GENERALIZED FUNCTION SOLUTIONS TO NONLINEAR WAVE EQUATIONS WITH DISTRIBUTION INITIAL DATA

DISSERTATION

Presented to the Graduate Council of the

University of North Texas in Partial

Fullfillment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

By

Jongchul Kim, A.S., M.S.

Denton, Texas

August, 1996

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In this study, we consider the generalized function solutions to nonlinear wave equation with distribution initial data. J. F. Colombeau shows that the initial value problem

$$u_{tt} - \Delta u = F(u)$$

$$u(x,0) = u_0$$

$$u_t(x,0) = u_1$$

where the initial data u_0 and u_1 are generalized functions, has a unique generalized function solution u. Here we take a specific F and specific distributions u_0, u_1 then inspect the generalized function representatives for the initial value problem solution to see if the generalized function solution is a distribution or is more singular. Using the numerical technics, we show for specfic F and specific distribution initial data u_0, u_1, there is no distribution solution.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION AND TERMINOLOGY

A theory of generalized functions more general than distributions has been developed by J. F. Colombeau ([1, 2, 5, 7, 8, 9, 20, 21]). Although considerably more general, these generalized functions have many of the properties of usual C^∞-functions. Let F be a slowly growing C^∞-function from $\mathbf{R}$ into $\mathbf{R}$. Now consider the following nonlinear wave initial value problem:

$$u_{tt} - \Delta u = F(u)$$

$$u(x,0) = u_0$$

$$u_t(x,0) = u_1.$$

J. F. Colombeau proves, for F obeying a growth restriction, that for generalized functions u_0, u_1, there is a unique generalized solution u. This raises the following question. Given distributions u_0 and u_1, could the solution to the initial value problem also be a distribution, or is it always a more singular generalized function? In order to explore this question, we take a specific F and specific distributions u_0 and u_1, then inspect the generalized function representative for the initial value problem solution to see if the generalized function solution is a distribution or is more singular. Here, we choose $F(u) = -|u|^{p-1}u$, $u_0 = 0$ and $u_1 = \delta(x)$, that is

$$u_{tt} - \Delta u = -|u|^{p-1}u$$

$$u(x,0) = 0$$

$$u_t(x,0) = \delta(x)$$

and we also examine the problem with $u_0 = \delta(x)$ and $u_1 = 0$.

Before we present Colombeau's method, let us define the generalized functions on $\mathbf{R}^n$.

Notation 1.1: Let $\mathcal{D}(\mathbf{R}^n)$ be the space of all C^∞-functions $\phi: \mathbf{R}^n \longrightarrow \mathbf{C}$ with compact support. For $q = 1, 2, 3, \ldots$ we set

$$\mathcal{A}_q = \left\{ \phi \in \mathcal{D}(\mathbf{R}^n) \text{ such that } \int_{\mathbf{R}^n} \phi(\lambda) d\lambda = 1, \int_{\mathbf{R}^n} \lambda^i \phi(\lambda) d\lambda = 0 \text{ for } 1 \leq |i| \leq q \right\}$$

here $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{R}^n$, $i = (i_1, i_2, \ldots, i_n) \in \mathbf{N}^n$, $|i| = i_1 + i_2 + \cdots + i_n$ and $\lambda^i = (\lambda_1)^{i_1} (\lambda_2)^{i_2} \cdots (\lambda_n)^{i_n}$.

Notation 1.2: If $\epsilon > 0$, $\lambda \in \mathbf{R}^n$ and $\phi \in \mathcal{D}(\mathbf{R}^n)$ we set $\phi_\epsilon(\lambda) = \frac{1}{\epsilon^n} \phi(\frac{\lambda}{\epsilon})$.

Notation 1.3: Let $\mathcal{E}[\mathbf{R}^n]$ denote the set of all the functions

$$R: \mathcal{A}_1 \times \mathbf{R}^n \longrightarrow \mathbf{C}$$

for which $R(\phi, x)$ is a C^∞-function in x for each fixed ϕ.

Definition 1.4: We say an element R of $\mathcal{E}[\mathbf{R}^n]$ is *moderate* if for every compact subset $\mathcal{K}$ of $\mathbf{R}^n$ and every derivation operator

$$D = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$$

there is an $N \in \mathbf{N}$ such that if $\phi \in \mathcal{A}_N$ then there exist $\eta > 0$, $c > 0$ such that

$$|DR(\phi_\epsilon, x)| \leq \frac{c}{\epsilon^N}$$

for all $x \in \mathcal{K}$ and $0 < \epsilon < \eta$. We denote by $\mathcal{E}_M[\mathbf{R}^n]$ the set of all moderate elements of $\mathcal{E}[\mathbf{R}^n]$.

Notation 1.5: We denote by Γ the set of all the increasing functions α from $\mathbf{N}$ into $\mathbf{R}^+$ such that $\alpha(q)$ tends to $+\infty$ when $q \longrightarrow +\infty$.

Definition 1.6: We say that an element R of $\mathcal{E}[\mathbf{R}^n]$ is *null* if for every compact subset $\mathcal{K}$ of $\mathbf{R}^n$ and every derivation operator D, there are $N \in \mathbf{N}$, $\alpha \in \Gamma$ such that if $\phi \in \mathcal{A}_q$, $q \geq N$, then there exist $\eta > 0$, $c > 0$ such that

$$|DR(\phi_\epsilon, x)| \leq c\epsilon^{\alpha(q)-N}$$

for all $x \in \mathcal{K}$ and $0 < \epsilon < \eta$. We denote by $\mathcal{N}[\mathbf{R}^n]$ the set of all the null elements of $\mathcal{E}[\mathbf{R}^n]$.

Definition 1.7: We define the generalized functions on $\mathbf{R}^n$ as the quotient space

$$\mathcal{G}(\mathbf{R}^n) = \frac{\mathcal{E}_M[\mathbf{R}^n]}{\mathcal{N}[\mathbf{R}^n]}.$$

In other words we define an equivalence relation in $\mathcal{E}_M[\mathbf{R}^n]$ by setting

$$R_1 \sim R_2 \quad \text{if and only if} \quad R_1 - R_2 \in \mathcal{N}[\mathbf{R}^n]$$

and so a generalized function is an equivalence class. Since $\mathcal{N}[\mathbf{R}^n]$ is a linear subspace of $\mathcal{E}_M[\mathbf{R}^n], \mathcal{G}(\mathbf{R}^n)$ is a linear space. Since $\mathcal{N}[\mathbf{R}^n]$ is an ideal of $\mathcal{E}_M[\mathbf{R}^n]$, $\mathcal{G}(\mathbf{R}^n)$ is an algebra. If D is any x-derivation operator and if $G \in \mathcal{G}(\mathbf{R}^n)$, then DG is defined as an element of $\mathcal{G}(\mathbf{R}^n)$ as follows. If $R \in \mathcal{E}_M[\mathbf{R}^n]$ is a representative of $G \in \mathcal{G}(\mathbf{R}^n)$ then $DG \in \mathcal{G}(\mathbf{R}^n)$ is defined as the class of $DR \in \mathcal{E}_M[\mathbf{R}^n]$; that class does not depend on the choice of R in the class G.

Definition 1.8: *Restrictions of generalized functions.*

Given an arbitrary element u of $\mathcal{G}(\mathbf{R}^4)$, we seek a natural concept defining the restriction $u|_{t=0}$ of u to $\mathbf{R}^3 = \mathbf{R}^3 \times \{0\} \subset \mathbf{R}^4$. For this let $\psi \in \mathcal{S}(\mathbf{R}^3)$ be given. We define symψ by

$$(\mathrm{sym}\psi)(x_1, x_2, x_3) = \frac{1}{3!} \sum_{\sigma \in G_3} \psi\left(x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3}\right)$$

where G_3 denotes the set of all 3! permutations of the set $\{1,2,3\}$. Then we define a function ψ_4 on $\mathbf{R}^4$ by

$$\psi_4(x,t) = \psi(x) \int_{\mathbf{R}^2} (\mathrm{sym}\psi)(\xi_1, \xi_2, t)\, d\xi_1 d\xi_2.$$

Now let u be an arbitrary element of $\mathcal{G}(\mathbf{R}^4)$. Let

$$R: \mathcal{A}_1(\mathbf{R}^4) \times \mathbf{R}^4 \longrightarrow \mathbf{C}$$

be a representative of u. We define a map

$$R': \mathcal{A}_1(\mathbf{R}^3) \times \mathbf{R}^3 \longrightarrow \mathbf{C}$$

by

$$R'(\psi, x) = R\,(\psi_4, (x, 0)) \quad \text{if} \quad \psi \in \mathcal{A}_1(\mathbf{R}^3).$$

We denote by

$$u|_{t=0} \in \mathcal{G}(\mathbf{R}^3)$$

the class of R', and we call $u|_{t=0}$ the restriction of u to $\mathbf{R}^3$.

Notation 1.9: Let us assume f, u_0 and u_1 are generalized functions with respective representatives

$$R_f \in \mathcal{E}_M[\mathbf{R}^4], \quad R_{u_0}, R_{u_1} \in \mathcal{E}_M[\mathbf{R}^3].$$

If $\psi \in \mathcal{A}_1(\mathbf{R}^4)$, let $\psi_3 \in \mathcal{A}_1(\mathbf{R}^3)$ be defined by

$$\psi_3(x) = \int_R \psi(x, t)dt.$$

Remark: By (Definition 1.8) and (Notation 1.9), it is immediate that

$$\psi_4 \in \mathcal{D}(\mathbf{R}^4) \quad \text{if} \quad \psi \in \mathcal{D}(\mathbf{R}^3)$$

$$\psi_4 \in \mathcal{S}(\mathbf{R}^4) \quad \text{if} \quad \psi \in \mathcal{S}(\mathbf{R}^3)$$

$$\psi_4 \in \mathcal{A}_q(\mathbf{R}^4) \quad \text{if} \quad \psi \in \mathcal{A}_q(\mathbf{R}^3) \quad \text{for} \quad q = 1, 2, 3 \ldots$$

Furthermore,

$$(\psi_3)_\epsilon = (\psi_\epsilon)_3 \quad \text{for} \quad 0 < \epsilon < 1,$$

because

$$\begin{aligned}
(\psi_\epsilon)_3(\vec{x}) &= \int_{-\infty}^{\infty} \frac{1}{\epsilon^4} \psi\left(\frac{\vec{x}}{\epsilon}, \frac{t}{\epsilon}\right) dt \\
&= \int_{-\infty}^{\infty} \frac{1}{\epsilon^3} \psi\left(\frac{\vec{x}}{\epsilon}, t\right) dt \\
&= \frac{1}{\epsilon^3} (\psi_3)\left(\frac{\vec{x}}{\epsilon}\right) \\
&= (\psi_3)_\epsilon(\vec{x}).
\end{aligned}$$

Also,

$$(\psi_4)_3 = \psi,$$

because

$$\begin{aligned}(\psi_4(x,t))_3 &= \left(\psi(x)\int_{\mathbf{R}^2} sym\psi(\xi_1,\xi_2,t)d\xi_1 d\xi_2\right)_3 \\ &= \int\left(\psi(x)\int_{\mathbf{R}^2} sym\psi(\xi_1,\xi_2,t)d\xi_1 d\xi_2\right)dt \\ &= \psi(x)\int\left(\int_{\mathbf{R}^2} sym\psi(\xi_1,\xi_2,t)d\xi_1 d\xi_2\right)dt \\ &= \psi(x).\end{aligned}$$

J. F. Colombeau's method for finding generalized function solutions to differential equations.

We consider the nonlinear wave equation.

$$u_{tt} - \Delta u = F(u)$$

in $\mathcal{G}(\mathbf{R}^4)$, where $F: \mathbf{R}^1 \longrightarrow \mathbf{R}^1$ is a C^∞-smooth function, such that $F(0) = 0$ and $\sup_{x\in R}|D^p F(x)| < \infty$, for all positive integer p.

Theorem 1.1: The initial value problem

$$u_{tt} - \Delta u = F(u) \tag{1.1}$$

$$u(0,x) = u_0(x) \tag{1.2}$$

$$u_t(0,x) = u_1(x) \tag{1.3}$$

has real valued generalized function solutions $u \in \mathcal{G}(\mathbf{R}^4)$ for every pair of real valued generalized functions $u_0, u_1 \in \mathcal{G}(\mathbf{R}^3)$.

(Proof) Suppose first that $u_0, u_1 \in C^\infty(\mathbf{R}^3)$. The problem (1.1) $\sim$ (1.3) is equivalent to solving the nonlinear integral equation

$$u(t,x) = v(t,x) + \int_0^t (t-s)M\left(F(u), s, x, t-s\right)ds \tag{1.1$'$}$$

where

$$M(f,s,x,\rho)=\frac{1}{4\pi}\int_{\xi\in\mathbf{R}^3,\|\xi\|=1} f(s,x+\rho\xi)d\xi$$

for

$$f\colon \mathbf{R}^4 \longrightarrow \mathbf{R}^1$$

sufficiently regular, while v is the solution of the following classical, homogenous initial value problem

$$v_{tt}-\Delta v=0 \tag{1.4}$$

$$v(0,x)=u_0(x) \tag{1.5}$$

$$v(0,x)=u_1(x). \tag{1.6}$$

Given the classical solution $v\in C^\infty(\mathbf{R}^4)$ of (1.4) $\sim$ (1.5), the solution $u\in C^\infty(\mathbf{R}^4)$ of (1.1)$'$ is obtained by iteration. We start with $u_1=v$ and continue according to

$$u_{\nu+1}(t,0)=v(t,0)+\int_0^t (t-x)M\left(F(u_\nu),x,s,t-s\right)\,ds \tag{1.7}$$

for $\nu\in\mathbf{N}^+$. Using estimates of the right hand term in (1.7), we obtain the unique solution $u\in C^\infty(\mathbf{R}^4)$ of (1.1)$'$, as a limit of u_ν, when $\nu\to\infty$, with the convergence being uniform on compact sets in $\mathbf{R}^4$.

Now we consider the case where u_0 and u_1 are arbitrary generalized functions in $\mathcal{G}(\mathbf{R}^3)$. We are looking for a generalized solution $u\in\mathcal{G}(\mathbf{R}^4)$ which will be well defined as soon as we obtain for it a representative

$$u=f+\mathcal{N}[\mathbf{R}^4]\in\mathcal{G}(\mathbf{R}^4)\quad\text{where}\quad f\in\mathcal{E}_M[\mathbf{R}^4]. \tag{1.8}$$

Let f_0 and f_1 be representatives of u_0 and u_1, that is,

$$u_i=f_i+\mathcal{N}[\mathbf{R}^3]$$

where $f_i\in\mathcal{E}_M[\mathbf{R}^3]$ for $i\in\{0,1\}$. Let us take an arbitrary

$$\phi\in\mathcal{A}_1(\mathbf{R}^4).$$

It follows that

$$f_0(\phi_3, \cdot), \quad f_1(\phi_3, \cdot) \in C^\infty(\mathbf{R}^3).$$

Let us consider the classical C^∞-smooth case (1.1) $\sim$ (1.3) with the initial values given by

$$u_0 = f_0(\phi_3, \cdot), \quad u_1 = f_1(\phi_3, \cdot).$$

Then according to the classical existence and uniquness result, we obtain a C^∞-smooth solution which we denote by

$$f(\phi, \cdot) \in C^\infty(\mathbf{R}^4). \tag{1.9}$$

In this way, we only have to show two things; first, that f in (1.9) satisfies $f \in \mathcal{E}_M[\mathbf{R}^4]$, and second, that the corresponding

$$u = f + \mathcal{N}[\mathbf{R}^4]$$

will satisfy (1.2) and (1.3). The mentioned estimates of the right hand term in (1.7) used in the classical C^∞-smooth proof directly yield the estimates needed in order to obtain $f \in \mathcal{E}_M[\mathbf{R}^4]$. In this way we obtain u in (1.8) which satisfies (1.1) in $\mathcal{G}(\mathbf{R}^4)$. Finally, we have to show that (1.2), (1.3) are satisfied.

In view of (1.8), we have

$$u|_{t=0} = g + \mathcal{N}[\mathbf{R}^3] \in \mathcal{G}(\mathbf{R}^3)$$

with $g(\psi, x) = f(\psi_4, (x, 0))$, $\psi \in \mathcal{A}_1(\mathbf{R}^3)$, $x \in \mathbf{R}^3$, according to Definition 1.8.

Since $f(\psi_4, (x, t))$ is the solution of (1.1) $\sim$ (1.3) with the initial values $u_0 = f_0((\psi_4)_3, \cdot)$ and $u_1 = f_1((\psi_4)_3, \cdot)$ we have

$$\begin{aligned} g(\psi, x) &= f(\psi_4, (\vec{x}, 0)) \\ &= f_0((\psi_4)_3, \vec{x}) \\ &= f_0(\psi, \vec{x}) \end{aligned}$$

where the last equality follows from $(\psi_4)_3 = \psi$. That is, $g = f_0$. That is, $u|_{t=0} = u_0$.

Similarly, $u_t|_{t=0} = h + \mathcal{N}[\mathbf{R}^3] \in \mathcal{G}(\mathbf{R}^3)$ where $h(\psi, \vec{x}) = \frac{\partial f}{\partial t}(\psi_4, (\vec{x}, 0))$, according to Definition 1.8. Since $f(\psi_4, (\vec{x}, t))$ is the solution of (1.1) $\sim$ (1.3) with initial values $u_0 = f_0((\psi_4)_3, \cdot)$ and $u_1 = f_1((\psi_4)_3, \cdot)$, we have

$$\begin{aligned} h(\psi, x) &= \frac{\partial f}{\partial t}(\psi_4, (\vec{x}, 0)) \\ &= f_1((\psi_4)_3, \vec{x}) \\ &= f_1(\psi, \vec{x}). \end{aligned}$$

That is, $h = f_1$. That is, $u_t|_{t=0} = u_1$.

Remark: In fact, the conclutions of Theorem 1.1 also hold when the nonlinearity F has unbounded derivatives but satisfies a growth condition. See ([8]). In particular, a nonlinearity of the form $F(u) = -|u|^{p-1}u$ with $1 \le p < 5$ (in three spatial dimensions) satisfies the growth condition. So for these pure power nonlinearities the initial value problem has generalized function solutions.

Distributions considered as generalized functions.

Distributions in $\mathcal{D}'(\mathbf{R}^n)$ may be considered to be generalized functions as follows. If $Q \in \mathcal{D}'(\mathbf{R}^n)$, then for $\psi \in \mathcal{A}_1$, we define

$$R_Q(\psi, x) = (\psi * Q)(x) = Q[\widetilde{\psi}_x]$$

where $\widetilde{\psi}_x(y) = \psi(x - y)$. Because $\psi \in \mathcal{D}(\mathbf{R}^n), \psi * Q$ is in $C^\infty(\mathbf{R}^n)$. Thus $R_Q(\psi, \cdot)$ is C^∞, so $R_Q \in \mathcal{E}[\mathbf{R}^n]$. In fact, R_Q is moderate, which we can see as follows. Let $D_x^{(k)} = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_2^{k_2}}$. Let K be a compact subset of $\mathbf{R}^n$. Because $Q \in D'(\mathbf{R}^n)$, we know that for all $\phi \in C_0^\infty(K)$,

$$Q[\phi] = \int_{\mathbf{R}^n} h(y) \left(D_y^{(l)} \phi \right) (y) d^n y$$

for some continuous slowly growing function $h : \mathbf{R}^n \longrightarrow \mathbf{C}$ and some fixed multiindex l. Therefore,

$$\begin{aligned}
D_x^{(k)}\left(R_Q(\psi_\epsilon, x)\right) &= D_x^{(k)}\left((\psi_\epsilon * Q)(x)\right) \\
&= D_x^{(k)} Q\left[(\widetilde{\psi_\epsilon})_x\right] \\
&= D_x^{(k)} \int_{\mathbf{R}^n} h(y) D_y^{(l)}\left[(\widetilde{\psi_\epsilon})_x(y)\right] d^n y \\
&= D_x^{(k)} \int_{\mathbf{R}^n} h(y) D_y^{(l)}\left[\psi_\epsilon(x-y)\right] \\
&= D_x^{(k)} \int_{\mathbf{R}^n} h(y) D_y^{(l)}\left[\frac{1}{\epsilon^n}\psi\left(\frac{x-y}{\epsilon}\right)\right] d^n y \\
&= \frac{(-1)^{|l|}}{\epsilon^{n+|l|+|k|}} \int_{\mathbf{R}^n} h(y)\left(D^{(k+l)}\psi\right)\left(\frac{x-y}{\epsilon}\right) d^n y.
\end{aligned}$$

Thus,

$$\left|D_x^{(k)}\left(R_Q(\psi_\epsilon, x)\right)\right| \leq \frac{C}{\epsilon^{n+|l|+|k|}}$$

where C is a constant and for x in K, that is, R_Q is moderate. The generalized function associated with the distribution Q is the equivalence class of R_Q in $\mathcal{G}[\mathbf{R}^n]$ where $R_Q(\psi, x) = (\psi * Q)(x)$.

Suppose we know $R : \mathcal{A}_1 \times \mathbf{R}^n \longrightarrow \mathbf{R}$ is associated with a distribution. We can recover the distribution from the generalized function representative R as follows. Suppose we are given a test function $\phi \in \mathcal{D}(\mathbf{R}^n)$. If R is indeed a representative of a distribution $Q \in \mathcal{D}'(\mathbf{R}^n)$ then we have $R(\psi, x) = (\psi * Q)(x) + \mathcal{N}$, so that

$$\begin{aligned}
lim_{\epsilon\to 0} \int_{\mathbf{R}^n} R(\psi_\epsilon, x)\phi(x) d^n x &= lim_{\epsilon\to 0} \int_{\mathbf{R}^n} (\psi_\epsilon * Q)(x)\phi(x) d^n x \\
&= Q[\phi]
\end{aligned}$$

because $\{\psi_\epsilon\}$ is a δ-sequence. Therefore, if R is a generalized function representative of the distribution $Q \in \mathcal{D}'(\mathbf{R}^n)$, then Q is recovered from R by

$$Q[\phi] = lim_{\epsilon\to 0} \int_{\mathbf{R}^n} R(\psi_\epsilon, x)\phi(x) d^n x.$$

We now consider the initial value problem

$$u_{tt} - \Delta u = f(u)$$

$$u(\vec{x}, 0) = u_0$$

$$u_t(\vec{x}, 0) = u_1$$

where u_0 and u_1 are distributions in $\mathcal{D}'(\mathbf{R}^3)$. Each distribution has an associated generalized function, with representative R_{u_0} and R_{u_1} given by

$$R_{u_0}(\psi, \vec{x}) = (\psi * u_0)(\vec{x})$$

$$R_{u_1}(\psi, \vec{x}) = (\psi * u_1)(\vec{x}).$$

J. F. Colombeau's generalized function solution to the initial value problem (with u_0 and u_1 regarded as generalized functions) has representative $R_u \in \mathcal{E}[\mathbf{R}^{3+1}]$. For $\phi \in \mathcal{A}_1$, $R_u(\phi, (\vec{x}, t))$ satisfies

$$u_{tt} - \Delta u = f(u)$$

$$u(\vec{x}, 0) = R_{u_0}(\phi_3, \vec{x}) = (\phi_3 * u_0)(\vec{x})$$

$$u_t(\vec{x}, 0) = R_{u_1}(\phi_3, \vec{x}) = (\phi_3 * u_1)(\vec{x}).$$

where $\phi_3(x) = \int_{-\infty}^{\infty} \phi(x, t) dt$.

Suppose it happened that the generalized function solution u was associated with a distribution $Q \in \mathcal{D}'(\mathbf{R}^{3+1})$. Then for fixed $\psi \in \mathcal{A}_1(\mathbf{R}^{3+1})$,

$$Q[\phi] = lim_{\epsilon \to 0} \int_{\mathbf{R}^{3+1}} R_u(\psi_\epsilon, x) \phi(x) d^{3+1}x$$

for all $\phi \in \mathcal{D}(\mathbf{R}^{3+1})$. Therefore, we consider the initial value problem satisfied by $u_\epsilon(\vec{x}, t) = R_u(\psi_\epsilon, (\vec{x}, t))$: It is

$$(u_\epsilon)_{tt} - \Delta u_\epsilon = f(u_\epsilon)$$

$$u_\epsilon(\vec{x}, 0) = ((\psi_\epsilon)_3 * u_0)(\vec{x})$$

$$(u_\epsilon)t(\vec{x}, 0) = ((\psi_\epsilon)_3 * u_1)(\vec{x}).$$

Because $(\psi_\epsilon)_3 = (\psi_3)_\epsilon$, the initial value problem becomes

$$(u_\epsilon)_{tt} - \Delta u = f(u_\epsilon)$$

$$u_\epsilon(\vec{x},0) = ((\psi_3)_\epsilon * u_0)(\vec{x})$$

$$(u_\epsilon)_t(\vec{x},0) = ((\psi_3)_\epsilon * u_1)(\vec{x}).$$

Therefore, to look for distribution solutions to the nonlinear initial value problem

$$u_{tt} - \Delta u = f(u)$$

$$u(\vec{x},0) = u_0$$

$$u_t(\vec{x},0) = u_1$$

where u_0 and u_1 are in $\mathcal{D}'(\mathbf{R}^3)$, we solve the initial value problem

$$(u_\epsilon)_{tt} - \Delta u_\epsilon = f(u_\epsilon)$$

$$u_\epsilon(\vec{x},0) = (\theta_\epsilon * u_0)(\vec{x})$$

$$(u_\epsilon)_t(\vec{x},0) = (\theta_\epsilon * u_1)(\vec{x})$$

where $\theta \in \mathcal{D}(\mathbf{R}^3)$. If it happens that the generalized function solution is associated with a distribution Q, then Q is obtained from

$$Q[\phi] = lim_{\epsilon\to 0} \int_{\mathbf{R}^{3+1}} u_\epsilon(\vec{x},t)\phi(\vec{x},t)d^3x dt.$$

We can summarize J. F. Colombeau's method in the case when the initial data are ordinary distributions. To find generalized function solutions to (1.1) $\sim$ (1.3), we regularize u_0 and u_1 by convolving them with families of D-functions that approximate the Dirac delta function. We then solve the initial value problem with the regularized (C^∞) initial data, to obtain a classical solution for each D-function. We then take those classical solutions to be representatives of a generalized function, which we regard as the generalized function solution. We apply this techique to specific examples in the following.

CHAPTER 2

INITIAL VALUE PROBLEMS

We consider the two initial value problems.

$$
\begin{aligned}
u_{tt} - \Delta u &= f(u) \\
u(x,0) &= \delta(x) \\
u_t(x,0) &= 0
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
u_{tt} - \Delta u &= f(u) \\
u(x,0) &= 0 \\
u_t(x,0) &= \delta(x)
\end{aligned}
\tag{2.2}
$$

where Δ is the Laplacian in x and the solution $u: \mathbf{R}^{n+1} \longrightarrow \mathbf{C}$ is complex valued function. Since $\mathcal{D}(\mathbf{R}^n)$, considered as a subspace of $\mathcal{D}'(\mathbf{R}^n)$, is dense in $\mathcal{D}'(\mathbf{R}^n)$, given $\psi \in \mathcal{D}(\mathbf{R}^n)$, with $\int \psi(x)dx = 1$, the sequence ψ_ϵ defined by $\psi_\epsilon(x) = \frac{1}{\epsilon^n}\psi(\frac{x}{\epsilon})$ converges to δ in $\mathcal{D}'(\mathbf{R}^n)$. To use J. F. Colombeau's method, we replace δ by $\psi_\epsilon * \delta = \psi_\epsilon$.

(Case 1) We consider the following initial value problem.

$$
\begin{aligned}
(u_\epsilon)_{tt} - \Delta u_\epsilon &= f(u_\epsilon) \\
u_\epsilon(x,0) &= \psi_\epsilon(x) = \frac{1}{\epsilon^n}\psi(\frac{x}{\epsilon}) \\
(u_\epsilon)_t(x,0) &= 0.
\end{aligned}
\tag{2.3}
$$

This problem has a solution $u_\epsilon(x,t)$ which is a representative of a generalized function because u_ϵ is a C^∞-function. Then there arises the following question. Does u_ϵ ever generate a distribution? That is, are there nonlinearities f for which

$$lim_{\epsilon\to 0}\int_{\mathbf{R}^{n+1}} u_\epsilon(x,t)\phi(x,t)d^n x dt \quad \text{exists for all} \quad \phi\in\mathcal{D}(\mathbf{R}^{n+1})? \tag{2.4}$$

We rewrite by introducing $v_\epsilon(x,t)$ such that $u_\epsilon(x,t)=\frac{1}{\epsilon^n}v_\epsilon(\frac{x}{\epsilon},\frac{t}{\epsilon^l})$ where l is to be determined, then $v_\epsilon(x,t)=\epsilon^n u_\epsilon(\epsilon x,\epsilon^l t)$. Next we find the initial value problem satisfied by v_ϵ. The initial condition in (2.3), $u_\epsilon(x,0)=\frac{1}{\epsilon^n}\psi(\frac{x}{\epsilon})$ implies

$$v_\epsilon(x,0)=\psi(x) \tag{2.5}$$

$(u_\epsilon)_t(x,0)=0$ implies

$$(v_\epsilon)_t(x,0)=0 \tag{2.6}$$

and from the equation $(u_\epsilon)_{tt}-\Delta u_\epsilon=f(u_\epsilon)$ we get

$$\frac{1}{\epsilon^n}\frac{1}{\epsilon^{2l}}(v_\epsilon)_{tt}(\frac{x}{\epsilon},\frac{t}{\epsilon^l})-\frac{1}{\epsilon^n}\frac{1}{\epsilon^2}(\Delta v_\epsilon)(\frac{x}{\epsilon},\frac{t}{\epsilon^l})=f\left(\frac{1}{\epsilon^n}v_\epsilon(\frac{x}{\epsilon},\frac{t}{\epsilon^l})\right). \tag{2.7}$$

From (2.5), (2.6) and (2.7) v_ϵ satisfies the following initial value problem

$$\begin{aligned}\frac{1}{\epsilon^{n+2l}}(v_\epsilon)_{tt}-\frac{1}{\epsilon^{n+2}}(\Delta v_\epsilon)&=f(\frac{1}{\epsilon^n}v_\epsilon)\\ v_\epsilon(x,0)&=\psi(x)\\ (v_\epsilon)_t(x,0)&=0.\end{aligned} \tag{2.8}$$

Now, to get the simplest possible situation, choose $l=1$ and take $f(w)=-|w|^{p-1}w$, then (2.8) becomes

$$\begin{aligned}(v_\epsilon)_{tt}-(\Delta v_\epsilon)&=\frac{\epsilon^2}{|\epsilon|^{n(p-1)}}(-|v_\epsilon|^{p-1}v_\epsilon)\\ v_\epsilon(x,0)&=\psi(x)\\ (v_\epsilon)_t(x,0)&=0.\end{aligned} \tag{2.9}$$

We notice that for the special value $p = 1 + \frac{2}{n}$, the initial value problem for v_ϵ is independent of ϵ. Then v_ϵ satisfies the following initial value problem

$$\begin{aligned}
(v_\epsilon)_{tt} - \Delta v_\epsilon &= -|v_\epsilon|^{p-1} v_\epsilon \quad \text{with} \quad p = 1 + \frac{2}{n} \\
v_\epsilon(x,0) &= \psi(x) \\
(v_\epsilon)_t(x,0) &= 0.
\end{aligned} \tag{2.10}$$

Since the solution of the initial value problem

$$\begin{aligned}
v_{tt} - \Delta v &= -|v|^{p-1} v \\
v(x,0) &= \psi(x) \\
v_t(x,0) &= 0
\end{aligned} \tag{2.11}$$

is unique, the solution v_ϵ of (2.10) actually does not depend on ϵ. That is, $v_\epsilon(y,s) = v(y,s)$ for all ϵ where $v(y,s)$ is the solution of (2.11). As a consequence the solution of u_ϵ of (2.3) is given by $u_\epsilon(x,t) = \frac{1}{\epsilon^n} v(\frac{x}{\epsilon}, \frac{t}{\epsilon})$. Then our question (2.4) becomes: Does

$$\lim_{\epsilon \to 0} \int_{\mathbf{R}^{n+1}} \frac{1}{\epsilon^n} v(\frac{x}{\epsilon}, \frac{t}{\epsilon}) \phi(x,t) d^n x dt$$

exist or not for all $\phi \in \mathcal{D}(\mathbf{R}^{n+1})$? where v is the solution of (2.11).

(Case 2) As in (case 1), we consider the following equation

$$\begin{aligned}
(u_\epsilon)_{tt} - \Delta u_\epsilon &= -|u_\epsilon|^{p-1} u_\epsilon \\
u_\epsilon(x,0) &= 0 \\
(u_\epsilon)_t(x,0) &= \frac{1}{\epsilon^n} \psi(\frac{x}{\epsilon})
\end{aligned} \tag{2.12}$$

where $\psi \in \mathcal{D}(\mathbf{R}^n)$. Define $v_\epsilon(x,t) = \epsilon^{n-1} u_\epsilon(\epsilon x, \epsilon t)$. Then

$$\begin{aligned}
(v_\epsilon)_{tt} - \Delta v_\epsilon &= \frac{\epsilon^{n+1}}{\epsilon^{(n-1)p}} (-|v_\epsilon|^{p-1}) v \\
v_\epsilon(x,0) &= 0 \\
(v_\epsilon)_t(x,0) &= \psi(x).
\end{aligned}$$

We notice that for the special value $p = 1 + \frac{2}{n-1}$, the initial value problem for v_ϵ is independent of ϵ. Thus $u_\epsilon(x,t) = \frac{1}{\epsilon^{n-1}} v(\frac{x}{\epsilon}, \frac{t}{\epsilon})$ where v satisfies the following initial value problem

$$\begin{aligned} v_{tt} - \Delta v &= -|v|^{p-1} v \quad \text{with} \quad p = 1 + \frac{2}{n-1} \\ v(x,0) &= 0 \\ v_t(x,0) &= \psi(x). \end{aligned} \tag{2.13}$$

As a consequence our question for Case 2 becomes: Does

$$lim_{\epsilon\to 0} \int_{\mathbf{R}^{n+1}} u_\epsilon(x,t)\phi(x,t) d^n x dt = lim_{\epsilon\to 0} \int_{\mathbf{R}^{n+1}} \frac{1}{\epsilon^{n-1}} v\left(\frac{x}{\epsilon}, \frac{t}{\epsilon}\right) \phi(x,t) d^n x dt \tag{2.14}$$

exist or not for all $\phi \in \mathcal{D}(\mathbf{R}^{n+1})$?

Remark: It is useful to consider the analog of our formulation in the linear case. Let $u_{0\epsilon}(x,t) = \frac{1}{\epsilon^n} v_0(\frac{x}{\epsilon}, \frac{t}{\epsilon})$ satisfy the following initial value problem

$$\begin{aligned} (u_{0\epsilon})_{tt} - \Delta u_{0\epsilon} &= 0 \\ (u_{0\epsilon})(x,0) &= \frac{1}{\epsilon^n} \psi(\frac{x}{\epsilon}) \\ (u_{0\epsilon})_t(x,0) &= 0. \end{aligned}$$

Because $\frac{1}{\epsilon^n} \psi\left(\frac{x}{\epsilon}\right)$ converges to $\delta(x)$ in $\mathcal{D}'(\mathbf{R}^n)$ as $\epsilon \longrightarrow 0$, this solution approaches the distribution solution to

$$\begin{aligned} (u_0)_{tt} - \Delta u_0 &= 0 \\ u_0(x,0) &= \delta(x) \\ (u_0)_t(x,0) &= 0. \end{aligned}$$

That is, we know

$$\lim_{\epsilon\to 0} \int_{\mathbf{R}^{n+1}} u_{0\epsilon}(x,t)\phi(x,t) d^n x dt = u_0[\phi]$$

for all $\phi \in \mathcal{D}(\mathbf{R}^{n+1})$. This is equivalent to

$$\lim_{\epsilon \to 0} \int_{\mathbf{R}^{n+1}} \frac{1}{\epsilon^n} v_0 \left(\frac{x}{\epsilon}, \frac{t}{\epsilon} \right) \phi(x,t) dx dt = u_0[\phi].$$

Thus, in the linear case, the analog to our question (2.4) has a positive answer.

Methods of Investigation

In this dissertation, we choose $n > 1$-dimension because there is no power p that makes v_ϵ independent of ϵ in Case 2 when $n = 1$ owing to $p = 1 + \dfrac{2}{n-1}$. Furthermore, when $n = 1$, the initial value problem for v in (case 1) is

$$v_{tt} - v_{xx} = -|v|^2 v$$

$$v(x,0) = \psi(x)$$

$$v_t(x,0) = 0.$$

The solutions to this equation have no dispersion and are oscillatory and not well suited to numerical computation.

When $n = 2$, there is no nice reduction of $v_{tt} - \Delta v = -|v|^{p-1} v$ to a one-space dimension equation. But, for $n = 3$ we can reduce $v_{tt} - \Delta v = -|v|^{p-1} v$ for spherically symmetric initial data to a one- space dimension equation that is numerically well behaved. So we choose $n = 3$. We consider the solutions to the problems (2.11) and (2.13) in $n = 3$ spatial dimensions, and investigate the limits (2.4) and (2.14) for $\phi \in \mathcal{D}(\mathbf{R}^{3+1})$.

The specific values of the exponent p in our problems are those at the borderline of the theory of scattering for $u_{tt} - \Delta u = -|u|^{p-1} u$, and estimates of the asymptotic behavior of u for large times have not yet been obtained for these values of p ([3, 4, 6, 11, 12, 13, 14, 16, 17, 19, 22, 23, 24, 25, 26]). Here we will numerically solve the initial value problems (2.11) and (2.13) and we will investigate the limits (2.4) and (2.14) based on the numerical solutions.

CHAPTER 3

GENERAL PROPERTIES OF SOLUTIONS

We consider

$$u_{tt} - \Delta = -|u|^{p-1}u. \tag{3.1}$$

We can find a conserved quantity for this equation by applying Noether's Theorem to the time translation invariance of the equation. Let

$$L[u] = \int_{\mathbf{R}^{n+1}} \left(\frac{1}{2}|u_t|^2 - \frac{1}{2}|\nabla u|^2 - F(u)\right) d^n x dt$$

where $F(u)$ is the primitive of $-|u|^{p-1}u$. Then the solutions of (3.1) are extrema of the functional L. Let $T_\epsilon u(x,t) = u(x, t-\epsilon)$; then

$$L(T_\epsilon u) = \int_{\mathbf{R}^{n+1}} \frac{1}{2}(u_t(x,t-\epsilon))^2 - \frac{1}{2}(\nabla u(x,t-\epsilon))^2 - F\left(u(x,t-\epsilon)\right) d^n x dt = L(u).$$

So L is invariant under the 1-parameter family of linear transformations T_ϵ. The generator of the family is

$$T_0' = \frac{\partial}{\partial \epsilon} T_\epsilon|_{\epsilon=0},$$

which we may compute as follows.

$$\frac{\partial}{\partial \epsilon} T_\epsilon u(x,t) = \frac{\partial}{\partial \epsilon} u(x,t-\epsilon) = -\frac{\partial}{\partial t} u(x,t-\epsilon)$$

so

$$T_0' u(x,t) = \frac{\partial}{\partial \epsilon} T_\epsilon|_{\epsilon=0} u(x,t) = -\frac{\partial}{\partial t} u(x,t)$$

hence we get the generator $T_0' = -\dfrac{\partial}{\partial t}$. Now we multiply $-\dfrac{\partial u}{\partial t}$ in (3.1).

We have

$$\begin{aligned}
\left(u_{tt} - \Delta u + |u|^{p-1}u\right)\left(-\frac{\partial u}{\partial t}\right) &= -\left(\frac{1}{2}\frac{\partial}{\partial t}(u_t)^2 - \Delta u \cdot u_t + \frac{\partial}{\partial t}(-F(u))\right) \\
&= -\frac{\partial}{\partial t}\left(\frac{1}{2}(u_t)^2 - F(u)\right) \\
&+ \vec{\nabla}\cdot\left(u_t\vec{\nabla}u\right) - (\vec{\nabla}u_t)(\vec{\nabla}u) \\
&= -\frac{\partial}{\partial t}\left(\frac{1}{2}(u_t)^2 + \frac{1}{2}|\nabla u|^2 - F(u)\right) \\
&+ \vec{\nabla}\cdot(u_t\cdot\vec{\nabla}u).
\end{aligned}$$

That is,

$$\left(u_{tt} - \Delta u + f(u)\right)u_t = \frac{\partial}{\partial t}e[u] + \vec{\nabla}\cdot\vec{p}[u]$$

where

$$\begin{aligned}
e[u] &= \frac{1}{2}(u_t)^2 + \frac{1}{2}|\nabla u|^2 - F(u) \\
\vec{p}[u] &= -u_t\vec{\nabla}u.
\end{aligned}$$

Thus, if u is a solution of (3.1), then

$$\frac{\partial}{\partial t}e[u] + \vec{\nabla}\cdot\vec{p}(u) = 0.$$

So,

$$\int_{\mathbf{R}^n}\frac{\partial}{\partial t}e[u](x,t) = 0$$

if $\lim_{|x|\to\infty}\vec{p}[u](x,t) = 0$.

Thus, for all u falling off fast enough at ∞,

$$\frac{\partial}{\partial t}E[u](x,t) = 0$$

where

$$E[u](t) = \int\left(\frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 - F(u)\right)d^nx,$$

so, $E[u](t)$ is independent of t.

Note that

$$E[u](t)=\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|\nabla u(t)\|_2^2+\frac{1}{p+1}\int_{\mathbf{R}^3}|u(x,t)|^{p+1}\,d^3x.$$

Because $E(u)$ is time independent, we have

$$\|u_t(t)\|_2^2+\|\nabla u(t)\|_2^2+\frac{2}{p+1}\int_{\mathbf{R}^3}|u(x,t)|^{p+1}\,d^3x=2E(u)(0)=\text{constant}.$$

Furthermore, each of the terms on the left side is nonnegative. Therefore, if the initial data have $E(u)(0)$ finite, then each of the terms is bounded by $2E(u)(0)$. This means that finite energy solutions are regular in the sense that all of the norms $\|u_t\|_2$, $\|\nabla u\|_2$, and $\|u\|_{p+1}$ remain bounded for all times t.

We also note that if the initial data for (3.1) have compact support in $\{\vec{x}\in\mathbf{R}^3 \mid |\vec{x}|\leq L\}$ then the solution u has support in $\left\{(x,t)\in\mathbf{R}^{3+1}\mid |\vec{x}|\leq L+|t|\right\}$.
Theorem 3.1: Suppose u is a classical solution of

$$u_{tt}-\Delta u=F'(u)$$

where $F(u)\leq 0$ for all u. Suppose that support of the initial data for u is contained in the ball $B_L=\{\vec{x}\in\mathbf{R}^n\mid |\vec{x}|\leq L\}$, that is, supp $u(\vec{x},0)\subset B_L$ and supp $u_t(\vec{x},0)\subset B_L$. Then the spatial support of $u(\vec{x},t)$ is contained in

$$B_{L+|t|}=\{\vec{x}\in\mathbf{R}^n\mid|\vec{x}|\leq L+|t|\}.$$

(Proof) Let $e[u]=\frac{1}{2}(u_t)^2+\frac{1}{2}|\nabla u|^2-F(u)$ and $\vec{p}[u]=-u_t\vec{\nabla}u$. We already know

$$\frac{\partial}{\partial t}e[u]+\vec{\nabla}\cdot\vec{p}[u]=0 \tag{3.2}$$

Integrating (3.2) over the interior of some truncated light cone V, we obtain

$$\begin{aligned}
0 &= \int_V \left(\frac{\partial}{\partial t} e[u] + \vec{\nabla}\cdot\vec{p}[u]\right) d^{n+1}x \\
&= \int_V D\cdot P d^{n+1}x \\
&= \int_{\partial V} P\cdot\widehat{n} ds \\
&= \int_T P\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \int_B P\cdot\begin{pmatrix}-1\\0\\0\\0\end{pmatrix} + \int_K P\cdot\widehat{n} ds \\
&= \int_T e[u]ds - \int_B e[u]ds + \int_K \frac{1}{\sqrt{2}}\left(e[u] + \widehat{x}\cdot\vec{p}[u]\right) ds
\end{aligned}$$

where

$$P = \begin{pmatrix} e[u] \\ \vec{p}[u] \end{pmatrix}$$

and the boundary of the truncated light cone V, that is ∂V, is composed of three parts, the top and bottom discs and the conical surface, T, B and K respectively, ds is the element of surface on ∂V, and third equality comes from divergence theorem.

Therefore,

$$\int_T e[u]ds - \int_B e[u]ds + \frac{1}{\sqrt{2}}\int_K \left(e[u] - u_t u_r\right) ds = 0.$$

Next write

$$e[u] - u_t u_r = \frac{1}{2}(u_t - u_r)^2 + \frac{1}{2}\left(|\nabla u|^2 - u_r^2\right) - F(u).$$

Therefore, the quantity $e[u] - u_t u_r \geq 0$ because $F(u) \leq 0$.

So,

$$\int_T e[u]ds = \int_B e[u]ds - \frac{1}{\sqrt{2}}\int_K \left(e[u] - u_t u_r\right) ds.$$

Now, apply this equality to a portion of a light cone with base B at time $t = 0$, with $B \cap B_L$ empty. Because the initial data vanish outside B_L, $e[u] = 0$ on the base B. Therefore,

$$\int_T e[u]ds = -\frac{1}{\sqrt{2}}\int_K \left(e[u] - u_t u_r\right) ds.$$

Therefore, because $e[u] = u_t u_r$ is nonnegative

$$\int_T e[u]ds = 0 = -\frac{1}{\sqrt{2}} \int \left(e[u] - u_t u_r\right) ds.$$

Since $e[u] \geq 0$ and $\int_T e[u]ds = 0$, we have $e[u] = 0$ on T. So $u_t = 0$ and $\vec{\nabla} u = \vec{0}$ and $F(u) = 0$ on T. This implies that $u = 0$ on T. Repeating for any cone outside the stated support of solution, we get the support of u is as stated. See ([22]).

Reduction of 3-dimension space problem to 1-dimension problem.

We consider the initial value problem

$$\begin{aligned} u_{tt} - \Delta u &= -k|u|^{p-1}u \\ u(x,0) &= f(|\vec{x}|) \\ u_t(x,0) &= g(|\vec{x}|) \end{aligned} \tag{3.3}$$

where $x \in \mathbf{R}^3$.

In (3.3), the initial data depend only on the radial variable $|\vec{x}| = r$. Then it follows that the solution $u(x,t)$ to (3.3) depends only on $|\vec{x}| = r$ and t, that is, $u(\vec{x},t) = u(r,t)$. Using the spherical coordinates, (3.3) is reduced to

$$\begin{aligned} u_{tt} - (u_{rr} + \frac{2}{r}u_r) &= -k|u|^{p-1}u \\ u(r,0) &= f(r) \\ u_t(r,0) &= g(r). \end{aligned} \tag{3.4}$$

Multiply by r in (3.4) and note that $(ru)_{rr} = r(u_{rr} + \frac{2}{r}u_r)$ so (3.4) becomes

$$\begin{aligned} (ru)_{tt} - (ru)_{rr} &= -k|u|^{p-1}u \\ (ru)(r,0) &= rf(r) \\ (ru)_t(r,0) &= rg(r). \end{aligned} \tag{3.5}$$

Now, let $w(r,t) = ru(r,t)$. Then (3.5) becomes

$$
\begin{aligned}
w_{tt} - w_{rr} &= -k \left| \frac{w}{r} \right|^{p-1} w \\
w(r,0) &= rf(r) \\
w_t(r,0) &= rg(r).
\end{aligned}
$$

We compute solutions to this reduced problem.

CHAPTER 4

NUMERICAL TECHNIQUE

We consider the reduced initial value problem for $w = ru$

$$\begin{aligned} w_{tt} - w_{rr} &= -k \left| \frac{w}{r} \right|^{p-1} w \\ w(r,0) &= rf(r) \\ w_t(r,0) &= rg(r) \end{aligned} \tag{4.1}$$

where k is a constant and $r \geq 0, t \geq 0$. We will take one of f or g to be zero and spatial boundary condition $w(0,t) = 0$ for all t because $w(r,t) = ru(r,t)$ implies that $w(0,t) = 0u(0,t) = 0$. We will encode this boundary condition in our algorithm.

Remark: Note that this is equivalent to solving the problem

$$\begin{aligned} w_{tt} - w_{xx} &= -k \left| \frac{w}{x} \right|^{p-1} w \\ w(x,0) &= xf(|x|) \\ w_t(x,0) &= xg(|x|) \end{aligned}$$

and then restricting to $x \geq 0$. This results in a solution $w(x,t)$ that is an odd function in x.

We use a centered-time and centered-space difference algorithm. For fixed Δr and Δt, we define

$$w_j^n = w(j\Delta r, n\Delta t).$$

Then

$$\frac{\partial^2 w}{\partial t^2}(j\Delta r, n\Delta t) = \frac{1}{(\Delta t)^2}\left(w_j^{n+1} - 2w_j^n + w_j^{n-1}\right) + O\left((\Delta t)^2\right)$$

$$\frac{\partial^2 w}{\partial r^2}(j\Delta r, n\Delta t) = \frac{1}{(\Delta r)^2}\left(w_{j+1}^n - 2w_j^n + w_{j-1}^n\right) + O\left((\Delta r)^2\right).$$

For simplity, we take $\Delta t = \Delta r$. We substitute into

$$w_{tt} - w_{rr} = -k\left|\frac{w}{r}\right|^{p-1} w$$

and solve for w_j^{n+1}:

$$w_j^{n+1} = -w_j^{n-1} + w_{j+1}^n + w_{j-1}^n + O\left((\Delta t)^4\right) + (\Delta t)^2\left(-k\left|\frac{w_j^n}{j\Delta r}\right|^{p-1} w_j^n\right)$$

for $n = 1, 2, 3, \dots$ and $j = 1, 2, 3, \dots$ We take boundary condition: $w_0^n = 0$ for all n. Define

$$f_j = (j\Delta r) f(j\Delta r) \quad \text{and} \quad g_j = (j\Delta r) g(j\Delta r).$$

Then one initial condition is $w_j^0 = f_j$ for $j = 0, 1, 2, \dots$ To find the discrete version of the initial condition $w_t(x, 0) = xg(|x|)$, we consider the Taylor series

$$\begin{aligned} w_j^1 &= w(j\Delta r, \Delta t) \\ &= w(j\Delta r, 0) + w_t(j\Delta r, 0)\Delta t + w_{tt}(j\Delta r, 0)\frac{(\Delta t)^2}{2} \\ &+ w_{ttt}(j\Delta r, 0)\frac{(\Delta t)^3}{3!} + O\left((\Delta t)^4\right). \end{aligned}$$

Since $w_{tt} = w_{rr} - k\left|\frac{w}{r}\right|^{p-1} w$, we have $w_{ttt} = (w_t)_{rr} - kp\left|\frac{w}{r}\right|^{p-1} w_t$. Therefore

$$w_{tt}(r, 0) = (rf(r))'' - k\left|\frac{rf(r)}{r}\right|^{p-1} rf(r)$$

and

$$w_{ttt}(r, 0) = (rg(r))'' - kp\left|\frac{rf(r)}{r}\right|^{p-1} rg(r).$$

Using the centered space derivatives

$$(rf(r))'' = \frac{1}{(\Delta r)^2}(f_{j+1} - 2f_j + f_{j-1}) + O\left((\Delta r)^2\right)$$

and

$$(rg(r))'' = \frac{1}{(\Delta r)^2}(g_{j+1} - 2g_j + g_{j-1}) + O\left((\Delta r)^2\right),$$

we find

$$\begin{aligned} w_j^1 =& f_j + g_j(\Delta t) \\ &+ \frac{1}{2}(\Delta t)^2\left(\frac{1}{(\Delta r)^2}(f_{j+1} - 2f_j + f_{j-1}) + O\left((\Delta r)^2\right) - k\left|\frac{f_j}{j(\Delta r)}\right|^{p-1} f_j\right) + \\ &\frac{1}{6}(\Delta t)^3\left(\frac{1}{(\Delta r)^2}(g_{j+1} - 2g_j + g_{j-1}) + O\left((\Delta r)^2\right) - kp\left|\frac{f_j}{j\Delta r}\right|^{p-1} g_j\right) \\ &+ O\left((\Delta t)^4\right). \end{aligned}$$

We use no right hand boundary condition since we take initial data with compact support in an interval $[0, L]$, so that at each time $t > 0$, the support of the solution is in $[0, L+t]$. Therefore, for each fixed t-value, we need only compute the solution at a finite number of j-values. That is, if the support of the initial data is in $[0, j\Delta r]$, then the spatial support of w at time $n\Delta t$ will be in $\left[0, (j + n\frac{\Delta t}{\Delta r})\Delta r\right]$.

Remark: If we use this scheme to integrate the linear equation with $(k = 0)$, the result is in fact exact (up to machine precision) because the solution of the wave equation obeys

$$w_j^{n+1} + w_j^{n-1} = w_{j+1}^n + w_{j-1}^n$$

exactly. We get this from D'Alembert's formula for the solution in the linear case:

$$2w(r,t) = w(r-t, 0) + w(r+t, 0) + \int_{r-t}^{r+t} w_t(y, 0)dy.$$

CHAPTER 5

NUMERICAL RESULTS

We apply the numerical technique of Chapter 4 to numerically solve problem (4.1) in the cases of interest outlined in Chapter 2:

(Case 1)

$$p = 1 + \frac{2}{n} = 1 + \frac{2}{3} = \frac{5}{3} \quad \text{and} \quad g(r) = 0$$

(Case 2)

$$p = 1 + \frac{2}{n-1} = 1 + \frac{1}{1} = 2 \quad \text{and} \quad f(r) = 0$$

For the nonvanishing initial data, Colombeau's formalism calls for a function in $\mathcal{A}_1$. Since the smoothness of the initial data beyond C^2 will not affect numerical solutions, we take the initial data to be a C^2-function ψ with support in the unit ball, integral equal to 1, and first moment equal to zero. Such a function on $\mathbf{R}^3$ is

$$\psi(\vec{x}) = \psi(|\vec{x}|) = \begin{cases} c(1 - |\vec{x}|^2)^3 & \text{if} \quad |\vec{x}| \le 1 \\ 0 & \text{if} \quad |\vec{x}| \ge 1 \end{cases} \tag{5.1}$$

where the constant c is determined by the condition

$$\int_{\mathbf{R}^3} \psi(\vec{x}) d^3x = 1 :$$

$$1 = 4\pi \int_0^1 c(1 - r^2)^3 r^2 dr \Longrightarrow c = \frac{315}{64\pi}$$

We note $\int_{\mathbf{R}^3} \vec{x}\psi(\vec{x}) d^3x = 0$ because ψ depends only on $|\vec{x}|$, so the integral is odd. Thus we numerically solve the following equations.

(Case 1)

$$
\begin{aligned}
w_{tt} - w_{rr} &= -k \left| \frac{w}{r} \right|^{\frac{2}{3}} w \\
w(r,0) &= r\psi(r) \\
w_t(r,0) &= 0 \\
w(0,t) &= 0
\end{aligned}
\tag{5.2}
$$

(Case 2)

$$
\begin{aligned}
w_{tt} - w_{rr} &= -k \left| \frac{w}{r} \right| w \\
w(r,0) &= 0 \\
w_t(r,0) &= r\psi(r) \\
w(0,t) &= 0
\end{aligned}
\tag{5.3}
$$

for $r \geq 0$, $t \geq 0$.

Since the support of ψ is in [0,1], we begin by taking $\Delta r = \frac{1}{10}$. We solve the problems both with $k = 0$ and $k = 1$ to allow us check our numerical solutions in the linear case. Once we have computed a solution to a large t-value (t on the order of 100), we investigate the limits (2.4) and (2.14) as follows: We have $v(\vec{x}, t) = \dfrac{w(|\vec{x}|, t)}{|\vec{x}|}$, so

(Case 1)

$$
u_\epsilon(\vec{x}, t) = \frac{1}{\epsilon^3} v\left(\frac{\vec{x}}{\epsilon}, \frac{t}{\epsilon}\right) = \frac{1}{\epsilon^3} \frac{\epsilon}{|\vec{x}|} w\left(\frac{|\vec{x}|}{\epsilon}, \frac{t}{\epsilon}\right) \tag{5.4}
$$

(Case 2)

$$
u_\epsilon(\vec{x}, t) = \frac{1}{\epsilon^2} v\left(\frac{\vec{x}}{\epsilon}, \frac{t}{\epsilon}\right) = \frac{1}{\epsilon^2} \frac{\epsilon}{|\vec{x}|} w\left(\frac{|\vec{x}|}{\epsilon}, \frac{t}{\epsilon}\right) \tag{5.5}
$$

For $\phi \in \mathcal{D}(\mathbf{R}^{3+1})$, define $\phi_s \in \mathcal{D}(\mathbf{R}^{1+1})$ by

$$
\phi_s(r, s) = \int_{S_1(0)} \phi(r, \vec{x}, t) d^2\widehat{x}
$$

where $\vec{x} = r\hat{x}$ are polar coordinates for $\vec{x}$ and the integration is over the unit sphere. Since $u_\epsilon(\vec{x}, t)$ is spherically symmetric in $\vec{x}$ for all times t, we have

(Case 1)

$$\int_{\mathbf{R}^{3+1}} u_\epsilon(\vec{x},t)\phi(\vec{x},t)d^3x dt = \frac{1}{\epsilon^2}\int_{-\infty}^{\infty} dt \int_0^\infty r^2 dr \frac{1}{r} w\left(\frac{r}{\epsilon}, \frac{t}{\epsilon}\right)\phi_s(r,t) \tag{5.6}$$

(Case 2)

$$\int_{\mathbf{R}^{3+1}} u_\epsilon(\vec{x},t)\phi(\vec{x},t)d^3x dt = \frac{1}{\epsilon}\int_{-\infty}^{\infty} dt \int_0^\infty r^2 dr \frac{1}{r} w\left(\frac{r}{\epsilon}, \frac{t}{\epsilon}\right)\phi_s(r,t) \tag{5.7}$$

Making the change of variables $r' = \dfrac{r}{\epsilon}$ and $t' = \dfrac{t}{\epsilon}$ and dropping the primes, we are led to consider the limits as $\epsilon \longrightarrow 0$ of

(Case 1)

$$\int_{\mathbf{R}^{3+1}} u_\epsilon(\vec{x},t)\phi(\vec{x},t)d^3x dt = \epsilon\int_{-\infty}^{\infty} dt \int_0^\infty r dr w(r,t)\phi_s(\epsilon r \epsilon t) \tag{5.8}$$

(Case 2)

$$\int_{\mathbf{R}^{3+1}} u_\epsilon(\vec{x},t)\phi(\vec{x},t)d^3x dt = \epsilon^2\int_{-\infty}^{\infty} dt \int_0^\infty r dr w(r,t)\phi_s(\epsilon r, \epsilon t) \tag{5.9}$$

To get indication of whether or not these quantities converges as $\epsilon \longrightarrow 0$, we choose a specific test function ϕ_s and approximate the integrals by sums. Although ϕ and ϕ_s are in C^∞, the numerical results will not depend on the smoothness of ϕ beyond continuity. So we take functions ϕ_s of the form

$$\phi_s(r,s) = \chi(a,b;r,t)P(c;r,t) \tag{5.10}$$

where $\chi(a,b;\cdot,\cdot)$ is the characteristic function of the rectangle $[0,a]\times[0,b]$, that is

$$\chi(a,b;r,t) = \begin{cases} 1 & \text{if} \quad 0 \le r \le a \quad \text{and} \quad 0 \le t \le b \\ 0 & \text{otherwise} \end{cases}$$

and where $P(c;r,t)$ is a homogeneous polynomial of degree c in r and t. In particular, we have

$$\phi_s(\epsilon r, \epsilon t) = \chi(a,b;\epsilon r, \epsilon t)P(c;\epsilon r, \epsilon t) = \chi\left(\frac{a}{\epsilon}, \frac{b}{\epsilon}; r, t\right)\epsilon^c P(c;r,t)$$

so, we consider

(Case 1)

$$\int_{\mathbf{R}^{3+1}} u_\epsilon(\vec{x},t)\phi(\vec{x},t)d^3x dt = \epsilon^{c+1}\int_0^{\frac{b}{\epsilon}} dt \int_0^{\frac{a}{\epsilon}} r dr w(r,t) P(c;r,t) \tag{5.11}$$

(Case 2)

$$\int_{\mathbf{R}^{3+1}} u_\epsilon(\vec{x},t)\phi(\vec{x},t)d^3x dt = \epsilon^{c+2}\int_0^{\frac{b}{\epsilon}} dt \int_0^{\frac{a}{\epsilon}} r dr w(r,t) P(c;r,t) \tag{5.12}$$

For convenience, we choose $a = 1$ and $b = 1$. Since we compute the values of $w(r,t)$ on a grid $\{(j\Delta x, n\Delta t) \mid j = 0,1,2,\ldots$ and $n = 0,1,2,\ldots\}$ we may approximate these integrals by sums:

$$\int_0^{\frac{b}{\epsilon}} dt \int_0^{\frac{a}{\epsilon}} r dr w(r,t) P(r,t) \approx (\Delta t)(\Delta x)^2 \sum_{n=0}^{M}\sum_{j=0}^{L} j w_j^n P(j\Delta x, n\Delta t)$$

where $\epsilon = \left(\frac{b}{\Delta t}\right)\frac{1}{M}$ and $L = INT[\left(\frac{a}{b}\right)\left(\frac{\Delta t}{\Delta r}\right)M]$.

Convergence of the sums

(Case 1)

$$A_1(M) \equiv \left(\frac{b^{c+1}(\Delta x)^2}{(\Delta t)^c}\right)\frac{1}{M^{c+1}}\sum_{n=0}^{M}\sum_{j=0}^{L} j w_j^n P(j\Delta x, n\Delta t) \tag{5.13}$$

(Case 2)

$$A_2(M) \equiv \left(\frac{b^{c+2}(\Delta x)^2}{(\Delta t)^{c+1}}\right)\frac{1}{M^{c+2}}\sum_{n=0}^{M}\sum_{j=0}^{L} j w_j^n P(j\Delta x, n\Delta t) \tag{5.14}$$

as $M \longrightarrow \infty$ is equivalent to convergence of (2.4) and (2.14), respectively.

We have computed numerical solutions to the initial value problems (5.2) and (5.3) with initial conditions given by (5.1) both in the linear case $k = 0$ and the nonlinear case $k = 1$. We find that the solutions to the nonlinear problems are very similar in general features to solutions of the corresponding linear problems. The major difference between linear and nonlinear results occurs for small values of r, where the nonlinear term $-\left|\frac{w}{r}\right|^{p-1} w$ is not negligible. In particular, the solutions to the linear problems vanish in the region $\{(r,t)|t \geq 1+r\}$, but the solutions to the nonlinear problems are nonzero in that region. This difference is responsible for the convergence properties of the sums (5.13) and (5.14) in the nonlinear versus linear cases.

Choice of test functions.

(Case 1)

For $t \geq 1$, the solution to linear problem (5.2) with $k = 0$ consists of a translated copy of the function $r\psi(r)$. Specifically, $w(r,t) = (r-t)\psi(r-t)$ for $t \geq 1$. Because $r\psi(r) > 0$ for $0 < r < 1$ and $r\psi(r) < 0$ for $-1 < r < 0$, the solution w to the linear Problem (5.2) will be positive in the strip $r-1 < t < r$ and will be negative in the strip $r < t < r+1$ for $t \geq 1$. We have found that solutions to the nonlinear problem (5.2) with $k = 1$ are similar to those of the linear problem. We choose a test function ϕ to avoid accidental cancellations in the integral $u_\epsilon[\phi]$. We take the polynomial $p(r,t) \equiv r - t$ in (5.10) in Case 1 so that ϕ has the same sign as the solution to the linear problem (5.2).

(Case 2)

The solution to the linear problem (5.3) with $k = 0$ is everywhere positive. In this case, we take the polynomial $p(r,t) \equiv t$ in (5.10) (We choose degree 1 for similarity to Case 1). Again, ϕ has the same sign as the solution to the linear problem (5.3).

Summary of results.

Figure 1 shows the sum $A_1(M)$ for the linear problem for $1 \leq M \leq 800$. This quantity is apparently asymtotically constant at approximate value 4×10^{-4}.

Our computed values show monotonic increase, with value $A_1(800) \approx 3.96 \times 10^{-4}$.

To check the accuracy of our computations, we have substituted the explicit solution formula

$$w(r,t) = \frac{1}{2}\left((r-t)\psi(r-t) + (r+t)\psi(r+t)\right).$$

for the linear problem (5.2) into (5.6). With our choice

$$\phi_s(r,s) = \chi(\Delta x, \Delta t; r, s)(r-t),$$

we can evaluate the integral explicitly.

Taking the limit $\epsilon \longrightarrow 0$ with $\Delta x = \Delta t = \frac{1}{10}$, we find

$$u_\epsilon[\phi] \longrightarrow \frac{1}{800\pi} \approx 3.979 \times 10^{-4}$$

which gives excellent agreement with our computed values of $A_1(\frac{1}{\epsilon})$. This asymptotic value is the value of $lim_{\epsilon \longrightarrow 0} u_\epsilon[\phi] = u[\phi]$ where u is the distribution solution to the linear problem (2.1) with $f(u) = 0$.

Figure 2 shows the sum $A_1(M)$ for the nonlinear problem, as a function of M for $1 \leq M \leq 800$. There is no numerical evidence to suggest that in the nonlinear case $A_1(M)$ converges as M tends to infinity.

Figure 3 shows the sum $A_2(M)$ for the linear problem (5.3), as a function of M for $1 \leq M \leq 800$. This quantity is apparently asymptotically constant, with value approximate 2.6×10^{-5}. Because $A_2(M)$ approximates $u_{(\frac{1}{M})}[\phi]$, we expect in the linear case that $A_2(M)$ approaches the constant value

$$lim_{\epsilon \longrightarrow 0} u_\epsilon[\phi] = u[\phi]$$

where u is the distribution solution to the linear problem (2.2) with $f(u) = 0$. Our numerical results confirm these expectations.

Figure 4 shows the sum $A_2(M)$ for the nonlinear problem, as a function of M for $1 \leq M \leq 800$. There is no numerical evidence to suggest that in the nonlinear case $A_2(M)$ converges as M tends to infinity. We next investigate our computed solutions in more detail, to illustrate the mechanisms for the divergence of $A_1(M)$ and $A_2(M)$ in the nonlinear problems.

Figure 5 shows the solution $w(r,t)$ to problem (5.2) in the linear case (k=0). The plotted quantity is $w_j^n \equiv w(j\Delta x, n\Delta t)$ for $0 \leq j \leq 30$ and $0 \leq n \leq 20$. This computed solution is exact, up to machine precision, as mentioned earlier. This solution vanishes on the lines $t = r - 1, t = r$ and $t = r + 1$.

Figure 6 shows snapshots of the solution $w(r,t)$ to the linear problem in Case 1. Shown are $w_j^n \equiv w(j\Delta x, n\Delta t)$ as functions of $j (0 \leq j \leq 30)$ for the times with $n = 0, 1, 2, \ldots, 14$.

Figure 7 shows the solution $w(r,t)$ to problem (5.2) in the nonlinear case (k=1). The plotted quantity is $w_j^n \equiv w(j\Delta x, n\Delta t)$ for $0 \leq j \leq 30$ and $0 \leq n \leq 20$. This computed solution is very close to the solution to the linear problem.

We can see a slight difference between the linear and nonlinear cases by comparing Figure 6 and Figure 8. Figure 8 shows w_j^n as a function of j for the time steps $n = 0, 1, 2, \ldots 14$ in the nonlinear case. Note that the traveling wave shape is slightly different, and that there is a small nonzero part of the solution in the region $t > r + 1$ seen in Figure 8 that is absent in Figure 6.

To see the difference between the linear and nonlinear prblems in Case 1 more clearly, we plot the negative part of $r(r-t)w(r,t)$ for the solution w to the nonlinear problem (5.2). The quantity $j(n-j)w_j^n$ appears in the summand of $A_1(M)$ and is responsible for the fact that $A_1(M)$ is not asymptotically constant as M tends to infinity in the nonlinear problem. The negative part of $r(r-t)w(r,t)$ is identically zero for the solution w to the linear problem (5.2).

Figure 9 shows the negative part of $r(r-t)w(r,t)$ for $0 \leq t \leq 200\Delta t$ and

$0 \leq r \leq 211\Delta r$ in the nonlinear problem. The integral of this quantity is responsible for the difference between the values of $A_1(200)$ in the linear and nonlinear cases.

We next consider the Case 2 problem (5.3).

Figure 10 shows the solution to the linear problem (5.3) (k=0).

Figure 11 shows the solution to the nonlinear problem (5.3) (k=1). Again, the solution are similar in structure.

Figure 12 shows snapshots of the solution to the linear problem in Case 2. Shown are w_j^n as functions of j $(0 \leq j \leq 30)$ for the times $n = 0, 1, 2, \ldots, 14$

Figure 13 shows snapshots of the solution to the nonlinear problem in Case 2, for the same j and n values as Figure 12. Note the small negative part of w_j^n that develops as n increases. This part of w_j^n is responsible for the divergence of $u_\epsilon[\phi]$ that we observe in Case 2.

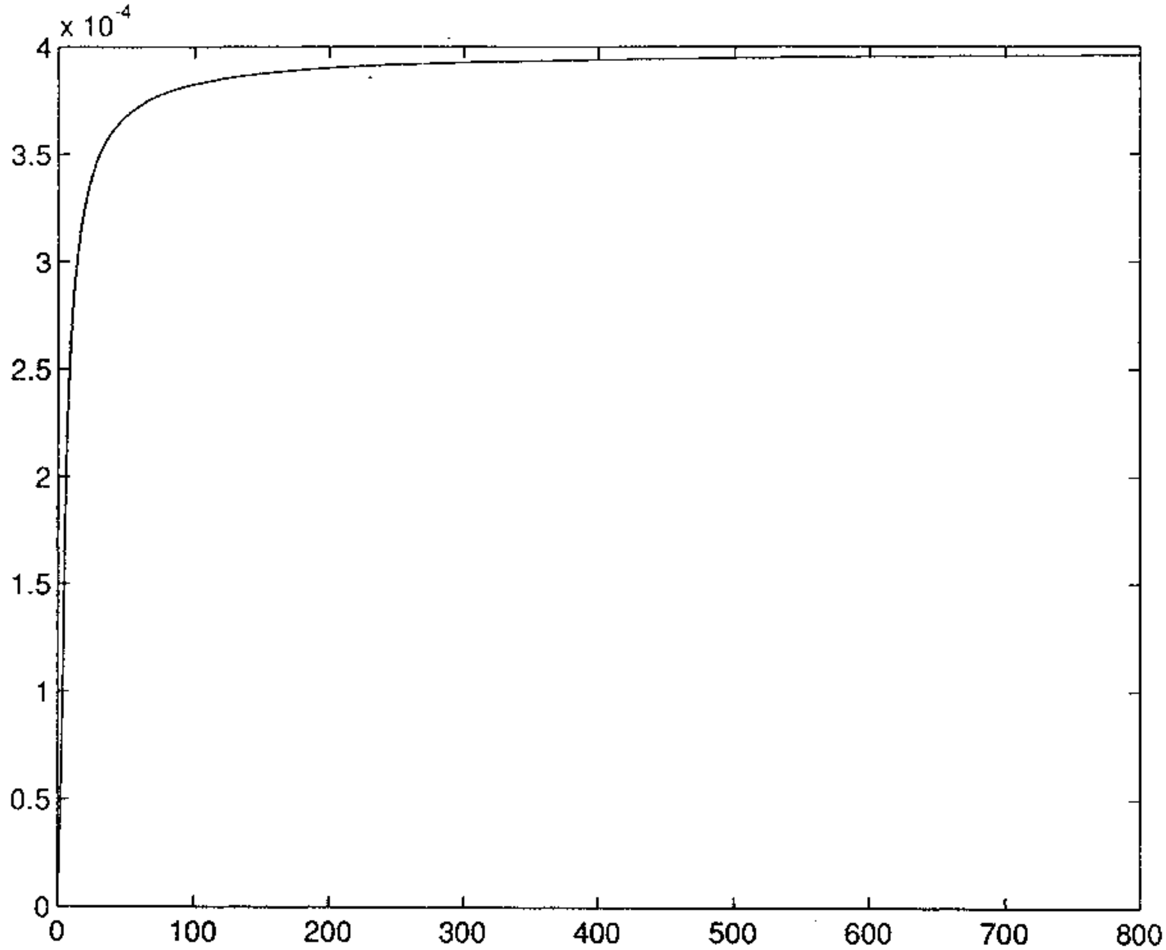
Figure 14 and Figure 15 show the negative part of $w(r,t)$ for the solution to the Case 2 nonlinear problem. This quantity is identically zero for the solution to the linear problem.

Figure 16 shows the solution $w(r,t)$ for large t and large r. Note that the solution is negative in the region $r < t - 1$. The solution to the linear problem is zero in that region.

Conclusions from numerical studies.

The divergence as $\epsilon \longrightarrow 0$ the action on test functions of generalized function solution representatives is due to a very slight difference between solutions to the nonlinear and linear problems, over the large spacetime region. This divergence comes from solution values in the region $|t| > |r| + 1$, where the linear solution is zero. The divergence appears to grow as a power of $\left(\frac{1}{\epsilon}\right)$ as ϵ tends to 0.

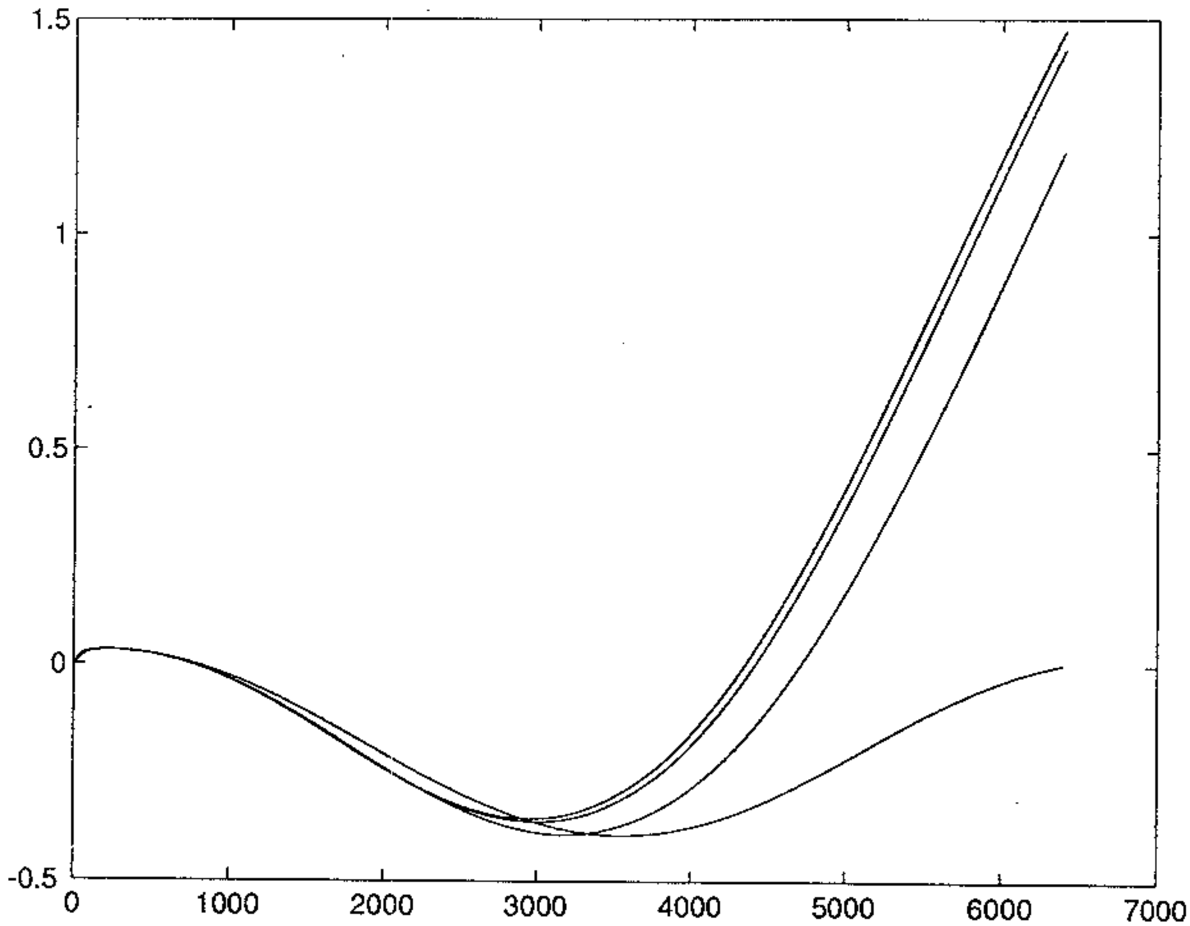
In summary, we have shown that for a test function ϕ, the quanties $u_\epsilon[\phi]$ apparently diverge as $\epsilon \longrightarrow 0$, in both Case 1 and Case 2. This implies that even our mildly singular problems have generalized function solutions that are more singular than distributions. Because we chose these problems as among the most likely to have (ordinary) distribution solutions to nonlinear partial differential equations with singular initial data will never have distribution actions, but will be more singular, and correspondingly more complicated, than distributions. This implies that generalized function solutions to nonlinear partial differential equations with singular initial data cannot be expected to have any interpretation as conventional solutions. The fact that we must view generalized function solutions as equivalence classes of divergent representatives greatly limits the practical utility of generalized functions for solving nonlinear partial differential equations.



(Figure 1)

This shows the action of u_ϵ on ϕ for the linear problem (k=0) in (Case 1),

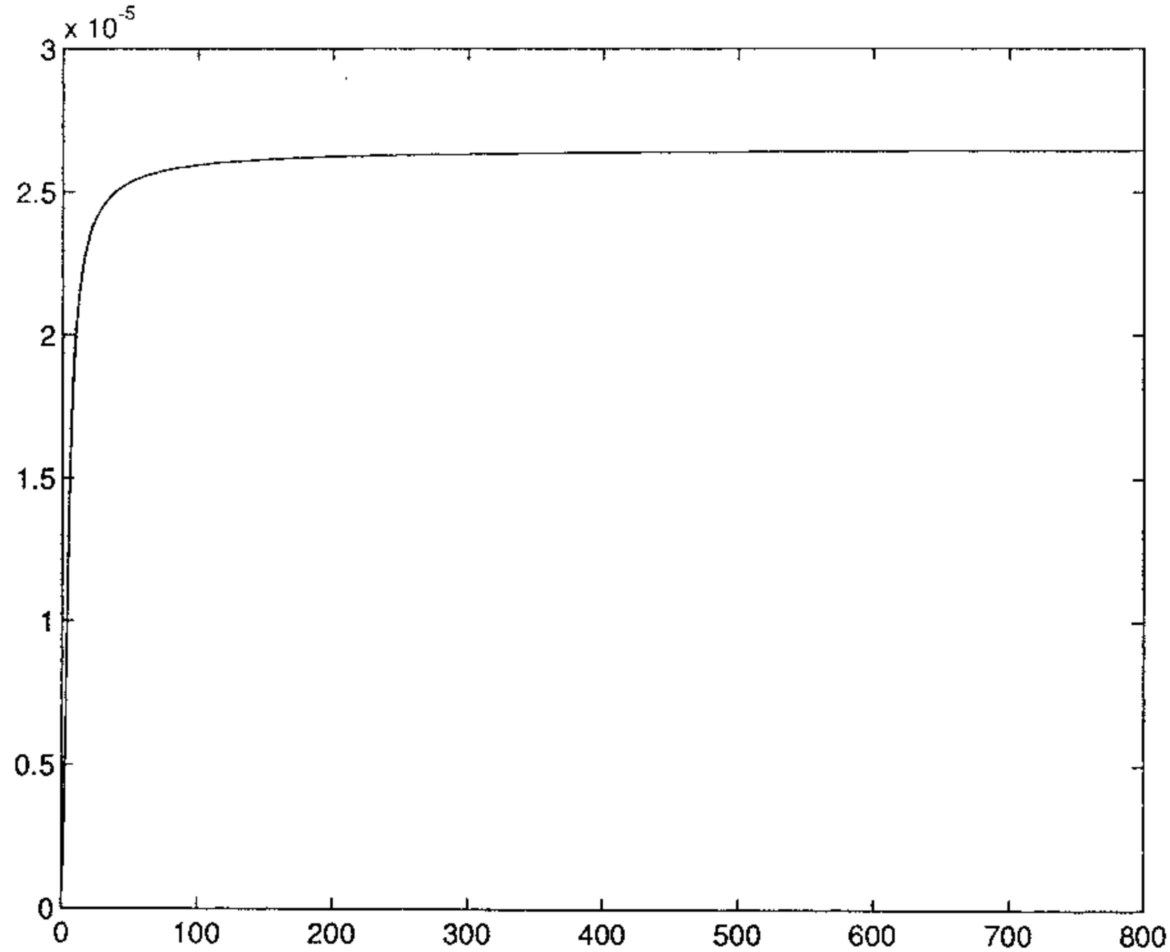
$u_\epsilon[\phi]$ where $\epsilon = \dfrac{10}{M}$, $1 \leq M \leq 800$ and $\phi = (t - x)$ on $[0,1] \times [0,1]$



(Figure 2)

This shows the action of u_ϵ on ϕ for the nonlinear problem in (Case 1),

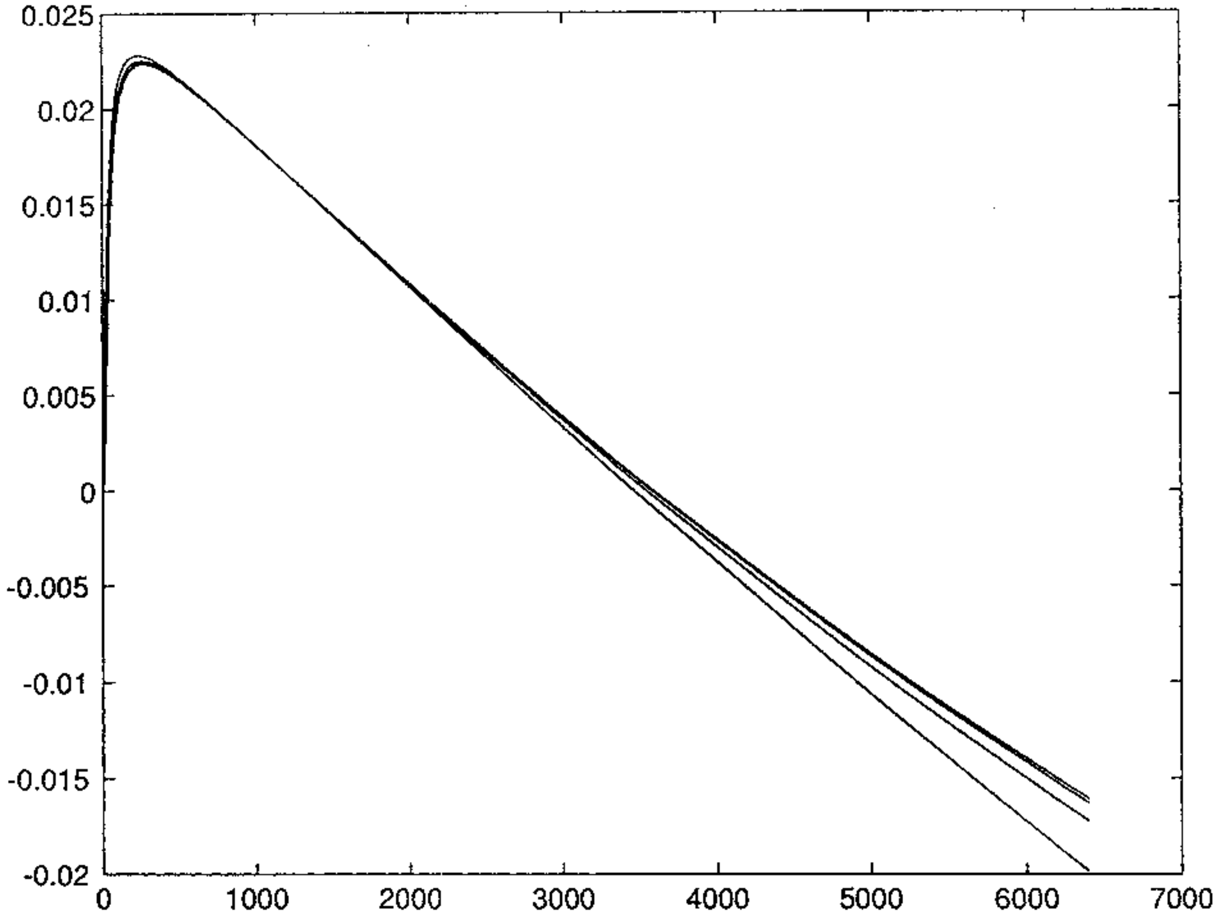
$$u_\epsilon[\phi] \quad \text{where} \quad \Delta t = \frac{1}{10}, \quad \frac{1}{20}, \quad \frac{1}{40}, \quad \frac{1}{80} \quad \text{and} \quad \epsilon = \frac{1}{M\Delta t}, \quad 1 \le M \le \frac{80}{\Delta t}$$



(Figure 3)

This shows the action of u_ϵ on ϕ for the linear problem (k=0) in (Case 2),

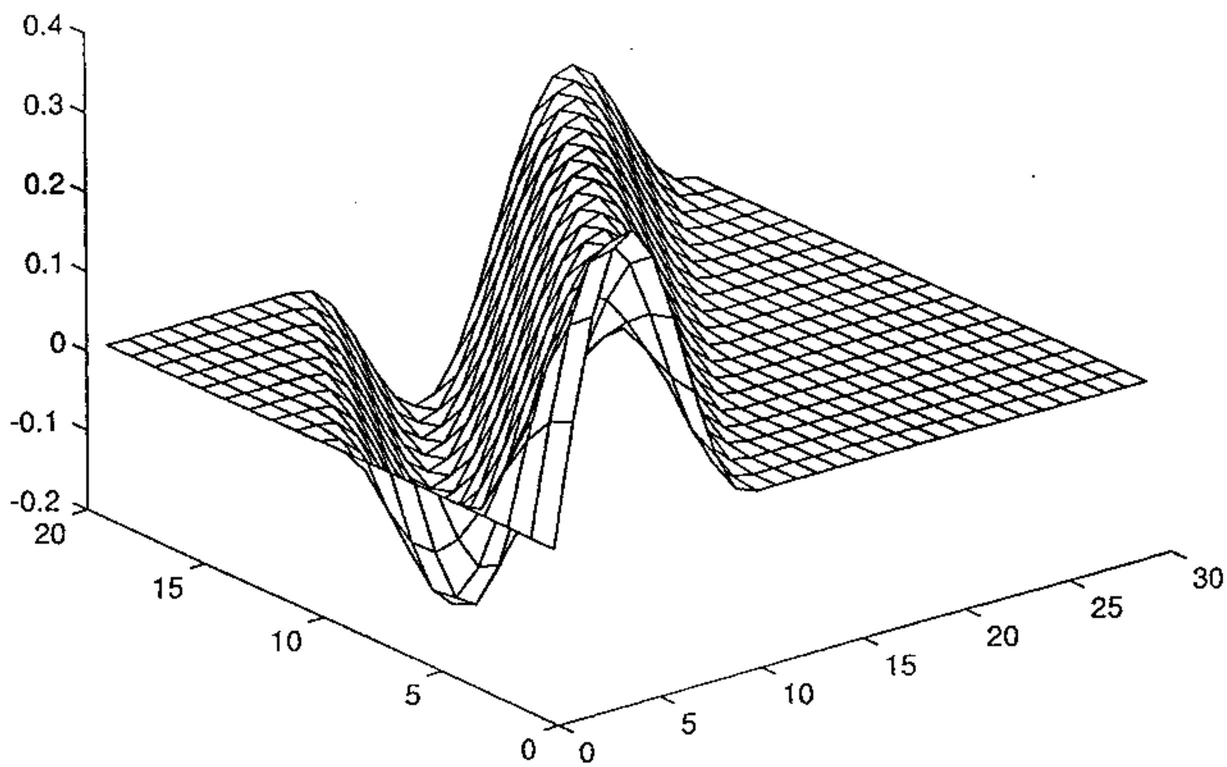
$$u_\epsilon[\phi] \quad \text{where} \quad \epsilon = \frac{10}{M}, \quad 1 \leq M \leq 800 \quad \text{and} \quad \phi = t \quad \text{on} \quad [0,1] \times [0,1]$$



(Figure 4)

This shows the action of u_ϵ on ϕ for the nonlinear problem in (Case 2),

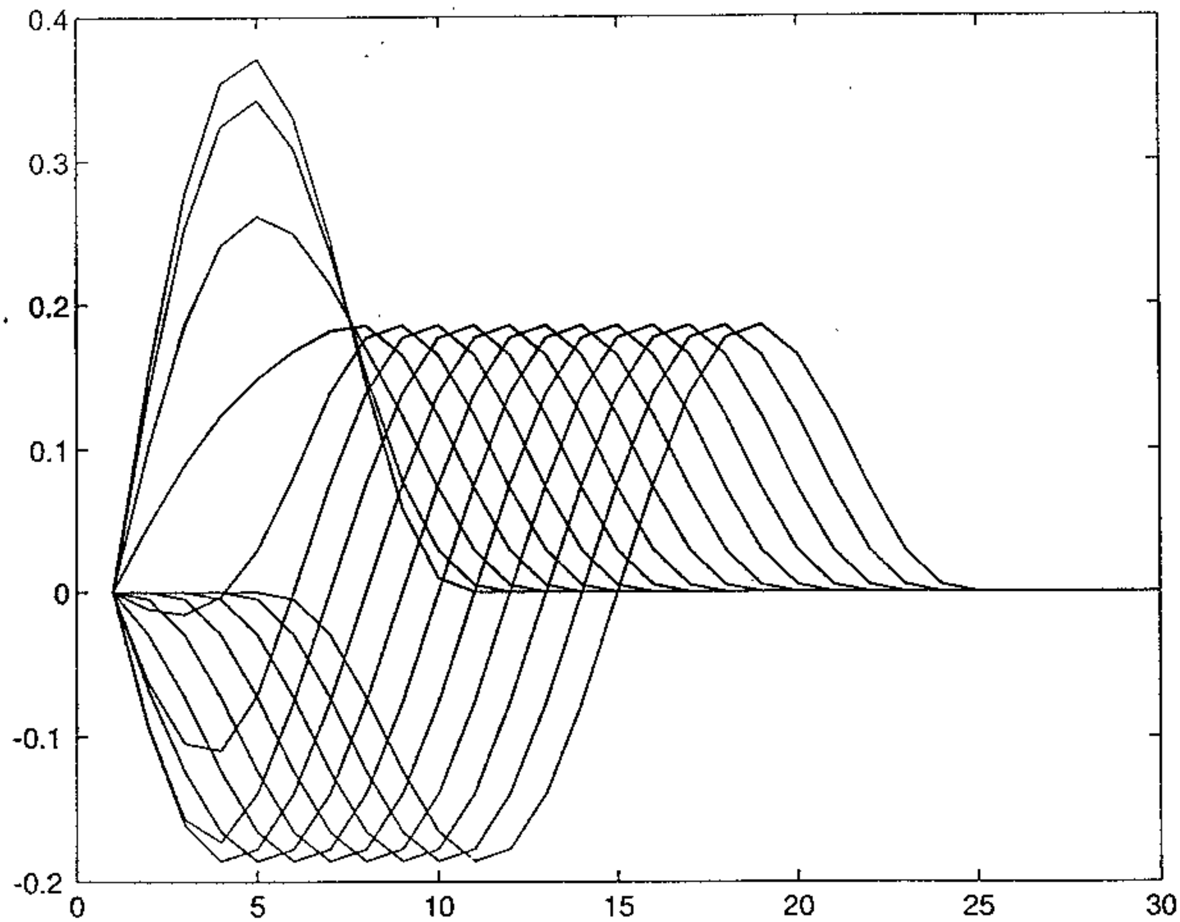
$$u_\epsilon[\phi] \quad \text{where} \quad \Delta t = \frac{1}{10}, \quad \frac{1}{20}, \quad \frac{1}{40}, \quad \frac{1}{80} \quad \text{and} \quad \epsilon = \frac{1}{M\Delta t}, \quad 1 \leq M \leq \frac{80}{\Delta t}$$



(Figure 5)

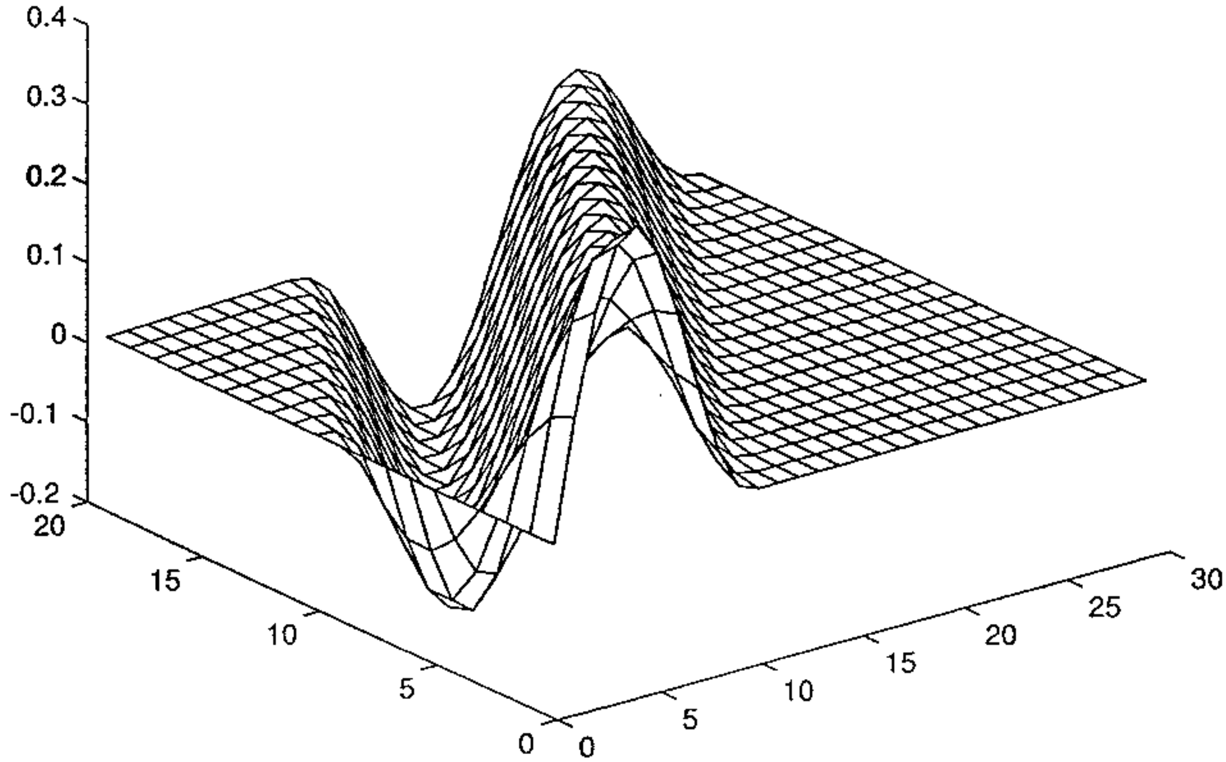
This shows $w(r,t)$ in the linear problem (k=0) in (Case 1),

$$w_j^n \equiv w(j\Delta x, n\Delta t) \quad \text{for} \quad 0 \le j \le 30, \quad 0 \le n \le 20$$



(Figure 6)

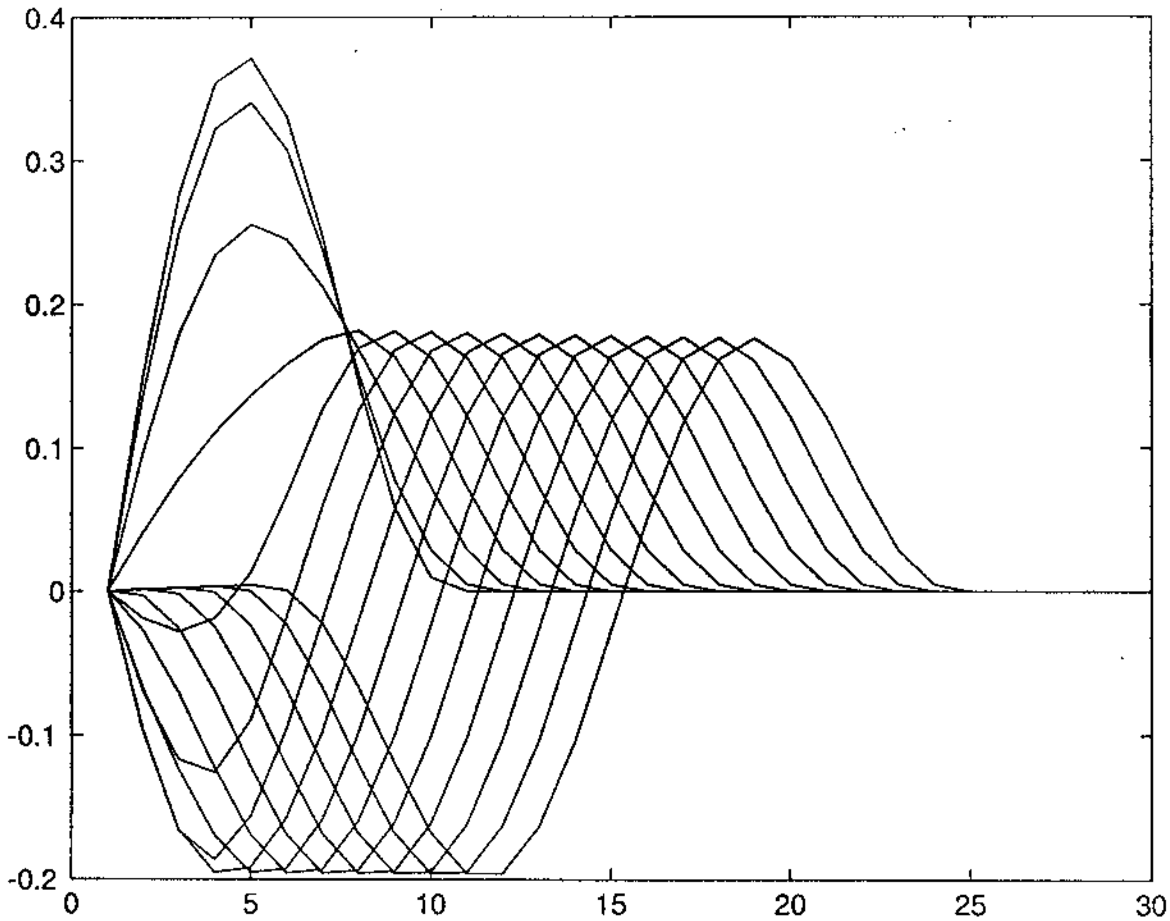
Snapshots of the solution $w(r,t)$ to the linear problem (k=0) in (Case 1)



(Figure 7)

This shows $w(r,t)$ in the nonlinear problem in (Case 1),

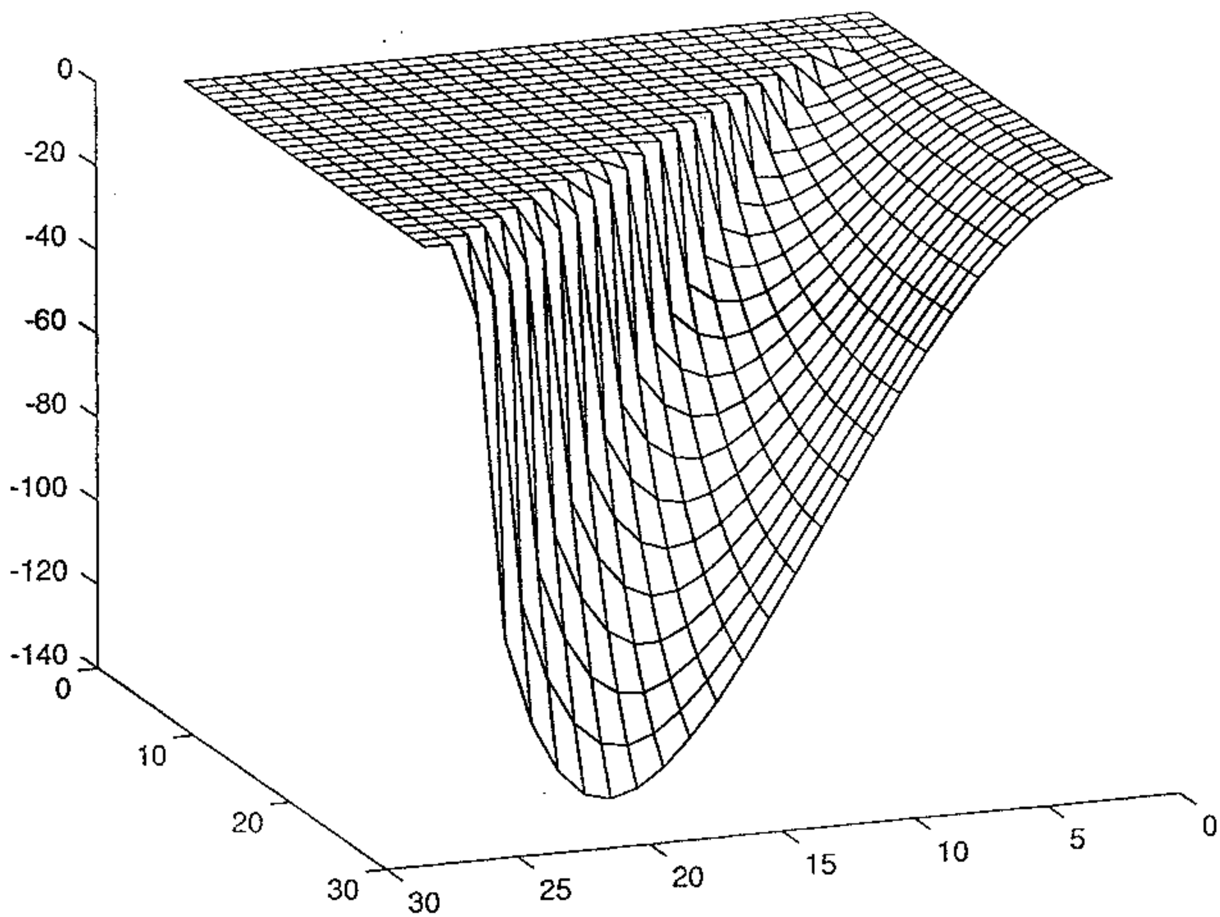
$$w_j^n \equiv w(j\Delta x, n\Delta t) \quad \text{for} \quad 0 \le j \le 30, \quad 0 \le n \le 20$$



(Figure 8)

Snapshots of the solution $w(r,t)$ to the nonlinear problem in (Case 1) for

$$0 \le t \le 14\Delta t$$

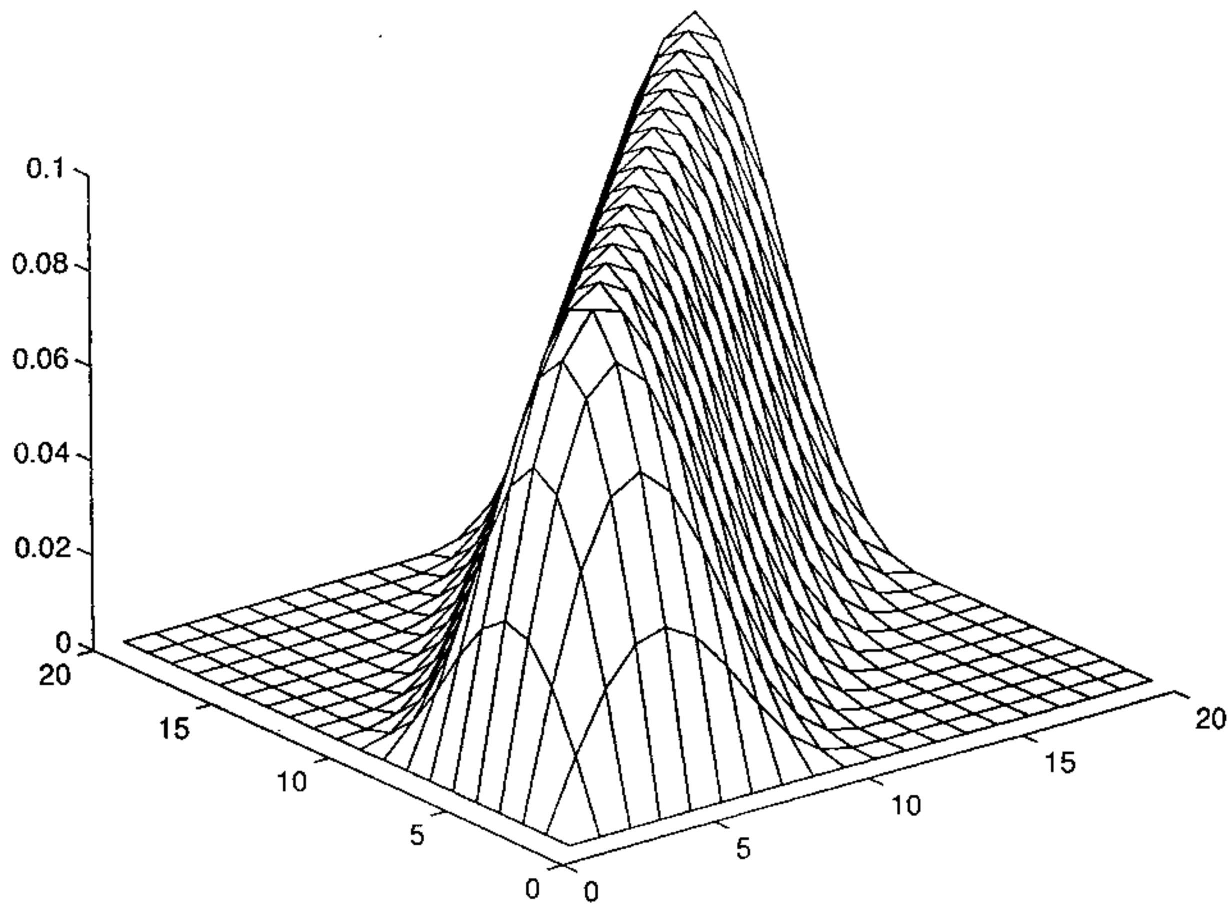


(Figure 9)

This shows the negative part of nonlinear (Case 1).

Explicity, this is a picture of $\frac{1}{2}[j(n-j)w_j^n - |j(n-jw_j^n|]$ for $0 \leq n \leq 200$,

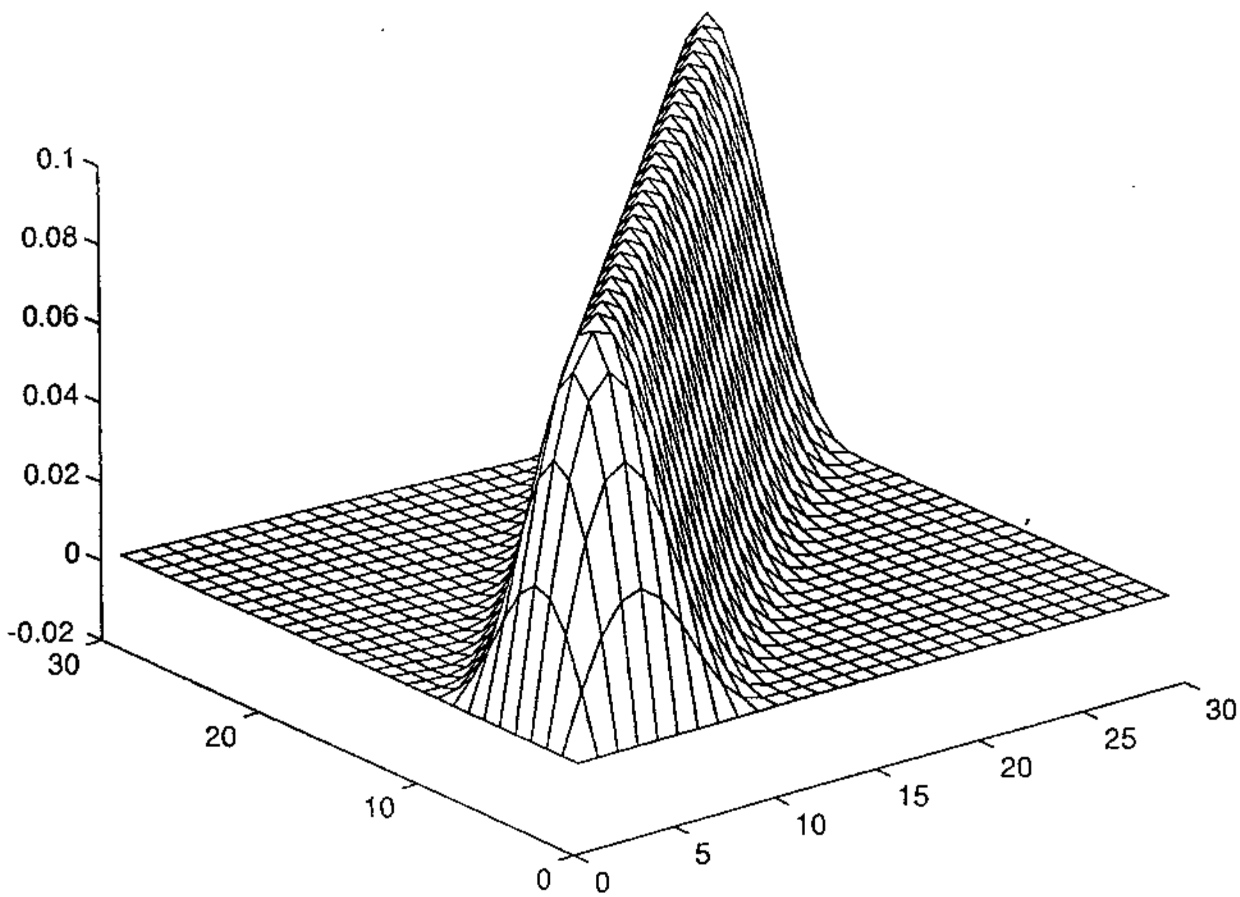
$0 \leq j \leq 211$



(Figure 10)

This shows $w(r,t)$ in the linear problem in (Case 2),

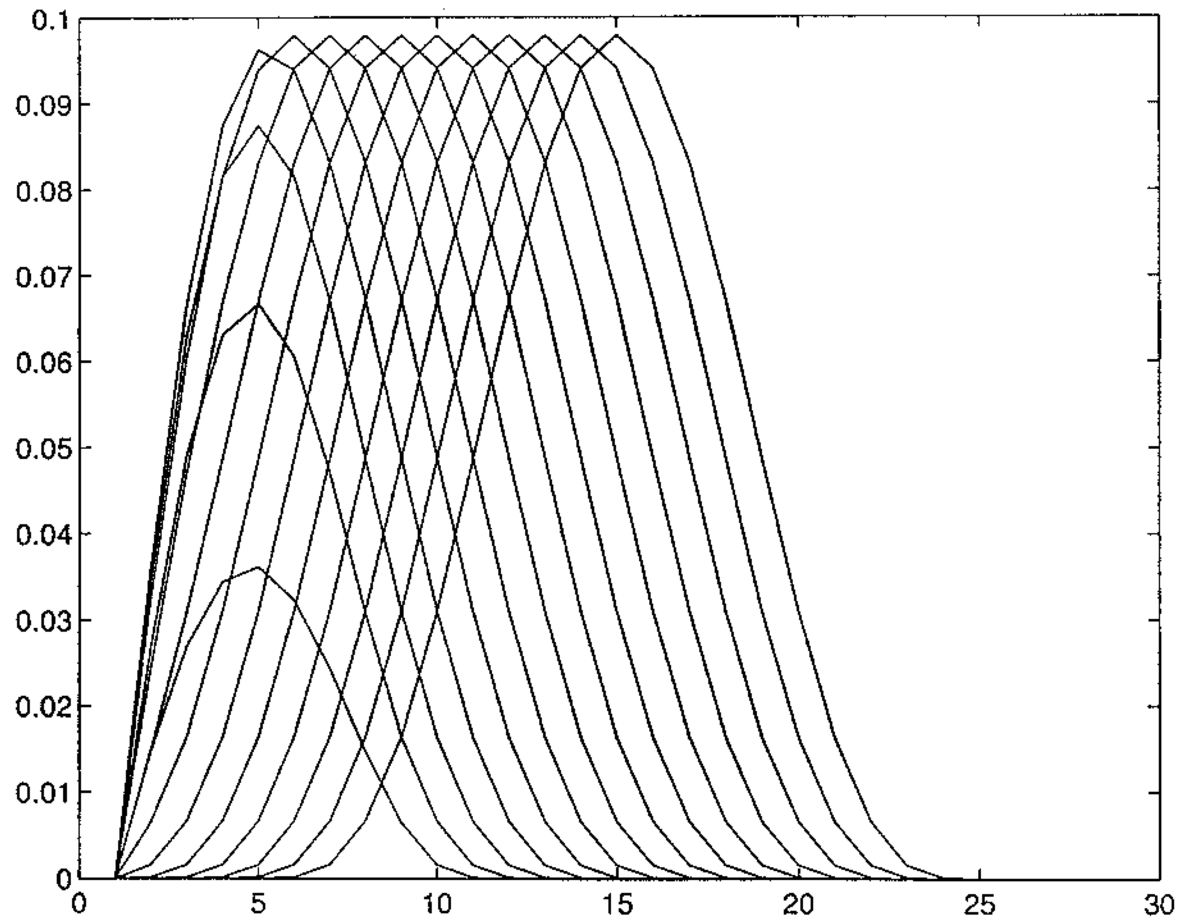
$$w_j^n \equiv w(j\Delta x, n\Delta t) \quad \text{for} \quad 0 \leq j \leq 20, \quad 0 \leq n \leq 20$$



(Figure 11)

This shows $w(r,t)$ in the nonlinear problem (k=1) in (Case 2),

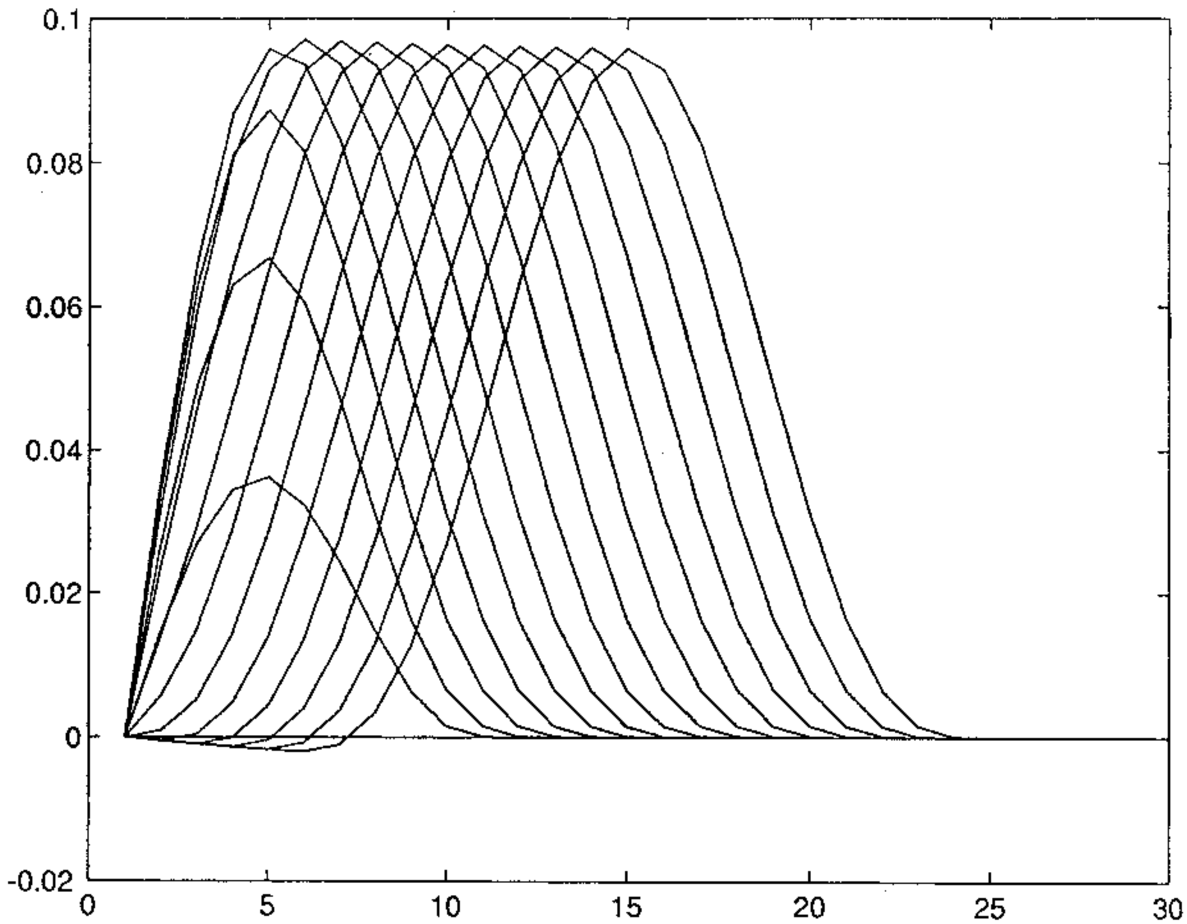
$$w_j^n \equiv w(j\Delta x, n\Delta t) \quad \text{for} \quad 0 \le j \le 30, \quad 0 \le n \le 30$$



(Figure 12)

Snapshots of the solution $w(r,t)$ to the linear problem (k=0) in (Case 2)

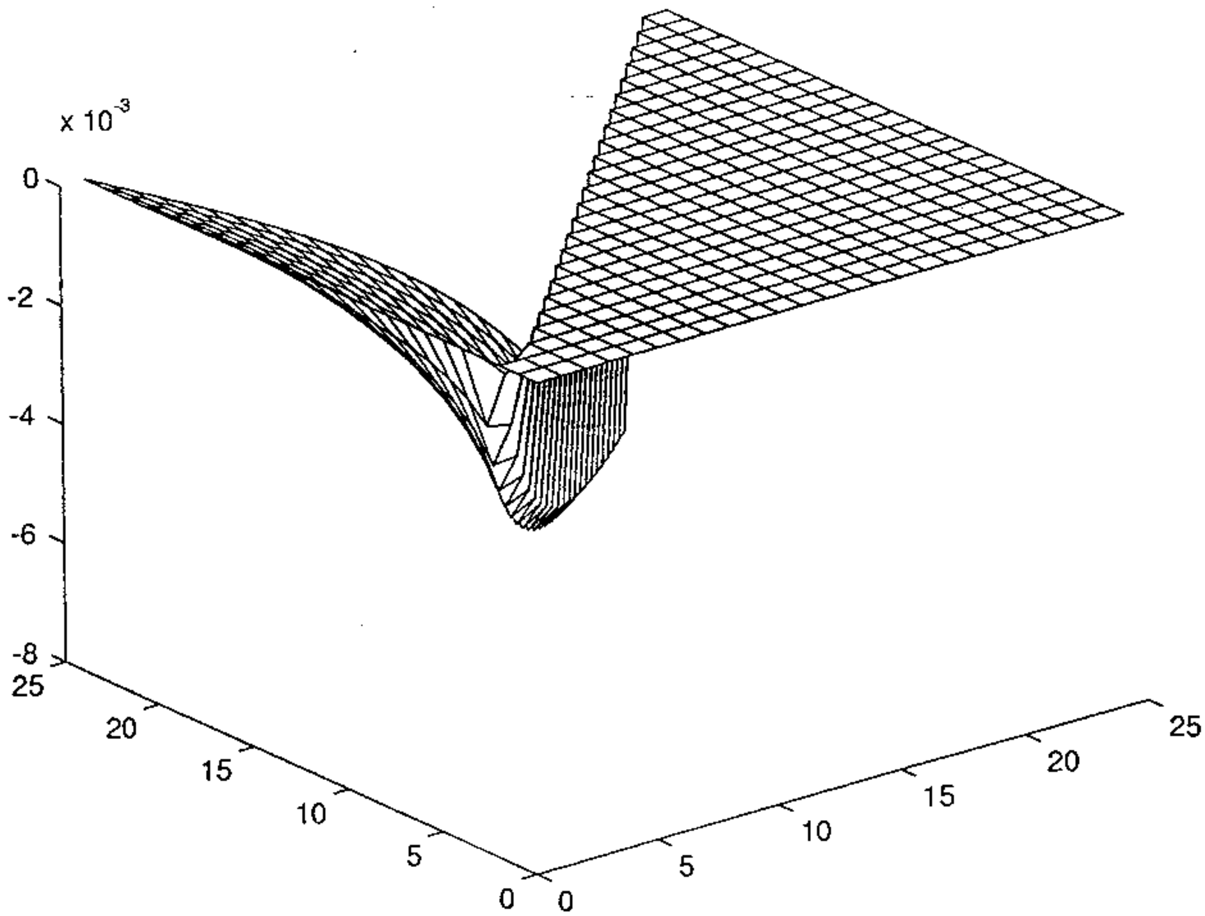
$$0 \leq t \leq 14\Delta t$$



(Figure 13)

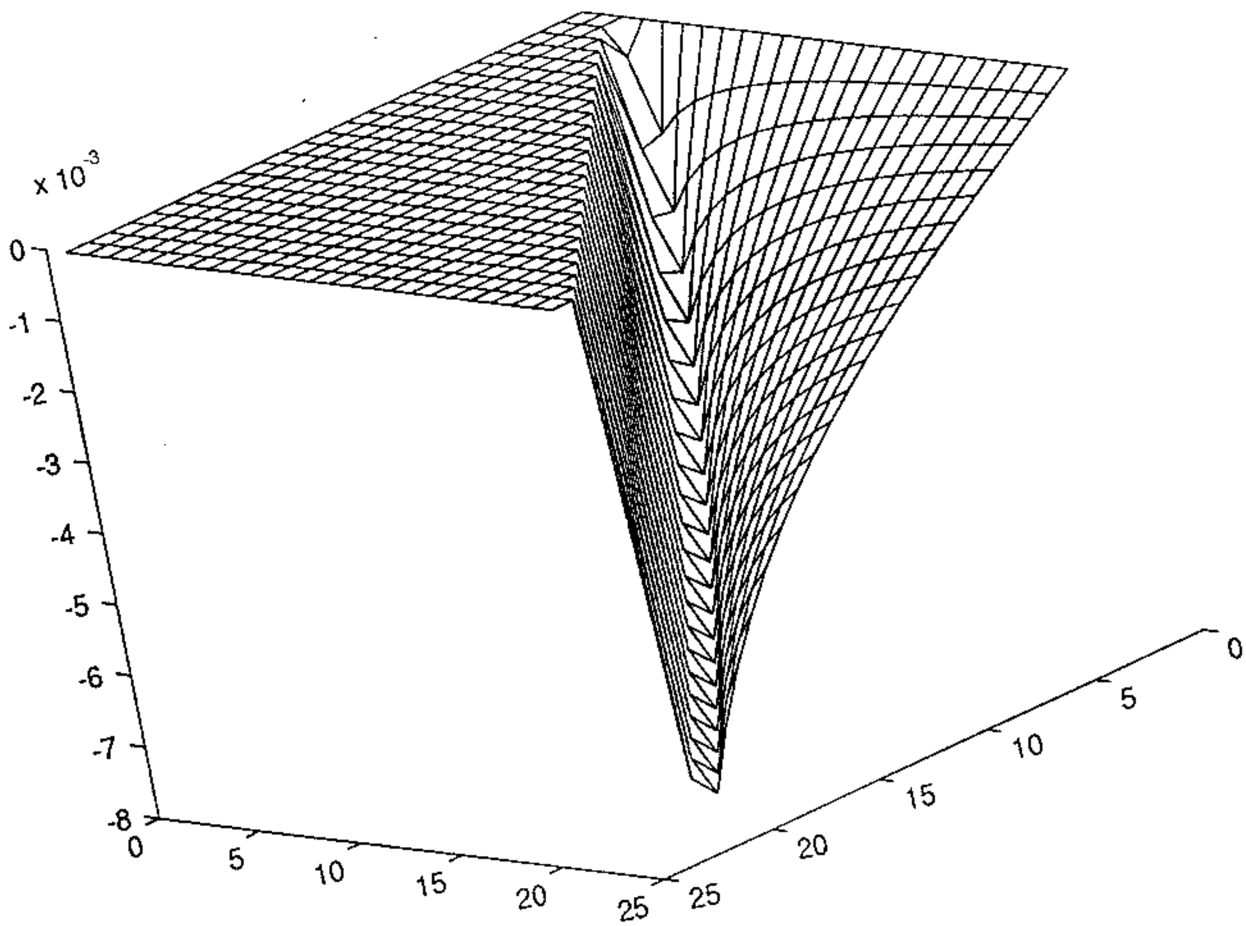
Snapshots of the solution $w(r,t)$ to the nonlinear problem in (Case 2)

$$0 \leq t \leq 14\Delta t$$



(Figure 14)

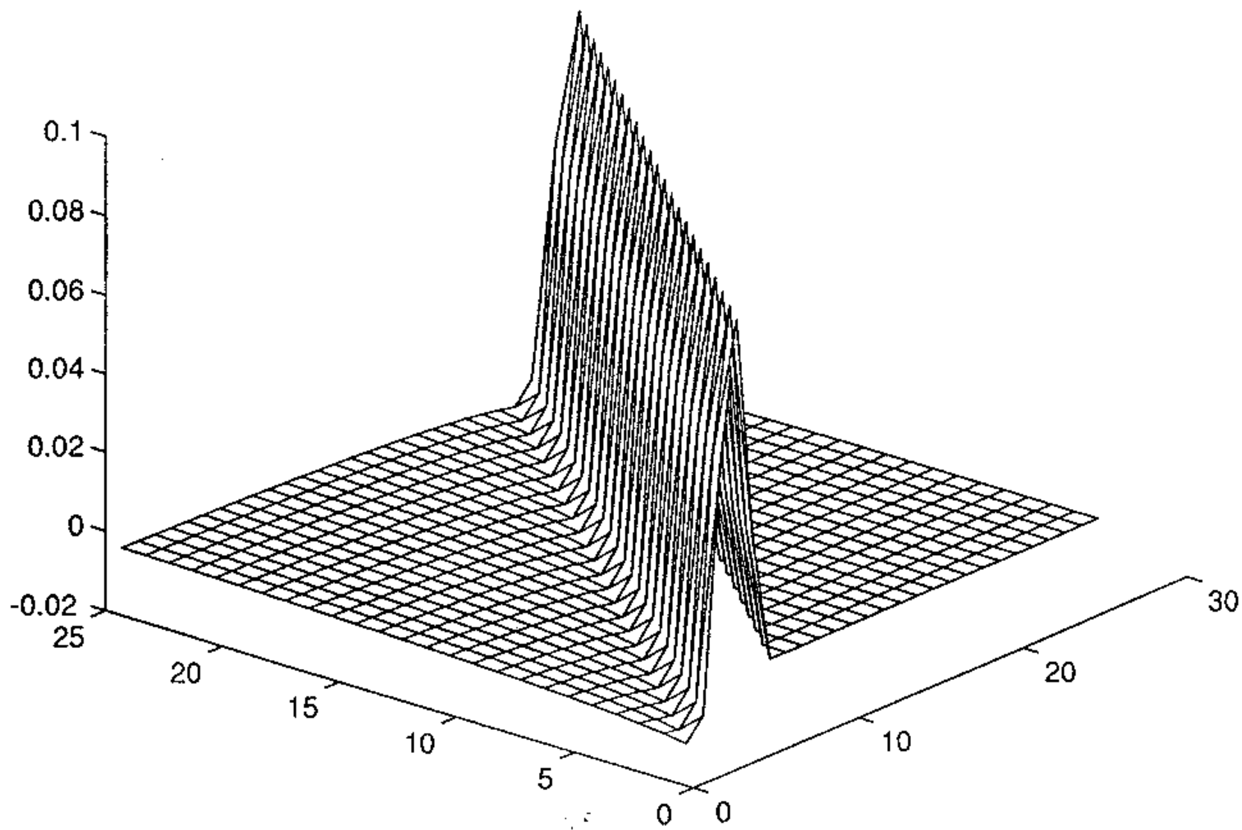
This shows the negative part of nonlinear problem (Case 2).



(Figure 15)

Another viewpoint for the graph of the negative part of $w(r,t)$ shown in

(Figure 14)



(Figure 16)

This shows the solution $w(r,t)$ in the nonlinear problem (k=1) in (Case 2) for

$$70 < t < 80 \quad \text{and} \quad 69 < r < 81.$$

REFERENCES

[1] H. A. Biagioni. The Cauchy Problem for Semilinear Hyperbolic Systems with Generalized Functions as Initial Conditions. Resultate Math. vol. 14, 231-241, 1988.

[2] H. A. Biagioni. A Nonlinear Theory of Generalized Functions vol. 1421. Spinger-Verlag, New York, 1990.

[3] P. Brenner. On L^p Decay and Scattering for Nonlinear Klein-Gordon Equations. Math. Scand. vol. 51, 333-360, 1982.

[4] P. Brenner. On Scattering and Everywhere defined Scattering Operators for Nonlinear Klein-Gordon Equations. Journal of Differential Equations vol. 56, 310-344, 1985.

[5] J. J. Cauret, J. F. Colombeau and A. Y. LeRoux. Discontinuous Generalized Solutions of Nonlinear Nonconservative Hyperbolic Equations. Journal of Mathematical Analysis and Application vol. 139, 552-573, 1983.

[6] T. Cazenane and F. B. Weissier. Rapidly Decaying Solutions of Nonlinear Schrödinger Equation. Commun. Math. Phys. vol. 147, 75-100, 1992.

[7] J. F. Colombeau. New Generalized Functions and Multiplication of Distribution vol. 84, Math. Studies, North-Holland, 1984.

[8] J. F. Colombeau. Elementary Introduction to New Generalized Function. vol. 113, Math. Studies, North-Holland, 1984.

[9] J. F. Colombeau and M. Langlais. Generalized Solutions of Nonlinear Parabolic Equations with Distributions as Initial Condition. Journal of Mathematical Analysis and Application vol. 145, 186-196, 1990.

[10] F. G. Folland. Introduction to Partial Differential Equations. Princeton University Press, 1976.

[11] R. T. Glassey. On the Asymtotic Behavior of Nonlinear Wave Equations. Transactions of American Mathematical Society vol. 182, 187-200, 1973.

[12] R. T. Glassey. Finite Time Blow up for Solutions of Nonlinear Wave Equation. Mathematische Zeitschrift vol. 177, 323-340, 1980.

[13] L. Hörmander. The Analysis of Linear Partial Differential Operators 1. Spinger-Verlag, 1983.

[14] F. John. Partial Differential Equations. Fourth-Edition, Springer-Verlag.

[15] F. John. Blow-up Solutions of Nonlinear Wave Equations in Three Space Dimensions. Menuscripta Mathematica vol. 28, 235-268, 1979.

[16] C. S. Morawetz and W. A. Strauss. Decay and Scattering Solutions of a Nonlinear Relativitic Wave Equation. Communications on pure and applied Mathematics, vol. 25, 1-31, 1972.

[17] H. Pecher. Decay and Asymptotics for Higher Dimensional Nonlinear Wave Equation. J.D.E. vol. 46, 103-151.

[18] M. Reed and B. Simom. Methods of Modern Mathematical Physics, Fourier Analysis, Self-Adjoint. Academic Press, 1975.

[19] M. Reed and B. Simom. Methods of Modern Mathematical Physics, Scattering Theory. Academic Presss, 1977.

[20] E. E. Rosinger. Generalized Solutions of Nonlinear Partial Differential Equations vol. 146. Math. Studies, North-Holland, 1987.

[21] E. E. Ronsinger. Nonlinear Partial Differential Equations vol. 164. Math. Studies, North-Holland, 1990.

[22] W. A. Strauss. Nonlinear Invariant Wave Equations. Springer Lecture Notes vol. 73, 197-249, 1977.

[23] W. A. Strauss. Nonlinear Scattering Theory. J.A. La Vita and J-P. Marchand(eds) Scattering Theory in Mathematical Physics, 53-78, 1974.

[24] W. A. Strauss. Nonlinear Scattering Theory at Low Energy. Journal of Functional Analysis vol. 141, 110-133, 1981.

[25] M. Tsutsumi. Scattering of Solutions of Nonlinear Klein-Gordon Equations in Three Space Dimensions. J. Math. Soc. Japan, vol. 35, 521-538, 1983.

[26] Y. Tsutsumi and K. Yajima. The Asymptotic Behaviour of Nonlinear Schrödinger Equations. Bull. AMS (new series) vol. 11, 186-187, 1984.