < \beta$. Otherwise, for all $\alpha < \lambda$ there is $\gamma < \lambda$ such that $f'_\alpha \lneqq_{J_{<\lambda}} f'_\gamma$. Let $C \subseteq \lambda$ be a c.u.b. set closed under the $\alpha \mapsto \gamma$ function. By lemma 2.1.2 there is an ordinal $\alpha \in C \cap S^\lambda_\kappa$ and $\langle \alpha_\gamma : \gamma < \kappa \rangle$ in C with $\sup_{\gamma<\kappa} \alpha_\gamma = \alpha$ and for all $\gamma < \kappa$ $\{\alpha_{\gamma'} : \gamma' < \gamma\} \in \mathcal{S}_{\alpha_{\gamma+1}}$. So, for all $\gamma < \kappa$ by the definition of C we get $f'_{\alpha_{\gamma+1}} \lneqq_{J_{<\lambda}} f'_{\alpha_{\gamma+2}}$. Since $\langle f_\alpha : \alpha < \lambda \rangle$ respects $\langle \mathcal{S}_\alpha : \alpha < \lambda \rangle$, $\text{ptsup}_{\gamma'<\gamma} f_{\alpha_{\gamma'}} \leq_{J_{<\lambda}} f_{\alpha_{\gamma+1}}$ and hence, $\text{ptsup}_{\gamma'<\gamma} f'_{\alpha_{\gamma'}} \leq_{J_{<\lambda}} f'_{\alpha_{\gamma+1}}$. Intersecting a set of positive measure and measure one, we have for all $\gamma < \kappa$ there exists $\delta_\gamma \in a$ such that

$$\sup_{\gamma'<\gamma} f'_{\alpha_{\gamma'}}(\delta_\gamma) \leq f'_{\alpha_{\gamma+1}}(\delta_\gamma) < f'_{\alpha_{\gamma+2}}(\delta_\gamma). \tag{2.2}$$

Since κ is regular and $\kappa > |a|$ there exists $\delta \in a$ and $H \subseteq \kappa$ where $|H| = \kappa$ such that $\delta_\gamma = \delta$ for all $\gamma \in H$. Enumerate a subset of H of size κ as follows. Let γ_0 be the least element of H. If η is a successor, let $\gamma_\eta = (\gamma_{\eta-1} + 2)^{+H}$, and if η is a limit, let $\gamma_\eta = (\sup_{\eta'<\eta} \gamma_{\eta'})^{+H}$. Then by equation 2.2 $\langle f'_{\alpha_{\gamma_\eta}}(\delta) : \eta < \kappa \rangle$ is a strictly increasing sequence of ordinals in $G^\beta(\delta)$, contradicting the fact that $|G^\beta(\delta)| < \kappa$. This finishes the claim, so the sequence $\langle f'_\alpha : \alpha < \lambda \rangle$ must stabilize.

$\beta = \kappa$: Repeat the limit case, replacing κ by κ^+.

For each $\delta \in a$ there exists $\beta_\delta < \kappa$ such that $g_\kappa(\delta) \in G^{\beta_\delta}(\delta)$. Let $\beta = \sup_{\delta\in a} \beta_\delta$. Then, $g_\kappa =_{J_{<\lambda}} g_\beta$, a contradiction. ☐

$\langle \mathcal{S}_\alpha : \alpha < \lambda \rangle$, *then g is also an exact upper bound.*

Proof. Let $g' <_{J_{<\lambda}} g$. If g is not an exact upper bound, then for each $\alpha < \lambda$ we have $g' \geq f_\alpha$ on a positive set P_α. Let $g''_\alpha = g' \restriction P_\alpha \cup g \restriction a - P_\alpha$. Since g is a least upper bound, there is $\beta > \alpha$ and positive $P'_\alpha \subseteq P_\alpha$ such that for each $\delta \in P'_\alpha$ $f_\alpha(\delta) \leq g'(\delta) < f_\beta(\delta)$. Let $C \subseteq \lambda$ be a c.u.b. subset closed under the $\alpha \mapsto \beta$ function. By lemma 2.1.2 there exists $\alpha \in C \cap S^\lambda_\kappa$ and sequence $\langle \alpha_\beta : \beta < \kappa \rangle$ in C such that $\sup_{\beta<\kappa} \alpha_\beta = \alpha$ and for all $\beta < \kappa$ $\{\alpha_{\beta'} : \beta' < \beta\} \in S_{\alpha_{\beta+1}}$. Since $\langle f_\alpha : \alpha < \lambda \rangle$ respects $\langle S_\alpha : \alpha < \lambda \rangle$, then

$$\operatorname*{ptsup}_{\beta'<\beta} f_{\alpha_{\beta'}} \leq_{J_{<\lambda}} f_{\alpha_{\beta+1}} \tag{2.3}$$

As before, for each $\beta < \kappa$ and $\delta \in P'_{\alpha_{\beta+1}}$

$$f_{\alpha_{\beta+1}}(\delta) \leq g'(\delta) < f_{\alpha_{\beta+2}}(\delta). \tag{2.4}$$

Intersecting a measure one set with a set of positive measure we have for each $\beta < \kappa$ there exists δ_β such that 2.4 and 2.3 hold at δ_β. Since κ is regular and $|a| < \kappa$, there is $\delta \in a$ and $H \subseteq \kappa$ with $|H| = \kappa$ such that $\delta_\beta = \delta$ for all $\beta \in H$. Choose β_1, β_2 in H with $\beta_1 + 2 < \beta_2$. Then

$$g'(\delta) < f_{\alpha_{\beta_1+2}}(\delta) \leq \sup_{\beta'<\beta_2} f_{\alpha_{\beta'}}(\delta) \leq f_{\alpha_{\beta_2+1}}(\delta) \leq g'(\delta),$$

a contradiction. □

2.2 Main Proof

The following proof is not new. Rather, it is a streamlined version of theorem 4.6 in [2]. Assume $|a|^{+++} < \min a$. We specifically do not assume $2^{|a|} < \min a$, as in other proofs of this result.

Theorem 2.2.1. *$J_{<\lambda^+}(a) = J_{<\lambda}(a)[b]$ for some $b \subseteq a$.*

Proof. Let $\langle S_\alpha : \alpha < \lambda \rangle$ be a Shelah sequence for $\kappa = |a|^+$. For some $\beta_0 \leq \kappa$ we will inductively define a two sequences of sequences of functions $\langle \langle f^\beta_\alpha : \alpha < \lambda \rangle : \beta < \beta_0 \rangle$ and $\langle \langle g^\beta_\alpha : \alpha < \lambda \rangle : \beta < \beta_0 \rangle$ in $\prod a$ as well as a sequence $\langle b^\beta : \beta < \beta_0 \rangle$ of subsets of a satisfying properties 2.5 2.10 until either we have found a generator, i.e. $J_{<\lambda^+} = J_{<\lambda}[b^{\beta_0}]$, or we have reached the κth step of the induction, i.e. $\beta_0 = \kappa$. To prove the theorem we will show the induction cannot continue κ steps. The inductive requirements are:

$$\alpha' < \alpha \implies f^{\beta}_{\alpha'} <_{J_{<\lambda}} f^{\beta}_{\alpha} \tag{2.5}$$

$$\beta' < \beta \implies f^{\beta'}_{\alpha} \leq_{\text{ev}} f^{\beta}_{\alpha} \tag{2.6}$$

$$\operatorname*{ptsup}_{\alpha' \in S} f^{\beta}_{\alpha'} \leq_{J_{<\lambda}} f^{\beta}_{\alpha} \text{ for all } S \in \mathcal{S}_\alpha \tag{2.7}$$

$$g^{\beta}_{\alpha} \leq_{J_{<\lambda}} f^{\beta+1}_{\alpha} \tag{2.8}$$

$$f^{\beta}_{\alpha} \leq_{J_{<\lambda}[b^\beta]} f^{\beta+1}_{0} \tag{2.9}$$

$$b^{\beta+1} - b^{\beta} \notin J_{<\lambda}. \tag{2.10}$$

To begin the induction let f^0_0 be an arbitrary element of $\prod a$. We require f^{β}_{α} to be a bound mod $J_{<\lambda}$ of $\{f^{\beta}_{\alpha'} + 1 : \alpha' < \alpha\}$. This is possible since $\alpha < \lambda$ and $J_{<\lambda}$ is λ directed. So 2.5 holds. For each $S \in \mathcal{S}_\alpha$ define $h^S_\alpha = \operatorname{ptsup}_{\alpha' \in S} f^{\beta}_{\alpha'}$, which is in $\prod a$ since $|S| \leq \kappa < \min a$. We also require f^{β}_{α} to be a bound mod $J_{<\lambda}$ of $\{h^S_\alpha : S \in \mathcal{S}_\alpha\}$. Again, this is possible since $|\mathcal{S}_\alpha| < \lambda$ and $J_{<\lambda}$ is λ directed. So 2.7 holds. Additionally, we require $\operatorname{ptsup}_{\beta' < \beta} f^{\beta'}_{\alpha} \leq_{\text{ev}} f^{\beta}_{\alpha}$, which is possible since $\beta < \min a$. So 2.6 holds. By lemma 2.1.4 $\langle f^{\beta}_{\alpha} : \alpha < \lambda \rangle$ has an exact upper bound mod $J_{<\lambda}$, say f^β. Define $b^\beta = \{\delta \in a : f^\beta(\delta) = \delta\}$. If b^β is a generator, the theorem is proved. Otherwise, there exists $d^\beta \in J_{<\lambda^+} - J_{<\lambda}[b^\beta]$, and by lemma 1.2.5 there is a scale $\langle g^{\beta}_{\alpha} : \alpha < \lambda \rangle$ mod $J_{<\lambda} \restriction d^\beta$. Furthermore, $\langle f^{\beta}_{\alpha} : \alpha < \lambda \rangle$ is a scale mod $J_{<\lambda} \restriction b^\beta$ and $\langle f^{\beta}_{\alpha} : \alpha < \lambda \rangle$ is bounded by f^β mod $J_{<\lambda}[b^\beta]$. To satisfy (2.8) and (2.9) we additionally require only that $f^\beta \leq_{J_{<\lambda}[b^\beta]} f^{\beta+1}_0$ and $g^{\beta}_{\alpha} \leq_{J_{<\lambda}} f^{\beta+1}_{\alpha}$. Note that since $\langle g^{\beta}_{\alpha} : \alpha < \lambda \rangle$ is a scale on d^β and 2.8 holds, then $d^\beta \subseteq_{J_{<\lambda}} b^{\beta+1}$, and hence 2.10 is satisfied.

Now we will show the induction cannot continue $\kappa = |a|^+$ steps. Let $g = \operatorname{ptsup}_{\beta < \kappa} f^{\beta}_{0}$. For each $\beta < \kappa$, since $\langle f^{\beta}_{\alpha} : \alpha < \lambda \rangle$ was a scale on b^β, there exists $\alpha_\beta < \lambda$ with $g <_{J_{<\lambda} \restriction b^\beta} f^{\beta}_{\alpha_\beta}$. Let $\alpha_\kappa = \sup_{\beta < \kappa} \alpha_\beta < \lambda$. For each $\beta < \kappa$ define $c^\beta = \{\delta \in a : f^{\beta}_{\alpha_\kappa}(\delta) > g(\delta)\}$. By 2.6, $\langle c^\beta : \beta < \kappa \rangle$ is an increasing sequence of subsets of a. For a contradiction we claim that this sequence is strictly increasing. Fix $\beta < \kappa$. On one hand by equation 2.9, $f^{\beta}_{\alpha_\kappa} <_{J_{<\lambda}[b^\beta]} f^{\beta+1}_{0} \leq_{\text{ev}} g$. In particular, since $b^{\beta+1} - b^\beta \notin J_{<\lambda}$, then $f^{\beta}_{\alpha_\kappa}(\delta) < g(\delta)$ for $J_{<\lambda}$ a.e. $\delta \in b^{\beta+1} - b^\beta$. On the other hand, $g <_{J_{<\lambda} \restriction b^{\beta+1}} f^{\beta+1}_{\alpha_\kappa}$. Hence, for $J_{<\lambda}$ a.e. $\delta \in b^{\beta+1} - b^\beta$, we have $f^{\beta}_{\alpha_\kappa}(\delta) < g(\delta) < f^{\beta+1}_{\alpha_\kappa}$, and

CHAPTER 3

Strong Partition Relation

Assume AD throughout this chapter.

3.1 Coding Subsets of ω_1

We describe a method to code an arbitrary subset of ω_1 by a real in ω^ω. Define WO^* to be $\{z \in \omega^\omega : z_0 \in \mathrm{WO} \wedge z_1 \in 2^\omega\}$. Intuitively, each $z \in \mathrm{WO}^*$ is a tagged wellorder, and α is an element of the set coded by z exactly when the rank of n in the wellorder coded by z_0 is α and $z_1(n) = 1$. Let T^{WO} on $\omega \times \omega_1$ be the Shoenfield tree of WO, that is $p[T^{\mathrm{WO}}] = \mathrm{WO}$ and $\forall \alpha < \omega_1 \exists x \in \omega^\omega\, |x| = \alpha \wedge T^{\mathrm{WO}}_x \restriction \alpha$ is illfounded. We say $z, z' \in \omega^\omega$ are *compatible codes* iff $z, z' \in \mathrm{WO}^*$ and for all $m, m' < \omega$ $|m|_{z_0} = |m'|_{z'_0} \implies z_1(m) = z'_1(m')$. So, the predicate S representing incompatibility of codes is

$$S(z, z') \iff [z, z' \in \mathrm{WO}^* \implies \exists m, m' < \omega\, |m|_{z_0} = |m'|_{z'_0} \wedge z_1(m) \neq z'_1(m')].$$

Note that S is $\mathbf{\Sigma}^1_1$, so there is a tree T^S on $\omega \times \omega \times \omega$ with $p[T^S] = S$. Define a strategy τ to be $\leq \alpha$ *invariant* if $\forall x, x' \in \mathrm{WO}\ (|x|, |x|' \leq \alpha) \implies \neg S(\tau(x), \tau(x'))$. Furthermore, τ is an *invariant strategy* if τ is $\leq \alpha$ invariant strategy for all $\alpha < \omega_1$. Say τ *codes* $B \subseteq \omega_1$ if

$$\begin{aligned}&\forall x \in \mathrm{WO}\ [\tau(x) \in \mathrm{WO}^* \wedge |\tau(x)_0| \geq |x| \wedge \\ &\quad \forall \beta < |\tau(x)_0| \forall m < \omega\ \beta = |m|_{\tau(x)_0} \implies (\beta \in B \iff \tau(x)_1(m) = 1)].\end{aligned}$$

Clearly, if τ codes $B \subseteq \omega_1$, then τ is invariant. Conversely, if τ is invariant and $\forall x \in \mathrm{WO}\ |\tau(x)_0| \geq |x|$, then there exists a unique $B \subseteq \omega_1$ which is coded by τ, namely $\beta \in B$ iff $\exists x \in \mathrm{WO} \exists m < \omega\ |m|_{\tau(x)_0} = \beta$ and $\tau(x)_1(m) = 1$. In an upcoming lemma (3.1.2) we will show each subset of ω_1 can be coded by some invariant strategy. Towards this end we define that of the Kunen tree (see lemma 1.3.1). Define T^J by

$$\begin{aligned}&(t, a, \alpha, a', \alpha', b, b', c) \in T^J \iff \\ &\quad (a, \alpha) \in T^{\mathrm{WO}} \wedge (a', \alpha') \in T^{\mathrm{WO}} \wedge ta = b \wedge ta' = b' \wedge (b, b', c) \in T^S,\end{aligned}$$

where, as in lemma 1.3.1, $ta = b$ means that in some reasonable manner t codes a partial strategy in a game applied to a partial run of the game a, and that b does not contradict t applied to a.

Lemma 3.1.1. *For all $\tau \in \omega^\omega$ and for all $\beta < \omega_1$, $T^V_\tau \restriction \beta$ is wellfounded iff τ is $\leq \beta$ invariant strategy.*

Proof. If $T^V_\tau \restriction \beta$ is illfounded, let $(x, \alpha, x', \alpha', z, z', w) \in [T^V_\tau \restriction \beta]$. So $(x, \alpha), (x', \alpha') \in [T^{WO}]$ and $x, x' \in \mathrm{WO}$ with $|x|, |x'| \leq \beta$. Also, $\tau(x) = z$ and $\tau(x') = z'$, where $(z, z', w) \in [T^S]$. Thus, $S(\tau(x), \tau(x'))$ holds, so τ cannot be $\leq \beta$ invariant.

Conversely, suppose τ is not $\leq \beta$ invariant. Then there exist $x, x' \in \mathrm{WO}$ with $|x|, |x'| \leq \beta$ and $S(\tau(x), \tau(x'))$. There exists $w \in \omega^\omega$ such that $(\tau(x), \tau(x'), w) \in [T^S]$. Choose α, α' such that for all n, $\alpha_n, \alpha'_n < \beta$ and $(x, \alpha), (x', \alpha') \in [T^{WO}]$. Then $(x, \alpha, x', \alpha', \tau(x), \tau(x'), w) \in [T^V_\tau \restriction \beta]$, showing $T^V_\tau \restriction \beta$ is illfounded. □

Lemma 3.1.2. *For all $B \subseteq \omega_1$ there exists an invariant strategy τ which codes B.*

Proof. Play the Solovay game where I plays x, II plays y and II wins iff

$$\begin{aligned} x \in \mathrm{WO} \implies & [y \in \mathrm{WO}^* \wedge |y_0| > |x| \wedge \\ & \forall \beta < |y_0| \forall m < \omega\, \beta = |m|_{y_0} \implies (\beta \in B \iff y_1(m) = 1)]. \end{aligned}$$

By boundedness I cannot have a winning strategy. So II must have a winning strategy τ. We claim that τ codes B, and hence is invariant. Let $x \in \mathrm{WO}$. Since τ is a winning strategy, $\tau(x) \in \mathrm{WO}^*$ and $|\tau(x)_0| > |x|$. By the last clause of the statement of the game τ codes B. □

Now suppose $\sigma \in \omega^\omega$ and the Kunen tree T^K_σ is wellfounded. Let C_σ be the c.u.b. subset of ω_1 of limit ordinals closed under $\alpha \mapsto |T^K_\sigma \restriction \alpha|$. The next lemma shows that such C_σ's form a base of the c.u.b. subsets of ω_1.

Lemma 3.1.3. *If $C \subseteq \omega_1$ is c.u.b., then there exists $\sigma \in \omega^\omega$ such that T^K_σ is wellfounded and $C_\sigma \subseteq C$.*

Proof. Let $\boldsymbol{A} \subseteq \omega^\omega$ be $\boldsymbol{\Sigma}^1_1$ complete. Play the Solovay game where I plays x, II plays y and II wins iff

$$x \in \mathrm{WO} \implies y \notin \boldsymbol{A} \wedge |T^{\boldsymbol{A}}_y| > |x|^{+C}.$$

By boundedness, I cannot have a winning strategy. So II has a winning strategy σ. As in the proof of lemma 1.3.1, T^K_σ is wellfounded, and $|T^{\boldsymbol{A}}_{\sigma(x)}| \leq |T^K_\sigma \restriction |x||$ for any $x \in \mathrm{WO}$. Let

$\gamma \in C_\sigma$ and choose $x \in \mathrm{WO}$ with $|x| < \gamma$. Then, $|x| < |x|^{+C} < |T^S_{\sigma(x)}| \leq |T^K_\sigma \restriction |x|| < \gamma$. So γ is a limit of points of C, hence in C. ☐

3.2 Simple Sets

Recall T^K is the linear order version of the Kunen tree (see lemma 1.3.2). So, for any $x \in \omega^\omega$, T^K_x is a linear order of a subset of ω_1. For any $\gamma < \omega_1$, $T^K_x \restriction \gamma \equiv T^K_x \cap \gamma$ is a linear order of a subset of γ. For any $\alpha, \delta < \omega_1$, $a = |T^K_x \restriction \gamma(\delta)|$ means if $\delta \in \mathrm{wf}(T^K_x \restriction \gamma)$, then $\alpha = |\delta|_{T^K_x \restriction \gamma} =$ the rank of δ in $T^K_x \restriction \gamma$.

We now define a notion of a *simple set* $S_{\tau,\sigma,x} \subseteq \omega_1$ coded by three reals $\tau, \sigma, x \in \omega^\omega$:

$$\begin{aligned}
\alpha \in S_{\tau,\sigma,x} \iff & \exists \text{ limit } \gamma \leq \alpha \\
& \forall \delta < \gamma (T^K_\sigma \restriction \delta \text{ is wellfounded} \wedge |T^K_\sigma \restriction \delta| < \gamma) \wedge \\
& \forall \delta < \gamma (T^J_\tau \restriction \delta \text{ is wellfounded} \wedge |T^J_\tau \restriction \delta| < \gamma) \wedge \\
& \exists \delta < \gamma \big(\delta \in \mathrm{wf}(T^K_x \restriction \gamma) \wedge \alpha = |T^K_x \restriction \gamma(\delta)| \wedge \\
& \qquad \exists y \in \mathrm{WO}_\gamma \exists n < \omega \; |n|_{\tau(y)_0} = \delta \wedge \tau(y)_1(n) = 1\big).
\end{aligned}$$

We say that τ, σ, x code $S_{\tau,\sigma,x}$. Intuitively, $\alpha \in S_{\tau,\sigma,x}$ iff α is the lift-up $|T^K_x \restriction \gamma(\delta)|$ by some γ in C_σ of some δ in the set coded by the invariant strategy τ. In lemma 3.2.3 we show that every subset of ω_1 is a countable union of simple sets.

We begin the trek to the strong partition relation on ω_1 with a quick proof that the diagonal intersection of normal measure one sets is still measure one, and then prove a general lemma about the coherence of sequences of subsets of ω_1. A sequence satisfying the conclusion lemma 3.2.2 is said to "cohere."

Lemma 3.2.1. *If for all $\alpha < \omega_1$ $D_\alpha \subseteq \omega_1$ contains a c.u.b. set, then the diagonal intersection $\triangle_{\alpha<\omega_1} D_\alpha = \{\gamma < \omega_1 : \forall \alpha < \gamma \;\; \gamma \in D_\alpha\}$ also contains a c.u.b. set.*

Proof. Without loss of generality $\langle D_\alpha : \alpha < \omega_1 \rangle$ is decreasing (otherwise take intersections). Partition pairs $\alpha \leq \gamma \leq \omega_1$ by whether or not $\gamma \in D_\alpha$. By the finite partition relation on ω_1 "not in" side, else for any $\alpha \in C$ we would have $D_\alpha \cap C \subseteq \alpha$, i.e. the intersection of two c.u.b. sets is countable. Now let C' be the limit points of C. We claim that $C' \subseteq \triangle_{\alpha<\omega_1} D_\alpha$. If $\gamma \in C'$ and $\alpha < \gamma$, then $\alpha^{+C} < \gamma$. Since C is homogeneous for the "in" side, then $\gamma \in D_{\alpha^{+C}} \subseteq D_\alpha$. ☐

Lemma 3.2.2. *For each sequence $\langle B_\gamma \subseteq \gamma : \gamma \in C\rangle$ where $C \subseteq \omega_1$ is c.u.b., there is a c.u.b. $D \subseteq C$ such that*

$$\forall \gamma_1, \gamma_2 \in D \quad \gamma_1 < \gamma_2 \implies B_{\gamma_1} = B_{\gamma_2} \cap \gamma_1.$$

Proof. For each $\alpha < \omega_1$ define $D_\alpha = \{\gamma \in C : \alpha \in B_\gamma\}$. One of the sets, D_α or $\omega_1 - D_\alpha$, contains a c.u.b. set, say D'_α. By lemma 3.2.1 $D = \triangle_{\alpha<\omega_1} D'_\alpha$ contains a c.u.b. set. Now suppose $\gamma_1, \gamma_2 \in D$ and $\gamma < \gamma_1 < \gamma_2$. By the definition of the diagonal intersection both $\gamma_1, \gamma_2 \in D'_\gamma$. If $D'_\gamma \subseteq D_\gamma$, then by definition of D_γ we have $\gamma \in B_{\gamma_1}$ and $\gamma \in B_{\gamma_2}$. Otherwise, $D'_\gamma \subseteq \omega_1 - D_\gamma$ in which case $\gamma \notin B_{\gamma_1}$ and $\gamma \notin B_{\gamma_2}$. Consequently, $B_{\gamma_1} = B_{\gamma_2} \cap \gamma_1$. ☐

Lemma 3.2.3. *Every subset of ω_1 is a countable union of simple sets.*

Proof. If not, let $A \subseteq \omega_1$ be a counterexample. Then the collection subsets of A which are countable unions of simple sets form a proper σ ideal on A. Let ν be a measure on A respecting this ideal, and let $f : \omega_1 \to \omega_1$ be a representative of the least equivalence class with respect to ν of a function which is nonconstant and increasing. Specifically, this means that f is not constant on any ν measure one set, and there exists a ν measure one set $A \subseteq \omega_1$ such that $\forall \alpha, \beta \in A(\alpha < \beta \implies f(\alpha) \leq f(\beta))$. Since the identity is such a function, f is necessarily nonstrictly pressing down. Also note that $f(\nu)$ is the normal measure W^1_1 on ω_1. Otherwise, there exist disjoint sets $C, A \subseteq \omega_1$ where C is c.u.b. and $f(\nu)(A) = 1$. If we define $g : A \to \omega_1$ by $g(\alpha) =$ the largest element of $C < \alpha$, then $g \circ f$ violates the minimality of f.

Fix $A_1 \subseteq A$ such that $\nu(A_1) = 1$ and $f \restriction A_1$ is increasing and nonstrictly pressing down. For each $\gamma < \omega_1$ define

$$g(\gamma) = \sup\{\alpha \in A_1 : f(\alpha) \leq \gamma\}.$$

If $\alpha \in A_1$, then $g(f(\alpha)) \geq \alpha$. Also, note that $g(\gamma) < \omega_1$ since f is nonconstant and increasing. By lemma 1.3.2 there exist $x \in \omega^\omega$ and c.u.b. $C_0 \subseteq \omega_1$ such that the section of the Kunen linear order T^K_x is wellordered, and for all $\gamma \in C_0$ we have $|T^K_x \restriction \gamma| > g(\gamma)$. For each $\gamma \in C_0$ define B_γ by

$$\delta \in B_\gamma \iff \delta < \gamma \wedge \exists \alpha \in A_1 \; \gamma < \alpha \wedge \alpha = |T^K \restriction \gamma(\delta)| \tag{3.1}$$

Let $C_1 \subseteq C_0$ be the c.u.b. set from lemma 3.2.2 such that the sequence $\langle B_\gamma \subseteq \gamma : \gamma \in C_1\rangle$ coheres. Let $B = \bigcup_{\gamma \in C_1} B_\gamma$ and by lemma 3.1.2 there exists an invariant strategy τ which codes B. By lemma 3.1.1 T^V_τ is wellfounded. Let c.u.b. $C_2 \subseteq C_1$ be closed under the function $\gamma \mapsto |T^V_\tau \restriction \gamma|$. By lemma 3.1.3 there is $\sigma \in \omega^\omega$ such that T^K_σ is wellfounded and $C_\sigma \subseteq C_2$. Consider the simple set $S_{\tau,\sigma,x}$.

First, we claim that $S_{\tau,\sigma,x} \subseteq A$. Let $\alpha \in S_{\tau,\sigma,x}$. By definition of $S_{\tau,\sigma,x}$ there is some $\gamma \in C_\sigma$ and $\delta \in B$ such that $\delta < \gamma \leq \alpha$ and $\alpha = |T_x^K \restriction \gamma(\delta)|$. Since B is a union, then $\delta \in B_{\gamma'}$ for some $\gamma' \in C_1$. By the definition of C_1 we have $\delta \in B_\gamma$, and by equation 3.1 it follows that $\alpha \in A_1 \subseteq A$.

Second, we claim that $A_1 \cap f^{-1}C_\sigma \subseteq S_{\tau,\sigma,x}$. If $\alpha \in A_1 \cap f^{-1}C_\sigma$, then $\gamma \equiv f(\alpha) \in C_\sigma$. Since $\alpha \in A_1$ and $\gamma \in C_0$, then $\alpha \leq g(\gamma) < |T_x^K \restriction \gamma|$. So, for some $\delta \in B_\gamma$, $\alpha = |T_x^K \restriction \gamma(\delta)|$. Since $\gamma \in C_1$, then $\delta \in B$. Finally, since $\gamma \in C_\sigma$ and $C_\sigma \subseteq C_2$, we have satisfied the definition for $\alpha \in S_{\tau,\sigma,x}$.

Putting these claims together, we have shown that $\nu(S_{\tau,\sigma,x}) = 0$ and $\nu(S_{\tau,\sigma,x}) = 1$, a contradiction. $\square$

Definition 3.2.4. A regular cardinal κ is *reasonable* if there is a nonselfdual pointclass $\mathbf{\Gamma}$ closed under $\exists^{\omega^\omega}$ and a map ϕ with domain ω^ω satisfying

1. $\forall x\, \phi(x) \subseteq \kappa \times \kappa$.
2. $\forall f\colon \kappa \to \kappa\, \exists x\, \phi(x) = f$.
3. $\forall \alpha, \beta < \kappa\, R_{\alpha,\beta} \in \mathbf{\Delta} \equiv \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$, where $x \in R_{\alpha,\beta} \iff \phi(x)(\alpha,\beta) \wedge \forall \beta' < \kappa\, (\phi(x)(\alpha,\beta') \implies \beta' = \beta)$.
4. If $\alpha < \kappa$, $A \in \exists^{\omega^\omega}\mathbf{\Delta}$, and $A \subseteq R_\alpha \equiv \{x : \exists \beta < \kappa\, R_{\alpha,\beta}(x)\}$, then $\exists \beta' < \kappa \forall x \in A \exists \beta < \beta'\, R_{\alpha,\beta}(x)$.

If $x \in R_\alpha$, let $\phi(x)(\alpha)$ be the unique β such that $\phi(x)(\alpha,\beta)$. Let $S_{\alpha,\beta} = \bigcap_{\alpha' \leq \alpha} \bigcup_{\beta' \leq \beta} R_{\alpha',\beta'}$ and $S_\alpha = \bigcap_{\alpha' \leq \alpha} R_{\alpha'}$.

Theorem 3.2.5 (Martin). *Every reasonable cardinal has the strong partition relation.*

Proof. Let $\mathbf{\Gamma}$ and $\phi\colon \omega^\omega \to \wp(\kappa \times \kappa)$ witness that κ is reasonable. Suppose p is a partition of functions from κ to κ of the correct type. Consider the game where I plays x, II plays y, and if there exists $\alpha < \kappa$ such that $y \in S_\alpha$ but $x \notin S_{\alpha+1}$, then II wins. Otherwise, both $\phi(x)$ and $\phi(y)$ determine functions from κ to κ, in which case II wins when $p(F_{xy}) = 1$, where

Assume II has a winning strategy τ. (The argument for I is slightly simpler.) For $\alpha, \beta < \kappa$ define $\ell(\alpha,\beta) = \sup\{\phi(y)(\alpha) : y \in \tau[S_{\alpha,\beta}]\}$. We note that $\mathbf{\Delta}$ is closed under $< \kappa$ unions and intersections (see the last paragraph of the proof of theorem 2.28 in [4]). So $S_{\alpha,\beta} \in \mathbf{\Delta}$, and hence $\tau[S_{\alpha,\beta}] \in \exists^{\omega^\omega}\mathbf{\Delta}$. Since κ is reasonable, $\ell(\alpha,\beta) < \kappa$. Let $C \subseteq \kappa$ be a c.u.b. set closed under ℓ, and let C' be the limit points of C. We claim for any $F\colon \kappa \to C'$ of the correct type, $p(F) = 1$. So suppose $F: \kappa \to C'$ is of the correct type. Fix $f: \kappa \to C$ such that for all $\alpha < \kappa$, $F(\alpha) = \sup_{\alpha' < \omega\alpha+\omega} f(\alpha')$, and fix $x \in \omega^\omega$ such that $f = \phi(x)$. Let $y = \tau(x)$. So, $\phi(y)$ is a function from κ to κ, and $\phi(y)(\alpha') \leq \phi(x)(\alpha'+1)$ for all $\alpha' < \kappa$. Thus, $F = F_{xy}$, and since τ was a winning strategy for II, $p(F) = 1$. □

3.3 Coding Functions

In the previous sections we showed how to code subsets of ω_1 by reals. In particular we showed that every subset of ω_1 is a countable union of simple sets. To show the strong partition relation on ω_1 we must code functions from ω_1 to ω_1. Of course, we could embed $\omega_1 \times \omega_1$ into ω_1 and view functions as subsets of ω_1 accordingly. The problem is that this coding does not seem to satisfy the conditions of Martin's theorem 3.2.5. Therefore, we will go through the two previous sections, sketching proofs of analogous results for functions from ω_1 to ω_1 and show that ω_1 is reasonable relative to this coding of functions and the pointclass of analytic sets. The proofs themselves are often identical to the proofs in the previous sections. Here, however, we highlight the new ideas needed to code functions and carefully point out the changes they bring about.

Let $\mathrm{WO}^{**} = \{z \in \omega^\omega : z_0 \in \mathrm{WO} \wedge z_1 \in 2^{\omega\times\omega}\}$. Formally, $\mathrm{WO}^{**} = \mathrm{WO}^*$, but we think of $z \in \mathrm{WO}^{**}$ as coding a subset of $\omega_1 \times \omega_1$. Namely, the pair (α, β) is in the subset coded by z iff the rank of m in the wellorder z_0 is α and the rank of n in the wellorder z_0 is β and $z_1(m,n) = 1$. We say $z, z' \in \omega^\omega$ are compatible codes iff $z, z' \in \mathrm{WO}^{**}$ and for all $m, m', n, n' < \omega (|m|_{z_0} = |m'|_{z_0'} \wedge |n|_{z_0} = |n'|_{z_0'}) \implies z_1(m,n) = z_1'(m',n')$. So, the predicate S representing incompatibility of codes is

$$S(z, z') \iff [z, z' \in \mathrm{WO}^{**} \implies \exists m, n, m', n' < \omega$$
$$|m|_{z_0} = |m'|_{z_0'} \wedge |n|_{z_0} = |n'|_{z_0'} \wedge z_1(m,n) \neq z_1(m', n')].$$

Clearly, $S \in \mathbf{\Sigma}^1_1$. Let T^S be a tree on $\omega \times \omega \times \omega$ which projects to S. Define strategy τ to be $\leq \alpha$ *invariant* if $\forall x, x' \in \mathrm{WO}\ |x|, |x|' \leq \alpha \implies \neg S(\tau(x), \tau(x'))$. Furthermore, τ is an

$$\forall x \in \mathrm{WO}\ [\tau(x) \in \mathrm{WO}^{**} \wedge |\tau(x)_0| \geq |x| \wedge$$
$$\forall \delta, \epsilon < |\tau(x)_0| \forall m, n < \omega\ (\delta = |m|_{\tau(x)_0} \wedge \epsilon = |n|_{\tau(x)_0}) \implies$$
$$((\delta, \epsilon) \in B \iff \tau(x)_1(m,n) = 1)].$$

The tree T^J representing noninvariance is defined the same as before.

Lemma 3.3.1. *For all $\tau \in \omega^\omega$ and for all $\alpha < \omega_1$, $T^J_\tau \upharpoonright \alpha$ is wellfounded iff τ is $\leq \alpha$ invariant strategy.*

Proof. Same as proof of lemma 3.1.1. ☐

Lemma 3.3.2. *For all $B \subseteq \omega_1 \times \omega_1$ there exists an invariant strategy τ which codes B.*

Proof. Same as proof of lemma 3.1.2. ☐

We now define a notion of *simple subset* $S_{\tau,\sigma,x} \subseteq \omega_1 \times \omega_1$ coded by three reals $\tau, \sigma, x \in \omega^\omega$:

$$\begin{aligned}
&(\alpha, \beta) \in S_{\tau,\sigma,x} \iff \exists \text{ limit } \gamma \leq \alpha \\
&\quad \forall \delta < \gamma (T^K_\sigma \upharpoonright \delta \text{ is wellfounded} \wedge |T^K_\sigma \upharpoonright \delta| < \gamma) \wedge \\
&\quad \forall \delta < \gamma (T^J_\tau \upharpoonright \delta \text{ is wellfounded} \wedge |T^J_\tau \upharpoonright \delta| < \gamma) \wedge \\
&\quad \exists \delta, \epsilon < \gamma \big[\delta, \epsilon \in \mathrm{wf}(T^K_x \upharpoonright \gamma) \wedge \alpha = |T^K_x \upharpoonright \gamma(\delta)| \wedge \beta = |T^K_x \upharpoonright \gamma(\epsilon)| \wedge \\
&\qquad \exists y \in \mathrm{WO}_\gamma \exists m, n < \omega \, |m|_{\tau(y)_0} = \delta \wedge |n|_{\tau(y)_0} = \epsilon \wedge \tau(y)_1(m, n) = 1\big].
\end{aligned}$$

This definition is the expected two dimensional analog of the definition of simple subset of ω_1, the only possible exception being in the first line where γ is required to be at most α, a more stringent requirement than $\gamma \leq \max(\alpha, \beta)$.

Lemma 3.3.3. *If for all $\alpha, \beta < \omega_1$ $D_{\alpha,\beta} \subseteq \omega_1$ contains a c.u.b. set, then the diagonal intersection $\triangle_{\alpha,\beta<\omega_1} D_{\alpha,\beta} = \{\gamma < \omega_1 : \forall \alpha, \beta < \gamma \;\; \gamma \in D_{\alpha,\beta}\}$ also contains a c.u.b. set.*

Proof. Without loss of generality $\{D_{\alpha,\beta} : \alpha, \beta < \omega_1\}$ is decreasing in both coordinates. Partition triples $\alpha < \beta < \gamma < \omega_1$ by whether or not $\gamma \in D_{\alpha,\beta}$. By the finite partition relation on ω_1 there is a homogeneous c.u.b. $C \subseteq \omega_1$ for this partition. C cannot be homogeneous for the "not in" side, else for any $\alpha, \beta \in C$ we would have $D_{\alpha,\beta} \cap C \subseteq \max(\alpha, \beta)$, i.e. the intersection of two c.u.b. sets is countable. Now let C' be the limit points of C. We claim that $C' \subseteq \triangle_{\alpha,\beta<\omega_1} D_{\alpha,\beta}$. If $\gamma \in C'$ and $\alpha, \beta < \gamma$ then $\alpha^{+C}, \beta^{+C} < \gamma$. Since C is homogeneous

Lemma 3.3.4. *For each sequence $\langle B_\gamma \subseteq \gamma \times \gamma : \gamma \in C \rangle$ where $C \subseteq \omega_1$ is c.u.b., there is a c.u.b. $D \subseteq C$ such that*

$$\forall \gamma_1, \gamma_2 \in D \quad \gamma_1 < \gamma_2 \implies B_{\gamma_1} = B_{\gamma_2} \cap (\gamma_1 \times \gamma_1).$$

Proof. For all $\alpha, \beta < \omega_1$ define $D_{\alpha,\beta} = \{\gamma \in C : (\alpha, \beta) \in B_\gamma\}$. One of the sets, $D_{\alpha,\beta}$ or $\omega_1 - D_{\alpha,\beta}$, contains a c.u.b. set, say $D'_{\alpha,\beta}$. By lemma 3.3.3 $D = \triangle_{\alpha,\beta<\omega_1} D'_{\alpha,\beta}$ contains a c.u.b. set. Now suppose $\gamma_1, \gamma_2 \in D$ and $\alpha, \beta < \gamma_1 < \gamma_2$. By the definition of the diagonal intersection both $\gamma_1, \gamma_2 \in D'_{\alpha,\beta}$. If $D'_{\alpha,\beta} \subseteq D_{\alpha,\beta}$, then by definition of $D_{\alpha,\beta}$ we have $\alpha, \beta \in B_{\gamma_1}$ and $\alpha, \beta \in B_{\gamma_2}$. Otherwise, $D'_{\alpha,\beta} \subseteq \omega_1 - D_{\alpha,\beta}$ in which case $\alpha, \beta \notin B_{\gamma_1}$ and $\alpha, \beta \notin B_{\gamma_2}$. Consequently, $B_{\gamma_1} = B_{\gamma_2} \cap (\gamma_1 \times \gamma_1)$. □

Lemma 3.3.5. *Every function $h : \omega_1 \to \omega_1$ is a countable union of simple partial functions from ω_1 to ω_1.*

Proof. If not, let $h : \omega_1 \to \omega_1$ be a counterexample. Then the collection of countable unions of simple partial functions which are subsets of h form a proper σ ideal on h. Let ν be a measure on h respecting this ideal. (The measure ν on $\omega_1 \times \omega_1$ can be identified with a measure on ω_1, namely for $A \subseteq \omega_1$, $\nu(A) = 1$ iff $\nu(\{(\alpha, \beta) : h(\alpha) = \beta\}) = 1$.) Let $f : \omega_1 \times \omega_1 \to \omega_1$ be a representative of the least equivalence class with respect to ν of a function which is nonconstant and increasing on the first coordinate. (Likewise, f can be identified with a function whose domain is ω_1 by $f(\alpha) = f(\alpha, \beta)$.) By this we mean that for all ν measure one sets A, f is nonconstant on A and that there exists $A \subseteq h$ such that $\nu(A) = 1$ and for all $(\alpha, h(\alpha)), (\alpha', h(\alpha')) \in A$ if $\alpha < \alpha'$, then $f(\alpha, h(\alpha)) \leq f(\alpha', h(\alpha'))$. Since the "identity" $\alpha \mapsto (\alpha, h(\alpha))$ is such a function, then f is nonstrictly pressing down, i.e. $\forall^*_\nu \alpha\ f(\alpha, h(\alpha)) \leq \alpha$. As before $f\nu$ is the normal measure on ω_1. Fix $A_1 \subseteq h$ such that $\nu(A_1) = 1$ and $f \restriction A_1$ is increasing on the first coordinate and nonstrictly pressing down. For each $\gamma < \omega_1$ define

$$g(\gamma) = \sup\{\max(\alpha, h(\alpha)) : (\alpha, h(\alpha)) \in A_1 \wedge f(\alpha, h(\alpha)) \leq \gamma\}.$$

If $(\alpha, h(\alpha)) \in A_1$, then $g(f(\alpha, h(\alpha))) \geq \max(\alpha, h(\alpha))$. Also, note that $g(\gamma) < \omega_1$ since f is nonconstant and increasing on the first coordinate. By lemma 1.3.2 there exist $x \in \omega^\omega$ and c.u.b. $C_0 \subseteq \omega_1$ such that the section of the Kunen linear order T^K_x is wellordered, and for all $\gamma \in C_0$ we have $g(\gamma) < |T^K_x \restriction \gamma|$. For each $\gamma \in C_0$ define $B_\gamma \subseteq \gamma \times \gamma$ by

$$\begin{aligned}(\delta, \epsilon) \in B_\gamma \iff & \delta, \epsilon < \gamma \wedge \exists(\alpha, h(\alpha)) \in A_1\ \gamma \leq \alpha \wedge \\ & \alpha = |T^K_x \restriction \gamma(\delta)| \wedge h(\alpha) = |T^K_x \restriction \gamma(\epsilon)|.\end{aligned} \tag{3.2}$$

Let $C_1 \subseteq C_0$ be the c.u.b. set from lemma 3.3.4 such that the sequence $\langle B_\gamma \subseteq \gamma \times \gamma : \gamma \in C_1\rangle$ coheres. Let $B = \bigcup_{\gamma \in C_1} B_\gamma$ and by lemma 3.3.2 there exists an invariant strategy τ which codes B. By lemma 3.3.1 T^J_τ is wellfounded. Let c.u.b. $C_2 \subseteq C_1$ be closed under the function $\gamma \mapsto |T^J_\tau \upharpoonright \gamma|$. By lemma 3.1.3 there is $\sigma \in \omega^\omega$ such that T^K_σ is wellfounded and $C_\sigma \subseteq C_2$. Consider the simple set $S_{\tau,\sigma,x}$.

First, we claim that $S_{\tau,\sigma,x} \subseteq h$. Let $(\alpha,\beta) \in S_{\tau,\sigma,x}$. By definition of $S_{\tau,\sigma,x}$ there is some $\gamma \in C_\sigma$ and $\delta, \epsilon \in B$ such that $\delta, \epsilon < \gamma \leq \alpha$ and $\alpha = |T^K_x \upharpoonright \gamma(\delta)|$ and $\beta = |T^K_x \upharpoonright \gamma(\epsilon)|$. Since B is a union, then $(\delta,\epsilon) \in B_{\gamma'}$ for some $\gamma' \in C_1$. By the definition of C_1 we have $(\delta,\epsilon) \in B_\gamma$, and by equation 3.2 it follows that $(\alpha,\beta) \in A_1 \subseteq h$.

Second, we claim that $A_1 \cap f^{-1}C_\sigma \subseteq S_{\tau,\sigma,x}$. If $(\alpha, h(\alpha)) \in A_1 \cap f^{-1}C_\sigma$, then $\gamma \equiv f(\alpha, h(\alpha)) \in C_\sigma$. Note that $\gamma \leq \alpha$. Since $(\alpha, h(\alpha)) \in A_1$ and $\gamma \in C_0$, then $\max(\alpha, h(\alpha)) \leq g(\gamma) < |T^K_x \upharpoonright \gamma|$. So, for some $(\delta,\epsilon) \in B_\gamma$, $\alpha = |T^K_x \upharpoonright \gamma(\delta)|$ and $h(\alpha) = |T^K_x \upharpoonright \gamma(\epsilon)|$. Since $\gamma \in C_1$, then $(\delta,\epsilon) \in B$. Finally, since $\gamma \in C_\sigma$ and $C_\sigma \subseteq C_2$, we have satisfied the definition for $(\alpha, h(\alpha)) \in S_{\tau,\sigma,x}$.

Putting these claims together, we have shown that $\nu(S_{\tau,\sigma,x}) = 0$ and $\nu(S_{\tau,\sigma,x}) = 1$, a contradiction. □

We are now able to give our coding of functions from ω_1 to ω_1 by reals. View each $z \in \omega^\omega$ as coding a countable sequence of reals $z_0, z_1, \ldots, z_p, \ldots$, each of which codes three reals τ_p, σ_p, x_p. Define $\phi(z) = \bigcup_{p<\omega} S_{\tau_p,\sigma_p,x_p}$. By Martin's theorem 3.2.5 all that remains to prove the strong partition relation on ω_1 is to verify that ω_1 (with this ϕ and $\mathbf{\Gamma} = \mathbf{\Sigma}^1_1$) is reasonable, i.e. satisfies the conditions in definition 3.2.4. Trivially, $\phi(z) \subseteq \omega_1 \times \omega_1$, and by lemma 3.3.5 for all $h\colon \omega_1 \to \omega_1$ there exists $z \in \omega^\omega$ with $\phi(z) = h$.

Lemma 3.3.6. *For all* $\alpha, \beta < \omega_1$

$$R_{\alpha,\beta} \equiv \{z \in \omega^\omega : \phi(z)(\alpha,\beta) \wedge \forall \beta' < \omega_1 \ \phi(z)(\alpha,\beta') \implies \beta = \beta'\} \in \mathbf{\Sigma}^1_1.$$

Proof. Fix $\alpha, \beta < \omega_1$ and consider the set $R_{\alpha,\beta}$. From the definition of simple set, we have

$$
\begin{aligned}
&x \in R_{\alpha,\beta} \iff \exists p < \omega \exists \gamma \leq \alpha \\
&\quad \forall \delta < \gamma \, T^K_{\sigma_p} \restriction \delta \text{ is wellfounded} \wedge |T^K_{\sigma_p} \restriction \delta| < \gamma \wedge \\
&\quad \forall \delta < \gamma \, T^J_{\tau_p} \restriction \delta \text{ is wellfounded} \wedge |T^J_{\tau_p} \restriction \delta| < \gamma \wedge \\
&\quad \exists \delta, \epsilon < \gamma \; \delta, \epsilon \in \mathrm{wf}(T^K_{x_p} \restriction \gamma) \wedge |T^K_{x_p} \restriction \gamma(\delta)| = \alpha \wedge |T^K_{x_p} \restriction \gamma(\epsilon)| = \beta \wedge \\
&\quad\quad \exists y \in \mathrm{WO}_\gamma \exists m, n < \omega \; |m|_{\tau_p(y)_0} = \delta \wedge |n|_{\tau_p(y)_0} = \epsilon \wedge \tau_p(y)_1(m,n) = 1 \wedge \\
&\quad \forall p' < \omega \forall \gamma' \leq \alpha \forall \delta', \epsilon' < \gamma' \\
&\quad \big[\forall \zeta' < \gamma' \, T^K_{\sigma_{p'}} \restriction \zeta' \text{ is wellfounded} \wedge |T^K_{\sigma_{p'}} \restriction \zeta'| < \gamma' \wedge \\
&\quad \forall \zeta' < \gamma' \, T^J_{\tau_{p'}} \restriction \zeta' \text{ is wellfounded} \wedge |T^J_{\tau_{p'}} \restriction \zeta'| < \gamma' \wedge \\
&\quad \delta' \in \mathrm{wf}(T^K_{x_{p'}} \restriction \gamma') \wedge |T^K_{x_{p'}} \restriction \gamma'(\delta')| = \alpha \wedge \\
&\quad \forall y' \in \mathrm{WO}_{\gamma'} \forall m', n' < \omega (|m'|_{\tau_{p'}(y')_0} = \delta' \wedge |n'|_{\tau_{p'}(y')_0} = \epsilon') \implies \tau_{p'}(y')_1(m', n') = 1\big] \implies \\
&\quad\quad\quad \epsilon' \in \mathrm{wf}(T^K_{x_{p'}} \restriction \gamma') \wedge |T^K_{x_{p'}} \restriction \gamma'(\epsilon')| = \beta.
\end{aligned}
$$

Lines 1-5 above show that $\phi(x)(\alpha, \beta)$ while lines 6-11 show if $\phi(x)(\alpha, \beta')$, then $\beta = \beta'$. We have written $R_{\alpha,\beta}$ as countable unions and intersections of Borel relations, most of which are of the form: a section of some tree is wellfounded of rank less than some countable ordinal, which by the standard tree computation is Borel. The quantifiers applied to WO_γ and $\mathrm{WO}_{\gamma'}$ (in lines 5 and 10) make this relation $\mathbf{\Sigma}^1_1$. Hence, $R_{\alpha,\beta}$ is $\mathbf{\Sigma}^1_1$. We parenthetically note that with a little more work we could show that $R_{\alpha,\beta}$ is in fact $\mathbf{\Delta}^1_1$, but this is not necessary. □

Lemma 3.3.7. *If $\alpha < \omega_1$, $A \subseteq R_\alpha \equiv \{x \in \omega^\omega : \exists \beta < \omega_1 \; x \in R_{\alpha,\beta}\}$, and $A \in \mathbf{\Sigma}^1_1$, then there exists $\beta' < \omega_1$ such that $\forall x \in A \exists \beta < \beta' \; x \in R_{\alpha,\beta}$.*

Proof. Let $\alpha < \omega_1$. We will define a prewellorder $x \prec x'$ which is equivalent to $x, x' \in A \wedge$

$\phi(z)(\alpha) < \phi(z')(\alpha)$. It is sufficient to show this prewellorder is $\mathbf{\Sigma}^1_1$. We define

$$\begin{aligned}
z \prec z' \iff\ & z, z' \in \boldsymbol{A} \wedge \exists p < \omega \exists \gamma < \alpha \\
& \forall \delta < \gamma\ T^K_{\sigma_p} \restriction \delta \text{ is wellfounded} \wedge |T^K_{\sigma_p} \restriction \delta| < \gamma \wedge \\
& \forall \delta < \gamma\ T^J_{\tau_p} \restriction \delta \text{ is wellfounded} \wedge |T^J_{\tau_p} \restriction \delta| < \gamma \wedge \\
& \exists \delta, \epsilon < \gamma\ \delta \in \mathrm{wf}(T^K_{x_p} \restriction \gamma) \wedge |T^K_{x_p} \restriction \gamma(\delta)| = \alpha \wedge \\
& \quad \exists y \in \mathrm{WO}_\gamma \exists m, n < \omega\ |m|_{\tau_p(y)_0} = \delta \wedge |n|_{\tau_p(y)_0} = \epsilon \wedge \tau_p(m, n) = 1 \wedge \\
& \exists p' < \omega \exists \gamma' < \alpha \\
& \forall \delta' < \gamma'\ T^K_{\sigma_{p'}} \restriction \delta' \text{ is wellfounded} \wedge |T^K_{\sigma_{p'}} \restriction \delta'| < \gamma' \wedge \\
& \forall \delta' < \gamma'\ T^J_{\tau_{p'}} \restriction \delta' \text{ is wellfounded} \wedge |T^J_{\tau_{p'}} \restriction \delta'| < \gamma' \wedge \\
& \exists \delta', \epsilon' < \gamma'\ \delta' \in \mathrm{wf}(T^K_{x_{p'}} \restriction \gamma') \wedge |T^K_{x_{p'}} \restriction \gamma'(\delta')| = \alpha \wedge \\
& \quad \exists y' \in \mathrm{WO}_{\gamma'} \exists m', n' < \omega |m'|_{\tau_{p'}(y')_0} = \delta' \wedge |n'|_{\tau_{p'}(y')_0} = \epsilon' \wedge \tau_{p'}(m', n') = 1 \wedge \\
& \quad\quad \text{“}|T^K_{x_p} \restriction \gamma(\epsilon)| < |T^K_{x_{p'}} \restriction \gamma'(\epsilon')|\text{”}.
\end{aligned}$$

First of all, note that lines 1-5 say that $\phi(z)(\alpha) = |T^K_{x_p} \restriction \gamma(\epsilon)|$. Likewise, lines 6-10 say $\phi(z')(\alpha) = |T^K_{x_{p'}} \restriction \gamma'(\epsilon')|$. Since $z, z' \in R_\alpha$, it follows from lines 1-5 and 6-10 that $\epsilon \in \mathrm{wf}(T^K_{x_p} \restriction \gamma)$ and $\epsilon' \in \mathrm{wf}(T^K_{x_{p'}} \restriction \gamma')$, so these predicates (which are $\mathbf{\Pi}^1_1$) are not included. The last line in quotation marks represents a $\mathbf{\Sigma}^1_1$ predicate which checks $|T^K_{x_p} \restriction \gamma(\epsilon)| < |T^K_{x_{p'}} \restriction \gamma'(\epsilon')|$, given that $\epsilon \in \mathrm{wf}(T^K_{x_p} \restriction \gamma)$ and $\epsilon' \in \mathrm{wf}(T^K_{x_{p'}} \restriction \gamma')$. Thus, this prewellorder is $\mathbf{\Sigma}^1_1$, and hence has some countable length β'. Clearly, if $z \in \boldsymbol{A}$, then $\phi(z)(\alpha) < \beta'$. Thus, if $z \in R_{\alpha,\beta}$, then $\beta < \beta'$. □

All the conditions of Martin's theorem 3.2.5 have been satisfied, so we may conclude that

Theorem 3.3.8. $\omega_1 \to \omega_1^{\omega_1}$.

CHAPTER 4

Collapsing Result

4.1 Partition Relations and Ultraproducts

In this final chapter we present some variants of known results, quote a theorem of Woodin, and prove our collapsing result.

Recall from chapter 1 that if k is an increasing, discontinuous sequence of cardinals of length μ, then $k \to k^\omega$ means that for all partitions $p\colon \prod_{\alpha<\mu}[k(\alpha)]^\omega \to 2$ there exists $b \in \prod_{\alpha<\mu}[k(\alpha)]^{k(\alpha)}$ such that $p \restriction \prod_{\alpha<\mu}[b(\alpha)]^\omega$ is constant. There is an equivalent c.u.b. version of the polarized partition relation. Let $\lambda = \sup_{\alpha<\mu} k(\alpha)$, and for each $\alpha < \mu$, let $b_\alpha = [\sup_{\alpha'<\alpha} k(\alpha'), k(\alpha))$ denote the αth block of λ. A function $f\colon \omega\mu \to \lambda$ is an *ω block function* if $\forall\alpha < \mu \forall n < \omega \;\; f(\omega\alpha + n) \in b_\alpha$. The *c.u.b. version of the polarized partition relation*, $k \xrightarrow{\text{c.u.b.}} k^\omega$, is the following: for all partitions $p\colon \{f\colon \omega\mu \to \lambda : f \text{ is an } \omega \text{ block function}\} \to 2$ there exists a block c.u.b. $C \subseteq \lambda$ such that $p \restriction \{f\colon \omega\mu \to C : f \text{ is an } \omega \text{ block function of the correct type}\}$ is constant. Exactly as in chapter 1, $k \to k^\omega$ is equivalent to $k \xrightarrow{\text{c.u.b.}} k^\omega$, and we will use these two notions interchangeably.

We will require a version of theorem 3.2.5 specifically for ω block functions. Towards this end we give conditions on a sequence k which are sufficient to prove $k \to k^\omega$, as in definition 5.2 of [1].

Definition 4.1.1. Let k be an increasing, discontinuous sequence of regular cardinals of length μ. We say the sequence k is *reasonable* if there is a sequence of nonselfdual pointclasses $\langle \mathbf{\Gamma}_\alpha : \alpha < \mu \rangle$ each closed under $\exists^{\omega^\omega}$, and a function ϕ with domain ω^ω satisfying:

1. $\forall x \in \omega^\omega \; \phi(x) \subseteq \omega\mu \times \lambda$

2. $\forall\, \omega$ block functions $f\colon \omega\mu \to \lambda \,\exists x \in \omega^\omega \; \phi(x) = f$

3. $\forall\alpha < \mu \; R_\alpha \in \mathbf{\Delta}_{\alpha+1}$, where $x \in R_\alpha \iff \forall\alpha' \le \alpha \forall n < \omega \exists\gamma \in b_{\alpha'} \; [\phi(x)(\omega\alpha' + n, \gamma) \wedge$

4. $\forall\alpha < \mu \forall n < \omega \forall\gamma \in b_\alpha \; R_{\alpha,n,\gamma} \in \mathbf{\Delta}_\alpha$, where $x \in R_{\alpha,n,\gamma} \iff [\phi(x)(\omega\alpha + n, \gamma) \wedge \forall\gamma' \in b_\alpha \, \phi(x)(\omega\alpha + n, \gamma')] \implies \gamma = \gamma'$.

5. If $\alpha < \mu$, $n < \omega$, $A \in \exists^{\omega^\omega} \mathbf{\Delta}_\alpha$, and $A \subseteq R_{\alpha,n} \equiv \bigcup_{\gamma \in b_\alpha} R_{\alpha,n,\gamma}$, then $\sup\{\phi(x)(\omega\alpha + n) : x \in A\} < k(\alpha)$.

Theorem 4.1.2. *If k is reasonable, then $k \to k^\omega$.*

The proof of 4.1.2 follows the proof of 3.2.5, mutatis mutandis.

Lemma 4.1.3. *Let k be a ω_1 length sequence of successors of limit Suslin cardinals of cofinality ω. Then $k \to k^\omega$.*

Proof. Let $\alpha < \omega_1$. Say $k(\alpha) = \lambda_\alpha^+$ where λ_α is a Suslin cardinal of cofinality ω. We will associate to each $k(\alpha)$ a pointclass $\mathbf{\Pi}_1^\alpha$, as defined below. Let $\mathbf{\Delta}$ be the pointclass of $< \lambda_\alpha$ Suslin sets. Then $\mathbf{\Delta}$ is closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$, $\neg$, finite union, and intersection. Let $\mathbf{\Sigma}_0$ be the pointclass of countable unions of $\mathbf{\Delta}$ sets, and $\mathbf{\Pi}_0$ be the dual of $\mathbf{\Sigma}_0$. Then every set in $\mathbf{\Sigma}_0$ is λ_α Suslin, and $\mathbf{\Sigma}_0$ is closed under countable union and $\exists^{\omega^\omega}$. Likewise, every set in $\mathbf{\Pi}_0$ is λ_α Suslin, and $\mathbf{\Pi}_0$ is closed under countable intersection and $\forall^{\omega^\omega}$. Furthermore, $\mathbf{\Sigma}_0$ has the scale property. To see this, let $A = \bigcup_{n<\omega} A_n \in \mathbf{\Sigma}_0$, where each $A_n \in \mathbf{\Delta}$, and let $\langle \phi_k^n : k < \omega \rangle$ be a $\mathbf{\Delta}$ scale on A_n into λ_α. For $x \in A$ define $\psi_0(x) =$ least n such that $x \in A_n$, and for each $k \geq 0$ define $\psi_{k+1}(x) = \langle \psi_0(x), \phi_k^{\psi_0(x)}(x) \rangle_{\text{lex}}$. To show $\langle \psi_k : k < \omega \rangle$ is a scale on A, suppose $\{x_m : m < \omega\} \subseteq A$, $\lim_{m \to \infty} x_m = x$, and for each $k < \omega$, $\psi_k(x_m)$ is eventually equal to an ordinal γ_k. In particular, $\psi_0(x_m)$ is eventually equal to γ_0, so eventually all $x_m \in A_{\gamma_0}$. Since, $\psi_{k+1}(x_m)$ is eventually constant, so is $\phi_k^{\gamma_0}(x_m)$, and since $\langle \phi_k^{\gamma_0} : k < \omega \rangle$ is a scale on A_{γ_0}, then $x \in A_{\gamma_0} \subseteq A$. So, $\psi_0(x) \leq \gamma_0$. If $\psi_0(x) < \gamma_0$, then $\psi_k(x) < \gamma_k$ since ψ_k was defined using the lexicographic ordering. Otherwise, $\phi_k^{\gamma_0}(x) \leq \lim_{m \to \infty} \phi_k^{\gamma_0}(x_m)$, so $\psi_k(x) \leq \gamma_k$ in this case, too. Thus, $\langle \psi_k : k < \omega \rangle$ is a scale on A. To show $\langle \psi_k : k < \omega \rangle$ is a $\mathbf{\Sigma}_0$ scale, note the starred relation $x \leq_k^* y \iff x \in A \wedge (y \notin A \vee \psi_k(x) \leq \psi_k(y))$ is equivalent to

$$\exists n (x \in A_n \wedge y \notin A_n \wedge \forall m < n \; x \notin A_m \wedge y \notin A_m) \vee$$
$$\exists n (x \in A_n \wedge y \in A_n \wedge \forall m < n [x \notin A_m \wedge y \notin A_m] \wedge \phi_k^n(x) \leq \phi_k^n(y)),$$

and likewise for $<_k^*$. Since all of the above conjuncts are $\mathbf{\Delta}$, both $\leq_k^*$ and $<_k^*$ are $\mathbf{\Sigma}_0$. Now define $\mathbf{\Sigma}_1^\alpha = \exists^{\omega^\omega} \mathbf{\Pi}_0$, $\mathbf{\Pi}_1^\alpha = \forall^{\omega^\omega} \mathbf{\Sigma}_0$, and $\mathbf{\Delta}_1^\alpha = \mathbf{\Sigma}_1^\alpha \cap \mathbf{\Pi}_1^\alpha$. $\mathbf{\Pi}_1^\alpha$ is closed under $\forall^{\omega^\omega}$, countable union and intersection, and by Moschovakis' second periodicity theorem has the scale property. Every set in $\mathbf{\Sigma}_1^\alpha$ is still λ_α Suslin. In fact, $\mathbf{\Sigma}_1^\alpha$ is the pointclass of λ_α is a $\mathbf{\Delta}_1^\alpha$ prewellorder of $\omega^\omega\}$, and any $\mathbf{\Pi}_1^\alpha$ prewellorder on a $\mathbf{\Pi}_1^\alpha$ complete set has length $\lambda_\alpha^+ = k(\alpha)$. Furthermore, the length of any $\mathbf{\Sigma}_1^\alpha$ prewellorder is $< k(\alpha)$. In this sense, $(\mathbf{\Pi}_1^\alpha, k(\alpha))$ is analogous to $(\mathbf{\Pi}_1^1, \omega_1)$.

For each $\alpha < \omega_1$ let P_α be a universal $\mathbf{\Pi}_1^\alpha$ and let $\langle \phi_0^\alpha : n < \omega \rangle$ be a $\mathbf{\Pi}_1^\alpha$ scale on P_α with $\phi_0^\alpha \colon P_\alpha \xrightarrow{\text{onto}} k(\alpha)$. It turns out that $\mathbf{\Pi}_1^\alpha$ and $\langle \phi_n^\alpha : n < \omega \rangle$ can be chosen uniformly in

α (see [10]). (Note: Steel's proof of this uniformity result assumes $V = L[\mathbb{R}]$. However, our result is true even without this assumption.) Let $U' \subseteq (\omega^\omega)^3$ be a universal $\mathbf{\Sigma}^1_2$ set, and let $U \subseteq (\omega^\omega)^2 \times \omega^\omega$ be a $\mathbf{\Sigma}^1_2$ uniformization of U'. To prove this lemma we will give a coding of block functions which satisfies the conditions of Martin's theorem 4.1.2. Borrowing an idea from Kechris' theory of generic codes of countable ordinals, for each $\alpha < \omega_1$, $\beta \in b_\alpha$, and $x \in \omega^\omega$ define

$$\phi(x)(\alpha, \beta) \iff \forall^* y \in \mathrm{WO}'_\alpha \exists z \in P_\alpha\, U(x, y, z) \wedge \phi^\alpha_0(z) = \beta,$$

where $\forall^* y \in \mathrm{WO}'_\alpha$ means for comeager many $y \in \mathrm{WO}'_\alpha$ in the sense of generic codes (see [7]).

First, we must show every block function is coded. Suppose $f : \omega_1 \to \lambda$ be an ω block function, and define $F : \omega_1 \to \wp(\omega^\omega)$ by $F(\alpha) = \{z \in P_\alpha : \phi^\alpha_0(z) = f(\alpha)\}$. By the Coding lemma relative to $\mathbf{\Sigma}^1_2$ (see lemma 7D.5 of [8]), there is a $\mathbf{\Sigma}^1_2$ choice set $C \subseteq \omega^\omega \times \omega^\omega$ such that if $(y, z) \in C$, then $y \in \mathrm{WO}$ and $z \in P_{|y|} \wedge \phi^{|y|}_0(z) = f(|y|)$, and for all $\alpha < \omega_1$ there exists $y \in \mathrm{WO}'_\alpha$ and $z \in \omega^\omega$ such that $(y, z) \in C$. Since U is $\mathbf{\Sigma}^1_2$ universal, there exists $x \in \omega^\omega$ such that $C = U_x$ and hence $f = \phi(x)$.

To show $R_{\alpha,\beta} \in \mathbf{\Delta}^\alpha_1$ note that

$$x \in R_{\alpha,\beta} \iff \forall^* y \in \mathrm{WO}'_\alpha[\exists z \in P_\alpha\, U(x, y, z) \wedge \phi^\alpha_0(z) = \beta].$$

Since $\langle \phi^\alpha_n : n < \omega \rangle$ is a $\mathbf{\Pi}^\alpha_1$ scale, $\mathbf{\Sigma}^1_2 \subseteq \mathbf{\Delta}^\alpha_1$, and the above z is unique, the predicate in the brackets is $\mathbf{\Delta}^\alpha_1$. $\mathbf{\Delta}^\alpha_1$ is closed under the category quantifier (see [8], page 262), so $R_{\alpha,\beta} \in \mathbf{\Delta}^\alpha_1$.

Lastly, we must show the boundedness property. So fix $\alpha < \lambda$ and suppose $S \in \exists^{\omega^\omega} \mathbf{\Delta}^\alpha_1$ and $S \subseteq R_\alpha$. We must find β in the αth block of k such that $S \subseteq R_{\alpha,\beta}$. To this end, define a prewellorder $\prec$ on S by $x_1 \prec x_2$ iff $x_1, x_2 \in S \wedge \phi(x_1)(\alpha) < \phi(x_2)(\alpha)$. Thus, by intersecting two comeager sets we get

$$x_1 \prec x_2 \iff \forall^* y \in \mathrm{WO}'_\alpha \exists z_1, z_2 \in P_\alpha\, U(x_1, y, z_1) \wedge U(x_2, y, z_2) \wedge \text{“}\phi^\alpha_0(z_1) < \phi^\alpha_0(z_2)\text{”},$$

where for the predicate in quotes we use the Σ^α_1 version of the inequality from a Π^α_1 scale. Note again that the z_1 and z_2 are unique. As before $\mathbf{\Sigma}^\alpha_1$ is closed under the category quantifier, which implies $\prec \in \mathbf{\Sigma}^\alpha_1$. So, the length of $\prec$, and hence the range of $\phi(x)(\alpha)$ for $x \in S$, is $< k(\alpha)$. □

Lemma 4.1.4. *Let k be a ω_1 length sequence of regular cardinals and μ a nonatomic measure* *on ω_1. If $k \to k^2$, then $\prod k/\mu$ is a regular cardinal.*

Proof. Let λ be the order type of $\prod k/\mu$. If λ is not regular, then there is $\rho < \lambda$ and an increasing, cofinal $\pi\colon \rho \to \lambda$. Partition elements of $[k]^2$ as follows: view each element of $[k]^2$ as two functions $f_0, f_1 \in \prod k/\mu$, and let α_i be the rank of f_i in $\prod k/\mu$. Define $P(f_0, f_1) = 0$ iff there exists some element in the range of π between α_0 and α_1. By assumption, $k \to k^2$, so there exists a block c.u.b. C which is homogeneous for block functions of correct type. C must be homogeneous for $P = 0$ since for any f_0 into C of correct type there exists $\rho' < \rho$ with $\alpha_0 < \pi(\rho')$ and $f_1\colon \lambda \to C$ of correct type with $\pi(\rho') < \alpha_1$. Let $\langle f_\beta : \beta < \rho^+\rangle$ be a strictly increasing sequence in $\prod k/\mu$. For each $\alpha < \omega_1$ and $\beta < \rho^+$ define $f'_\beta(\alpha) = \omega(f_\beta(\alpha) + 1)$th element of $(C \cap k(\alpha))$. So, $f'_\beta\colon \omega_1 \to C$ is a block function of correct type. In addition, for any distinct $\beta', \beta < \rho^+$ there is an element in the range of π between $[f'_{\beta'}]$ and $[f'_\beta]$, contradicting the fact that the range of π has order type ρ. Thus, λ is regular. ☐

Lemma 4.1.5. *Suppose λ is a singular cardinal of cofinality κ and $g\colon \kappa \to \lambda$ is cofinal and normal. Then there exists a c.u.b. $c \subseteq \kappa$ such that* $\max\text{pcf}\{(g(\delta))^+ : \delta \in c\} = \lambda^+$.

Proof. For any $d \subseteq \kappa$ define $\hat{d} = \{(g(\delta))^+ : \delta \in d\}$. Let $J_{<\lambda} = J_{<\lambda}(\hat{\kappa})$. Since λ is singular, $J_{<\lambda} = J_{\leq\lambda}$. By theorem 2.2.1, there exists $b \subseteq \kappa$ such that $J_{\leq\lambda^+} = J_{\leq\lambda}[\hat{b}]$. By lemma 1.2.5, $\max\text{pcf}(\hat{b}) = \lambda^+$.

We claim that there exists a c.u.b. $c \subseteq b$. If so, then $\max\text{pcf}(\hat{c}) = \lambda^+$. The proof of this claim is by contradiction. So, assume $\kappa - b = \text{s}$ is stationary. Since $\hat{b}$ is a generator, any λ^+ sequence in $\prod \hat{\text{s}}$ is bounded mod $J_{<\lambda}$. Note that if $\hat{d} \in J_{<\lambda}$, then d is bounded below κ. So, the nonstationary ideal on κ, $\mathcal{I}_{\text{NS}}$, contains $J_{<\lambda}$, and it immediately follows that any λ^+ sequence in $\prod \hat{\text{s}}$ is bounded mod $\mathcal{I}_{\text{NS}}$. Claim 4.1.6 finishes the proof of this claim. ☐

Claim 4.1.6. *There does not exists stationary $\text{s} \subseteq \kappa$ such that any λ^+ sequence in $\prod \hat{\text{s}}$ is bounded mod $\mathcal{I}_{\text{NS}}$.*

Proof. Suppose not, and fix stationary $\text{s} \subseteq \kappa$ such that any λ^+ sequence in $\prod \hat{\text{s}}$ is bounded mod $\mathcal{I}_{\text{NS}}$.

be a c.u.b. subset of β of cardinality $\text{cof}(\beta) < \lambda$. Define $\mathcal{S}_\alpha = \{d_\beta \cap \alpha : \beta < \lambda^+,\ \beta \text{ limit}\}$. Clearly, for each $\alpha < \lambda^+$, $|\mathcal{S}_\alpha| \leq \lambda^+$ and $\mathcal{S}_\alpha \subseteq \wp_{<\lambda}(\alpha)$. Further, $\langle \mathcal{S}_\alpha : \alpha < \lambda^+\rangle$ is increasing in the sense that for all $\alpha' < \alpha < \lambda^+$, $\mathcal{S}_{\alpha'} = \{d \cap \alpha' : d \in \mathcal{S}_\alpha\}$.

Now we inductively define $\langle f_\alpha : \alpha < \lambda^+\rangle$ in $\prod \hat{\text{s}}$ such that for all $\alpha < \lambda^+$, $f_\alpha <_{\text{ev}} f_{\alpha+1}$ and for all $d \in \mathcal{S}_\alpha$, $\text{ptsup}_{\alpha' \in d} f_{\alpha'} \leq_{\mathcal{I}_{\text{NS}}} f_\alpha$. Given $\langle f_{\alpha'} : \alpha' < \alpha\rangle$ satisfying the inductive hypothesis up to α, for each $d \in \mathcal{S}_\alpha$, $\sup_{\alpha' \in d} f_{\alpha'}(\delta) < \delta$ for each $\delta \in \hat{s}$ with $\delta >$ some δ_d. So $\mathrm{ptsup}_{\alpha' \in d} f_{\alpha'} \in \prod \hat{s}/\mathcal{I}_{\mathrm{NS}}$. Since we are assuming λ^+ sequences are bounded mod $\mathcal{I}_{\mathrm{NS}}$, let f_α be a bound of $\{\mathrm{ptsup}_{\alpha' \in d} f_{\alpha'} : d \in \mathcal{S}_\alpha\}$ mod $\mathcal{I}_{\mathrm{NS}}$. The condition $f_\alpha <_{\mathrm{ev}} f_{\alpha+1}$ can be satisfied trivially. This completes the inductive definition. As in lemma 2.1.3, $\langle f_\alpha : \alpha < \lambda^+ \rangle$ has a least upper bound, say h.

For any $\delta \in \hat{s}$, $\mathrm{cof}(h(\delta)) < \delta$, so by Fodor's theorem there exist a stationary $s' \subseteq \hat{s}$ and a regular cardinal $\rho < \lambda$ such that for all $\delta \in s'$, $\mathrm{cof}(h(\delta)) = \rho$. For each $\delta \in s'$ let c_δ be a c.u.b. subset of $h(\delta)$ of order type ρ, and for each $\alpha < \lambda^+$ define $f'_\alpha(\delta) = (f_\alpha(\delta))^{+c_\delta}$. Clearly, for all $\alpha' < \alpha < \lambda^+$ we have $f_{\alpha'} <_{\mathrm{ev}} f'_{\alpha'} \leq_{\mathcal{I}_{\mathrm{NS}}} f'_\alpha$ and for all $d \in \mathcal{S}_\alpha$, $\mathrm{ptsup}_{\alpha \in d} f'_{\alpha'} \leq_{\mathcal{I}_{\mathrm{NS}}} f'_\alpha$. Since h is a least upper bound of $\langle f_\alpha : \alpha < \lambda^+ \rangle$, then $\langle f'_\alpha : \alpha < \lambda^+ \rangle$ cannot stabilize (otherwise, the stable value would be a smaller than h on s'). Thus $\forall \alpha < \lambda^+ \exists \beta < \lambda^+ \, f'_{\alpha'} \lneqq_{\mathcal{I}_{\mathrm{NS}}} f'_\beta$. Let c.u.b. $c_0 \subseteq \lambda^+$ be closed under $\alpha \mapsto \beta$. Define c_1 to be the first ρ^+ elements of c_0, and let $\alpha_0 = \sup c_1$. So, $\mathrm{cof}(\alpha_0) = \rho^+$. Let $c_2 = c_1 \cap d_{\alpha_0}$ which is c.u.b. in α_0. For all $\alpha \in c_2$, $\mathrm{ptsup}_{\alpha' \in c_2 \cap \alpha} f'_{\alpha'} \leq_{\mathcal{I}_{\mathrm{NS}}} f'_\alpha$. Also, for all $\alpha', \alpha \in c_0$ if $\alpha' < \alpha$, then $f'_{\alpha'} \lneqq_{\mathcal{I}_{\mathrm{NS}}} f'_\alpha$. Hence for each $\alpha \in c_2$ there exists $\delta \in s'$ such that

$$\sup_{\alpha' \in c_2 \cap \alpha} f'_{\alpha'}(\delta) \leq f'_\alpha(\delta) < f'_{\alpha^{+c_2}}(\delta). \tag{4.1}$$

By the regularity of ρ^+ there exist $c_3 \subseteq c_2$ with $|c_3| = \rho^+$ and $\delta^* \in s'$ such that for all $\alpha \in c_3$ equation 4.1 holds at $\delta = \delta^*$. Lastly, choose $c_4 \subseteq c_3$ such that $|c_4| = \rho^+$ and between any two elements of c_4 there is an element of c_2. Then $\{f'_\alpha(\delta^*) : \alpha \in c_4\}$ is a subset of d_{δ^*} of size ρ^+, a contradiction. □

4.2 Main Theorem

At this point we quote a theorem of Woodin (theorem 1.3 of [11]).

Theorem 4.2.1 (Woodin). *Assume the nonstationary ideal on ω_1 is ω_2 saturated, there exist ω many Woodin cardinals with a measurable cardinal above them all, and X is a bounded subset of $\Theta^{L[\mathbb{R}]}$ of cardinality ω_1. Then there exists a set $Y \in L[\mathbb{R}]$ of cardinality ω_1 in $L[\mathbb{R}]$*

We are now ready to prove the main result of this thesis.

Theorem 4.2.2. *If the nonstationary ideal on ω_1 is ω_2 saturated and there exist ω many Woodin cardinals with a measurable cardinal above them all, then some regular cardinal $< \aleph_{\omega_2}$ in $L[\mathbb{R}]$ collapses in V.*

Proof. Let $C_0 \subseteq \aleph_{\aleph_1}$ be a c.u.b. set of limit Suslin cardinals of size ω_1. Let $\boldsymbol{A}_0 = \{\kappa^+ : k \in C_0\}$. By theorem 4.1.3 $\prod \boldsymbol{A}_0$ has the countable and hence finite polarized partition relation. By theorem 4.1.5 there is a c.u.b. $C_1 \subseteq C_0$ such that $\max \mathrm{pcf}\{\kappa^+ : \kappa \in C_1\} = \aleph_{\aleph_1+1}$. Let $\boldsymbol{A}_1 = \{\kappa^+ : \kappa \in C_1\}$ and $\kappa_1 = \aleph_{\aleph_1+1}$.

In $L[\mathbb{R}]$ there are $\aleph_2$ many functions $f : \omega_1 \to \boldsymbol{A}_1$. For each such function f by theorem 4.1.4 $\prod \mathrm{range}(f)/W_1^1$ is a regular cardinal. So there are at least $\aleph_2$ regular cardinals in $L[\mathbb{R}]$ of the form $\prod \mathrm{range}(f)/W_1^1$. Among these fix some regular $\kappa_2 > \kappa_1$, where $\kappa_2 = \prod \mathrm{range}(f)/W_1^1$ for some $f : \omega_1 \to \boldsymbol{A}_1$. Let $B = \mathrm{range}\, f$.

Now in $\mathbf{V}$, ${W_1^1}^{L[\mathbb{R}]}$ is not a measure, but at least a filter base for the subsets of ω_1. By Zorn's lemma this filter base can be extended to an ultrafilter U. Since $\max \mathrm{pcf}(\boldsymbol{A}_1) = \kappa_1$, then $\mathrm{cof}(\prod B/U) \leq \kappa_1$. (In fact, equality holds here.) Towards a contradiction suppose κ_2 is regular in $\mathbf{V}$. Let $\langle f_\alpha : \alpha < \kappa_1 \rangle$ be some cofinal sequence in $\prod B/U$.

For a contradiction we will define a cofinal map $\pi : \kappa_1 \to \kappa_2$. For each $\alpha < \kappa_1$ by theorem 4.2.1 there is some $F \in L[\mathbb{R}]$ such that $f_\alpha \leq_{\mathrm{ev}} F$. For $\alpha < \kappa_1$ define $\pi(\alpha) = \min\{[F]_{W_1^1} : F \in L[\mathbb{R}] \wedge f_\alpha \leq_{\mathrm{ev}} F\}$. Note that if $F, F' \in L[\mathbb{R}]$ and $F =_{W_1^1} F'$, then $[F]_{W_1^1} = [F']_{W_1^1}$. If $\beta < \kappa_2$, then $\beta = [G]_{W_1^1}$ for some $G \in L[\mathbb{R}]$. Of course, $G \in \mathbf{V}$, and hence there exists $\alpha < \kappa_1$ such that $G <_U f_\alpha \leq_U F$. But both G and F are in $L[\mathbb{R}]$, so it must be the case that $G <_{W_1^1} F$. Thus, $\beta = [G]_{W_1^1} < [F]_{W_1^1} = \pi(\alpha)$, showing π is cofinal. This contradicts the assumption that κ_2 was regular in $\mathbf{V}$. $\square$

BIBLIOGRAPHY

[1] A. Apter, J. Henle, S. Jackson, "The calculus of partition sequences, changing cofinalities, and a question of Woodin," *Trans. Amer. Math. Soc.* 352 (1999) 969-1003.

[2] M. Burke, M. Magidor, "Shelah's pcf theory and its applications," *Ann. Pure Appl. Logic* 50 (1990) 207-254.

[3] S. Jackson, "A new proof of the strong partition relation on ω_1," *Trans. Amer. Math. Soc.* 320 (1990) 737-745.

[4] —, "Structural consequences of AD," to appear in the *Handbook of Set Theory*, M. Foreman, A. Kanamore, M. Magidor eds.

[5] T. Jech, "Singular cardinal problem: Shelah's theorem on $2^{\aleph_\omega}$," *Bull. London Math. Soc.* 24 (1992) 127-139.

[6] A. Kechris, "AD and infinite exponent partition relations," unpublished manuscript, Dec. 1977.

[7] A. Kechris and H. Woodin, "Generic codes of uncountable ordinals, partition properties, and elementary embeddings," unpublished manuscript.

[8] D. A. Martin, Unpublished notes on the strong partition relation on ω_1.

[9] Y. N. Moschovakis, *Descriptive Set Theory*, North-Holland, Amsterdam, 1980.

[10] J. Steel, "Scales in $L[\mathbb{R}]$," Cabal Seminar 79–81, *Lecture Notes in Mathematics*, 1019, 107–156, Springer-Verlag, 1983.

[11] W. H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, de Gruyter Series in Logic and Its Applications, 1999.