CONTINUOUS, NOWHERE-DIFFERENTIABLE FUNCTIONS
WITH NO FINITE OR INFINITE ONE-SIDED
DERIVATIVE ANYWHERE

THESIS

Presented to the Graduate Council of the
University of North Texas in Partial
Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

Jae S. Lee, B.S.
Denton, Texas
December, 1994
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In this paper, we study continuous functions with no finite or infinite one-sided derivative anywhere. In 1925, A. S. Besicovitch published an example of such a function. Since then we call them Besicovitch functions. This construction is presented in chapter 2. The example was simple enough to clear the doubts about the existence of Besicovitch functions. In 1932, S. Saks showed that the set of Besicovitch functions is only a meager set in $C[0,1]$. Thus the Baire category method for showing the existence of Besicovitch functions cannot be directly applied. A. P. Morse in 1938 constructed Besicovitch functions with the following properties (we will call these Morse-Besicovitch Functions):

$$\limsup_{x \to z^-} \left| \frac{f(z) - f(x)}{x - z} \right| = \infty \text{ for every } z \in (0,1]$$

and

$$\limsup_{x \to z^+} \left| \frac{f(z) - f(x)}{x - z} \right| = \infty \text{ for every } z \in [0,1).$$

In 1984, Malý revived the Baire category method by finding a non-empty compact subspace of $(C[0,1],\| \cdot \|)$ with respect to which the set of Morse-Besicovitch functions is comeager.
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In this paper, we study continuous functions with no finite or infinite one-sided derivative anywhere. Recall Weierstrass' nowhere differentiable continuous function [BR] or Tagaki's [Ta]. But these functions have infinite one-sided derivative at some points. In 1925, A. S. Besicovitch published an example of continuous functions with no finite or infinite one-sided derivative anywhere [Be]. Since then we call such functions Besicovitch functions. Later E. D. Pepper examined the same construction and simplified the proof [Pe]. A detailed presentation of Besicovitch's construction and Pepper's argument is given in chapter 2. The construction was simple enough to clear the doubts about the existence of Besicovitch functions. Banach and Mazurkiewicz showed the class of Weierstrass' nowhere differentiable functions is comeager in the space of continuous functions [Ba]. In 1932, S. Saks [Sa] showed that the set of Besicovitch functions is only a meager set in $C[0,1]$. Thus, the Baire category method for showing the existence of Besicovitch functions cannot be directly applied. Not until much later was the category method used to prove the existence of Besicovitch functions. Meanwhile, many different approaches to the Besicovitch functions were developed including constructions of Besicovitch functions satisfying more strict conditions. A. P. Morse in 1938 [Mo] constructed Besicovitch functions with the following properties:

$$\limsup_{x \to z^-} \left| \frac{f(x) - f(z)}{x - z} \right| = \infty \text{ for every } z \in (0,1]$$

and

$$\limsup_{x \to z^+} \left| \frac{f(x) - f(z)}{x - z} \right| = \infty \text{ for every } z \in [0,1).$$
Thus, at points \( x \), only one of the right Dini derivatives is infinite and only one of the left Dini derivatives is infinite. We will call Besicovitch functions with these properties, *Morse-Besicovitch functions*. The original function constructed by Besicovitch does not have the above property: for example,

\[
\limsup_{x \to 0^+} \left| \frac{f(x) - f(0)}{x - 0} \right| = 2 \quad \text{and} \quad \liminf_{x \to 0^+} \left| \frac{f(x) - f(0)}{x - 0} \right| = 0.
\]

The construction and the properties of Morse-Besicovitch functions are presented in chapter 4. As the original construction of Besicovitch made its mark as the first example, Morse's construction provided motives for the study of various properties of Besicovitch functions. Most of all, Morse-Besicovitch functions were used by J. Malý [Mal] in 1984 in the revival of the Baire category method for proving the existence of Besicovitch functions. Malý found a non-empty compact subspace of \((C[0,1], \| \cdot \|)\) with respect to which the set of Morse-Besicovitch functions is comeager. This is discussed in chapter 5.
CHAPTER 2

CONSTRUCTION OF A BESICOVITCH FUNCTION

2.1 Construction of a Besicovitch function.

We first construct a Cantor set in $[0, 1]$ [Fig. 2.1]. Then we symmetrize this set about 1 so that we have a Cantor set in $[0, 2]$. Secondly, we define a Lebesgue singular function on $[0, 2]$ which is symmetric about 1 [Fig. 2.2 on page 7]. Then we use a contraction mapping from $[0, 2]$ to each complementary interval of the Cantor set to define "the second stage" Lebesgue singular function on $[0, 2]$ [Fig. 2.3 on page 7]. We denote the Lebesgue measure of $A \subseteq \mathbb{R}$ by $m(A)$.

Basic definitions and construction of a Cantor set. Define a sequence $\{a_n\}_{n=0}^{\infty}$ in $[0, 1]$ by the following: let $a_0 = 1$ and $a_n = \frac{1}{2}[a_{n-1} - (\frac{1}{2})^n]$ for $n \geq 1$. Then $\forall n \geq 0$, we have $a_n = \frac{2^n+1}{2^{n+1}}$.

Figure 2.1: Construction of a Cantor set.

Let $J_{0, 1}$ denote the closed interval $[0, 1]$. Let $J_{1, 1} = [0, a_1]$ and $J_{1, 2} = [1 - a_1, 1]$. Let $J_1$ be the collection of intervals $J_{1, 1}$ and $J_{1, 2}$. Suppose that for each $k \leq n - 1$ we have $J_k$...
defined. Then we define \( \mathcal{I}_n \) by the following: \( \mathcal{I}_n \) will be the collection intervals \( \{I_{n,i}\}_{i=1}^{2^n} \) such that if \([\alpha, \beta]\) is the interval \( I_{n-1,k} \) then \( I_{n,2k-1} = [\alpha, \alpha + a_n] \) and \( I_{n,2k} = [\beta - a_n, \beta] \). Thus the number of intervals in \( \mathcal{I}_n \) is \( 2^n \) for each \( n \geq 0 \), and the length of each interval in \( \mathcal{I}_n \) is \( a_n \). Let

\[
C = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^n} I_{n,k}
\]

which is a Cantor set.

We also give notations to complementary intervals of \([0,1] \setminus \bigcup_{k=1}^{2^n} J_{n,k}\), for all \( n \). Let \( I_{n,k} = J_{n-1,k} \setminus (J_{n,2k-1} \cup J_{n,2k}) \) and let \( \mathcal{I}_n = \{I_{n,k}\}_{k=1}^{2^{n-1}} \). Then the length of each interval in \( \mathcal{I}_n \) is \( |J_{n-1,k}| - 2|J_{n,2k}| = a_{n-1} - 2a_n = \left(\frac{1}{4}\right)^n \).

**Claim 2.1.1** Let \( L_n := \bigcup_{k=1}^{2^{n-1}} I_{n,k} \) for each \( n \) and let \( L := \bigcup_{n=1}^{\infty} L_n \). Then \( L = [0,1] \setminus C \) and

\[
m(L) = m(C) = \frac{1}{2}.
\]

**Proof.** For each \( n \)

\[
m(L_n) = \sum_{k=1}^{2^{n-1}} m(I_{n,k}) = \sum_{k=1}^{2^{n-1}} \left(\frac{1}{4}\right)^n = 2^{n-1} \left(\frac{1}{4}\right)^n = \frac{1}{2^{n+1}}
\]

and

\[
m(L) = \sum_{n=1}^{\infty} m(L_n) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{4} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2}.
\]

Also, since \( C = [0,1] \setminus L \), \( m(C) = \frac{1}{2} \).

Now we symmetrize the Cantor set \( C \) about 1. Let \( E_0 = C \cup \{x \in [1,2] : 2 - x \in C\} \). Let \( \mathcal{I}^{(0,0)} = \bigcup_{n=1}^{\infty} \mathcal{I}_n \) and let \( \mathcal{I}^{(0,1)} = \{(\alpha, \beta) \leq (1,2) : (2 - \beta, 2 - \alpha) \in \mathcal{I}^{(0,0)}\} \). This defines the symmetrization of the complementary intervals of \([0,1] \setminus \bigcup_{k=1}^{2^n} J_{n,k}\), for all \( n \), to the interval \([1,2]\). Let \( I^{(0,0)}_{n,k}(0,2) = I_{n,k} \) and \( I^{(0,1)}_{n,k}(0,2) = \{x \in (1,2) : 2 - x \in I_{n,k}\} \). Then \( \mathcal{I}^{(0,0)} = \{I^{(0,0)}_{n,k}(0,2) : 1 \leq k \leq 2^{n-1}, n = 1,2,\ldots\} \), and \( \mathcal{I}^{(0,1)} = \{I^{(0,1)}_{n,k}(0,2) : 1 \leq k \leq 2^{n-1}, n = 1,2,3,\ldots\} \).
In each of the intervals $I_{n,k}^{(0,1)}(0,2)$ we define a scaled version of $E_0$. We give notations to the complementary intervals resulted by the scaling. Define $\mathcal{I}^{(1,t)}$ for $t \in \{0, 1\}$ by

$$\mathcal{I}^{(1,0)} = \left\{ I_{n,k}^{(1,0)}(\alpha, \beta) : (\alpha, \beta) \text{ is an interval in } I_{n,k}^{(0,0)} \cup I_{n,k}^{(0,1)}, \right.$$  

such that $x \in I_{n,k}^{(1,0)}(\alpha, \beta)$ if and only if $x = \frac{2\alpha - 2y}{\beta - \alpha}$ for $y \in I_{n,k}^{(0,0)}(0,2)$

and

$$\mathcal{I}^{(1,1)} = \left\{ I_{n,k}^{(1,1)}(\alpha, \beta) : (\alpha, \beta) \text{ is an interval in } I_{n,k}^{(0,0)} \cup I_{n,k}^{(0,1)}, \right.$$  

such that $x \in I_{n,k}^{(1,1)}(\alpha, \beta)$ if and only if $x = \frac{2\alpha - 2y}{\beta - \alpha}$ for $y \in I_{n,k}^{(0,1)}(0,2)$.

Thus, for a fixed interval $(\alpha, \beta)$ in $I_{n,k}^{(0,0)} \cup I_{n,k}^{(0,1)}$, the intervals $I_{n,k}^{(1,t)}(\alpha, \beta)$ are the scaled images of the intervals $I_{n,k}^{(0,1)}(0,2)$. We now iterate this procedure.

Suppose $\mathcal{I}^{(m,t)}$ for $t \in \{0, 1\}$ and $m \in \{0, \ldots, l\}$ have been defined. Then define $\mathcal{I}^{(l+1,t)}$ for $t \in \{0, 1\}$ by

$$\mathcal{I}^{(l+1,0)} = \left\{ I_{n,k}^{(l+1,0)}(\alpha, \beta) : (\alpha, \beta) \text{ is an interval in } I_{n,k}^{(l,0)} \cup I_{n,k}^{(l,1)}, \right.$$  

such that $x \in I_{n,k}^{(l+1,0)}(\alpha, \beta)$ if and only if $x = \frac{2\alpha - 2y}{\beta - \alpha}$ for $y \in I_{n,k}^{(0,0)}(0,2)$

and

$$\mathcal{I}^{(l+1,1)} = \left\{ I_{n,k}^{(l+1,1)}(\alpha, \beta) : (\alpha, \beta) \text{ is an interval in } I_{n,k}^{(l,0)} \cup I_{n,k}^{(l,1)}, \right.$$  

such that $x \in I_{n,k}^{(l+1,1)}(\alpha, \beta)$ if and only if $x = \frac{2\alpha - 2y}{\beta - \alpha}$ for $y \in I_{n,k}^{(0,1)}(0,2)$.
Then $E_0 = [0, 2] \setminus \{(\alpha, \beta) : (\alpha, \beta) \in \mathcal{I}^{(0,0)} \cup \mathcal{I}^{(0,1)}\}$. Let $E_m = [0, 2] \setminus \{(\alpha, \beta) : (\alpha, \beta) \in \mathcal{I}^{(m,0)} \cup \mathcal{I}^{(m,1)}\}$. Note that we have defined a nested sequence of Cantor sets, $E_0 \subset E_1 \subset E_2 \subset \ldots$.

Similarly, we define $J^{(m,t)}_{n,2k-1}(\alpha, \beta)$ and $J^{(m,t)}_{n,2k+1}(\alpha, \beta)$ to be the closed intervals on the left and on the right of $J^{(m,1)}_{n,k}(\alpha, \beta)$, respectively, for all $m \geq 1$ and for all $t \in \{0, 1\}$ such that $x \in J^{(m,t)}_{n,k}(\alpha, \beta)$ if and only if $x = \frac{2\alpha - 2y}{\beta - \alpha}$ for some $y \in J^{(m,t)}_{n,k}$.

**Observation 2.1.2** If $(\alpha, \beta)$ is an interval in $\mathcal{I}^{(m-1,t)}$, then $J^{(m,t)}_{n,k}(\alpha, \beta)$ is an interval of length $(\frac{1}{4})^n (\frac{\beta - \alpha}{2})$.

**Definition of a Besicovitch Function $\psi$.** We first define $\varphi_0 : [0, 2] \to [0, 1]$ by

$$
\varphi_0(x) = \begin{cases} 
2m(E_0 \cap [0, x]) & \text{if } x \in [0, 1); \\
2m(E_0 \cap [0, 2 - x]) & \text{if } x \in [1, 2] \text{ (see fig. 2.2 on page 7).}
\end{cases}
$$

Then we define a sequence of functions $\{\varphi_i\}_{i=1}^\infty$ as follows: define $\varphi_1 : [0, 2] \to [0, 1]$ by

$$
\varphi_1(x) = \begin{cases} 
0 & \text{if } x \in E_0; \\
h_{n,k}^{1,0} \varphi_0 \left(\frac{2x - 2\alpha}{\beta - \alpha}\right) & \text{if } x \in (\alpha, \beta) \text{, (see fig. 2.3 on page 7).}
\end{cases}
$$

where $(\alpha, \beta) \subseteq (0, 2)$ is an interval $J^{(0,0)}_{n,k}(0, 2)$ or $J^{(0,1)}_{n,k}(0, 2)$ and $h_{n,k}^{1,0} = \frac{\varphi_0(1)}{2^n} = \frac{1}{2^n}$. The constant $h_{n,k}^{1,0}$ controls the height of the scaled version of $\varphi_0$ so that the height is reduced by $\frac{1}{2^n}$ while the width is reduced by $\frac{1}{2^n}$, for each $n$. Note that at each stage we use a scaled version of $\varphi_0$. We now iterate this procedure.

Suppose $\varphi_1, \ldots, \varphi_m$ are defined. Define $\varphi_{m+1} : [0, 2] \to [0, 1]$ by

$$
\varphi_{m+1}(x) = \begin{cases} 
0 & \text{if } x \in E_m; \\
h_{n,k}^{\gamma,m} \varphi_0 \left(\frac{2x - 2\alpha}{\beta - \alpha}\right) & \text{if } x \in (\alpha, \beta),
\end{cases}
$$

where $(\alpha, \beta) \subseteq (\alpha', \beta')$ is an interval $J^{(m,0)}_{n,k}(\alpha', \beta')$ or $J^{(m,1)}_{n,k}(\alpha', \beta')$ and $(\alpha', \beta') \in \mathcal{I}^{(m-1,0)}$ or $\mathcal{I}^{(m-1,1)}$ and $\gamma = \frac{\alpha' + \beta'}{2}$ and $h_{n,k}^{\gamma,m} = \frac{\varphi_0(1)}{2^{m+1}}$. Thus, $\varphi_m$ over $(\alpha, \beta) \in \mathcal{I}^{(m-1,t)}$, where $t = 0, 1$, is a scaled version of $\varphi_0$. 
Figure 2.2: The Function $\varphi_0$.

Figure 2.3: The Function $\varphi_1$. 
Define \( \Phi : [0, 2] \rightarrow [0, 1] \) by \( \Phi(x) := \sum_{i=0}^{\infty} (-1)^i \varphi_i(x) \).

Figure 2.4: The Function \( \Phi \).

Lemma 2.1.3 For all \( m \geq 0 \) and for all \( x \in [0, 2] \), \( 0 \leq \varphi_m(x) \leq \frac{1}{2^m} \).

Proof. Let \( (\alpha_m, \beta_m) \) be the largest interval in \( I^{(m,0)} \) for all \( m \geq 0 \). Then for all \( m \geq 1 \), \( (\alpha_{m-1}, \beta_{m-1}) \supseteq (\alpha_m, \beta_m) \). Then \( \beta_m - \alpha_m = \left( \frac{1}{2} \right) \left( \frac{1}{4} \right) m = \frac{1}{2^{m+1}} \). Let \( \gamma = \frac{\alpha_{m-1} + \beta_{m-1}}{2} \) and \( H_m := h^{(\gamma,m-1)}_{1,1} \) for all \( m \geq 1 \). Then \( H_m = \frac{1}{2^m} \frac{1}{2} = \frac{1}{2^m} \), and

\[
0 \leq \varphi_m(x) \leq H_m \varphi_{m-1}(x) = \frac{1}{2^m} \varphi_{m-1}(x) \leq \frac{1}{2^m}
\]

\( \forall m \geq 1. \blacksquare \)

Claim 2.1.4 \( \Phi \) is well defined as a continuous function on the interval \([0, 2]\).

Proof. Let \( \psi_k(x) := \sum_{i=0}^{k} (-1)^i \varphi_i(x) \). Then \( \psi_k \) is continuous on \([0, 2]\). From the lemma we have \( 0 \leq \varphi_m(x) \leq \frac{1}{2^m} \) for all \( m \geq 0 \). Then for all \( m \),

\[
|\psi_m(x) - \psi_{m+1}(x)| \leq |\varphi_{m+1}(x)| \leq \frac{1}{2^m+1}.
\]
Then we also have
\[ |\psi_m(x) - \psi_{m+p}(x)| \leq \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \cdots + \frac{1}{2^{m+p}} < \frac{1}{2^m}. \]

Thus \( \{\psi_k\}_{k=1}^\infty \) is uniformly Cauchy on the interval \([0, 2]\). So, \( \{\psi_k\}_{k=1}^\infty \) is uniformly convergent and \( \lim_{k \to \infty} \psi_k = \Phi \). Thus \( \Phi \) is continuous. \[ \blacksquare \]

2.2 One-sided derivatives.

In this section we show that \( \Phi \) does not have a finite or infinite one-sided derivative anywhere in the interval \([0, 2]\). We utilize the following Dini derivative notations:

\[
\Phi^+(x) := \limsup_{y \to x^+} \frac{\Phi(y) - \Phi(x)}{y - x}, \quad \Phi_+(x) := \liminf_{y \to x^+} \frac{\Phi(y) - \Phi(x)}{y - x} \\
\Phi^-(x) := \limsup_{y \to x^-} \frac{\Phi(y) - \Phi(x)}{y - x}, \quad \Phi_-(x) := \liminf_{y \to x^-} \frac{\Phi(y) - \Phi(x)}{y - x}.
\]

We first consider the one-sided derivatives of points in \( E := \bigcup_{m=0}^\infty E_m \), and secondly in \([0, 2]\) \( E \). We also limit our efforts to the points in \([0, 1]\) since \( \Phi \) is symmetric about \( x = 1 \).

One-sided derivatives in \( E \cap \[0, 1\] \) Let \( E_{0,L}, E_{0,R} \) be left, right end points of intervals in \( T^{(0,0)} \), respectively. There are three different kinds of points in \( E \cap \[0, 1\] : E_0 \setminus (E_{0,L} \cap E_{0,R}) \), \( E_{0,L} \), or \( E_{0,R} \).

Claim 2.2.1 If \( x \in E_0 \setminus E_{0,L} \) and \( x \neq 1 \), then

\[ \Phi_+(x) \leq 0 \leq \Phi_+(x) \quad \text{and} \quad \Phi^+(x) - \Phi_+(x) \geq 1. \]

Claim 2.2.2 If \( x \in E_0 \setminus E_{0,R} \) and \( x \neq 0 \), then

\[ \Phi^-(x) \geq 1, \quad \Phi_-(x) \leq 2 \quad \text{and} \quad \Phi^-(x) - \Phi_-(x) \geq 1. \]
Claim 2.2.3 If \( x \in E_{0,L} \), then
\[
\Phi_+(x) \leq 0 \leq \Phi_+(x) \quad \text{and} \quad \Phi_+(x) < \Phi^+(x).
\]

Claim 2.2.4 If \( x \in E_{0,R} \), then
\[
\Phi_-(x) \leq 0 \leq \Phi_-(x) \quad \text{and} \quad \Phi_-(x) < \Phi^-(x).
\]

Thus if \( x \in E_0 \) then one-sided derivatives do not exist. Then \( \forall m \geq 0 \), if \( x \in E_m \), \( \Phi \) does not have one sided derivatives at \( x \).

Proofs of the Claims

Claim 2.2.5 If \( x \in E_{0 \setminus E_{0,L}} \) and \( x \neq 1 \), then
\[
\Phi_+(x) \leq 0 \leq \Phi_+(x) \quad \text{and} \quad \Phi_+(x) - \Phi_+(x) \geq 1.
\]

Proof. Let \( h > 0 \). Let \( x \in E_{E_{0,L}} \). Let \( (\alpha, \beta) \) be the largest interval such that \( (\alpha, \beta) = I_{n,k}^{(0,0)}(0,2) \in \mathcal{I}^{(0,0)} \) and \( (\alpha, \beta) \subseteq (x, x + h) \). Then \( z \in J_{n,2k-1} \). Let \( p \) be the left end point of \( J_{n,2k-1} \) and let \( \gamma = \frac{x+\beta}{2} \) (see Fig. 2.5 on page 11). Note that \( \Phi \) is non-decreasing on \( J_{n,2k-1} \). Then
\[
\frac{\Phi(\alpha) - \Phi(x)}{\alpha - x} \geq 0, \quad \text{and} \quad \Phi(\alpha) - \Phi(x) \leq \Phi(\alpha) - \Phi(p).
\]

Since \( J_n = \{J_{n,k}\}_{k=1}^{2^n} \) and \( \Phi(1) - \Phi(0) = 1 \),
\[
\Phi(\alpha) - \Phi(p) = \frac{\Phi(1) - \Phi(0)}{2^n} = \frac{1}{2^n}.
\]

Also note that
\[
\Phi(\alpha) - \Phi(\gamma) = \varphi(\gamma) = h_{n,k}^{1,0} = \frac{\varphi(1)}{2^n}.
\]

Thus
\[
\frac{\Phi(\gamma) - \Phi(x)}{\gamma - x} = \frac{\Phi(\gamma) - \Phi(\alpha) + \Phi(\alpha) - \Phi(x)}{\gamma - x} \leq \frac{-\frac{1}{2^n} + \frac{1}{2^n}}{\gamma - x} = 0.
\]
Therefore
\[ \Phi^+(x) \geq 0 \geq \Phi_+(x). \]

Also,
\[ \frac{\Phi(\alpha) - \Phi(x)}{\alpha - x} > \frac{\Phi(\gamma) - \Phi(x)}{\gamma - x} = \frac{1}{2^n} \frac{1}{\gamma - x} \geq \frac{1}{2^n} \frac{1}{\gamma - p}, \]
and
\[ \gamma - p = m(J_{n,2k-1}) + \frac{1}{2} (\beta - \alpha) = \frac{2^n + 1}{2^{n+1}} + \frac{1}{2} \left( \frac{1}{4} \right)^n = \frac{2^n + 2}{2^{n+1}}. \]

Then
\[ \frac{\Phi(\alpha) - \Phi(\gamma)}{\gamma - x} \geq \frac{1}{2^n} \frac{1}{\gamma - p} = \frac{1}{2^n} \frac{2^{n+1}}{2^n + 2} = \frac{2^{n+1}}{2^n + 2} = \frac{2^n}{2^n - 1} \geq \frac{2^n}{2^n} = 1. \]
Thus
\[ \Phi^+(x) - \Phi_+(x) \geq 1. \]

\[ J_{n,2k-1}^{(0,0)}(0,2) \quad I_{n,k}^{(0,0)}(0,2) \]

\[ 0 \quad x_0 \quad \alpha \quad \gamma \quad \beta \quad x_0+h \]

Figure 2.5: Illustration for Claim 2.2.5
Claim 2.2.6 If \( x \in E_0 \setminus E_{0,R} \) \( x \neq 0 \), then

\[
\Phi^- (x) \geq 1, \quad \Phi_+ (x) \leq 2 \quad \text{and} \quad \Phi^- (x) - \Phi_+ (x) \geq 1.
\]

Proof. Let \( h > 0 \). Let \( x \in E_0 \setminus E_{0,R} \). Let \( (\alpha, \beta) \) be the largest interval such that \((\alpha, \beta) = I_{n,k}^{(0,0)}(0,2) \in I_{n,2k}^{(0,0)} \) and \( (\alpha, \beta) \subseteq (x - h, x) \). Then \( x \in J_{n,2k} \). Let \( p \) be the right end point of \( J_{n,2k} \) and \( \gamma = \frac{\alpha + \beta}{2} \) (see Fig. 2.6 on page 12). Then

\[
\Phi(x) - \Phi(\alpha) = \frac{1}{x - \alpha} 2m \left( E \cap (\alpha, x) \right).
\]

Thus

\[
0 \leq \frac{\Phi(x) - \Phi(\alpha)}{x - \alpha} \leq 2.
\]

Also,

\[
\frac{\Phi(x) - \Phi(\gamma)}{x - \gamma} - \frac{\Phi(x) - \Phi(\alpha)}{x - \alpha} > \frac{\Phi(\alpha) - \Phi(\gamma)}{x - \gamma}
\]
Thus

\[ \frac{\Phi(x) - \Phi(\gamma)}{x - \gamma} \leq 1. \]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2.7.png}
\caption{Illustration for Claim 2.2.7}
\end{figure}

Claim 2.2.7 If \( x \in E_{0,L} \), then

\[ \Phi_+(x) \leq 0 \leq \Phi^+(x) \text{ and } \Phi_+(x) < \Phi^+(x). \]

Proof. Let \( \alpha \in E_{0,L} \). Let \( \beta \) be the right end point of the corresponding interval. (Let \((\alpha, \beta) = R^{(0,0)}_{m,k}(0,2)\). See Fig. 2.7 on page 13.) Then

\[ \Phi(\alpha) - \Phi\left(\frac{\alpha + \beta}{2}\right) = h^{1,0}_{n,k} = \Phi_0(1) \frac{1}{2^n} = 1 \cdot \frac{1}{2^n}. \]

Let \( h > 0 \) such that \( \alpha + h < \frac{\alpha + \beta}{2} \). Let \((a, b)\) be the largest interval \( R^{(1,0)}_{m,d}(\alpha, \beta) \) in \( I^{(1,0)} \) such that \((a, b) \subseteq (\alpha, \alpha + h)\). Since \( \Phi \) is non increasing on \( R^{(1,0)}_{m,2^{l-1}}(\alpha, \beta) \),

\[ \frac{\Phi(b) - \Phi(b)}{b - a} \leq 0. \]
We also have

\[
\Phi(\alpha) - \Phi(a) = \frac{1}{2^m} \left( \Phi(\alpha) - \Phi \left( \frac{\alpha + \beta}{2} \right) \right) = \frac{1}{2^m} \cdot \frac{1}{2^n} = \frac{1}{2^{m+n}}
\]

and

\[
\Phi \left( \frac{a + b}{2} \right) - \Phi(a) = h_{m,i} = \varphi_1 \left( \alpha + \beta \right) \frac{1}{2^m} = \frac{1}{2^m} = \frac{1}{2^{m+n}}.
\]

Thus

\[
\frac{\Phi \left( \frac{a+b}{2} \right) - \Phi(a)}{\frac{a+b}{2} - \alpha} = \frac{\Phi \left( \frac{a+b}{2} \right) - \Phi(a) + \Phi(a) - \Phi(\alpha)}{\frac{a+b}{2} - \alpha} = \frac{\frac{1}{2^m + \left( -\frac{1}{2^n} \right)}}{\frac{a+b}{2} - \alpha} = 0,
\]

and

\[
\Phi \left( \frac{a+b}{2} \right) - \Phi(a) - \frac{\Phi(a) - \Phi(\alpha)}{a - \alpha} \geq \frac{\Phi \left( \frac{a+b}{2} \right) - \Phi(a)}{\frac{a+b}{2} - \alpha}
\]

\[
= \frac{1}{2^{m+n}} \cdot \frac{1}{2^{m+n}}
\]

\[
= \frac{2^{m+1}2^n}{2^{m+n}(2^m + 2)} = \frac{2^{m+1}2^n}{2^{m+n+1}2^n} \cdot \frac{1}{2}
\]

\[
= \frac{2^m+1}{2^m} \cdot \frac{2^m}{2^m+1} = 2^n \cdot \frac{2^m}{2^m+1} \geq 2^m.
\]

The proof of Claim 2.2.4 is similar to the proof of Claim 2.2.3.

One Sided Derivatives in \([0,1]\setminus E\).

**Theorem 2.2.8** Let \(x_0 \in [0,1]\setminus E\). Then

\[
\Phi_-(x_0) \leq 0 \leq \Phi^-(x_0) \text{ and } \Phi^-(x_0) - \Phi_-(x_0) \geq 1.
\]

Also,

\[
\Phi_+(x_0) \leq 0 \leq \Phi^+(x_0) \text{ and } \Phi^+(x_0) - \Phi_+(x_0) \geq \frac{1}{2}.
\]
Proof. Let $x_0 \in [0,1] \backslash E$ and $h > 0$. Let $m > 1$ be even and $t \in \{0,1\}$ such that $x_0 \in (\alpha, \beta) = I^{(m+1,0)}_{n,k}(\alpha', \beta') \in I^{(m,t)}$ and $(\alpha, \beta) \subseteq (x_0 - h, x_0 + h)$. Suppose $x_0 \in (a, b) = J^{(m+1,0)}_{n,k}(\alpha, \beta) \in I^{(m+1,0)}$. Let $\gamma = \frac{a + \beta}{2}$. Let $p$ denote the left endpoint of $J^{(m+1,0)}_{n,2k-1}(\alpha, \beta)$ (see Fig. 2.8 on page 15). Then

$$
\Phi(a) - \Phi(x_0) \leq \Phi(a) - \Phi\left(\frac{a + b}{2}\right) = h^{m}_{n,k} \varphi_m(\gamma) \frac{i}{2^n} = \Phi(a) - \Phi(p).
$$

Thus

$$
\frac{\Phi(x_0) - \Phi(p)}{x_0 - p} \geq 0 \quad \text{and} \quad \frac{\Phi(x_0) - \Phi(a)}{x_0 - a} \leq 0.
$$

Thus

$$
\Phi^-(x_0) \leq 0 \leq \Phi^-(x_0).
$$
Also, noting \( n = m + 1 > 2 \), we have

\[
\frac{\Phi(x_0) - \Phi(p)}{x_0 - p} - \frac{\Phi(x_0) - \Phi(a)}{x_0 - a} \geq \frac{\Phi(a) - \Phi(p)}{x_0 - p}
\]

\[
= \varphi_m(\gamma) \frac{1}{2^n} \frac{1}{x_0 - p} \geq \varphi_m(\gamma) \frac{1}{2^n} \frac{1}{b - p}
\]

\[
= \varphi_m(\gamma) \frac{1}{2^n} \frac{1}{m \left( J_{n,2k-1}^{(m+1,0)}(\alpha,\beta) \right) + m \left( J_{n,k}^{(m+1,0)}(\alpha,\beta) \right)}
\]

\[
= \varphi_m(\gamma) \frac{1}{2^n} \frac{1}{(\gamma - \alpha) a_n + (\gamma - \alpha) \left( \frac{1}{4} \right)^n}
\]

\[
= \varphi_m(\gamma) \frac{1}{2^n} \frac{1}{(\gamma - \alpha) \frac{2^n}{2^n+1} + \frac{1}{2^n}} = \varphi_m(\gamma) \frac{2^{n+1}}{2^n (\gamma - \alpha) 2^n + 3}
\]

\[
= \frac{\varphi_m(\gamma) 2^{n+1}}{\gamma - \alpha 2^n + 3} > \varphi_m(\gamma) > 1.
\]

Thus \( \Phi^-(x_0) - \Phi^-(x_0) \geq 1 \).

For the right derivatives at \( x_0 \), we consider \( \Phi(\gamma) - \Phi(x_0) \) and \( \Phi(\beta) - \Phi(x_0) \). Since \( x_0 \in J_{n,k}^{(m+1,0)}(\alpha,\beta) \),

\[
\Phi(x_0) < \Phi(b) \leq \Phi(\gamma) \text{ and } \Phi(x_0) > \Phi(\alpha) = \Phi(\beta).
\]

Thus

\[
\frac{\Phi(\gamma) - \Phi(x_0)}{\gamma - x_0} > 0 \text{ and } \frac{\Phi(\beta) - \Phi(x_0)}{\beta - x_0} < 0
\]

Also, we have

\[
\frac{\Phi(\gamma) - \Phi(x_0)}{\gamma - x_0} - \frac{\Phi(\beta) - \Phi(x_0)}{\beta - x_0} > \frac{\Phi(\gamma) - \Phi(\beta)}{\beta - x_0} = \varphi_m(\gamma) \frac{1}{\beta - x_0} \geq \varphi_m(\gamma) \frac{1}{\beta - \alpha} \geq \frac{1}{2}.
\]

Thus

\[
\Phi^+(x_0) \leq 0 \leq \Phi^+(x_0) \text{ and } \Phi^+(x_0) - \Phi^+(x_1) \geq \frac{1}{2}.
\]
CHAPTER 3

THE SET OF BESICOVITCH FUNCTIONS IS MEAGER IN $\mathcal{C}$

3.1 Main results on the meagerness of the set of Besicovitch functions.

Define the Saks' set $S := \{ f \in C(0,1) : f^+(x) = f_+(x) = +\infty \text{ for } x \in P, |P| = c \}$. We first claim that $S$ is an analytic set and therefore satisfies the Baire property. Then we present Saks' Theorem 3.1.3 which states that $S$ is comeager in $C(0,1)$. This proves that the set of Besicovitch functions is meager in $C(0,1)$.

Lemma 3.1.1 $Q := \{(t, f) \in (0,1) \times C(0,1) : f^+(t) = +\infty \} \subseteq (0,1) \times C(0,1)$ is $F_{\sigma\delta}$.

Proof. Let

$$Q_{mn} = \left\{ (t, f) : 0 < h \leq \frac{1}{m} \Rightarrow \frac{f(t + h) - f(t)}{h} \geq n \right\}.$$

Then

$$Q = \bigcap_{n} \bigcup_{m} Q_{mn}.$$

Theorem 3.1.2 (Kuratowski) If $X$ and $Y$ are complete separable spaces and $I$ is an analytic subset of $X \times Y$, then the set $A$ of the points $y$ for which the $\{ x : (x, y) \in I \}$ is uncountable is analytic.

Let $P_f := \{ t \in (0,1) : (t, f) \in Q \}$. Let $R := \{ f \in C(0,1) : |P_f| > k_0 \}$. By Mazurkiewicz and Sierpinski (or Kuratowski) $R$ is analytic. Thus $S$ satisfies the Baire property.

Theorem 3.1.3 (Saks) If $B \subseteq C(0,1)$ is an open ball and $\{ A_n \}_{n=1}^{\infty}$ is a sequence of nowhere-dense sets in $C(0,1)$, then $(B \setminus \bigcup_{n=1}^{\infty} A_n) \cap S \neq \emptyset$. Thus $S$ is comeager.
Proof. Let $B$ be an open ball in $C(0,1)$ and $\{A_n\}_{n=1}^{\infty}$ a sequence of nowhere-dense subsets of $C(0,1)$. Define a sequence of open balls $\{B_j\}_{j=0}^{\infty}$, a dyadic system of subintervals $\{I_{n_1n_2...n_j} = (a_{n_1n_2...n_j}, b_{n_1n_2...n_j})\}$, and a sequence of continuous functions $\{f_j(t)\}_{j=1}^{\infty}$, where $n_i \in \{0, 1\}$, by the following conditions:

a. $B_0 = B, B_j \subset B_{j-1}, \overline{B_j} \cap \overline{A_j} = \emptyset$ for $j \geq 1$;

b. $f_j(t)$ is the center of $B_j$ for $j = 0, 1, 2, \ldots$;

c. $I_{n_1n_2...n_{j-1}0} \subseteq I_{n_1n_2...n_{j-1}}, I_{n_1n_2...n_{j-1}0} \cap I_{n_1n_2...n_{j-1}1} = \emptyset, |I_{n_1n_2...n_{j-1}}| \leq \frac{1}{j}$ for $j \geq 1$;

d. $f_j(t)$ is linear with the coefficient $j$ in each interval $I_{n_1n_2...n_j}$ i.e. $f_j(t) = jt + c_{n_1n_2...n_j}$;

e. if $1 < i < j$, $t \in I_{n_1n_2...n_i...n_j} \subseteq I_{n_1n_2...n_i} = (a_{n_1n_2...n_i}, b_{n_1n_2...n_i})$, and $u \in (b_{n_1n_2...n_{i+1}}, b_{n_1n_2...n_i})$ then

$$\frac{f_j(u) - f_j(t)}{u - t} > j - 2 > i - 2.$$

Suppose that the functions $f_j$, balls $B_j$ and subintervals $I_{n_1n_2...n_j}$ have been determined for $j = 1, 2, \ldots, r$ by the above conditions (a) through (e). We define a continuous linear function $f_{r+1}$, a ball $B_{r+1}$ and intervals $\{I_{n_1n_2...n_{r+1}} : n_i = 0, \text{ or } 1\}$. Since $A_{r+1}$ is nowhere-dense, $\forall N_r$, a neighborhood of $f_r$ in $C(0,1)$, $\exists g \in N_r \cap B_r \setminus \overline{A_{r+1}}$. Let $\varepsilon = 1$. Let $l_r$ denote the length of the interval $I_{n_1n_2...n_r}$, which is less than or equal to $\frac{1}{r}$. Let $N_r$ be a neighborhood of $f_r$ such that if $g \in N_r$ then

$$\|f_r - g\| < \frac{1}{8} l_r.$$

Then $\exists g_r \in N_r \cap B_r \setminus \overline{A_{r+1}}$ such that

$$\|g_r - f_r\| < \frac{1}{8} l_r.$$
Suppose $f_r(t) = rt + c_{n_1 n_2 \ldots n_r}$ for $t$ in each of intervals $I_{n_1 n_2 \ldots n_{r-1} 0}$ and $I_{n_1 n_2 \ldots n_{r-1} 1}$. If $I_{n_1 n_2 \ldots n_r} = (a, b)$ then

$$|g_r(t) - rt - c_{n_1 n_2 \ldots n_r}| < \frac{1}{8} l_r = \frac{b - a}{8} \epsilon$$

for $a \leq t \leq b$. Then $\exists \ s_1, s_2 \in I_{n_1 n_2 \ldots n_r}$ such that $s_1 < s_2$ and

$$g_r(t) - g_r(s_i) > (r - 1)(t - s_i)$$

for $s_i < t < b$ (by Lemma 3.2.2). Let $b_{n_1 n_2 \ldots n_r 0} = s_1$ and $b_{n_1 n_2 \ldots n_r 1} = s_2$.

**Claim:** We can find $h \in N_r \cap \overline{B_r \setminus \overline{A_{r+1}}}$ such that $h$ is linear with coefficient $r + 1$ in the subintervals $b_{n_1 n_2 \ldots n_r} \setminus b_{n_1 n_2 \ldots n_r}$ of $I_{n_1 n_2 \ldots n_r}$ whose right ends are the points $b_{n_1 n_2 \ldots n_r 0}, b_{n_1 n_2 \ldots n_r 1}$ respectively and whose left ends are chosen so that the length of the subintervals is smaller than $\frac{1}{r+1}$ in length, and

$$h(t) - h(b_{n_1 n_2 \ldots n_r n_{r+1}}) > (r - 1)(t - b_{n_1 n_2 \ldots n_r n_{r+1}})$$

for $b_{n_1 n_2 \ldots n_r n_{r+1}} < t < b_{n_1 n_2 \ldots n_r}$.

Proof of the claim is given in 3.2.1.

Let $f_{r+1}(t) := h(t)$. Let $I_{n_1 n_2 \ldots n_{r+1}, 0} := \delta_{n_1 n_2 \ldots n_r}$ and $I_{n_1 n_2 \ldots n_{r+1}, 1} := \delta_{n_1 n_2 \ldots n_r}$. Let $B_{r+1}$ be the ball with center $f_{r+1}(t)$ and the radius $\frac{1}{r+1}$ so that $B_{r+1} \subset B_r$ and $\overline{B_{r+1}} \cap \overline{A_{r+1}} = \emptyset$.

Conditions (a) through (d) can be verified immediately. Then condition (e) can be verified by the following argument. Let $(\alpha, \beta) = I_{n_1 n_2 \ldots n_{r+1}}$ and $\gamma = b_{n_1 n_2 \ldots n_r}$. Suppose $\alpha < t < \beta < u < \gamma$. Then

$$\frac{f_{r+1}(u) - f_{r+1}(\beta)}{u - \beta} > r - 1, \quad \frac{f_{r+1}(\beta) - f_{r+1}(t)}{\beta - t} = r + 1, \quad \text{and} \quad \beta - \alpha < \frac{1}{r + 1}.$$
\[
\frac{(r-1)(u-\beta) + (r+1)(\beta - t)}{u-t} \\
= \frac{r(u-t) - u + 2\beta - t}{u-t} = r + \frac{u-t - 2u + 2\beta}{u-t} \\
= r + 1 - 2 \frac{u-\beta}{u-t} > r + 1 - 2 = r - 1.
\]

By conditions (a) and (b), the sequence \(f_j\) converges uniformly to \(f \in B \setminus \bigcup_{n=1}^{\infty} A_n\).

Now we claim that \(f_+(t) = \infty\) for \(t \in I_{n_1} \cap I_{n_1 n_2} \cap I_{n_1 n_2 n_3} \cap \cdots \). By the condition (e), if \(b_{n_1 n_2 \cdots n_i n_{i+1}} < u < b_{n_1 n_2 \cdots n_i}\) and \(1 < i\), then for \(j > i\)

\[
\frac{f_j(u) - f_j(t)}{u-t} > i - 2.
\]

Then as \(j \to \infty\), for \(b_{n_1 n_2 \cdots n_i n_{i+1}} < u < b_{n_1 n_2 \cdots n_i}\),

\[
\frac{f(u) - f(t)}{u-t} \geq i - 2.
\]

Now as \(i \to \infty\), \(u \to t^+\) and \(f_+(t) \geq \infty\). Thus \(f_+(t) = +\infty\).

The set of points \(t\) where \(f_+(t) = \infty\) is a perfect set. \(\blacksquare\)

3.2 Proofs of the Lemmas and the Claims.

Claim 3.2.1 Let \(I = (a, b)\) a subinterval of \((0, 1)\) with length less than or equal to \(\frac{1}{r}\), and \(f_r(t) = rt + c\) for \(t \in I\). Let \(N_r\) be a neighborhood of \(f_r\) such that if \(g \in N_r\) then

\[
\|f_r - g\| < \frac{b-a}{8}.
\]

Let \(g_r \in N_r\). Suppose that \(\exists b_0\) and \(b_1\) are in \(I\) such that \(b_0 < b_1\) and

\[
g_r(t) - g_r(b_i) > (r - 1)(t - b_i)
\]

where \(b_i < t < b\), for each \(i \in \{0, 1\}\). Then we can find \(h \in N_r\), and \(a_0\) and \(a_1\) in \(I\) such that \(a < a_0 < b_0 < a_1 < b_1 < b\), \(\max\{b_0 - a_0, b_1 - a_0\} < \frac{1}{r+1}\), \(h\) is linear in each of \((a_0, b_0)\)
and \((a_i, b_i)\), and

\[ h(t) - h(b_i) > (r - 1)(t - b_i) \]

where \(b_i < t < b\) for each \(i \in \{0, 1\}\).

Proof. Let \(a_0\) and \(a_1\) in \(I\) such that \(a < a_0 < b_0 < a_1 < b_1 < b\). Define \(h\) on \((a, b)\) to be a continuous extension of

\[ h(t) := \begin{cases} 
(r + 1)t + g(b_0) - (r + 1)b_0 & \text{if } t \in (a_0, b_0) \\
(r + 1)t + g(b_1) - (r + 1)b_1 & \text{if } t \in (a_1, b_1) 
\end{cases} \]

We need to give \(a_0\) and \(a_1\) some more conditions in order to prove the claim, i.e., pick \(a_0\) and \(a_1\) so that

\[ |h(t) - rt + ct| < \frac{b - a}{8} \quad \text{and} \quad h(t) - h(b_i) > (r - 1)(t - b_0) \]

for \(b_i < t < b\).

Let \(y_0 = f(b_0) - \frac{b - a}{8}\) and let \(y_1 = f(b_1) - \frac{b - a}{8}\) (see Fig. 3.1 on page 22). Then

\[
\begin{align*}
\text{Case 1: } b_1 > s.
\end{align*}
\]

Let \(a_0 > b_0 - (h(b_0) - y_0)\). But for \(a_1\) there are more conditions to be satisfied.
Figure 3.1: Choice of $a_0$.

Figure 3.2: Choice of $a_1$. 
Case 2: $b_1 \leq s$.

We want to choose $a_1$ so that if $a_1 < t < b_1$ then

$$(r - 1)(t - b_0) + g_r(b_0) < (r + 1)(t - b_1) + g_r(b_1).$$

i.e.,

$$\frac{h(t) - h(b_0)}{t - b_0} > r - 1.$$  

Then

$$(r + 1)t - (r - 1)t > -(r - 1)b_0 + (r + 1)b_1 + g_r(b_0) - g_r(b_1)$$

$$\Rightarrow 2t > b_0 + b_1 + r(b_1 - b_0) + g_r(b_0) - g_r(b_1)$$

$$\Rightarrow t > \frac{b_0 + b_1}{2} + \frac{g_r(b_0) - g_r(b_1)}{2} = \frac{b_0 + b_1}{2} + \frac{1}{2}[(g_r(b_0) - y_0) - (g_r(b_1) - y_1)].$$

Let $a_1 > \max\left\{b_1 - (h(b_1) - y_1), \frac{b_0 + b_1}{2} + \frac{1}{2}[(g_r(b_0) - y_0) - (g_r(b_1) - y_1)]\right\}$. Certainly, this choice of $a_0$ and $a_1$ satisfies the claim. \qed

Lemma 3.2.2 Let $f(t)$ be continuous in an interval $(a, b)$ and let $|f(t) - mt - n| < \frac{b-a}{8\epsilon}$ $(m > 0, \epsilon > 0)$ for every $a \leq t \leq b$. Then there exists in the interval $(a, \frac{a+b}{2})$ an uncountable set of points $c$ with the property that

$$f(t) - f(c) > (m - \epsilon)(t - c)$$

for every $c \leq t \leq b$.

Proof. Let $C := \{c \in (a, \frac{a+b}{2}) : f(t) - f(c) \geq (m - \epsilon)(t - c) \text{ for every } c \leq t \leq b\}$. Suppose $C$ is countable. Let $\{c_i\}$ be an enumeration of elements of $C$. Let $\delta > 0$. Let

$$T := \left\{ t \in [a, b] : f(t) - f(a) \leq (m - \epsilon)(t - a) + \sum_{c_n < t_m} \frac{\delta}{2^n} \right\}.$$  

Let $t_0 := \sup T$. Note that $T$ is closed. Let $\{t_i\} \subseteq T$ and $t_i \to s$. Then since $f(t)$ is continuous $f(t_i) \to f(s)$. Let $\eta > 0$. Let $m > 0$ such that if $i \geq m$ then $|f(t_i) - f(s)| < \eta$.

Then

$$f(s) - f(a) = f(s) - f(t_m) + f(t_m) - f(a) < \eta + (m - \epsilon)(t_m - a) + \sum_{c_n < t_m} \frac{\delta}{2^n}.$$
Thus $t_0 \in T$.

**Claim:** $t_0 \geq \frac{a+b}{2}$.

Then $t_0 \in T$ and $f(t_0) - f(a) \leq (m - \varepsilon)(t_0 - a) + \delta$ and

\[
\frac{b-a}{4} \geq |(f(t_0) - mt_0 - n) - (f(a) - ma - n)| \text{ (hypothesis)}
\]

\[
= | - m(t_0 - a) + f(t_0) - f(a) |
\]

\[
= | m(t_0 - a) - (f(t_0) - f(a)) |
\]

\[
\geq m(t_0 - a) - (f(t_0) - f(a))
\]

\[
\geq \varepsilon(t_0 - a) - \delta
\]

\[
\geq \varepsilon \left( \frac{b + a}{2} - a \right) - \delta = \varepsilon \left( \frac{b - a}{2} \right) - \delta.
\]

This is a contradiction. \(\blacksquare\)

**Proof of the claim in 3.2.2**

Suppose $t \in (a, \frac{a+b}{2}) \cap T$. We will show that $\exists u > t$ such that $u \in T$.

**Case 1. Suppose $t \notin C$.**

Then $\exists u \in (t, b]$ such that $f(u) - f(t) < (m - \varepsilon)(u - t)$. Then

\[
f(u) - f(a) = f(u) - f(t) + f(t) - f(a)
\]

\[
< (m - \varepsilon)(u - t) + (m - \varepsilon)(t - a) + \sum_{n \leq t} \frac{\delta}{2^n}
\]

\[
= (m - \varepsilon)(u - a) + \sum_{n \leq u} \frac{\delta}{2^n}
\]

\[
\leq (m - \varepsilon)(u - a) + \sum_{n \leq u} \frac{\delta}{2^n}
\]

So, $u \in T$.

**Case 2. Suppose $t \in C$.**
Write $t = c_k$. Since $f$ is continuous, there exist a point $u > c_k$ such that $f(u) - f(c_k) < \frac{\delta}{2^k}$.

Then

$$f(u) - f(a) = f(u) - f(c_k) - f(c_k) - f(a)$$

$$\leq (m - \varepsilon)(c_k - a) + \sum_{\forall_{n,c_n < c_k}} \frac{\delta}{2^n} + \frac{\delta}{2^k}$$

$$\leq (m - \varepsilon)(c_k - a) + \sum_{\forall_{n,c_n < u}} \frac{\delta}{2^n}$$

$$\leq (m - \varepsilon)(u - a) + \sum_{\forall_{n,c_n < u}} \frac{\delta}{2^n}$$

So, $u \in T$.

Thus $t_0 \geq \frac{a + b}{2}$.
CHAPTER 4

MORSE-BESICOVITCH FUNCTIONS

4.1 Characteristics of Morse's Construction

A. P. Morse in 1938 constructed Besicovitch functions with the following properties (we will call these Morse-Besicovitch Functions):

\[
\limsup_{x \to z^-} \left| \frac{f(x) - f(z)}{x - z} \right| = \infty \text{ for every } z \in (0, 1]
\]

and

\[
\limsup_{x \to z^+} \left| \frac{f(x) - f(z)}{x - z} \right| = \infty \text{ for every } z \in [0, 1).
\]

Moreover,

\[
\liminf_{x \to z^-} \left| \frac{f(x) - f(z)}{x - z} \right| = \liminf_{x \to z^+} \left| \frac{f(x) - f(z)}{x - z} \right| = 0
\]

for \( z \in \mathcal{P} \) where \( \mathcal{P} \) is a residual subset of \([0, 1]\) with \( 0 < m(\mathcal{P}) < 1 \).

4.2 Notation and Convention

We define notations and make some conventions. We use \([x_1, x_2]\), \((x_1, x_2)\) for the closed, open interval between \(x_1\) and \(x_2\) respectively. When we say “set”, we will always mean the set of real numbers. Every function is real-valued defined on a set of real numbers. Lebesgue measure of a set \(A\) is denoted by \(m(A)\). We use the phrase “\((\alpha, \beta)\) is a maximal interval of a set \(A\)” to mean “\((\alpha, \beta) \subset A\) and \(\{\alpha, \beta\} \subset \overline{\mathbb{R} \setminus A}\).” We also define the following sets:

\[ K(f) := \operatorname{int} \left\{ x \in \operatorname{Dom}(f) : \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = 0 \right\} , \]

\[ H(f) := \operatorname{Dom}(f) \setminus K(f), \quad Z(f) := \{ x \in \mathbb{R} : f(x) = 0 \}, \quad P(f) := \{ x \in \mathbb{R} : f(x) > 0 \} , \]
$A(r) := \{z \in \mathbb{R} : z = \pm 1 \text{ or } z \in (a, b), \text{ where } (a, b) \text{ is a maximal interval of } A \text{ and } b - a > r\}.$

4.3 The Sequence $\{\lambda_n\}$, the Function $\theta$, the Functions of Class $A$, and the Fundamental Operation.

In this section we define essential terms and discuss the immediate consequences. Let $\{\lambda_n\}_{n=-\infty}^{\infty}$ be the sequence of real numbers such that

$$\lambda_n = \frac{1}{2} + \frac{n}{2(|n| + 3)}.$$  

Note that $\lim_{n \to \infty} \lambda_n = 1$ and $\lim_{n \to -\infty} \lambda_n = 0$ and $\lambda_n + \lambda_{-n} = 1$. Also, the following properties can be observed.

**Observation 4.3.1** $\forall n \in \mathbb{Z}$

$$0 < \lambda_{n+1} - \lambda_n < \lambda_n^2 < \lambda_n$$

**Proof.** We first note that $\forall n \in \mathbb{Z}, |n|(n+1) - |n+1||n = 0$. Then $\forall n \in \mathbb{Z}$ we have

$$\lambda_{n+1} - \lambda_n = \frac{1}{2} \left( \frac{n + 1}{|n+1|+3} - \frac{n}{|n|+3} \right)$$

$$= \frac{1}{2} \frac{(n+1)(|n|+3)-n(|n+1|+3)}{(n+1)(|n|+3)}$$

$$= \frac{1}{2} \frac{(|n+1| - n) + 3n + 3 - 3n}{(|n+1|+3)(|n|+3)}$$

$$= \frac{1}{2} \frac{3}{(|n+1|+3)(|n|+3)}$$

$$= \frac{3}{2(|n+1|+3)(|n|+3)}$$

$$= \lambda_n^2 \frac{3}{(|n+1|+3)(n+|n|+3)\frac{n+|n|+3}{n+|n|+3}} < \lambda_n^2.$$  

We also have $\lambda_n < \lambda_{n+1}$ and $\lambda_n^2 < \lambda_n$. ■
Definition 4.3.2 (The Function θ) Let \( \{\gamma_n\} \) be the sequence defined as follows:

\[
\gamma_n = \begin{cases} 
\lambda_n & \text{for } n \text{ odd;} \\
(\lambda_n)^{\frac{1}{2}} & \text{for } n \text{ even}
\end{cases}
\]

Let \( D \) be a nowhere-dense closed subset of \([0, 1]\) with \( m(D) > 0 \) (for example, the set \( E \) defined in Chapter 2), and \( m([\lambda_n - h, \lambda_n] \cap D) > 0 \) and \( m([\lambda_n, \lambda_n + h] \cap D) > 0 \) for \( h > 0 \) and \(-\infty < n < \infty\). We define \( \theta : [0, 2] \to \mathbb{R} \) by

\[
\theta(x) := \gamma_n + (\gamma_{n+1} - \gamma_n) \frac{m([\lambda_n, x] \cap D)}{m([\lambda_n, \lambda_{n+1}] \cap D)}
\]

for \( \lambda_n \leq x \leq \lambda_{n+1} \), and

\[
\begin{align*}
\theta(1) &:= 1; \\
\theta(0) &= \theta(2) = 0; \\
\theta(x) &:= \theta(2 - x) \quad \text{for } 1 < x < 2 \text{ (see Fig. 4.1)}.
\end{align*}
\]

Then we make the following observation from the definition of the function \( \theta \).
Observation 4.3.3 \( \{\lambda_n\}_{n=-\infty}^{\infty} \subseteq H(\theta) \) and \( 1 \in H(\theta) \).

Proof.

\[
\limsup_{h \to 0^+} \frac{\theta(\lambda_n + h) - \theta(\lambda_n)}{h} = \limsup_{h \to 0^+} \frac{(r_{n+1} - r_n)}{m([\lambda_n, \lambda_{n+1}])} \frac{m([\lambda_n, \lambda_{n+1}] \cap D)}{h} = \frac{r_{n+1} - r_n}{m([\lambda_n, \lambda_{n+1}])} \limsup_{h \to 0^+} \left[ \frac{1}{h} m([\lambda_n, \lambda_{n+1}] \cap D) \right] > 0.
\]

Thus \( \lambda_n \in H(\theta) \). Since \( H(\theta) \) is closed and \( \lambda_n \to 1 \) as \( n \to \infty \), then \( 1 \in H(\theta) \).

Definition 4.3.4 (Functions of class \( A \)) A function \( f \) on \((-1,1)\) is said to belong to the class \( A \) if it possesses the following properties:

a. (1) \( f \) is continuous

(2) \( 0 \leq f(-x) = f(x) \) for \( x \in (-1,1) \)

b. (1) \( K(f) \) is dense in \((-1,1)\)

(2) \( P(f) \) is dense in \((-1,1)\)

c. if \( (\alpha, \beta) \) is a maximal interval of \( P(f) \) then there exists a number \( h_{\alpha, \beta} \geq \sqrt{\frac{\beta - \alpha}{2}} \) such that \( x \in (\alpha, \beta) \) implies

\[
f(x) = h_{\alpha, \beta} \theta \left( \frac{2x - 2\alpha}{\beta - \alpha} \right).
\]

Observation 4.3.5

a. \( P(f) = f^{-1}(\{x \in (-1,1): f(x) > 0\}) \) is open since \( f \) is continuous.

b. \( K(f) \subset P(f) \). (Let \( x \in K(f) \). Suppose \( f(x) = 0 \). Then since \( K(f) \) is open we can find \( \epsilon > 0 \) such that \( (x - \epsilon, x + \epsilon) \subset K(f) \). Then \( f(y) = 0 \) for all \( y \in (x - \epsilon, x + \epsilon) \).

This is a contradiction since \( P(f) \) is dense. Thus \( x \in P(f) \).)
c. If $(\alpha, \beta)$ is a maximal interval of $P(f)$, then $f(\alpha) = f(\beta) = 0$ and $f\left(\frac{\alpha + \beta}{2}\right) = \overline{h}_{\alpha,\beta}.$

d. $x \mapsto \frac{2x - 2\alpha}{\beta - \alpha}$ maps $(\alpha, \beta)$ onto $(0, 2)$ linearly.

e. If $f(x) = 0$, then $x \in H(f)$.

**Definition 4.3.6 (The Fundamental Operation on the class $A$)** The Fundamental Operation on the class $A$ is the association of $\overline{f}$ with a function $f \in A$, where $\overline{f}$ on $(-1, 1)$ by the following: if $x \in H(f)$ then

$$\overline{f}(x) = 0,$$

else if $x \in K(f)$ then

$$\overline{f}(x) = \overline{h}_{\alpha,\beta} \theta\left(\frac{2x - 2\alpha}{\beta - \alpha}\right),$$

where $(\alpha, \beta)$ is the interval of $K(f)$ and $x \in (\alpha, \beta)$ and $\overline{h}_{\alpha,\beta}$ is the lesser of the numbers

$$\overline{h}'_{\alpha,\beta} = \sqrt{\inf \left\{K(f)(\beta - \alpha) \cap \left[\frac{\alpha + \beta}{2}, 1\right]\right\} - \sup \left\{K(f)(\beta - \alpha) \cap \left[-1, \frac{\alpha + \beta}{2}\right]\right\}},$$

and $\overline{h}''_{\alpha,\beta} = \frac{1}{\sqrt{2}} \left|f\left(\frac{\alpha + \beta}{2}\right)\right|.$

**Observation 4.3.7**

a. Since $H(f) \subset Z(\overline{f})$ and $K(f) \subset P(\overline{f})$, $H(f) = Z(\overline{f})$ and $K(f) = P(\overline{f})$.

b. If $f(x) = 0$, then by the part b of Observation 4.3.5 $x \in H(f)$ and $\overline{f}(x) = 0$.

c. If $(\alpha, \beta)$ is an interval of $K(f)$, then $\overline{f}(\alpha) = \overline{f}(\beta) = 0$ and $\overline{f}\left(\frac{\alpha + \beta}{2}\right) = \overline{h}_{\alpha,\beta}$.

We devote the next two sections to the study of this operation.
4.4 The Function $\theta(x+1)$ is of Class $A$.

In this section, we show that the function $\theta(x+1)$ belongs to the class $A$. We have separate subsections to show that the function $\theta$ satisfies each of the properties of the functions of class $A$.

The property a.

Lemma 4.4.1 $\theta$ is continuous on $(0,2)$.

Proof. $\theta$ is clearly continuous on $[\lambda_n, \lambda_{n+1}]\forall n$. $\theta$ is continuous at $x = 1$ for the following reasons: If $\gamma_{n+1} - \gamma_n < 0$ then

$$\lambda_n < \lambda_{n+1} \leq \gamma_{n+1} = \theta(\lambda_{n+1}) \leq \theta(x) \leq \theta(\lambda_n) = \gamma_n \leq \lambda_n^{\frac{1}{2}} < \lambda_{n+1}^{\frac{1}{2}},$$

and if $\gamma_{n+1} - \gamma_n > 0$ then

$$\lambda_n \leq \gamma_n = \theta(\lambda_n) \leq \theta(x) \leq \theta(\lambda_{n+1}) = \gamma_{n+1} \leq \lambda_{n+1}^{\frac{1}{2}}.$$

Thus we have

$$\lambda_n \leq \theta(x) \leq \lambda_{n+1}^{\frac{1}{2}} \text{ for } \lambda_n \leq x \leq \lambda_{n+1}. \quad (4.1)$$

Thus we also have

$$\lim_{n \to \infty} \lambda_n = \lim_{n \to \infty} \lambda_n^{\frac{1}{2}} = 1. \quad (4.2)$$

Then $\lim_{y \to 1^-} \theta(y) = 1$ (also $\lim_{y \to 1^+} \theta(y) = 1$ since $\theta(y) = \theta(2 - y)$ for $1 < y < 2$). Thus $\theta$ is continuous at 1; hence continuous on $(0,2)$. ■

Thus $\theta(x+1)$ satisfies the property a-(1) of Definition 4.3.4. Also, from the last part of the definition of the function $\theta$ [4.3.2], we conclude that $\theta(x + 1)$ is even and non-negative on $(-1,1)$.
The property b. By the definition of $\theta$ [4.3.2], $K(\theta)$ is dense in the interval $(0,2)$. So, $\theta(x+1)$ satisfies the property b-(1) of Definition 4.3.4. The following lemma is used to show that $\theta(x+1)$ also satisfies the property b-(2).

**Lemma 4.4.2** If $0 < x < 1$ then

$$\frac{x}{2} < \theta(x) < \frac{x+3}{4}.$$  

**Proof.** From Lemma 4.3.1, we get $\lambda_{n+1} - \lambda_n < \lambda_n$ and $\lambda_{n+1} < 2\lambda_n$. Then by (4.1) in Lemma 4.4.1 we get

$$\theta(x) > \lambda_n > \frac{\lambda_{n+1}}{2} \geq \frac{x}{2}$$

and

$$\theta(x) \leq \sqrt{\lambda_{n+1}} = \sqrt{1 - \frac{\lambda_{n-1}}{2}} < \sqrt{1 - \frac{\lambda_n}{2}} = \sqrt{\frac{2 - \lambda_n}{2}} = \sqrt{\frac{1 + \lambda_n}{2}} = \sqrt{1 - \frac{1 + \lambda_n}{2}}$$

$$\leq \frac{1}{2} \left(1 + \frac{1 + \lambda_n}{2}\right) = \frac{13 + \lambda_n}{2}$$

$$\leq \frac{3 + x}{4}$$

Thus $P(\theta(x+1)) = (-1,1)$.

The property c. Since $(-1,1) = (\alpha, \beta)$ is the maximal interval of $P(\theta(x+1))$, by letting $h_{\alpha,\beta} = 1$, $\theta$ satisfies the property c of Definition 4.3.4.

4.5 The Fundamental Operation on the Functions of Class A.

The property a of $\bar{f}$

**Lemma 4.5.1** $\lim_{y \to 0^+} \theta(y) = 0$. 
Proof. From the first paragraph of the section 4.3,

$$\lim_{n \to -\infty} \lambda_n = \lim_{n \to -\infty} \lambda_n^{\frac{1}{2}} = 0.$$  

Then by (4.1) in the proof of Lemma 4.4.1,

$$\lim_{y \to 0^+} \theta(y) = 0.$$

\[\square\]

**Lemma 4.5.2** If $f \in A$ and $(\alpha, \beta)$ is a maximal interval of $K(f)$, then

$$\lim_{y \to \alpha^+} \overline{I}(y) = \overline{I}(\alpha) = 0.$$

i.e., $\overline{I}$ is right continuous at $\alpha$.

Proof. Since $H(f) = \mathcal{Z}(\overline{I})$, $\overline{I}(\alpha) = 0$. And, by Lemma 4.5.1,

$$\lim_{y \to \alpha^+} \overline{I}(y) = \overline{h}_{\alpha, \beta} \lim_{y \to \alpha^+} \theta\left(\frac{2y - 2\alpha}{\beta - \alpha}\right) = \overline{h}_{\alpha, \beta} \lim_{x \to 0^+} \theta(x) = 0.$$

\[\square\]

**Lemma 4.5.3** If $f \in A$ and $(\alpha, \beta)$ is a maximal interval of $K(f)$ such that $\{\alpha, \beta\} \subseteq (-1, 1)$ and if $x \in (\alpha, \beta)$, then

$$0 < \overline{I}(x) \leq \overline{h}_{\alpha, \beta} = \overline{I}\left(\frac{\alpha + \beta}{2}\right) \leq \frac{1}{\sqrt{2}} f(x).$$

Proof. Since $K(f) = P(\overline{I})$, $\overline{I}(x) > 0$. Then by Observation 4.3.7

$$\overline{I}(x) = \overline{h}_{\alpha, \beta} \theta\left(\frac{2x - 2\alpha}{\beta - \alpha}\right) \leq \overline{h}_{\alpha, \beta} \leq \overline{h}_{\alpha, \beta} = \frac{1}{\sqrt{2}} f\left(\frac{\alpha + \beta}{2}\right) = \frac{1}{\sqrt{2}} f(x).$$

\[\square\]
Lemma 4.5.4 If \( f \in A \), then for \(-1 < x < 1\),
\[
0 \leq \overline{f}(x) \leq \frac{1}{\sqrt{2}} f(x).
\]

**Proof.** Since \( f \) is nonnegative, for \( x \in H(f) = Z(f) \),
\[
\overline{f}(x) = 0 \leq \frac{1}{\sqrt{2}} f(x).
\]

Then by Lemma 4.5.3 the statement is true. \( \blacksquare \)

Define

\[
M(r) := \sup_{-1 < x < 1} \left\{ \inf \left\{ K(f)^{(r)} \cap [-1, 1] \right\} - \sup \left\{ K(f)^{(r)} \cap [-1, 1] \right\} \right\} \tag{4.3}
\]

**Lemma 4.5.5** \( \lim_{r \to \infty} M(r) = 0. \)

**Proof.** Let \( \varepsilon > 0 \). Let \(-1 = a_0 < a_1 < \ldots < a_N = 1\) such that \( a_i - a_{i-1} < \varepsilon/2 \) for \( i = 1, \ldots, N \). Since \( K(f) \) is dense in \((-1, 1), \exists y_i \in K(f) \cap (a_{i-1}, a_i) \). Since \( K(f) \) is open, \( \exists \delta_i > 0 \) such that \( (y_i - \delta_i, y_i + \delta_i) \subseteq K(f) \cap (a_{i-1}, a_i) \). Then \( y_i \in K(f)^{\delta_i} \cap (a_{i-1}, a_i) \). Thus \( K(f)^{\delta_i} \cap (a_{i-1}, a_i) \neq \emptyset \). Let \( \delta = \min \{ \delta_i \} \). Then \( K(f)^{\delta} \supseteq K(f)^{\delta_i} \) for all \( i \). Suppose \( 0 < r < \delta \).

Let \( t \in [a_{i-1}, a_i] \). Then either

\[
(a_{i-1}, t] \bigcap K(f)^{(r)} \neq \emptyset \tag{4.4}
\]

or

\[
[t, a_i) \bigcap K(f)^{(r)} \neq \emptyset \tag{4.5}
\]

In the case of 4.4,

\[
\sup \left\{ K(f)^{(r)} \cap [-1, 1] \right\} > a_{i-1} \text{ and } \inf \left\{ K(f)^{(r)} \cap [t, 1] \right\} < a_{i+1}.
\]

Then

\[
\inf \left\{ K(f)^{(r)} \cap [t, 1] \right\} - \sup \left\{ K(f)^{(r)} \cap [-1, 1] \right\} < a_{i+1} - a_{i-1} < \varepsilon.
\]
In the case of 4.5,

\[ \inf \left\{ K(f)^{(r)} \cap [t, 1] \right\} > a_i \text{ and } \sup \left\{ K(f)^{(r)} \cap [-1, t] \right\} > a_{i-2}. \]

Then

\[ \inf \left\{ K(f)^{(r)} \cap [t, 1] \right\} - \sup \left\{ K(f)^{(r)} \cap [-1, t] \right\} < a_i - a_{i-2} < \epsilon. \]

Thus \( M(\tau) < \epsilon. \]

Lemma 4.5.6 If \( f \in A \) and \( x \in H(f) \cap P(f) \), then

\[ \lim_{y \to x^+} \bar{f}(y) = \bar{f}(x) = 0. \]

Proof. Let \( x \in H(f) \cap P(f) \) be fixed. Suppose \( x \) is not a cluster point of \( H(f) \cap [x, 1] \).
Then \( \exists \epsilon > 0 \) such that \( (x, x + \epsilon) \cap H(f) = \emptyset \). Then \( x \) is the left endpoint of some maximal interval of \( K(f) \). Then by 4.5.2, the statement (a) is true. Now suppose \( x \) is a cluster point of \( H(f) \cap [x, 1] \). We will show that, for an arbitrary sequence \( \{x_n\} \to x^+ \), \( \{\bar{f}(x_n)\} \to \bar{f}(x) = 0 \). Since \( \bar{f}(y) = 0 \) for all \( y \in H(f) \), we will take \( \{y_n\} \subseteq K(f) \) such that \( y_n \to x^+ \).

Let \( (\alpha_n, \beta_n) \) be the maximal interval of \( K(f) \) such that \( y_n \in (\alpha_n, \beta_n) \).

Claim: \( \beta_n \to x^+ \) as \( n \to \infty \).

Proof: Let \( \epsilon > 0 \). Let \( N \in \mathbb{N} \) such that \( y_n - x < \epsilon \). Then \( \alpha_n > x \) since \( x \) is cluster point of \( H(f) \cap [x, 1] \). Then \( \exists N' > 0 \) such that if \( m \geq N' \) then \( \beta_m < \alpha_n \), i.e. \( \beta_m - x < \epsilon \).

Then \( (\beta_n - \alpha_n) \to 0 \) as \( n \to \infty \). So, by Lemma 4.5.5, \( M(\beta_n - \alpha_n) \to 0 \) as \( n \to \infty \). Also,

\[
\bar{f}\left(\frac{\alpha_n + \beta_n}{2}\right) = \bar{h}_{\alpha_n, \beta_n} \leq \bar{h}_{\alpha_n, \beta_n}
\]

\[
= \sqrt{\inf \left\{ K(f)^{(\beta_n-\alpha_n)} \cap \left[ \frac{\alpha_n + \beta_n}{2}, 1 \right] \right\} - \sup \left\{ K(f)^{(\beta_n-\alpha_n)} \cap \left[ -1, \frac{\alpha_n + \beta_n}{2} \right] \right\}}
\]

\[
\leq \sqrt{M(\beta_n - \alpha_n)}
\]
and by Lemma 4.5.3
\[ f(y_n) \leq f\left(\frac{\alpha_n + \beta_n}{2}\right). \]
Thus \( f(y_n) \to 0 \) as \( n \to \infty \).

\[ \square \]

**Lemma 4.5.7** If \( f \in A \) and \( (\alpha, \beta) \) is a maximal interval of \( K(f) \) such that \( \{\alpha, \beta\} \subseteq (-1, 1) \) and if \( x \in (\alpha, \beta) \), then \( f(-x) = \overline{f}(x) \).

**Proof.** We use the evenness of \( f \). Note \((\beta, -\alpha)\) is a maximal interval of \( K(f) \), too. If \( x \in (\alpha, \beta) \) then,
\[ \overline{f}(x) = \overline{h}_{\alpha, \beta} \theta\left(\frac{2x - 2\alpha}{\beta - \alpha}\right) \]
and
\[ \overline{f}(-x) = \overline{h}_{-\beta, -\alpha} \theta\left(\frac{2(-x) + 2\beta}{-\alpha + \beta}\right). \]

We first note that \( \overline{h}_{\alpha, \beta} = \overline{h}_{-\beta, -\alpha} \) by the following:
\[ \overline{h}'_{-\beta, -\alpha} = \sqrt{-\alpha - (-\beta)} = \sqrt{\beta - \alpha} = \overline{h}'_{\alpha, \beta} \]
and
\[ \overline{h}''_{-\beta, -\alpha} = \frac{1}{\sqrt{2}} f\left(\frac{(-\beta) + (-\alpha)}{2}\right) = \frac{1}{\sqrt{2}} f\left(\frac{\alpha + \beta}{2}\right) = \overline{h}''_{\alpha, \beta}. \]

Also,
\[ \theta\left(\frac{-2x + 2\beta}{-\alpha + \beta}\right) = \theta\left(2 - \frac{-2x + 2\beta}{-\alpha + \beta}\right) = \theta\left(\frac{-2\alpha + 2\beta + 2x - 2\beta}{-\alpha + \beta}\right) = \theta\left(\frac{2x - 2\alpha}{\beta - \alpha}\right). \]
Thus \( \overline{f}(-x) = \overline{f}(x) \). \[ \square \]

**Lemma 4.5.8** If \( f \in A \) and \( x \in (-1, 1) \), then \( \overline{f}(-x) = \overline{f}(x) \).
Proof. Since $f$ is even, $-x \in H(f)$ when $x \in H(f)$. But we know $H(f) = Z(\overline{f})$; thus
$$\overline{f}(x) = 0 = \overline{f}(-x) \text{ for } x \in H(f).$$
Thus $\overline{f}$ is even.

**Lemma 4.5.9** If $f \in A$, then $\overline{f}$ is continuous.

Proof.

- $\overline{f}$ is right continuous on $H(f)$.

By Lemma 4.5.6 $\overline{f}$ is right continuous on $H(f) \cap P(f)$. By Lemma 4.5.4 $0 \leq \overline{f}(x) \leq \frac{1}{\sqrt{2}} f(x)$.

Since $f$ is continuous, $\lim_{y \to x} f(y) = f(x)$. Now if $x_0 \in H(f) \cap Z(f)$ then
$$\overline{f}(x_0) = 0 \leq \lim_{y \to x_0^+} \overline{f}(y) \lim_{y \to x_0^+} \frac{1}{\sqrt{2}} f(x) = \frac{1}{\sqrt{2}} f(x_0) = 0.$$
Thus $\overline{f}$ is right continuous on $H(f) \cap Z(f)$.

- $\overline{f}$ is right continuous on $K(f)$

If $z \in (\alpha, \beta)$ where $(\alpha, \beta)$ is a maximal interval of $K(f)$, then $\overline{f}(z) = \overline{h}_{\alpha, \beta}\left(\frac{2z-3\alpha}{\beta-\alpha}\right)$. By Lemma 4.4.1, $\overline{h}$ is continuous on $(0, 2)$. So $\overline{f}$ is right continuous on $K(f)$.

By Lemma 4.5.8 $\overline{f}$ is even, so $\overline{f}$ is also left continuous.

The property b of $\overline{f}$ Since $K(\overline{h})$ is dense in $(0, 2)$, then $K(\overline{f})$ is dense in $K(f)$. Thus $K(\overline{f})$ is dense in $(-1, 1)$. Also by definition $P(\overline{f}) = K(f)$ is dense.

The property c of $\overline{f}$

**Lemma 4.5.10** If $(\alpha, \beta)$ is a maximal interval of $K(\overline{h})$, then $\overline{h}\left(\frac{\alpha+\beta}{2}\right) > \sqrt{\beta - \alpha}$. 
Proof. Let \((\alpha, \beta)\) be a maximal interval of \(K(\theta)\). Then by Observation 4.3.3, we can assume that \((\alpha, \beta) \subseteq [\lambda_n, \lambda_{n+1}]\) for some \(n\). Then, by (4.1) in Lemma 4.4.1 and Observation 4.3.1,

\[
\theta\left(\frac{\alpha + \beta}{2}\right) \geq \lambda_n > \sqrt{\lambda_{n+1} - \lambda_n} \geq \sqrt{\beta - \alpha}.
\]

Lemma 4.5.11 If \(f \in A\) and \((\alpha, \beta)\) is a maximal interval of \(K(f)\) such that \(\alpha, \beta \subseteq (-1, 1)\), then \(f\left(\frac{\alpha + \beta}{2}\right) > \sqrt{\beta - \alpha}\).

Proof. By Observation 4.3.5, \(K(f) \subseteq P(f)\).

Claim: \(\{\alpha, \beta\} \subseteq P(f)\)

Let \(y = f(x)\) for \(x \in (\alpha, \beta)\). Then \(y > 0\). Then \(f(\alpha) = y = f(\beta)\) by the continuity of \(f\). Thus \(\{\alpha, \beta\} \subseteq P(f)\).

Since \(P(f)\) is open \([4.3.4]\), \(\exists \epsilon > 0\) such that \((\alpha - \epsilon, \beta + \epsilon) \subseteq P(f)\). Let \((\alpha', \beta')\) be the maximal interval of \(P(f)\) such that \(\alpha' < \alpha < \beta < \beta'\). Now let \(\mu := \frac{\alpha + \beta}{2}\) and \(\mu' := \frac{\alpha' + \beta'}{2}\).

Let \(g : (\alpha', \beta') \to (0, 2)\) by

\[
g(x) = \frac{2x - 2\alpha'}{\beta' - \alpha'}.
\]

Let

\[
\alpha'' := g(\alpha) = \frac{2\alpha - 2\alpha'}{\beta' - \alpha'} \quad \text{and} \quad \beta'' := g(\beta) = \frac{2\beta - 2\alpha'}{\beta' - \alpha'}
\]

and

\[
\mu'' := g(\mu) = g\left(\frac{\alpha + \beta}{2}\right) = \frac{\alpha'' + \beta''}{2} \quad \text{and} \quad g(\mu') = 1.
\]

By the part c of Definition 4.3.4 we observe that \(f(x) = h_{\alpha, \beta} \theta(g(x))\) for \(x \in (\alpha', \beta')\) where \(h \leq \sqrt{\frac{\beta' - \alpha'}{2}}\).

Claim: \((\alpha'', \beta'')\) is a maximal interval of \(K(\theta)\).

We know that \((\alpha, \beta)\) is a maximal interval of \(K(f)\) and \(f'(x) = h_{\alpha, \beta} \theta'(g(x))g'(x) = \).
$2h_{\alpha, \beta} \theta'(g(x))$. So $f'(x) = 0$ for $x \in (\alpha, \beta)$. Then $\theta'(g(x)) = 0$ for $x \in (\alpha, \beta)$. Then $\theta'(x) = 0$ for $x \in (g(\alpha), g(\beta)) = (\alpha'', \beta'')$. Thus $(\alpha'', \beta'')$ is a maximal interval of $K(\theta)$.

Then by above claim and by Lemma 4.5.10,

$$f(\mu) = f\left(\frac{\alpha + \beta}{2}\right) = h_{\alpha, \beta} \theta(g(\mu))$$

$$> h_{\alpha, \beta} \sqrt{\beta'' - \alpha''} \geq \sqrt{\frac{\beta' - \alpha'}{2}} \sqrt{\beta'' - \alpha''}$$

$$= \sqrt{\frac{\beta' - \alpha'}{2} \cdot \left(\frac{2\beta - 2\alpha}{\beta' - \alpha'} - 2\alpha - 2\alpha'\right)} = \sqrt{\beta - \alpha}.$$

\[ \blacksquare \]

**Lemma 4.5.12** If $f \in \mathcal{A}$ and $(\alpha, \beta)$ is a maximal interval of $K(f)$ such that $\{\alpha, \beta\} \subset (-1, 1)$, then

$$\mathcal{J}\left(\frac{\alpha + \beta}{2}\right) > \frac{\sqrt{\beta - \alpha}}{2} \quad \text{(i.e. } h_{\alpha, \beta} > \frac{\sqrt{\beta - \alpha}}{2}\text{)}).$$

*Proof.* Let

$$\alpha_0 = \sup \left\{ K(f)^{\beta - \alpha} \bigcup \left[-1, \frac{\alpha + \beta}{2}\right] \right\} \text{ and } \beta_0 = \inf \left\{ K(f)^{\beta - \alpha} \bigcup \left[\frac{\alpha + \beta}{2}, 1\right] \right\}.$$ 

Then

$$-\alpha_0 = \inf \left\{ K(f)^{\beta - \alpha} \bigcup \left[-\frac{\alpha + \beta}{2}, 1\right] \right\} \text{ and } -\beta_0 = \sup \left\{ K(f)^{\beta - \alpha} \bigcup \left[-\frac{\alpha + \beta}{2}, 1\right] \right\}.$$ 

Notice that $\alpha_0 \leq \alpha$ and $\beta \leq \beta_0$. Then

$$h_{\alpha, \beta}' := \sqrt{\beta_0 - \alpha_0} \geq \sqrt{\beta - \alpha} > \sqrt{\frac{\beta - \alpha}{2}}$$

and by noting $K(f) = P(\mathcal{J})$ and by Lemma 4.5.11

$$h_{\alpha, \beta}'' := \frac{1}{\sqrt{2}} \left| f\left(\frac{\alpha + \beta}{2}\right) \right| = \frac{1}{\sqrt{2}} f\left(\frac{\alpha + \beta}{2}\right)$$

$$> \frac{1}{\sqrt{2}} \sqrt{\beta - \alpha} = \sqrt{\frac{\beta - \alpha}{2}}.$$
Then \( \overline{f}(\frac{a+b}{2}) = \overline{h}_{a,b} > \sqrt{\frac{b-a}{2}} \).

Then \( \overline{f} \) satisfies the property \( c \) of class \( \mathcal{A} \).
4.6 The Morse-Besicovitch Function $F$.

Let $\{F_n\}$ be the sequence of functions on $(-1,1)$ determined by $F_0(x) = \theta(x + 1)$ for $-1 < x < 1$, and $F_{n+1} = \overline{F}_n$ for $n \geq 1$. Define $F : [-1,1] \rightarrow \mathbb{R}$ by

$$F(x) = \sum_{n=0}^{\infty} (-1)^n F_n(x).$$

Then by Lemma 4.5.4,

$$0 \leq F_{n+1}(x) \leq \frac{1}{\sqrt{2}} F_n(x).$$

By recursion,

$$F_n(x) \leq \left(\frac{1}{\sqrt{2}}\right)^n F_0(x).$$

Thus

$$F = \sum_{n=0}^{\infty} (-1)^n F_n \leq \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\sqrt{2}}\right)^n F_0 = F_0 \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = F_0 \frac{1}{1 + \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{1 + \sqrt{2}} F_0$$

and $F$ is well defined.

Morse-Besicovitch Function $F$ is even and continuous. By discussions in previous sections, each $F_n$ belongs to the class $\mathcal{A}$. Then each $F_n$ is even and continuous. So, $F$ is even.

Claim 4.6.1 $F = \sum_{n=1}^{\infty} (-1)^n F_n$ is uniformly convergent.

Proof.

$$|F(x) - S_n(x)| = \left| \sum_{i=n+1}^{\infty} (-1)^i F_i(x) \right| \leq F_{n+1}(x) \leq \frac{F_n(x)}{\sqrt{2}} \leq \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right)^n F_0(x) \leq \left(\frac{1}{\sqrt{2}}\right)^{n+1}.$$
Notations and Immediate Observations

We employ the following notations.

\[ S_n := \sum_{i=0}^{n} (-1)^i F_i \quad R_n := \sum_{i=n+1}^{\infty} (-1)^i F_i \]
\[ P_n := P(F_n) \quad Z_n := Z(F_n) \]
\[ P := \bigcap_{n=0}^{\infty} (P_n) \quad Z := \bigcup_{n=0}^{\infty} (Z_n) \]

Observation 4.6.2

a. If \( x \in Z_{n+1} \) (i.e. \( F_{n+1}(x) = 0 \)), then \( R_n(x) = 0 \).

b. \( P_n \cup Z_n = (-1,1) \) (\( \forall \ n, H(F_n) \cup K(F_n) = (-1,1) \). Since \( H(F_n) = Z(F_{n+1}) \) and \( K(F_n) = P(F_{n+1}) \), \( Z(F_{n+1}) \cup P(F_{n+1}) = (-1,1) \).

c. \( P \cup Z = (-1,1) \)

d. \( Z_n \subset Z_{n+1}, \ P_n \supset P_{n+1}, \) and \( P_0 = (-1,1) \) (\( P(F_{n+1}) = K(F_n) \subseteq P(F_n) \Rightarrow P_{n+1} \subseteq P_n \Rightarrow Z_n \subset Z_{n+1} \).
4.7 One-sided Derivatives of the Morse Function $F$

Lower One-sided Derivatives in $\mathcal{P}$

Theorem 4.7.1 If $x \in \mathcal{P}$, then

$$
\liminf_{y \to x+} \left| \frac{F(y) - F(x)}{y - x} \right| = 0.
$$

Let $x_0 \in \mathcal{P}$ then $x_0 \in P_n$ for all $n$. Let $(\alpha_n, \beta_n)$ be the interval of $P_n$ such that $x_0 \in (\alpha_n, \beta_n)$. Let $\mu_n = \frac{\alpha_n + \beta_n}{2}$. Then if $0 \leq i \leq n - 1$ then $(\alpha_n, \beta_n) \subset P_n \subset P_{i+1} = K(F_i)$.

Then $0 \leq i \leq n - 1$ implies

$$
F_i(\mu_n) - F_i(x_0) = F_i(\beta_n) - F_i(x_0) = 0 \quad (4.6)
$$

and $S_i(\mu_n) - S_i(x_0) = S_i(\beta_n) - S_i(x_0) = 0$.

Claim 4.7.2 For $n$ even and positive $F(\beta^N) - F(x_0) \leq 0$ while for $n$ odd and positive $F(\beta^N) - F(x_0) \leq 0$.

Proof.

Case 1: $n$ is odd and positive.

$$
F(\beta_n) - F(x_0) = F_n(x_0) - R_n(x_0) \geq F_n(x_0) - |R_n(x_0)| \geq F_n(x_0) - \frac{1}{\sqrt{2}} F_n(x_0) \geq 0.
$$

Case 2: $n$ is even and positive.

$$
F(\beta_n) - F(x_0) = [S_{n-1}(\beta_n) + F_n(\beta_n) + R_n(\beta_n)] - [S_{n-1}(x_0) + F_n(x_0) + R_n(x_0)]
$$

$$
= [S_{n-1}(\beta_n) - S(n+1)(x_0)] + [F_n(\beta_n) + R_n(\beta_n)] - [F_n(x_0) + R_n(x_0)]
$$

$$
= 0 + [F_n(\beta_n) + R_n(\beta_n)] - [F_n(x_0) + R_n(x_0)]
$$

$$
= -F_n(x_0) - R_n(x_0) \leq -F_n(x_0) + |R_n(x_0)| \leq -F_n(x_0) + \frac{1}{\sqrt{2}} F_n(x_0)
$$

$$
= \left( \frac{1}{\sqrt{2}} - 1 \right) F_n(x) \leq 0.
$$
The consequence of the claim is that we have for $n$ odd

$$F(\beta_n) - F(x_0) \geq 0 \geq F(\beta_{n+1}) - F(x_0)$$

and for $n$ even

$$F(\beta_n) - F(x_0) \leq 0 \leq F(\beta_{n+1}) - F(x_0).$$

Then

$$F(\beta_{n+1}) \leq F(x_0) \leq F(\beta_n) \text{ for } n \text{ odd, and } F(\beta_n) \leq F(x_0) \leq F(\beta_{n+1}) \text{ for } n \text{ even.}$$

Since $F$ is continuous, by IVT $\exists y_n$ such that $\beta_{n+1} \leq y_n \leq \beta_n$ and $F(x_0) = F(y_n)$. This defines a sequence \{y_n\}. Now, by Lemma 4.5.12 and the facts $F_n = F_{n-1}$ and $P(F_n) = K(F_{n-1}),$

$$0 < y_n - x_0 \leq \beta_n - x_0 < \beta_n - \alpha_n$$

$$< 2 \left( F_n \left( \frac{\alpha_n + \beta_n}{2} \right) \right)^2$$

$$\leq 2 \left( \left( \frac{1}{\sqrt{2}} \right)^{n+1} \right)^2 = \left( \frac{1}{2} \right)^n. \tag{4.7}$$

Then

$$\liminf_{y \to x_0^+} \frac{F(y) - F(x_0)}{y - x_0} \leq \liminf_{n \to \infty} \left| \frac{F(y_n) - F(x_0)}{y_n - x_0} \right| = \liminf_{n \to \infty} \frac{0}{y_n - x_0} = 0.$$  

Upper One-sided Derivatives in $\mathcal{P}$

**Lemma 4.7.3** If $f \in A$ and $(\alpha, \beta)$ is a maximal interval of $K(f)$, then \{\alpha, \frac{\alpha + \beta}{2}, \beta\} $\subset H(f)$.

**Proof.** Define a sequence \{y_n\} by

$$y_n = \alpha + \lambda_n \left( \frac{\beta - \alpha}{2} \right).$$
Then by Lemma 4.3.3.

\[ \mathcal{F}(y_n) = \Re_{\alpha, \beta} \left( \frac{2y_n - 2\alpha}{\beta - \alpha} \right) = \Re_{\alpha, \beta} \theta(\lambda_n). \]

Since \( \lambda_n \in H(\theta) \), \( y_n \in H(\mathcal{F}) \). Note that \( H(\mathcal{F}) \) is closed by its definition and \( \{y_n\} \subset H(\mathcal{F}) \).

Since \( \alpha = \lim_{n \to \infty} y_n, \alpha \in H(\mathcal{F}) \). Also since \( \frac{a + \beta}{2} = \lim_{n \to \infty} y_n, \frac{a + \beta}{2} \in H(\mathcal{F}) \). By the symmetry of \( \theta \), \( \beta \in H(\mathcal{F}) \).

**Theorem 4.7.4** If \( x \in \mathcal{P} \), then

\[ \limsup_{y \to x^+} \left| \frac{F(y) - F(x)}{y - x} \right| = \infty. \]

**Proof.** Let \( x_0 \in \mathcal{P} \). Let \( (\alpha_n, \beta_n) \) be the maximal interval of \( P_n \) such that \( x_0 \in (\alpha_n, \beta_n) \). Let \( \mu_n = \frac{a + \beta_n}{2} \). Let \( n \) be fixed.

**Case 1: \( \alpha_0 < x_0 < \mu_n \) for \( n \) even and positive.**

\[ |F(\mu_n) - F(x_0)| = |S_{n-1}(\mu_n) + F_n(\mu_n) + R_n(\mu_n) - S_{n-1}(x_0) - F_n(x_0) - R_n(x_0)| \]
\[ \geq |S_{n-1}(\mu_n) - S_{n-1}(x_0)| + |F_n(\mu_n) - F_n(x_0) - R_n(x_0)| \]
\[ \geq |F_n(\mu_n) - F_n(x_0) - R_n(x_0)| \]
\[ \geq F_n(\mu_n) - F_n(x_0) \]
\[ \geq F_n(\mu_n) - F_n(\mu_n) \left[ \frac{2x_0 - 2\alpha_n}{\beta_n - \alpha_n} \right] \]
\[ = F_n(\mu_n) \left\{ 1 - \theta \left( \frac{2x_0 - 2\alpha_n}{\beta_n - \alpha_n} \right) \right\} \]
\[ > F_n(\mu_n) \left\{ 1 - \frac{1}{4} \left( \frac{2x_0 - 2\alpha_n}{\beta_n - \alpha_n} + 3 \right) \right\} \]
\[ = F_n(\mu_n) \left\{ \frac{4(\beta_n - \alpha_n) - 2x_0 + 2\alpha_n - 3(\beta_n - \alpha_n)}{4(\beta_n - \alpha_n)} \right\} \]
\[ = F_n(\mu_n) \left\{ \frac{\alpha_n + \beta_n - 2x_0}{4(\beta_n - \alpha_n)} \right\} \]
\[ F_n(\mu_n) = \frac{\mu_n - x_0}{2(\beta_n - \alpha_n)} \]

\[ > \left( \frac{\beta_n - \alpha_n}{2} \right)^{1/2} \cdot \left( \frac{\mu_n - x_0}{2\beta_n - 2\alpha_n} \right) \]  

(4.13)

\[ = \frac{1}{2\sqrt{2}} \cdot \frac{\mu_n - x_0}{\sqrt{\beta_n - \alpha_n}} \]

\[ \geq \frac{1}{2\sqrt{2}} \cdot \frac{1}{(\frac{1}{2})^{\frac{1}{2}}} \cdot (\mu_n - x_0) \]

\[ = (2)^{n-\frac{3}{2}+\text{order}} \cdot (\mu_n - x_0). \]  

(4.14)

We provide some reasons to this formula.

- (4.8) \( P(F_n) = K(F_{n-1}) \) and, by Lemma 4.7.3, \( \mu_n \in H(F_{n-1}) = H(F_n) = Z_{n+1} \). So, 
  
  \( R_n(\mu_n) = 0. \)

- (4.9) Equation 4.6 in Theorem 4.7.1 on page 43.

- (4.10) \( R_n(x_0) \leq 0 \) for \( n \) even.

- (4.11) \( F_n \in A. \)

- (4.12) \( \alpha_n < x_0 < \mu_n \) from Lemma 4.4.2

- (4.13) \( F_n = \overline{F}_{n-1} \) and \( P(F_n) = K(F_{n-1}) \) from Lemma 4.5.12

- (4.14) Equation 4.7 in Theorem 4.7.1 on page 43.

Case 2: \( \mu_n \leq x_0 < \beta_n \) with \( n \) even and positive.

\[ |F(\beta_n) - F(x_0)| = |F_n(x_0) + R_n(x_0)| \]

(4.15)

\[ \geq F_n(x_0) - |R_n(x_0)| \]

\[ \geq F_n(x_0) - \frac{1}{\sqrt{2}} F_n(x_0) \]

(4.16)

\[ \geq \frac{1}{4} F_n(x_0) \]
\[
= \frac{1}{4} F_n(\mu_n) \phi \left( \frac{2x_0 - 2\alpha_n}{\beta_n - \alpha_n} \right) \\
= \frac{1}{4} F_n(\mu_n) \phi \left( \frac{2\beta_n - 2x_0}{\beta_n - \alpha_n} \right) \\
> \frac{1}{4} F_n(\mu_n) \cdot \frac{\beta_n - x_0}{\beta_n - \alpha_n} \\
> \frac{1}{4} \left( \frac{\beta_n - \alpha_n}{2} \right)^{\frac{1}{3}} \cdot \frac{\beta_n - x_0}{\beta_n - \alpha_n} \\
= \frac{1}{4\sqrt{2}} (\beta_n - \alpha_n)^{-\frac{1}{3}} (\beta_n - x_0) \\
> 2^{-\frac{1}{3}} \cdot 2^{\frac{5}{6}} (\beta_n - x_0) \\
= 2^{\frac{2}{3}} (\beta_n - x_0).
\]

We provide some reasonings to this formula.

- (4.15) Claim 4.7.2 in the proof of Theorem 4.7.1.
- (4.16) Proof of Claim 4.6.1.
- (4.17) Lemma 4.4.2 and 0 < \( \frac{2\beta_n - 2x_0}{\beta_n - \alpha_n} < 1 \).
- (4.18) Equation 4.7.

We know from 4.7 in the proof of Theorem 4.7.1 that \( \beta_n \to x_0^+ \) and \( \mu_n \to x_0^+ \) as \( n \to \infty \). So we have \( \{y_n\} \to x_0^+ \) where \( y_n = \beta_n \) or \( \mu_n \). Then

\[
\limsup_{y \to x_0^+} \left| \frac{F(y) - F(x_0)}{y - x_0} \right| \geq \limsup_{y \to \infty} \left| \frac{F(y_n) - F(x_0)}{y_n - x_0} \right| \geq \limsup_{n \to \infty} 2^{\frac{2\nu_0}{\nu}} = \infty.
\]

One-sided Derivatives in \( \mathcal{Z} \)

Lemma 4.7.5

\[
\lim_{y \to x_0^+} \left| \frac{\theta(y) - \theta(x)}{y - x_0} \right| < \infty \text{ for } 0 < x_0 < 2.
\]
Proof.

From (4.1) in Lemma 4.4.1, \( \lambda_n \leq \theta(x) \leq \sqrt{\lambda_{n+1}} \). From 4.3.1, \( \lambda_{n+1} < 2\lambda_n \) and \( \lambda_n = 1 - \lambda - n \).

Then, for \( \lambda_n \leq y \leq \lambda_{n+1} \),

\[
|\theta(1) - \theta(y)| = 1 - \theta(y) \leq 1 - \lambda_n = \lambda_n
\]

\[
\leq 2\lambda_{n-1} = 2(1 - \lambda_{n+1}) = 2(1 - \lambda_{n+1}) \leq 2(1 - y)
\]

So,

\[
\limsup_{y \to 1-} \left| \frac{\theta(y) - \theta(1)}{y - 1} \right| \leq \limsup_{y \to 1-} \left| \frac{2(1 - y)}{y - 1} \right| = 2.
\]

By the symmetry of \( \theta \) about 1,

\[
\limsup_{y \to 1+} \left| \frac{\theta(y) - \theta(1)}{y - 1} \right| \leq 2.
\]

Now, for all other points in \((0, 2)\), we observe the following: if \( \{x_1, x_2\} \subset [\lambda_n, \lambda_{n+1}] \), then

\[
|\theta(x_2) - \theta(x_1)| = \frac{|r_{n+1} - r_n|}{m([\lambda_n, \lambda_{n+1}] \cap D)} \left| m([\lambda_n, x_2] \cap D) - m([\lambda_n, x_1] \cap D) \right|
\]

\[
\leq \frac{1}{m([\lambda_n, \lambda_{n+1}] \cap D)} |x_2 - x_1|.
\]

Then, for \( x_1 \) and \( x_2 \) in \([\lambda_n, \lambda_{n+1}]\),

\[
\frac{|\theta(x_2) - \theta(x_1)|}{|x_2 - x_1|} \leq \frac{1}{m([\lambda_n, \lambda_{n+1}] \cap D)} < \infty.
\]

Thus, for \( 0 < x_0 < 2 \),

\[
\limsup_{y \to x_0+} \left| \frac{\theta(y) - \theta(x_0)}{y - x_0} \right| < \infty.
\]

Lemma 4.7.6 If \( f \in A \) and \((\alpha, \beta)\) is a maximal interval of \( K(f) \), then

\[
0 \leq \liminf_{y \to \alpha^+} \frac{\overline{f}(y) - \overline{f}(\alpha)}{y - \alpha} < \limsup_{y \to \alpha^+} \frac{\overline{f}(y) - \overline{f}(\alpha)}{y - \alpha} = \infty.
\]
Proof. We utilize the sequence \( \{y_n\} \) defined in Lemma 4.7.3. We immediately note the following:

\[
\frac{\bar{f}(y_n)}{y_n - \alpha} = \frac{\bar{h}_{\alpha, \beta}(\lambda_n)}{\lambda_n} \cdot \frac{2}{\beta - \alpha} = \frac{\theta(\lambda_n)}{\lambda_n} \cdot \frac{2\bar{h}_{\alpha, \beta}}{\beta - \alpha} = \begin{cases} 
\frac{2\bar{h}_{\alpha, \beta}}{\beta - \alpha} & \text{for } n \text{ odd; } \\
\frac{2\bar{h}_{\alpha, \beta}}{\beta - \alpha} & \text{for } n \text{ even.}
\end{cases}
\]

Also note that \( \bar{f}(\alpha) = 0 \) since \( \alpha \in H(f) = Z(f) \) and that by Lemma 4.5.4 \( \bar{f}(x) \geq 0 \). Then

\[
\liminf_{y \rightarrow y_n} \frac{\bar{f}(y) - \bar{f}(\alpha)}{y - \alpha} = \liminf_{y \rightarrow y_n} \frac{\bar{f}(y)}{y - \alpha} \geq 0.
\]

Since \( \{y_n\} \subset H(f) \),

\[
\liminf_{y \rightarrow y_n} \frac{\bar{f}(y) - \bar{f}(\alpha)}{y - \alpha} \leq \liminf_{n \rightarrow \infty} \frac{\bar{f}(y_n) - \bar{f}(\alpha)}{y_n - \alpha} = \frac{2\bar{h}_{\alpha, \beta}}{\beta - \alpha} < \infty
\]

and

\[
\limsup_{y \rightarrow y_n} \frac{\bar{f}(y) - \bar{f}(\alpha)}{y - \alpha} \geq \limsup_{n \rightarrow \infty} \frac{\bar{f}(y_n) - \bar{f}(\alpha)}{y_n - \alpha} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\lambda_n}} \cdot \frac{2\bar{h}_{\alpha, \beta}}{\beta - \alpha} = \infty.
\]

\[\Box\]

Lemma 4.7.7 If \( f \in A \) and \( x \in H(f) \cap P(f) \), then

\[
0 = \liminf_{y \rightarrow y_n} \frac{\bar{f}(y) - \bar{f}(x)}{y - x} < \limsup_{y \rightarrow y_n} \frac{\bar{f}(y) - \bar{f}(x)}{y - x} = \infty.
\]

Proof. Let \( x \in H(f) \cap P(f) \) be fixed. Suppose \( x \) is not a cluster point of \( H(f) \cap [x, 1] \). Then \( \exists \varepsilon > 0 \) such that \( (x, x + \varepsilon) \cap H(f) = \emptyset \). Then \( x \) is the left endpoint of some maximal interval of \( K(f) \). Then by Lemma 4.7.6 the statement is true. Now suppose \( x \) is a cluster point of \( H(f) \cap [x, 1] \). We employ the notation \((4.3)\) in page 34.

Claim 1: \( K(f) = \bigcup_{r > 0} K(f)^{(r)} \).

If \( y \in K(f) \), then since \( K(f) \) is open, we can find an interval \((a, b)\) such that \( y \in (a, b) \). Then \( y \in K(f)^{(r)} \) for \( r < b - a \). Also note that \( K(f)^{(r)} \subseteq K(f)^{(r')} \) if \( r > r' \). Thus \( K(f)^{(r)} \not\subset K(f) \) as \( r \rightarrow 0^+ \).
Claim 2: There exists \( \{(\alpha_n, \beta_n)\} \), a sequence of maximal intervals of \( K(f) \), with \( \frac{\alpha_n + \beta_n}{2} = m_n \) such that

a. \( m_n \to x+ \) as \( n \to \infty \);

b. \([x, m_n] \cap K(f)^{(b_n-a_n)} = \emptyset \) for \( n = 1, 2, 3, \ldots \);

c. \( \sqrt{2M(\beta_n-a_n)} < f(m_n) \) for \( n = 1, 2, 3, \ldots \).

Let \( G \) be the disjoint collection of all maximal intervals of \( K(f) \cap [x, 1] \). Then \( G \) is countable and the lengths of intervals are bounded by \( 1 - x \). Thus \( \exists \) an interval \( (a_1, b_1) \in G \) such that \( b_1 - a_1 = \max \{ \beta - \alpha : (\alpha, \beta) \in G \} \). Let \( x_1 = \min \{ x + \frac{1-x}{2}, a_1 \} \). Let \( (a_2, b_2) \) be the largest interval in \( K(f) \cap [x, x_1] \). Let \( x_2 = \min \{ x = \frac{1-x}{2^2}, a_2 \} \). Suppose \( (a_k, b_k) \) and \( x_k \) have been chosen for \( k = 1, \ldots, n \). Let \( (a_{n+1}, b_{n+1}) \) be the largest interval in \( K(f) \cap [x, x_n] \) and let \( x_{n+1} = \min \{ x + \frac{1-x}{2^n}, a_{n+1} \} \). This procedure determines \( \{(a_n, b_n)\} \) satisfying above conditions.

a. \( b_n - x \leq \frac{1-x}{2^n} \to 0 \) as \( n \to \infty \). So, \( m_n \to x+ \) as \( n \to \infty \).

b. Since \( (a_n, b_n) \) was the largest interval in \( K(f) \cap [x, x_{n-1}], [x, m_n] \cap K(f)^{(b_n-a_n)} = \emptyset \).

c. Since \( f \) is continuous and \( f(x) > 0, \exists N_1 > 0 \) such that if \( k \geq N_1 \), then

\[
|f(m_k) - f(x)| < \frac{f(x)}{2} \quad \left( i.e., \frac{f(x)}{2} < f(m_k) < \frac{3}{2}f(x) \right).
\]

Since \( \lim_{r \to \infty} M(r) = 0 \) by Lemma 4.5.5, \( \exists \delta > 0 \) such that if \( 0 < r < \delta \) then \( \sqrt{2M(r)} < \frac{f(x)}{2} \). Then \( \exists N_2 > 0 \) such that if \( k \geq N_2 \) then \( b_k - a_k < \delta \).

Let \( N = \max \{ N_1, N_2 \} \). Then if \( k \geq N \) then

\[
\sqrt{2M(b_k-a_k)} < \frac{f(x)}{2} < f(m_k).
\]

Let \( \alpha_n = a_{n+N} \) and \( \beta_n = b_{n+N} \). Then \( \{(\alpha_n, \beta_n)\} \) is the sequence that satisfies the claim.
Let \( \{(a_n, b_n)\} \) be the sequence that satisfies the claim with \( m_n = \frac{a_n + b_n}{2} \). Then by part (c) of Claim 2,
\[
\overline{f}(m_n) = \overline{f}\left(\frac{a_n + b_n}{2}\right) \leq \sqrt{M(b_n - a_n)} < \frac{f(m_n)}{\sqrt{2}}.
\]
Then \( \overline{f}(m_n) = h_{a_n, b_n} = h'_{a_n, b_n} \). By part (b) of Claim 2,
\[
\sup \left\{ [-1, m_n] \cap K(f)^{(b_n-a_n)} \right\} < x.
\]
Then
\[
\frac{\overline{f}(m_n)}{m_n - x} > \frac{\sqrt{m_n - x}}{m_n - x} = \frac{1}{\sqrt{m_n - x}}.
\]
Thus \( \overline{f}(m_n) > \sqrt{m_n - x} \). So,
\[
\frac{\overline{f}(m_n)}{m_n - x} > \frac{\sqrt{m_n - x}}{m_n - x} = \frac{1}{\sqrt{m_n - x}}.
\]
Then by the facts \( m_n \in H(\overline{f}) \) from Lemma 4.7.3 and \( \overline{f}(x) = 0 \),
\[
\limsup_{y \to x+} \frac{\overline{f}(y) - \overline{f}(x)}{y - x} \geq \limsup_{n \to \infty} \frac{\overline{f}(m_n) - \overline{f}(x)}{m_n - x} \geq \limsup_{n \to \infty} \frac{1}{\sqrt{m_n - x}} = \infty,
\]
and, by the facts that \( a_n \in H(f) \) by Lemma 4.7.3,
\[
\liminf_{y \to x+} \frac{\overline{f}(y) - \overline{f}(x)}{y - x} \leq \liminf_{n \to \infty} \frac{\overline{f}(a_n) - \overline{f}(x)}{a_n - x} = 0.
\]

**Theorem 4.7.8** If \( x \in \mathcal{E} \) then
\[
0 \leq \liminf_{y \to x+} \left| \frac{F(y) - F(x)}{y - x} \right| < \limsup_{y \to x+} \left| \frac{F'(y) - F(x)}{y - x} \right| = \infty.
\]
Proof. Let \( x_0 \in \mathbb{Z} \). Let \[
abla f(y) = \frac{f(y) - f(x_0)}{y - x_0}.
\]

Let \( N \in \mathbb{N} \) such that \( x_0 \in \mathbb{Z}_N \) but \( x_0 \notin \mathbb{Z}_{N-1} \). Since \( P_0 = (-1, 1) \), \( x_0 \notin \mathbb{Z}_0 \). So \( N \geq 1 \).

Then \( x_0 \in \mathbb{Z}_N \cap P_{N-1} \). Then \( x_0 \in P_n \) for \( 0 \leq n \leq N - 1 \). Let \((\alpha_n, \beta_n)\) be the maximal interval of \( P_n \) which contains \( x_0 \) for \( 0 \leq n \leq N - 1 \). Then, by (c) of Definition 4.3.4, for \( y \in (\alpha_n, \beta_n) \) and \( 0 \leq n \leq N - 1 \),

\[
F_n(y) = h_{\alpha_n, \beta_n} \left( \frac{2y - 2\alpha_n}{\beta_n - \alpha_n} \right).
\]

Fix \( n \) such that \( 0 \leq n \leq N - 1 \). Then

\[
\nabla F_n(y) = \frac{F_n(y) - F_n(x_0)}{y - x_0} = \frac{h_{\alpha_n, \beta_n} \left( \frac{2y - 2\alpha_n}{\beta_n - \alpha_n} \right) - h_{\alpha_n, \beta_n} \left( \frac{2\alpha_n - 2\alpha_n}{\beta_n - \alpha_n} \right)}{(2y - 2\alpha_n - 2\alpha_n - 2\alpha_n) \beta_n - \alpha_n}
\]

Then, by Lemma 4.7.5,

\[
\limsup_{y \to x_0^+} |\nabla F_n(y)| = \frac{2h_{\alpha_n, \beta_n}}{\beta_n - \alpha_n} \limsup_{y \to x_0^+} \left| \frac{\theta \left( \frac{2y - 2\alpha_n}{\beta_n - \alpha_n} \right) - \theta \left( \frac{2\alpha_n - 2\alpha_n}{\beta_n - \alpha_n} \right)}{2y - 2\alpha_n - 2\alpha_n - 2\alpha_n} \right| < \infty.
\]

Then

\[
\limsup_{y \to x_0^+} |\Delta S_{N-1}(y)| = \limsup_{y \to x_0^+} \left| \frac{1}{y - x_0} \sum_{i=0}^{N-1} (-1)^i F_i(y) - \sum_{i=0}^{N-1} (-1)^i F_i(x_0) \right|
\]

\[
= \limsup_{y \to x_0^+} \left| \frac{1}{y - x_0} \sum_{i=0}^{N-1} (-1)^i (F_i(y) - F_i(x_0)) \right|
\]

\[
\leq \sum_{i=0}^{N-1} \limsup_{y \to x_0^+} \left| \frac{F_i(y) - F_i(x_0)}{y - x_0} \right|
\]

\[
= \sum_{i=0}^{N-1} \limsup_{y \to x_0^+} |\nabla F_i(y)| < \infty. \tag{4.19}
\]

On the other hand, if \( y \in \mathbb{Z}_{N+1} \) then \( R_N(y) = 0 \) and \( R_N(x_0) = 0 \) and

\[
\nabla R_N(y) = \frac{R_N(y) - R_N(x_0)}{y - x_0} = 0. \tag{4.20}
\]
Since \( x_0 \in Z_n \cap P_{N-1} = H(F_{N_n}) \cap P_{N-1} \) and \( F_N = \overline{F_{N_n}} \), by Lemma 4.7.7

\[
0 = \lim_{y \to x_0^+} \liminf_{y \in H(F_{N_n}) \cap H(F_N) = z_{N+1}} \Delta F(y) < \lim_{y \to x_0^+} \limsup_{y \in z_{N+1}} \Delta F(y) = \infty. \tag{4.21}
\]

Then, using above equations in the order of (4.20), (4.21), and (4.19),

\[
\lim_{y \to x_0^+} \inf \left| \Delta F(y) \right| \leq \lim_{y \to x_0^+} \inf_{y \in z_{N+1}} \left| \Delta F(y) \right|
= \lim_{y \to x_0^+} \inf_{y \in z_{N+1}} \left| \Delta S_{N-1}(y) + (-1)^N \Delta F_N(y) + \Delta R_N(y) \right|
\leq \lim_{y \to x_0^+} \inf_{y \in z_{N+1}} \left( \left| \Delta S_{N-1}(y) \right| + \left| \Delta F_N(y) \right| + \left| \Delta R_N(y) \right| \right)
\leq \lim_{y \to x_0^+} \sup_{y \in z_{N+1}} \left| \Delta S_{N-1}(y) \right| + \lim_{y \to x_0^+} \inf_{y \in z_{N+1}} \left| \Delta F_N(y) \right|
\leq \lim_{y \to x_0^+} \sup_{y \in z_{N+1}} \left| \Delta S_{N-1}(y) \right| < \infty
\]

And,

\[
\lim_{y \to x_0^+} \sup \left| \Delta F(y) \right| \leq \lim_{y \to x_0^+} \sup_{y \in z_{N+1}} \left| \Delta F(y) \right|
= \lim_{y \to x_0^+} \sup_{y \in z_{N+1}} \left| \Delta S_{N-1}(y) + (-1)^N \Delta F_N(y) + \Delta R_N(y) \right|
\leq \lim_{y \to x_0^+} \sup_{y \in z_{N+1}} \left( \left| (-1)^N \Delta F_N(y) \right| - \left| \Delta S_{N-1}(y) \right| \right)
= \lim_{y \to x_0^+} \sup_{y \in z_{N+1}} \left| \Delta F_N(y) \right| - \lim_{y \to x_0^+} \sup_{y \in z_{N+1}} \left| \Delta S_{N-1}(y) \right|
\leq \lim_{y \to x_0^+} \sup_{y \in z_{N+1}} \left| \Delta F_N(y) \right| - \lim_{y \to x_0^+} \sup_{y \in z_{N+1}} \left| \Delta S_{N-1}(y) \right| = \infty.
\]

\[\blacksquare\]

4.8 Miscellaneous Facts

Observation 4.8.1

a. For \( n \) odd, \( R_n(x) \leq F_{n+1}(x) \).
b. For \( n \) even, \(-F_{n+1}(x) \leq R_n(x)\).

c. For \( n \) odd, \(0 \leq R_n(x) \leq F_{n+1}(x)\).

d. For \( n \) even, \(-F_{n+1} \leq R_n(x) \leq 0\).
CHAPTER 5

EXISTENCE OF MORSE-BESICOVITCH FUNCTION BY CATEGORY METHOD

Since Saks showed that the set of Besicovitch functions is meager in the space of continuous functions, most studies of Besicovitch functions have been relied upon the explicit constructions of such functions. The category method failed to show the existence of Besicovitch functions in its direct application. In 1984, Malý successfully defended the category method of showing the existence of Besicovitch functions by considering a subspace of $C([0,1], ||.||)$. This argument was motivated by the existence of Morse-Besicovitch functions.

5.1 Preliminary

Basic Definitions Let $f : [0,1] \to \mathbb{R}$ be a function. We denote sup norm of $f$, $||f|| = \sup\{|f(x)| : x \in [0,1]\}$ and Lipschitz constant $\text{Lip } f = \sup \left\{|\frac{f(x) - f(y)}{x-y}| : x, y \in [0,1]\} \right\}$. We also utilize the notation $f^+, f_+, f^-, f_-$ for Dini derivatives. We denote the space of continuous functions on the interval $[0,1]$ with the sup norm $|| \cdot ||$ as $C$. Define

$$K = \{ f \in C : f(0) = f(1) = 0, \text{Lip } f \leq 1 \}.$$

Note that $K$ is a compact metric space. Since $K$ is a closed subset of a complete space $C$, it is complete. If $\varepsilon > 0$ and $\frac{1}{N} < \frac{\varepsilon}{2}$ then $K$ is totally bounded by the set $\{ f_n : n = 1, \ldots, N \}$ where $f_n(x) = \frac{1}{2N} x$ if $x \in [0, \frac{1}{2})$, or $f_n(x) = \frac{1}{2N} (1 - x)$ if $x \in [\frac{1}{2}, 1]$. Thus $K$ is compact.

Then we note that $K^N$, the collection of all sequences of functions in $K$, is a compact metrizable space. Let $E$ be a subset of $K^N$ consisting of all sequences $u = \{u_n\}$ satisfying the following conditions:

$$u_1 \geq u_2 \geq \cdots \geq 0$$

(5.1)
and

for every interval $I \subseteq [0,1]$ and $n \in \mathbb{N}$, if $u_{n+1} > 0$ on $I$, then $u_n$ is constant on $I$ (5.2)

A Compact Subspace $E$ of $\prod_{n \in \mathbb{N}} K$. By viewing $K^n$ as the product of countably many $K_j$, we regard $E$ as a subset of a product space $\prod_{n \in \mathbb{N}} K$.

Observation 5.1.1 $E$ is closed.

Proof. Since $\prod_{n \in \mathbb{N}} K$ is a metrizable space, we can consider a sequence $\{u^k\}_{k=1}^\infty$ in $E$.

Suppose $u^k$ converges to $v \in \prod_{n \in \mathbb{N}} K$, i.e., $\forall \varepsilon > 0$ and $\forall m > 0$ $\exists k_0$ such that if $k \geq k_0$ then $\|u^k - v\| < \varepsilon$ for all $i \leq m$.

Suppose that $v$ does not satisfy the property 5.1. Let $i, j \in \mathbb{N}$ and $x \in (0,1)$ such that $i < j$ and $v_i(x) > v_j(x)$. Let $\varepsilon = \frac{v_i(x) - v_j(x)}{2}$. Let $k_i \in \mathbb{N}$ and $k_j \in \mathbb{N}$ such that $\varepsilon < \varepsilon$ and $\|u^{k_i} - v_i\| < \varepsilon$.

Let $k_0 = \max\{k_i, k_j\}$. Then

$$u^{k_i}_{j}(x) > v_i(x) - \varepsilon \quad \text{and} \quad u^{k_j}_{i}(x) < v_j(x) + \varepsilon.$$ 

Thus $u^{k_j}_{i}(x) < u^{k_i}_{j}(x)$. This is a contradiction. Thus $v_i \leq v_j$.

Now suppose that $v$ does not satisfy the property 5.2 of $E$. Let $v_{n+1} > 0$ on some interval $I$ and $\{t_1, t_2\} \in I$ such that $v_n(t_1) \neq v_n(t_2)$. Then $\inf v_{n+1}(x): t_1 \leq x \leq t_2 > 0$.

Then $\exists$ an interval $I' \subseteq I$ such that $\inf \{v_{n+1}(x): x \in I'\} > 0$ and $t_1, t_2 \in I'$. Let

$$\varepsilon = \min \left\{ \inf \{v_{n+1}(x): x \in I'\}, \frac{|v_n(t_2) - v_n(t_1)|}{2} \right\}.$$ 

Then $\varepsilon > 0$. Let $k_\varepsilon \in \mathbb{N}$ such that $\|u^{k_\varepsilon} - v_i\| < \varepsilon$. 


for \( i \leq n + 1 \). Then \( u_{n+1} > 0 \) on \( I' \) and \( u_n \) is constant on \( I' \). Then

\[
\varepsilon \leq \frac{1}{2} |u_n(t_2) - u_n(t_1)| \leq \frac{1}{2} (|u_n(t_2) - u_n(t_2)| + |u_n(t_1) - u_n(t_1)|) < \frac{1}{2} (\varepsilon + \varepsilon) = \varepsilon.
\]

This is a contradiction. Thus \( v_n \) is constant on \( I' \).

5.2 A Compact Subspace \( A_\varphi(E) \) of \( C \)

In this section we construct a compact subspace of \( C \) where Morse Besicovitch functions are typical.

**Definition 5.2.1** Let \( \varphi : [0, 1] \to \mathbb{R} \) be a continuous increasing function such that \( \varphi(0) = 0 \). Let \( A_\varphi \) be the map from \( E \) to \( C \) defined by

\[
A_\varphi(u) = \sum_{n=1}^{\infty} (-1)^{n+1} \varphi \circ u_n.
\]

**Compactness of \( A_\varphi(E) \).** We prove that \( A_\varphi : E \to C \) is continuous; therefore, \( A_\varphi(E) \) is compact since \( E \) is compact. We utilize the following lemma.

**Lemma 5.2.2** If \( u \in E \), then \( \|u_n\| < \frac{1}{n} \) for all \( n \in \mathbb{N} \).

**Proof.** We have \( u_n(0) = u_n(1) = 0 \). Suppose \( \exists x_0 \in [0, 1) \) such that \( u_n(x_0) \geq \frac{1}{n} \). Then \( \exists x_1 \in [0, x_0) \) such that \( u_n(x_1) = 0 \) and \( u_n \) is strictly positive on \((x_1, x_0] \).

If there is no such \( x_1 \in [0, x_0) \). Then \( u_n \) is strictly positive on \((0, x_0] \) and \( u_{n-1} \) is constant on \((0, x_0] \). Since \( u_n(x_0) \geq \frac{1}{n} \), \( u_{n-1}(x) \geq \frac{1}{n} \) for \( x \in (0,x_0] \). Then \( u_{n-1} \) is not continuous at \( 0 \). This is a contradiction.

Let \( x_1 = \sup \{ t \in [0, x_0) : u_n(t) = 0 \} \). Then \( x_1 < x_0 \); otherwise, \( u_n \) is not continuous at \( x_0 \). Since \( \text{Lip} u_n \leq 1 \), we deduce \( x_1 \leq x_0 - \frac{1}{n} \) by the following:

\[
\text{Lip} u_n \leq 1 \Rightarrow \left| \frac{u_n(x_0) - u_n(x_1)}{x_0 - x_1} \right| \leq 1 \Rightarrow \left| \frac{\frac{1}{n} - 0}{x_0 - x_1} \right| \leq 1 \Rightarrow \frac{1}{n} \leq x_0 - x_1 \Rightarrow x_1 \leq x_0 - \frac{1}{n}.
\]
And (5.2) on page 56 says that \( u_{n-1} \) is constant on \([z_1, z_0]\); thus (using (5.2)) \( u_{n-1}(x_1) = u_{n-1}(x_0) \geq u_n(x_0) \geq \frac{1}{n} \). By induction we can find points \( x_i, i = 0, \ldots, n-1 \) such that

\[
x_0 > x_1 > \cdots > x_{n-1} \geq 0,\ x_i \leq x_0 - \frac{i}{n} \text{ and } u_{n-i}(x_i) \geq \frac{1}{n}.
\]

In particular,

\[
x_{n-1} \leq x_0 - \frac{n-1}{n} < 1 - \frac{n-1}{n} = \frac{n-n+1}{n} = \frac{1}{n} \text{ and } u_1(x_{n-1}) \geq \frac{1}{n}, \text{ and } v_1(0) = 0.
\]

So \( \text{Lip} u_1 > 1 \), which is a contradiction. \( \blacksquare \)

**Theorem 5.2.3** \( A_\varphi \) is continuous.

**Proof.** Fix \( \varepsilon > 0 \). Find \( m \in \mathbb{N} \) and \( \delta > 0 \) such that \( \varphi \left( \frac{1}{m} \right) < \varepsilon \) and such that if \( s, t \in [0, 1] \) and \( |s-t| < \delta \), then

\[
|\varphi(s) - \varphi(t)| < \varepsilon /m.
\]

Suppose \( u, v \in \mathcal{E} \) such that \( ||u_i - v_i|| \leq \delta \) for every \( i = 1, \ldots, m \). By 5.1, \( \varphi \circ u_1 \geq \varphi \circ u_2 \geq \cdots \geq 0 \), and \( \varphi \circ v_1 \geq \varphi \circ v_2 \geq \cdots \geq 0 \). Then, by Lemma 5.2.2,

\[
||\varphi \circ u_{m+1}|| \leq \varphi \left( \frac{1}{m+1} \right) < \varepsilon \text{ and } ||\varphi \circ v_{m+1}|| \leq \varphi \left( \frac{1}{m+1} \right) < \varepsilon.
\]

Hence

\[
||A_\varphi u - A_\varphi v||
\leq \left\| \sum_{i=1}^{m} (-1)^{i+1}(\varphi \circ u_i - \varphi \circ u_i) \right\| + \left\| \sum_{i=m+1}^{\infty} (-1)^{i+1}\varphi \circ u_i \right\| + \left\| \sum_{i=m+1}^{\infty} (-1)^{i+1}\varphi \circ v_i \right\|
\leq \sum_{i=1}^{m} \frac{\varepsilon}{m} + ||\varphi \circ u_{m+1}|| + ||\varphi \circ v_{m+1}|| \leq 3\varepsilon.
\]

\( \blacksquare \)
A property of $A_\varphi(E)$. The subspace $A_\varphi(E)$ of $C$ has the following property.

**Theorem 5.2.4** Let $\varphi : [0, 1] \to \mathbb{R}$ be a continuous increasing map and $\varphi(0) = 0$. Suppose that $\varphi$ has a finite derivative at every point of $(0, 1)$ and $\varphi_+(0) < \infty$. If $f \in A_\varphi(E)$, then $f_+(x) < \infty$ and $f_+(x) > -\infty$ for each $x \in (0, 1)$.

Let $\varphi : [0, 1] \to \mathbb{R}$ be a continuous increasing map with $\varphi'(x) < \infty$ for $x \in (0, 1)$, $\varphi(0) = 0$ and $\varphi_+(0) < \infty$. Let $u \in E$ and $A_\varphi(u) = f$. Let $z_0 \in [0, 1)$ be fixed. The proof will be divided into the following cases.

**Case 1.** Suppose $u_n(z_0) > 0$ for every $n \in \mathbb{N}$.

Define a sequence $\{b_n\}_{n=1}^\infty$ by $b_n = \inf\{x \in (z_0, 1] : u_n(x) = 0\}$. Then by the property (5.1) of $E$, $\{b_n\}$ is nonincreasing. Note that $z_0 \neq b_n$ since $u_n$ is continuous. Then for each $n$, $u_n > 0$ on $[z_0, b_n)$. By the property (5.2) of $E$, we have for each $n \in \mathbb{N}$ that each of $u_{n-1}, u_{n-2}, \ldots, u_2, u_1$ is constant on $[z_0, b_n]$. Specifically, we have $u_i(z_0) = u_i(b_n)$ if $1 \leq i \leq n - 1$. Also, if $i \geq n$ then $u_i(b_n) = 0$. Then for each $n \in \mathbb{N}$,

$$f(z_0) = \sum_{i=1}^{\infty} (-1)^{i+1} \varphi(u_i(z_0)) \geq \sum_{i=1}^{2n} (-1)^{i+1} \varphi(u_i(z_0)) = \sum_{i=1}^{2n} (-1)^{i+1} \varphi(u_i(b_{2n+1})) = f(b_{2n+1})$$

Note that the first inequality is obtained by the fact that $\varphi$ is increasing, $(-1)^{2n+1} = -1$, and $u_i(z_0) \geq u_{i+1}(z_0)$. We can give a similar argument to get

$$f(z_0) \leq \sum_{i=1}^{2n-1} (-1)^{i+1} \varphi(u_i(z_0)) = \sum_{i=1}^{2n-1} (-1)^{i+1} \varphi(u_i(b_n)) = f(b_n).$$

Combining these two inequalities, we get for each $n$,

$$f(b_{2n+1}) \leq f(z_0) \leq f(b_{2n}).$$

Let $b = \lim_{n \to \infty} b_n$. If $b = z_0$, then

$$f_+(z_0) \leq \lim_{n \to \infty} \frac{f(b_{2n+1}) - f(z_0)}{b_{2n+1} - z_0} \leq \lim_{n \to \infty} \frac{f(b_{2n+1}) - f(b_{2n+1})}{b_{2n+1} - z_0} = 0$$
and

\[ f^+(z_0) \geq \lim_{n \to \infty} \frac{f(b_{2n}) - f(z_0)}{b_{2n} - z_0} \geq \lim_{n \to \infty} \frac{f(b_{2n}) - f(b_{2n})}{b_{2n} - z_0} = 0. \]

If \( z_0 < b \), then for each \( n \in \mathbb{N} \), \( u_n \) is constant on the interval \([z_0, b] \); thus, \( f \) is constant on \([z_0, b] \) and \( f^+(z_0) = f_+(z_0) = 0. \)

**Case 2:** Suppose \( u_n(z_0) = 0 \) for some \( n \in \mathbb{N} \).

Let \( m \geq 1 \) be the least integer such that \( u_m(z_0) = 0 \).

**Subcase 2-a:** Suppose \( m = 1 \)

Then \( u_n(z_0) = 0 \) for all \( n \in \mathbb{N} \). If \( x > z_0 \) then \( u_1(x) = u_1(x) - u_1(z_0) \leq x - z_0 \) and

\[
|f(x)| = \left| \sum_{i=1}^{\infty} (-1)^{i+1} \varphi(u_i(x)) \right| \leq \varphi(u_1(x)) \leq \varphi(x - z_0).
\]

Thus

\[
\liminf_{x \to z_0^+} \frac{|f(x) - f(z_0)|}{x - z_0} \leq \liminf_{x \to z_0^+} \frac{\varphi(x - z_0)}{x - z_0} = \varphi_+(0) < \infty.
\]

**Subcase 2-b:** Suppose \( m > 1 \) and \( u_{m-1} \) is constant and positive on the interval \([z_0, z_0 + \varepsilon] \) for some \( \varepsilon > 0 \).

Then \( u_1, u_2, \ldots, u_{m-1} \) are constant and positive on \([z_0, z_0 + \varepsilon] \) and \( u_m(z_0) = u_{m+1}(z_0) = \ldots = 0 \). Then for \( x \in [z_0, z_0 + \varepsilon] \) we have \( u_m(x) = u_m(z_0) - u_m(z_0) \leq x - z_0 \) and

\[
|f(x) - f(z_0)| = \left| \sum_{i=1}^{\infty} (-1)^{i+1} \varphi(u_i(x)) - \sum_{i=1}^{m-1} (-1)^{i+1} \varphi(u_i(z_0)) \right|
\]

\[
= \left| \sum_{i=m}^{\infty} (-1)^{i+1} \varphi(u_i(x)) + \sum_{i=1}^{m-1} (-1)^{i+1} [\varphi(u_i(x)) - \varphi(u_i(z_0))] \right|
\]

\[
= \left| \sum_{i=m}^{\infty} (-1)^{i+1} \varphi(u_i(x)) \right| \leq \varphi(u_m(x)) \leq \varphi(x - z_0).
\]

Thus

\[
\liminf_{x \to z_0^+} \frac{|f(x) - f(z_0)|}{x - z_0} \leq \liminf_{x \to z_0^+} \frac{\varphi(x - z_0)}{x - z_0^+} = \varphi_+(0) < \infty.
\]
Subcase 2-e: Suppose \( m > 1 \) and \( u_{m-1} \) is not constant on \([z_0, z_0 + \epsilon]\) for any \( \epsilon > 0 \). Then \( u_m(z_0) = u_{m+1}(z_0) = \cdots = 0 \) and \( \exists \) an interval \( I \) such that \( u_i \) is constant and positive on \( I \) for \( i < m - 1 \). Let \( \epsilon_1 > 0 \) be such that \( \forall i < m - 1, u_i \) is constant and positive on \([z_0 - \epsilon_1, z_0 + \epsilon_1]\). Let \( \{\delta_n\}_{n=1}^{\infty} \subseteq (0, \epsilon) \) be such that \( \lim_{n \to \infty} \delta_n = 0 \) and \( \delta_1 \geq \delta_2 \geq \cdots \geq \delta_n \geq \delta_{n+1} \geq \cdots \geq 0 \). Since \( u_{m-1} \) is not constant on \([z_0, z_0 + \delta_n]\) for any \( n \geq 1 \), for each \( n \) we have \( x_n \in [z_0, z_0 + \delta_n] \) such that \( u_m(x_n) = u_{m+1}(x_n) = \cdots = 0 \). Then we have a sequence \( \{x_n\}_{n=1}^{\infty} \) that converges to \( z_0 \) from the right and for each \( n \) and each \( i < m - 1 \) we have \( u_i(x_n) = u_i(z_0) \). Then, for each \( n \),

\[
|f(x_n) - f(z_0)| = \left| \sum_{i=1}^{\infty} (-1)^{i+1} \varphi(u_i(x_n)) - \sum_{i=1}^{\infty} (-1)^{i+1} \varphi(u_i(z_0)) \right|
\]

\[
= \left| \sum_{i=1}^{m-2} (-1)^{i+1} \varphi(u_i(z_0)) + (-1)^m \varphi(u_{m-1}(x_n)) - \sum_{i=1}^{m-1} (-1)^{i+1} \varphi(u_i(z_0)) \right|
\]

\[
= |(-1)^m \varphi(u_{m-1}(z_0)) - (-1)^m \varphi(u_{m-1}(x_n))|
\]

\[
= |\varphi(u_{m-1}(z_0)) - \varphi(u_{m-1}(x_n))|.
\]

Then

\[
\limsup_{n \to \infty} \frac{|f(x_n) - f(z_0)|}{x_n - z_0} = \limsup_{n \to \infty} \frac{|\varphi(u_{m-1}(x_n)) - \varphi(u_{m-1}(z_0))|}{u_{m-1}(x_n) - u_{m-1}(z_0)} \cdot \frac{|u_{m-1}(x_n) - u_{m-1}(z_0)|}{x_n - z_0}
\]

\[
\leq \limsup_{y \to u_m(z_0)^+} \frac{|\varphi(y) - \varphi(y_m(z_0))|}{y - y_m(z_0)} \limsup_{n \to \infty} \frac{|u_{m-1}(x_n) - u_{m-1}(z_0)|}{x_n - z_0}
\]

\[
= \varphi'(u_m(z_0)) \limsup_{n \to \infty} \frac{|u_{m-1}(x_n) - u_{m-1}(z_0)|}{x_n - z_0}
\]

\[
\leq \varphi'(u_m(z_0)) \text{Lip} u_{m-1} < \infty.
\]

5.3 A meager subset of \( A_\varphi(E) \).

**Theorem 5.3.1 (Malý)** Let \( \varphi \) and \( \psi \) be continuous increasing functions on the interval \([0, 1]\) such that \( \psi \) is concave and \( \varphi(0) = \psi(0) = 0 \). If \( \lim_{x \to 0^+} \varphi(x) = \infty \), then

\[
M := \{ f \in A_\varphi(E) : \exists z_0 \in [0, 1) \text{ such that } \limsup_{x \to z_0^+} \frac{|f(x) - f(z_0)|}{\psi(x - z_0)} < \infty \}
\]
is of the first category in \( A_\varphi(E) \).

Let \( M_k = \{ f \in A_\varphi(E) : \exists z_0 \in [0, 1 - \frac{1}{k}] \text{ such that } |f(x) - f(z_0)| \leq k\psi(x - z_0) \} \) for every \( x \in [z_0, z_0 + \frac{1}{k}] \). Then by the lemma \( M_k \) is closed in \( A_\varphi(E) \) [Lemma 5.4.1] and \( M \subseteq \bigcup_{k=1}^{\infty} M_k \). To prove the meagerness of \( M \) it suffices to show that \( M_k \) is nowhere dense. Fix \( u \in E, k \in \mathbb{N}, \) and \( \varepsilon \in (0, \frac{1}{2}) \). Let \( f = A_\varphi(u) \). We will show there is some \( u^* \in E \setminus A_\varphi^{-1}(M_k) \) in any neighborhood of \( u \).

We briefly argue why this proves the theorem. Suppose that \( N \) is a neighborhood of \( f \). Then \( A_\varphi^{-1}(N) \) is an open neighborhood of \( u \) since \( A_\varphi \) is continuous. Then we find \( u^* \in [E \setminus A_\varphi^{-1}(M_k)] \cap A_\varphi^{-1}(N) \) so that \( f = A_\varphi(u^*) \in [A_\varphi(E) \setminus M_k] \cap N \). Thus \( M_k \) is nowhere dense in \( A_\varphi(E) \).

Now we switch back to showing the existence of \( u^* \). Let \( \varepsilon > 0 \). First, let \( m \in \mathbb{N} \) be such that \( \frac{1}{m} < \varepsilon \). Define \( v = \{ v_n \} \) by \( v_n = (1 - 2\varepsilon) \max \{ 0, u_n - \frac{n}{m^2} \} \). Then we have the following claims.

**Claim 5.3.2**

a. \( v \in E \)

b. \( \text{Lip} v_n \leq 1 - 2\varepsilon \) for every \( n \in \mathbb{N} \)

c. \( ||v_n - u_n|| < 3\varepsilon \) for every \( n \in \mathbb{N} \)

d. \( \forall n \in \mathbb{N} \) and \( x \in [0, 1] \) either \( v_{n+1}(x) = 0 \) or \( v_n(x) \geq v_{n+1}(x) + \frac{1}{m^2}(1 - 2\varepsilon) \).

**Claim 5.3.3**

\[
\limsup_{p \in \mathbb{N}} \frac{\varphi(\varepsilon/p)}{\psi(2/p)} = \infty.
\]

Secondly, we define \( \lambda : [0, 1] \to [0, 1] \) by the following. Let \( p \in \mathbb{N} \) such that

\[
p \geq 2k, \quad \frac{\varepsilon}{p} < \frac{1}{m^2}(1 - 2\varepsilon), \quad \frac{1}{p} < \varepsilon, \quad \text{and} \quad 8k\psi(\frac{2}{p}) \leq \varphi(\frac{\varepsilon}{p}).
\]
Let $b_j = \frac{j - \varepsilon}{p}$, $c_j = \frac{j - \varepsilon}{p}$, and $a_j = \frac{j - 2\varepsilon}{p}$ for $j = 1, \cdots, p$, and $b_0 = 0$. Then we have a partition of the interval $[0,1]$ such that

$$0 = b_0 < a_1 < c_1 < b_1 < a_2 < c_2 < b_2 < \cdots < a_p < c_p < b_p = 1.$$

Let

$$\lambda(x) = \begin{cases} \frac{1-2\varepsilon}{2\varepsilon}(x - b_{j-1}) + b_{j-1} & \text{if } x \in [b_{j-1}, a_j] \\ b_j & \text{if } x \in [a_j, b_j]. \end{cases}$$

Then $\lambda(0) = 0$, $\lambda(b_{j-1}) = b_{j-1}$, $\lambda(a_j) = b_j$ and $\lambda$ is linear on $[b_{j-1}, a_j]$ and constant on $[a_j, b_j]$ for $j = 1, \cdots, p$.

Thirdly, we define $w = \{w_n\}$ by $w_n = v_n \circ \lambda$ for all $n \in \mathbb{N}$.

**Claim 5.3.4**

a. $w \in E$;

b. $|v_n(x) - w_n(x)| < \varepsilon$ for any $x \in [0,1]$ and $n \in \mathbb{N}$.

Finally we define $u^* = \{u^*_n\}$ by the following. Let $m_j = \min\{n \in \mathbb{N} : v_n(b_j) < \frac{\varepsilon}{p}\}$. If $x \in [a_j, b_j]$, we let

$$\theta_{m_j}(x) = \begin{cases} v_{m_j}(b_j) + \frac{\varepsilon}{p} - |x - c_j| & \text{if } \varphi(v_{m_j}(b_j)) \leq 4k\psi(\frac{2}{p}); \\ v_{m_j}(b_j) - \min\{v_{m_j}(b_j), \frac{\varepsilon}{p} - |x - c_j|\} & \text{if } \varphi(v_{m_j}(b_j)) > 4k\psi(\frac{2}{p}). \end{cases}$$

Let

$$u^*_n(x) = \begin{cases} \theta_{m_j}(x) & \text{if } n = m_j \text{ and } x \in [a_j, b_j] \text{ for } j = 1, \cdots, p; \\ w_n(x) & \text{otherwise, i.e., } n \neq m_j \text{ or } x \not\in [b_{j-1}, a_j] \text{ for } j = 1, \cdots, p. \end{cases}$$

**Claim 5.3.5**

a. $u^* \in E$;

b. $f^* = A_\varphi(u^*) \notin M_k$;
c. \( u^* \in E \setminus A_{\psi}^{-1}(M_k) \);

d. \( \|u_n^* - u_n\| < 5\varepsilon \).

## 5.4 Proofs of Claims

**Lemma 5.4.1** Let \( \psi \) be a continuous increasing function on the interval \([0, 1]\) such that \( \psi(0) = 0 \). Then \( M_k = \{ f \in C([0, 1]) : \exists z \in [0, 1 - \frac{1}{k}] \text{ such that } |f(x) - f(z)| \leq k\psi(x - z) \text{ for every } x \in [z, z + \frac{1}{k}] \} \) is closed in \( C([0, 1]) \).

**Proof.** Let \( \{f_n\} \) be a sequence of functions in \( M_k \). Suppose \( f_n \) converges to \( f \in C([0, 1]) \). For each \( n \), let \( z_n \in [0, 1 - \frac{1}{k}] \) such that \( |f_n(x) - f_n(z_n)| \leq k\psi(x - z_n) \) for every \( x \in [z_n, z_n + \frac{1}{k}] \).

We can assume that \( \{z_n\} \) converges because otherwise we can take a convergent subsequence \( \{z_{n_k}\} \) and the corresponding \( \{f_{n_k}\} \). Let \( z_0 \) be the limit of \( \{z_n\} \). Suppose \( x_0 \in [z_0, z_0 + \frac{1}{k}] \).

Let \( \varepsilon > 0 \). Let \( N \in \mathbb{N} \) such that if \( n \geq N_1 \) then \( \|f_n - f\| < \varepsilon/4 \). Let \( \delta > 0 \) such that if \( |x_2 - x_1| < \delta \) then \( |f(x_2) - f(x_1)| < \varepsilon/4 \) for all \( x_2, x_1 \in [0, 1] \). Let \( N_2 \in \mathbb{N} \) such that if \( n \geq N_2 \) then \( |z_n - z_0| < \delta \). Let \( N_3 \in \mathbb{N} \) such that if \( n \geq N_3 \) then \( z_n < x_0 \).

**Case 1.** Suppose \( x_0 = z_0 + \frac{1}{k} \).

Let \( N = \max\{N_1, N_2, N_3\} \). Then we have for \( n \geq N \)

\[
|f(x_0) - f(z_0)| = |f\left(z_0 + \frac{1}{k}\right) - f(z_0)| \\
\leq \left| f\left(z_0 + \frac{1}{k}\right) - f\left(z_n + \frac{1}{k}\right) \right| + \left| f\left(z_n + \frac{1}{k}\right) - f_n\left(z_n + \frac{1}{k}\right) \right| \\
+ \left| f_n\left(z_n + \frac{1}{k}\right) - f_n(z_n) \right| + \left| f_n(z_n) - f(z_n) \right| + \left| f(z_n) - f(x_0) \right| \\
< \varepsilon/4 + \varepsilon/4 + k\psi\left(\frac{1}{k}\right) + \varepsilon/4 + \varepsilon/4 \\
= k\psi\left(\frac{1}{4}\right) + \varepsilon.
\]
Case 2. Suppose $x_0 < z_0 + \frac{1}{k}$.

Let $N_4 \in \mathbb{N}$ such that if $n \geq N_4$ then $|z_n - x_0| < x_0 + \frac{1}{k} - z_0$, i.e., $x_0 \in [z_n, z_n + \frac{1}{k}]$.

Let $N_5 \in \mathbb{N}$ such that if $n \geq N_5$ then $|\psi(x_0 - z_n) - \psi(x_0 - z_0)| < \frac{\varepsilon}{4k}$. Then we have for $n \geq \max\{N_5, N_4, N\}$

$$|f(x_0) - f(z_0)| \leq |f(x_0) - f_n(z_0)| + |f_n(x_0) - f_n(z_n)| + |f_n(z_n) - f(z_n)| + |f(z_n) - f(z_0)|$$

$$< \varepsilon/4 + k\psi(x_0 - z_n) + \varepsilon/4 + \varepsilon/4$$

$$< \frac{3\varepsilon}{4} + k \cdot \frac{\varepsilon}{4k} = \varepsilon.$$

Proof. (of Claim 5.3.2) First note that $\text{Lip} u_n \leq 1 - 2\varepsilon$ for every $n \in \mathbb{N}$ because $\text{Lip} u_n \leq 1$ for every $n$. Also $v_n \geq v_{n+1}$ for every $n \in \mathbb{N}$ since $u_n \geq u_{n+1}$ for every $n \in \mathbb{N}$. Suppose $v_{n+1} > 0$ on an interval $I$. Then $u_{n+1} > \frac{n+1}{m^2} > 0$ and $u_n$ is constant on $I$. Then $v_n$ is constant on $I$. Thus $v \in E$.

To prove that $||v_n - u_n|| < 3\varepsilon$ for every $n \in \mathbb{N}$ we note that if $n > m$ then $v_n = 0$ because for every $n \in \mathbb{N}$, $||u_n|| < \frac{1}{n}$ which implies

$$u_n - \frac{n}{m^2} < \frac{1}{n} - \frac{n}{m^2} = n \left( \frac{1}{n^2} - \frac{1}{m^2} \right) < 0.$$ 

Thus if $n > m$ then

$$||v_n - u_n|| = ||u_n|| < \frac{1}{n} < \frac{1}{m} < \varepsilon.$$

Or, if $n \leq m$ then

$$||v_n - u_n|| \leq \left| (1 - 2\varepsilon)(u_n - \frac{n}{m^2}) - u_m \right| = \left| 2\varepsilon u_n - \frac{(1 - 2\varepsilon) n}{m^2} \right|$$

$$= \left| 2\varepsilon(u_n - \frac{n}{m^2}) + \frac{n}{m^2} \right| \leq 2\varepsilon \left| u_n - \frac{n}{m^2} \right| + \frac{1}{m} \cdot \frac{n}{m}$$

$$\leq 2\varepsilon + \frac{1}{m} < 3\varepsilon.$$
Suppose $v_{n+1}(x) \neq 0$. Then

$$v_{n+1}(x) = (1 - 2\epsilon) \left( u_{n+1}(x) - \frac{n+1}{m^2} \right)$$

$$= (1 - 2\epsilon) \left( u_{n+1}(x) - \frac{n}{m^2} \right) - (1 - 2\epsilon) \frac{1}{m^2}$$

$$\leq (1 - 2\epsilon) \left( u_n(x) - \frac{1}{m^2} \right) - (1 - 2\epsilon) \frac{1}{m^2}$$

$$\leq v_n(x) - (1 - 2\epsilon) \frac{1}{m^2}.$$

---

**Proof.** (of Claim 5.3.3) Let $\frac{2}{p} \leq z \leq \frac{2}{p-1}$. Then $\psi(z) \leq \psi(x)$ since $\psi$ is increasing. For $\epsilon > 0$, we have $\frac{2\epsilon}{4} \leq \frac{2\epsilon}{3(p-1)} \leq \frac{\epsilon}{p}$ for $p \leq 2$. Since $\varphi$ is increasing, $\varphi(\frac{2\epsilon}{4}) \leq \varphi(\frac{\epsilon}{p})$. Thus

$$\frac{\varphi(\frac{\epsilon}{p})}{\psi(\frac{2}{p})} \geq \frac{\varphi(\frac{2\epsilon}{4})}{\psi(\frac{\epsilon}{p})}.$$

By the concavity of $\psi$, for $0 \leq t \leq 1$ and $a, b \in [0, 1]$ where $a < b$, we have

$$\psi(ta + (1-t)b) \geq t\psi(a) + (1-t)\psi(b).$$

Let $t = \epsilon/4$, $a = x$, $b = 0$. Then

$$\psi(\frac{\epsilon x}{4}) \geq \frac{\epsilon x}{4} \psi(x).$$

Then

$$\frac{\epsilon}{4} \frac{1}{\psi(\frac{\epsilon x}{4})} \leq \frac{1}{\psi(x)}.$$

Thus

$$\frac{\varphi(\frac{\epsilon x}{4})}{\psi(x)} \geq \frac{1}{4} \frac{\varphi(\frac{\epsilon x}{4})}{\psi(\frac{\epsilon x}{4})}.$$

By letting $p \to \infty$, we have $x \to 0^+$ and

$$\limsup_{p \in \mathbb{N}} \frac{\varphi(\epsilon/q)}{\psi(2/p)} \geq \limsup_{x \to 0^+} \frac{\varphi(\epsilon x/4)}{\psi(x)} \geq \frac{1}{4} \epsilon \limsup_{x \to 0^+} \frac{\varphi(\epsilon x/4)}{\psi(\epsilon x/4)} = +\infty.$$
Proof. (of Claim 5.3.4) Since \( \text{Lip } \lambda = \frac{1}{1 - 2\epsilon} \), for every \( n \in \mathbb{N} \),

\[
\text{Lip } w_n \leq \text{Lip } v_n \cdot \text{Lip } \lambda \leq 1.
\]

Also, note

\[
w_n(0) = v_n(\lambda(0)) = v_n(0) = w_n(1) = v_n(1) = 0.
\]

Thus \( w_n \in \mathcal{C} \). Since \( \lambda \) is nondecreasing function from \([0, 1]\) onto \([0, 1]\), for every \( n \in \mathbb{N} \), \( w_n \geq w_{n+1} \geq 0 \). If \( I \) is an interval, then \( \lambda(I) \) is an interval or a point, so obviously \( w_{n+1} > 0 \) on \( I \) implies \( w_n \) being constant on \( I \). Thus \( w \in E \).

To prove the second assertion of the claim, we let \( x \in [0, 1] \) and \( n \in \mathbb{N} \), then for \( x \in [b_{j-1}, b_j] \)

\[
|v_n(x) - w_n(x)| = |v_n(x) - v_n(\lambda(x))| \leq \text{Lip } v_n |x - \lambda(x)| \leq \text{Lip } \lambda_n |\lambda(a_j) - a_j| = \text{Lip } v_n (b_j - a_j) = \text{Lip } v_n \frac{2\epsilon}{p} \leq \frac{2\epsilon}{p} \leq \frac{1}{p} < \epsilon.
\]

Proof. of Claim 5.3.5 For all \( j \in \{1, \cdots, p\} \), if \( x \in [a_j, c_j] \), then

\[
\left| \frac{\epsilon}{p} - |x - c_j| \right| = |x - a_j|;
\]

or if \( x \in [c_j, b_j] \) then

\[
\left| \frac{\epsilon}{p} - |x - c_j| \right| = |x - b_j| \leq |x - a_j|.
\]

Also, \( v_n(b_j) = v_n(\lambda(b_j)) = v_n(\lambda(a_j)) = w_n(a_j) \) for all \( j \). If \( n = m_j \) and \( x, y \in [a_j, b_j] \), then

\[
\left| \frac{u_n^*(x) - u_n^*(y)}{x - y} \right| < \frac{1}{|x - y|} \frac{1}{|x - a_j|} \leq \frac{1}{|x - y|} |x - y| = 1.
\]

If \( n = m_j \) and \( x, y \in [b_{j-1}, a_j] \), then

\[
\left| \frac{u_n^*(x) - u_n^*(y)}{x - y} \right| = \left| \frac{w_n(x) - w_n(y)}{x - y} \right| \leq 1.
\]
Suppose \( n = m_j \) and \( x \in [b_{j-1}, a_j) \) and \( y \in [a_j, b_j] \). Suppose \( u_n^*(y) = v_n(b_j) + \frac{\varepsilon}{p} - |y - c_j| \).

Then

\[
|u_n^*(x) - u_n^*(y)| = \left| w_n(x) - v_n(b_j) - \frac{\varepsilon}{p} + |y - c_j| \right|
\]
\[
\leq |w_n(x) - v_n(b_j)| + \left| \frac{\varepsilon}{p} - |y - c_j| \right|
\]
\[
= |w_n(x) - w_n(a_j)| + \left| \frac{\varepsilon}{p} - |y - c_j| \right|
\]
\[
\leq |x - a_j| + \left| \frac{\varepsilon}{p} - |y - c_j| \right|
\]
\[
\leq |x - a_j| + |y - a_j| = |x - y|.
\]

Suppose \( u_n^*(y) = v_n(b_j) - \min\{v_n(b_j), \frac{\varepsilon}{p} - |y - c_j|\} \). So, if \( v_n(b_j) > \frac{\varepsilon}{p} - |y - c_j| \) then

\[
|w_n^*(x) - u_n^*(y)| \leq |w_n(x) - v_n(b_j)| + \left| \frac{\varepsilon}{p} - |y - c_j| \right| \leq |x - y|.
\]

If \( v_n(b_j) \leq \frac{\varepsilon}{p} - |y - c_j| \), then \( v_n(b_j) \leq |y - a_j| \); hence,

\[
|u_n^*(x) - u_n^*(y)| = w_n(x) = v_n(\lambda(x)) \leq v_n(\lambda(x)) + |y - a_j| - v_n(b_j)
\]
\[
= v_n(\lambda(x)) - v_n(\lambda(a_j)) + |y - a_j|
\]
\[
\leq (1 - 2\varepsilon)|\lambda(x) - \lambda(a_j)| + |y - a_j| \leq |x - a_j| + |y - a_j| = |x - y|.
\]

Thus if \( n = m_j \) then \( \text{Lip} u_n^* \leq 1 \) and if \( n \neq m_j \) then \( \text{Lip} u_n^* = \text{Lip} w_n \leq 1 \).

(The property 5.1)

From Claim 5.3.2(d) we have either \( v_{n+1}(x) = 0 \) or \( v_n(x) > v_{n+1}(x) + (1 - 2\varepsilon)\frac{1}{m^2} \) for \( x \in [0, 1] \) and \( n \in \mathbb{N} \). Let \( m_j \) be as defined above for each \( j \in \{1, \cdots, p\} \). Then \( v_{m_j}(b_j) < \frac{\varepsilon}{p} \).

Suppose \( v_{m_j+1}(b_j) \neq 0 \). Then

\[
v_{m_j+1}(b_j) < v_{m_j}(b_j) - (1 - 2\varepsilon)\frac{1}{m^2} < \frac{\varepsilon}{p} - (1 - 2\varepsilon)\frac{1}{m^2} < 0.
\]

Thus \( v_{m_j+1}(b_j) = 0 \). So, if \( x \in [a_j, b_j] \) then

\[
u_{m_j+1}(x) = w_{m_j+1}(x) = u_{m_j+1}(\lambda(x)) = v_{m_j+1}(b_j) = 0.
\]
Also, by definition of $m_j$, we have $v_{m_j-1}(b_j) \geq \frac{\varepsilon}{p}$. Then for $x \in [a_j, b_j]$ we have

$$v_{m_j-1}(x) = v_{m_j-1}(b_j) \geq \frac{\varepsilon}{p} > v_{m_j}(b_j),$$

and $v_{m_j}(b_j) \neq 0$ implies that

$$v_{m_j-1}(b_j) > v_{m_j}(b_j) + (1 - 2\varepsilon)\frac{1}{m^2} > v_{m_j}(b_j) + \frac{\varepsilon}{p}.$$

Thus $u^*_{m_j-1}(x) \geq u^*_{m_j}(x)$ for $x \in [a_j, b_j]$.

(The property 5.2)

Since $w$ and $v$ satisfy this property, so does $u$.

Now we show that $f^* = A_\varphi(u^*) \notin M_k$. Let $z_0 \in [0, 1 - \frac{1}{p}]$. Let $j \in \{2, \ldots, p\}$ such that $b_{j-2} \geq z < b_{j-1}$ and let $n = m_j$. Then $u_n^*(b_j) = u_n(b_j)$. If $\varphi(u_n(b_j)) \leq 4k\psi(\frac{2}{p})$, then

$$\varphi(u_n^*(c_j)) = \varphi(u_n(b_j) + \varepsilon/p) \geq \varphi(\varepsilon/p) \geq 8k\varphi(2/p).$$

Otherwise, $\varphi(u_n^*(c_j)) = 0$. Thus in either case we have

$$|\varphi(u_n^*(c_j)) - \varphi(u_n^*(b_j))| \geq 4k\psi(2/p).$$

Since $u_i^*$ are constant on $[a_j, b_j]$ for every $i \neq m_j$, we have

$$|f^*(c_j) - f^*(b_j)| = |\varphi(u_n^*(c_j)) - \varphi(u_n^*(b_j))| \geq 4k\psi(2/p).$$

**Case 1:** Suppose $|f^*(z) - f^*(b_j)| \leq 2k\psi(\frac{2}{p})$.

Then $|f^*(z) - f^*(c_j)| \geq |f^*(c_j) - f^*(b_j)| - |f^*(b_j) - f^*(z)| \geq 4k\psi(\frac{3}{p}) - 2k\psi(\frac{2}{p}) = 2k\psi(\frac{2}{p}).$

**Case 2:** Suppose $|f^*(z) - f^*(b_j)| > 2k\psi(\frac{3}{p})$. 
In either case there is \( z \in [c_j, b_j] \) such that \( |f^*(z) - f^*(x)| \geq 2k\psi(\frac{2}{p}) \). Also, for any \( z \in [c_j, b_j] \) we have \( x - z \geq \frac{2}{p} < \frac{1}{k} \) thus, \( \psi(x-z) \geq \psi(\frac{2}{p}) \). Then

\[
\frac{|f^*(z) - f^*(x)|}{\psi(x-z)} \geq \frac{2k\psi(\frac{2}{p})}{\psi(\frac{2}{p})} = 2k.
\]

Thus \( f^* \notin M_k \).

Therefore \( u^* \in E \setminus A_{\varphi}^{-1}(M_k) \).

Note that \( \|u_n^* - u_n\| \leq \epsilon/p < \epsilon \). From claim 5.3.4(b) \( \|v_n - w_n\| < \epsilon \). From claim 5.3.4(b) \( \|v_n - u_n\| < 3\epsilon \). Thus \( \|u^* - u\| \leq \|u_n^* - w_n\| + \|w_n - v_n\| + \|v_n - u_n\| < 5\epsilon \). □

### 5.5 Morse-Besicovitch functions in \( A_\varphi(E) \).

Let \( \varphi : [0,1] \to \mathbb{R} \) be a continuous increasing function such that \( \varphi(0) = 0 \) and \( \varphi'(x) < \infty \) for all \( x \in (0,1) \), and \( \varphi_+(0) = \varphi^+(0) = \infty \). Then by Theorem 5.2.4 for every \( f \in A_\varphi(E), f_+(x) < \infty \) if \( x \in [0,1] \). Note that if \( f \in A_\varphi(E) \) then \( f(1-z) \in A_\varphi(E) \) since \( u_n(1-z) \in E \) if \( u_n(x) \in E \). Then for every \( f \in A_\varphi(E), f_-(x) < \infty, ([f^-(x) > -\infty]) \) if \( x \in [0,1] \). To apply Theorem 5.2.3 we let \( \psi(x) = x \). Then

\[
M_1 = \left\{ f \in A_\varphi(E) : \exists z_0 \in [0,1] \text{ such that } \limsup_{x \to z_0^+} \frac{|f(x) - f(z_0)|}{x - z_0} < \infty \right\}
\]

is meager in \( A_\varphi(E) \). Let \( S^+ := A_\varphi(E) \setminus M_1 \). Then

\[
S^+ = \left\{ f \in A_\varphi(E) : f_+(x) < f^+(x) = \infty \text{ for every } x \in [0,1] \right\}
\]

is comeager. Consider also \( \psi_2(x) = \psi_1(1-x) = 1-x \) and \( \varphi(1-x) \). Then by Theorem 5.2.3

\[
M_2 = \left\{ f \in A_\varphi(E) : \exists z_0 \in (0,1] \text{ such that } \lim_{x \to z_0^-} \frac{|f(x) - f(z_0)|}{x - z_0} < \infty \right\}
\]
is meager in $A_\varphi(E)$. Let $S^- = A_\varphi(E) \setminus M_2$. Then

$$S^- = \{ f \in A_\varphi(E) : f_-(x) < f^-(x) = \infty \text{ for every } x \in (0,1) \}$$

is comeager. Then $S^+ \cap S^-$ is dense in $A_\varphi(E)$ and if $f \in S^+ \cap S^-$ then $f$ is a Morse-Besicovitch function.
CHAPTER 6

OTHER RESULTS ON BESICOVITCH FUNCTIONS

There are many other results on various aspects of Besicovitch functions.

**Definition 6.0.1** We say $f$ has symmetrical derivatives at a point $x$ if $f^{+}(x) = f^{-}(x)$ and $f_+(x) = f_-(x)$. We say $f$ has asymmetrical derivatives at $x$ if $f^{+}(x) \neq f^{-}(x)$ or $f_+(x) \neq f_-(x)$. We also define $\text{Knot}(f) := \{ x : f^{+}(x) = f^{-}(x) = \infty \text{ and } f_+(x) = f_-(x) = -\infty \}$.

W. H. Young proved in 1908 that the set $\text{Knot}(f)$ for a Besicovitch function $f$ is residual and the set of all points where $f$ has asymmetrical derivatives is non-empty and is everywhere dense [Yo]. A. N. Singh proved that the complement of $\text{Knot}(f)$ is uncountable in every interval [Si1]. K. M. Garg proved that there exists a Besicovitch function which have asymmetrical derivatives almost everywhere and the Knot points of such a function forms a residual set of measure zero [Ga]. Garg also proved that the set of points where a Besicovitch function $f$ has asymmetrical derivatives has a positive measure in every interval (Corollary of Prop. 2 in [Ga]). In 1990, J. Bobok proved that almost all continuous $\mu$-measure preserving functions, in category sense, have a knot point at $\nu$-almost every point, where $\mu$, $\nu$ are continuous probability measures on $[0, 1]$ [Bo]. Thus we can conclude from these two results that Besicovitch functions form a meager set in the space of continuous measure preserving functions. In 1988, M. Hata [Ha] showed the existence of a Besicovitch function $f$ with Hölder constant $1 - \varepsilon$ and $m(\text{Knot}(f)) = \alpha$, for any $\alpha \in [0, 1]$ and $\varepsilon \in (0, 1)$. Mauldin [Mau2][Mau3] proved that Besicovitch functions form coanalytic non-Borel subset in the space of all real valued continuous functions on $[0, 1]$. In 1989, T. I. Rasamujh published a result on the study of the complexity of Besicovitch functions [Ra].
Brownian Motion and Besicovitch functions. Brownian motion can be described as a probability space whose elements are all continuous functions defined on the whole real line and vanishing at the origin [DEK]. Dvoretzky, Erdős, and Kakutani proved that the probability of the set of functions which increase at least at one point is zero [DEK]. They also deduced the following:

**Theorem 6.0.2** Almost all Brownian paths have everywhere lower derivative $-\infty$ and upper derivative $+\infty$, that is,

\[
P \left\{ \liminf_{h \to 0} \frac{z_\omega(t + h) - z_\omega(t)}{h} = -\infty, \limsup_{h \to 0} \frac{z_\omega(t + h) - z_\omega(t)}{h} = +\infty \text{ for all } t \right\} = 1,
\]

where $P$ is a probability measure on Brownian motion.

So, if a path has an one-sided derivative, it will be infinite ($+\infty$ or $-\infty$). But, almost every path in Brownian motion is not a Besicovitch function. This claim is due to R. D. Mauldin [Maul]. Mauldin wrote, in a letter to Kakutani dated July 28th, 1982,

In Sherbrooke, I raised the following problem: Is there a reasonable class of stochastic processes such that almost every path in such a process is a function of Besicovitch? ....... I think that quite a few people are under the impression that your results with Erdős and Dvoretzky imply that almost every path in Brownian motion is a function of Besicovitch. One argument that this is not so is the following:

First, if $x$ is a continuous function such that $D^+x(t) \geq 0$ for almost every $t$ and $D^+x(t) > -\infty$ for all $t$, then $x$ is nondecreasing. This follows from Theorem 7.3 of Sak’s book (Chapter IV, page 204), Theory of Integral. So, if $x$ is a continuous path in Brownian motion such that (1) $x$ is not nondecreasing, and (2) $D^+x(t) \geq 0$ for almost every $t$, then $D^+x(t) = -\infty$ for some $t$. Thus,
$x'_4(t) = -\infty$, for some $t$ (in fact, uncountably many). Since almost every path in Brownian motion has the properties (1) and (2), almost every path in Brownian motion is not a function of Besicovitch.

We state the Theorem 7.3 of Chapter IV in Theory of Integral as a reference.

**Theorem 6.0.3** Suppose that $f$ is a continuous function and $g$ a integrable function of a real variable, and that, further, we have

1. $f^+(x) \geq g(x)$ at almost all points $x$ and
2. $f^+(x) > -\infty$ at every point $x$, except at most those of an enumerable set;

then

$$f(b) - f(a) \geq \int_a^b g(x) dx$$

for every pair of points $a$ and $b$ such that $a < b$.

From this theorem, by taking $g(x) = 0$, we see the argument in the letter.
REFERENCES


