COUNTABLE ADDITIVITY, EXHAUSTIVITY, AND THE STRUCTURE OF CERTAIN BANACH LATTICES

DISSERTATION

Presented to the Graduate Council of the University of North Texas in Partial Fulfillment of the Requirements For the Degree of

DOCTOR OF PHILOSOPHY

By

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Denton, Texas
August, 1999

The notion of uniform countable additivity or uniform absolute continuity is present implicitly in the Lebesgue Dominated Convergence Theorem and explicitly in the Vitali-Hahn-Saks and Nikodým Theorems, respectively. V.M. Dubrovsky studied the connection between uniform countable additivity and uniform absolute continuity in a series of papers, and Bartle, Dunford, and Schwartz established a close relationship between uniform countable additivity in $ca(\Sigma)$ and operator theory for the classical continuous function spaces $C(K)$. Numerous authors have worked extensively on extending and generalizing the theorems of the preceding authors. Specifically, we mention Bilyeu and Lewis as well as Brooks and Drewnowski, whose efforts molded the direction and focus of this paper.

This paper is a study of the techniques used by Bell, Bilyeu, and Lewis in their paper on uniform exhaustivity and Banach lattices to present a Banach lattice version of two important and powerful results in measure theory by Brooks and Drewnowski. In showing that the notions of exhaustivity and continuity take on familiar forms in certain Banach lattices of measures they show that these important measure theory results follow as corollaries of the generalized Banach lattice versions. This work uses their template to generalize results established by Bator, Bilyeu, and Lewis.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>DEFINITIONS AND ELEMENTARY PROPERTIES OF BANACH LATTICES</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>EXHAUSTIVITY AND ABSOLUTE CONTINUITY IN BANACH LATTICES</td>
<td>23</td>
</tr>
<tr>
<td>4</td>
<td>EXHAUSTIVITY AND $l^\infty$</td>
<td>39</td>
</tr>
<tr>
<td>5</td>
<td>EXHAUSTIVITY AND A GENERALIZATION OF GATEAUX DIFFERENTIABILITY</td>
<td>44</td>
</tr>
<tr>
<td>6</td>
<td>APPLICATIONS TO SPACES OF MEASURES</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>BIBLIOGRAPHY</td>
<td>64</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

If $\Sigma$ is a $\sigma$-algebra of subsets of a universal space $\Omega$, then $\text{ca}(\Sigma)$ is defined to be the space of all countably additive real valued set functions defined on $\Sigma$. The space $\text{ca}(\Sigma)$ with its usual total variation norm is a Banach space when equipped with the standard operations of vector addition and scalar multiplication. A subset $F$ of $\text{ca}(\Sigma)$ is said to be uniformly countably additive provided that

$$
\lim_{n \to \infty} \mu(\bigcup_{i=n}^{\infty} A_i) = 0
$$

uniformly for $\mu \in F$ whenever $(A_i)$ is a pairwise disjoint sequence from $\Sigma$. If $0 \leq \lambda \in \text{ca}(\Sigma)$, then $F$ is said to be uniformly absolutely continuous with respect to $\lambda$ provided that if $\epsilon > 0$ then there is a $\delta > 0$ so that $|\mu(A)| < \epsilon$ for each $\mu \in F$ whenever $A \in \Sigma$ and $\lambda(A) < \delta$.

The notion of uniform countable additivity or uniform absolute continuity is present implicitly in one of the staples of a standard measure and integration theory class:

**Theorem 1.1 (Lebesgue Dominated Convergence Theorem)** If $(\Omega, \Sigma, \lambda)$ is a measure space, $g$ is $\lambda$-integrable, $(f_n)_{n=0}^{\infty}$ is a sequence of measurable functions so that $(f_n)_{n=1}^{\infty} \rightarrow f_0$ $\lambda$-almost everywhere, and $|f_n(x)| \leq g(x)$ for $n \geq 1$ and almost all $x$, then $\lim_{n} \int_{\Omega} f_n d\lambda = \int_{\Omega} f_0 d\lambda$. 

1
If $\nu_n(A) = \int_A f_n d\lambda$ for each $n$ and $A \in \Sigma$, then the domination in the preceding theorem ensures that $\{\nu_n : n \in N\}$ is uniformly countably additive and uniformly absolutely continuous with respect to $\lambda$. Further, if the inequality in Lebesgue's theorem is replaced with the assumption that $\{\nu_n : n \in N\}$ is uniformly absolutely continuous with respect to $\lambda$, then one obtains the same conclusion. In fact, with either assumption, one obtains that

$$\int_{\Omega} |f_n - f_0| d\lambda \to 0.$$ 

Thus $\int_A f_n d\lambda \to \int_A f_0 d\lambda$ for each $A \in \Sigma$.

Conversely, if one assumes that $\int_A f_n d\lambda \to \int_A f_0 d\lambda$ for each $A \in \Sigma$, then an application of the classical Vitali-Hahn-Saks theorem [15, p.158] shows that $\{\nu_n : n \in N\}$ is uniformly absolutely continuous with respect to $\lambda$. Thus this set is also uniformly countably additive. Of course, uniform absolute continuity is present explicitly in the statement of the Vitali-Hahn-Saks theorem.

**Theorem 1.2 (Vitali-Hahn-Saks Theorem)** If $(\Omega, \Sigma, \lambda)$ is a measure space and $(\mu_n)$ is a sequence of $\lambda$-continuous vector or scalar valued measures on $\Sigma$ such that $\lim_n \mu_n(A)$ exists for each $A \in \Sigma$, then $\{\mu_n : n \in N\}$ is uniformly absolutely continuous with respect to $\lambda$.

V.M.Dubrovsky studied the connection between uniform countable additivity and uniform absolute continuity in a series of papers. Specifically, Dubrovsky showed that the two ideas were equivalent in [14].
A firm link among uniform absolute continuity, uniform countable additivity, and the structure of certain classical Banach spaces was established in a seminal paper by Bartle, Dunford, and Schwartz [1]. The norm on \( \text{ca}(\Sigma) \) is (again) the total variation norm.

**Theorem 1.3** (Bartle, Dunford, and Schwartz) The following statements are equivalent:

(i) The subset \( K \) of \( \text{ca}(\Sigma) \) is relatively weakly compact.

(ii) The subset \( K \) of \( \text{ca}(\Sigma) \) is bounded in norm, and \( K \) is uniformly countably additive.

(iii) The subset \( K \) of \( \text{ca}(\Sigma) \) is bounded in norm, and \( K \) is uniformly absolutely continuous with respect to a non-negative member of \( \text{ca}(\Sigma) \).

Numerous authors have worked extensively on extending and generalizing the Vitali-Hahn-Saks theorem and the Bartle-Dunford-Schwartz theorem. We shall mention selected efforts in this direction which have molded the focus and direction of this paper.

If \( x \) and \( y \) belong to the Banach space \( X \), then the norm is said to be Gateaux differentiable at \( x \) in the direction \( y \) if

\[
\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}
\]

exists. The Gateaux derivative of the norm at \( x \) in the direction \( y \) is denoted by \( D(x,y) \). Bilyeu and Lewis [5,Theorem 2.5] showed that \( K \) is a relatively weakly compact subset of \( \text{ca}(\Sigma) \) if and only if there is an element \( \mu \in \text{ca}(\Sigma) \) so the \( D(\mu,\nu) \)
exists uniformly for $v \in K$. This same paper also contains examples which show that uniform Gateaux differentiability does not characterize relative weak compactness for arbitrary Banach spaces. However, Lemma 3.1 of [5] does show that if $D(x,y)$ exists for each $y$ in $K$ and $K$ is a relatively norm compact subset of $X$, then $D(x,y)$ exists uniformly for $y \in K$. This result will be investigated in the setting of Banach lattices in a subsequent chapter of this paper.

In 1971 James Brooks [6] obtained a significant extension of some of the principal results of [1] on two fronts:

(1) the $\sigma$-algebra of Bartle-Dunford-Schwartz theorem was replaced by a ring of sets and

(2) the assumption that the measures in [1] were countably additive was replaced in Brooks's paper by the assumption that the set function was finitely additive and strongly bounded.

If $R$ is a ring of sets, $X$ is a Banach space, and $\mu : R \rightarrow X$ is a finitely additive set function so that $(\mu(A_i)) \rightarrow 0$ whenever $(A_i)$ is a pairwise disjoint sequence from $R$, then $\mu$ is said to be strongly bounded [16]. (Other terms used to describe this property of $\mu$ include "strongly additive" and "exhaustive.")

**Theorem 1.4 (6)** If $R$ is a ring of subsets of $\Omega$ and $\mu : R \rightarrow X$ is finitely additive, then $\mu$ is strongly bounded if and only if there is a positive finitely additive bounded set function $\nu$ defined of $R$ so that

$$\lim_{\nu(E) \rightarrow 0} \mu(E) = 0.$$
Brooks and Jewett were able to use the idea of strong boundedness to significantly extend the Vitali-Hahn-Saks theorem in [10].

**Theorem 1.5** If $R$ is a $\sigma$-ring of sets, $X$ is a Banach space, $\nu_n : R \to X$ is finitely additive and strongly bounded for each $n$, $(\nu_n(A))$ converges for each $A \in R$, $\nu$ is a non-negative (possibly infinite) finitely additive set function on $R$, and $\nu_n << \nu$ for each $n$, then $\nu_n << \nu$ uniformly for $n \in N$.

Drewnowski [13] subsequently showed that the Brooks-Jewett theorem and the Vitali-Hahn-Saks theorem were equivalent. Further, Brooks [7] and Drewnowski [12] independently established versions of the following fundamental result [5, Theorem 3.3].

**Theorem 1.6** (Brooks-Drewnowski Theorem). Suppose that $R$ is a ring of sets, $K$ is a uniformly strongly bounded subset of $ba(R)$, and $0 \leq \lambda$ is a finitely additive (possibly infinite) measure on $R$ such that $\mu << \lambda$ for each $\mu \in K$. Then $\mu << \lambda$ uniformly for $\mu \in K$.

The relationship of the Brooks-Drewnowski Theorem to the Vitali-Hahn-Saks Theorem is the subject of a major portion of [5]. From the perspective of this paper, it is appropriate to note that the proof of Theorem 3.3 in [5] is lattice theoretic in nature.

Bell, Bilyeu, and Lewis extended the idea of strong boundedness-exhaustivity to the Banach lattice setting in [4]. That paper, together with a paper of Bator, Bilyeu, and Lewis [2] which treats parameterized versions of the Vitali-Hahn-Saks and
Brooks-Drewnowski theorems, are treated in some detail in this paper.

Although the relationship between strong boundedness and operator theory is not studied (per se) in this paper, it seemed appropriate to conclude the introduction with a brief overview of this connection. This detour also helps explain why Drewnowski's choice of "exhaustivity" was given preference over Brook's "strong boundedness" in the title of this paper.

If $X$ is a Banach space, $K$ is a compact Hausdorff space, $\Sigma$ is a $\sigma$-algebra of Borel subsets of $K$, and $T: C(K) \to X$ is a continuous linear transformation (=operator) from the continuous function space $C(K)$ into $X$, then there is a finitely additive vector measure $m$ defined on $\Sigma$ with values in the bidual $X^{**}$ of $X$ so that $\langle x^*, m \rangle : \Sigma \to \mathbb{R}$ is countably additive and regular for each $x^* \in X^*$,

$$T(f) = \int_K f dm, \quad f \in C(K),$$

and

$$\sup\{\|\sum_{i=1}^{n} \alpha_i m(A_i)\| : (A_i)_i^n \text{ is a } \Sigma\text{-partition of } K \text{ and } |\alpha_i| \leq 1 \text{ for each } i \} = \|T\|.$$

Bartle, Dunford, and Schwartz [1] showed that the operator $T$ above is weakly compact if and only if $m$ takes its values in $X$ and is countably additive. (The countable additivity of $m$ is equivalent to the uniform countable additivity of $\{(x^*, m) : x^* \in X^*, \|x^*\| \leq 1\}$.)

In [9] Brooks and Lewis established the following result.

**Theorem 1.7** If each of $X$ and $Y$ is a Banach space, $K$ is a compact Hausdorff space, and $T: C(K) \to Y$ is a continuous linear transformation (=operator) from the continuous function space $C(K)$ into $Y$, then there is a finitely additive vector measure $m$ defined on $\Sigma$ with values in the bidual $Y^{**}$ of $Y$ so that $\langle x^*, m \rangle : \Sigma \to \mathbb{R}$ is countably additive and regular for each $x^* \in Y^*$,
space, and \( T: C(K, X) \to Y \) is an operator, then there is a representing measure \( m: \Sigma \to B(X, Y^{**}) \) so that \( T(f) = \int_X f \, dm \), \( f \in C(K, X) \). Further, if \( X^* \) and \( X^{**} \) have the Radon-Nikodym property, then \( T \) is weakly compact if and only if \( \hat{m}(A_i) \to 0 \) whenever \( (A_i) \) is a pairwise disjoint sequence from \( \Sigma \) and \( m(A) \) is weakly compact for each \( A \).

In this setting, \( \hat{m}(A) = \sup \{ \| \sum_{i=1}^n m(B_i)x_i \| : (B_i) \) is a finite partition of \( A \), \( x_i \in X \), and \( \| x_i \| \leq 1 \} \). If \( (\hat{m}(A_i)) \to 0 \) for each pairwise disjoint sequence \( (A_i) \) in \( \Sigma \), then Brooks and Lewis said that the representing measure and corresponding operator were strongly bounded. Saab [17] has shown that the restriction of \( X^* \) and \( X^{**} \) in the preceding theorem is necessary. In a recent paper, Slavens and Bator [3] have shown that if \( Y \) is an arbitrary infinite dimensional Banach space, then there is a representing measure \( m \) with values in \( B(Y, c_0) \) so that \( m \) is countably additive and not strongly bounded.
DEFINITIONS AND ELEMENTARY PROPERTIES OF BANACH LATTICES

The exposition in this chapter begins with a presentation of the basic Banach lattice properties which are necessary to provide the setting in which Bell, Bilyeu, and Lewis [4] extended the two results of Brooks [6], [7] and Drewnowski [12] mentioned in the introduction. Some of the material in [4] is repeated in detail herein so that the presentation will be more complete and the subsequent discussion will be better motivated.

Definition 2.1 A vector space $X$ over a field $F$ is an additive group together with an operation $m: F \times X \rightarrow X$ written as $m(\alpha, x) = \alpha x$ which satisfy the following four conditions:

i) $\alpha(x + y) = \alpha x + \alpha y$ \hspace{1cm} $\alpha \in F$, $x, y \in X$

ii) $(\alpha + \beta)x = \alpha x + \beta x$ \hspace{1cm} $\alpha, \beta \in F$, $x \in X$

iii) $\alpha(\beta x) = (\alpha \beta)x$ \hspace{1cm} $\alpha, \beta \in F$, $x \in X$

iv) $1x = x$ \hspace{1cm} $1 \in F$, $x \in X$.

Definition 2.2 Let $L$ be a vector space over $R$ which is endowed with an order structure defined by a reflexive, transitive, and anti-symmetric binary relation $\leq$; $L$ is called an ordered vector space over $R$ if the following axioms are satisfied:

i) $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in L$. 

8
ii) $x \leq y$ implies $\lambda x \leq \lambda y$ for all $x, y \in L$ and $\lambda > 0$.

**Definition 2.3** Let $V$ designate an ordered vector space. For each pair $(x,y)$ in $V \times V$ define $x \lor y$ and $x \land y$ as sup${\{x,y}\}$ and inf${\{x,y}\}$, respectively. (That is, $x \lor y \geq x$, $x \lor y \geq y$, and if $z \geq x$ and $z \geq y$, then $z \lor y \leq z$.)

**Definition 2.4** A **vector lattice** is defined to be an ordered vector space $E$ over $R$ such that for each pair $(x,y) \in E \times E$, $x \lor y$ and $x \land y$ exist.

**Note 2.5** For the remainder of this paper, all lattices will be assumed to be vector lattices.

**Theorem 2.6** If $x$, $y$, $z$ are in $L$ then $z + (x \lor y) = (z + x) \lor (z + y)$.

**PROOF.** If $x$, $y$, and $z$ belong to $L$, then $x \leq x \lor y$, $y \leq x \lor y$, $z + x \leq z + (x \lor y)$, and $z + y \leq z + (x \lor y)$. Therefore, $(z + x) \lor (z + y) \leq z + (x \lor y)$. Conversely, let $u = (x + z) \lor (y + z)$. Certainly, $u \geq x + z$ implies $u - z \geq x$, and $u \geq y + z$ implies $u - z \geq y$. Hence, $u - z \geq (x \lor y)$, $u \geq (x \lor y) + z$, and $(x + z) \lor (y + z) \geq (x \lor y) + z$. Thus, $(x \lor y) + z = (x + z) \lor (y + z)$.

**Theorem 2.7** If $x$, $y$, $z \in L$, then $(x \land y) + z = (x+z) \land (y+z)$.

**PROOF.** Suppose $x$, $y$, $z \in L$. Certainly, $x \geq x \land y$, $x + z \geq (x \land y) + z$, $y \geq x \land y$, and $y + z \geq (x \land y) + z$. Therefore, $(x + z) \land (y + z) \geq (x \land y) + z$. Now, let $u = (x + z) \land (y + z)$. Then, $u \leq x + z$, and $u - z \leq x$, $u \leq y + z$, and $u - z \leq y$. Therefore, $x \land y \geq u - z$, $x \land y + z \geq u$, and $x \land y + z \geq (x + z) \land (y + z)$. Hence, $x \land y + z = (x + z) \land (y + z)$. Q.E.D.
Theorem 2.8 If \( x \) and \( y \) belong to \( L \), then \( x + y = x \lor y + x \land y \).

PROOF. Suppose \( x, y \in L \). Then, \((x \lor y) + (x \land y) = [x + (x \land y)] \lor [y + (x \land y)] \leq [x + y] \lor [y + x] = x + y\). Also, \((x \lor y) + (x \land y) = [x + (x \lor y)] \land [y + (x \lor y)] \geq [x + y] \land [y + x] = x + y\). Therefore, \((x \lor y) + (x \land y) = x + y\). \(\Box\).

Lemma 2.9 If \( x \leq y \) then \( x \lor a \leq y \lor a \) for \( x, y, a \in L \).

PROOF. Suppose \( x, y, a \in L \) and \( x \leq y \). Then \( x \leq y \leq y \lor a \) and \( a \leq y \lor a \). Therefore, \( x \lor a \leq y \lor a \). \(\Box\).

Definition 2.10 If \( x \in L \), define \( x^+ \) and \( x^- \) by \( x \lor 0 \) and \( (-x) \lor 0 \), respectively.

The reader should note that \( x^+ \geq 0 \) and \( x^- \geq 0 \).

Theorem 2.11 If \( x \in L \) then \( x = x^+ - x^- \).

PROOF. Suppose \( x \in L \). An application of Theorem 7 shows that \((-x) \lor 0) + x = (-x + x) \lor (0 + x)\). Hence, \( x^- + x = 0 \lor x \), \( x^- + x = x^+ \), and \( x = x^+ - x^- \). \(\Box\).

The reader should note that if \( x \geq y \), then \( x \lor 0 \geq y \lor 0 \) or \( x^+ \geq y^+ \). Also, in general, lattices are not distributive. However, as the following theorem indicates, the lattice operations are distributive in ordered vector spaces.

Theorem 2.12 If \( x, y, \) and \( z \in L \), then \((x \lor y) \land z = (x \land z) \lor (y \land z)\), and \((x \land y) \lor z = (x \lor z) \land (y \lor z)\).

PROOF. Suppose \( x, y, \) and \( z \in L \). Obviously, \((x \lor y) \land z \geq x \land z\), and \((x \land y) \lor z \geq y \land z\).

Therefore, \((x \land z) \lor (y \land z) \leq (x \lor y) \land z\). Now, \([[(x \land z) \lor (y \land z)] + (x \lor y \lor z) =\)
\[(x \land z) + (x \lor y \lor z)] \lor [(y \land z) + (x \lor y \lor z)] = [x + (x \lor y \lor z)] \land [z + (x \lor y \lor z)] \lor [y + (x \lor y \lor z)] \land [z + (x \lor y \lor z)] \geq (x + z) \land (z + x) \lor (y + z) \land (z + y) = (x + z) \lor (y + z) = (x \lor y) + z = [(x \lor y) \land z] + [(x \lor y) \lor z]. \] Therefore, \((x \land z) \lor (y \land z) + (x \lor y \lor z) \geq (x \lor y) \land z + (x \lor y) \lor z,\) and \((x \land z) \lor (y \land z) \geq (x \lor y) \land z. \) Hence, \((x \land z) \lor (y \land z) = (x \lor y) \land z. \) Similarly, \(\lambda (x \land y) \lor z = (x \lor y) \land (y \lor z). \quad \text{Q.E.D.}\)

In fact, more can be said. See Lemma 2.29.

**Theorem 2.13** If \(x \in L,\) then \(x^+ \land x^- = 0.\)

**Proof.** Suppose \(x \in L.\) Begin by noting that \(x^+ \land x^- = (x \lor 0) \land x^- = (x \land x^-) \lor (0 \land x^-) = (x \land x^-) \lor 0 = (x \land (x \lor 0)) \lor 0 = ((x \land x^-) \lor (x \land 0)) \lor 0.\) Since \(x \land x^- \leq x\) and \(x \land x^- \leq -x,\) \(2(x \land x^-) = (x \land x^-) + (x \land x^-) \leq 0.\) Thus, \(x \land -x \leq 0.\) Consequently, since \(x \land 0 \leq 0,\) \(x^+ \land x^- = (x \land -x) \lor (x \land 0) \lor 0 = 0.\) \(\text{Q.E.D.}\)

**Theorem 2.14** If \(x, y \in L\) then \(- (x \lor y) = -x \land -y.\)

**Proof.** Suppose \(x, y \in L.\) Note that \((-x \land -y) + (x \lor y) = [x + (x \lor y)] \land [-y + (x \lor y)]\)
\[= [(x + x) \lor (x + y)] \land [-y + x] \lor (-y + y)] = (y - x)^+ \land (y - x)^- = 0.\] Therefore, \((-x \land -y) = -(x \lor y). \quad \text{Q.E.D.}\)

**Definition 2.15** If \(x \in L,\) then \(|x| = x^+ + x^-\).

**Theorem 2.16** If \(\lambda > 0\) and \(x, y \in L,\) then \(\lambda (x \lor y) = \lambda x \lor \lambda y,\) and \(\lambda (x \land y) = \lambda x \land \lambda y.\)

**Proof.** Suppose \(\lambda > 0\) and \(x, y \in L.\) Since \(x \lor y \geq x,\) and \(x \lor y \geq y,\) \(\lambda (x \lor y) \geq \lambda x,\) and \(\lambda (x \lor y) \geq \lambda y.\) Therefore, \(\lambda x \lor \lambda y \leq \lambda (x \lor y).\) Similarly, \(\lambda x \lor \lambda y \geq \lambda x,\) and
\( \lambda x \lor \lambda y \geq \lambda y \). Also, \( \lambda^{-1}(\lambda x \lor \lambda y) \geq \lambda^{-1}(\lambda x) \), and \( \lambda^{-1}(\lambda x \lor \lambda y) \geq \lambda^{-1}(\lambda y) \). Therefore, \\
\( (\lambda^{-1} \lambda x) \lor (\lambda^{-1} \lambda y) \leq \lambda^{-1}(\lambda x \lor \lambda y) \), and \( x \lor y \leq \lambda^{-1}(\lambda x \lor \lambda y) \). Thus, \( \lambda x \lor \lambda y = \lambda(x \lor y) \)
Q.E.D.

**Definition 2.17** If \( X \) a real vector space, then a norm on \( X \) is a function

\[ || \cdot || : X \longrightarrow R \]

which satisfies the following conditions:

1. \( || 0 || = 0; || x || > 0, x \neq 0. \)
2. \( || x + y || \leq || x || + || y ||, x, y \in X. \)
3. \( || \alpha x || = |\alpha| || x ||, \alpha \in R, x \in X. \)

**Definition 2.18** A normed vector lattice is a normed vector space with a lattice structure such that \( |x| \leq |y| \) implies \( ||x|| \leq ||y|| \). If the normed vector lattice is complete with respect to the metric topology, then it is called a Banach lattice.

Recall that \( |x| = |x^+| + |x^-| = (x^+ + x^-)^+ + (x^+ + x^-)^- = x^+ + x^- \). Since \( x^+ \geq 0 \), and \( x^- \geq 0 \), \( x^+ + x^- \geq 0 \). Therefore, \( (x^+ + x^-)^+ = x^+ + x^- \). Also, \( -(x^+ + x^-) \leq 0 \). Hence, \( -(x^+ + x^-) \lor 0 = 0 \), or \( (x^+ + x^-)^- = 0 \). Thus, the absolute value of the absolute value is the absolute value, i.e. \( |x| = |x| \). Consequently, the monotonicity of the norm gives us the following: \( ||x|| = ||x|| \).

**Lemma 2.19** If \( a, b, \) and \( c \) are in \( L \), then \( (a + b) \lor c \leq (a + b^+) \lor (c + b^+) \), and \( (a + b) \land c \leq (a + b^+) \land (c + b^+) \).

**Proof.** Suppose that \( a, b, c \in L \). Now, \( a + b \leq a + (b \lor 0) = a + b^+ \leq (a + b^+) \lor (c + b^+) \), and \( c + 0 \leq c + (b \lor 0) = c + b^+ \leq (a + b^+) \lor (c + b^+) \).
Therefore, \((a+b) \lor (c+0) \leq (a+b^+) \lor (c+b^+)\). Next, note that \((a+b^+) \land (c+b^+) = (a+(b \lor 0)) \land (c+(b \lor 0)) = [(a+b) \lor (a+0)] \land [(c+b) \lor (c+0)] = [(a+b) \lor a] \land [(c+b) \lor c] \). Further, \(a+b \leq (a+b) \lor a\), and \(c \leq (c+b) \lor c\). Also, \((a+b) \land c \leq a+b \leq (a+b) \lor a\), and \((a+b) \land c \leq c \leq (c+b) \lor c\). Therefore, \((a+b) \land c \leq [(a+b) \lor a] \land [(c+b) \lor c] = (a+b^+) \land (c+b^+)\). Q.E.D.

Lemma 2.20 If \(a, b \in L\), and \(a \leq b^+\), then \(a^+ \leq b^+\).

PROOF. Suppose that \(a, b \in L\) and \(a \leq b^+\). Then, \(a \leq b \lor 0, 0 \leq b \lor 0\), and \(a \lor 0 \leq b \lor 0\). That is, \(a^+ \leq b^+\). Q.E.D.

Theorem 2.21 If \(x, x', y \in L\), then \(\|x \lor y - x' \lor y\| \leq \|x - x'\|\), and \(\|x \land y - x' \land y\| \leq \|x - x'\|\). Consequently, \(\land\) and \(\lor\) are continuous with respect to the norm.

PROOF. Suppose \(x, x', y \in L\). Note that \(x \lor y = (x' + (x-x')) \lor y \leq (x' + (x-x')^+) \lor (y + (x-x')^+) = (x' \lor y) + (x-x')^+\). Therefore, \((x \lor y) - (x' \lor y) \leq (x-x')^+\), and \(((x \lor y) - (x' \lor y))^+ \leq (x-x')^+\). Similarly, \(((x \lor y) - (x' \lor y))^- \leq (x-x')^-\), \(((x \lor y) - (x' \lor y))^+ + ((x \lor y) - (x' \lor y))^- \leq (x-x')^+ + (x-x')^-\), and \(\|(x \lor y) - (x' \lor y)\| \leq \|x - x'\|\). Consequently, \(\|x \lor y - (x' \lor y)\| \leq \|x - x'\|\). A similar technique yields the second inequality. The final conclusion of the theorem follows immediately from the preceding two inequalities. Q.E.D.

Theorem 2.22 If \(\lambda > 0\) and \(u, x \in L\), then \(u \land x = 0\) if and only if \(u \land \lambda x = 0\).

PROOF. Suppose \(\lambda > 0\) and \(u, x \in L\). The argument proceeds by first showing that \(u \land x = 0\) guarantees that \(u \land \lambda x = 0\). If \(\lambda = 1\), then \(u \land \lambda x = u \land x = 0\). If
\[ 0 < \lambda < 1, \text{ then } 0 \leq u \land \lambda x \leq u \land x = 0. \text{ Therefore, } u \land \lambda x = 0. \text{ If } \lambda > 1, \text{ then } 0 \leq u \land \lambda x = \lambda(\frac{1}{\lambda} u \land x) \leq \lambda(u \land x) = 0. \text{ Therefore, } u \land \lambda x = 0. \]

Now suppose that \( u \land \lambda x = 0 \). If \( \lambda = 1 \), then \( u \land x = u \land \lambda x = 0 \). If \( \lambda > 1 \), then \( 0 \leq u \land x \leq u \land \lambda x = 0 \). Therefore, \( u \land x = 0 \). If \( \lambda < 1 \), then \( 0 = u \land \lambda x = \frac{1}{\lambda}(u \land \lambda x) = \frac{1}{\lambda}u \land x \). Since \( 0 \leq u \land x \leq \frac{1}{\lambda}u \land x, u \land x = 0 \). Q.E.D.

**Definition 2.23** A lattice \( S \) is said to be \( \sigma \)-complete if every countable subset bounded above has a least upper bound.

**Lemma 2.24** If \( a, b, c, d \in L \), \( a \leq c \), and \( b \leq d \), then \( a \land b \leq c \land d \).

**Proof.** If \( a, b, c, d \in L \), \( a \leq c \), and \( b \leq d \), then \( a \land b \leq a \leq c \) and \( a \land b \leq b \leq d \). Therefore, \( a \land b \leq c \land d \). Q.E.D.

**Lemma 2.25** If \( x, y, \) and \( z \) are positive members of \( L \), then \( (x + y) \land z \leq (x \land z) + (y \land z) \).

**Proof.** Suppose \( x, y, \) and \( z \) are positive members of \( L \). Note that \( (x \land z) + (y \land z) = [x + (y \land z)] \land [z + (y \land z)] = (x + y) \land (x + z) \land (z + y) \land (z + z) \). Now, since \( 0 \leq x, z \leq x + z \). Also, since \( 0 \leq y, z \leq y + z \). And, since \( 0 \leq z, z \leq z + z \).

By the previous fact, since \( x + y \leq x + y \) and \( z \leq (x + y) \land (y + z) \land (z + z) \), \( (x + y) \land z \leq (x + y) \land (x + z) \land (y + z) \land (z + z) = (x \land z) + (y \land z) \). Q.E.D.

**Lemma 2.26** If \( L \) is an arbitrary lattice, \( A \subseteq X \), and \( \text{sup}(A) \) exists, then \( x + \text{sup}(A) = \text{sup}(x + A) \) for all \( x \in X \).
PROOF. Suppose $L$ is an arbitrary lattice, $A \subseteq X$, and $\text{sup}(A)$ exists. Let $u = \text{sup}(A)$. Then, $a \leq u$ for all $a \in A$, and $x + a \leq x + u$, i.e. $x + u$ is an upper bound for $x + A$. Suppose $w$ is an upper bound for $x + A$. Then for each $a$ in $A$, $a + x \leq w, a \leq w - x, u \leq w - x$, and $u + x \leq w$. Therefore, $x + u = \text{sup}(x + A)$, i.e. $x + \text{sup}(A) = \text{sup}(x + A)$. Q.E.D.

Lemma 2.27 If $L$ is a vector lattice, $(x_\alpha)_{\alpha \in A}$ is a family in $L$, and $x = \text{sup}_\alpha x_\alpha$, then $x \land z = \text{sup}_\alpha (x_\alpha \land z)$ for each $z$ in $E$.

PROOF. Suppose $L$ is a vector lattice, $(x_\alpha)_{\alpha \in A}$ is a family in $L$, and $x = \text{sup}_\alpha x_\alpha$. Since $x_\alpha \leq x$, $x_\alpha \land z \leq x \land z$ for each $\alpha \in A$. Hence, $x \land z$ is an upper bound. Suppose now that $u$ is any other upper bound for this set. Then note that $x_\alpha \land z \leq u$, and by Theorem 9, $x_\alpha + z - x_\alpha \lor z \leq u, x_\alpha + z \leq u + x_\alpha \lor z$ and $u + x_\alpha \lor z \leq u + x \lor z$. From the two preceding steps, we conclude that $x_\alpha + z \leq u + x \lor z$, and $x_\alpha \leq u + x \lor z - z$. Furthermore, since $x$ is the least upper bound, $x \leq u + x \lor z - z, x + z - x \lor z \leq u$, and, again by Theorem 9, $x \land z \leq u$. Therefore, $x \land z$ is the least upper bound, i.e. $x \land z = \text{sup}_\alpha (x_\alpha \land z)$. Q.E.D.

Note that $x \in X^+$ means that $x \in X$ and $x > 0$.

Definition 2.28 If $x \geq 0$ and $y \geq 0$ and $x, y \in S$, a $\sigma$-complete lattice, define $P_x(y)$ by $P_x(y) = \lor_n (nx \land y), n \in N$.

The reader should note that $P_x(0) = \lor_n (nx \land 0) = 0$.

Theorem 2.29 If $S$ is a $\sigma$-complete lattice and $x$ and $y$ are nonnegative elements of $S$, then $P_x(y)$ exists and $0 \leq P_x(y) \leq y$. 

PROOF. Suppose $x \geq 0$, $y \geq 0$ and $x, y \in \mathbb{S}$, a $\sigma$-complete lattice. Let $z_n = nx \land y$.

Then $z_n = nx \land y \leq y$ for all $n \in \mathbb{N}$. Since $\mathbb{E}$ is $\sigma$-complete, $\bigvee_n z_n = \bigvee_n nx \land y = P_x(y)$ exists and $P_x(y) \leq y$. Also, $0 \leq x$ implies $x \leq x + x$ or $x \leq 2x$. Therefore, $x \land y \leq x, x \land y \leq 2x, x \land y \leq y$, and $2x \land y \geq x \land y$. In general,

$$0 \leq x \land y \leq 2x \land y \ldots \leq nx \land y \leq \ldots$$

Therefore, $P_x(y) \geq 0$. Q.E.D.

**Definition 2.30** Suppose $\mathbb{X}$ is a $\sigma$-complete lattice. If $x \in \mathbb{X}^+$ and $y \in \mathbb{X}$, define $P_x(y)$ by $P_x(y) = P_x(y^+) - P_x(y^-)$.

**Lemma 2.31** If $\mathbb{X}$ is a $\sigma$-complete lattice and $P_x$ is defined as above, then $P_x$ is linear.

PROOF. Suppose $\mathbb{X}$ is a $\sigma$-complete lattice, $y, z$ are in $\mathbb{X}$, $y, z \geq 0$. Note that an application of 2.19 shows that $nx \land (y + z) \leq (nx \land y) + (nx \land z) \leq P_x(y) + P_x(z)$.

Therefore, $P_x(y+z) \leq P_x(y) + P_x(z)$. Further, $(nx \land y) + (mx \land z) = [nx + (mx \land z)] \land [y + (mx \land z)] = [(n + m)x \land (nx + z)] \land [(y + mx) \land (y + z)] \leq (nx + z) \land (y + mx) \land (y + z) \leq nx \land (y + z) \leq P_x(y+z)$. Therefore, $nx \land y \leq P_x(y + z) - mx \land z$. Fix m and note that the preceding holds for each n. Therefore, $P_x(y) \leq P_x(y + z) - mx \land z$. Now,

$$mx \land z \leq P_x(y + z) - P_x(y).$$

Hence, $P_x(z) \leq P_x(y + z) - P_x(y)$, and $P_x(y) + P_x(z) \leq P_x(y + z)$. Therefore, $P_x(y + z) = P_x(y) + P_x(z)$.

Next suppose $y, z$ are arbitrary in $\mathbb{X}$. Note that $(y+z)^+ - (y+z)^- = y+z = y^+ - y^- + z^+ - z^-$. Therefore, $(y + z)^+ - (y + z)^- = (y^+ - y^-) + (z^+ - z^-)$, $(y + z)^+ +
\[ y^- + z^- = y^+ + z^+ + (y + z)^-, \ P_x((y + z)^+ + y^- + z^-) = P_x(y^+ + z^+ + (y + z)^-), \]
\[ P_x((y + z)^+) + P_x(y^-) + P_x(z^-) = P_x(y^+) + P_x(z^+) + P_x((y + z)^-), \]
and \( P_x((y + z)^-) = P_x(y^+) + P_x(z^+) - P_x(y^-) - P_x(z^-). \) Thus, \( P_x(y + z) = P_x(y) + P_x(z), \) and \( P \) is additive. If \( y \geq 0 \) and \( \lambda \geq 0, \) then 2.16 ensures that \( \lambda P_x(y) = P_x(\lambda y). \)

Now suppose \( \lambda > 0 \) and \( y \) is arbitrary. Begin by observing that \( (\lambda y)^+ = \lambda y \lor 0 = \lambda y \lor \lambda 0 = \lambda(y \lor 0) = \lambda y^+ \) and \( (\lambda y)^- = -\lambda y \lor 0 = \lambda(-y) \lor 0 = \lambda(-y \lor 0) = \lambda y^- \). Therefore, \( P_x(\lambda y) = P_x((\lambda y)^+) - P_x((\lambda y)^-) = P_x(\lambda(y^+)) - P_x(\lambda(y^-)) = \lambda P_x(y^+) - \lambda P_x(y^-) = \lambda(\lambda P_x(y)). \) Next, suppose \( \lambda < 0 \) and \( y \) is arbitrary. Recall that \( 0 = P_x(0) = P_x(y - y) = P_x(y) + P_x(-y). \) Therefore, \( P_x(-y) = -P_x(y) \) and, \( P_x(\lambda y) = P_x(-\lambda(-y)) = -\lambda(P_x(-y)) = -\lambda(-P_x(y)) = \lambda P_x(y). \) Thus, \( P \) is linear.

Q.E.D.

Lemma 2.32 If \( X \) is a lattice, \( u \in X, \ u = p - q \) and \( p \land q = 0, \) then \( p = u^+ \) and \( q = u^- \).

PROOF. Suppose \( X \) is a lattice, \( u \in X, \ u = p - q, \) and \( p \land q = 0. \) Now \( p + q = p \lor q. \)

Hence, \( p = p + q - q = (p \lor q) - q = (p - q) \lor (q - q) = (p - q) \lor 0 = (p - q)^+ = u^+. \)

Thus, \( u = p - q = u^+ - q. \) Also, \( u = u^+ - u^- \). Therefore, \( q = u^- \). Q.E.D.

Lemma 2.33 If \( X \) is a \( \sigma \)-complete lattice, \( u, \ v, \ x \geq 0 \) are in \( X \) and \( u \land v = 0, \) then \( P_x(u) \land P_x(v) = 0. \)

PROOF. Suppose \( X \) is a \( \sigma \)-complete lattice, \( u, \ v, \ x \geq 0 \) are in \( X \) and \( u \land v = 0. \) Note that \( P_x(u) \land P_x(v) = P_x(u) \land (\lor_{n} nx \land v) = \lor_{n} nx \land u \land P_x(v) = \lor_{n} nx \land (v \land \lor_{m} mx \land u) = \lor_{n} nx \land (\lor_{m} mx \land u \land v) = \lor_{n} nx \land 0 = 0. \) Q.E.D.
Lemma 2.34 If $X$ is a $\sigma$-complete lattice and $y \in X$, then $|P_x(y)| = P_x(|y|)$.

PROOF. Suppose $X$ is a $\sigma$-complete lattice and $y \in X$. Recall that $y^+ \land y^- = 0$. Therefore, $P_x(y^+) \land P_x(y^-) = 0$. Also, $P_x(y) = P_x(y^+ - y^-) = P_x(y^+) - P_x(y^-)$. Hence, $P_x(y^+) = (P_x(y))^+$, and $P_x(y^-) = (P_x(y))^-$.

Further, $|P_x(y)| = (P_x(y))^+ + (P_x(y))^-$ = $P_x(y^+) + P_x(y^-) = P_x(y^+ + y^-) = P_x(|y|)$. Q.E.D.

Note 2.35 If $y \in S$ then $\|P_x(y)\| = \|P_x(y)\| = \|P_x(|y|)\|$.

Lemma 2.36 If $y$ and $z$ belong to a $\sigma$-complete lattice $X$, then $\|P_x(y) - P_x(z)\| \leq \|y - z\|$.

PROOF. Suppose $X$ is a $\sigma$-complete lattice, and $y, z \in X$. Note that $\|P_x(y) - P_x(z)\| = \|P_x(y - z)\| = \|P_x(|y - z|)\| = \|P_x(|y - z|)\| \leq |y - z|$. By the monotonicity of the norm, $\|P_x(|y - z|)\| \leq \|y - z\|$. Q.E.D.

Lemma 2.37 If $S$ is a $\sigma$-complete lattice, $x \geq 0$, $y, z \in S$, and $y \leq z$ then $P_x(y) \leq P_x(z)$.

PROOF. First suppose that $y \geq 0, z \geq 0$, and $y \leq z$. Then, $nx \land y \leq nx \land z \leq P_x(z)$ for each $n$. Therefore, $P_x(y) \leq P_x(z)$. Now suppose that $y \leq z$. Then, $y \lor 0 \leq z \lor 0$, i.e. $y^+ \leq z^+$, and $P_x(y^+) \leq P_x(z^+)$. Also, if $y \leq z$, then $0 \leq z - y, -z \leq -y$, and $(-z) \lor 0 \leq (-y) \lor 0$. Therefore, $z^- \leq y^-, P_x(z^-) \leq P_x(y^-)$, and $-P_x(y^-) \leq -P_x(z^-)$.

Hence, $P_x(y^+) - P_x(y^-) \leq P_x(z^+) - P_x(z^-)$, i.e. $P_x(y) \leq P_x(z)$. Q.E.D.

Lemma 2.38 If $S$ is a $\sigma$-complete lattice, $x \geq 0$, and $y \in S$, then $P_x(P_x(y)) = P_x(y)$. 

PROOF. Suppose first that $S$ is a $\sigma$-complete lattice, $x \geq 0$, and $y \geq 0$. Fix $m$ and observe that $mx \land P_x(y) = \bigvee_{n=1}^{m-1} (mx \land nx \land y) = \bigvee_{n=1}^{m} (mx \land nx \land y) = mx \land y \leq P_x(y)$ for each $m$. Therefore, $P_x(P_x(y)) \leq P_x(y)$. Now fix $m$. If $n < m$, then $\bigvee_{n=1}^{m-1} mx \land nx \land y < mx \land y$.

If $n \geq m$, $mx \land y \leq \bigvee_{n=1}^{m} (mx \land nx \land y) = mx \land y = mx \land P_x(y) \leq P_x(P_x(y))$ for each $m$. Therefore, $P_x(y) \leq P_x(P_x(y))$. Hence, $P_x(y) = P_x(P_x(y))$.

Now consider an arbitrary $y \in S$. By definition, $P_xP_x(y) = P_xP_x(y^+) - P_xP_x(y^-) = P_x(y^+) - P_x(y^-) = P_x(y)$. Q.E.D.

The above lemmas establish the following important properties of $P$. $S^+$ will denote \{ $x \in S$: $x \geq 0$ \}.

**Theorem 2.39** If $S$ is a normed $\sigma$-complete lattice, $x \in S^+$, $y$, $y'$, $z \in S$, then

i) $P_x$ is linear.

ii) $P_x$ is monotone increasing.

iii) $P_x$ is idempotent, i.e. $P_x(P_x(y)) = P_x(y)$.

iv) $\| P_x(y) - P_x(y') \| \leq \| y - y' \|$.

v) If $y \land z = 0$, then $P_x(y) \land P_x(z) = 0$.

**Lemma 2.40** If $S$ is a $\sigma$-complete lattice, and $(x_i)$ and $(y_j)$ are bounded sequences in $S$, then $\sup(x_i + y_j) = \sup(x_i) + \sup(y_j)$.

PROOF. Suppose that $S$ is a $\sigma$-complete lattice and that $(x_i)$ and $(y_j)$ are bounded sequences in $S$. Let $x = \sup(x_i)$ and $y = \sup(y_j)$. Hence, $x_i \leq x$, and $y_j \leq y$. Thus, $x_i + y_j \leq x + y$ for each $i$ and $j$. Hence, $\sup(x_i + y_j) \leq x + y$. Let $w$ be any upper bound of \{ $x_i + y_j$ \} for each $i$ and $j$. Fix $j$. Then, $x_i + y_j \leq w, x_i \leq w - y_j$ for
each $i$. Thus, $\sup(x_i) \leq w - y_j$. Also, $y_j \leq w - \sup(x_i)$ for each $j$. Consequently, 
$\sup(y_j) \leq w - \sup(x_i)$, $\sup(y_j) + \sup(x_i) \leq w$, i.e. $x + y \leq w$. Therefore, $\sup(x_i + y_j) = \sup(x_i) + \sup(y_j)$. Q.E.D.

**Theorem 2.41** If $S$ is a $\sigma$-complete lattice, $x, x' \in S^+$, $y \in S$, and $x \land x' = 0$, then 
$P_{x+x'}(y) = P_x(y) + P_{x'}(y) = P_{x\lor x'}(y)$.

**PROOF.** Suppose $S$ is a $\sigma$-complete lattice, $x, x' \in S^+$, $y \in S$, and $x \land x' = 0$. Note that $n(x + x') \land y = (nx + nx') \land y = [(nx \land nx') + (nx \lor nx')] \land y = [n(x \land x') + (nx \lor nx')] \land y = [0 + (nx \lor nx')] \land y = (nx \land y) \lor (nx' \land y)$. Also, note that $(nx \land y) \land (nx' \land y) = n(x \land x') \land y = 0$. Therefore, $n(x + x') \land y = (nx \land y) \lor (nx' \land y) + (nx \land y) \land (nx' \land y) = (nx \land y) + (nx' \land y)$. By the above lemma, 
$P_{x+x'}(y) = P_x(y) + P_{x'}(y)$. Also, $x + x' = x \lor x' + x \land x' = x \lor x'$. Therefore, 
$P_{x+x'}(y) = P_{x\lor x'}(y)$. Q.E.D.

**Lemma 2.42** If $S$ is a $\sigma$-complete lattice, $x \geq 0$, and $y \in S$, then $P_x(y) = 0$ if and only if $x \land y = 0$.

**PROOF.** Suppose $S$ is a $\sigma$-complete lattice, $x \geq 0$, $y \in S$, and $P_x(y) = 0$. Since $0 \leq x \land y \leq nx \land y \leq P_x(y) = 0$, $x \land y = 0$. Conversely, suppose $x \land y = 0$. Then, $nx \land y = 0$ for each $n$. Hence, $P_x(y) = 0$. Q.E.D.

**Theorem 2.43** If $S$ is a $\sigma$-complete lattice, $x \geq 0$, and $y \in S$, then $x \land (y - P_x(y)) = 0$. 

21

PROOF. Suppose that $S$ is a $\sigma$-complete lattice, $x \geq 0$, and $y \in S$. Note that $P_x(y - P_x(y)) = P_x(y) - P_x(P_x(y)) = P_x(y) - P_x(y) = 0$. Therefore, $x \wedge (y - P_x(y)) = 0$. Q.E.D.

**Theorem 2.44** If $S$ is a $\sigma$-complete lattice, $x \geq 0$, and $y \geq 0$, then $2P_x(y) \wedge y = P_x(y)$.

PROOF. Suppose that $S$ is a $\sigma$-complete lattice and that $x, y \in S^+$. Then $(2P_x(y) \wedge y = (P_x(2y)) \wedge y = (\vee_n nx \wedge 2y) \wedge y = \vee_n nx \wedge (2y \wedge y) = \vee_n nx \wedge y = P_x(y)$.

Q.E.D.

**Theorem 2.45** If $S$ is a $\sigma$-complete lattice and $x$, $y \in S^+$, then $P_x(y) \wedge (y - P_x(y)) = 0$.

PROOF. Suppose $S$ is a $\sigma$-complete lattice, $x \geq 0$, $y \geq 0$, and $x$ and $y$ are in $S$. Note that $P_x(y) \wedge (y - P_x(y)) = [2P_x(y) - P_x(y)] \wedge [y - P_x(y)] = [2P_x(y) \wedge y] - P_x(y) = P_x(y) - P_x(y) = 0$. Q.E.D.

**Theorem 2.46** If $S$ is a $\sigma$-complete lattice, and $x$, $y \in S^+$, then $P_{P_x(y)} = P_xP_y = P_{x \wedge y}$.

PROOF. Suppose $S$ is $\sigma$-complete and that $x$, $y$, $z \in S^+$. Fix $m \in \mathbb{N}$. Then $mx \wedge (\forall n y \wedge z) = \vee_n mx \wedge ny \wedge z \leq \vee_n nx \wedge ny \wedge z = P_{x \wedge y}(z)$. Thus, $\vee_m mx \wedge P_y(z) \leq P_{x \wedge y}(z)$, i.e. $P_x(P_y(z)) \leq P_{x \wedge y}(z)$. Also, $m(x \wedge y) \wedge z = mx \wedge (my \wedge z) \leq mx \wedge (P_y(z)) \leq P_x(P_y(z))$ for each $m$. Hence, $P_{x \wedge y}(z) \leq P_{x}(P_y(z))$. Therefore, $P_{x \wedge y} = P_xP_y$.

Next fix $n \in \mathbb{N}$, and note that $nP_x(y)) \wedge z = P_x(ny) \wedge z = (\forall m x \wedge ny \wedge z = \vee_m mx \wedge ny \wedge z = P_{x \wedge y}(z)$. Consequently, $P_{P_x(y)}(z) \leq P_{x \wedge y}(z)$. Further,
\[ n(x \land y) \land z = (nx \land ny) \land z \leq \bigvee_m mx \land ny \land z = P_x(ny) \land z = nP_x(y) \land z \leq P_{x,y}(z). \]

Since this inequality holds for each \( n \), \( P_{x,y}(z) \leq P_{x,y}(z) \), and \( P_{x,y}(y) = P_{x,y} = P_{x,y} \).

Q.E.D.

**Lemma 2.47** If \( X \) is a lattice, \( x \) and \( y \) are in \( X \), \( x \geq 0 \), and \( y \geq 0 \), then \( x \land y = y - (x - y)^- \).

**PROOF.** Suppose \( X \) is a lattice, and \( x, y \in X^+ \). Note that \( x \land y - y = (x - y) \land \)
\( (y - y) = (x - y) \land 0 = -(x - y) \land 0 = -(x - y)^- \). Therefore, \( x \land y = y - (x - y)^- \). Q.E.D.

**Lemma 2.48** If \( X \) is a lattice and \( x, y \in X^+ \), then \( x \lor y = (x - y)^+ + y \).

**PROOF.** Suppose that \( X \) is a lattice, \( x \) and \( y \) are in \( X \), \( x \geq 0 \), and \( y \geq 0 \). Note that \( x \lor y - y = (x - y) \lor (y - y) = (x - y) \lor 0 = (x - y)^+ \). Therefore, \( x \lor y = y + (x - y)^+ \).

Q.E.D.

**Theorem 2.49** If \( X \) is a normed vector lattice and \( x \land y = 0 \), then \( \| x - y \| = \| x+y \| \).

**PROOF.** Suppose \( X \) is a normed vector lattice and \( x \land y = 0 \). Note that \( (x \land y) + (x \lor y) = x + y \), \( (x \lor y) = x + y \), \( (x \lor y) = y + (x - y)^+ \), \( x + y = y + (x - y)^+ \), and \( x = (x - y)^+ \). Also, note that \( x \land y = y - (x - y)^- \), \( 0 = y - (x - y)^- \), and \( (x - y)^- = y \).

Therefore, \( |x - y| = (x - y)^+ + (x - y)^- = x + y \). Hence, \( \|x - y\| = \|x - y\| = \|x + y\| \).

Q.E.D.
CHAPTER 3

EXHAUSTIVITY AND ABSOLUTE CONTINUITY IN BANACH LATTICES

As indicated in a preceding chapter, Bell, Bilyeu, Lewis extended the measure theoretic notion of strong boundedness to the setting of a real $\sigma$-complete Banach lattice $X$ in [4]. The lattice theoretic concept which corresponds to strong boundedness is termed exhaustivity. Brooks’s theorem and the Brooks-Drewnowski theorem then follow as simple corollaries from the Banach lattice results. Throughout this chapter $X$ will denote a real Banach lattice which is $\sigma$-complete.

Definition 3.1 Let $\mathcal{O}$ denote \{${P_x : x \geq 0}$\}.

(i) A sequence $(P_i)$ from $\mathcal{O}$ is said to be disjoint (or pairwise disjoint) if $P_iP_j = 0$ for $i \neq j$.

(ii) A subset $K$ of $X$ is said to be (uniformly) continuous or (uniformly) absolutely continuous with respect to an element $m \in X$ if $P_i(u) \to 0$ (uniformly) for $u \in K$ whenever $(P_i)$ is a sequence from $\mathcal{O}$ such that $P_i(m) \to 0$. The notation $K \ll m$ is used to denote this property.

(iii) A subset $K$ of $X$ is said to be (uniformly) exhaustive if $P_i(u) \to 0$ (uniformly) for $u \in K$ whenever $(P_i)$ is a disjoint sequence from $\mathcal{O}$. If $K$ is a singleton (i.e. $K = \{k\}$), then we say that the element $k$ is exhaustive.
Bell, Bilyeu, and Lewis failed to note that the above definition of continuity is equivalent to the usual \((\epsilon, \delta)\) definition with respect to \(O\). More specifically, in the lattice setting \(\nu << \mu\) if and only if \((*)\) given \(\epsilon > 0\) there exists a \(\delta > 0\) such that if \(P_x \in O\) and \(\|P_x(\mu)\| < \delta\), then \(\|P_x(\nu)\| < \epsilon\).

In fact, suppose \(\nu << \mu\) and \((*)\) fails. Let \(\epsilon > 0\) such that no matter what \(\delta > 0\) one finds, there exists and \(x_\delta \in X^+\) such that \(\|P_{x_\delta}(\mu)\| < \delta\) and \(\|P_{x_\delta}(\nu)\| \geq \epsilon\). Find \(P_{x_n} \in O\) such that \(\|P_{x_n}(\mu)\| < \frac{1}{n}\) and \(\|P_{x_n}(\nu)\| \geq \epsilon\). Therefore, \(P_{x_n}(\mu) \rightarrow 0\) and \((P_{x_n}(\nu))\) does not converge to 0. Since we have reached a contradiction, the lattice setting definition implies \((*)\).

Conversely, suppose that \((*)\) holds. Suppose \(\epsilon > 0\) and \(\|P_{x_i}(\mu)\| \rightarrow 0\). Then there exists a corresponding \(\delta > 0\) and there exists an \(N\) such that \(\|P_{x_i}(\nu)\| < \delta\) for \(i \geq N\). Therefore, \(\|P_{x_i}(\nu)\| < \epsilon\) for \(i \geq N\), and \((P_{x_i}(\nu)) \rightarrow 0\). Therefore, the two definitions are equivalent.

Furthermore, in the lattice setting when we say \(\nu_n << \mu_n\) uniformly in \(n\) we mean

\((1)\) \(\nu_n << \mu_n\) for each \(n\) and if \((P_{x_n}(\mu_n)) \rightarrow 0\), then \(\|P_{x_n}(\nu_i)\| \rightarrow 0\) uniformly in \(i\). Considering the equivalence established above, it is natural to conjecture that \(\nu_n << \mu_n\) uniformly in \(n\) if and only if \((2)\) for each \(\epsilon > 0\) there is a \(\delta > 0\) such that if \(n \in N, P_x \in O\), and \(\|P_x(\mu_n)\| < \delta\), then \(\|P_x(\nu_n)\| < \epsilon\).

Indeed, suppose that \((1)\) holds and \((2)\) does not. Let \(\epsilon > 0\) such that if \(\delta > 0\) then there exists \(n_\delta \in N\) and \(x_\delta \in X^+\) such that \(\|P_{x_\delta}(\mu_{n_\delta})\| < \delta\) and \(\|P_{x_\delta}(\nu_{n_\delta})\| \geq \epsilon\). Choose \(x_1 \in X^+\) and \(n_1 \in N\) such that \(\|P_{x_1}(\mu_{n_1})\| < 1\) and \(\|P_{x_1}(\nu_{n_1})\| \geq \epsilon\). Now use
the fact that $v_n << \mu_n$ and the preceding claim to find $n_2 > n_1$ and $x_2 \in X^+$ such that $\|P_{x_2}(\mu_{n_2})\| < \frac{1}{2}$ and $\|P_{x_2}(\nu_{n_2})\| \geq \epsilon$. Continue this construction inductively. Let $y_1 = y_2 = \cdots = y_{n_1} = 0$, $y_{n_1} = x_1$, $y_{n_1+1} = \cdots = y_{n_2-1} = 0$, $y_{n_2} = x_2$, $\ldots$. Therefore, $\|P_{y_1}(\mu_i)\| \rightarrow 0$ and $\|P_{y_{n_1}}(\nu_{n_2})\| \geq \epsilon$. This contradicts (1). Certainly (2) implies (1).

Also in the lattice setting when we say $v << \mu$ uniformly for $(\nu, \mu) \in H \times K$ we mean (1) given $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in X^+$, $(\nu, \mu) \in H \times K$, and $\|P_x(\mu)\| < \delta$, then $\|P_x(\nu)\| < \epsilon$.

The following sequential version of (a) will prove to be helpful in subsequent arguments. (b) If $(P_{x_i})$ is a sequence in $\mathcal{O}$, $(\mu_i)$ is a sequence in $K$, and $\|P_{x_i}(\mu_i)\| \rightarrow 0$, then $\|P_{x_i}(\nu)\| \rightarrow 0$ uniformly for $\nu \in H$.

Certainly (a) implies (b). Suppose (b) holds and (a) does not. Let $\epsilon > 0$ so that no $\delta > 0$ can be found which satisfies (a). Find $\delta_i \rightarrow 0$, $x_i \in X^+$ for each $i$, and $(\nu_i, \mu_i) \in H \times K$ such that $\|P_{x_i}(\mu_i)\| < \delta_i$ and $\|P_{x_i}(\nu_i)\| \geq \epsilon$ for each $i$. Clearly this contradicts (b). Therefore, (b) implies (a).

Further, observe that if $v << \mu$ uniformly for $(\nu, \mu) \in H \times K$, $(P_{y_j})$ is a sequence in $\mathcal{O}$, and $\epsilon > 0$, then there exists a $\delta > 0$ such that if $\xi \in K$ and $\|P_{y_j}(\xi)\| < \delta$ for all sufficiently large $j$, then $\|P_{y_j}(\nu)\| < \epsilon$ for all sufficiently large $j$ and all $\nu \in H$. To see this apply (1) above.

Also, note that if $H << K$, $x \in X^+$, and $\inf\{\|P_x(\mu)\| : \mu \in k\} = 0$, then $P_x(\nu) = 0$ for each $\nu \in H$. (Let $x_i = x$ for each $i$. Choose $\mu_i$ such that $\|P_{x_i}(\mu_i)\| \rightarrow 0$. Then $P_{x_i}(\nu) \rightarrow 0$ for each $\nu$.)
The next lemma is technically helpful in discussing the exhaustive elements of X.

**Lemma 3.2** If \( x, y \in L \), then \( x \lor y = \frac{x + y + |x - y|}{2} \).

**PROOF.** Suppose \( x, y \in L \). Note that \( |x - y| = (x - y)^+ + (x - y)^- = (x - y) \lor 0 + [- (x - y) \lor 0] = [(x - y) \lor -(x - y)] + 0 = (x - y) \lor (y - x) \). Therefore, \( (x + y) + |x - y| = (x - y) \lor (y - x) + (x + y) = [(x - y) + (x + y)] \lor [(y - x) + (x + y)] = 2x \lor 2y = 2(x \lor y) \). Hence, \( \frac{x + y + |x - y|}{2} = (x \lor y) \). **Q.E.D.**

**Definition 3.3** A subset \( K \) of \( X \) is a sublattice of \( X \) if \( K \) is closed with respect to the lattice operations, i.e. if \( x, y \in K \) then \( x \lor y \in K \) and \( x \land y \in K \).

**Theorem 3.4** The set of all exhaustive elements of \( X \) is a sublattice.

**PROOF.** Let \( \mathcal{E}(X) \) denote the exhaustive elements of \( X \), and suppose that \( k_1, k_2 \in \mathcal{E}(X) \). Let \( (P_i) \) be a disjoint sequence from \( \mathcal{O} \). Note that \( \|P_i(k_1 \lor k_2)\| = \|P_i\left(\frac{k_1 + k_2 + |k_1 - k_2|}{2}\right)\| \leq \|P_i(k_1)\| + \|P_i(k_2)\| + \|P_i(|k_1 - k_2|)\| = \|P_i(k_1)\| + \|P_i(k_2)\| + \|P_i(k_1 - k_2)\| \leq \|P_i(k_1)\| + \|P_i(k_2)\| + \|P_i(k_1)\| + \|P_i(k_2)\| \leq 2\|P_i(k_1)\| + 2\|P_i(k_2)\| \). Therefore, for \( \varepsilon > 0 \) choose \( N \) such that if \( i \geq N \) then \( \|P_i(k_j)\| \leq \frac{\varepsilon}{4} \) for \( j = 1, 2 \). Then, \( P_i(k_1 \lor k_2) \longrightarrow 0 \). Therefore, \( k_1 \lor k_2 \in K \). Also, if \( k_1, k_2 \in K \), then \( -k_1, -k_2 \in K \). Consequently, \( (-k_1) \lor (-k_2) \in K \) and \( -[(k_1) \lor (k_2)] \in K \). Hence, \( k_1 \land k_2 = -[(k_1) \lor (k_2)] \) is in \( K \). Therefore, \( K \) is a sublattice.

**Theorem 3.5** The exhaustive elements of \( X \) form a topologically closed subspace.
PROOF. Suppose $K$ is exhaustive, $(x_n) \subset K$ and $x_n \to x$. Suppose $x \notin K$. Then there exists $(P_i)$ disjoint, $\epsilon > 0$ such that $\|P_i(x)\| > \epsilon$ for each $i$. Pick $N$ such that $\|x_N - x\| \leq \frac{\epsilon}{2}$. Since $x_N$ is exhaustive, pick $i_0$ such that $\|P_{i_0}(x_N)\| < \frac{\epsilon}{2}$. Therefore, $\|P_{i_0}(x)\| = \|P_{i_0}(x_N) + P_{i_0}(x) - P_{i_0}(x_N)\| \leq \|P_{i_0}(x_N)\| + \|P_{i_0}(x) - P_{i_0}(x_N)\| < \frac{\epsilon}{2} + \|x_N - x\| < \epsilon$. Since this is a contradiction, $x$ must be in $K$. Therefore, $K$ is closed. Now, let $k_1, k_2 \in K, \alpha \in \mathbb{R}$ and $(P_i)$ be a disjoint sequence in $O$. Note that $\|P_i(\alpha k_1 + k_2)\| \leq |\alpha|\|P_i(k_1)\| + \|P_i(k_2)\|$. Also, note that for $\epsilon > 0$ there exists $N$ such that if $i \geq N$ then $\|P_i(k_1)\| < \frac{\epsilon}{2|\alpha| + 1}$ and $\|P_i(k_2)\| < \frac{\epsilon}{2}$. Therefore, $P_i(\alpha k_1 + k_2) \to 0$ and $\alpha k_1 + k_2 \in K$. Hence, $K$ is a subspace. Q.E.D.

Hence the exhaustive elements of $X$ form a closed linear subspace which is a sublattice. The next theorem shows that the range of each element of $O$ is also a closed linear subspace which is a sublattice.

Theorem 3.6 If $P = P_x$ for some $x \in X^+$ and $R(P)$ is the range of $P$, then $R(P)$ is a closed linear subspace of $X$ which is also a sublattice.

PROOF. Suppose $(x_n) = (P(y_n))$, and $x_n \to x$, i.e. $P(y_n) \to x$. Then $P(x_n) = P(P(y_n)) = P(y_n)$, where, on the one hand, $P(x_n) \to P(x)$, and, on the other, $P(y_n) \to x$. Therefore, $P(x) = x$ or $x \in R(P)$. Hence, $R(P)$ is closed (topologically). Now suppose $x = P(x_1), y = P(y_1), \alpha \in \mathbb{R}$. Note that $\alpha x + y = \alpha P(x_1) + P(y_1) = P(\alpha x_1 + y_1)$. Therefore, $\alpha x_1 + y_1 \in R(P)$. Hence, $R(P)$ is a subspace. Note also that $x \lor y = \frac{x + y + |x - y|}{2} = \frac{x + y + (x - y)^+ + (y - x)^-}{2}$. Therefore, we need to show that if $z \in R(P)$, then $z^+ \in R(P)$. Suppose $z = P(z_1)$. Note that $z^+ = z \lor 0 \geq z$. 

Therefore, $P(z^+) \geq P(z) = P(P(z_1)) = P(z_1) = z$. Hence, $P(z^+) \lor 0 \geq z \lor 0$, and $P(z^+) \geq z^+$. We know that $P(z^+) \leq z^+$. Therefore, $P(z^+) = z^+$ or $z^+ \in R(P)$. Hence, $x \lor y \in R(P)$. Similarly, $x \land y \in R(P)$. Therefore, $R(P)$ is a sublattice.

Q.E.D.

Lemma 3.7 i) If $x$ and $y$ are in $X$ and $0 \leq y \leq x$, then there is a $z$ in $X^+$ so that $P_x - P_y = P_z$.

ii) If $K$ is a uniformly exhaustive subset of $X$, then $|K| = \{|x| : x \in K\}$ is uniformly exhaustive.

PROOF. i) Suppose that $x$ and $y$ belong to $X$ and $0 \leq y \leq x$. Recall that $P_y(x) \land (x - P_y(x)) = 0$. Note that $P_x = P_y(x) + (x - P_y(x)) = P_y(x) + P_x - P_y(x)$, and $P_y(x) = P_x P_y = P_{x \land y}$. But, by hypothesis $x \land y = y$. Thus, $P_x = P_y + P_x - P_y(x)$. Consequently, $P_x - P_y = P_z$, where $z = x - P_y(x)$.

ii) Suppose that $K$ is uniformly exhaustive and $(P_i) = (P_{x_i})$ is a disjoint sequence from $O$. Let $u \in K$. Note that $\|P_i(\{u\})\| = \|P_i(u)\| \longrightarrow 0$ uniformly for $u \in K$. Therefore $|K|$ is uniformly exhaustive. Q.E.D.

The next theorem is the Bell, Bilyeu, and Lewis [4] Banach lattice version of the Brooks-Drewnowski theorem. The authors remark in [4] that their argument is an adaptation of the one given in Drewnowski [12, Theorem 6.1]. A rather detailed presentation of their proof is included for completeness. Techniques used in this argument will also be prevalent in the Banach lattice treatment of several theorems from Bator, Bilyeu, and Lewis [2].
Theorem 3.8 Suppose that $K \subseteq X$.

i) If $K$ is uniformly exhaustive, $\varepsilon > 0$, and $(P_{x_i})$ is a sequence from $O$, then there is a positive integer $N$ so that if $k \geq n \geq N$, then $\|(P_{x_k} - P_{x_k \land \bigvee_{i=1}^{\infty} x_i})(u)\| < \varepsilon$ for all $u \in K$.

ii) If $K$ is uniformly exhaustive and $K$ is continuous with respect to $m \in X^+$, then $K$ is uniformly continuous with respect to $m$.

PROOF. i) Suppose the result is false. Therefore, there exists an $\varepsilon > 0$ so that no matter what $N_i \in N$ we choose there exists $k_i \geq n_i \geq N_i$ and $u_i \in K$ such that $\|P_{x_{k_i}} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}(u_i)\| > \varepsilon$. Note that we may pick $N_{i+1} \geq k_i$. Further for $i = 2, 3,...$ we let $k_i = l_i x_{k_i}$ and $y = x_{k_i}$. We can replace $P_{x_{k_i}} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}(u_i)$ by $P_{x_{k_i} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}(u_i)}$. Hence, $x_{k_i} \land \bigvee_{j=1}^{n_i} x_j \subseteq x_{k_i} \land \bigvee_{j=1}^{n_i} x_j$

\[
x_{k_i} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}(x_{k_i}) \leq x_{k_i} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}(x_{k_i})
\]

Recall that in the norm we can replace $|u_i|$ by $u_i$, and $P_{x-P}(x) = P_x - P_y$. If we let $x = x_{k_i}$ and $y = x_{k_i} \land \bigvee_{j=1}^{n_i} x_j$ we can replace $P_{x_{k_i} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}(x_{k_i})}$ by $P_{x_{k_i} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}}$. Hence,

\[
\|P_{x_{k_i}} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}(u_i)\| = \|P_{x_{k_i} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}(x_{k_i})}(u_i)\|
\]

\[
\geq \|P_{x_{k_i} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}(x_{k_i})}(u_i)\|
\]

\[
= \|P_{x_{k_i} - P_{x_{k_i} \land \bigvee_{j=1}^{n_i} x_j}(u_i)}\| > \varepsilon.
\]
Hence, in general, there exists an $\epsilon > 0$, an increasing sequence $(n_i)$ of positive integers, and a sequence $(u_i)$ from K such that $\|P_{x_{n_i}} - P_{x_{n_i} \bigvee_{k=1}^{n_i-1} x_k} (u_i)\| > \epsilon$ for $i > 2$. Now we claim that this sequence of difference of projections is pairwise disjoint.

Suppose $i \neq j$. Without loss of generality, we may assume $2 < i < j$. Note that $x_{n_i}$ is included in the max of the $x_k$'s up to $n_{j-1}$ since $n_i < n_j$. Therefore, $n_i \leq n_{j-1}$ and $x_{n_i} \leq \bigvee_{k=1}^{n_{j-1}} x_k$. Also note that $\bigvee_{k=1}^{n_{j-1}} x_k \leq \bigvee_{k=1}^{n_j} x_k$. Using the definition of the difference of two functions, the linearity of $P_x$, and, the property $P_x P_y = P_{x \wedge y}$ we obtain the following equalities:

\[
\left( P_{x_{n_i}} - P_{x_{n_i} \bigvee_{k=1}^{n_i-1} x_k} \right) \left( P_{x_{n_j}} - P_{x_{n_j} \bigvee_{k=1}^{n_j-1} x_k} \right) (z) = 0.
\]

Therefore, $\left( P_{x_{n_i}} - P_{x_{n_i} \bigvee_{k=1}^{n_i-1} x_k} \right)$ is pairwise disjoint. Furthermore, these differences belong to $\mathcal{O}$. Thus, we have contradicted the uniform exhaustivity of K and (i) follows.

ii) Suppose, now, that $m \in X^+$ and that K is a uniformly exhaustive subset of X so that K is continuous with respect to $m$ but not uniformly continuous with respect to $m$. Thus, there is a sequence $(x_i)$ from $X^+$ and an $\epsilon > 0$ and a sequence $(y_i)$ from
K such that

$$\|P_{z_i}(m)\| < \frac{\epsilon}{2} \text{ but } \|P_{z_i}(y_1)\| > 2\epsilon.$$ 

Therefore, $$\sum_{i=1}^{\infty} \|P_{z_i}(m)\| < \epsilon \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i < \frac{\epsilon}{2} < \infty.$$ Hence, we have

(I) $$\sum_{i=1}^{\infty} \|P_{z_i}(m)\| < \infty$$

and

(II) $$\|P_{z_i}(y_1)\| > 2\epsilon$$

for all i. Applying part (i) above, let $$n_1$$ be a positive integer so that if $$n \geq n_1$$, then

$$\|P_{x_n} - P_{x_n \wedge \sqrt[n]{\bigvee_{k=1}^{n_1} x_k}}(u)\| < \frac{\epsilon}{2}$$

for each $$u \in K$$. Let $$z_1 = \sqrt[n_1]{\bigvee_{k=1}^{n_1} x_k}$$. Then from (II) and the immediately preceding inequality,

$$\|P_{x_n}(y_1)\| - \|P_{x_n \wedge z_1}(y_1)\| \leq \|(P_{x_n} - P_{x_n \wedge z_1})(y_1)\| < \frac{\epsilon}{2};$$

$$2\epsilon < \|P_{x_n}(y_1)\| < \|P_{x_n \wedge z_1}(y_1)\| + \frac{\epsilon}{2},$$

and

$$2\epsilon - \frac{\epsilon}{2} < \|P_{x_n \wedge z_1}(y_1)\| \quad \text{for } n \geq n_1.$$ 

Let $$a_1 = z_1 \wedge x_{n_1}, a_2 = z_1 \wedge x_{n_2 + 1}, \ldots$$. Note from above that

$$2\epsilon - \frac{\epsilon}{2} < \|P_{a_1}(y_1)\|,$$
and

$$2\varepsilon - \frac{\varepsilon}{2} < \|P_{a_n}(y_{n+1})\|,$$

for each $n$. Applying part (i) above to $(P_{a_n})$, let $n_2 (> n_1)$ be a positive integer so that if $n \geq n_2$ then

$$\|P_{a_n} - P_{\bigwedge_{k=1}^{n_2} a_k}(u)\| < \frac{\varepsilon}{4},$$

for each $u \in K$. Consequently, (arguing as above) there is a sequence $(b_n)$ in $K$ $(b_n = y_{n+n-1})$ such that

$$\|P_{a_n}(b_n)\| - \|P_{\bigwedge_{k=1}^{n_2} a_k}(b_n)\| \leq \|P_{a_n} - P_{\bigwedge_{k=1}^{n_2} a_k}(b_n)\| \leq \frac{\varepsilon}{4},$$

and

$$2\varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} \leq \|P_{a_n}(b_n)\| - \frac{\varepsilon}{4} \leq \|P_{\bigwedge_{k=1}^{n_2} a_k}(b_n)\|,$$

for $n > n_2$. Let $z_2 = \bigvee_{k=1}^{n_2} a_k$. Then $2\varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} \leq \|P_{a_n \wedge z_2}(b_n)\| \leq \|P_{z_2}(b_n)\|$. Notice that

$$\|P_{z_2}(b_n)\| = \|P_{z_2}(|b_n|)\| > 2\varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{4}$$

for $n > n_2$, and $z_2 \leq z_1$ since $a_n = z_1 \wedge x_{n_1 + (n-1)} \leq z_1$ for all $n$

and $z_2 = \bigvee_{n=1}^{n_2} a_k \leq z_1$. Continue inductively to manufacture a sequence $(z_k)$ from $X^+$

and a sequence $(d_k)$ (a subsequence of $(y_n)$) from $K$ such that
(III) \[ z_{k+1} \leq z_k \]

and

(IV) \[ \|P_{z_k}(|d_k|)\| > \epsilon \]

for each \( k \). Now observe that if \( q_1, q_2 \), and \( u \in X^+ \), then

\[
n(q_1 \lor q_2) \land u = (nq_1 \lor nq_2) \land u
= (nq_1 \land u) \lor (nq_2 \land u)
= (nq_1 \land u) + (nq_2 \land u) - (nq_1 \land u) \land (nq_2 \land u)
\leq (nq_1 \land u) + (nq_2 \land u).
\]

Therefore, \( P_{q_1 \lor q_2} \leq P_{q_1} + P_{q_2} \), or, inductively, \( P_{q_1}(u) \leq \sum_{i=1}^{i=t} P_{q_i}(u) \) for \( \{q_1, ..., q_t\} \subseteq X^+ \). Therefore, using (I) and the fact that the sequence \((z_k)\) was defined inductively in terms of \((x_k)\) \( (z_k = \bigvee_{i=n_k-1}^{n_k} x_i, \ n_k \rightarrow \infty) \),

\[
0 \leq \|P_{z_k}(m)\| \leq \|P_{\bigvee_{i=n_k-1}^{n_k} x_i}(m)\| \leq \sum_{i=n_k-1}^{n_k} \|P_{x_i}(m)\| \longrightarrow 0 \text{ as } k \rightarrow \infty. \text{ Therefore,}
\]

(V) \[ \|P_{z_k}(m)\| \longrightarrow 0. \]

Now pick \( K_1 \in N \). Since \( P_{z_k}(m) \longrightarrow 0 \), \( P_{z_k}(|d_{K_1}|) \longrightarrow 0 \). Therefore, for \( \epsilon > 0 \) there exists \( K_2 > K_1 \) such that, if \( k \geq K_2 \) then \( \|P_{z_k}(|d_{K_1}|)\| < \frac{\epsilon}{2} \). In particular, \( \|P_{z_{K_2}}(|d_{K_1}|)\| < \frac{\epsilon}{2} \). Consider \( d_{K_2} \). Note that \( P_{z_k}(|d_{K_2}|) \longrightarrow 0 \), i.e. there exists \( K_3 > K_2 \) such that \( \|P_{z_{K_3}}(|d_{K_2}|)\| < \frac{\epsilon}{2} \). In general, for all \( i \in N \) there exists \((K_i)\) such that \( \|P_{z_{K_{i+1}}}(|d_{K_i}|)\| < \frac{\epsilon}{2} \). Now consider the sequence \((P_{z_{K_i}} - P_{z_{K_{i+1}}})\). We want to show that \((P_{z_{K_i}} - P_{z_{K_{i+1}}})\) is a disjoint sequence from \( \mathcal{O} \). Without loss of generality,
we may assume that \( i < j \). Therefore, \( i + 1 \leq j \) and \( i + 1 < j + 1 \). Recall from (III) that \( z_{k+1} \leq z_k \). Note that

\[
\left( P_{z_{K_i}} - P_{z_{K_{i+1}}} \right) \left( P_{z_{K_j}} - P_{z_{K_{j+1}}} \right) = P_{z_{K_i} \wedge z_{K_j}} - P_{z_{K_i} \wedge z_{K_{j+1}}} - P_{z_{K_{i+1}} \wedge z_{K_j}} + P_{z_{K_{i+1}} \wedge z_{K_{j+1}}}
\]

\[
= P_{z_{K_j}} - P_{z_{K_{j+1}}} - P_{z_{K_j}} + P_{z_{K_{j+1}}}
\]

\[
= 0.
\]

Consider, now, \( \| P_{z_{K_i}} - P_{z_{K_{i+1}}} (|d_{K_i}|) \| \). Note that

\[
\varepsilon - \frac{\varepsilon}{2} \leq \| P_{z_{K_i}} (|d_{K_i}|) \| - \| P_{z_{K_{i+1}}} (|d_{K_i}|) \| \leq \| P_{z_{K_i}} - P_{z_{K_{i+1}}} (|d_{K_i}|) \|.
\]

Thus we have contradicted the uniform exhaustivity of \( |K| \). Q.E.D.

The next theorem is a Banach lattice interpretation of a bi-sequential version of a fundamental theorem on uniform absolute continuity as stated in Theorem 1.1 of [2].

As indicated earlier, its proof uses the sliding hump technique of the previous theorem.

The corollary and theorem which follow are Banach lattice versions of the consequent corollaries in [2]. These three results constitute the principal new contributions of this chapter.

**Theorem 3.9** If \( (\nu_n) \) is a uniformly exhaustive sequence in \( X \) and \( (\mu_n) \) is a sequence in \( X \) so that \( \nu_n \ll \mu_n \) and

\[
P_x (\nu_n) \xrightarrow{k} 0 \text{ as } P_x (\mu_k) \xrightarrow{k} 0 \text{ for each } n,
\]

then \( \nu_n \ll \mu_n \) uniformly in \( n \).
PROOF. Suppose \((\nu_n)\) is a uniformly exhaustive sequence in \(X\) and \((\mu_n)\) is a sequence in \(X\) so that \(\nu_n \ll \mu_n\) and

\[ P_{x_k}(\nu_n) \xrightarrow{k} 0 \text{ as } P_{x_k}(\mu_k) \xrightarrow{k} 0 \text{ for each } n. \]

Deny the conclusion. Let \(\epsilon > 0\) so that if \(\delta > 0\) then there is a positive integer \(n\) and an element \(x_n\) in \(X^+\) so that \(\|P_{x_n}(\mu_n)\| < \delta\) and \(\|P_{x_n}(\nu_n)\| > 2\epsilon\). Therefore we may (and do) suppose that

\[ \|P_{x_n}(\mu_n)\| < \frac{1}{n^2} \text{ and } \|P_{x_n}(\nu_n)\| > 2\epsilon, \quad n \in \mathbb{N}. \]

(Note that \((P_{x_n}(\nu_k)) \xrightarrow{n} 0 \text{ for each } k.) \) Pass to a subsequence \((y_i)\) of \((\nu_n)\) and obtain \((y_i)\), and \((x_i)\) so that \(\|P_{x_i}(y_k)\| < \frac{1}{n^2}\) and the following two inequalities hold:

\[ \text{(I)} \quad \sum_{i=k+1}^{\infty} \|P_{x_i}(y_k)\| < \frac{\epsilon}{2k+1} \]

\[ \text{(II)} \quad \|P_{x_i}(y_k)\| > 2\epsilon \quad \text{for all } i. \]

Using the same construction as that in part (ii) of the previous theorem, we continue inductively to manufacture a sequence \((z_k)\) from \(X^+\) and a sequence \((d_k)\) \((d_k) \subset (y_i) \subset (\nu_n))\) such that

\[ \text{(III)} \quad z_{k+1} \preceq z_k \]

and

\[ \text{(IV)} \quad \|P_{x_k}(d_k)\| > \epsilon. \]

Note that since \(\|P(|z|)\| = \|P(z)\|\) we can replace \(|d_k|\) by \(d_k\) in the previous proof, part (ii), for each \(k\). Therefore, using (I) and the fact that the sequence \((z_k)\) was...
defined inductively in terms of \((x_k), (z_k \leq n_k \sqrt[n_k]{x_k}, n_k \rightarrow \infty), 0 \leq \|P_{x_k}(y_k)\| \leq \|P_{x_{n_k}}(y_{n_k})\| \leq \sum_{i=n_{k-1}}^{n_k} \|P_{x_i}(y_i)\| \rightarrow 0\) as \(k \rightarrow \infty\).

Therefore for each \(i\),

\[(V) \quad \|P_{x_i}(y_i)\| \rightarrow 0.\]

Now pick \(K_1 \in N\). Since

\[P_{x_k}(y_k) \rightarrow 0, \quad P_{x_k}(d_{K_1}) \rightarrow 0.\]

Therefore for \(\epsilon > 0\) there exists \(K_2 > K_1\) such that if \(k \geq K_2\) then \(\|P_{x_k}(d_{K_1})\| < \frac{\epsilon}{2}\).

In particular, \(\|P_{x_{K_2}}(d_{K_1})\| < \frac{\epsilon}{2}\).

Now consider \(d_{K_2}\). Note that \(P_{x_k}(d_{K_2}) \rightarrow 0\), i.e. there exists \(K_3 > K_2\) such that \(\|P_{x_{K_3}}(d_{K_2})\| < \frac{\epsilon}{2}\). In general, for all \(i \in N\) there exists \((K_i)\) such that \(\|P_{x_{K_{i+1}}}(d_{K_i})\| < \frac{\epsilon}{2}\). Now, consider the sequence \((P_{x_{K_i}} - P_{x_{K_{i+1}}})\). As shown in the previous proof, \((P_{x_{K_i}} - P_{x_{K_{i+1}}})\) is a disjoint sequence from \(\mathcal{O}\). Consider, now, \(\|P_{x_{K_i}} - P_{x_{K_{i+1}}}(d_{K_i})\|\). One obtains

\[\epsilon - \frac{\epsilon}{2} \leq \|P_{x_{K_i}}(d_{K_i})\| - \|P_{x_{K_{i+1}}}(d_{K_i})\| \leq \|P_{x_{K_i}} - P_{x_{K_{i+1}}}(d_{K_i})\|.\]

Thus we have contradicted the uniform exhaustivity of \((\nu_n)\). Q.E.D.

**Corollary 3.10** Suppose that \(K \subset X^+\) and \(H\) is uniformly exhaustive.

(i) If \(\nu << \mu\) uniformly for \(\mu \in K\) whenever \(\nu \in H\), then \(\nu << \mu\) uniformly for \((\nu, \mu) \in H \times K\).

(ii) If \(K\) is compact and \(\nu << \mu\) for \((\nu, \mu) \in H \times K\), then \(\nu << \mu\) uniformly for \((\nu, \mu) \in H \times K\).
PROOF. (i) Suppose that the uniform absolute continuity fails in $H \times K$. Let $(\mu_n)$ be a sequence in $K$, and let $(x_n)$ be a sequence in $X^+$ so that $\|P_{x_n}(\mu_n)\| > \epsilon$ and $\|P_{x_n}(\mu_n)\| \to 0$. Certainly the second condition of Theorem 3.3 must fail for the sequence $(\nu_n, \mu_n, x_n)$ of triples. Therefore there exists $p \in \mathbb{N}$ and a subsequence $(n_k)$ of positive integers and $\delta > 0$ so that $\|P_{x_{n_k}}(\nu_p)\| > \delta$ for each $k$. However, this contradicts the hypothesis since $\nu_p << \mu$ uniformly for $\mu \in K$.

(ii) In view of (i), it clearly suffices to say that if $\nu \in H$ such that $\nu << \mu$ uniformly for $\mu \in K$ then the result follows. Suppose, to the contrary, that this uniformity does not hold. Let $\nu \in H$, let $(\mu_n)$ be a sequence from $K$, let $\epsilon > 0$, and let $(x_n)$ be a sequence from $X^+$ so that $\|P_{x_n}(\mu_n)\| \to 0$ and $\|P_{x_n}(\nu)\| > \epsilon$ for each $n$. Certainly $(\mu_n)$ must cluster at some point $\mu$ in $K$. Therefore, suppose $\|\mu_n - \mu\| \to 0$. Then $\|P_{x_n}(\mu_n) - P_{x_n}(\mu)\| \leq \|\mu_n - \mu\| \to 0$. Since $P_{x_n}(\mu_n) \to 0$, $P_{x_n}(\mu) \to 0$. Thus, $P_{x_n}(\nu) \to 0$. Since we have reached a contradiction, the result follows. Q.E.D.

The appropriate Banach lattice definition yields one more lattice version of a theorem from the Bator, Bilyeu and Lewis paper [2].

Definition 3.11 Suppose $H$ and $K$ are subsets of $X$. One says that $H$ is continuous with respect to $K$, (denoted by $H << K$) if for each sequence $(P_{x_i})$ from $\mathcal{O}$ such that $(P_{x_i}(k_{i})) \to 0$ for some sequence $(k_{i})$ from $K$, then $(P_{x_i}(h_i)) \to 0$ for each $h \in H$.

Note that $h << k$ for each $(h,k) \in H \times K$.

Theorem 3.12 Suppose $H,K \subseteq X$ and that $K$ is relatively compact. The following are equivalent:
(i) $H \ll K$

(ii) $\nu \ll \mu$ for $(\nu, \mu) \in H \times K$

In addition, if $H$ is uniformly exhaustive, then each of (i) and (ii) imply

(iii) $\nu \ll \mu$ uniformly for $(\nu, \mu) \in H \times K$.

Finally (iii) implies each of (i) and (ii).

**PROOF.** Suppose that (i) holds and let $\mu \in \bar{K}$. Let $(P_{x,t})$ be a sequence from $O$ so that $P_{x,t}(\mu) \to 0$. Let $(\mu_i)$ be a sequence from $K$ so that $\|\mu_i - \mu\| \to 0$. Therefore, $\|P_{x,t}(\mu_i)\| = \|P_{x,t}(\mu_i)\| \to 0$. Hence, $(P_{x,t}(h)) \to 0$ for each $h \in H$.

Conversely, suppose (ii) holds. Let $(P_{x,t})$ be a sequence from $O$, and let $(\mu_i)$ be a sequence from $K$ so that $\|P_{x,t}(\mu_i)\| \to 0$. Without loss of generality, suppose $(\mu_i) \to \mu \in \bar{K}$. Then, $\|P_{x,t}(\mu)\| \to 0$. Since $\nu \ll \mu$ for each $\nu \in H$, $\|P_{x,t}(\nu)\| \to 0$ for each $\nu \in H$. Hence, $H \ll K$.

If $H$ is uniformly exhaustive, then the previous corollary shows that (ii) implies (iii).

Finally, suppose that $\mu \in \bar{K}$, and let $\epsilon > 0$. Choose $\delta > 0$ so that $\|P_x(\nu)\| < \epsilon$ whenever $\|P_x(\xi)\| < \delta$ and $(\nu, \xi) \in H \times K$. If $\|P_x(\mu)\| < \frac{\delta}{2}$, then choose $\xi \in K$ so that $\|P_x(\xi)\| < \delta$. Thus, $\|P_x(\nu)\| < \epsilon$ for all $\nu \in H$, and (iii) implies (ii). Q.E.D.
CHAPTER 4

EXHAUSTIVITY AND $l^\infty$

In this chapter a connection between the existence of non-exhaustive elements and the presence of $l^\infty$ in certain Banach lattices is verified. The presentation makes strong use of Theorem 2.5 of [4]. For the convenience of the reader, the proof of this result is presented in detail. Again, $X$ is assumed to be a $\sigma$-complete Banach lattice.

**Definition 4.1** If $(P_x)$ is a pairwise disjoint sequence from $\mathcal{O}$, $A$ is a non-empty subset of $N$, and $x \in X$, define $S(x, (x_i)) (A) = \bigvee_{i \in A} P_{x_i}(x)$, and define $S(\emptyset)$ to be 0. If $x$ and $(x_i)$ are understood, we simply denote $S(x, (x_i))$ by $S$.

**Theorem 4.2** (*4, Theorem 2.5*) The element $x$ is exhaustive if and only if each mutually disjoint infinite sequence $E = (a_n)$ in $X^+$ contains an infinite subsequence $E'$ so that $S$ is countably additive on the $\sigma$-algebra $\Sigma$ of all subsets of $E'$.

**Proof.** First, suppose that $x$ is exhaustive, $E = (a_n)$ is as in the hypothesis, and $S$ is as defined above with respect to the sequence $(P_{a_n})$ and $x$. Let $A_1, A_2, \ldots$ be a “partition” of $(a_n)$ into an infinite number of mutually disjoint infinite subsequences. (For example, separate the sequence into evens and odds. Relabel and separate again into evens and odds, etc.) Next let $\|S\| (A_i) = \sup \{\|S(B)\| : B \subseteq A_i\}$. Since $x$ is exhaustive, $\|S\| (A_i) \rightarrow 0$. Therefore, we may (and shall) assume $\|S\| (A_i) < 1$. Relabel $A_1$ as $A_{11}$. Partition $A_{11}$ into an infinite number of mutually disjoint infiniti-
nite subsequences $A_{21}, A_{22}$, such that $\|S\|(A_{21}) < \frac{1}{2}$ and $A_{21}$ does not contain the first element of $A_{11}$. Partition $A_{21}$ into an infinite number of mutually disjoint infinite subsequences $A_{31}, A_{32}$, such that $\|S\|(A_{31}) < \frac{1}{3}$ and $A_{31}$ does not contain the first element of $A_{21}$. Continue this process inductively, obtaining an infinite sequence $A_{11}, A_{21}, A_{31}, \ldots$ of subsequences of $E$ such that $\|S\|(A_{n1}) < \frac{1}{n}$, and $A_{n1}$ does not contain the first element of $A_{ki}$ if $i > k$. Let $b_i$ be the first term of $A_{ii}$ for $i = 1, 2, \ldots$. From the finite additivity of $S$ and the construction above, it follows that $S$ is countably additive on the collection of all subsets of $\{b_i : i = 1, 2, \ldots\}$.

(Consider $\|S\left(\bigcup_{i=1}^{\infty} (b_i)\right) - \sum_{i=1}^{n} S(b_i)\| = \|S\left(\bigcup_{i=n+1}^{\infty} b_i\right)\|$, where $A_{n+1} \supseteq \bigcup_{i=n+1}^{\infty} b_i$, and $\|S\left(\bigcup_{i=n+1}^{\infty} b_i\right)\| < \|S\|(A_{n+1}) < \frac{1}{n+1}$.)

By way of contradiction, conversely suppose that $S$ is countably additive on appropriate subsequences and $x$ is not exhaustive. Then there is a disjoint infinite sequence $(P_{x_i})$ from $\mathcal{O}$ and a positive number $\epsilon$ so that $\|P_{x_i}(x_i)\| > \epsilon$. Let $E = (x_i)$. But then $\|S\{x_i\}\| = \|P_{x_i}(x_i)\| > \epsilon$ for each $i$ and $S$ cannot be countably additive on any infinite subsequence of $E$. Therefore we have the desired contradiction.

Q.E.D

The set function $S$ and a classical theorem from Schaefer [18] yield new results about the structure of certain Banach lattices.

**Definition 4.3** A normed linear lattice $X$ is said to have **order continuous norm** if, for each upwards directed family $\{x_d\}$ which has an upper bound, $\{x_d\}$ is Cauchy in $X$.

**Theorem 4.4** If $X$ is a $\sigma$-order complete Banach lattice with order continuous norm
then every element is exhaustive.

Proof. Suppose that $X$ is $\sigma$-order complete with order continuous norm. But suppose there exists an $x \in X^+$ such that $x$ is not exhaustive. Then for $\epsilon > 0$ there exists $(P_{x_i})$, a pairwise disjoint sequence from $\mathcal{O}$, so that $\|P_{x_i}(x)\| > \epsilon$ for each $i$. Since $P_{x_i}(x) \leq x$ for each $i$, the $\sigma$-order completeness of $X$ implies then $\bigvee_{i \in A} P_{x_i}(x)$ exists for each non-empty subset of $N$. For such a set $A$ define $S(A) = \bigvee_{i \in A} P_{x_i}(x)$ and define $S(\emptyset) = 0$. It has been noted that $S$ is finitely additive. Consequently, if $A \neq B$ it follows that $\|S(A) - S(B)\| = \|S(A \setminus B) + S(B \cap A) - \{S(B \setminus A) + S(A \setminus B)\}\| = \|S(A \setminus B) - S(B \setminus A)\| > \epsilon$. Define $z_n = \bigvee_{i=n}^{\infty} P_{x_i}(x) = S(A_n)$ for each $n \in N$, where $A_i = \{i, i+1, \ldots\}$. Note that $(z_n)$ is a decreasing sequence in $X^+$ and $\|z_n - z_{n+1}\| = \|S(A_n) - S(A_{n+1})\| > \epsilon$ for all $n \in N$. Choose any $\delta > 0$ and define $v_n = \frac{1+\delta}{\epsilon} (z_n - z_{n+1})$. The sequence $(v_n)$ satisfies the hypothesis of Theorem II.5.13 [18]. Thus there exists a disjoint normalized sequence $(y_n)$ in $X^+$ which is majorized by $x_0 = \frac{1+\delta}{\epsilon} z_1$. Moreover, since $\sum_{n=1}^{k} y_n \leq x_0$ for all $k \in N$, and since $X$ is $\sigma$-order complete, $w = \bigvee_{n=1}^{\infty} \alpha_n y_n$ exists in $X$ whenever $\alpha = (\alpha_n) \in l^{\infty}$ and $\alpha_n \geq 0$ for each $n$. Further, $\bigvee_{n=1}^{k} \alpha_n y_n \longrightarrow w$ and $\sum_{n=1}^{k} \alpha_n y_n = \bigvee_{n=1}^{k} \alpha_n y_n$. Thus, $\left(\sum_{n=1}^{k} \alpha_n y_n\right) \longrightarrow w$. But, this is impossible if $\liminf |\alpha_n| > 0$, and we have a contradiction. Q.E.D.

**Corollary 4.5** If $X$ is a $\sigma$-order complete Banach lattice with order continuous norm and $x \in X^+$, then each mutually disjoint infinite sequence $E = (a_n)$ in $X^+$ contains an infinite subsequence $E'$ so that $S_{(x,(a_n))}$ is countably additive on the $\sigma$-algebra $\Sigma$ of all subsets of $E'$.
Finally, the set function $S$ can be used in conjunction with the Diestel-Faires Theorem [11] to detect the presence of $l^\infty$ in certain Banach lattices.

**Theorem 4.6** Suppose $X$ is a $\sigma$-order complete Banach lattice. If $x$ is a positive and non-exhaustive element in $X$, then there is an embedding (topological isomorphism) of $l^\infty$ into $X$. Conversely, if there is a topological and lattice isomorphism $\phi : l^\infty \hookrightarrow X$, then $X$ contains an element which is not exhaustive.

**Proof.** Suppose that $X$ is $\sigma$-order complete and suppose there exists an $x \in X$ such that $x$ is not exhaustive. Then there exists a pairwise disjoint sequence $(P_{x_i})$ from $\mathcal{O}$ and a $\epsilon > 0$ so that $\|P_{x_i}(x)\| > \epsilon$ for each $i$. Since $P_{x_i}(x) \leq x$ for each $i$, the $\sigma$-order completeness of $X$ implies that $\bigvee_{i \in A} P_{x_i}(x)$ exists for each non-empty subset of $\mathbb{N}$.

For such a set $A$ define $S(A) = \bigvee_{i \in A} P_{x_i}(x)$. Define $S(\emptyset) = 0$. Consequently, if $A \neq B$ it follows that $\|S(A) - S(B)\| = \|S(A \setminus B) + S(B \cap A) - \{S(B \setminus A) + S(A \cap B)\}\| = \|S(A \setminus B) - S(B \setminus A)\| = \|S(A \setminus B) + S(B \setminus A)\| > \epsilon$. Let $Y = \{A : A \subseteq \mathbb{N}\}$.

Then $S : Y \rightarrow X$ is a bounded vector measure which is not strongly additive. By the Diestel-Faires Theorem [11] there is a topological isomorphism $T : l^\infty \rightarrow X$ and a sequence $(E_n)$ of disjoint members of $Y$ such that $T(e_n) = S(E_n)$. Consequently, $S(Y)$ contains the image under $T$ of all the $\{0,1\}$-valued sequences in $l^\infty$.

Conversely, suppose that $T : l^\infty \rightarrow X$ is a lattice isomorphism. Let $y = (1,1,1,\ldots) \in l^\infty$. Certainly $T(y) > 0$ since $y > 0$, and $T(e_i) > 0$. Furthermore, $T(e_i) \wedge T(e_j) = T(e_i \wedge e_j) = T(0) = 0$, if $i \neq j$. Let $x_i = T(e_i)$, $i \in \mathbb{N}$. As noted above, $(x_i)$ is pairwise disjoint and $\|P_{x_i}(T(y))\| = \bigvee_n n x_i \wedge T(y) \| > \|x_i \wedge T(y)\| = \epsilon$. \hfill $\blacksquare$
\[ \|T(e_i) \land T(y)\| = \|T(e_i \land y)\| = \|T(e_i)\| \geq c \text{ for some } c, \text{ since } T \text{ is an isomorphism.} \]

\((c\|x\| \leq \|T(x)\| \leq C\|x\|.) \text{ Therefore, } T(y) \text{ is not exhaustive.} \quad \text{Q.E.D.} \]
CHAPTER 5

EXHAUSTIVITY AND A GENERALIZATION OF GATEAUX DIFFERENTIABILITY

As mentioned in the introduction, Bilyeu and Lewis [5] extended the Bartle, Dunford, and Schwartz theorem by showing that a subset $K$ of $ca(\Sigma)$ is relatively weakly compact if and only if there is an element $\mu \in ca(\Sigma)$ so the $D(\mu, \nu)$ exists uniformly for $\nu \in K$. While examples are presented in this same paper which show that uniform Gateaux differentiability and relative weak compactness are not equivalent for Banach spaces in general, the authors do show that if $K$ is a relatively compact subset of $X$ and $D(x,y)$ exists for each $y \in K$ then $D(x,y)$ exists uniformly for $y \in K$ [5, Lemma 3.1]. This result is extended to the setting of Banach lattices in this section. In fact, this result is strengthened in this context. The techniques used make strong use of Theorem 2.4 of [4]. An expanded version of the proof of this theorem will be presented in this chapter.

Many of these calculations in this proof are made possible by (ii), a result which is closely akin to some of the major theorems in section 5 of Chapter 2 of Schaefer [18]. The reader may also compare (iii) with characterizations of strong additivity given in Chapter I of Diestel and Uhl [11] as well as Proposition II.1.9 of Schaefer [18].

Definition 5.1 A subset $A$ of a vector lattice $E$ is called solid if $x \in A, y \in E,$ and $|y| \leq |x|$ imply $y \in A$. A solid vector subspace of $X$ is called an ideal.
Definition 5.2 If $A$ is a subset then $\hat{A}$ denotes the solid hull of $A$, i.e. $\hat{A} = \{y \in X : |y| \leq |x| \text{ for some } x \in A\}$

Theorem 5.3 i) A subset $K$ is uniformly exhaustive if and only if each disjoint sequence in $\hat{K}$ converges to zero. Furthermore, if $I$ is an ideal in $X$ and $K$ is a subset of $I$ so that $P_{x_i}(k) \to 0$ uniformly for $k \in K$ whenever $(x_i)$ is a disjoint sequence from $I^+$, then $K$ is uniformly exhaustive in $X$.

ii) A positive element $k$ of $X$ is exhaustive if and only if the norm is countably order continuous on the order interval $[0, k]$. Consequently, the norm in $X$ is order continuous if and only if each positive element of $X$ is exhaustive.

iii) A positive element $k$ of $X$ is exhaustive if and only if the series $\sum_{i=1}^{\infty} P_{x_i}(k)$ converges unconditionally whenever $(P_{x_i})$ is a pairwise disjoint sequence from $O$. Furthermore, if $M$ is a maximal pairwise disjoint subset of $O$, then $\sum_{M} P_{x}(k) = k$.

iv) If the subset $K$ of $X$ is uniformly exhaustive then there is a positive element $m \in X$ such that $K$ is uniformly continuous with respect to $m$.

v) If $K$ is a relatively norm compact subset of $X$ such that each element in $K$ is exhaustive, then $K$ is uniformly exhaustive.

vi) If $K$ is a subset of the exhaustive elements of $X$ then $K$ is continuous with respect to some exhaustive element in $X$ if and only if each pairwise disjoint subset of $\hat{K}$ is countable.

PROOF of i). First, suppose that $K$ is uniformly exhaustive and that $(x_i)$ is a disjoint sequence in $\hat{K}$, i.e. $|x_i| \wedge |x_j| = 0$ if $i \neq j$. Suppose further that $y_i \in K$ and $|x_i| \leq |y_i|$. 


Then, since $P_u(u) = u$ and $P_u$ is monotone on each $u \in X^+$, it follows that $0 \leq |x_i| = P_{|x_i|}(|x_i|) \leq P_{|x_i|}(|y_i|)$. Also, recall $\|P_{|x_i|}(y_i)\| = \|P_{x_i}(y_i)\|$ and $P_{x_i}P_{x_j} = P_{x_i \wedge x_j}$.

Since $|x_i| \wedge |x_j| = 0$, $P_{x_i}P_{x_j} = 0$, i.e. $(P_{|x_i|})$ is a disjoint sequence from $O$. Thus, the uniform exhaustivity of $K$ insures that $\|P_{|x_i|}(y_i)\| \to 0$. Consequently, the monotonicity of the lattice norm (i.e. $\|x_i\| = \|x_j\| = \|P_{|x_i|}(x_i)\| \leq \|P_{x_i}(y_i)\| \to 0$) guarantees that $\|x_i\| \to 0$.

Now, conversely, suppose that each disjoint sequence in $\hat{K}$ converges to $0$, $(P_{x_i})$ is a disjoint sequence from $O$, and suppose that $(u_i)$ is a sequence in $K$. Then, $|x_i| \wedge |x_j| = P_{|x_i|}(|x_i| \wedge |x_j|) = P_{|x_i|}P_{|x_j|}(|x_i| \wedge |x_j|) = 0$, since $(P_{x_i})$ is disjoint. Hence, $(x_i)$ is a pairwise disjoint sequence from $X^+$. Now suppose that $i \neq j$, and let $z = P_{|x_i|}(u_i) \wedge P_{|x_j|}(u_j)$. Recall that $z = P_z(z)$. Furthermore, $P_z(z) = P_{|x_i|}(u_i) \wedge P_{|x_j|}(u_j)(z) = P_{P_{|x_i|}(u_i)}(z) = P_{|x_i|}(P_{x_i}(z)) = P_{|x_i|}(z) = P_0(z) = 0$. Thus, $(P_{x_i}(u_i))$ is a disjoint sequence. But, $P_{x_i}(|u_i|) \in \hat{K}$ for each $i$, since the definition of $P_x$ ensures that $P_{x_i}(|u_i|) \leq |u_i|$. Therefore, $\|P_{x_i}(u_i)\| = \|P_{x_i}(|u_i|)\| \to 0$. Hence, it follows that $K$ is uniformly exhaustive.

We pause here to make a comment on strategy. Note that if $K$ is not uniformly exhaustive then there is a disjoint sequence $(P_{x_i})$ and a sequence $(u_i)$ from $K$ such that $\|P_{x_i}(u_i)\| \not\to 0$. Hence, to prove uniform exhaustivity we will show that $\|P_{x_i}(u_i)\| \to 0$ for each disjoint sequence $(P_{x_i})$ and sequence $(u_i)$ from $K$.

Therefore, we next suppose that $I$ is an ideal in $X$, $K$ is a subset of $I$ satisfying the hypothesis of the final statement in (i), and $(\xi_i)$ (or $P_{\xi_i}$) is a disjoint sequence
in $X^+$ (or $\mathcal{O}$). Let $(u_i)$ be an arbitrary sequence from $K$. Then $\xi_i \land |u_i| \leq |u_i|$. Therefore, $\xi_i \land |u_i| \in I_+$ for each $i$. Now $[\xi_i \land (|u_i|)] \land [\xi_j \land (|u_j|)] = [\xi_i \land \xi_j] \land [\xi_i \land |u_i|] \land [\xi_j \land |u_j|] = 0 \land (|u_i| \land |u_j|) = 0$. Hence, $(\xi_i \land |u_i|)$ is a disjoint sequence. Also, $P_{\xi_i}(|u_i|) = \bigvee_n n \xi_i \land |u_i| = \bigvee_n (n \xi_i \land n |u_i| \land |u_i|) = \bigvee_n (n (\xi_i \land |u_i|) \land |u_i|) = P_{\xi_i \land |u_i|}(|u_i|)$. Thus, $\|P_{\xi_i}(u_i)\| = \|P_{\xi_i}(|u_i|)\| = \|P_{\xi_i \land |u_i|}(|u_i|)\| \to 0$, and it follows that $K$ is uniformly exhaustive. This completes the proof of (i).

Before we begin the proof of (ii) we will cite a relevant definition from Schaefer [18].

**Definition 5.4** $D$ is **upwards directed** if and only if for any $f, g \in D$, there exists an $h \in D$ such that $h \geq f \lor g$.

**PROOF of (ii).** Suppose, first, that the norm is countably order continuous on $[0, k], \ k \in X^+$. Furthermore, suppose that $k$ is not exhaustive. Then, there is a disjoint sequence $(P_{x_i})$ from $\mathcal{O}$ and an $\epsilon > 0$ such that $\|P_{x_i}(k)\| > \epsilon$ for all $i$. Let $l_n = P_{\bigvee_{i=1}^n} (k)$ for $n = 1, 2, \ldots$, and let $l = \bigvee_n l_n$. Now the definition of the projection operators in $\mathcal{O}$ guarantees that $l_n \leq k$ for all $n$ since $l_n = P_{\bigvee_{i=1}^n} (k) \leq k$. Thus $l \leq k$. Due to the order continuity of the norm on $[0, k]$, we have $\|l_n - l\| \to 0$. Therefore, $(l_n)$ is a Cauchy sequence. However, recall that if $(P_{x_i})$ is a disjoint sequence then $(x_i)$ is disjoint. Using the definition of $(l_n)$ and properties of $P$ we obtain the following: $l_n = P_{\bigvee_{i=1}^n} (k) = P_{\sum_{i=1}^n z_i} (k) = \sum_{i=1}^n P_{z_i} (k)$. Consequently, $\|l_{n+1} - l_n\| = \|\sum_{i=1}^{n+1} P_{z_i} (k) - \sum_{i=1}^n P_{z_i} (k)\| = \|P_{z_{n+1}} (k)\| > \epsilon$. Hence, $(l_n)$ is not Cauchy. Thus we have a contradiction, and $k$ is exhaustive.
Conversely, suppose that \{k\} is exhaustive and that the norm is not countably order continuous on \([0, k]\). Then, (following Schaefer [18], p.94), there is an \(\epsilon > 0\), and a decreasing sequence \((z_n)\) from \([0, k]\), so that \(|z_n - z_{n+1}| > \epsilon\) for each \(n\). Let \(\delta\) be a positive number, and put \(\nu_n = \frac{1+\delta}{\epsilon}(z_n - z_{n+1})\). Thus, \(\nu_n \leq \frac{1+\delta}{\epsilon} k\) for each \(n\) and \(\sum_1^\infty \nu_n = \frac{1+\delta}{\epsilon}(z_1 - z_{n+1}) \leq \frac{1+\delta}{\epsilon} (k)\). Note that \(|\nu_n| \geq \frac{(1+\delta)}{\epsilon}(\epsilon) > (1+\delta)\). Also note that \(|\sum_1^n \nu_n| \leq \frac{(1+\delta)}{\epsilon} ||k||\). Hence, the hypothesis of the lemma on p.92 of Schaefer [18] is satisfied. Consequently, by this lemma, there is a disjoint sequence \((x_i)\) from \(X^+\) and a subsequence \((\nu_n)\) of \((\nu_n)\) so that \(x_i \leq \nu_n\) and \(|x_i| \geq 1\) for each \(i\). Now recall that \(k\) is exhaustive and consider \([0, \frac{1+\delta}{\epsilon} k]\). If \(z \in [0, \frac{1+\delta}{\epsilon} k]\) and \((P_i)\) is a disjoint sequence, then \(|P_i(z)| \leq |P_i\left(\frac{1+\delta}{\epsilon} k\right)| = \frac{1+\delta}{\epsilon} |P_i(k)| \rightarrow 0\). Therefore, \([0, \frac{1+\delta}{\epsilon} k]\) is uniformly exhaustive. Since \(0 < x_i \leq \nu_n < \frac{1+\delta}{\epsilon} k\) and \((P_{x_i})\) is disjoint, we have that \(|x_i| = |P_{x_i}(x_i)| \rightarrow 0\), i.e. we have arrived at a contradiction.

We now suppose that each positive element is exhaustive and let \((u_n)\) be a decreasing sequence from \(X\) with infimum 0. (See 5.10(d) of Schaefer, [18]). Then \(\{u_n\} \subseteq [0, u_1]\) and by the first part of ii) the norm is countably order continuous on \([0, u_1]\). Therefore, the sequence norm converges to 0. By Theorem 5.10(a) of Schaefer [18], \(X\) is order continuous.

Conversely, if the norm on \(X\) is order continuous, \(k \in X^+\) and \((P_{x_i})\) is a disjoint sequence from \(\mathcal{O}\), then \(\sum_{i=1}^n P_{x_i}(k) = \bigvee_{i=1}^n P_{x_i}(k) + \bigwedge_{i=1}^n P_{x_i}(k) = \bigvee_{i=1}^n P_{x_i}(k) \leq k\) for each \(n\). Then, \(\sum_{i=1}^\infty P_{x_i}(k)\) is Cauchy in \(X\). Therefore, for \(\epsilon > 0\) there exists \(N\) such that if \(n \geq N\), then \(|\sum_{i=1}^{n+1} P_{x_i}(k) - \sum_{i=1}^{n} P_{x_i}(k)| = |P_{x_{n+1}}(k)| < \epsilon\), i.e. \(|P_{x_i}(k)| \rightarrow 0\) or
k is exhaustive.

The following lemma is helpful in the proof of (iii) of Theorem 5.3.

**Lemma 5.5** If $x_i \in X^+$ and $1 \leq i \leq n$, then $P_{\bigvee_{i=1}^n x_i}(k) = \bigvee_{i=1}^n P_{x_i}(k)$.

**Proof.** Suppose $x_i \in X^+$. Fix $n$ and consider $m \geq n$. Note that $n(x_1 \lor x_2) \land k = (nx_1 \lor nx_2) \land k = (nx_1 \land k) \lor (nx_2 \land k) \leq (nx_1 \land k) \lor (nx_2 \land k) \lor P_{x_1}(k)$. Therefore, $P_{x_1 \lor x_2}(k) \leq P_{x_1}(k) \lor P_{x_2}(k)$. Now fix $m$ and consider $n \geq m$. In this case, note that $(nx_1 \land k) \lor (nx_2 \land k) \leq (nx_1 \land k) \lor (nx_2 \land k) = (nx_1 \lor nx_2) \land k \leq P_{x_1 \lor x_2}(k)$. Therefore, $P_{x_1}(k) \lor (nx_2 \land k) \leq P_{x_1 \lor x_2}(k)$, and $P_{x_1}(k) \lor P_{x_2}(k) \leq P_{x_1 \lor x_2}(k)$. Thus $P_{x_1 \lor x_2}(k) = P_{x_1}(k) \lor P_{x_2}(k)$. A simple induction argument finishes the proof. Q.E.D.

**Proof of (iii).** First, suppose that $\sum P_{x_i}(k)$ converges unconditionally whenever $(P_{x_i})$ is a disjoint sequence from $\mathcal{O}$. Then, clearly $P_{x_i}(k) \to 0$ and $k$ must be exhaustive.

Conversely, suppose that $\{k\}$ is exhaustive and that $(P_{x_i})$ is a disjoint sequence from $\mathcal{O}$. Suppose further that $(P_{x_i})$ is a permutation of this sequence so that $\sum P_{x_i}(k)$ fails to converge. Then there is an $\varepsilon > 0$ and a sequence $(n_i', n_i'')$ of pairs of integers such that $n_i' \leq n_i'' < n_i'+1$ and $\|\sum_{j=n_i'}^{n_i''} P_{x_j}(k)\| > \varepsilon$ for all $i$. Recall that if $(P_{x_i})$ is disjoint so is $(x_i)$. Define $y_i = \bigvee_{j=n_i'}^{n_i''} x_j = \sum_{j=n_i'}^{n_i''} x_j$ and put $T_i = P_{y_i}$ for all $i$. Then the pairwise disjointness of $(P_{x_i})$ (and consequently of $(x_i)$) implies that $(T_i)$ is a disjoint sequence from $\mathcal{O}$. Again repeated applications of the properties of
P show that \( \|T_i(k)\| = \|P_{\sum_{j=1}^{n_i'} x_j}^{n_i'}(k)\| = \|P_{\sum_{j=1}^{n_i'} x_j}^{n_i'}(k)\| = \|\sum_{j=1}^{n_i'} x_j P_x(k)\|. \) Consequently, \( \|\sum_{j=1}^{n_i'} x_j P_x(k)\| = \|T_i(k)\| \to 0 \) as \( i \to \infty \) and we have a contradiction.

We now suppose that \( M \) is a maximal pairwise disjoint subset of \( \mathcal{O} \) and that the non-negative element \( k \) is exhaustive. Then \( M(n) = \{ P \in M : \|P(k)\| > \frac{1}{n} \} \) must be finite. (Otherwise there exists a disjoint sequence \( (P_i) \) such that \( P_i(k) \to 0 \) and \( k \) is not exhaustive.) Therefore, \( \{ P \in M : \|P(k)\| > 0 \} = \bigcup_n M(n) \) must be countable. Let \( (P_{x_i}) \) be an enumeration of this set. Now it is possible that \( \sum \|x_i\| \) does not converge. However, note that for \( k \in \mathbb{N}, \sum_{m=1}^{\infty} \frac{k}{n} x \wedge u = \sum_{n=1}^{\infty} nx \wedge u \), or \( P_{\frac{k}{n} x} = P_x \). Therefore, for each \( x \in X^+ \) and each \( k \in \mathbb{N} \) we may pick an appropriate factor and assume that \( \sum \|x_i\| < \infty \). By the preceding paragraphs of the proof, \( \sum P_{x_i}(k) \) is unconditionally convergent. Further, by (ii) the countable order continuity of the norm on \([0, k]\) ensures that since \( \sum_{i=1}^{n} P_{x_i}(k) = \sum_{i=1}^{n} \frac{k}{n} P_x(k) \),

\[
\bigvee_{i=1}^{n} P_{x_i}(k) \to \bigvee_{i=1}^{\infty} P_{x_i}(k) = \sum_{i=1}^{\infty} P_{x_i}(k) \text{ as } n \to \infty.
\]

Let \( y = \sum_{i=1}^{\infty} P_{x_i}(k) \). Since \( \bigvee_{i=1}^{\infty} P_{x_i}(k) \to y \) as \( n \to \infty \) and \( P_{x_i}(k) \leq k \) for each \( n \), it follows that \( y \leq k \). We assert that \( y = k \). In order to obtain this equality we shall assume that \( y < k \) and use the maximality of \( M \) to obtain a contradiction. To this end, consider the case where \( P_u \in M \), and \( P_u(k) = 0 \). Note that \( 0 \leq u \wedge (k - y) \leq nu \wedge k \). Therefore, if \( P_u(k) = 0 \) then \( u \wedge (k - y) = 0 \). Hence, \( P_u \) and \( P_{k-y} \) are disjoint. Now we consider the case where \( P_w \in M \), and \( P_w(k) > 0 \). Note that \( P_w(k) \leq y, -y \leq -P_w(k) \), and \( k - y \leq k - P_w(k) \). Then, \( 0 \leq w \wedge (k - y) \leq w \wedge (k - P_w(k)) \). But, by a property of
\[ P, w \land (k - P_w(k)) = 0. \] Therefore, \( P_w P_{k-y} = P_0 = 0. \) Consequently, \( P_{k-y} \) is disjoint from each member of \( M \) in either case. The maximality of \( M \) forces \( P_{k-y} \) to be in \( M. \) Thus, \( P_{k-y} P_{k-y} = P_{k-y} = 0. \) But \( P_{k-y}(k - y) = k - y \) and we assumed that \( y < k. \) This contradiction guarantees that \( k=y \) and (iii) follows.

**PROOF of iv).** Suppose that \( K \) is a uniformly exhaustive subset of \( X. \) Then \( |K| = \{ |k| : k \in K \} \) is also uniformly exhaustive. Let \( M \) be a maximal pairwise disjoint subset of \( O \) and put \( M(K, n) = \{ P \in M : \|P(k)\| > \frac{1}{n} \text{ for some } k \in K \} \) for \( n = 1, 2, \ldots. \)

We assert that \( M(K, n) \) is finite for each \( n. \) For, if not, there exists a positive integer \( n_0, \) a disjoint sequence \( (P_{x_i}) \) in \( M(K, n_0), \) and a sequence \( k_i \) in \( K \) such that \( \|P_{x_i}(k_i)\| > \frac{1}{n_0} \) for each \( i. \) But this clearly violates the assumption that \( K \) is uniformly exhaustive. Thus \( M(K, n) \) is finite for each \( n, \) and \( M_0 = \{ P \in M : \|P(u)\| > 0 \text{ for some } u \in K \} \) is countable. Let \( (P_{x_i}) \) be an enumeration of \( M_0 \) where again we assume without loss of generality that \( \sum \|x_i\| < \infty. \) Put \( x = \sum x_i \) and suppose that \( (P_{q_j}) \) is a sequence such that \( P_{q_j}(x) \longrightarrow 0. \) From the monotonicity of the lattice norm, we see that \( \|P_{q_j}(x_i)\| \longrightarrow 0 \) uniformly in \( i, \) i.e. \( \|P_{q_j}(x_i)\| \leq \|P_{q_j}(x)\| \) for each \( i \) and \( j. \) Now suppose \( \epsilon > 0 \) and let \( k \) be an arbitrary member of \( K. \) From (iii) we know that \( \sum P_{x_i}(|k|) = |k| = P_{\vee x_i}(|k|) \leq P_{\sum x_i}(|k|) \leq |k|. \) Let \( N \) be a positive integer so that \( \sum_{i>N} P_{x_i}(|k|) \ll \epsilon. \) Let \( N_1 \) be a positive integer and use the definition of \( P_{x_i}(|k|) \) and the countable order continuity of the norm on \([0, |k|] \) to obtain

\[
\sum_{i=1}^{N} \|N_1 x_i \land |k| - P_{x_i}(|k|)\| < \epsilon. \]

Observing that \( P_{q_j}(N_1 x_i \land |k|) \leq P_{q_j}(N_1 x_i \land N_1|k|) \leq N_1 P_{q_j}((x_i) \land |k|) \leq N_1 P_{q_j}(x_i) \) for each \( i \) and \( j, \) we then choose \( J \) so that if \( j \geq J \) then
\[ \sum_{i=1}^{N} \| P_{q_j}(N_1 x_i \wedge |k|) \| < \sum_{i=1}^{N} \| N_1 P_{q_j}(x_i) \| < N_1 N \frac{\varepsilon}{N_1 N} = \varepsilon, \text{ where } \| P_{q_j}(x_i) \| < \frac{\varepsilon}{N_1 N} \]

uniformly in i. Therefore, if \( j \geq J \), then

\[ \| P_{q_j}(|k|) \| = \| P_{q_j} \left( \sum P_{x_i}(|k|) \right) \| = \left\| P_{q_j} \left( \sum_{i=1}^{N} P_{x_i}(|k|) + \sum_{i>N} P_{x_i}(|k|) \right) \right\| \]

\[ \leq \| P_{q_j} \left( \sum_{i=1}^{N} P_{x_i}(|k|) - N_1 x_i \wedge |k| \right) + \sum_{i=1}^{N} N_1 x_i \wedge |k| \right) + P_{q_j} \left( \sum_{i>N} P_{x_i}(|k|) \right) \| < \]

\[ \| P_{q_j} \left( \sum_{i=1}^{N} P_{x_i}(|k|) - N_1 x_i \wedge |k| \right) \| + \sum_{i=1}^{N} \| P_{q_j} \left( N_1 x_i \wedge |k| \right) \| + \| P_{q_j} \left( \sum_{i>N} P_{x_i}(|k|) \right) \| \]

\[ < 3\varepsilon. \] Since \( k \) was arbitrary and the choice of \( J \) did not depend on \( k \), the uniform continuity follows.

**PROOF of (v).** Let \( M = \{ P_a : a \in D \} \) be a maximal pairwise disjoint subset of \( O \), let \( D \) be the \( \sigma \)-algebra of all subsets of \( D \), and let \( ca(D, X) \) be the space of all countably additive measures \( \mu : D \rightarrow X \). Note that if \( \mu \) is countably additive, then \( \sum \mu(E_i) = \mu(\cup E_i) \) for any pairwise disjoint sequence, \( (E_i) \), i.e. \( \sum \mu(E_i) \) converges in norm to \( \mu(\cup E_i) \). Therefore, \( \mu \) is strongly additive. Furthermore, from corollary 19, p.9 of Diestel and Uhl [11], if \( \mu \in ca(D, X) \) then \( \mu(D) = \{ \mu(A) : A \in D \} \) is a norm bounded set in \( X \). In fact, \( ca(D, X) \) is a Banach space when equipped with the \( \sup \) norm \( (\| \mu \| = \sup \{ \| \mu(A) \| : A \in D \}) \). Let \( ca(D, X) \) be so normed and consider the mapping \( \pi \) on the exhaustive elements of \( X \) defined by \( \pi(x) = \mu_x \), where

\[ \mu_x(A) = \sum_{a \in A} P_a(x), \quad A \in D. \]

Note that if \( (A_i) \) is a pairwise disjoint sequence, then

\[ \mu_x(\cup A_i) = \sum_{a \in \cup A_i} P_a(x), \text{ and } \sum_i \mu_x(A_i) = \sum_i \sum_{a \in A_i} P_a(x), \]

where the unconditional convergence of the series established in (iii) above guarantees that \( \mu_x \in ca(D, X) \).

Suppose that \( A = \{ a_1, \ldots, a_n \} \) is a finite set in \( D \) and recall \( a_i \wedge a_j = 0, \text{ if } i \neq j. \) Then

\[ \| \mu_x(A) \| = \| \sum_{a} P_{a}(x) \| = \| P_{\sum_{a} (x)} \| = \| P_{\sum_{a} (|x|)} \| \leq \| x \| = \| x \|. \] Therefore,
\[\|\mu_x(A)\| \leq \|x\| \text{ for each } A \in \mathcal{D}. \] (Recall that if \(x\) is exhaustive \(\{P \in M : \|P(x)\| > 0\}\) is countable.) But, \(\mu_x(\mathcal{D}) = x\) by (iii). Consequently, \(\|\mu_x\| = \|x\|\), i.e. \(\|\pi(x)\| = \|x\|\).

The linearity of \(P_a\) for each \(a \in A\) makes it clear that \(\pi\) is linear; therefore, \(\pi\) is an isometric isomorphism from the exhaustive elements of \(X\) into \(\text{ca}(\mathcal{D},X)\). Now suppose that \(K\) is a relatively compact (i.e. the closure of \(K\) is compact) subset of exhaustive elements and that \(K\) fails to be uniformly exhaustive. Let \(\epsilon > 0\), and let \(\langle P_{x_i} \rangle\) be a pairwise disjoint sequence from \(\mathcal{O}\) such that \(\|P_{x_i}(u_i)\| > \epsilon\) for some sequence \(\langle u_i \rangle\) in \(K\). Suppose \(\langle u_i \rangle\) is one such sequence. Using the Hausdorff Maximal Principle, let \(M = \{P_a : a \in \mathcal{A}\}\) be a maximal pairwise disjoint subset of \(\mathcal{O}\) such that \(M\) contains \(\langle P_{x_i} \rangle\), and suppose that \(\pi\) and \(\text{ca}(\mathcal{D},X)\) are defined as above. Then \(\pi(K) = \{\pi(k) : k \in K\}\) is a relatively compact subset of \(\text{ca}(\mathcal{D},X)\). Therefore, by Theorem 4.1 of Brooks and Dinculeanu [8], \(\pi(k)(A_i) \longrightarrow 0\) uniformly for \(k \in K\) whenever \(\langle A_i \rangle\) is a disjoint sequence from \(\mathcal{D}\). But this is a contradiction since \(\|P_{x_i}(u_i)\| > \epsilon\) for each \(i\). Thus, \(K\) is uniformly exhaustive. Q.E.D.

PROOF of vi). Let \(\mathcal{E}(X)\) denote the exhaustive elements in \(X\). Suppose that \(K \subseteq \mathcal{E}(X)\). Let \(\lambda \in \mathcal{E}(X)\) so that \(K\) is continuous with respect to \(\lambda\). (Note that \(\dot{K}\) is continuous with respect to \(x\) if and only if \(K\) is continuous with respect to \(x\). In fact, if \(\dot{K}\) is continuous with respect to \(x\), then certainly \(K\) is continuous with respect to \(x\).

On the other hand, if \(K\) is continuous with respect to \(x\), and \(\langle P_i \rangle\) is a sequence from \(\mathcal{O}\) such that \(\langle P_i(x) \rangle \longrightarrow 0\), then \(P_i(k) \longrightarrow 0\) for each \(k \in K\). Let \(\dot{k} \in \dot{K}\). Then there exists \(k \in K\) such that \(|\dot{k}| \leq |k|\). The monotonicity of \(P_i\) and the norm guarantees
that $P_i(k) \to 0$.) Without loss of generality, suppose that $\lambda > 0$, i.e. consider $\lambda^+, \lambda^-$. Let $A$ be a pairwise disjoint subset of $\hat{K}\setminus 0$, i.e. $|x| \land |y| = 0$ for $x, y \in A, x \neq y$. Our first assertion is that $P_{\eta i}(\lambda) > 0$ for each $x \in A$.

By way of contradiction, suppose that $\eta \in A$ and $P_{\eta i}(\lambda) = 0$. Then let $x_i = |\eta|$ for $i=1,2,...$. Thus $P_{x_i}(\lambda) \to 0$, and, due to the continuity, $P_{x_i}(\eta) \to 0$. But $P_{x_i}(\eta) = \eta \neq 0$ and our assertion is verified. Now suppose $r \in \mathbb{R}$ and $r > 0$. Let $A_r = \{x \in A : \|P_{x i}(\lambda)\| > r\}$. The set $A_r$ cannot be infinite. If it were, then there would exist $(x_i)_{i=1}^{\infty}$, an infinite sequence of distinct pairwise disjoint elements in $A_r$, and by (iii) $\sum_{i=1}^{\infty} P_{|x_i|}(\lambda)$ converges. However, the sum is not Cauchy. Consequently, the unconditional convergence of the series in (iii) tells us that $A$ must be countable.

For the second part of the proof of (vi), suppose that each pairwise disjoint subset of $\hat{K}$ is countable, and let $M$ be a maximal pairwise disjoint subset of $\mathcal{E}(\mathcal{X})_+$. Then, let $B = \{\lambda \in M : P_{\lambda}(x) \neq 0 \text{ for some } x \in K\}$. Next, we claim that $B$ is countable. Let $\lambda_1$ and $\lambda_2$ be distinct elements of $M$ such that $P_{\lambda_1}(x) \neq 0$ and $P_{\lambda_2}(y) \neq 0$ for some $x, y \in K$. The disjointness of $\lambda_1$ and $\lambda_2$ guarantee that $|P_{\lambda_1}(x)| \land |P_{\lambda_2}(y)| = 0$. Therefore, fix $n$ and consider $m \geq n$. Note that $0 \leq (n \lambda_1 \land |x|) \land (m \lambda_2 \land |y|) \leq m(\lambda_1 \land \lambda_2) \land |x| \land |y| = 0$. Therefore, $0 \leq (n \lambda_1 \land |x|) \land P_{\lambda_2}(|y|) = 0$, and $0 \leq (P_{\lambda_1}(|x|) \land P_{\lambda_2}(|y|) = 0$. Further, since $|P_{\lambda_1}(x)| = P_{\lambda_1}(|x|) \leq |x|$ and $|P_{\lambda_2}(y)| = P_{\lambda_2}(|y|) \leq |y|$, it follows that $P_{\lambda_1}(x)$ and $P_{\lambda_2}(y) \in \hat{K}$. Thus, if $B$ were uncountable $\hat{K}$ would necessarily contain an uncountable pairwise disjoint subset, a contradiction to our hypothesis. Let $(x_i)$ be an enumeration of $B$, and put $x = \sum_i \frac{x_i}{3^i(1+\|x_i\|)}$. Since $\mathcal{E}(X)$ is a closed linear subspace of
$X, x \in \mathcal{E}(X)$ and the proof of (iii) demonstrates that $P_x(k) = k$ for each $k \in \hat{K}$. Now, recall that if $k$ is exhaustive, then $\{ P \in M : \|P(k)\| > 0 \}$ is countable. Therefore, if $F$ is any finite subset of $M$, then $\sum_{u \in F} P_u(k) = P_{\sum_{u \in F}}(k) \leq P_x(k) \leq k$. Therefore, $\sum_K P_u(k) \leq P_x(k) \leq k$. In fact, if we let $y = \sum_K P_u(k)$, then as in the proof of iii, we consider $P_{k-y}$, where $k-y$ is exhaustive in order to arrive at a contradiction. Consequently, $k = \sum_K P_u(k) \leq P_x(k) \leq k$. Further, the proof of (iv) shows that $K$ is continuous with respect to $x$. Q.E.D.

In an abstract L-space Bilyeu and Lewis [5, Theorem 2.4] note that $D(f,g)$ exists uniformly for $g \in K$ if and only if $\|nf - g - |g|\| \to 0$ uniformly for $g \in K$. (In an arbitrary Banach space $X$, it is an easy exercise in inequalities to show that $D(f,g)$ exists uniformly for $g \in K$ if and only if $\|nf + g\| + \|nf - g\| - 2\|n\|f\| \to 0$ uniformly for $g \in K$.) In an L-space, $D(f,g)$ exists uniformly for $g \in K$ if and only if $D(|f|, |g|)$ exists uniformly for $g \in K$ if and only if $\|nf + g\| + |nf - g| - 2\|n\|f\| \to 0$ uniformly for $g \in K$ if and only if $2\|g\| - |nf| \land |g| \to 0$ uniformly for $g \in K$.

A key ingredient in the proof of Theorem 2.4 of [5] is Lemma 2.3 of the same paper. This lemma has a simple extension to a general Banach lattice, and this extension plays an important role in the remainder of this chapter.

**Lemma 5.6** If $X$ is a Banach lattice and $x, y \in X$. Then

$$|x + y| + |x - y| = 2(|x| \lor |y|) = 2|x| + 2(|y| - |x| \land |y|).$$

Proof. If $x$ and $y$ belong to $X$, then

$$|x| + |y| = (x \lor -x) + |y|$$
\[= (x + |y|) \lor (-x + |y|)\]
\[= (x + (y \lor -y)) \lor (-x + (y \lor -y))\]
\[= (x + y) \lor (x - y) \lor (-x + y) \lor (-x - y)\]
\[= ((x + y) \lor (-x - y)) \lor ((x - y) \lor (-x + y))\]
\[= |x + y| \lor |x - y|.
\]

Substituting \(x + y\) for \(x\) and \(x - y\) for \(y\), one obtains
\[|x + y| + |x - y| = 2|x| \lor 2y = 2(|x| \lor |y|).
\]

Further, since \(|x| + |y| = |x| \lor |y| + |x| \land |y|\), one immediately sees that
\[2(|x| \lor |y|) = 2(|x| + |y| - |x| \land |y|)
\[= 2|x| + 2(|y| - |x| \land |y|).
\]

Q.E.D.

In the setting of a \(\sigma\)-order complete countably order continuous Banach lattice, (uniform) differentiability of the norm can be tied to the projection function \(P_x\) via the following definition.

**Definition 5.7** If \(X\) is a \(\sigma\)-complete Banach lattice, \(K \subseteq X\), and \(x \in X^+\), then \(P_x\) is said to exist uniformly for \(y \in K\) provided that for each \(\epsilon > 0\) there is an \(M \in \mathbb{N}\) so that
\[||nx \land y - P_x(y)|| < \epsilon\]
whenever \(n \geq M\) and \(y\) is any element of \(K\).
Theorem 5.8 If $X$ is a $\sigma$-order complete and countably order continuous Banach lattice, $K \subseteq X$, and $f \in X$, then $P_f(|g|) = |g|$ uniformly for $g \in K$ if and only if 
$$\|nf+g|+|nf-g|-2nf\| \longrightarrow 0 \text{ uniformly for } g \in K.$$ Moreover, if $K$ is a relatively norm compact subset of $X^+$, then there is an element $x \in X^+$ so that 
$$P_x(k) = k$$
uniformly for $k \in K$.

Proof. A direct application of Lemma 5.6 shows that $|nf + g| + |nf - g| - 2nf| = 2(|g| - |nf| \wedge |g|)$. Since $|nf| \wedge |g| \longrightarrow P_f(|g|)$ in a $\sigma$-complete countably order continuous Banach lattice, $P_f(|g|) = |g|$ uniformly in $g$ if and only if $\|nf+g|+|nf-g|-2nf\| \longrightarrow 0 \text{ uniformly in } g$.

Now suppose that $K$ is a relatively compact subset of $X^+$. By theorem 5.3, part ii), each positive element of $X$ is exhaustive. By part v), $K$ is uniformly exhaustive. In the proof of part iv), we have an $x \in X$ such that $P_x(k) = k$ for each $k \in K$.

We claim that $P_x(k) = k$ uniformly. Suppose not. Then, there exists an $\epsilon > 0$ such that for each $n \in N$ there exists $k_n \in K$ such that $\|nx \wedge k_n - k_n\| > \epsilon$. Without loss of generality, we may assume that there exists $k \in K$ such that $k_n \longrightarrow k$. Since $P_x$ is continuous, $P_x(k_n) \longrightarrow P_x(k)$. Hence, for this $\epsilon$, there exists an $M$ in $N$ such that if $n \geq M$ then $\|k_n - k\| < \frac{\epsilon}{2}$. Also, recall that $P_x(k_n) = k_n$. Therefore, note that 
$$\|nx \wedge k_n - k_n\| = \|nx \wedge k_n - P_x(k_n)\| = \|nx \wedge k_n - P_x(k) + P_x(k) - P_x(k_n)\| < \|nx \wedge k_n - P_x(k)\| + \|P_x(k) - P_x(k_n)\| < \|k_n - P_x(k)\| + \|k_n - k_n\| < \|P_x(k_n) - P_x(k)\| + \frac{\epsilon}{2} < \|k_n - k\| + \frac{\epsilon}{2} < \epsilon.$$ Since this contradicts the above inequality, $P_x(k) = k$ uniformly.
for \( k \in K \). Q.E.D.

The next result relates Theorem 5.8 to the notion of absolute continuity studied in Chapter 3 of this paper. Again, suppose that \( X \) is \( \sigma \)-order complete and countably order continuous.

**Theorem 5.9** If \( P_{|\mu|}(|\nu|) = |\nu| \) uniformly for \( \nu \in K \), then \( |\nu| \ll |\mu| \) uniformly for \( \nu \in K \).

Proof. Suppose that \( P_{|\mu|}(|\nu|) = |\nu| \) uniformly for \( \nu \in K \) but that \( |K| = \{ |\nu| : \nu \in K \} \) is not uniformly absolutely continuous with respect to \( |\mu| \). Let \((x_i)\) be a sequence in \( X^+ \), \((\nu_i)\) be a sequence in \( K \), and \( \epsilon \) be a positive number so that \( (P_{x_i}(\mu_i)) \rightarrow 0 \) and \( \|P_{x_i}(\nu_i)\| > \epsilon \) for each \( i \). Fix \( n \) such that \( \|\nu - n|\mu| \wedge |\nu|\| < \frac{\epsilon}{2} \) for each \( \nu \in K \). Since \( P_{x_i}(x) \leq x \) whenever \( x \geq 0 \), \( P_{x_i}(|\nu_i| - n|\mu| \wedge |\nu_i|) \leq |\nu_i| - n|\mu| \wedge |\nu_i| \); therefore \( \|P_{x_i}(|\nu_i| - n|\mu| \wedge |\nu_i|)\| \leq \|\nu_i| - n|\mu| \wedge |\nu_i|\| < \frac{\epsilon}{2} \). Next use the fact that \( \|P_{x_i}(\nu_i)\| = \|P_{x_i}(\nu_i)\| > \epsilon \) and the triangle inequality to conclude that

\[
\|P_{x_i}(n|\mu| \wedge |\nu_i|)\| > \frac{\epsilon}{2}.
\]

Since \( P_{x_i}(n|\mu|) \geq P_{x_i}(n|\mu| \wedge |\nu_i|) \), it follows that

\[
\|P_{x_i}(n|\mu|\| > \frac{\epsilon}{3}
\]

for each \( i \). This is a contradiction since \( (P_{x_i}(n|\mu|)) = (nP_{x_i}(|\mu|)) \rightarrow 0 \). Q.E.D.

We conclude this chapter by noting that Theorem 5.8 fails for relatively weakly compact sets. Let \( X = l^2 \) with its usual pointwise ordering, let \( x = (x_i) \in l^2 \) so that \( x_i > 0 \) for all \( i \) and let \((e_i)_{i=1}^\infty\) be the canonical basis for \( l^2 \). Then \( P_x(y) = y \) for each
$y \in l^2$, $\{e_i : i \in N\}$ is relatively weakly compact, and it is not true that $P_x(e_i)$ is $e_i$ uniformly in $i$. Further, there is no element $x \in (l^2)_+$ so that $P_x(e_i) = e_i$ uniformly in $i$. 
CHAPTER 6

APPLICATIONS TO SPACES OF MEASURES

This chapter shows that the concepts of exhaustivity and continuity (or absolute continuity) which have been discussed in the context of Banach lattices take familiar forms in spaces of measures. If \( R(\Sigma) \) is a ring (\( \sigma \)-algebra) of sets, then \( \text{ba}(R) \) (\( \text{ca}(\Sigma) \)) denotes the space of all bounded and finitely additive (countable additive) real valued measures defined on \( R(\Sigma) \). Both \( \text{ba}(R) \) and \( \text{ca}(\Sigma) \) are Banach spaces when endowed with the usual total variation norm. In fact, each of these spaces is a complete Banach lattice when \( "\wedge" \) is defined as follows: if \( \mu, \nu, \in \text{ba}(R) \) and \( A \in R \), then \( \mu \wedge \nu(A) = \inf\{\mu(B) + \nu(C) : B, C \in R, \ B \cap C = \emptyset, \ B \cup C = A \} \). The reader may consult III.7 of Dunford and Schwartz [15] for a discussion of lattice properties of these spaces.

We remark that a subset \( K \) of \( \text{ca}(\Sigma) \) is uniformly strongly bounded if and only if \( K \) is uniformly countable additive [11,Corollary 1.18]. Additionally, a finitely additive set function (even a Banach space valued measure) must be bounded if it is strongly bounded, e.g. see the proof of Corollary 1.19 of [11]. Chapter I of [11] contains many results concerning strongly bounded (or strongly additive) set functions.

**Theorem 6.1** i) A subset \( K \) of \( \text{ba}(R) \) is uniformly strongly bounded if and only if it is uniformly exhaustive in the lattice \( \text{ba}(R) \).

ii) If \( K \subseteq \text{ba}(R) \) and \( 0 \leq \nu \in \text{ba}(R) \) then \( K \) is uniformly absolutely continuous with respect to \( \nu \) in the usual \((\epsilon, \delta)\)-sense if and only if \( K \) is uniformly continuous.
with respect to \( \nu \) in the lattice sense.

**Proof.** i) First, suppose that \( K \) is a uniformly exhaustive subset of \( \text{ba}(R) \) which fails to be uniformly strongly bounded. Then there exists a disjoint sequence \( \{A_i\} \) from \( R \), a sequence \( \{\mu_i\} \) from \( K \), and a positive number \( \epsilon \) such that \( |\mu_i(A_i)| > \epsilon \) for each \( i \). But certainly \( |\mu|(A) \geq |\mu(A)| \) for each \( \mu \in \text{ba}(R) \) and each \( A \in R \). Therefore, \( |\mu_i|(A_i) > \epsilon \) for each \( i \). Let \( \nu_i \) be the non-negative measure defined on \( R \) by \( \nu_i(B) = |\mu_i|(B \cap A_i) \), \( i = 1,2,... \). Since \( \nu_i \) is concentrated on \( A_i \) (i.e. \( \nu_i(C) = 0 \) if \( C \cap A_i = \emptyset \)) and the sequence \( \{A_i\} \) is pairwise disjoint, it follows that \( \nu_i \wedge \nu_j = 0 \) for \( i \neq j \). Therefore, \( \{\nu_i\} \) is a disjoint sequence from \( \mathcal{O} \) and the uniform exhaustivity of \( K \) guarantees that \( P_{\nu_i}(\mu) \rightarrow 0 \) uniformly for \( \mu \in K \). But the lattice structure of \( \text{ba}(R) \) ensures that the absolute value of an element \( \mu \in \text{ba}(R) \) is the total variation measure \( |\mu| \). Consequently, \( |\mu_i| \in \hat{K} \) for each \( i \) and \( P_{\nu_i}(|\mu_i|) \) is a disjoint sequence in \( \hat{K} \) and thus converges to \( 0 \) in norm. Now \( \nu_i \leq |\mu_i| \) by definition; thus, \( \nu_i = \nu_i \wedge |\mu_i| \leq \bigvee_i \nu_i \wedge |\mu_i| = P_{\nu_i}(|\mu_i|) \). Since \( \nu_i(A_i) = |\mu_i|(A_i) > \epsilon \) for each \( i \), it follows that \( P_{\nu_i}(|\mu_i|) \) does not converge to \( 0 \), and we have a contradiction. Therefore, \( K \) is uniformly strongly bounded.

The converse of (i) uses the following lemma from Bilyeu and Lewis [5, Theorem 4.1].

**Lemma 6.2** If \( \epsilon > 0 \) and \( \{\mu_n\} \) is a disjoint sequence of positive members of \( \text{ba}(R) \) such that \( |\mu_n| > \epsilon \), then there is a \( \delta > 0 \), a disjoint sequence \( \{A_i\} \) from \( R \) and a subsequence \( \{\mu_{n_i}\} \) of \( \{\mu_n\} \) so that \( \mu_{n_i}(A_i) > \delta \) for all \( i \).
By way of contradiction, suppose that $K$ is not uniformly exhaustive. Then there is a positive number $\varepsilon$, a disjoint sequence $(\nu_i)$ in $\text{ba}(\mathbb{R})_+$ and a sequence $(\mu_i)$ from $K$ such that $\|P_{\nu_i}(\mu_i)\| = \|P_{\nu_i}(|\mu_i|)\| > \varepsilon$ for each $i$. Since $(\nu_i)$ is a disjoint sequence, $(P_{\nu_i}(|\mu_i|))$ is pairwise disjoint. From the lemma above, there is a subsequence from $(P_{\nu_i}(|\mu_i|))$ (without loss of generality, suppose it is the full sequence), a positive number $\delta$, and a disjoint sequence $(A_i)$ from $\mathbb{R}$ such that $P_{\nu_i}(|\mu_i|)(A_i) > \delta$ for each $i$. But, $|\mu_i| \geq n\nu_i \wedge |\mu_i|$ for all $n$. Thus, $|\mu_i|(A_i) \geq P_{\nu_i}(|\mu_i|)(A_i) > \delta$ for all $i$. Therefore, (by the definition of $|\mu_i| = \sup \text{ etc}$) there is a set $B_i \subseteq A_i$, $B_i \in \mathbb{R}$ such that $|\mu_i(B_i)| > \frac{\delta}{2}$, $i = 1, 2, ...$. However, $(\mu_i)$ fails to be uniformly bounded and the proof of (i) is complete.

**PROOF of ii).** First, by way of contradiction, suppose that $K$ is uniformly continuous with respect to the positive measure $\nu$ and that $K$ fails to be uniformly absolutely continuous with respect to $\nu$ in the $(\epsilon, \delta)$-sense. Then, as in the proof of i) above, there is a positive number $\varepsilon$, a sequence $(\mu_i)$ from $K$ and a sequence $(A_i)$ from $\mathbb{R}$ such that $\nu(A_i) \to 0$, and $|\mu_i(A_i)| > \varepsilon$ for each $i$. Let $\nu_k$ be defined as in part i), i.e. $\nu_k(B) = |\mu_k|(B \cap A_k)$. Then, $\nu_k$ is concentrated on $A_k$ and $((n\nu_k) \wedge \nu)(B) \leq \nu(B \cap A_k)$ for all $k, n \in \mathbb{N}$ and $B \in \mathbb{R}$. Therefore, $\|P_{\nu_k}(\nu)\| \leq \nu(A_k) \to 0$, $k = 1, 2, ..., \text{i.e.} P_{\nu_k}(\nu) \to 0$. Thus the uniform continuity of $K$ with respect to $\nu$ implies $\|P_{\nu_k}(\mu_k)\| = \|P_{\nu_k}(|\mu_k|)\| \to 0$. But, $P_{\nu_k}(|\mu_k|) = \vee_n n\nu_k \wedge |\mu_k| \geq \nu_k \wedge |\mu_k|$. Consequently, $\|P_{\nu_k}(|\mu_k|)\| \geq (\nu_k \wedge |\mu_k|)(A_k) = |\mu_k|(A_k) \geq |\mu_k|(A_k) > \varepsilon$, and we have contradicted the uniform continuity of $K$.

Conversely, by way of contradiction, suppose that $K \subseteq \text{ba}(\mathbb{R})$ $0 \leq \nu \in \text{ba}(\mathbb{R})$
and \( \mu \ll \nu \) uniformly for \( \mu \in K \) in the \((\epsilon, \delta)\)-sense. We claim that \( K \) is uniformly continuous with respect to \( \nu \). Suppose not. Then choose a positive number \( \epsilon \), a sequence \((\xi_i)\) from \( ba(R) \) and a sequence \((\mu_i)\) from \( K \) such that \( \|P_{\xi_i}(\nu)\| \to 0 \) and \( \|P_{\xi_i}(\mu_i)\| = \|P_{\xi_i}(|\mu_i|)\| > \epsilon \) for each \( i \). Let \( \delta > 0 \) \( (\delta < \frac{\epsilon}{4}) \) such that \( |\mu_i(A)| < \frac{\delta}{8} \) for all \( \mu \in K \) whenever \( \nu(A) < \delta \). Since \( |\mu|(A) = \sup\{|\mu(B) - \mu(C)| : B, C \in R, B \cap C = \emptyset, B \cup C \subseteq A\} \) [15, p.97] it follows that \( |\mu|(A) \leq \frac{\delta}{8} + \frac{\epsilon}{8} \leq \frac{\epsilon}{4} \) whenever \( \nu(A) < \delta \). Let \( i \) be a positive integer so that \( \|P_{\xi_i}(\nu)\| < \delta \). Certainly, \( n\xi_i \wedge \nu \leq P_{\xi_i}(\nu) \) for each \( n \). Thus, \( \|n\xi_i \wedge \nu\| \leq \|P_{\xi_i} \wedge \nu\| < \delta \) for each \( n \). Since each bounded scalar valued measure is strongly bounded (otherwise we would have a disjoint sequence of sets where a finite sum of the measures would exceed the bound) and therefore exhaustive by (i), the norm is countably order continuous on \([0, |\mu_i|]\) and \( \|n\xi_i \wedge |\mu_i| - P_{\xi_i}(|\mu_i|)\| \to 0 \) as \( n \to \infty \). Let \( n \) be a positive integer so that \( \|n\xi_i \wedge |\mu_i|\| > \epsilon \). Now choose \( D \) from \( R \), and recall that \( \|n\xi_i \wedge \nu\| < \delta \). Hence let \( A \) and \( B \) be disjoint subsets of \( D \) such that \( n\xi_i(A) + \nu(B) < \delta \); therefore \( n\xi_i(A) < \delta \) and \( \nu(B) < \delta \). By the uniform absolute continuity assumption we then have \( |\mu|(B) \leq \frac{\delta}{4} \). Hence by the lattice definition of "\( \wedge \)" in \( ba(R) \) we have \( (n\xi_i \wedge |\mu|)(D) \leq n\xi_i(A) + |\mu|(B) < \delta + \frac{\epsilon}{4} < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon \). Since \( D \) is arbitrary, it follows that \( \|n\xi_i \wedge |\mu|\| \leq \epsilon \) for each \( \mu \in K \). Therefore we have a contradiction. Q.E.D.
BIBLIOGRAPHY


