PRIMITIVE SUBSTITUTIVE NUMBERS ARE CLOSED UNDER RATIONAL MULTIPLICATION

THESIS

Presented to the Graduate Council of the

University of North Texas in Partial

Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

Pallavi S. Ketkar, B.S.

Denton, Texas

August, 1998

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Lehr(1991) proved that, if $M(q,r)$ denotes the set of real numbers whose expansion in base-r is q-automatic i.e., is recognized by an automaton $\mathcal{A} = (A_q, A_r, a_o, \delta, \phi)$ (or is the image under a letter to letter morphism of a fixed point of a substitution of constant length q) then $M(q,r)$ is closed under addition and rational multiplication. Similarly if we let $M(r)$ denote the set of real numbers α whose base-r digit expansion is ultimately primitive substitutive, i.e., contains a tail which is the image (under a letter to letter morphism) of a fixed point of a primitive substitution then in an attempt to generalize Lehr's result we show that the set $M(r)$ is closed under multiplication by rational numbers. We also show that $M(r)$ is not closed under addition.

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CHAPTER 1

PRELIMINARIES

This chapter consists of the notation and definitions of some of the concepts used throughout as well as some preliminary results about them which are not proved in this manuscript. It is by no means an exhaustive presentation of the relevant mathematical topics, but is simply intended to prepare the reader for the chapters that will follow.

1.1 Notation and Definitions

In what follows, for $a \in \mathbb{R}$ we denote the integer part of a by $\lfloor a \rfloor = max\{x \in \mathbf{Z} | x \leq a\}$ and the fractional part of a by $(a)_f = a - \lfloor a \rfloor$.

Next we recall some definitions from abstract algebra. A **monoid** consists of a non-empty set A and a binary relation

$$\begin{aligned} *: \ A \times A \to & \ A \\ (a,b) \to & \ a * b \end{aligned}$$

We shall use the notation ab for $a * b$. The binary relation * satisfies the following axioms:

Associativity: For any $a, b, c \in A$

$$(ab)c = a(bc).$$

Unit: There exists an element $1 \in A$ such that

$$1a = a = a1, \quad \forall a \in A.$$

Given any set A, the **free monoid** A^+ with base A is defined as follows. The elements of A^+ are $n - tuples$

$$s = (a_1, a_2, \cdots, a_n), \quad (n \geq 0) \tag{1.1}$$

of elements of A. The integer n is called the **length** of s and is denoted by $|s|$. If $t = (b_1, b_2, \cdots, b_m)$ is another element of A^+, the product st is defined by concatenation, i.e.,

$$st = (a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m).$$

This clearly produces a monoid with unit () (the only 0-tuple). Clearly, $|st| = |s| + |t|$ and $|()| = 0$.

We shall agree to write a instead of the 1-tuple (a). In this way (1.1) may be written as

$$s = a_1 a_2 \cdots a_n$$

if $n > 0$. Because of this, the element s is called a **word** of length n, () is called the **empty word** while $a \in A$ is called a **letter**, and A itself is called an **alphabet**.

Note that the convention $a = (a)$ permits us to treat A as a subset of A^+.

Now, let $s \in A^+$. An element $t \in A^+$ is called a **segment** of s if $s = utv$ for some $u, v \in A^+$. If u is the empty word, then t is an **initial segment;** if v is the empty word then, t is a **terminal segment**. If A and A' are monoids, a **morphism** $\tau : A \to A'$

is a function on A satisfying the following conditions.

$$\tau(ab) = \tau(a)\tau(b),$$

and

$$\tau(1_A) = \tau(1_{A'}).$$

A morphism $\phi : A \to B$ (where A and B are alphabets) that maps every letter in A to a letter in B, is referred to as a **letter to letter** morphism. A **substitution** on A is a morphism $\tau : A \to A^+$ that takes every element of A to a word in A. A substitution τ on A is said to be of **constant length** q provided that $\forall a \in A, \tau(a)$ is a word of length q.

1.2 Fixed Points

If we have a substitution τ on an alphabet A and some $a \in A$ with the property that $\tau(a)$ is a word that begins with the letter a then, for some $w \in A^+$ then $\tau(a) = aw$ which would imply that for all $n \in \mathbb{N}$,

$$\begin{aligned} \tau^n(a) &= \tau^n(a)\tau^n(w) \\ &= \tau^{n-1}(\tau(a))\tau^n(w) \\ &= \tau^{n-1}(aw)\tau^n(w) \\ &= \tau^{n-1}(a)\tau^{n-1}(w)\tau^n(w) \end{aligned}$$

Thus, for every natural number n, $\tau^n(a)$ has as initial segment $\tau^{n-1}(a)$. Therefore, the $\lim_{n\to\infty} \tau^n(a)$ can be written as an infinite sequence that has, $\forall n \in \mathbb{N}$, the initial segment $\tau^n(a)$. The above limit in such cases is denoted as $\tau^\infty(a)$ and is called a **fixed point** of τ. It can be verified that, if we have $a \in A$ such that $\tau(a)$ begins in a then, $\tau^\infty(a)$ can be constructed by sequentially concatenating to $\tau(a)$, τ applied to every letter occurring in the sequence, beginning with the second letter of $\tau(a)$.

For example, if we define a substitution τ on the alphabet $A = \{1, 2, 3\}$ such that:

$$\begin{aligned}\tau(1) &= 132\\ \tau(2) &= 12\\ \tau(3) &= 31.\end{aligned}$$

Then $\tau^\infty(1) = 13231123113213212311 32\cdots\cdots$

Equivalently given a substitution τ on an alphabet A and an arbitrary sequence $\omega \in A^{\mathbb{N}}$, ω is a fixed point of τ if $\tau(\omega) = \omega$.

Note: A substitution may have no fixed points or more than one fixed point. (The substitution in the example above has two fixed points.)

1.3 q-Automatic Sequences

A **finite deterministic automaton** $\mathcal{A}$ is defined to be composed of a finite set of states A, a set of inputs Z, an initial state $z_0 \in A$ and a transition function $\delta : A \times Z \to A$.

For example, let $A = \{0, 1, 2, 3, 4\}, Z = \{0, 1\}, z_0 = 4$ and let $\delta : A \times Z \to A$ be defined by the formula,

$$\delta(a, z) = (a + z) \bmod 4.$$

Then given the input sequence $101 \in Z^3$ we follow the steps:

$\delta(4,1) = 1, \quad \delta(1,0) = 1, \quad \delta(1,1) = 2.$

Therefore 2 is our final output.

Remark 1

- The input sequence may be any word in Z of finite length.
- The automaton defined above is denoted as $\mathcal{A} = (A, Z, z_0, \delta)$.
- An automaton may also be composed of a labelling function ϕ which simply maps the final output to a letter in a new alphabet, which then becomes the new final output. The automaton can then be denoted more generally as $\mathcal{A} = (A, Z, z_0, \delta, \phi)$, where ϕ may be the identity function.

Let $A_q = \{0, \cdots, (q-1)\}$. A sequence $\omega \in A_q^{\mathbb{N}}$ is said to be ***q*-automatic** if there exists an automaton with $A = A_q$ as described above such that for every natural number i, if $\hat{\imath}$ denotes its base q representation then, $\hat{\imath}$ considered as an input sequence gives a final output which is the ith entry of the sequence ω. (i.e the automaton **recognizes** the sequence). Alternately, a sequence $\omega \in A^{\mathbb{N}}$ is also called q-automatic if it is the image, under a letter to letter morphism, of a fixed point of a substitution of constant length q.

The **Thue-Morse** sequence generated in the following fashion is an example of a 2-automatic sequence.

Let $\tau : \{a, b\} \to \{a, b\}^+$ be the substitution of constant length 2 defined by

$$\tau(a) = ab$$

$$\tau(b) = ba.$$

The fixed point $abbabaabbaababbabaababbaabbabaabbaab\cdots\cdots$ of this substitution is called the Thue-Morse sequence and is 2-automatic.

The other fixed point of τ, $baababbaabbababababbabaabbaabbaabba\cdots\cdots$ is also 2-automatic.

1.4 Primitive Substitutive Sequences

A substitution τ on A is said to be **primitive** if, $\exists n \in \mathbb{N}$ such that $\forall a \in A$ every element of A occurs in the word $\tau^n(a)$. For example, the substitution τ defined on the alphabet $A = \{1, 2, 3\}$ as follows:

$$\begin{aligned} \tau(1) &= 312 \\ \tau(2) &= 12 \\ \tau(3) &= 31 \end{aligned}$$

is primitive, since $\tau^2(1), \tau^2(2)$ and $\tau^2(3)$ all contain every letter of the alphabet.

A sequence on an alphabet A is called **primitive substitutive** if it is the image under a letter to letter morphism of a fixed point of a primitive substitution.

Thus, the fixed point $31312313121231312 31\cdots\cdots$ of the above substitution is primitive substitutive. If we define the letter to letter morphism $\phi : \{1, 2, 3\} \to \{a, b, c\}$ such that,

$$\phi(1) = a$$

$$\phi(2) = b$$

$$\phi(3) = c$$

then, the image $cacabcacababcacabca\cdots\cdots$ of the above fixed point under ϕ is also primitive substitutive. Equivalently, a sequence $\omega \in A^{\mathbb{N}}$ is also said to be primitive substitutive if every word in the sequence occurs in bounded gaps. A sequence $\omega \in A^{\mathbb{N}}$ is said to be **ultimately primitive** substitutive if it has a primitive substitutive tail.

CHAPTER 2

STATEMENT OF THE MAIN RESULT

2.1 Historical Remarks

In [CKFR], Christol et al. proved that for any prime p, a sequence

$$x = x_1 x_2 x_3 \cdot \cdots \cdots \in A_p^{\mathbf{N}}$$

is p-automatic iff the formal power series $x(t) = \sum_{k=0}^{\infty} \hat{x_k} t^k$ is algebraic over the function field $K(t)$ over some finite field K of characteristic p, where $\hat{x_k}$ is the image of x_k under some injective map from A_p into K. It follows that, within the field $K[[t]]$ of formal power series, sums and products of these elements $x(t)$ are again p-automatic. If we replace t by the reciprocal of an integer $r \geq 2$ then it was proved by Lehr in [Le] that the set of real numbers naturally obtained in the same way is closed under addition and rational multiplication. Moreover, he proved that if we let $\mathbf{M(q, r)}$ denote the set of real numbers whose fractional part in base-r is q-automatic then $M(q, r)$ is a $\mathbb{Q}$-vector space.(Recall that, a $\mathbb{Q}$-vector space is a space that is closed under addition and rational multiplication.) However, if we let $\mathbf{M(r)}$ denote the set of real numbers whose base-r digit expansion is ultimately primitive substitutive, we prove that $M(r)$ is closed under multiplication by rational number, but not under addition.

2.2 Statement of Main Theorem

The main result of this manuscript can be stated in the following theorem:

Theorem 1 *The set $M(r)$ is closed under multiplication by $\mathbb{Q}$, but is not closed in general, under addition.*

CHAPTER 3

PRELIMINARY LEMMAS

This chapter consists of the lemmas used to prove our main result. Of these lemmas only the last one is proved in this manuscript.

In [Le],Lehr's proof relies in part on a theorem of J.-P. Allouche, M. Mendès France:

Lemma 1 (J.-P. Allouche, M. Mendès France, [AlMe]) *Let $\star$ be an associative binary operation on a finite set $\mathcal{A}$ and let $\omega = \omega_1\omega_2\omega_3\ldots$ be a q-automatic sequence in $\mathcal{A}^{\mathbf{N}}$. Then the induced sequence of partial products*

$$\omega_1, \omega_1 \star \omega_2, \omega_1 \star \omega_2 \star \omega_3, \omega_1 \star \omega_2 \star \omega_3 \star \omega_4, \ldots$$

is q-automatic.

In order to prove that $M(r)$ is closed under rational multiplication we will use the following analogue of Lemma 1

Lemma 2 (C. Holton, L.Q. Zamboni, [HoZa]) *Let $\star$ be a binary operation on a finite set $\mathcal{A}$ and let $\omega = \omega_1\omega_2\omega_3\ldots$ be an ultimately primitive substitutive sequence in $\mathcal{A}^{\mathbf{N}}$. Then the induced sequence of partial products*

$$\omega_1, \omega_1 \star \omega_2, (\omega_1 \star \omega_2) \star \omega_3, ((\omega_1 \star \omega_2) \star \omega_3) \star \omega_4, \ldots$$

is ultimately primitive substitutive.

Lemma 3 *Let $n \in \mathbb{N}$. If $\{\xi_k\} \in A_r^{\mathbb{N}}$ is an ultimately primitive substitutive sequence which is not ultimately periodic then, there is a positive integer $M = M(n)$ such that for each $k \geq 0$*

$$\left\lfloor n \sum_{i=1}^{\infty} \frac{\xi_{k+i}}{r^i} \right\rfloor = \left\lfloor n \left(\frac{\xi_{k+1}}{r} + \frac{\xi_{k+2}}{r^2} + \ldots + \frac{\xi_{k+M}}{r^M} \right) \right\rfloor .$$

Proof. Fix a positive integer l so that $r^l \geq n$, and set $(a)_f = a - \lfloor a \rfloor$ for each $a \in \mathbb{R}$. Then

$$\left\lfloor n \sum_{i=1}^{\infty} \frac{\xi_{k+i}}{r^i} \right\rfloor = \left\lfloor n \left(\frac{\xi_{k+1}}{r} + \frac{\xi_{k+2}}{r^2} + \ldots + \frac{\xi_{k+l}}{r^l} \right) \right\rfloor + \left\lfloor S_k + n \sum_{i=l+1}^{\infty} \frac{\xi_{k+i}}{r^i} \right\rfloor$$

where $S_k = \left(n \left(\frac{\xi_{k+1}}{r} + \frac{\xi_{k+2}}{r^2} + \ldots + \frac{\xi_{k+l}}{r^l} \right) \right)_f$.

Note that for $k \geq 0$, $\left\lfloor S_k + n \sum_{i=l+1}^{\infty} \frac{\xi_{k+i}}{r^i} \right\rfloor$ is either 0 or 1. Let $\mathcal{S} = \{S_k \,|\, k \geq 1\}$. Then $\mathrm{Card}(\mathcal{S}) \leq r^l$. For each $s \in \mathcal{S}$ there exist words V_s, $U_s \in A_r^+$ such that the base-r digit expansion of $\frac{1}{n}(1-s) \in \mathbb{Q}$ is given by $V_s U_s U_s U_S U_s \cdots$. Since the sequence $\{\xi_k\}$ is not ultimately periodic, for each $s \in \mathcal{S}$ there is a positive integer m_s so that the sequence $\{\xi_k\}$ does not contain the subword $U_s^{m_s}$. Set $M_s = |V_s| + m_s|U_s|$ and M' =max$\{M_s \,|\, s \in \mathcal{S}\}$. Then for each $k \geq 0$ we have

$$\left\lfloor S_k + n \sum_{i=l+1}^{\infty} \frac{\xi_{k+i}}{r^i} \right\rfloor = 1$$

if and only if

$$\sum_{i=l+1}^{\infty} \frac{\xi_{k+i}}{r^i} > \frac{1}{n}(1 - S_k)$$

if and only if

$$\frac{\xi_{k+l+1}}{r^{l+1}} + \frac{\xi_{k+l+2}}{r^{l+2}} + \ldots + \frac{\xi_{k+l+M'}}{r^{l+M'}} > \frac{1}{n}(1 - S_k).$$

Thus $M = l + M'$ satisfies the conclusion of Lemma 3. ∎

CHAPTER 4

PROOF OF MAIN THEOREM

In this chapter we prove the main theorem of this manuscript. Section 4.1 consists of proving that $M(r)$ is closed under rational multiplication. In section 4.2 we show that $M(r)$ is not closed under addition by providing a counterexample.

4.1 Closed Under Multiplication by a Rational

We begin by observing that $\mathbb{Q} \subset M(r)$, since the digit expansion of a rational number is ultimately periodic, i.e., it has a periodic tail and in a periodic sequence any word occurs within a uniform and (therefore bounded) gap. Hence a periodic sequence is primitive substitutive. Let $\xi \in M(r)$. In order to show that $M(r)$ is closed under rational multiplication we need to show that for positive integers n and p, $\frac{n}{p}\xi \in M(r)$. We prove separately that $n\xi \in M(r)$ and $\frac{\xi}{p} \in M(r)$. In each case we can assume that $0 < \xi < 1$ and that $\xi \notin \mathbb{Q}$. Hence we can write $\xi = \sum_{k=1}^{\infty} \xi_k r^{-k}$ with $\xi_k \in A_r = \{0, 1, \ldots, r-1\}$. The sequence $\{\xi_k\}$ is then ultimately primitive substitutive but not ultimately periodic. We begin by showing that $y = \frac{\xi}{p} \in M(r)$. We write $y = \sum_{k=1}^{\infty} y_k r^{-k}$ with $y_k \in A_r$. Then following [Le] we have

$$\begin{aligned} y_k &= \left\lfloor \frac{\xi}{p} r^k \right\rfloor \bmod r \\ &= \left\lfloor \frac{r^k}{p} \sum_{i=1}^{\infty} \xi_i r^{-i} \right\rfloor \bmod r \end{aligned}$$

$$= \left\lfloor \frac{1}{p}\sum_{i=1}^{\infty}\xi_i r^{k-i}\right\rfloor \bmod r$$
$$= \left\lfloor \frac{1}{p}\sum_{i=1}^{k}\xi_i r^{k-i} + \frac{1}{p}\sum_{i=k+1}^{\infty}\xi_i r^{k-i}\right\rfloor \bmod r.$$

Since,

$$\frac{1}{p}\sum_{i=1}^{k}\xi_i r^{k-i} = \frac{m}{p}$$

for some natural number m, and

$$\frac{1}{p}\sum_{i=k+1}^{\infty}\xi_i r^{k-i} < \frac{1}{p}$$

we obtain

$$y_k = \left\lfloor \frac{1}{p}\sum_{i=1}^{k}\xi_i r^{k-i}\right\rfloor \bmod r = \left\lfloor \frac{\left(\sum_{i=1}^{k}\xi_i r^{k-i}\right) \bmod pr}{p}\right\rfloor \bmod r$$

(which can be verified by the division algorithm).

Consider the sequence $\{(\xi_k, r)\}_{k=1}^{\infty}$ in the alphabet $A_{pr} \times A_{pr}$. Since the sequence $\{\xi_k\}$ is ultimately primitive substitutive, the same is true of the sequence $\{(\xi_k, r)\}$. Let $\star$ denote the associative binary operation on $A_{pr} \times A_{pr}$ given by

$$(a,\alpha) \star (b,\beta) = (a\beta + b \bmod pr, \alpha\beta \bmod pr).$$

(There is a typographical error in the definition of the binary operation $*$ given in [Le]. It should be the same as $\star$.) For each $k \geq 1$ we set

$$x_k = (\xi_1, r) \star (\xi_2, r) \star \ldots \star (\xi_k, r) = \left(\sum_{i=1}^{k}\xi_i r^{k-i} \bmod pr, r^k \bmod pr\right).$$

By Theorem 2 the sequence $\{x_k\}$ is ultimately primitive substitutive, and hence so is the sequence $\{y_k\}$ as required.

We next show that $n\xi \in M(r)$. Let $z = n\xi$. Then we can write $(z)_f = \sum_{k=1}^{\infty} z_k r^{-k}$. It suffices to show that the sequence $\{z_k\}$ is ultimately primitive substitutive. Let M be as in Lemma 3. Then for $k \geq 0$ we have

$$\begin{aligned}
z_k &= \lfloor n\xi r^k \rfloor \bmod r \\
&= \left\lfloor nr^k \sum_{i=1}^{\infty} \xi_i r^{-i} \right\rfloor \bmod r \\
&= \left\lfloor n \sum_{i=1}^{k-1} \xi_i r^{k-i} + n\xi_k + n \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \right\rfloor \bmod r \\
&= \left\lfloor n\xi_k + n \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \right\rfloor \bmod r \\
&= \lfloor n\xi_k \rfloor \bmod r + \left\lfloor n \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \right\rfloor \bmod r \\
&= \lfloor n\xi_k \rfloor \bmod r + \left\lfloor n \left(\frac{\xi_{k+1}}{r} + \frac{\xi_{k+2}}{r^2} + \ldots + \frac{\xi_{k+M}}{r^M} \right) \right\rfloor \bmod r.
\end{aligned}$$

Now since $\{\xi_k\}$ is ultimately primitive substitutive, the same is true of the sequence $\{(\xi_k, \xi_{k+1}, \ldots, \xi_{k+M})\}_{k=1}^{\infty}$. In fact if a tail of $\{\xi_k\}$ is the image of a fixed point of a primitive substitution ζ, then the corresponding tail of $\{(\xi_k, \xi_{k+1}, \ldots, \xi_{k+M})\}$ is the image of a fixed point of the primitive morphism ζ_{M+1} defined in [Qu] (see Lemma V.11 and Lemma V.12 in [Qu]). Define $\phi : A_r^{M+1} \to A_r$ by

$$\phi(a_1, a_2, \ldots, a_{M+1}) = \lfloor na_1 \rfloor \bmod r + \left\lfloor n \left(\frac{a_2}{r} + \frac{a_3}{r^2} + \ldots + \frac{a_{M+1}}{r^M} \right) \right\rfloor \bmod r.$$

Then the sequence $\{z_k\} = \{\phi(\xi_k, \xi_{k+1}, \ldots, \xi_{k+M})\}_{k+1}^{\infty}$ is ultimately primitive substitutive as required.

4.2 Counterexample to addition

It remains to show that $M(\tau)$ is not closed under addition. Let τ be the primitive morphism defined by

$$\begin{aligned} 1 &\mapsto 1211 \\ 2 &\mapsto 2112. \end{aligned}$$

Let $a = \{a_i\}$ denote the fixed point of τ beginning in 1 and $b = \{b_i\}$ the fixed point of τ beginning in 2. Let $\alpha = \sum_{i=1}^{\infty} a_i(10)^{-i}$ and $\beta = \sum_{i=1}^{\infty} b_i(10)^{-i}$. Then α and β are each in $M(10)$ but $\alpha + \beta \notin M(10)$. In fact, the digit 3 occurs an infinite number of times in the decimal expansion of $\alpha + \beta$ but not in bounded gap. We note that for each $n \geq 1$ the sequence a begins in $\tau^n(12)\tau^n(1)$ and b begins in $\tau^n(21)\tau^n(1)$. Since $|\tau^n(12)| = |\tau^n(21)|$ it follows that for each $N \geq 1$ we can find $k = k(N)$ so that $a_k a_{k+1} \ldots a_{k+N} = b_k b_{k+1} \ldots b_{k+N}$. If $c = \{c_i\}$ denotes the decimal expansion of $\alpha + \beta$ then the block $c_k c_{k+1} \ldots c_{k+N}$ consists only of the digits 2 and 4. At the same time, for each $n \geq 1$ the sequence a begins in $\tau^n(121)\tau^n(2)$ while b begins in $\tau^n(211)\tau^n(2)$. Since $|\tau^n(121)| = |\tau^n(211)|$ it follows that $a_j \neq b_j$ (and hence $c_j = 3$) for infinitely many values of j. Thus no tail of the decimal expansion of $\alpha+\beta$ is a minimal sequence. In particular $\{c_i\}$ is not ultimately primitive substitutive. ∎

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