Symmetry-induced intermittency in a stochastic reflexive model

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Abstract

We discuss a route to intermittency based on the concept of reflexivity, namely on the interaction between an observer and a stochastic system. A simple model mirroring the essential aspects of this interference is shown to generate perennial out of equilibrium condition, intermittency and $1/f$-noise. The intermittent properties of this model is induced by a scaling symmetry that turns isolated equilibrium points into equilibrium manifold with a non-constant escape rate. The permanence times distribution in the basin of attraction of equilibrium manifold can be analytically predicted through the adoption of a first-passage time technique. Finally we discuss the possible generalization of our approach and its implications in different fields of complex intermittent systems.

Key words: Reflexive models, Intermittency, First return time
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1. Introduction

Intermittency is an ubiquitous phenomenon in nature, ranging from physics [1] to astrophysics [2], from geophysics [3, 4] to neurophysiology [5] and from turbulence [6] to financial market [7]. In all these cases there are rare bursts of activities and the time distance between two consecutive bursts, τ, is given by a distribution density $\psi(\tau)$ with the inverse power law structure $\psi(\tau) \propto 1/\tau^{\mu}$. In spite of being so widespread, the origin of intermittency is still the object of research, debates and conjectures. According to an increasing number of investigators [8, 9] the brain lives at criticality, namely, in a condition of transition from chaos to order. This condition generates intermittency, but the value of μ is either confined to $\mu = 1.5$, obtained as the first return time of ordinary diffusion [10], or derived from the experimental observation, yielding either $\mu = 1.65$ [5] or $\mu = 2.05$ [11]. A theoretical attempt to depart from the trivial index $\mu = 1.5$ to explain $\mu = 1.7$ in the case of Blinking Quantum Dots (BQD) [1] was recently done by running a model of interaction between cooperating units [12]. The value of $\mu = 2.05$ generated by the analysis of neuro-physiological data remains unexplained as well as the value of $\mu = 1.65$ of an earlier investigation [5].

In this paper we propose an alternative route towards intermittency based on the idea of reflexivity [13]. The model we present mimicks the reflexive interaction between a system and

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an observer that, according to the results of his/her measurements, make some choices perturbing the same dynamics of the system [13, 14]

The outline of the paper is the following: in section 2 we present the model and we numerically describe its intermittent behavior. In section 4 we show an approximated version of the model dynamics valid during the laminar zones. Thanks to the approsimation we analitically show that the first passage times of the approximated version of the model follows a power law tailed distribution with a power law exponent $\mu \simeq 2$ like the non-approximated version of the model. In section 4 we present the conlusions.

2. The reflexive model

In this paper we follow the perspective advocated by financier Soros [13] for the financial market. In his vision the system (the market) is observed by different agents that analyze it through the cognitive function, y that depends on the state of the system r,

$$y = f(r). \tag{1}$$

The agents behaves according to the value of $f(r)$ and their action perturbs the system through the participating function [13]. Indeed the state of the system r is influenced by the participant's perceptions y according to

$$r = \Phi(y). \tag{2}$$

The interaction between the observer and the system generates the following dynamical system

$$\begin{cases} y(t) = f(r(t)) \\ r(t+1) = g(r(t)) + \Phi(y(t), r(t)), \end{cases} \tag{3}$$

where we added a term $g(r(t))$ describing the dynamics of the system when the participating function vanishes (i.e. the case with no observer). In our model

$$g(r(t)) = (1 + \sigma_0)\eta(t), \tag{4}$$

where $\eta(t)$ is assumed to be a random Gaussian fluctuation with $\langle \eta^2 \rangle = 1$, so that $r(t+1) = (1+\sigma_0)\eta(t)$ would represent the random time evolution of the system in the absence of observer ($\Phi(y, r) = 0$). According to Soros [13] we have to plug the first of Eq. (3) into the second so as to yield an equation of motion for the situation r alone. Following Ref.[14] we set $\mathbf{r} = (r(t), r(t-1), ..., r(t-N+1))$ and $\Phi(y(t), \mathbf{r}) = q(f(\mathbf{r})) \cdot h(\mathbf{r})$ with

$$q(f(\mathbf{r})) = |f(\mathbf{r})|, \; h(r) = [-\eta(t) + f(\mathbf{r})]r(t), \tag{5}$$

thereby obtaining

$$r(t+1) = [1 - |f(\mathbf{r})| + \sigma_0]\eta(t) + |f(\mathbf{r})|f(\mathbf{r})r(t). \tag{6}$$

This is the Reflexive Model (RM) proposed in the earlier work of one of us [14]. The reflection function $f(\mathbf{r})$ has a subtle connection with the cooperation between units [12]. In fact the single unit [13] observes the last $N-1$ values of $r(t)$, records them by means of the vector $\tilde{\mathbf{r}}(t) = (r(t), r(t-1), ..., r(t-N+2))$ and assigns to $f(\mathbf{r})$ the form

$$f(\mathbf{r}) = \frac{\tilde{\mathbf{r}}(t) \cdot \tilde{\mathbf{r}}(t-1)}{||\tilde{\mathbf{r}}(t)||||\tilde{\mathbf{r}}(t-1)||}, \tag{7}$$

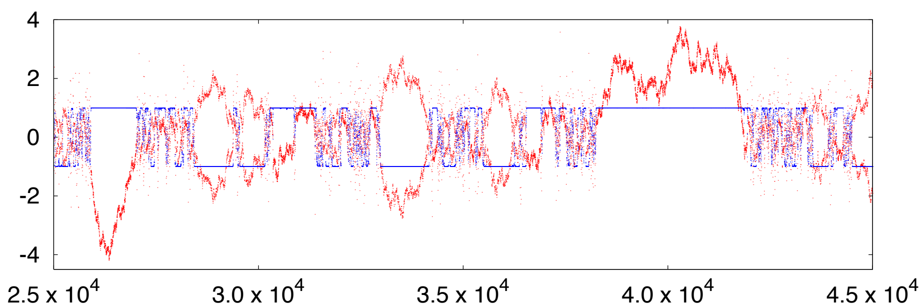

Figure 1: (Color online) Example of the behavior of $r(t)$ (red) and $f(\mathbf{r})$ (blue, the dots having the ordinate $\in [-1 : 1]$ forming horizontal straight lines) with $N = 6$ and $\sigma_0 = 0.05$.

the normalized auto-correlation function of $r(t)$ fulfilling the condition $-1 \leq f(\mathbf{r}) \leq 1$. N is the time scale the agents use to observe the past outcomes of the system. We suppose that this is the same for all the agents.

Fig. 1 illustrates a single realization of the RM of Eq. (6). We see that the variable $f(\mathbf{r})$ lies for most of time in either the state $f = 1$ or the state $f = -1$ and, from time to time, as an effect of a sudden burst of activity, it makes a transition from the one to the other state. The variable $r(t)$, on the contrary, has a diffusion-like nature and its modulus can get values significantly larger than 1. When $|r(t)| > 1$, the rest states $f = \pm 1$ last for extended times. For a transition to occur, the variable f must cross the origin $f = 0$. However, noise may make $|r|$ significantly larger than 1 and the larger the departure from the condition $|r| = 0$ the smaller the probability of escaping from the rest states. We can imagine the rest conditions as two-parameter states $|s(f, r)\rangle \equiv |f = \pm 1, r\rangle$, with an r-dependent stability. The celebrated Kramers theory [15] would correspond to a rate of escape independent of r, thereby making $\psi(\tau)$ an exponential function of τ. Here, the larger the departure of r from $|r| = 1$, the larger the Kramer's barrier. This is the reason why the Poisson $\psi(\tau)$ emerging from the Kramers theory is replaced by an inverse power law distribution density. It is remarkable that the time average of r, $\bar{r}$, allows us to interpret the states $|f = 1, r\rangle$ and $|f = -1, r\rangle$ as the "light on" and "light off" BQD states [1], respectively. In fact, $|f = 1, r\rangle$ corresponds to a single trajectory $r(t)$, with $|\bar{r}| > 0$, whereas $|f = -1, r\rangle$ generates two symmetric trajectories, one positive and one negative, and consequently $\bar{r} = 0$. If $r(t)$ is the logarithmic return of prices like in Ref. [14], the single trajectories with $r(t) > 1$ can be identified with the Soros bubbles [13].

It is worth noticing that reason why the RM has non-isolated equilibrium is the scaling symmetry of the trajectories. Indeed if one neglects the stochastic term putting $\sigma_0 = 0$, the RM trajectories should obey to the scaling symmetry driving $r(t) \to \lambda r(t)\ \forall t$ where $\lambda \in \mathbb{R}$. Thanks to this simmetry the RM has two equilibrium lines in the N-dimensional phase space given by all the points like (r, r, r, r, r) $(f(r) = 1)$ and $(r, -r, r, -r, r, -r)$ $(f(r) = -1)$. Consequently the Poisson escape rate from an isolated equilibrium point turns into a power law tail distribution due to the fact that the escape rate from the equilibrium lines depends on the value of r.

In Fig. 2 we show the logical scheme of simmetry-induced intermittency. The scaling simmetry of the model with $\sigma_0 = 0$ leads to the presence of an equilibrium manifold (in the figure is bottom of the "valley") corresponding to the line $r = 0$. The variable y refers to the distance in the phase-space from the line $r = 0$ while the z-axis represents the "potential" $V(r, y)$ driving the deterministic motion of the particles. Along the bottom of the valley $r = 0$ the potential is $V(r, 0) = \alpha \log(r)$ that leads to the deterministic term $-\alpha/r$ shown after in Eq.(13). Different particles are driven by the gradient of $V(r, y)$ and by the stochastic terms. Some of the trajectories enter in the valley and, as the trajectory labelled as 1, are driven by the sthochastic term to the region of the valley with higher hills on the two sides with vanishing escape rate. Other trajectories, like that labelled as 2, reaches the zone $r \simeq 0$ at the beginning of the valley. In this region the slope of the valley (due to the term $-\alpha/x$ shown later in Eq.(13)) eventually drives the particles outside the laminar zone.

This schematic picture suggests that the escape rate averaged over several particles (i.e. several laminar zones) is not constant in time because after a while all trajectories behaving like the 1-particle survive making the escape rate lower and lower exactly as it happens for a permanence time with a power law tail. We refer to this route of intermittency as *symmetry-induced intermittency*.

The simmetry-induced power law index μ emerges immediately from the numerical iteration of Eqs. (6) and (7). The time durations of the states $f = 1$ and $f = -1$ are evaluated numerically by identifying the former state with $f(\mathbf{r}) > 1 - \epsilon$ and the latter with $f(\mathbf{r}) < \epsilon - 1$. Fig. 3 illustrates the results of this numerical analysis for $\epsilon = 0.02$, but the adoption of different values of ϵ does not affect them if $\epsilon \ll 1$. The fit of the curves of Fig. 3 with a power law decay, for the values N, σ_0 shown in the figure, moving from top to bottom, yields $\mu = 2.00, 2.05, 2.19, 2.09, 2.13, 2.29$. All these values are close to $\mu = 2$ and turn out to be closer and closer to it by decreasing σ_0 and N.

3. Analytical calculation of the power law index μ

The key property $\psi(\tau) \propto 1/\tau^{\mu}$ is confirmed by theoretical arguments resting on the fact that when $f(\mathbf{r}) \simeq 0$ we have $|r(t)| \simeq 1 + \sigma_0 \simeq 1$. This condition forces the system to fall in the basin of attraction of one of the states $f(\mathbf{r}) = \pm 1$, thereby making the sojourn in the states $f(\mathbf{r}) = \pm 1$ begin when $|r(t)| \simeq 1$ and end when $f(\mathbf{r}) \simeq 0$ and consequently $|r(t)| \simeq 0$.

To analytically evaluate the sojourn distribution $\psi(\tau)$ we thus use a first return time perspective. We write down the approximated equation driving the dynamics of the trajectories when $f(\mathbf{r}) \simeq 1$ and we notice that the trajectory enters in the basin of attraction of the state $f(\mathbf{r}) = 1$ with $|r(t)| \simeq 1$ and exits only when $|r(t)| \to 0$. Consequently the sojourn time in the laminar zones of the RM model should be equivalent to the first passage to the origin of the stochastic model driving the value of $r(t)$ when the approximation $f(\mathbf{r}) \simeq 1$ is valid (the case $f(\mathbf{r}) \simeq -1$ is completely symmetric).

Thus, the first passage time method [17] is applied to evaluate the distribution of the first times t_{fp} giving $r(t_{fp}) = 0$ under the condition $r(0) = 1$. The need to approximate the RM model equation is due to the fact that with the full equation an analytical calculation is impossible.

We start noticing that when $f(\mathbf{r}) \simeq 1$ Eq. (6) is approximated by

$$r(t+1) = \sigma_0 \eta(t) + f(\mathbf{r})^2 r(t), \tag{8}$$

thereby allowing us to generate the Stochastic Differential Equation (SDE)

$$\Delta r(t+1) = r(t+1) - r(t) = [f(\mathbf{r})^2 - 1] r(t) + \sigma_0 \eta(t). \tag{9}$$

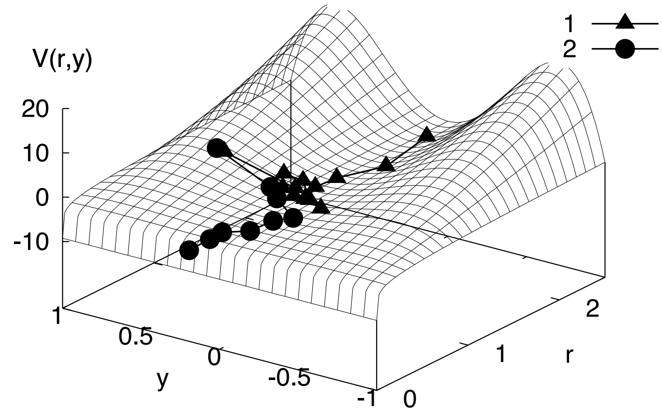

Figure 2: Logical scheme of simmetry-induced intermittency. The figure show the basin of attraction of the equilibrium manifold (the bottom of the valley, i.e. the line $y = 0$) with two possible trajectories: the first that enters in the regione with vanishing escape rate and the second that exits the laminar zone. The z-axis corresponds to the potential $V(r, y)$. Along the line $y = 0$ we have $V(r, 0) = \alpha \log r$.

Due to $f(\mathbf{r}) \simeq 1$, we may neglect the deterministic component of this equation, thereby getting $\mu = 3/2$, a well known result established by Sparre-Andersen [10]. We go beyond Sparre-Andersen adopting a reliable expression for the deterministic component of Eq.(9): We run the prescription of Eq.(7) with the vector $\tilde{\mathbf{r}}(t) = (r(t), r(t-1), ..., r(t-N+2))$ replaced by the expression

$$\tilde{\mathbf{r}} \simeq (r, r + \sigma_0 \eta(1), \ldots, r + \sigma_0 \sum_{j=1}^{N-1} \eta(j)), \tag{10}$$

resting on the approximation that for the whole time duration of the state $f(\mathbf{r}) \simeq 1$ the variable r changes only as an effect of the σ_0-fluctuations with no dependence on the still unknown deterministic component. A straightforward expansion to first order in σ_0/r yields $\langle [f(\mathbf{r})^2 - 1] \rangle \simeq \sigma_0^2/r^2$. We check this theoretical prediction numerically by comparing, in Fig. 4, the numerical result to

$$\langle [f(\mathbf{r})^2 - 1] \rangle \simeq \frac{\alpha}{r^2}, \text{ with } \alpha = (1/2)\sigma_0^2. \tag{11}$$

The factor of $1/2$ in $\alpha = (1/2)\sigma_0^2$ corrects the earlier crude approximation $\alpha = \sigma_0^2$, yields the excellent agreement with the numerical of Fig. 4, and, as we shall see hereby, has the remarkable property of establishing $\mu = 2$.

To analytically calculate the sojourn times distribution $\psi(\tau)$ under the approximation that led to Eq.(9) and to Eq. (11), let us proceed as follows. We plug $f(r)^2 - 1 = -\alpha/r^2$ into Eq. (9) and we obtain

$$\frac{dr}{dt} \simeq r_{t+1} - r_t = -\frac{\alpha}{r_t} + \sigma_0 \eta(t). \tag{12}$$

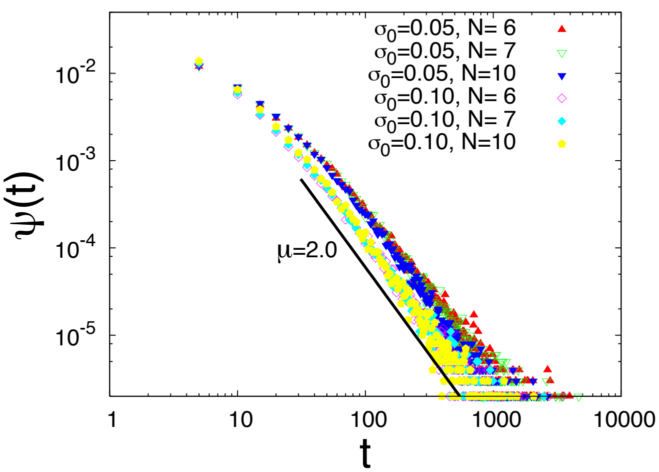

Figure 3: (Color online) Waiting time distribution in the state $f(\mathbf{r}) \simeq 1$ for $N = 6, 7, 10$ and $\sigma_0 = 0.05, 0.10$.

This SDE explains why the variable r exhibits the diffusion-like behavior illustrated by Fig. 1 and why the variable $r(t)$ from time to time comes back to the origin. Note that the first term on the r.h.s. of Eq. (12) has the effect of preventing the trajectory from freely departing from the origin so as to realize the Sparre-Andersen prescription $\mu = 3/2$. However, this control becomes weaker and weaker as the trajectory increases its distance r from the origin. If we imagine that this term is due to the gradient of a potential function V we should have $V = \alpha \log x$ like in Fig.2.

From a formal viewpoint the problem to solve is the evaluation of the first times of return to the origin of the following SDE

$$dx = -\frac{\alpha}{x}dt + \sigma_0 dW(t), \quad x(0) = x_0, \tag{13}$$

where $dW(t)$ are the increments of a Wiener process. The Fokker-Planck equation corresponding to Eq.(13) reads

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\alpha \frac{\rho(x,t)}{x}\right) + \frac{1}{2}\sigma_0^2 \frac{\partial^2}{\partial x^2}\rho(x,t), \tag{14}$$

with $\rho(x, 0) = \delta(x - x_0)$ and $\rho(0, t) = 0$. By Laplace transforming this equation, we obtain

$$s\hat{\rho}(x,s) - \rho(x,0) = -\frac{\alpha}{x^2}\hat{\rho}(x,s) + \frac{\alpha}{x}\hat{\rho}(x,s)' + \frac{1}{2}\sigma_0^2 \hat{\rho}(x,s)'', \tag{15}$$

the primes denoting spatial differentiation. Defining

$$\beta = 2\alpha/\sigma_0^2 > 0, \quad \bar{s} = 2s/\sigma_0^2 \tag{16}$$

we rewrite Eq. (15) as

$$x^2\hat{\rho}(x,s)'' + \beta x \hat{\rho}(x,s)' - (\beta + \bar{s}x^2)\hat{\rho}(x,s) = 0, \tag{17}$$

valid when $x \neq x_0$. This is the linear second-order differential equation of a modified Bessel kind. The use of the notation of [16] allows us to write two linearly independent solutions $\rho_<$ and $\rho_>$ as

$$\hat{\rho}_<(x,\bar{s}) = x^{1-\nu} I_\nu(\sqrt{\bar{s}}x), \;\; \hat{\rho}_>(x,\bar{s}) = x^{1-\nu} K_\nu(\sqrt{\bar{s}}x), \tag{18}$$

where $I_\nu(z), K_\nu(z)$ are the modified Bessel functions with index

$$\nu = (1+\beta)/2. \tag{19}$$

For $x \to +\infty$ only $\hat{\rho}_>(x,\bar{s})$ has a regular behavior. Thus for $x > x_0$ we set $\hat{\rho}(x,s) = \hat{\rho}_>(x,\bar{s})$. For $x \to 0$ only $\hat{\rho}_<(x,\bar{s})$ does fit the condition $\rho(0,s) = 0$, thereby making us set $\hat{\rho}(x,s) = \hat{\rho}_<(x,\bar{s})$, for $x < x_0$. The matching conditions are

$$\hat{\rho}_<(x_0,\bar{s}) = \hat{\rho}_>(x_0,\bar{s}), \;\; \hat{\rho}_<(x_0,\bar{s})' - \hat{\rho}_>(x_0,\bar{s})' = 2/\sigma_0^2. \tag{20}$$

We note that the second condition is obtained integrating from $x_0 - \epsilon$ to $x_0 + \epsilon$ Eq.(15). We impose both of them to the solution

$$\hat{\rho}(x,s) = \begin{cases} A\,\hat{\rho}_<(x,\bar{s}) \text{ if } \;x < x_0 \\ B\,\hat{\rho}_>(x,\bar{s}) \text{ if } \;x < x_0, \end{cases} \tag{21}$$

yielding, after some algebra

$$A = 2x_0^\nu/\sigma_0^2 K_\nu(\sqrt{\bar{s}}x_0) \tag{22}$$

where we use the following properties of the Bessel function [16]:

$$\begin{array}{lll} I'_\nu(z) = I_{\nu-1}(z) - \nu/z I_\nu(z), & K'_\nu(z) = -K_{\nu-1}(z) - \nu/z K_\nu(z), & I_\nu = (1/2z)^\nu \sum_{k=0}^{\infty} \frac{(1/4z^2)^k}{k!\Gamma(\nu+k+1)}, \\ K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, & K_\nu(z)I_{\nu-1}(z) + K_{\nu-1}(z)I_\nu(z) = 1/z, & \Gamma(\nu)\Gamma(1-\nu) = \pi/\sin(\nu\pi). \end{array} \tag{23}$$

Thus, for $x < x_0$

$$\hat{\rho}_<(x,\bar{s}) = \frac{2x_0^\nu}{\sigma_0^2} K_\nu(\sqrt{\bar{s}}x_0) x^{1-\nu} I_\nu(\sqrt{\bar{s}}x). \tag{24}$$

To evaluate the survival probability $\Psi(t) = \int_0^\infty \rho(x,t)\mathrm{d}x$ [17] we integrate all the terms of Eq.(14) and use $\psi(t) = -d\Psi(t)/dt$ to obtain for the Laplace transform of $\psi(t)$ the following expression:

$$\hat{\psi}(s) = \lim_{x\to 0}\left[\alpha\frac{\hat{\rho}(x,\bar{s})}{x} + \frac{\sigma_0^2}{2}\partial_x\hat{\rho}(x,\bar{s})\right] = \nu\sigma_0^2\partial_x\hat{\rho}(0,\bar{s})\big|_{x\to 0}. \tag{25}$$

For $\bar{s} \ll 1/x_0^2$, and thus for $t \gg 2x_0^2/\sigma_0^2$, we have

$$\begin{aligned} \hat{\psi}(s) \;\; &= \nu\sigma_0^2 \cdot \partial_x\hat{\rho}(x,\bar{s})\big|_{x\to 0} = 2\nu x_0^\nu K_\nu(\sqrt{\bar{s}}x_0)\bar{s}^{\nu/2}\tfrac{1}{2^\nu\Gamma(\nu+1)} \\ &= \nu\left(\tfrac{x_0}{2}\right)^\nu \tfrac{\bar{s}^{\nu/2}}{\Gamma(1+\nu)}\tfrac{\pi}{\sin(\nu\pi)}\left\{\left(\tfrac{1}{2}\sqrt{\bar{s}}x_0\right)^{-\nu}\sum_{k=0}^{\infty}\tfrac{(1/4\bar{s}x_0^2)^k}{k!\Gamma(1-\nu+k)} - \left(\tfrac{1}{2}\sqrt{\bar{s}}x_0\right)^{\nu}\sum_{k=0}^{\infty}\tfrac{(1/4\bar{s}x_0^2)^k}{k!\Gamma(1+\nu+k)}\right\} \\ &\simeq \tfrac{\pi\nu}{\Gamma(1+\nu)\sin(\nu\pi)}\left\{\tfrac{1}{\Gamma(1-\nu)} + \tfrac{\bar{s}x_0^2}{4\Gamma(2-\nu)} - \tfrac{(x_0/2)^{2\nu}}{\Gamma(1+\nu)}\bar{s}^\nu + \ldots\right\} \\ &\simeq 1 - \tfrac{x_0^2}{2(\nu-1)\sigma_0^2}s + \tfrac{\Gamma(2-\nu)}{(\nu-1)\Gamma(\nu+1)}s^\nu + O(s^2). \end{aligned} \tag{26}$$

Using the Tauberian properties of Laplace transform [19] and Eq. (19) we find that $\hat{\psi}(s)$ is the Laplace transform of an inverse power law distribution with μ and mean time value $\langle\tau\rangle$ given by

$$\mu = 1 + \nu = \frac{3}{2} + \frac{\beta}{2}, \tag{27}$$

If $\mu < 2$, $\langle\tau\rangle \to +\infty$ while if $\mu > 2$ we have

$$\langle\tau\rangle = \frac{x_0^2}{2(\nu - 1)\sigma_0^2}. \tag{28}$$

Notice that if $\beta = 0$ we have $\mu = 3/2$ recovering the well known result on the first passage times of a random walk [10, 17].

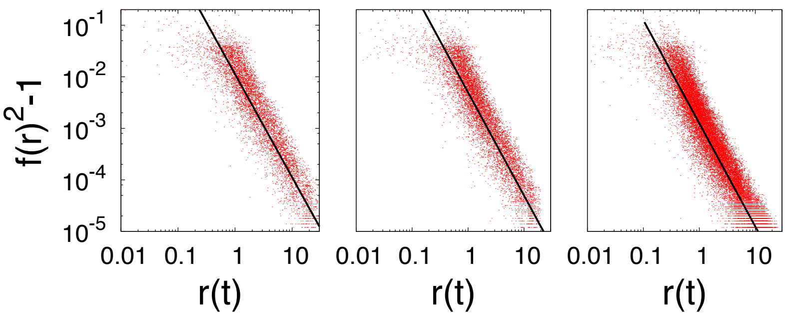

Figure 4: (Color online) $f(\mathbf{r})^2 - 1$ as a function of r for $N = 6$ and from top to bottom $\sigma_0 = 0.15, 0.10, 0.05$. The straight lines correponds to Eq.(11) with $\alpha = 1/2 \cdot \sigma_0^2$.

This result regard the first passage properties to the origin of Eq.(13) indipendently from the fact that this equation approximate the RM model. We now come back to the original problem of the permanence times statistics in the laminar zones of the RM model and we study the case of small values of σ_0 used to derive the results of Fig. 4. It is now evident that the choice made in Eq.(11) goes much beyond the remarkable result of fitting the numerical results of Fig. 3. In fact we see that plugging $\alpha = (1/2)\sigma_0^2$ into Eq. (16), yields $\beta = 1$ making μ of Eq.(27) exactly identical to 2. We note that the approximations adopted to replace the RM of Eq. (6) with the SDE of Eq.(12) may generate trajectories departing from those produced by Eq.(6) while maintaining however the same asymptotic properties and thus explaining why the RM model for smaller and smaller values of σ_0 yields results closer and closer to the condition $\mu = 2$. It is worth noticing that for small values of σ_0 when the approximated analytical calculation are valid the exponent μ is practically independent on σ_0. This show that the power law tail exponent in the RM model is robust under external additive noise that has the effect, as the reader can easily see in Eq.(6), of simply increasing σ_0.

4. Conclusions

The main result of this paper is the fact that a simple stochastic model, the RM, show intermittent behavior with an exponent $\mu \simeq 2$. We show that the intermittent behavior is related to the scaling simmetry of the system. The route towards intermittency proposed is alternative to the deterministic scheme adopted by Manneville in its well known paper [18]. We think that the main difference between our model and that of Manneville is that the equilibrium manifold is in our case stable along all different directions in the phase space but one along which it is neutral. On the contrary in the Manneville map the main ingredient is a marginally unstable equilibirum point. This difference is very important because in our case if we change the noise intensity σ_0 the intermittent behavior is not perturbed (and if the change in σ_0 is not so large also the exponent is not affected because σ_0 is present in both α and β). In the Manneville map it is sufficient a very low intensity external noise to destroys the asymptotic power law tail in the distribution of residence times that shows an exponential cutoff.

From the point of view of first return and first passage time the important result of this paper is that the model of Eq.(6) affords a way of exploring the region $\mu > 1.5$. The analytical derivation of $\mu = 2$ is of remarkable importance. Indeed the range near $\mu \simeq 2$ is important in different fields of research. For example the recent neurophysiological work of Ref. [20] proves that the global complexity of the brain, when mental activity is in action, generates intermittent events with $\mu = 2.05$. On the other end, the theoretical article of Aquino *et al*. [21] shows that the efficiency of the transport of information from one to another complex system becomes maximal when both systems are the sources of an ideal $1/f$ noise, namely, when both of them are characterized by intermittency with $\mu = 2$. Being the interaction of observer and system related to the problem of cognition his paper also supports the conjecture [8] of close connection between cognition in action and intermittency, insofar as experimental observation [11] proves that cognition in action make the brain yield $1/f$-noise.

The reader could ask why we are trying to extend the results obtained in a simple low-dimensional model to so different fields of research. The reason is that the main ingredient for our route towards intermittency is the presence of an equilibrium manifold with a non-constatnt rate of escape. We expect that this situation is not so rare because it is sufficient that the deterministic terms of the system to obey a symmetry property that isolated equilibrium points turn into equilibrium manifold. Being symmetry a very common property of dynamical system we think that our approach can be generalized to different fields of investigations.

A question to answer has to do with the nature of intermittency associated to the RM. Are criticality and intermittency generated by phase-transition processes? Bailly and Longo [22] argue that in biology the critical points are not isolated but belong to a dense subset of values. We have seen that changing the pair (N, σ_0) in the RM of Eq. (6) yields a set of critical values of μ close to $\mu = 2$, whereas the simplified model of Eq. (13), with α and β assumed to be independent variables makes μ range from 1.5 to $\mu = 2$ and beyond. For this reason we are inclined to believe that cognition-induced criticality is a form of extended criticality. The authors gratefully acknowledge financial support from Welch and ARO through Grant No. B-1577 and W911NF-11-1-0478, respectively.

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