TENSOR PRODUCTS OF BANACH SPACES

DISSERTATION

Presented to the Graduate Council of the

University of North Texas in Partial

Fulfillment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

By

James Philip Ochoa, B.M.E., M.S.

Denton, Texas

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Tensor products of Banach Spaces are studied. An introduction to tensor products is given. Some results concerning the reciprocal Dunford-Pettis Property due to Emmanuele are presented. Pełczyński's property (V) and (V)-sets are studied. It will be shown that if X and Y are Banach spaces with property (V) and every integral operator from X into Y^* is compact, then the (V)-subsets of $(X\check{\otimes}Y)^*$ are weak* sequentially compact. This in turn will be used to prove some stronger convergence results for (V)-subsets of $C(\Omega, X)^*$. Finally, it will be shown that if the Banach space X has a basis and f is a member of $C(\Omega, X)$, then there exists a unique sequence (f_n) in $C(\Omega)$ such that

$$f = \sum_{n=1}^{\infty} f_n \otimes x_n.$$

This representation will be used to show that representing measures for operators from $C(\Omega, X)$ into Y take thier values in $\mathcal{L}(X,\ Y)$ if and only if the operator is the pointwise limit of a sequence of weakly compact operators and the representing measure is the pointwise limit of the corresponding sequence of representing measures.

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CHAPTER 1

INTRODUCTION

This paper studies various aspects tensor products of Banach Spaces. An introduction to the tensor products is given in Chapter 2. Most of the material in this chapter comes from the mongraph by Diestel and Uhl [DU, Chapter VIII]. The proofs presented here generally provide more details then the proofs in thier book. Furthermore, some of thier proofs have been modified and simplified. For example, the use of the Stone Representation Theorem in the proof of the factorization theorem for integral operators (Theorem VIII.1.9 in [DU], Theorem 2.19 here) has been abandoned in favor of a more basic argument.

In Chapter 3, the reciprocal Dunford-Petts property on the projective tensor product of two Banach spaces is studied. The main results are due to Emmanuele [EM2]. A detailed presentation is given. The proof of Theorem 3.9 provides the motivation for the main results in Chapter 4.

Property (V) and (V)-sets are introduced in Chapter 4. The (V)-subsets of the contiuous dual of the injective tensor product of Banach spaces are studied. Sufficient conditions for weak* sequential compactness, a necessary condition for weak compactness, are given. This is used to provide stronger convergence result for (V)-subsets of the space $C(\Omega, X)^*$. Additionally, a new proof of a well known theorem is presented.

In Chapter 5, a representation theorem for the space $C(\Omega, X)$ is presented when it is assumed that X has a basis. This result is used to give a charactorization of representing measures for operators from $C(\Omega, X)$ into Y which take thier values in $\mathcal{L}(X, Y)$.

Most definitions of terms and symbols are provided in the paper as needed. The symbol $\square$ at the end of a line indicates the end of a proof. Definitions of any terms or symbols not given in this paper may be found in [DU]. Royden [RDN, Chapter 10] and Diestel and Uhl [DU, Chapters I, II, and VI] provide a sufficient background in functional analysis for this paper.

CHAPTER 2

TENSOR PRODUCTS OF BANACH SPACES

Let X and Y be vector spaces over the real numbers and let $B(X, Y)$ be the vector space of all bilinear forms on $X \times Y$. For $(x, y) \in X \times Y$, let $x \otimes y$ be the member of $B(X, Y)^\sharp$, the algebraic dual of $B(X, Y)$, defined by

$$(x \otimes y, f) = f(x, y)$$

for all $f \in B(X, Y)$. The linear span in $B(X, Y)^\sharp$ of $\{x \otimes y : \ x \in X, \ y \in Y\}$ will be denoted by $X \otimes Y$. Members of $X \otimes Y$ satisfy the following properties:

$$(x + z) \otimes y = x \otimes y + z \otimes y,$$

$$x \otimes (y + w) = x \otimes y + x \otimes w,$$

$$\alpha x \otimes y = x \otimes \alpha y.$$

The proofs of these properties are an easy exercise and are ommitted. Further information on the algebraic properties of $X \otimes Y$ can be found in any standard algebra text (see, for example, Hungerford [HUN]).

The remainder of this chapter involves the study of tensor products of Banach spaces based on material from Diestel and Uhl [DU, Chapter 8]. A detailed presentation is given here. Let X and Y be Banach spaces. For each member (x, y) of $X \times Y$ define $\|(x, y)\|$ by

$$\|(x, y)\| = \max \{\|x\|, \ \|y\|\}.$$

This defines a norm on $X \times Y$. The subspace of $B(X, Y)$ of continuous bilinear forms on $(X \times Y, \|\cdot\|)$ will be denoted by $\mathcal{B}(X, Y)$. Each member of $X^* \otimes Y^*$ defines in a natural way a member of $\mathcal{B}(X, Y)$. Let $v = \sum_{i=1}^n x_i^* \otimes y_i^*$ be a member of $X^* \otimes Y^*$ and (x, y) be a member of $X \times Y$. Then $v(x, y)$ is defined by

$$v(x, y) = \sum_{i=1}^{n} x_i^*(x) y_i^*(y).$$

Definition 2.1 *Let X and Y be Banach spaces. A norm α on $X \otimes Y$ is called a reasonable crossnorm if the following two conditions hold:*

R1 $\alpha(x \otimes y) \leq \|x\| \, \|y\|$ *for all* $x \in X$, $y \in Y$,

R2 *if $x^* \in X^*$ and $y^* \in Y^*$, then $x^* \otimes y^*$ defines a member of $(X \otimes Y)^*$ and has functional norm no larger than $\|x^*\| \, \|y^*\|$.*

Proposition 2.2 *Suppose α is a reasonable crossnorm on $X \otimes Y$. Then*

1. $\alpha(x \otimes y) = \|x\| \, \|y\|$,
2. *if $x^* \in X^*$ and $y^* \in Y^*$ then the norm of $x^* \otimes y^*$ as a member of $(X \otimes Y, \alpha)^*$ is $\|x^*\| \, \|y^*\|$,*
3. *if α^* is the norm on $X^* \otimes Y^*$ as a subspace of $(X \otimes Y, \alpha)^*$, then α^* is a reasonable crossnorm on $X^* \otimes Y^*$.*

Proof. To prove (1), let $x \in X$ and $y \in Y$. Choose $x^* \in X^*$ and $y^* \in Y^*$, each of norm one, such that $x^*(x) = \|x\|$ and $y^*(y) = \|y\|$. By R2 of the definition, $x^* \otimes y^*$

is a member of $(X \otimes Y,\ \alpha)^*$ and the functional norm of $x^* \otimes y^*$ is no more than one. Thus

$$\begin{aligned} \|x\|\ \|y\| &= |x^*(x)y^*(y)| \\ &= |(x^* \otimes y^*)(x \otimes y)| \\ &\leq \alpha(x \otimes y). \end{aligned}$$

R1 of the definition gives the reverse inequality.

To prove (2), let $x^* \in X^*$ and $y^* \in Y^*$ Choose sequences (x_n) and (y_n) from X and Y respectively such that $\|x_n\| = \|y_n\| = 1$, $\|x^*\| = \lim_n x^*(x_n)$, and $\|y^*\| = \lim_n y^*(y_n)$. Then

$$\begin{aligned} \|x^*\|\ \|y^*\| &= \lim_n |x^*(x_n)|\ |y^*(y_n)| \\ &= \lim_n |(x^* \otimes y^*)(x \otimes y)| \\ &\leq \limsup_n \alpha(x_n \otimes y_n)\mathrm{norm}(\mathrm{x}^* \otimes \mathrm{y}^*) \\ &\leq \mathrm{norm}(\mathrm{x}^* \otimes \mathrm{y}^*), \\ &\leq \|x^*\|\ \|y^*\|. \end{aligned}$$

The last inequality follows from R2 of the definition. It follows that the functional norm of $x^* \otimes y^*$ is $\|x^*\|\ \|y^*\|$.

Finally, to prove (3), let $x^* \in X^*$ and $y^* \in Y^*$. Then $\alpha^*(x^* \otimes y^*)$ is the functional norm of $x^* \otimes y^*$. Therefore condition R1 of the definition is satisfied. Thus it must be shown that if x^{**} is a member of X^{**} and y^{**} is a member of Y^{**}, then $x^{**} \otimes y^{**}$

is a member of $(X^* \otimes Y^*,\ \alpha^*)^*$ and the functional norm of $x^{**} \otimes y^{**}$ is no more than $\|x^{**}\|\ \|y^{**}\|$.

Let $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$. Choose nets (x_β) in X and (y_γ) in Y such that $\|x_\beta\| \le \|x^{**}\|$, $\|y_\gamma\| \le \|y^{**}\|$, $\lim_\beta x_\beta = x^{**}$, and $\lim_\gamma y_\gamma = y^{**}$, where the limits occur in the weak* topologies on X^{**} and Y^{**} respectively. Let $u^* = \sum_{i=1}^n x_i^* \otimes y_i^*$ be a member of $X^* \otimes Y^*$. Then

$$\begin{aligned}
|(x^{**} \otimes y^{**})(u^*)| &= \left|\sum_{i=1}^n x^{**}(x_i^*)y^{**}(y_i^*)\right| \\
&= \left|\sum_{i=1}^n \lim_\beta x_i^*(x_\beta)\ \lim_\gamma y_i^*(y_\gamma)\right| \\
&= \lim_\beta\ \lim_\gamma \left|\sum_{i=1}^n x_i^*(x_\beta)y_i^*(y_\gamma)\right| \\
&\le \limsup_{\beta,\gamma} |(x_\beta \otimes y_\gamma)(u^*)| \\
&\le \limsup_{\beta,\gamma} \|x_\beta\|\ \|y_\gamma\|\, \alpha^*(u^*) \\
&\le \|x^{**}\|\ \|y^{**}\|\, \alpha^*(u^*).
\end{aligned}$$

Part (3) follows. This completes the proof of the proposition. □

Let $u \in X \otimes Y$. Define $\lambda(u)$ by

$$\lambda(u) = \sup\{|(x^* \otimes y^*)(u)| :\ x^* \in X^*,\ y^* \in Y^*,\ \|x^*\|\ , \|y^*\| \le 1\}.$$

It is easily seen that λ defines a norm on $X \otimes Y$.

Proposition 2.3 *The norm λ is a reasonable crossnorm on $X \otimes Y$.*

Proof. Let $x \in X$ and $y \in Y$. Then

$$\lambda(x \otimes y) = \sup\{|(x^* \otimes y^*)(x \otimes y)| :\ x^* \in X^*,\ y^* \in Y^*,\ \|x^*\|,\ \|y^*\| \le 1\}$$

$$= \sup\{|x^*(x)y^*(y)| : \ x^* \in X^*, \ y^* \in Y^*, \ \|x^*\|, \ \|y^*\| \le 1\}$$

$$\le \ \|x\| \, \|y\| .$$

This shows λ satisfies the R1 of the definition.

Let $x^* \in X^*$ and $y^* \in Y^*$. For any $u \in X \otimes Y$,

$$\begin{aligned} |(x^* \otimes y^*)(u)| &\le \|x^*\| \, \|y^*\| \, |(\,(x^*/\|x^*\|) \otimes (y^*/\|y^*\|)\,)(u)| \\ &\le \|x^*\| \, \|y^*\| \, \lambda(u), \end{aligned}$$

the last inequality resulting from the definition of λ. Thus R2 is also satisfied and λ is a reasonable crossnorm. The proposition follows. □

The completion of $(X \otimes Y, \lambda)$ will be denoted by $X \check{\otimes} Y$ and called the injective tensor product of X and Y. The norm on $X \check{\otimes} Y$ will still be denoted by λ.

Proposition 2.4 *Let Ω be a compact Hausdorff space and X be a Banach space. The space $C(\Omega) \check{\otimes} X$ is linearly isometric to the Banach space $C(\Omega, X)$ of continuous functions $f \colon \Omega \longmapsto X$ equipped with norm $\|f\|_\infty = \sup\{\|f(\omega)\| : \ \omega \in \Omega\}$.*

Proof. Define $J : \ C(\Omega) \otimes X \longmapsto C(\Omega, X)$ by

$$J(\sum_{i=1}^n f_i \otimes x_i)(\omega) = \sum_{i=1}^n f_i(\omega) x_i .$$

Then

$$\left\| J \left(\sum_{i=1}^n f_i \otimes x_i \right) \right\|_\infty$$

$$= \ \sup \left\{ \left\| \sum_{i=1}^n f_i(\omega) x_i \right\| : \ \omega \in \Omega \right\}$$

$$
\begin{aligned}
&= \sup\left\{\left|\langle x^*, \sum_{i=1}^n f_i(\omega)x_i\rangle\right| :\ x^* \in X^*,\ \|x^*\| \le 1,\ \omega \in \Omega\right\} \\
&= \sup\left\{\left\|\sum_{i=1}^n \langle x^*, x_i\rangle f_i\right\|_\infty :\ x^* \in X^*,\ \|x^*\| \le 1\right\} \\
&= \sup\left\{\langle \nu, \sum_{i=1}^n \langle x^*, x_i\rangle f_i\rangle :\ \nu \in C(\Omega)^*,\ x^* \in X^*,\ \|\nu\| \le 1, \|x^*\| \le 1\right\} \\
&= \sup\left\{\left|\sum_{i=1}^n \langle \nu, f_i\rangle\langle x^*, x_i\rangle\right| :\ \nu \in C(\Omega)^*,\ x^* \in X^*,\ \|\nu\| \le 1, \|x^*\| \le 1\right\} \\
&= \lambda\left(\sum_{i=1}^n f_i \otimes x_i\right).
\end{aligned}
$$

It follows that J extends to a linear isometry from $X\check{\otimes}Y$ into $C(\Omega, X)$.

Now suppose g is a member of $C(\Omega, X)$. The range of g is a compact subset of X. Let $\epsilon > 0$. Choose $\omega_1,\ \omega_2,\ \ldots,\ \omega_n$ in Ω such that for each ω in Ω, there is an i, $1 \le i \le n$, for which $\|g(\omega) - g(\omega_i)\| \le \epsilon/2$. For each i, put $U_i = \{\omega \in \Omega : \|g(\omega) - g(\omega_i)\| < \epsilon\}$. The set $\mathcal{U} = \{U_1,\ U_2,\ \ldots,\ U_n\}$ is a finte open cover of Ω. Let $\{g_1,\ g_2,\ \ldots,\ g_n\}$ be a finite partition of unity subordinate to $\mathcal{U}$ [WIL]. That is, each g_i is continuous, $\sum_{i=1}^n g_i(\omega) = 1$ for all ω in Ω, $0 \le g_i(\omega) \le 1$ for all ω in Ω, and $g_i(\omega) = 0$ if ω is not a member of U_i. Define h: $\Omega \longmapsto X$ by

$$h(\omega) = \sum_{i=1}^n g_i(\omega)g(\omega_i).$$

Then

$$h = J\left(\sum_{i=1}^n g_i \otimes g(\omega_i)\right)$$

and

$$
\begin{aligned}
\|h(\omega) - g(\omega)\| &= \left\|\left(\sum_{i=1}^n g_i(\omega)g(\omega_i)\right) - g(\omega)\right\| \\
&= \left\|\left(\sum_{i=1}^n g_i(\omega)g(\omega_i)\right) - \left(\sum_{i=1}^n g_i(\omega)\right) g(\omega)\right\|
\end{aligned}
$$

$$\begin{aligned} &\leq \sum_{i=1}^{n} g_i(\omega) \, \|g(\omega_i) - g(\omega)\| \\ &< \sum_{i=1}^{n} g_i(\omega)\epsilon \;=\; \epsilon. \end{aligned}$$

Thus the range of J is a closed subspace (J is an isometry) of $C(\Omega, X)$ and the range of J is dense in $C(\Omega, X)$. It follows that the range of J is all of $C(\Omega, X)$. Thus J is a surjection. □

Let $u \in X \otimes Y$. Define $\gamma(u)$ by

$$\gamma(u) = \sup \{ |\psi(u)| : \; \psi \in \mathcal{B}(X, \; Y), \|\psi\| \leq 1 \} \, .$$

Then γ defines a seminorm on $X \otimes Y$.

Proposition 2.5 *The norm γ is a reasonable crossnorm on $X \otimes Y$. Futhermore, if u is a member of $X \otimes Y$ then $\lambda(u) \leq \gamma(u)$.*

Proof. First note that $X^* \check{\otimes} Y^*$ is isometric to a closed linear subspace of $\mathcal{B}(X^{**}, Y^{**})$. Thus $\|x^* \otimes y^*\|_{\mathcal{B}(X^{**}, \; Y^{**})} = \lambda(x^* \otimes y^*) = \|x^*\| \; \|y^*\|$. Consequently the restriction $(x^* \otimes y^*)|_{X \otimes Y}$ of $x^* \otimes y^*$ to $X \otimes Y$ satisfies

$$\|(x^* \otimes y^*)|_{X \otimes Y}\|_{\mathcal{B}(X, \; Y)} \leq \|x^* \otimes y^*\|_{\mathcal{B}(X^{**}, \; Y^{**})} = \|x^*\| \; \|y^*\| \, .$$

Thus if $u \in X \otimes Y$ then

$$\begin{aligned} \lambda(u) &= \sup \{ |x^* \otimes y^*(u)| : \; x^* \in X^*, \; y^* \in Y^*, \; \|x\|, \; \|y^*\| \leq 1 \} \\ &\leq \sup \{ |\psi(u)| : \; \psi \in \mathcal{B}(X, \; Y), \; \|\psi\| \leq 1 \} \\ &= \gamma(u) \, . \end{aligned}$$

This shows that $\lambda(u) \leq \gamma(u)$.

Now suppose x and y are nonzero members of X and Y respectively. Then

$$\begin{aligned}\gamma(x \otimes y) &= \sup\{|\psi(x,y)| : \psi \in \mathcal{B}(X, Y), \|\psi\| \leq 1\} \\ &= \sup\{\|x\| \, \|y\| \, |\psi(x/\|x\|, y/\|y\|)| : \psi \in \mathcal{B}(X, Y), \|\psi\| \leq 1\} \\ &= \|x\| \, \|y\|.\end{aligned}$$

This shows that γ satisfies R1. Since γ dominates λ, for $x^* \in X^*$ and $y^* \in Y^*$, it follows that $x^* \otimes y^*$ is a member of $(X \otimes Y, \gamma)^*$ and has functional norm no greater than $\|x^*\| \, \|y^*\|$. Thus γ satisfies R2 and first statement of the proposition is proven. □

The completion of $(X \otimes Y, \gamma)$ will be denoted $X \hat{\otimes} Y$ and called the projective tensor product of X and Y. The norm on $X \hat{\otimes} Y$ will still be denoted by γ. The following proposition gives a useful alternative way to consider γ.

Proposition 2.6 *If u is member of $X \otimes Y$, then*

$$\gamma(u) = \inf\left\{\sum_{i=1}^{n} \|x_i\| \, \|y_i\| : x_i \in X, \ y_i \in Y, \ u = \sum_{i=1}^{n} x_i \otimes y_i\right\}.$$

If u is a member of $X \hat{\otimes} Y$ and $\epsilon > 0$, then there exist sequences (x_n) in X and (y_n) in Y such that $\lim_n x_n = 0 = \lim_n y_n$, $u = \sum_{n=1}^{\infty} x_n \otimes y_n$ in γ norm, and such that

$$\gamma(u) \leq \sum_{n=1}^{\infty} \|x_n\| \, \|y_n\| \leq \gamma(u) + \epsilon.$$

Proof. To prove the first statement, let

$$\alpha(u) = \inf\left\{\sum_{i=1}^{n} \|x_i\| \, \|y_i\| : x_i \in X, \ y_i \in Y, \ u = \sum_{i=1}^{n} x_i \otimes y_i\right\}.$$

Clearly, $\alpha(x \otimes y) \leq \|x\| \; \|y\|$. If $u = \sum_{i=1}^{n} x_i \otimes y_i$ then

$$\gamma(u) \leq \sum_{i=1}^{n} \gamma(x_i \otimes y_i) = \sum_{i=1}^{n} \|x_i\| \; \|y_i\| .$$

It follows $\gamma(u) \leq \alpha(u)$ for all $u \in X \otimes Y$ and that α is a reasonable crossnorm on $X \otimes Y$.

Let $u \in X \otimes Y$. Choose $\phi \in (X \otimes Y, \alpha)^*$ such that $\phi(u) = \|u\|$ and $\|\phi\|_{(X \otimes Y, \alpha)^*} = 1$. Define ψ on $X \times Y$ by

$$\psi(x, y) = \phi(x \otimes y).$$

Then

$$\begin{aligned} |\psi(x, y)| &= |\phi(x \otimes y)| \\ &\leq \alpha(x \otimes y) \, \|\phi\|_{(X \otimes Y, \alpha)^*} \\ &= \alpha(x \otimes y) \\ &= \|x\| \; \|y\| \end{aligned}$$

It follows that ψ is a member of $\mathcal{B}(X, \; Y)$ and $\|\psi\| \leq 1$. Hence

$$\alpha(u) = |\phi(u)| = |\psi(u)| \leq \gamma(u)$$

and $\gamma = \alpha$. This proves the first statement.

To prove the second statement, select a sequence (u_n) in $X \otimes Y$ such that $\gamma(u - u_n) < \epsilon/2^{n+3}$ for each natural number n. Using the first statement of the proposition, write $u_1 = \sum_{i=1}^{i(1)} x_i \otimes y_i$, where

$$\sum_{i=1}^{i(1)} \|x_i\| \; \|y_i\| \leq \gamma(u_1) + \epsilon/2^4 \leq \gamma(u) + \epsilon/2^3.$$

The last inequality follows from a simple calculation. For each $n \geq 1$,

$$\begin{aligned} \gamma(u_{n+1} - u_n) &\leq \gamma(u - u_{n+1}) + \gamma(u - u_n) \\ &\leq \epsilon/2^{n+4} + \epsilon/2^{n+3} < \epsilon/2^{n+2}. \end{aligned}$$

Using this inequality and the first statement of the proposition, for each $n \geq 1$, write

$$u_{n+1} - u_n = \sum_{i=i(n)+1}^{i(n+1)} x_i \otimes y_i,$$

where $\sum_{i=i(n)+1}^{i(n+1)} \|x_i\| \, \|y_i\| < \epsilon/2^{n+2}$. Thus

$$\begin{aligned} \gamma\left(u - \sum_{i=1}^{i(n+1)} x_i \otimes y_i\right) &= \gamma\left(u - \left(\sum_{i=1}^{i(1)} x_i \otimes y_i + \sum_{k=1}^{n} \sum_{i=i(k)+1}^{i(k+1)} x_i \otimes y_i\right)\right) \\ &= \gamma\left(u - \left(u_1 + \sum_{k=1}^{n}(u_{k+1} - u_k)\right)\right) \\ &= \gamma(u - u_n) < \epsilon/2^{n+3}. \end{aligned}$$

Hence $\sum_{i=1}^{\infty} x_i \otimes y_i$ converges absolutely to u and clearly, using the triangle inequality,

$$\gamma(u) \leq \sum_{i=1}^{\infty} \|x_i\| \, \|y_i\| .$$

Also,

$$\begin{aligned} \sum_{i=1}^{\infty} \|x_i\| \, \|y_i\| &= \sum_{i=1}^{i(1)} \|x_i\| \, \|y_i\| + \sum_{i=i(1)+1}^{\infty} \|x_i\| \, \|y_i\| \\ &\leq \gamma(u) + \epsilon/2^3 + \sum_{k=1}^{\infty} \sum_{i=i(k)+1}^{i(k+1)} \|x_i\| \, \|y_i\| \\ &\leq \gamma(u) + \epsilon/2^3 + \sum_{k=1}^{\infty} \epsilon/2^{k+2} \\ &< \gamma(u) + \epsilon. \end{aligned}$$

All that remains to be proved is that (x_n) and (y_n) may be chosen so that $\lim_n \|x_n\| = 0 = \lim_n \|y_n\|$. Suppose (x_n) and (y_n) are the sequences obtained above and, without lose of generality, assume all of the terms are nonzero. Let i be a natural number. Choose n such that

$$i(n)+1 \leq i \leq i(n+1).$$

Then $\|x_i\| \; \|y_i\| < \epsilon/2^{n+1}$. Choose α_i such that $\|y_i\| / \alpha_i = \sqrt{\epsilon/2^{n+1}}$. Then

$$\alpha_i \|x_i\| \sqrt{\frac{\epsilon}{2^{n+2}}} = \|x\| \; \|y\| < \frac{\epsilon}{2^{n+1}}.$$

Therefore,

$$\|\alpha_i x_i\| < \sqrt{\epsilon/2^{n+2}}.$$

The sequences $(\alpha_n x_n)$ and (y_n/α_n) satisfy the conclusion of the proposition. □

Attention is now turned to the continuous duals of $X\hat{\otimes}Y$ and $X\check{\otimes}Y$. Let $\mathcal{L}(X,\ Y)$ denote the space of bounded linear operators from X into Y with the usual operator norm. The next theorem provides some natural linearly isometric representations of $(X\hat{\otimes}Y)^*$.

Theorem 2.7 *Let X and Y be Banach spaces. The spaces $(X\hat{\otimes}Y)^*$, $\mathcal{B}(X,\ Y)$, and $\mathcal{L}(X, Y^*)$ are all linearly isometric.*

Proof. Let $\psi \in \mathcal{B}(X,\ Y)$ and $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. Put $\hat{\psi}(u) = \sum_{i=1}^{n} \psi(x_i, y_i)$. The definition of the tensor product guarantees that $\hat{\psi}$ is well defined. Futhermore,

$$\left|\hat{\psi}(u)\right| = \left|\sum_{i=1}^{n} \psi(x_i, y_i)\right| \leq \|\psi\| \sum_{i=1}^{n} \|x_i\| \; \|y_i\|.$$

Therefore $\hat{\psi}$ is continuous on $(X \otimes Y, \gamma)$ and by 2.6 the norm of $\hat{\psi}$ is no greater than $\|\psi\|$. Extend $\hat{\psi}$ to all of $X\hat{\otimes}Y$ to obtain a member of $(X\hat{\otimes}Y)^*$.

Now suppose $\hat{\psi} \in (X\hat{\otimes}Y)^*$. For each (x,y) in $X \times Y$ put $\psi(x,y) = \hat{\psi}(x \otimes y)$. Then

$$|\psi(x,y)| = \left|\hat{\psi}(x \otimes y)\right| \leq \left\|\hat{\psi}\right\| \gamma(x \otimes y) = \left\|\hat{\psi}\right\| \|x\| \, \|y\| .$$

Therefore ψ defines a member of $\mathcal{B}(X,\ Y)$ and $\|\psi\| \leq \left\|\hat{\psi}\right\|$. It follows that the map $\psi \mapsto \hat{\psi}$ defines a linear isometry from $\mathcal{B}(X,\ Y)$ onto $(X\hat{\otimes}Y)^*$.

Let $\psi' \in \mathcal{L}(X,\ Y^*)$ and $(x,y) \in X \times Y$. Put $\psi(x,y) = \langle\psi'(x), y\rangle$. Then

$$|\psi(x,y)| = |\langle\psi'(x), y\rangle| \leq \|\psi'\| \, \|x\| \, \|y\| .$$

Therefore ψ is a member of $\mathcal{B}(X,\ Y)$ and the functional norm of ψ is no greater than $\|\psi'\|$.

Now suppose $\psi \in \mathcal{B}(X,\ Y)$, $x \in X$, and $y \in Y$. Put $\langle\psi'(x), y\rangle = \psi(x,y)$. Then

$$|\langle\psi'(x), y\rangle| = |\psi(x,y)| \leq \|\psi\| \, \|(x,y)\| .$$

Therefore, ψ' is member of $\mathcal{L}(X,\ Y^*)$ and the operator norm of ψ' is no greater than $\|\psi\|$. It follows the map $\psi' \mapsto \psi$ defines a linear isometry from $\mathcal{L}(X,\ Y^*)$ onto $\mathcal{B}(X,\ Y)$.

Thus

$$(X\hat{\otimes}Y)^* \cong \mathcal{B}(X,\ Y) \cong \mathcal{L}(X,\ Y)$$

under the correspondence

$$\hat{\psi} \leftrightarrow \psi \leftrightarrow \psi'.$$

The theorem follows. □

There is a natural map, η, from $Y \otimes X$ onto $X \otimes Y$, given by

$$\eta\left(\sum_{i=1}^{n} y_i \otimes x_i\right) = \sum_{i=1}^{n} x_i \otimes y_i,$$

which establishes a linear isometry between $Y\hat{\otimes}X$ and $X\hat{\otimes}Y$ (and between $Y\check{\otimes}X$ and $X\check{\otimes}Y$). Thus the adjoint, η^*, is a linear isometry between the respective duals. Let ψ be a member of $\mathcal{B}(X, Y)$. Then $\eta^*(\psi)$ is the member of $\mathcal{B}(Y, X)$ defined by

$$\eta^*(\psi)(y, x) = \psi(x, y)$$

for all $y \in Y,\ x \in X$. Let T be the member of $\mathcal{L}(X, Y^*)$ for which

$$\langle T(x), y\rangle = \psi(x, y)$$

for each $x \in X,\ y \in Y$. Thinking of η^* as an isometry between $\mathcal{L}(X, Y^*)$ and $\mathcal{L}(Y, X^*)$ note that

$$\langle \eta^*(T)(y), x\rangle \ = \ \eta\psi(y, x) \ = \ \psi(x, y).$$

Let J be the natural embedding of Y into Y^{**}. Then

$$\begin{aligned} \langle T^* J(y), x\rangle &= \langle T(x), y\rangle \\ &= \psi(x, y) \\ &= \eta^*(y, x) \\ &= \langle \eta^*(T)(y), x\rangle. \end{aligned}$$

These remarks are summarized in the next corollary.

Corollary 2.8 *Let $J\colon Y \longmapsto Y^{**}$ be the natural embedding. The map $T \mapsto T^* \circ J$ is a linear isometry between $\mathcal{L}(X, Y^*)$ and $\mathcal{L}(Y, X^*)$.*

Attention is now turned to the dual of $(X\check{\otimes}Y)$. Let $S = (B_{X^*} \times B_{Y^*}, w^* \times w^*)$. Let $u = \sum_{i=1}^n x_i \otimes y_i$ be a member of $X\check{\otimes}Y$. For each $(x^*, y^*) \in S$, define $\hat{u}(x^*, y^*)$ by

$$\hat{u}(x^*, y^*) = \sum_{i=1}^{n} x^*(x_i)y^*(y_i).$$

The map $u \mapsto \hat{u}$ defines a linear isometry from $X\check{\otimes}Y$ into $C(S)$.

Theorem 2.9 (Grothendieck) *A continuous bilinear functional ψ defines a member of $(X\check{\otimes}Y)^*$ if and only if there exists a regular Borel measure μ on S such that for all $x \in X$ and for all $y \in Y$,*

$$\psi(x, y) = \int_S x^*(x)y^*(y)\, d\mu(x^*, y^*).$$

In this case, μ may be chosen so that the norm of ψ as member of $(X\check{\otimes}Y)^*$ equals $|\mu|(S)$ where $|\mu|$ is the variation of μ.

Proof. Let ψ be a member of $(X\check{\otimes}Y)^*$. Thinking of $X\check{\otimes}Y$ as a closed linear subspace of $C(S)$, let $\tilde{\psi}$ be a Hahn-Banach extension of ψ to all of $C(S)$. Using the Riesz Representation Theorem, obtain a regular Borel measure μ on S such that

$$\tilde{\psi}(f) = \int_S f\, d\mu$$

for all $f \in C(S)$ and such that $|\mu|(S) = \left\|\tilde{\psi}\right\| = \|\psi\|_{(X\check{\otimes}Y)^*}$. Thus

$$\psi(x, y) = \tilde{\psi}(x, y) = \int_S x^*(x)y^*(y)\, d\mu(x^*, y^*).$$

Now suppose ψ is a continuous bilinear functional on $X \times Y$ and has a representation

$$\psi(x,y) = \int_S x^*(x)y^*(y)\,d\mu(x^*,y^*),$$

where μ is a regular Borel measure on S. Define $\tilde{\psi}$ on $X \otimes Y$ by

$$\tilde{\psi}(u) = \sum_{i=1}^{n} \psi(x_i, y_i)$$

for all $u = \sum_{i=1}^{n} x_i \otimes y_i$ in $X \otimes Y$. Then for each $u = \sum_{i=1}^{n} x_i \otimes y_i$ in $X \otimes Y$,

$$\begin{aligned}
\left|\hat{\psi}(u)\right| &= \left|\sum_{i=1}^{n} \psi(x_i, y_i)\right| \\
&= \left|\sum_{i=1}^{n} \int_S x^*(x_i)y^*(y_i)\,d\mu(x^*,y^*)\right| \\
&\leq \int_S \left|\sum_{i=1}^{n} x^*(x_i)y^*(y_i)\,d\mu(x^*,y^*)\right| \\
&\leq \left\|\sum_{i=1}^{n} x_i \otimes y_i\right\|_\infty \; |\mu|(S) \\
&= \lambda(u)\;|\mu|(S).
\end{aligned}$$

The last inequality pertains to u as a member of $C(S)$. Thus $\tilde{\psi}$ extends to a continuous linear functional on $X\check{\otimes}Y$ with $\left\|\tilde{\psi}\right\| \leq |\mu|(S)$. An appeal to the first half of the argument guarantees that μ may be chosen so that the norm of ψ is $|\mu|(S)$. The theorem follows. □

Theorem 2.9 inspires the following definition.

Definition 2.10 *A continuous bilinear form ψ on $X \times Y$ is said to be integral whenever ψ defines a member of $(X\check{\otimes}Y)^*$. The norm of ψ as a member of $(X\check{\otimes}Y)^*$ will be called the integral norm of ψ and denoted by $\|\psi\|_{int}$.*

The space of integral bilinear forms on $X \times Y$ with the integral norm will be denoted by $\hat{\mathcal{B}}(X, Y)$.

Theorem 2.9 guarantees that

$$(X \check{\otimes} Y)^* \cong \hat{\mathcal{B}}(X,\ Y).$$

Definition 2.11 *An operator $T\colon\ X \longmapsto Y$ is said to be an integral operator if the bilinear functional τ on $X \times Y^*$ defined by*

$$\tau(x, y^*) = \langle T(x), y^* \rangle$$

is a member of $\hat{\mathcal{B}}(X,\ Y^*)$. The integral norm of T is defined to be $\|\tau\|_{int}$ and is denoted by $\|T\|_{int}$.

Suppose W, X, Y, and Z are Banach spaces and suppose $T\colon\ X \longmapsto W$ and $S\colon Y \longmapsto Z$ are bounded linear operators. Let $\sum_{i=1}^{n} x_i \otimes y_i$ be a member of $X \otimes Y$. Define $(T \otimes S)(u)$ by

$$(T \otimes S)(u) = \sum_{i=1}^{n} T(x_i) \otimes S(y_i).$$

Lemma 2.12 *The map $T \otimes S$ is a well defined bounded linear operator from $X \check{\otimes} Y$ into $W \check{\otimes} Z$. Futhermore,*

$$\|T \otimes S\| \leq \|T\|\ \|S\|.$$

Proof. Suppose $u \in X \otimes Y$ and has representations

$$u = \sum_{i=1}^{n} x_i \otimes y_i = \sum_{j=1}^{m} u_j \otimes v_j.$$

Define $T \times S : X \times Y \longmapsto W \times Z$ by

$$(T \times S)(x, y) = (T(x), S(y)).$$

Let $\theta \in \mathcal{B}(W,\ Z)$. Then $\theta \circ (T \times S)$ is a member of $\mathcal{B}(X,\ Y)$. Thus

$$\begin{aligned} \theta\left(\sum_{i=1}^{n} T(x_i) \otimes S(y_j)\right) &= \sum_{i=1}^{n} \theta(\,T(x_i), S(y_i)\,) \\ &= \sum_{i=1}^{n} \theta(T \times S)(x_i, y_i) \\ &= \sum_{j=1}^{m} \theta(T \times S)(u_j, v_j) \\ &= \theta\left(\sum_{j=1}^{m} (\,T(u_j), S(v_j)\,)\right). \end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} T(x_i) \otimes S(y_i) = \sum_{j=1}^{m} T(u_j) \otimes S(v_j).$$

The proves the first statement. The second statement follows from a simple calculation. □

Theorem 2.13 *Let W, X, Y, and Z be Banach spaces and let $T\colon\ W \longmapsto X$, $S\colon X \longmapsto Y$, and $R\colon\ Y \longmapsto Z$ be bounded linear operators with S integral. Then $RST\colon W \longmapsto Z$ is integral and $\|RST\|_{int} \le \|R\|\ \|S\|_{int}\ \|T\|$.*

Proof. Using 2.12, the map $T \otimes R^*\colon W \check{\otimes} Z^* \longmapsto X \check{\otimes} Y^*$ is well defined. Thus,

$$(T \otimes R^*)^*\colon \hat{\mathcal{B}}(X,\ Y^*) \longmapsto \hat{\mathcal{B}}(W,\ Z^*).$$

Let $\psi \in \hat{\mathcal{B}}(X,\ Y^*)$ such that $\langle S(x),\ y^* \rangle = \psi(x, y^*)$ and $\|S\|_{int} = \|\psi\|_{int}$. Then $(T \otimes R^*)^*(\psi)$ is a member of $\hat{\mathcal{B}}(W,\ Z^*)$. Let $w \in W$ and $z^* \in Z^*$. Then

$$(T \otimes R^*)^*(\psi)(w, z^*) \quad = \quad \psi(T \otimes R^*)(w, z^*)$$

$$= \psi(T(w), R^*(z^*))$$

$$= \langle ST(w), z^* \rangle$$

$$= \langle RST(w), z^* \rangle.$$

Thus RST is integral. Note that

$$\begin{aligned} \|RST\|_{int} &= \|(T \otimes R^*)^*(\psi)\|_{int} \\ &\leq \|(T \otimes R^*)^*\| \; \|\psi\|_{int} \\ &\leq \|T\| \; \|R^*\| \; \|\psi\|_{int} \\ &= \|R\| \; \|S\|_{int} \; \|T\|. \end{aligned}$$

The theorem follows. □

Theorem 2.14 *Let X and Y be Banach spaces, $T\colon X \longmapsto Y$ be a bounded linear operator, and $J\colon Y \longmapsto Y^{**}$ be the natural embedding. Then T is integral if and only if JT is integral. In this case, $\|JT\|_{int} = \|T\|_{int}$.*

Proof. If T is integral then, by 2.13, so is JT, and $\|JT\|_{int} \leq \|J\| \; \|T\|_{int} = \|T\|_{int}$. Suppose JT is integral. Let ψ be the member of $\hat{\mathcal{B}}(X,\ Y^{***})$ corresponding to JT. Let J_* be the natural embedding of Y^* into Y^{***} and I_X be the identity on X. Then $(I_X \otimes J_*)\colon X\check{\otimes}Y^* \longmapsto X\check{\otimes}Y^{***}$ is continuous and

$$(I_X \otimes J_*)^*\colon \hat{\mathcal{B}}(X,\ Y^*) \longmapsto \hat{\mathcal{B}}(X,\ Y^{***}).$$

Let $(x, y^*) \in X \times Y^*$. Then

$$\langle (I_X \otimes J_*)^*(\psi), (x, y^*) \rangle = \langle \psi, (I_X \otimes J_*)(x, y^*) \rangle$$

$$
\begin{aligned}
&= \psi\left(x, J_*(Y^*)\right) \\
&= \langle JT(x), J_*(y^*)\rangle \\
&= \langle T(x), y^*\rangle.
\end{aligned}
$$

This shows T is integral. Futhermore,

$$
\begin{aligned}
\|T\|_{int} &= \|(I_X \otimes J_*)^*(\psi)\|_{int} \\
&\leq \|(I_X \otimes J_*)^*\| \, \|\psi\|_{int} \\
&\leq \|\psi\|_{int} = \|JT\|_{int}.
\end{aligned}
$$

The theorem follows. □

The following lemma will be used to prove that the natural inclusion map from $L_\infty(\mu)$ into $L_1(\mu)$ and is integral and has integral norm equal to $|\mu|(\Omega)$.

Lemma 2.15 *Let K be a finite dimensional subspace of $L_\infty(\mu)$ and let $\{[f_1], [f_2], \ldots, [f_n]\}$ be a basis for K. For each i, $1 \leq i \leq n$, let f_i be a representative from $[f_i]$. Then there exists a μ-null subset N of Ω such that for each n-tuple $(r_1, r_2, \ldots, r_n)$ of real numbers,*

$$\left\|\sum_{i=1}^{n} r_i[f_i]\right\|_\infty = \sup_{\omega \in \Omega\setminus N} \left|\sum_{i=1}^{n} r_i f_i(\omega)\right|.$$

Proof [LEW]. For each n-tuple $(q_1, q_2, \ldots, q_n)$ of rational numbers, choose a μ-null subset $N(q_1, \ldots, q_n)$ of Ω such that

$$\left\|\sum_{i=1}^{n} q_i[f_i]\right\|_\infty = \sup\left\{\left|\sum_{i=1}^{n} q_i f_i(\omega)\right| : \omega \in \Omega\setminus N(q_1, \ldots, q_n)\right\}.$$

Let

$$N = \bigcup_{(q_1,\ldots,q_n)\in Q^n} N\left((q_1,\,\ldots,\,q_n\right).$$

Since N is the countable union of null sets, N is also a null set. Thus, for any n-tuple $(q_1,\,\ldots,\,q_n)$ of rational numbers,

$$\left\|\sum_{i=1}^{n} q_i[f_i]\right\|_\infty = \sup_{\omega\in\Omega\setminus N}\left|\sum_{i=1}^{n} q_i f_i(\omega)\right|.$$

Now suppose $(r_1,\,r_2,\,\ldots,\,r_n)$ is an n-tuple of real numbers and let $\epsilon > 0$. Note that $\sup_{\omega\in\Omega\setminus N}|\sum_{i=1}^{n} r_i f_i(\omega)|$ is finite since $\sup_{\omega\in\Omega\setminus N}|\sum_{i=1}^{n} f_i(\omega)|$ is finite. Choose a rational n-tuple $(q_1,\,q_2,\,\ldots,\,q_n)$ such that

$$\sup_{\omega\in\Omega\setminus N}\left|\sum_{i=1}^{n}(r_i - q_i)f_i(\omega)\right| < \epsilon.$$

Then

$$\begin{aligned}
\left\|\sum_{i=1}^{n} r_i[f_i]\right\|_\infty &\le \sup_{\omega\in\Omega\setminus N}\left|\sum_{i=1}^{n} r_i f_i(\omega)\right| \\
&\le \sup_{\omega\in\Omega\setminus N}\left|\sum_{i=1}^{n} q_i f_i(\omega)\right| + \sup_{\omega\in\Omega\setminus N}\left|\sum_{i=1}^{n}(r_i - q_i)f_i(\omega)\right| \\
&< \sup_{\omega\in\Omega\setminus N}\left|\sum_{i=1}^{n} q_i f_i(\omega)\right| + \epsilon \\
&= \left\|\sum_{i=1}^{n} q_i[f_i]\right\|_\infty + \epsilon \\
&\le \left\|\sum_{i=1}^{n} r_i[f_i]\right\|_\infty + \left\|\sum_{i=1}^{n}(r_i - q_i)[f_i]\right\|_\infty + \epsilon \\
&\le \left\|\sum_{i=1}^{n} r_i[f_i]\right\|_\infty + \sup_{\omega\in\Omega\setminus N}\left|\sum_{i=1}^{n}(r_i - q_i)f_i(\omega)\right| + \epsilon \\
&< \left\|\sum_{i=1}^{n} r_i[f_i]\right\|_\infty + 2\epsilon.
\end{aligned}$$

Specifically,

$$\left\|\sum_{i=1}^{n} r_i[f_i]\right\|_\infty \leq \sup_{\omega\in\Omega\setminus N}\left|\sum_{i=1}^{n} r_i f_i(\omega)\right| < \left\|\sum_{i=1}^{n} r_i[f_i]\right\|_\infty + 2\epsilon.$$

It follows that

$$\left\|\sum_{i=1}^{n} r_i[f_i]\right\|_\infty = \sup_{\omega\in\Omega\setminus N}\left|\sum_{i=1}^{n} r_i f_i(\omega)\right|.$$

The lemma follows. □

Proposition 2.16 *Let (Ω, Σ, μ) be a finite measure space. Let $I\colon L_\infty(\mu) \longmapsto L_1(\mu)$ be the natural inclusion. Then I is integral and $\|I\|_{int} = |\mu|(\Omega)$.*

Proof. It will suffice to prove the proposition with the assumption $|\mu|(\Omega) = 1$. Let $\phi \in \mathcal{B}(L_\infty(\mu), L_\infty(\mu))$ such that

$$\phi(f, g) = \langle I(f), g\rangle = \int_\Omega fg\, d\mu$$

for all $f, g \in L_\infty(\mu)$. Suppose $u = \sum_{i=1}^{n}[f_i]\otimes[g_i]$ is a member of $L_\infty(\mu)\otimes L_\infty(\mu)$ with $\lambda(u) = 1$. For each j, $1 \leq j \leq n$, choose representatives $f_j \in [f_j]$ and $g_j \in [g_j]$. Let $\{[h_1],\ [h_2],\ \ldots,\ [h_k]\}$ be a maximal linearly independent subset of the set

$$\{[f_1],\ [f_2],\ \ldots,\ [f_n],\ [g_1],\ [g_2],\ \ldots,\ [g_n]\}$$

and let $\{h_1,\ \ldots,\ h_k\}$ be the corresponding linearly independent subset of

$$\{f_1, \ldots, f_n, g_1, \ldots, g_n\}.$$

Using 2.15 let N be a μ-null subset of Ω such that

$$\left\|\sum_{i=1}^{n} r_i[h_i]\right\|_\infty = \sup_{\omega\in\Omega\setminus N}\left|\sum_{i=1}^{n} r_i h_i(\omega)\right|.$$

for all n-tuples $(r_1, \ldots, r_n)$ of real numbers. Then the following isometry results:

$$(\text{span}\{\mathrm{h}_1, \ldots, \mathrm{h}_k\}, \text{sup norm on } \Omega\backslash\mathrm{N}) \cong (\text{span}\{[\mathrm{h}_1], \ldots, [\mathrm{h}_k]\}, \|\cdot\|_\infty).$$

For each $\omega \in \Omega\backslash N$, let δ_ω be the member of $(\text{span}\{\mathrm{h}_1, \ldots, \mathrm{h}_k\})^*$ defined by

$$\langle\delta_\omega, \sum_{i=1}^{k} r_i h_i\rangle = \sum_{i=1}^{k} r_i h_i(\omega).$$

Note that $\|\delta_\omega\| \leq 1$. Furthermore, δ_ω may be considered as a bounded linear functional on span $\{[\mathrm{h}_1], \ldots, [\mathrm{h}_k]\}$. Using the Hahn-Banach theorem, extend δ_ω to all of $L_\infty(\mu)$. Thus

$$\begin{aligned}
\left|\langle\phi, \sum_{i=1}^{n}[f_i]\otimes[g_i]\rangle\right| &= \left|\sum_{i=1}^{n}\phi([f_i],[g_i])\right| \\
&= \left|\sum_{i=1}^{n}\int_\Omega f_i g_i\, d\mu\right| \\
&= \left|\int_\Omega\left(\sum_{i=1}^{n} f_i g_i\right) d\mu\right| \\
&\leq \left\|\sum_{i=1}^{n}[f_i][g_i]\right\|_\infty |\mu|(\Omega) \\
&= \sup_{\omega\in\Omega\backslash N}\left|\sum_{i=1}^{n} f_i(\omega) g_i(\omega)\right| |\mu|(\Omega) \\
&= \sup_{\omega\in\Omega\backslash N}\left|\sum_{i=1}^{n}\langle\delta_\omega, f_i\rangle\langle\delta_\omega, g_i\rangle\right| |\mu|(\Omega) \\
&= \sup_{\omega\in\Omega\backslash N}\left|\sum_{i=1}^{n}\langle\delta_\omega, [f_i]\rangle\langle\delta_\omega, [g_i]\rangle\right| |\mu|(\Omega) \\
&\leq \lambda\left(\sum_{i=1}^{n}[f_i]\otimes[g_i]\right)|\mu|(\Omega).
\end{aligned}$$

It follows that ϕ is continuous on $(L_\infty(\mu)\otimes L_\infty(\mu), \lambda)$. Therefore, I is integral. Moreover,

$$\|\phi\|_{int} \leq |\mu|(\Omega).$$

Using the Hahn Decomposition Theorem [RDN, Proposition 11.5.21], write

$$\Omega = A \bigcup B$$

where A and B are disjoint measurable sets,

$$\mu(C) \geq 0,$$

for any measurable subset C of A, and

$$\mu(D) \leq 0$$

for any measuralble subset D of B. Then

$$\begin{aligned} |\langle \phi, \chi_A + \chi_B \otimes \chi_A - \chi_B \rangle| &= \left| \int_\Omega (\chi_A + \chi_B)(\chi_A - \chi_B)\, d\mu \right| \\ &= \left| \int_\Omega (\chi_A - \chi_B)\, d\mu \right| \\ &= |\mu(A) - \mu(B)| \;=\; |\mu|(\Omega). \end{aligned}$$

Thus

$$\|I\|_{int} = \|\phi\|_{int} = |\mu|(\Omega).$$

The proposition follows. □

Recall that a operator is absolutely summing if it sends weakly unconditionally Cauchy (wuC) series onto absolutely converging series. The following characterization for absolutely summing operators can be found in Diestel and Uhl [DU, Proposition VI.3.2].

Theorem 2.17 *Let $T\colon X \longmapsto Y$ be a bounded linear operator. Then T is absolutely summing if and only if there exists a $K > 0$ such that for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X the following inequality holds:*

$$\sum_{i=1}^{n} \|x_i\| \leq K \sup \left\{ \sum_{i=1}^{n} |\langle x^*, x_i \rangle| : x^* \in X^*, \|x^*\| \leq 1 \right\}.$$

The next proposition, coupled with Theorem 2.19, will show that integral operators are absolutely summing.

Proposition 2.18 *Let (Ω, Σ, μ) be a finite measure space. Let $I\colon L_\infty(\mu) \longmapsto L_1(\mu)$ be the natural inclusion. Then I is an absolutely summing operator.*

Proof. Suppose $\sum_{i=1}^{\infty} [f_i]$ is a wuC series in $L_\infty(\mu)$. Let n be a fixed natural number. Assume, also, that for each i, $1 \leq i \leq n$, $[f_i]$ is nonnegative μ-almost everywhere. The general case will follow. For each i, $1 \leq i \leq n$, let $f_i \in [f_i]$. Using 2.15, let N be a μ-null subset of Ω such that for each n-tuple $(r_1, r_2, \ldots, r_n)$ of real numbers,

$$\left\| \sum_{i=1}^{n} r_i [f_i] \right\|_\infty = \sup_{\omega \in \Omega \setminus N} \left| \sum_{i=1}^{n} r_i f_i(\omega) \right|.$$

For each ω of $\Omega \setminus N$ let δ_ω be the member of span $\{[f1], [f_2], \ldots, [f_n]\}^*$ defined by

$$\langle \delta_\omega, \sum_{i=1}^{n} r_i [f_i] \rangle = \sum_{i=1}^{n} r_i f_i(\omega)$$

for each $\sum_{i=1}^{n} r_i [f_i] \in$ span $\{[f_1], \ldots, [f_n]\}$. Thus

$$\begin{aligned} \sum_{i=1}^{n} \|[f_i]\|_1 &= \sum_{i=1}^{n} \int_\Omega |f_i| \, d\,|\mu| \\ &= \sum_{i=1}^{n} \int_\Omega f_i \, d\,|\mu| \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \sum_{i=1}^n f_i \, d|\mu| \\
&\leq |\mu|(\Omega) \left\| \sum_{i=1}^n f_i \right\|_\infty \\
&= |\mu|(\Omega) \sup_{\omega \in \Omega \setminus N} \left| \sum_{i=1}^n f_i(\omega) \right| \\
&= |\mu|(\Omega) \sup_{\omega \in \Omega \setminus N} \left| \sum_{i=1}^n \langle \delta_\omega, [f_i] \rangle \right| \\
&\leq |\mu|(\Omega) \sup \left\{ \sum_{i=1}^n |\langle \nu, [f_i] \rangle| : \nu \in (L_\infty(\mu))^*, \|\nu\| \leq 1 \right\}.
\end{aligned}
$$

For the general case, write

$$[f_i] = [f_i^+] - [f_i^-],$$

where $[f_i^+]$ and $[f_i^-]$ are both nonnegative μ-almost everywhere and

$$|[f_i]| = [f_i^+] + [f_i^-].$$

Apply the above argument to the set span $\left\{[f_1^+], \ldots, [f_n^+], [f_1^-], \ldots, [f_n^-]\right\}$. Note that

$$\sup \left\{ \sum_{i=1}^n f_i^+(\omega) : \omega \in \Omega \setminus N \right\} \leq \sup \left\{ \sum_{i=1}^n |f_i(\omega)| : \omega \in \Omega \setminus N \right\}$$

and

$$\sup \left\{ \sum_{i=1}^n f_i^-(\omega) : \omega \in \Omega \setminus N \right\} \leq \sup \left\{ \sum_{i=1}^n |f_i(\omega)| : \omega \in \Omega \setminus N \right\}.$$

Thus

$$\sum_{i=1}^n \|[f_i]\|_1 \leq 2\,|\mu|(\Omega) \sup \left\{ \sum_{i=1}^n |\langle \nu, [f_i] \rangle| : \nu \in (L_\infty(\mu))^*, \|\nu\| \leq 1 \right\}.$$

It follows from 2.17 that I is absolutely summing. $\square$

Theorem 2.19 *An operator* $T\colon X \longmapsto Y$ *is integral if and only if* JT *admits a factorization*

$$\begin{array}{ccccc} X & \xrightarrow{\;T\;} & Y & \xrightarrow{\;J\;} & Y^{**} \\ S\big\downarrow & & & & \big\uparrow Q \\ L_\infty(\mu) & & \xrightarrow[\;I\;]{} & & L_1(\mu) \end{array}$$

where $J\colon Y \longmapsto Y^{**}$ *is the natural embedding,* μ *is a finite regular Borel measure on a compact Hausdorff space* Ω, $I\colon L_\infty(\mu) \longmapsto L_1(\mu)$ *is the natural inclusion, and* $S\colon X \longmapsto L_\infty(\mu)$ *and* $Q\colon L_1(\mu) \longmapsto Y^{**}$ *are bounded linear operators. In this case,* Ω, μ, Q, *and* S *can be chosen so that* $\|S\|, \|Q\| \le 1$ *and* $\|T\|_{int} = |\mu|(\Omega)$.

Proof. Suppose T is integral. Let ψ be the member of $\hat{\mathcal{B}(X, Y)}$ induced by T. Let $\Omega = (B_{X^*} \times B_{Y^*}, w^* \times w^*)$. Choose a regular Borel measure μ on Ω such that

$$\langle T(x), y^* \rangle = \psi(x, y^*) = \int_\Omega x^*(x) y^*(y)\, d\mu(x^*, y^*)$$

and

$$|\mu|(\Omega) = \|\psi\|_{int} = \|T\|_{int}.$$

Define $S\colon X \longmapsto L_\infty(\mu)$ by

$$S(x)(x^*, y^{**}) = x^*(x)$$

and $R\colon Y^* \longmapsto L_\infty(\mu)$ by

$$R(x)(x^*, y^{**}) = y^{**}(y)$$

for all $x \in X$ and $y \in Y$. Then S and R are bounded linear operators, $\|S\| \leq 1$,and $\|R\| \leq 1$. Let $x \in X$ and $y^* \in Y^*$. Then

$$\begin{aligned}
\langle T(x), y^* \rangle &= \int_\Omega x^*(x) y^{**}(y^*)\, d\mu(x^*, y^{**}) \\
&= \int_\Omega S(x)(x^*, y^{**}) R(y^*)(x^*, y^{**})\, d\mu(x^*, y^{**}) \\
&= \int_\Omega IS(x)(x^*, y^{**}) R(y^*)(x^*, y^{**})\, d\mu(x^*, y^{**}) \\
&= \langle IS(x), R(y^*) \rangle \\
&= \langle R^* IS(x), y^* \rangle.
\end{aligned}$$

Let $Q = R^*$. Then $JT = QIS$ is the desired factorization.

The converse follows from 2.15 and 2.13. □

Corollary 2.20 *A bounded linear operator* $T\colon X \longmapsto Y$ *is integral if and only if the adjoint* $T^*\colon Y^* \longmapsto X^*$ *is integral. In this case,* $\|T\|_{int} = \|T^*\|_{int}$.

Proof. Suppose T is integral. Using the factorization in 2.19 and taking adjoints produces the commutative diagram

$$\begin{array}{ccccc}
Y^{***} & \xrightarrow{J^*} & Y^* & \xrightarrow{T^*} & X^* \\
Q^* \big\downarrow & & & & \big\uparrow S^* \\
L_\infty(\mu) = L_1(\mu)^* & & \xrightarrow[I^*]{} & & L_\infty(\mu)^* = L_1(\mu)^{**}
\end{array}$$

where J, μ, S, Q, and I are as in 2.19.

Let L be the natural embedding of $L_1(\mu)$ into $L_\infty(\mu)^*$. It is a simple exercise to show that $I^* = LI$. Let K be the natural embedding of Y^* into Y^{***}. Thus the diagram

$$
\begin{array}{ccccccc}
Y^* & \xrightarrow{K} & Y^{***} & \xrightarrow{J^*} & Y^* & \xrightarrow{T^*} & X^* \\
 & & \Big\downarrow {\scriptstyle Q^*} & & & & \Big\uparrow {\scriptstyle S^*} \\
 & & L_\infty(\mu) = L_1(\mu)^* & \xrightarrow[I^*]{} & L_1(\mu) & \xrightarrow[L]{} & L_\infty(\mu)^* = L_1(\mu)^{**}
\end{array}
$$

is obtained. Therefore,

$$T^* = T^*J^*K = S^*LIQ^*K.$$

Since I is integral, it follows by 2.13 that T^* is also integral. Finally, using 2.16,

$$\|T^*\|_{int} = \|S^*LIQ^*K\|_{int} \leq \|I\|_{int} = |\mu|(\Omega) = \|T\|_{int}.$$

Specifically,

$$\|T^*\|_{int} \leq \|T\|_{int}.$$

Now suppose T^* is integral. Let R be the natural embedding of X into X^{**}. Using the first part of the argument, T^{**} is also integral. Accordingly, the diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{T} & Y & \xrightarrow{J} & Y^{**} \\
\Big\downarrow {\scriptstyle R} & & & \nearrow {\scriptstyle T^{**}} & \\
X^{**} & & & &
\end{array}
$$

is obtained. Thus JT is integral. It follows that T is integral and

$$\|T\|_{int} = \|JT\|_{int} = \|T^{**}R\|_{int} \leq \|T^{**}\|_{int} \leq \|T^*\|_{int}.$$

Therefore $T$is integral and

$$\|T\|_{int} = \|T^*\|_{int}.$$

The corollary follows. □

Let X and Y be Banach spaces. The space of integral operators from X into Y will be denoted $I(X,\ Y)$. The next corollary shows that $(\,I(X,\ Y^*),\ \|\cdot\|_{int}\,)$ is linearly isometric to $(\,\hat{\mathcal{B}}(X,\ Y),\ \|\cdot\|_{int}\,)$.

Corollary 2.21 *A continuous bilinear functional ψ on $X \times Y$ is integral if and only if the continuous linear operator $T_\psi\colon\ X \longmapsto Y^*$ defined by $T_\psi(x)(y) = \psi(x,y)$ is integral. In this case, $\|\psi\|_{int} = \|T\|_{int}$.*

Proof. Let ψ be a member of $\mathcal{B}(X,\ Y^*)$ and suppose T_ψ is integral. Let τ be the member of $\hat{\mathcal{B}}(X,\ Y^{**})$ induced by T. Let $J\colon\ Y \longmapsto Y^{**}$ be the natural embedding and let I_X be the identity on X. Then

$$(I_X \otimes J)\colon\ X\check{\otimes}Y \longmapsto X\check{\otimes}Y^{**},$$

has operator norm one, and

$$(I_X \otimes J)^*\colon\ \hat{\mathcal{B}}(X,\ Y^{**}) \longmapsto \hat{\mathcal{B}}(X,\ Y).$$

Thus it will suffice to show $\psi = (I_X \otimes J)^*(\tau)$. Let $(x,y) \in X \times Y$. Then

$$\begin{aligned}
(I_X \otimes J)^*(\tau)(x,y) &= \tau\,(I_X \otimes J)\,(x,y) \\
&= \tau\,(I_X(x), J(y)) \\
&= \tau\,(x, J(y))
\end{aligned}$$

$$
\begin{aligned}
&= \langle T_\psi(x), J(y)\rangle \\
&= \langle T_\psi(x), y\rangle \; = \; \psi(x,y).
\end{aligned}
$$

It follows $\psi = (I_X \otimes J)^*(\tau)$. Futhermore,

$$
\begin{aligned}
\|\psi\|_{int} &= \|(I_X \otimes J)^*(\tau)\|_{int} \\
&\leq \|(I_X \otimes J)^*\| \, \|\tau\|_{int} \\
&= \|I_X \otimes J\| \, \|\tau\|_{int} \\
&= \|\tau\|_{int} \, .
\end{aligned}
$$

Now suppose ψ is integral. Let Ω be the space $(B_{X^*} \times B_{Y^*}, w^* \times w^*)$. Let μ be a regular Borel measure on Ω such that

$$\psi(x,y) = \int_\Omega x^*(x) y^*(y) \, d\mu \, (x^*, y^*)$$

for all $(x,y) \in X \times Y$ and $\|\psi\|_{int} = |\mu| \, (\Omega)$. Define R: $X \longmapsto L_\infty(\mu)$ by

$$R(x)(x^*, y^*) = x^*(x),$$

for all $x \in X$ and define S: $Y \longmapsto L_\infty(\mu)$ by

$$S(y)(x^*, y^*) = y^*(y)$$

for all $y \in Y$. Then for $(x,y) \in X \times Y$,

$$
\begin{aligned}
\langle T_\psi(x), y\rangle &= \psi(x,y) \\
&= \int_\Omega x^*(x) y^*(y) \, d\mu \, (x^*, y^*)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\Omega R(x)(x^*, y^*)S(x)(x^*, y^*)\, d\mu\,(x^*, y^*) \\
&= \langle R(x), IS(x)\rangle \\
&= \langle (IS)^* R(x), y\rangle,
\end{aligned}
$$

where I: $L_\infty(\mu) \longmapsto L_1(\mu)$ is the natural inclusion. Thus

$$T_\psi = (IS)^* R = S^* I^* R$$

and T_ψ is integral. Futhermore,

$$\|T_\psi\|_{int} = \|S^* I^* R\|_{int} \leq \|S^*\|\, \|I^*\|_{int}\, \|R\| \leq \|I^*\|_{int} = \|I\|_{int} = |\mu|\,(\Omega) = \|\psi\|_{int}\,.$$

The last inequality follows from 2.16. The corollary follows. □

CHAPTER 3

THE RECIPROCAL DUNFORD-PETTIS PROPERTY ON $X\hat{\otimes}Y$

This chapter deals with some results on tensor products of Banach spaces due to Emmanuele [EM2]. Recall that an operator on a Banach space is said to be Dunford-Pettis (or completely continuous) if it sends weakly Cauchy sequences onto norm convergent sequences. Such an operator will be called a DP operator. It is an easy exercise to show that T is a DP operator if and only if T sends weakly convergent sequences onto norm convergent sequences.

Definition 3.1 *A Banach space X is said to have the reciprocal Dunford-Pettis property (RDPP) if every DP operator on X is weakly compact.*

Definition 3.2 *Let X be a Banach space. A bounded subset K of X^* is an L-set if for each weakly null sequence (x_n) in X,*

$$\lim_n \sup_{x^* \in K} |\langle x^*, x_n\rangle| = 0.$$

The next theorem gives a useful charactorization of the RDPP.

Theorem 3.3 (Leavelle, [LV]) *A Banach space X has the RDPP if and only if every L-set in X^* is weakly compact.*

Proof. Suppose each L-set in X^* is weakly compact. Let $T\colon X \longmapsto Y$ be a DP operator. To show that T is weakly compact it will suffice to show $T^*(B_{Y^*})$ is an

L-set. Suppose (x_n) is a weakly null sequence in X and $y^* \in B_{Y^*}$. Then

$$|\langle T^*(y^*), x_n\rangle| = |\langle y^*, T(x_n)\rangle| \leq \|T(x_n)\| \xrightarrow{n} 0.$$

Thus B_{Y^*} is an L-set.

Now suppose X has the RDPP . Let K be an L-subset of X^*. Let $B(K)$ be the Banach space of all bounded real valued functions on K equipped with the supremum norm. Define T: $X \longmapsto B(K)$ by

$$T(x)(x^*) = \langle x^*, x\rangle$$

for all $x \in X$ and $x^* \in K$. Then T is a DP operator. To see this, let (x_n) be a weakly null sequence in X. Then

$$\|T(x_n)\|_\infty = \sup_{x^* \in K} |\langle T(x_n), x^*\rangle| = \sup_{x^* \in K} |\langle x^*, x\rangle| \xrightarrow{n} 0,$$

since K is an L-set. Thus T is a DP operator. Hence T and T^* are weakly compact. For $x^* \in K$ and $f \in B(K)$ put

$$\delta_{x^*}(f) = f(x^*).$$

Then δ_{x^*} defines a member of $B(K)^*$ with norm no greater than one. Note $T^*(\delta_{x^*}) = x^*$. Hence K is a subset of $T^*(B_{B(K)^*})$, a relatively weakly compact set. The theorem follows. □

The next theorem is due to Odell [ROS2, page 377].

Theorem 3.4 *A Banach space X does not contain an isomorphic copy of l_1 if and only if every DP operator mapping X into another Banach space is compact.*

Proof. If X does not contain an isomorphic copy of l_1 then by Rosenthal's l_1 Theorem [ROS1], B_X is weakly conditionally compact (i.e. every sequence has a weakly Cauchy subsequence). It follows that every DP operator on X is compact.

Conversely, suppose X contains an isomorphic copy of l_1. Let (e_n) be a copy of the canonical unit vector basis of l_1 in X and let (r_n) be the sequence of Radamacher functions in $L_\infty[0,1]$. That is for each natural number n and each real number t, $0 \leq t \leq 1$,

$$r_n(t) = \mathrm{sgn}(\sin(2^n \pi t))$$

where $\mathrm{sgn}(t) = t/|t|$ for $t \neq 0$ and $\mathrm{sgn}(t) = 0$ for $t = 0$. For each $\alpha = \sum_{i=1}^{n} \alpha_i e_i$ in $\overline{\mathrm{span}}\,\{e_n\}$ define $T(\alpha)$ by

$$T(\alpha) = \sum_{i=1}^{n} \alpha_i r_i.$$

Then T is a bounded linear operator from $\overline{\mathrm{span}}\,\{e_n\}$ into $L_\infty[0,1]$. Using the fact that $L_\infty[0,1]$ is injective, T can be extended to a map, still called T, on all of X. Now let I: $L_\infty[0,1] \longmapsto L_1[0,1]$ be the natural inclusion. Since $L_\infty[0,1]$ is linearly isometric to a $C(\Omega)$ space for some compact Hausdorff space Ω and I is weakly compact, it follows that I is also DP (see [DU][Corollary 17, p. 160]). Thus $I \circ T$ is DP; however, it is not compact. The theorem follows. □

Theorem 3.4 will be used to prove the next theorem.

Theorem 3.5 (Emmanuele, [EM1]) *A Banach space X does not contain an isomorphic copy of l_1 if and only if every L-subset of X^* is relatively compact.*

Proof. Suppose X does not contain an isomorphic copy of l_1. Let K be an L-subset of X^*. Let $B(K)$ be the space of bounded real value functions on K. Following [LV], Define T: $X \longmapsto B(K)$ by

$$T(x)(x^*) = \langle x^*, x\rangle$$

for all x in X and for all x^* in X^*. The argument in 3.3 shows that T is DP. Since T is DP, T is compact by 3.4. Therefore T^* is also compact. It follows from the proof of 3.3 that K is a subset of $T^*(B_{B(K)^*})$, a relatively compact set.

Now suppose every L-subset of X^* is relatively compact. Let T: $X \longmapsto Y$ be a DP operator and let $K = T^*(B_{Y^*})$. Hence K is a L-set and T^* and T are compact. The theorem follows. □

Lemma 3.6 *Suppose (x_n^*) is a sequence in X^* with the property that for each weakly null sequence (x_n) in X*

$$\lim_n \langle x_n^*, x_n\rangle = 0.$$

Then $\{x_n^* : n \in \mathbf{N}\}$ is an L-set.

Proof. The proof will consist of three steps.

Step 1. If ϕ is a permutation of the natural numbers then the sequence $(x^*_{\phi(n)})$ also satisfies the hypothesis of the lemma. To see this, let (x_n) be a weakly null sequence in X. For each natural number n, let $z_n = x_{\phi^{-1}(n)}$. Then (z_n) is also weakly null. Thus $|x_n^*(z_n)| \xrightarrow{n} 0$. Let $\epsilon > 0$. Choose a natural number N such that for all $n \geq N$,

$|x_n^*(z_n)| < \epsilon$. Next choose $M \geq N$ such that for all $n \geq M$,

$$\sup_{1 \leq k \leq N} \phi(k) < \phi(n).$$

Then for $n \geq M$, $\left|x_{\phi(n)}^*(z_{\phi(n)})\right| < \epsilon$. That is, $\left|x_{\phi(n)}^*(x_n)\right| < \epsilon$.

Step 2. If $(x_{n_i}^*)$ is a subsequence of (x_n^*), then $(x_{n_i}^*)$ satisfies the hypothesis of the lemma. To see this, let (x_i) be a weakly null sequence in X. For each natural number n, define z_n to be x_i if $n = n_i$ and to be the zero vector otherwise. Then (z_n) is a weakly null sequence. Thus $(x_{n_i}^*(x_i))$ is a subsequence of $(x_n^*(z_n))$, a null sequence.

Step 3. Finally, to show the set $\{x_n^* : n \in \mathbf{N}\}$ is an L-set, let (z_n^*) be a sequence in the set. Note that

$$\{z_n^* : n \in \mathbf{N}\} \subseteq \{x_n^* : \mathbf{N}\}.$$

Let (x_n) be a weakly null sequence in X. Suppose $(|z_n^*(x_n)|)$ does not converge to 0. A moment's reflection reveals that this implies $\{z_n^* : n \in \mathbf{N}\}$ must be an infinite set. Thus, it may be assumed, upon passing to a subsequence and relabeling if necessary, that there is $\epsilon > 0$ such that $|z_n^*(x_n)| > \epsilon$ for each n and such that $z_i^* \neq z_j^*$ whenever $i \neq j$. Now for some subsequence (w_n^*) of (x_n^*) and some permutation ϕ of the natural numbers, $z_n = w_{\phi(n)}$. Using the first two steps of the argument, it follows that $|z_n^*(x_n)| \xrightarrow{n} 0$, a contradiction. The lemma follows. □

A sequence satisfing the hypothesis of 3.6 will be call an L-sequence. The space of all compact linear operators from the Banach space X into the Banach space Y will be denoted $K(X, Y)$, and the space of all compact weak* to weak continuous linear operators from X^* into Y will be denoted $K_{w^*}(X^*, Y)$. The next lemma establishes a linear isometry between $K(X, Y)$ and $K_{w^*}(X^{**}, Y)$.

Lemma 3.7 *Let X and Y be Banach spaces. Then the spaces $K(X, Y)$ and $K_{w^*}(X^{**}, Y)$ are linearly isometric.*

Proof. Let T be a member of $K(X, Y)$. Then

$$T^{**}\colon X^{**} \longmapsto J(Y) \subseteq Y^{**}$$

where $J\colon\ Y \longmapsto Y^{**}$ is the natural embedding (see [DS, Theorem VI.4.2]). Thus $J^{-1} \circ T^{**}$ maps X^{**} into Y (here $J^{-1}\colon\ J(Y) \longmapsto Y$). Since T^{**} is weak* to weak* continuous and $(J(Y), w^*)$ and (Y, w) are linearly homeomorphic, it follows that $J^{-1} \circ T^{**}$ is a compact weak* to weak continuous operator. Thus the map $T \mapsto J^{-1} \circ T^{**}$ is a linear isometric embedding of $K(X, Y)$ into $K_{w^*}(X^{**}, Y)$.

Now suppose S is a member of $K_{w^*}(X^{**}, Y)$. Let $T = S \circ I$, where I is the natural embedding of X into X^{**}. Then $T^{**} = (S^{**} \circ I^{**})$. Thus for $x^{**} \in X^{**}$ and $y^* \in Y^*$,

$$\begin{aligned}\langle J^{-1}T^{**}(x^{**}), y^*\rangle &= \langle T^{**}(x^{**}), y^*\rangle \\ &= \langle S^{**}I^{**}(x^{**}), y^*\rangle \\ &= \langle x^{**}, I^*S^*(y^*)\rangle.\end{aligned}$$

Now let (x_α) be a net in X such that $\lim_\alpha I(x_\alpha) = x^{**}$ in the weak* topology on X^{**}. Then

$$\begin{aligned}\langle x^{**}, I^*s^*(y^*)\rangle &= \lim_\alpha \langle I(x_\alpha), I^*S^*(y^*)\rangle \\ &= \lim_\alpha \langle x_\alpha, I^*S^*(y^*)\rangle\end{aligned}$$

$$= \lim_{\alpha} \langle I(x_\alpha), S^*(y^*) \rangle$$
$$= \langle x^{**}, S^*(y^*) \rangle.$$

Thus

$$\langle J^{-1}T^{**}(x^{**}), y^* \rangle = \langle x^{**}, S^*(y^*) \rangle$$
$$= \langle S(x^{**}), y^* \rangle.$$

Therefore,

$$J^{-1} \circ T^{**} = S.$$

It follows that the map $T \mapsto J^{-1} \circ T^{**}$ defines a linear isomentry from $K(X, Y)$ onto $K_{w^*}(X^{**}, Y)$. The lemma follows. □

The following theorem, due to Ruess [RSS, 4.1.4], will be used in the proof of the main theorem (Theorem 3.9) of this chapter.

Theorem 3.8 *A bounded sequence (T_n) in $K_{w^*}(X^*, Y)$ converges weakly to T in $K_{w^*}(X^*, Y)$ if and only in $\langle T_n(x^*), y^* \rangle \xrightarrow{n} \langle T(x^*), y^* \rangle$ for all $x^* \in X^*$ and $y^* \in Y^*$.*

Theorem 3.9 (Emmanuele, [EM2]) *Let X be a Banach space not containing an isomorphic copy of l_1 and let Y be a Banach space with the RDPP. If $\mathcal{L}(X, Y^*) = K(X, Y^*)$, then $X \hat{\otimes} Y$ has the RDPP.*

The proof of the theorem will contain several numbered italicized assertions and thier proofs.

Proof. Let M be an L-subset of $(X \hat{\otimes} Y)^*$. Using the linear isometries established in 2.7, M will be considered as a subset of $K(X, Y^*)$. Let (h_n) be a sequence in M.

The goal is to show that (h_n) has a wealky convergent subsequence. To this end, let H be the closed linear span of $\{h_n(x) : x \in X, n \in \mathbf{N}\}$. Since each h_n is compact, H is separable. Let A be a countable weak* dense subset of H^*.

3.10 *By passing to a subsequence, it may be assumed that $(h_n(r))$ is convergent for each r in A.*

Proof of 3.10. First note that the sequence $(h_n^*(r))$ is an L-sequence in X^* for all r in H^*. To see this, let (x_n) be a weakly null sequence in X and let r be a member of H^*. Thus

$$\begin{aligned} |\langle h_n^*(r), x_n\rangle| &= |\langle r, h_n(x_n)\rangle| \\ &\leq \|r\| \, \|h_n(x_n)\| \,. \end{aligned}$$

3.11 *The sequence $(\|h_n(x_n)\|)$ converges to 0.*

Proof of 3.11. Suppose not. Choose $\epsilon > 0$ and a subsequence $h_{n_i}(x_{n_i})$ of $h_n(x_n)$ such that

$$\|h_{n_i}(x_{n_i})\| > \epsilon$$

for each i. Next, choose a sequence (z_i) in B_Y such that

$$|\langle h_{n_i}(x_{n_i}), z_i\rangle| > \epsilon$$

for each i. However, if T is a member of $K(X, Y^*)$, then

$$|\langle T(x_{n_i}), z_i\rangle| \leq \|T(x_{n_i})\| \xrightarrow{i} 0$$

since T is compact and (x_{n_i}) is weakly null. It follows that $(x_{n_i} \otimes z_i)$ is a weakly null sequence in $X\hat{\otimes}Y$. Since $(h_n^*(r))$ is an L-sequence,

$$|\langle h_{n_i}(x_{n_i}), z_i\rangle| \xrightarrow{i} 0.$$

However, this is a contradiction. The claim 3.11 follows.

From 3.11 it follows that

$$|\langle h_n^*(r), x_n\rangle| \xrightarrow{n} 0.$$

Hence $(h_n^*(r)$ is an L-sequence in X^*. Since X does not contain a copy of l_1, it follows from 3.5 that $(h_n^*(r))$ has a convergent subsequence. Since A is countable, a diagnalization argument finishes the proof. The claim 3.10 follows.

Now let x^{**} be a member of X^{**} and, using the fact each h_n is compact, consider $(h_n^{**}(x^{**}))$ as a sequence in Y^*.

3.12 *The sequence* $(h_n^{**}(x^{**}))$ *is an L-sequence.*

Proof of 3.12. Let (y_n) be a weakly null sequence in Y. For each natural number n,

$$\begin{aligned} |\langle h_n^{**}(x^{**}), y_n\rangle| &= |\langle x^{**}, h_n^*(y_n)\rangle| \\ &\leq \|x^{**}\| \, \|h_n^*(y_n)\|. \end{aligned}$$

By 2.8, (h_n^*) is an L-sequence in $K(Y^* X)$. Consequently, $\|h_n^*(y_n)\| \xrightarrow{n} 0$. This proves the claim 3.12.

Since Y has the RDPP, by 3.3, $\{h_n^{**}(x^{**}) : n \in \mathbf{N}\}$ is a relatively weakly compact subset of Y^* for each x^{**} in X^{**}. Since each h_n is compact and takes its range in H, the set $\{h_n^{**}(x^{**}) : n \in \mathbf{N}\}$ may be considered as a subset of H.

A weak limit for (h_n) will now be constructed. Fix x^{**} in X^{**}. Using the fact that $(h_n^{**}(x^{**}))$ is a sequence in a relatively weakly compact subset of H, let w and z be two weak-sequential cluster points of $(h_n^{**}(x^{**}))$, and let $(h_{n_i}^{**}(x^{**}))$ and $h_{n_j}^{**}(x^{**}))$ be subsequences converging weakly to w and z respectively. Let r be a member of A. Then, since $h_n^*(r)$ is a convergent sequence, it follows that

$$\begin{aligned}
\langle w, r\rangle &= \lim_i \langle h_{n_i}^{**}(x^{**}), r\rangle \\
&= \lim_i \langle x^{**}, h_{n_i}^*(r)\rangle \\
&= \lim_n \langle x^{**}, h_n^*(r)\rangle \\
&= \lim_j \langle x^{**}, h_{n_j}^*(r)\rangle \\
&= \lim_j \langle h_{n_j}^{**}(x^{**}), r\rangle \\
&= \langle z, r\rangle.
\end{aligned}$$

Therefore,

$$\langle w, r\rangle = \langle z, r\rangle$$

for all y in A (a weak* dense subset of H^*), and

$$w = z.$$

Hence $(h_n^{**}(x^{**}))$ is weakly convergent for every x^{**} in X^{**}.

For each x^{**} in X^{**} define $\tilde{h}(x^{**})$ by

$$\tilde{h}(x^{**}) = w - \lim_n h_n^{**}(x^{**}).$$

Then $\tilde{h}$ defines a bounded linear operator from X into H or from X into Y^{**}.

3.13 *$\tilde{h}$ is weak* to weak* continuous.*

Proof of 3.13. Let $(x_\alpha^{**})_\alpha$ be a weak* null net in X^{**} and let y be a member of Y. Thinking of y as a member of Y^{**}, $(h_n^*(y))$ is an L-sequence in X^*. By 3.5, $(h_n^*(y))$ has a subsequence $(h_{n_i}^*(y))$ converging to some x^* in X^*. Thus

$$\begin{aligned}
\lim_\alpha \langle \tilde{h}(x_\alpha^{**}), y\rangle &= \lim_\alpha \left(\lim_i \langle h_{n_i}^{**}(x_\alpha^{**}), y\rangle \right) \\
&= \lim_\alpha \left(\lim_i \langle x_\alpha^{**}, h_{n_i}^*(y)\rangle \right) \\
&= \lim_\alpha \langle x_\alpha^{**}, x^*\rangle \;=\; 0,
\end{aligned}$$

and 3.13 follows.

Let $h = \tilde{h} \circ I$, where I is the natural embedding of X into X^{**}. Then h is a compact operator from X into Y^*.

3.14 $h^{**} = \tilde{h}$.

Proof of 3.14. Let x^{**} be a member of X^{**} and let $(x_\alpha)_\alpha$ be a bounded net in X converging to x^{**} in the weak* topology on X^{**}. Then, using the fact that adjoints are weak* to weak* continuous,

$$\begin{aligned}
h^{**}(x^{**}) &= w^* - \lim_\alpha h^{**}(x_\alpha) \\
&= w^* - \lim_\alpha h(x_\alpha) \\
&= w^* - \lim_\alpha \tilde{h}(x_\alpha) \\
&= \tilde{h}(x^{**}),
\end{aligned}$$

and 3.14 follows.

Since $h^{**} = \tilde{h}$,

$$\lim_n \langle h_n^{**}(x^{**}), y^{**} \rangle = \langle \tilde{h}(x^{**}), y^{**} \rangle = \langle h^{**}(x^{**}), y^{**} \rangle$$

for all $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$. Thus, using 3.8, (h_n^{**}) converges to h^{**} in the weak topology on $K_{w^*}(X^{**}, Y^*)$. Hence by 3.7, (h_n) converges weakly to h in $\mathcal{L}(X, Y^*)$. The theorem follows. $\square$

CHAPTER 4

PROPERTY (V) ON $X\check{\otimes}Y$ AND (V)-SUBSETS OF $(X\check{\otimes}Y)^*$

In this chapter unconditionally converging operators on tensor products of Banach spaces are studied.

Definition 4.1 *Let X and Y be Banach spaces. An operator $T\colon X \longmapsto Y$ is said to be unconditionally converging if T sends weakly unconditionally Cauchy (wuC) series onto unconditionally converging (uc) series.*

Lemma 4.2 *Suppose $\sum_{n=1}^{\infty} x_n$ is a wuC series in X and (y_n) is a bounded sequence in Y. Then $\sum_{n=1}^{\infty} x_n \otimes y_n$ is a wuC series in $X\check{\otimes}Y$.*

Proof. Let T be a member of $(X\check{\otimes}Y)^*$. Using the isometries established in Chapter 2, T may be considered to be a member of $I(X, Y^*)$. Hence by the remarks preceeding Proposition 2.18, T is an absolutely summing operator. Let $M = \sup_n \|y_n\|$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} |T(x_n \otimes y_n)| &= \sum_{n=1}^{\infty} |\langle T(x_n), y_n\rangle| \\ &\leq \sum_{n=1}^{\infty} \|T(x_n)\| \; \|y_n\| \\ &\leq M \sum_{n=1}^{\infty} \|T(x_n)\| \; < \; \infty. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} x_n \otimes y_n$ is wuC. □

One should note that if the roles of (x_n) and (y_n) in the lemma are reversed, the series $\sum_{n=1}^{\infty} x_n \otimes y_n$ is still wuC.

It is easily seen, in view of the Orlicz-Pettis theorem, that weakly compact operators are unconditionally converging. It is not the case, however, that every unconditionally converging operator is weakly compact. For example, the identity operator on l_1 is certainly unconditionally converging but not weakly compact. This motivates the next definition.

Definition 4.3 (Pełczyński , [PEL]) *A Banach space X is said to have property (V) if every unconditionally converging operator on X is weakly compact.*

Among the spaces with property (V) is the space $C(\Omega)$ where Ω is a compact Hausdorff space (see [DU, Corollary VI.2.16]). Pełczyński studied Banach spaces with property (V) and published his results in 1962 (see [PEL]). The question whether the space $C(\Omega, X)$ has property (V) whenever X has property (V) remains open. Pełczyński has given an affirmative answer when X is reflexive. Cembranos, Kalton, Saab, and Saab [CKSS] have shown that if X has the so called property (u) and does not contain an isomorphic copy of l_1 then $C(\Omega, X)$ has property (V). Finally, N. Randrianantoanina [RAND] has recently announced that if X is separable and has property (V), then $C(\Omega, X)$ has property (V).

Definition 4.4 *Let X be a Banach space. A bounded subset K of X^* is called a (V)-set if for each wuC series $\sum_{n=1}^{\infty} x_n$ in X,*

$$\lim_n \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0.$$

The proof of the next lemma is almost identical to the proof of 3.6 and is omitted.

Lemma 4.5 *Suppose (x_i^*) is a sequence in X^* with the property that for each wuC series $\sum_{n=1}^{\infty} x_n$ in X*

$$\lim_n \sup_m |\langle x_m^*, x_n \rangle| = 0.$$

Then the set $\{x_n^* : n \in \mathbf{N}\}$ is a (V)-set.

A sequence in X^* satisfying the hypothesis of Lemma 4.5 will be called a (V)-sequence.

The next theorem due to Pełczyński gives a charactorization of property (V) in terms of (V)-sets.

Theorem 4.6 (Pełczyński , [PEL]) *Let X be a Banach space. Then X has property (V) if and only if every (V)-subset of X^* is relatively weakly compact.*

Let X and Y be Banach spaces. The space of compact integral operators from X into Y will be denoted $KI(X, Y)$. In the next proposition, sufficient conditions are given so that $KI(X, Y) = I(X, Y)$.

Proposition 4.7 *Let X and Y be Banach spaces and suppose X does not contain an isomorphic copy of l_1. Then*

$$KI(X, Y) = I(X, Y).$$

Proof. Let T: $X \longmapsto Y$ be an integral operator. Then JT has a factorization

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xrightarrow{J} & Y^{**} \\ \Big\downarrow S & & & & \Big\uparrow Q \\ L_\infty(\mu) & & \xrightarrow[I]{} & & L_1(\mu) \end{array}$$

where $J\colon Y \longmapsto Y^{**}$ is the natural embedding, μ is a regular Borel measure on a compact Hausdorff space Ω, $|\mu|(\Omega) = \|T\|_{int}$, $I\colon L_\infty(\mu) \longmapsto L_1(\mu)$ is the natural inclusion, and $S\colon X \longmapsto L_\infty(\mu)$ and $Q\colon L_1(\mu) \longmapsto Y^{**}$ are bounded linear operators with $\|S\| \leq 1$ and $\|Q\| \leq 1$. The map I is weakly compact and hence, using the fact $L_\infty(\mu)$ is linearly isometric to a space $C(\Lambda)$ for some compact Hausdorff space Λ, T is also DP (see [DU, Corollary VI.2.17]). Accordingly, the map QIS is also DP.

Since X does not contain a copy of l_1, by Rosenthal's l_1-theorem [D, page 201], the unit ball of X is weakly precompact, that is, every sequence has a weakly Cauchy subsequence. Thus SIR is a compact operator. Since $JT = SIR$, JT is also compact. It follows that T must be compact. The proposition follows. □

Theorem 4.8 *Let X and Y be Banach spaces with property (V) and suppose that $I(X, Y^*) = KI(X, Y^*)$. Then every (V)-set in $(X\check{\otimes}Y)^*$ is relatively weak * sequentially compact.*

The proof is similar to the proof of 3.9. Several assertions will again be numbered and italicized.

Proof. Let K be a (V)-subset of $(X\check{\otimes}Y)^*$. Using the isometries established in Chapter 2, K may be considered as a subset of $I(X, Y^*)$. Let (h_n) be a sequence in K and let $H = \overline{\text{span}}\,\{h_n(x) : x \in X, n \in \mathbb{N}\}$. Then H is a separable subspace of Y^*. Let A be a countable dense subset of H^*.

4.9 *The sequence $(h_n^*(r))$ is a (V)-sequence for each $r \in A$.*

Proof of 4.9. Suppose not. Let $r \in A$ and let $\sum_{n=1}^{\infty} x_n$ be a wuC series in X such that $\lim_n \sup_m |\langle x_n, h_m^*(r)\rangle|$ is not zero. Let $\epsilon > 0$ and assume (passing to a subsequence of (x_n) if necessary) that (h_{m_n}) is a subsequence of (h_n) such that

$$\left|\langle x_n, h_{m_n}^*(r)\rangle\right| > \epsilon.$$

Let y^{**} be a member of Y^{**} such that

$$y^{**}|_H = r$$

and

$$\|y^{**}\| = \|r\|.$$

For each natural number n, choose y_n in Y such that

$$|\langle h_{m_n}(x_n), y^{**} - y_n\rangle| < 1/2^n.$$

Then

$$\begin{aligned}
\left|\langle x_n, h_{m_n}^*(r)\rangle\right| &= \left|\langle x_n, h_{m_n}^*(x^{**})\rangle\right| \\
&= |\langle h_{m_n}(x_n), y^{**}\rangle| \\
&\leq |\langle h_{m_n}(x_n), y^{**} - y_n\rangle| + |\langle h_{m_n}(x_n), y_n\rangle| \\
&< 1/2^n + |\langle h_{m_n}(x_n), y_n\rangle| \xrightarrow{n} 0,
\end{aligned}$$

since $\sum_{n=1}^{\infty} x_n \otimes y_n$ is wuC and (h_{m_n}) is a (V)-sequence. However, this is a contradiction, and 4.9 follows.

Using the fact that A is countable and 4.9, it will be assumed that $(h_n^*(r))$ is weakly convergent for every $r \in A$.

Now let x^{**} be a member of X^{**} and consider the sequence $(h_n^{**}(x^{**}))$ in Y^* (h_n^{**} may be considered as a map from X^{**} into Y^*). An argument similar to that of 4.9 shows that $(h_n^{**}(x^{**}))$ is a (V)-sequence. It follows that the set

$$\{h_n^{**}(x^{**}) : n \in \mathbf{N}\}$$

is a relatively compact subset of Y^*; in fact, it is a relatively weakly compact subset of H.

A weak* limit for (h_n) is now constructed. Let x^{**} be a member of X^{**}. Using the that fact that $(h_n^{**}(x^{**}))$ is a sequence in a relatively weakly compact subset of H, let w and z be two weak sequential cluster points of $(h_n^{**}(x^{**}))$, and let $(h_{n_i}^{**}(x^{**}))$ and $(h_{n_j}^{**}(x^{**}))$ be subsequences of $(h_n^{**}(x^{**}))$ converging to w and z respectively. Let r be a member of A. Then

$$\begin{aligned} \langle w, r\rangle &= \lim_i \langle h_{n_i}^{**}(x^{**}), r\rangle \\ &= \lim_i \langle x^{**}, h_{n_i}^{*}(r)\rangle \\ &= \lim_n \langle x^{**}, h_{n}^{*}(r)\rangle \\ &= \lim_j \langle x^{**}, h_{n_j}^{*}(r)\rangle \\ &= \lim_j \langle h^{**}(x^{**}), r\rangle \\ &= \langle z, r\rangle. \end{aligned}$$

It follows that $w = z$. Thus $(h_n^{**}(x^{**}))$ is weakly convergent for all x^{**} in X^{**}. For each x^{**} in X^{**}, define $\tilde{h}(x^{**})$ by

$$\tilde{h}(x^{**}) = w - \lim_n h_n^{**}(x^{**}).$$

4.10 *The map $\tilde{h}$ is weak* to weak* continuous.*

Proof of 4.10. Let (x_α^{**}) be a weak* null net in X^{**} and let y be a member of Y. Then, thinking of y as a member of Y^{**}, $(h_n^*(y))$ is a (V)-sequence in X^*. To see this, note that if $\sum_{n=1}^\infty x_n$ is a wuC series in X then $\sum_{n=1}^\infty x_n \otimes y$ is a wuC series in $X\check{\otimes}Y$. Thus,

$$\lim_n \sup_m |\langle h_m^*(y), x_n\rangle| = \lim_n \sup_m |h_m(x_n \otimes y)| \xrightarrow{n} 0.$$

Hence $(h_n^*(y))$ has a weakly convergent subsequence $(h_{n_i}^*(y_i))$ converging to some x^* in X^*. Thus

$$\begin{aligned}
\lim_\alpha \langle \tilde{h}(x_\alpha^{**}), y\rangle &= \lim_\alpha \left(\lim_n \langle h_n^*(x_\alpha^{**}), y\rangle\right) \\
&= \lim_\alpha \left(\lim_i \langle h_{n_i}^{**}(x_\alpha^{**}), y\rangle\right) \\
&= \lim_\alpha \left(\lim_i \langle x_\alpha^{**}, h_{n_i}^*(y)\rangle\right) \\
&= \lim_\alpha \langle x_\alpha^{**}, x^*\rangle \;=\; 0.
\end{aligned}$$

The claim 4.10 follows.

Now let $h = \tilde{h} \circ I$, where I is the natural embedding of X into X^{**}. Then

$$h^{**} = \tilde{h}.$$

4.11 *The map h: $X \longmapsto Y^*$ is integral.*

Proof of 4.11. First note that

$$h(x) = h^{**}I(x) = w - \lim_n h_n^{**}I(x) = w - \lim_n h_n(x)$$

for all x in$X$. Let $\epsilon > 0$ and let $u = \sum_{i=1}^{k} x_i \otimes y_i$ be a member of $X \otimes Y$. Choose a natural number N such that for all $n \geq N$ and for each i, $1 \leq i \leq k$,

$$|\langle h(x_i), y_i\rangle - \langle h_n(x_i), y_i\rangle| < \epsilon/k.$$

Then

$$|h(u) - h_N(u)| < \epsilon.$$

Let $M = \sup_n \|h_n\|_{int}$. Then

$$\begin{aligned} |h(u)| &\leq |h_N(u)| + \epsilon \\ &\leq \|h_N\|_{int} \lambda(u) + \epsilon \\ &\leq M\lambda(u) + \epsilon. \end{aligned}$$

It follows that h is continuous on $(X \otimes Y, \lambda)$. Hence h is continuous on $X\check{\otimes}Y$. Therefore, h is integral and 4.11 follows.

Note that since h is integral, it is also a compact operator. Futhermore, if u is a member of $X \otimes Y$, then $(h_n(u))$ converges to $h(u)$. It follows that $(h_n(u))$ converges to $h(u)$ for all u in $X\check{\otimes}Y$. Hence K is relatively w*-compact. □

The next corollary uses the set theoretic containment of $KI(X, Y^*)$ as a subset of $K(X, Y^*)$.

Corollary 4.12 *Suppose X and Y have property (V) and $I(X, Y^*) = KI(X, Y^*)$. If K is a (V)-subset of $I(X, Y^*)$, then K is relatively weakly compact in $(K(X, Y^*), w)$.*

Proof. If h is a member of $KI(X, Y^*)$, then h^{**} is a member of $K_{w^*}(X^{**}, Y^*)$. Suppose K is a (V)-subset of $KI(X, Y^*)$. Let (h_n) be a sequence in K, and, using

4.8, assume (h_n) converges in the weak* topology to h, where h is the limit constructed in the proof of Theorem 4.8. Then $(h_n^{**}(x^{**}))$ converges weakly to $h^{**}(x^{**})$ in Y^* for each x^{**} in X^{**}. Thus the sequence $(\langle h_n(x^{**}), y^{**}\rangle)$ converges to $\langle h(x^{**}), y^{**}\rangle$ for all x^{**} in X^{**} and for all y^{**} in Y^{**}. Thus, by Therorem 3.8, (h_n^{**}) converges in the weak topology on $K_{w^*}(X^{**}, Y^{**})$ to h^{**}. It follows that (h_n) converges weakly to h in $K(X, Y^*)$. □

Let Ω be a compact Hausdorff space and let Σ be the σ-algebra of Borel subsets of Ω. The uniform closure of Σ-simple functions taking values in the Banach space X will be denoted $\mathcal{U}(\Sigma, X)$. Recall that the dual of $C(\Omega, X)$ is the space $M(\Omega, X^*)$ of X^*-valued regular Borel measures of bounded variation equipped with the variation norm (see [DU, Chapter VI]). Since $C(\Omega, X)$ and $C(\Omega)\check{\otimes}X$ are linearly isometric, it follows that $M(\Omega, X^*)$ and $I(C(\Omega), X^*)$ are also linearly isometric. Let T be a member of $I(C(\Omega), X^*)$ and let μ be the coresponding member of $M(\Omega, X^*)$. Then for each $u = \sum_{i=1}^{n} f_i \otimes x_i$ in $C(\Omega \otimes X)$,

$$\langle T, u\rangle = \sum_{i=1}^{n}\langle T(f_i), x_i\rangle$$

and

$$\begin{aligned}\langle \mu, u\rangle &= \sum_{i=1}^{n}\int_\Omega f_i x_i \, d\mu \\ &= \sum_{i=1}^{n}\langle \int_\Omega f_i \, d\mu, x_i\rangle.\end{aligned}$$

It follows that

$$T(f) = \int_\Omega f \, d\mu$$

for all f in $C(\Omega)$; that is, μ is the representing measure for T.

Theorem 4.13 ([LEW]) *Suppose X has property (V) and*

$$I(C(\Omega), X^*) = KI(C(\Omega), X^*).$$

Then every (V)-set in $M(\Omega, X^*)$ is sequentially compact in the $\mathcal{U}(\Sigma, X^{**})$-topology on $M(\Omega, X^*)$.

Proof. Let K be a (V)-subset of $M(\Omega, X^*)$ and let (μ_n) be a sequence in K. Using Theorem 4.8, it will be assumed that (μ_n) converges to μ in the weak* topology on $M(\Omega, X^*)$, where μ is the limit constructed in the proof of Theorem 4.8. Thinking of M as a subset of $KI(C(\Omega), X^*)$, by Corollary 4.12, (μ_n) converges weakly to μ in $K(C(\Omega), X^*)$ and in $\mathcal{L}(C(\Omega), X^*)$.

Let A be a Borel subset of Ω and let x^{**} be a member of X^{**}. Then $\chi_A x^{**}$ defines a member of $\mathcal{L}(C(\Omega), X^*)^*$ by

$$\langle \chi_A x^{**}, \nu \rangle = \langle \nu(A), x^{**} \rangle$$

for all ν in $\mathcal{L}(C(\Omega), X^*)$ (see [DU, Theorem VI.2.1]). Thus

$$\langle \chi_A x^{**}, \mu_n \rangle \xrightarrow{n} \langle \chi_A x^{**}, \mu \rangle.$$

It follows that if θ is a member of $\mathcal{U}(\Sigma, X^{**})$, then

$$\langle \mu_n, \theta \rangle \xrightarrow{n} \langle \mu, \theta \rangle.$$

The theorem follows. □

Let K be a bounded subset of $M(\Omega, X^*)$. Define $|K|$ by

$$|K| = \{|\mu| : \mu \in K\}.$$

Also, recall that $|K|$ is said to be uniformly countably additive if for each pairwise disjoint sequence (A_n) of Borel subsets of Ω

$$\lim_m \sup_{\mu \in K} \sum_{n=m}^{\infty} |\mu| (A_n) = 0.$$

The next proposition, is well known (see [PEL]); however, the proof presented will use the results in this chapter and the following theorem.

Theorem 4.14 ([BL], [BOM]) *Let X be a Banach space and let K be a (V)-subset of $M(\Omega, X)$. Then $|K|$ is uniformly countably additive.*

Proposition 4.15 *Suppose X has property (V) and that X and X^* have the Radon-Nikodým Property. Then $C(\Omega, X)$ has property (V).*

Proof. First note that under this hypothesis, $I(C(\Omega), X^*) = KI(C(\Omega), X^*)$. In fact, every integral operator from $C(\Omega)$ into X^* is nuclear (see [DU], Chapter VI.4). Let K be a (V)-subset of $M(\Omega, X^*)$. Using [DU, Theorem I.2.4], let m be a control measure for $|K|$. That is, let m be a nonnegative countably additive measure on the Borel subsets of Ω such that

$$\lim_{m(E) \longrightarrow 0} |\mu| = 0$$

uniformly for μ in $|K|$. Let (μ_n) be a sequence in K and, using 4.13, assume (μ_n) converges to μ in the $\mathcal{U}(\Sigma, X^{**})$-topology on $M(\Omega, X^*)$. Using the fact X^* has the

Radon-Nikodým Property, choose a sequence (f_n) and f in $L_1(m, X^*)$ such that for each Borel subset E of Ω

$$\mu_n(E) = \int_E f_n \, dm$$

for each natural number n and

$$\mu(E) = \int_E f \, dm.$$

Now suppose $g = \sum_{i=1}^{\infty} \chi_{E_i} x_i^{**}$ where $\|x_i^{**}\| \le 1$ and (E_i) is a pairwise disjoint sequence of Borel subsets of Ω. Then g is a member of $L_\infty(m, X^{**})$,

$$\begin{aligned} \langle f_n, g \rangle &= \int_\Omega f_n g \, dm \\ &= \sum_{i=1}^{k} \langle \mu_n(E_i), x_i^{**} \rangle + \int_{\bigcup_{j>k} E_j} f_n g \, dm \end{aligned}$$

for each natural number n, and

$$\begin{aligned} \langle f, g \rangle &= \int_\Omega f g \, dm \\ &= \sum_{i=1}^{k} \langle \mu(E_i), x_i^{**} \rangle + \int_{\bigcup_{j>k} E_j} f g \, dm. \end{aligned}$$

Futhermore,

$$\begin{aligned} \left| \int_{\bigcup_{j>k} E_j} f g \, dm \right| &\le \left\| \chi_{\bigcup_{j>k} E_j} f_n \right\|_1 \|g\|_\infty \\ &\le \int_{\bigcup_{j>k} E_j} \|f_n\| \, dm \\ &= |\mu_n| \left(\bigcup_{j>k} E_j \right) \xrightarrow{k} 0 \end{aligned}$$

uniformly in n. Therefore,

$$\langle f_n, g \rangle \xrightarrow{n} \langle f, g \rangle.$$

It follows that (f_n) converges to f in the weak topology on $L_1(m, X^*)$. Therefore (μ_n) converges weakly to μ. □

CHAPTER 5

A REPRESENTATION THEOREM FOR $C(\Omega, X)$

In this chapter, a representation for members of $C(\Omega, X)$ will be given when X has a basis. The fact that $C(\Omega, X)$ can be expressed as a tensor product will be used. This representation will be used to characterize when the representing measure of a bounded linear operator from $C(\Omega, X)$ into Y takes its values in $\mathcal{L}(X, Y)$. In this chapter, Ω will be a compact Hausdorff space and Σ will be the σ–algebra of Borel subsets of Ω,

Let X be a Banach space. Recall that sequence (x_n) in X is called a Schauder basis (or just a basis) if for each x in X there exists a unique sequence (α_n) of real numbers such that

$$x = \sum_{n=1}^{\infty} \alpha_n x_n.$$

Futhermore, if a sequence (x_n) is a basis, then there exists a positive real number K such that for each sequence (α_n) of real numbers and each pair of integers n and m with $n \leq m$,

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq K \left\| \sum_{i=1}^{m} \alpha_i x_i \right\|.$$

The least such K is called the basis constant. Finally, for each n let x_n^* be the member of X^* defined by

$$\langle x_n^*, x \rangle = \alpha_n$$

for each $x = \sum_{n=1}^{\infty} \alpha_n x_n$ in X. The sequence (x_n^*) is called the sequence of coefficient functionals in X^*. Each member x of X may be written

$$x = \sum_{n=1}^{\infty} x_n^*(x) x_n.$$

Theorem 5.1 *Let X be a Banach space with basis (x_n) and let Ω be compact Hausdorff space. Then for each f in $C(\Omega, X)$ there exists a unique sequence (f_n) in $C(\Omega)$ such that*

$$f = \sum_{n=1}^{\infty} f_n x_n.$$

Proof. Using the linear isometry established in example 2.4, it will suffice to show that if f is a member of $C(\Omega)\check{\otimes}X$, then there exists a unique sequence (f_n) in $C(\Omega)$ such that

$$f = \sum_{n=1}^{\infty} f_n \otimes x_n.$$

The first step will be to show that every member of $C(\Omega) \otimes X$ has such a representation. To this end, let $g = \sum_{i=1}^{k} h_i \otimes z_i$ be a member of $C(\Omega) \otimes X$. Each z_i has a representation

$$z_i = \sum_{n=1}^{\infty} x_n^*(z_i) x_n$$

where (x_n^*) is the sequence of coefficient funtionals in X^*. Thus

$$\begin{aligned} g &= \sum_{i=1}^{k} h_i \otimes z_i \\ &= \sum_{i=1}^{k} h_i \otimes \left(\sum_{n=1}^{\infty} x_n^*(z_i) x_n \right) \\ &= \sum_{i=1}^{k} \sum_{n=1}^{\infty} h_i \otimes x_n^*(z_i) x_n \end{aligned}$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{k} x_n^*(z_i) h_i \otimes x_n$$
$$= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k} x_n^*(z_i) h_i \right) \otimes x_n.$$

For each natural number n let

$$g_n = \sum_{i=1}^{k} x_n^*(z_i) h_i.$$

Then

$$g = \sum_{n=1}^{\infty} g_n \otimes x_n.$$

Thus a representation exists for each member of $C(\Omega) \otimes X$.

Now let f be a member of $C(\Omega)\check{\otimes}X$. Let (g_i) be a sequence in $C(\Omega)\otimes X$ converging in λ-norm to f. Using the first half of the argument each g_i has a representation

$$g_i = \sum_{n=1}^{\infty} g_{i,n} \otimes x_n.$$

Let K be the basis constant for (x_n), let m be a natural number, and let ω be a member of Ω. Then

$$\begin{aligned}
\|g_{i,m}(\omega)x_m - g_{j,m}(\omega)x_m\| &= \|(g_{i,m}(\omega) - g_{j,m}(\omega))\, x_m\| \\
&\leq \left\| \sum_{n=1}^{m} (g_{i,n}(\omega) - g_{j,n}(\omega))\, x_n \right\| \\
&\quad + \left\| \sum_{n=1}^{m-1} (g_{i,n}(\omega) - g_{j,n}(\omega))\, x_n \right\| \\
&\leq 2K \left\| \sum_{n=1}^{\infty} (g_{i,n}(\omega) - g_{j,n}(\omega))\, x_n \right\| \\
&= 2K \left\| \sum_{n=1}^{\infty} g_{i,n}(\omega)x_n - \sum_{n=1}^{\infty} g_{j,n}(\omega)x_n \right\|.
\end{aligned}$$

Accordingly,

$$\lambda\left(g_{i,m} \otimes x_m - g_{j,m} \otimes x_m\right) \leq 2K\lambda\left(g_i - g_j\right) \xrightarrow{i,j} 0.$$

Therefore, the sequence $(g_{i,m})_i$ is Cauchy. For each natural number n, let

$$f_n = \lim_i g_{i,n}.$$

Let $\epsilon > 0$. Choose a natural number N such that for $i, j \geq N$,

$$\lambda\left(f - g_i\right) < \epsilon/2$$

and

$$\lambda\left(g_i - g_j\right) < \epsilon/4K.$$

Next choose a natural number L such that for all $l \geq L$,

$$\lambda\left(f - \sum_{n=1}^{l} g_{N,n} \otimes x_n\right) < \epsilon/2.$$

Fix $l \geq L$. For $i \geq N$,

$$\begin{aligned} \lambda\left(f - \sum_{n=1}^{l} g_{i,n} \otimes x_n\right) &\leq \lambda\left(f - \sum_{n=1}^{l} g_{N,n} \otimes x_n\right) + \lambda\left(\sum_{n=1}^{l} g_{N,n} \otimes x_n - \sum_{n=1}^{l} gi,n \otimes x_n\right) \\ &< \epsilon/2 + 2K\lambda\left(g_N - g_i\right) < \epsilon. \end{aligned}$$

Now choose $M_l \geq N$ such that for all $i \geq M_l$,

$$\lambda\left(\sum_{n=1}^{l} g_{N,n} \otimes x_n - \sum_{n=1}^{l} g_{i,n} \otimes x_n\right) < \epsilon.$$

Then

$$\begin{aligned} \lambda\left(f - \sum_{n=1}^{l} f_n \otimes x_n\right) &\leq \lambda\left(f - \sum_{n=1}^{l} g_{N,n} \otimes x_n\right) + \lambda\left(\sum_{n=1}^{l} g_{N,n} \otimes x_n - \sum_{n=1}^{l} g_{M_l,n} \otimes x_n\right) \\ &\quad + \lambda\left(\sum_{n=1}^{l} g_{M_l,n} \otimes x_n - \sum_{n=1}^{l} f_n \otimes x_n\right) \\ &< 3\epsilon. \end{aligned}$$

Thus f has a representation

$$f = \sum_{n=1}^{\infty} f_n \otimes x_n.$$

To establish the uniqueness of the representation, suppose

$$f = \sum_{n=1}^{\infty} f_n \otimes x_n = \sum_{n=1}^{\infty} g_n \otimes x_n.$$

Then for each ω in Ω,

$$f(\omega) = \sum_{n=1}^{\infty} f_n(\omega) x_n \sum_{n=1}^{\infty} g_n(\omega) x_n.$$

It follows that, for each natural number n and for each ω in Ω,

$$f_n(\omega) = g_n(\omega).$$

Therefore,

$$f_n = g_n.$$

The theorem follows. □

There is a useful way to obtain the sequence (f_n) in Theorem 5.1. Let f be a member of $C(\Omega, X)$ and let $f = f_n \otimes x_n$ be representation of f given in the theorem. If ω is a member of Ω then

$$\begin{aligned} f(\omega) &= \sum_{n=1}^{\infty} x_n^* f(\omega) x_n \\ &= \sum_{n=1}^{\infty} f_n(\omega) x_n. \end{aligned}$$

Therefore,

$$f_n = x_n^* \circ f.$$

The next theorem, [DIN], provides a means of representing a bounded linear operator on $C(\Omega, X)$ as a vector valued measure. Let $m\colon \Sigma \longmapsto \mathcal{L}(X,\ Y^{**})$ be a vector measure. Then $m_x\colon \Sigma \longmapsto Y^{**}$ is the vector measure defined by

$$m_x(A) = m(A)(x)$$

for each x in X and every A in Σ, and $m_{y^*}\colon \Sigma \longmapsto X^*$ is the vector measure defined by

$$m_{y^*}(A) = \langle m(A)(\cdot), y^* \rangle$$

for each y^* in Y^* and every A in Σ. Finally, $\|m\|$ is the set function defined by

$$\|m\|(A) = \sup\{|m_{y^*}(A)| : y^* \in Y^*,\ \|y^*\| \le 1\}$$

for all A in Σ.

Theorem 5.2 (Dinculeanu-Singer) *Let $T\colon C(\Omega, X) \longmapsto Y$ be a bounded linear operator. Then there exists a unique vector measure $m\colon \Sigma \longmapsto \mathcal{L}(X,\ Y^{**})$ such that*

1. *m is finitely additive and $\|m\|(\Omega) < \infty$;*

2. *m is weakly regular, that is m_{y^*} is regular for each $y^* \in Y^*$;*

3. *the mapping $y^* \mapsto m_{y^*}$ is weak* to weak* continuous from Y^* into $C(\Omega, X)^*$;*

4. *$T(f) = \int_\Omega f\, dm$ for all $f \in C(\Omega, X)$;*

5. *$\|m\|(\Omega) = \|T\|$; and*

6. *$T^*(y^*) = m_{y^*}$ for all $y^* \in Y^*$.*

Conversley, any vector $m\colon \Sigma \longmapsto \mathcal{L}(X,\ Y^{**})$ *that satisfies 1, 2, and 3 defines a bounded linear operator* $T\colon C(\Omega, X) \longmapsto Y$ *by 4 and satisfies 5 and 6.*

Let $T\colon C(\Omega, X) \longmapsto Y$ be a bounded linear operator with representing measure m. Let x be a member of X and x^* be a member of X^*. Define $T_{x,x^*}\colon C(\Omega, X) \longmapsto Y$ by

$$T_{x,x^*}(f) = T((x^* \circ f) \otimes x)$$

for all f in $C(\Omega, X)$. Define $m_{x,x^*}\colon \Sigma \longmapsto \mathcal{L}(X,\ Y^{**})$ by

$$m_{x,x^*}(A)(u) = x^*(u)m(A)(x)$$

for all u in X and A in Σ.

Lemma 5.3 ([LEW]) *Let* $T: C(\Omega, X) \longmapsto Y$ *be a bounded linear operator with representing measure* m, $x \in X$, *and* $x^* \in X^*$. *Then* m_{x,x^*} *is the representing measure for* T_{x,x^*}.

Proof. Clearly, m_{x,x^*} is finitely additive. Suppose $u \in X$ and $y^* \in Y^*$. Then

$$\begin{aligned} |\langle m_{x,x^*}(A)(u), y^* \rangle| &= |\langle x^*(u)m(A)(x), y^* \rangle| \\ &\leq |x^*(u)|\ \|m(A)\|\ \|x\|\ \|y^*\| \\ &\leq \|x^*\|\ \|m\|(A)\,\|y^*\| \ < \ \infty \end{aligned}$$

for all $A \in \Sigma$. It follows that

$$\|m_{x,x^*}\|(\Omega) \leq \|x\|\ \|x^*\|\ \|m\|(\Omega)$$

and that $(m_{x,x^*})_{y^*}$ is regular and countably additive (see [DU, VI.2.14,VI.2.5]). Thus m is weakly regular.

Let $\phi = \sum_{i=1}^{n} \chi_{A_i} u_i$ be an X-valued Borel simple function. Then

$$\begin{aligned}
\int_\Omega \phi \, dm_{x,x^*} &= \sum_{i=1}^{n} m_{x,x^*}(A_i) u_i \\
&= \sum_{i=1}^{n} x^*(u_i) m(A_i)(x) \\
&= \sum_{i=1}^{n} \int_\Omega x^*(u_i) \chi_{A_i} \otimes x \, dm \\
&= \int_\Omega (x^* \circ \phi) \otimes x \, dm.
\end{aligned}$$

Therefore,

$$\int_\Omega f \, dm_{x,x^*} = \int_\Omega (x^* \circ f) \otimes x \, dm$$

for all f in $C(\Omega, X)$. Thus,

$$\begin{aligned}
T_{x,x^*}(f) &= T\left((x^* \circ f) \otimes x\right) \\
&= \int_\Omega (x^* \circ f) \otimes x) \, dm \\
&= \int_\Omega f \, dm_{x,x^*}.
\end{aligned}$$

The lemma follows. □

Theorem 5.4 ([LEW]) *Let X be a Banach space with basis (x_n) and suppose T: $C(\Omega, X) \longmapsto Y$ is a bounded linear operator with representing measure m. Then m takes its values if $\mathcal{L}(X,\ Y)$ if and only if there exists a sequence of weakly compact operators (T_n) from $C(\Omega, X)$ into Y and a corresponding sequence of representing measures (m_n), taking their values in $\mathcal{L}(X,\ Y)$, such that*

1. $T_n(f) \xrightarrow{n} T(f)$ *for every* $f \in C(\Omega, X)$ *and*

2. $m_n(A)(x) \xrightarrow{n} m(A)(x)$ *for all* $A \in \Sigma$ *and all* $x \in X$.

Proof. Assume m takes its values in $\mathcal{L}(X, Y)$. To prove Condition 1, several operators on $C(\Omega, X)$ and vector measures on Σ will need to be defined. To this end, fix $f = \sum_{k=1}^{\infty} f_n \otimes x_n$ in $C(\Omega)\check{\otimes}X$ and a natural number n. Note that

$$f_n = x_n^* \circ f$$

where (x_n^*) is the sequence of coefficient functionals associated with (x_n). Let K be the basis constant for (x_n). Define $T_n(f)$ by

$$T_n(f) = \sum_{i=1}^{n} T_{x_i, x_i^*}(f).$$

Then

$$\begin{aligned} \|T_n(f)\| &= \left\| T(\sum_{i=1}^{n} f_i \otimes x_i \right\| \\ &\leq \|T\| \, K\lambda(f). \end{aligned}$$

By lemma 5.3

$$T_{x_n, x_n^*}(f) = \int_\Omega f \, dm_{x_n, x_n^*}.$$

Define bounded linear operators S_n: $C(\Omega)\check{\otimes}X \longmapsto C(\Omega)$ and R_n: $C(\Omega) \longmapsto Y$ by

$$S_n(f) = x_n^* \circ f$$

for f in $C(\Omega)\check{\otimes}X$ and

$$R_n(g) = \int_\Omega g \, dm_{x_n}$$

for $h \in C(\Omega)$. Since m_{x_n} takes its values in $\mathcal{L}(X,\ Y)$, R_n is weakly compact (see [DU, VI.2.5]). Moreover,

$$T_{x_n, x_n^*} = R_n S_n.$$

Therefore, T_n is weakly compact and $(T_n(f))$ converges to $T(f)$. Condition 1 follows.

To prove Condition 2 holds, let $A \in \Sigma$ and $x \in X$. Then

$$\sum_{i=1}^{n} m_{x_i, x_i^*}(A)(x) = \sum_{i=1}^{n} x_i^*(x) m(A) x_i,$$

$$\sum_{i=1}^{n} x_i^*(x) m(A) x_i \xrightarrow{n} \sum_{i=1}^{\infty} x_i^* m(A) x_i,$$

and

$$\sum_{i=1}^{\infty} x_i^* m(A) x_i = m(A) \left(\sum_{i=1}^{\infty} x_i^*(x) x_i \right) = m(A)(x).$$

Let $m_n = \sum_{i=1}^{n} m_{x_i, x_i^*}$. Note that m_n is the representing measure for T_n for each natural number n and that $(m_n(A))$ converges pointwise to $m(A)$ for all A in Σ.

The converse is obvious. □

It should be noted that the second condition in Theorem 5.4 cannot be removed. For example, the identity operator $I\colon\ C[0,1] \longmapsto C[0,1]$ is the pointwise limit of compact operators. This follows from the fact that $C[0,1]$ has a basis. However, if m is the representing measure for I, then

$$m(A) = \chi_A$$

for every subset A of $[0,1]$. Thus $m(A) \in C[0,1]$ if and only in $A = [0,1]$ or A is the empty set.

Dobrakov [DBK] has provided an example of a non-weakly compact operator which satisfies the conclusion of Theorem 5.4.

BIBLIOGRAPHY

[BL] Bator, E. and Lewis, P., Properties (V) and (wV) on $C(\Omega, X)$, Math. Proc. Camb. Phil. Soc. 117 (1995), 469-477.

[BOM] Bombal, F., On (V^*) sets and Pełczyński's Property (V^*), Glasgow J. Math. 32 (1990), 109-120.

[CKSS] Cembranos, P., Kalton, N., Saab, E., Saab, P, Pełczyński's Property (V) on $C(\Omega, E)$ spaces, Math. Ann. 271 (1985), 91-97.

[D] Diestel, J., Sequences and Series in Banach Spaces, Springer-Verlag, New York (1984).

[DU] Diestel, J. and Uhl, J., Vector Measures, American Mathematical Society, Providence (1977).

[DIN] Dinculeanu, N., Vector Measures, Pergamon Press, New York (1967).

[DBK] Dobbrakov, I., On Integration in Banach Spaces, Czech. Math. J. 95 (1970), 511-536.

[DS] Dunford, N. and Schwartz, J., Linear Operators, Part I: General Theory, John Wiley and Sons, New York (1988).

[EM1] Emmanuele, G., A Dual Charactorization of Banach Spaces Not Containing l_1, Bull. Polish Acad. 34 (1986), 155-160.

[EM2] Emmanuele, G., On the Reciprocal Dunform-Pettis Property in Projective Tensor Products, Math. Proc. Camb. Phil. Soc. 109 (1991), 161-166.

[HUN] Hungerford, T., Algebra, Springer-Verlag, New York (1974).

[LV] Leavelle, T., The Reciprocal Dunford-Pettis Property, preprint.

[LEW] Lewis, Paul, Private communication.

[PEL] Pełczyński , A., On Banach Spaces on Which Every Unconditionally Converging Operator is Weakly Compact, Bull. Acad. Polon. Sci. 10 (1962), 641- 648.

[RAND] Randrianantoanina, N, Pełczyński's Property (V) on Spaces of Vector-Valued Funtions, to appear.

[ROS1] Rosenthal, H. P., A Characterization of Banach Spaces Not Containing l_1, Proc. Nat. Acad. Sci., U. S. A. 71 (1971), 2411-2413.

[ROS2] Rosenthal, H. P., Pointwise Compact Subsets of the First Baire Class, Amer. J. Math. 99 (1977), 362-378.

[RDN] Royden, H. L., Real Analysis, Macmillan, New York (1988).

[RSS] Ruess, Duality and Geometry of Spaces of Compact Operators, Functional Analysis: Surveys and Recent Results, Studies in Math. and its Applications 90 (1984).

[S] Schaefer, H., Topological Vector Spaces, Springer-Verlag, New York (1971).

[WIL] Willard, S., General Topology, Addison-Wesley, Reading, Mass. (1970).