
Laczkovich proved from ZF that, given a countable sequence of Borel sets on a perfect Polish space, if the limit superior along every subsequence was uncountable, then there was a particular subsequence whose intersection actually contained a perfect subset. Komjath later expanded the result to hold for analytic sets. In this paper, by adding AD and sometimes \( V=L(R) \) to our assumptions, we will extend the result further. This generalization will include the increasing of the length of the sequence to certain uncountable regular cardinals as well as removing any descriptive requirements on the sets.
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Throughout this paper we are assuming the axioms of ZF. We also equivocate between \( \mathbb{R} \) and \( \omega^\omega \).

**Definition 1.** Given a limit ordinal \( \lambda \), an unbounded \( S \subseteq \lambda \), and a sequence of sets \( \langle A_\alpha \rangle_{\alpha \in S} \), define the **limit superior** of \( \langle A_\alpha \rangle_{\alpha \in S} \) by

\[
\limsup_{\alpha \in S} A_\alpha = \{ x : \forall \alpha < \lambda \exists \alpha' > \alpha \ (\alpha' \in S \land x \in A_{\alpha'}) \}.
\]

That is, the limit superior is simply all those objects that are in cofinally many \( A_\alpha \)'s.

**Definition 2.** Let \( X \) be a set of ordinals and \( \kappa \) a cardinal. Define \( [X]^\kappa = \{ f : \kappa \to X : \forall \alpha < \beta \ f(\alpha) < f(\beta) \} \). The elements of \( [X]^\kappa \) can also be identified with their ranges. We are very abusive of this ambiguity. When viewing the elements as subsets of \( X \) we denote them with capital letters. When viewing them as functions we use standard functions variables, such as \( f, g \), etc. We say two functions \( f, f' \in [X]^\kappa \) are \( E_0 \)-equivalent if their ranges are the same for some point in \( X \) on. A set \( C \subseteq [X]^\kappa \) is \( E_0 \)-invariant if it is closed under \( E_0 \) equivalence.

We first start with the already established results on limit superiors of sets of reals.

**Theorem 1** (Laczkovich, 1977). [1] If \( \langle A_n \rangle_{n \in \omega} \) is a sequence of Borel sets of reals such that \( \forall f \in [\omega]^{\omega} \ (\limsup_{n \in \omega} A_{f(n)} \text{ is uncountable}) \), then for some \( f \in [\omega]^{\omega} \ (\bigcap_{n \in \omega} A_{f(n)} \text{ has a perfect subset}) \).

In 1982, Komjath extended Laczkovich’s result to hold for analytic sets (as well as to all sets when assuming Martin’s Axiom) [2]. Our main goal in this paper is, by strengthening the axiomatic assumptions, to generalize this phenomenon. Before we state our goal, however, we must first introduce some more preliminary concepts.
Definition 3. Consider a game between two players, traditionally referred to as I and II, with the following rules: Before the game starts, a set $A \subseteq \omega^\omega$, known as the payoff set, is fixed. I goes first, playing an integer $n_0$. II then plays an integer $n_1$, which is followed by I playing an integer $n_2$, etc. This continues for $\omega$-many turns. At the end of time, the two players have together built up a real $x = \langle n_0, n_1, n_2, \ldots \rangle$. We say that I wins the game if $x \in A$ and II wins otherwise. We say that the game is determined, or more often we refer to the set $A$ itself being determined, if one of the players has a winning strategy. To be clear, a winning strategy for I would be a function $\sigma$ from the set of all even length, finite sequences of integers into $\omega$ such that $\forall \langle n_0, n_1, n_2, \ldots \rangle \in \omega^\omega$ if $\forall k \sigma(\langle n_0, \ldots, n_{2k-1} \rangle) = n_{2k}$, then $\langle n_0, n_1, n_2, \ldots \rangle \in A$. The notion of a winning strategy for II is defined similarly. The Axiom of Determinacy, abbreviated AD, is the statement that every set of reals is determined. A winning strategy for one of the players can also be viewed as a continuous function. That is, if $\sigma$ is a winning strategy for I and $y \in \omega^\omega$, let $\sigma(y) = \text{the real that } \sigma \text{ demands I play when II plays } y$. A similar notion is defined for winning strategies for II.

Definition 4. Define the ordinal (actually, the cardinal) $\Theta$ as the supremum of the ranks of the prewellorderings on subsets of $\mathbb{R}$. If AC and CH hold, then $\Theta = \omega_2$. Once we assume AD, however, $\Theta$ becomes quite a large cardinal.

Theorem 2. Assume AD.

(i) Every set of reals has the perfect set property. That is, if $A \subseteq \omega^\omega$ is uncountable, then $A$ has a perfect subset. [3]

(ii) The well-ordered union of countable sets of reals is countable.

(iii) There is no uncountable, well-ordered sequence of distinct, countable sets of reals.

(iv) $\omega_1$ and $\omega_2$ are regular, but if $n \geq 3$, then $\text{cof}(\omega_n) = \omega_2$. [4]

(v) If $\kappa < \Theta$ has uncountable cofinality, then there is some $S \subseteq \mathbb{R}$ and a prewellordering $\prec$ on $S$ with length $\kappa$ that has the $\Sigma^1_1$-boundedness property. That is, if $A \subseteq S$ is $\Sigma^1_1$, then $\exists \alpha < \kappa \forall y \in A |y| < \alpha$, where $|y|$ is $y$'s rank in $\prec$. [5]
(vi) Suppose $\mathcal{X}, \mathcal{Y}$ are perfect Polish spaces and $R \subseteq \mathcal{X} \times \mathcal{Y}$ is such that $\forall x \in \mathcal{X} \exists y \in \mathcal{Y} \ R(x, y)$. Then $R$ has a continuous uniformizing function on some comeager subset of $\mathcal{X}$.

(vii) The Axiom of Countable Choice holds for sets of reals.

**Definition 5.** Informally, Godel’s *constructible universe*, denoted by $L$, is the smallest inner model of set theory. That is, it is the smallest model of set theory that contains $ON$, the class of ordinal numbers. There is another model that is of great interest to descriptive set theorists, namely $L(\mathbb{R})$, which is the smallest inner model that contains all of the reals.

**Definition 6.** A predicate $R \subseteq \omega^{n_1} \times (\omega^{n_2}) \times (\mathcal{P}(\omega^{n_3})$ is **projective** if is of the form $R(\vec{n}, \vec{x}, \vec{A}) \leftrightarrow \exists f_1, f_2, \ldots, f_m \in \omega^\omega \forall k \in \omega \ \varphi(\vec{n}, \vec{x}, \vec{A}, f_1, \ldots, f_m, k)$, where $\varphi$ is recursive with respect to some oracle machine. To make the definition clear, the predicate $\psi(x, A) \leftrightarrow x \in A$ is considered recursive. A predicate $R \subseteq \omega^{n_1} \times (\omega^{n_2}) \times (\mathcal{P}(\omega^{n_3})$ is $\Sigma^2_1$ if it is of the form $R(\vec{n}, \vec{x}, \vec{A}) \leftrightarrow \exists B \in \mathcal{P}(\omega^{n_2}) \ \varphi(\vec{n}, \vec{x}, \vec{A}, B)$, where $\varphi$ is projective. A predicate is $\Delta^2_1$ if both it and its complement are $\Sigma^2_1$.

The model $L(\mathbb{R})$ has some fascinating descriptive properties:

**Theorem 3.** Assume $AD$ and $V=L(\mathbb{R})$.

(i) Let $R \subseteq \mathcal{P}(\mathbb{R})$ be a projective relation (that is, a relation that only allows for real quantification). If $R$ is a non-empty relation, then there is some $\Delta^2_1$ set $A \subseteq \mathbb{R}$ such that $R(A)$.

(ii) Let $\mathcal{X}, \mathcal{Y}$ be perfect Polish spaces and $R \subseteq \mathcal{X} \times \mathcal{Y}$ a $\Delta^2_1$ ($\Sigma^2_1$) relation such that $\forall x \in \mathcal{X} \exists y \in \mathcal{Y} \ R(x, y)$. Then $R$ possesses a $\Delta^2_1$ ($\Sigma^2_1$) uniformizing function. [6]

(iii) $A \subseteq \omega^{n_2}$ is $\Sigma^2_1$ iff $A$ is $\kappa$-Suslin for some $\kappa < \Theta$. [6]

(iv) The Axiom of Dependent Choice, DC, is true. [4]
Definition 7. One of the topological spaces that we will make use of in chapter 2 is the Ellentuck space, \(\langle [\omega]^\omega, \varepsilon \rangle\). The basic open sets are of the form \([t, Z] = \{ X \in [\omega]^\omega : X \upharpoonright \ell(t) = t \land X \setminus t \subseteq Z \}\), where \(t \in 2^{<\omega}\) and \(Z \in [\omega]^\omega\).

Definition 8. A \(\subseteq [\omega]^\omega\) is completely Ramsey if for every \([t, Z]\) there is some \(Y \in [Z]^\omega\) such that either \([t, Y] \subseteq A\) or \([t, Y] \cap A = \emptyset\).

The following foundational facts are known about this space:

Theorem 4. (i) \([\omega]^\omega\) is a Baire space. [7]
(ii) A \(\subseteq [\omega]^\omega\) is \(\varepsilon\)-Baire iff it is completely Ramsey. [7]
(iii) (AD + V=L(R)) All subsets of \([\omega]^\omega\) are completely Ramsey. [8]

Definition 9. For \(n \geq 1\) let \(\delta_n^1 = \sup\{ |\prec| : \prec \text{ is a } \Delta_n^1 \text{ prewellordering on a subset of } \mathbb{R} \}\).

Definition 10. A regular cardinal \(\kappa\) has the strong partition property if, given any function \(h : [\kappa]^\kappa \to \{0, 1\}\) there is an \(H \in [\kappa]^\kappa\) and an \(i \in \{0, 1\}\) such that \(\forall f \in [H]^\kappa\) \(h(f) = i\).

Definition 11. For this purposes of this paper, when referring to a measure \(\mu\) on a regular, uncountable cardinal \(\kappa\) we will assume that \(\mu\) only takes values in \(\{0, 1\}\) and that its domain is \(\mathcal{P}(\kappa)\). If \(\varphi\) represents some unary formula on \(\kappa\), then the statement \(\mu(\{\alpha < \kappa : \varphi(\alpha)\}) = 1\) will be written as \(\forall^* \alpha \varphi(\alpha)\). We write \(f =_\mu f'\) if \(\mu(\{\alpha : f(\alpha) = f'(\alpha)\}) = 1\). Let \([\kappa]^\kappa/\mu\) be the naturally induced ultrapower, which will be another cardinal. A measure is normal if, given any function \(f : \kappa \to \kappa\) such that \(\forall^* \alpha f(\alpha) < \alpha\) (a "pressing down" function), there is some \(\gamma < \kappa\) such that \(\forall^* \alpha f(\alpha) = \gamma\).

A few introductory known results:

Theorem 5. Suppose AD is true. For \(n \geq 0\),

\footnote{It is still an open problem as to whether this fact follows from AD alone.}
(i) $\delta_{2n+2}^1 = (\delta_{2n+1}^1)^+ [9]\]

(ii) $\delta_{2n+1}^1$ has the strong partition property. [10]

(iii) If we define the ordinal $\omega(n)$ recursively by $\omega(0) = 1$ and $\omega(n + 1) = \omega^\omega(n)$, then $\delta_{2n+1}^1 = (\delta_{\omega(2n-1)}^1)^+$. In particular, $\delta_1^1 = \omega_1$, $\delta_3^1 = \omega_{\omega_1}$, and $\delta_5^1 = \omega_{\omega_{\omega_1}}$. [10]\

(v) The cardinals with the strong partition property go cofinally up into $\Theta$. [4]\

**Theorem 6 (Steel).** $(AD+V=L(\mathbb{R}))$ Let $\kappa$ be an uncountable, regular cardinal. Define $\mu_\omega : \mathcal{P}(\kappa) \to \{0, 1\}$ by $\mu_\omega(A) = 1$ iff $A$ possesses an unbounded subset that is closed under supremums of length $\omega$. $\mu_\omega$ is a normal measure. [11]\

**Definition 12.** Given an uncountable cardinal $\kappa$, a subset $F \subseteq \kappa$ is called **closed and unbounded**, or a **club**, if it is unbounded in $\kappa$ and is closed under supremums less than $\kappa$.\

**Definition 13.** A function $f \in [\omega_1]^{\omega_1}$ has **uniform cofinality $\omega$** if there is a function $g : \omega_1 \times \omega \to \omega_1$ such that $\forall \alpha < \omega_1 \ (\forall n \in \omega \ g(\alpha, n) < g(\alpha, n + 1)$ and $f(\alpha) = \sup g(\alpha, n))$. A function $f$ has **uniform cofinality $\alpha$** if there is a function $g : \{\langle \alpha, \beta \rangle : \beta < \alpha < \omega_1 \} \to \omega_1$ such that $\forall \beta < \beta' < \alpha < \omega_1 \ (g(\alpha, \beta) < g(\alpha, \beta'))$ and $f(\alpha) = \sup g(\alpha, \gamma)$. A function $f$ is said to be **of the correct type** if it is everywhere discontinuous (i.e. $f(\alpha) > \sup f(\beta)$) and has uniform cofinality $\omega$. A function $f$ is said to be **of the other type** if it is everywhere discontinuous and has uniform cofinality $\alpha$. Given a club $F \subseteq \omega_1$, let $[F]^{\omega_1}_{ic} = \{f \in [F]^{\omega_1} : f$ is of the correct type} and $[F]^{\omega_1}_o = \{f \in [F]^{\omega_1} : f$ is of the other type}. In Chapter 4 we will rely heavily upon the fact that almost all of the functions in $[\omega_1]^{\omega_1}$ are essentially one of the two types, which we will now show.\

**Theorem 7 (Kunen).** $[13](AD)$ There is a tree $T$ on $\omega \times \omega_1$ such that $\forall f \in [\omega]^{\omega_1}$ $\exists x \in \omega^\omega$ such that $T_x$ is well-founded and for all infinite $\alpha < \omega_1 \ f(\alpha) \leq |T_x \upharpoonright \alpha|$. $T$ is called the **Kunen tree**.\

**Proof.** Let $S$ be a recursive tree on $\omega \times \omega$ such that $p[S]$ is a $\Sigma^1_1$-complete set. Define a tree $U$ on $\omega \times \omega_1$ by $\langle \langle a_0, ..., a_{n-1} \rangle, \langle \alpha_0, ..., \alpha_{n-1} \rangle \rangle \in U$ iff $\forall i, j, k < n \ (a_{i,j}) = 1 \to \alpha_i <$
\[ \alpha_j \land [(a_{i,j} = 1 \land a_{j,k} = 1) \rightarrow a_{i,j} = 1], \text{ where } \langle \cdot \rangle : \omega^2 \rightarrow \omega \text{ is some canonical recursive bijection. Therefore, } p[U] = \text{WF}, \text{ the set of all reals coding well-founded relations on } \omega. \]

Now define a tree \( V \) on \( \omega \times \omega \times \omega \times \omega \) by \( \langle \vec{s}, \vec{a}, \vec{b}, \vec{c} \rangle \in V \) iff \( \langle \vec{a}, \vec{a} \rangle \in U, \langle \vec{b}, \vec{c} \rangle \in S, \) and there is some \( \tau \in \omega^\omega \) extending \( s \) such that \( \tau(\vec{a}) = \vec{b} \) (here viewing \( \tau \) as a strategy for player II in some game).

We claim that, after coding all the integer coordinates into a single integer, \( V \) will be our \( T \). Consider any \( f \in [\omega_1]^{\omega_1} \). We look at the game where I plays \( x \in \text{WF} \) and II plays \( y \in \omega^\omega \). II will win iff \( S_y \) is well-founded and \( |S_y| > \sup \{ f(\beta) : \beta \leq |x| \} \).

Suppose first that I has a winning strategy, \( \sigma \). \( \sigma[\omega^\omega] \) is a \( \Sigma^1_1 \) subset of \( \text{WF} \) and thus the ranks of its elements are bounded by some \( \gamma < \omega_1 \). Since \( p[S] \) is \( \Sigma^1_1 \)-complete, it is not Borel. Therefore, \( \sup \{ |S_y| : S_y \text{ is well-founded} \} = \omega_1 \). II can then defeat I's use of \( \sigma \) by playing any \( y \) with \( |S_y| \) sufficiently high.

Therefore, II must have a winning strategy, \( \tau \). \( V_T \) must be well-founded since, being a winning strategy for II, \( \tau \) guarantees that \( S_y \) is well-founded.

Fix some \( \alpha < \omega_1 \) and let \( x \in \text{WF} \) have rank \( \alpha \). We can find some \( \vec{a} \in \alpha^\omega \) such that \( \langle x, \vec{a} \rangle \in [U] \). \( |V_T \upharpoonright \alpha| \geq |S_{\tau(x)}| > \sup \{ f(\beta) : \beta \leq |x| \} \geq f(|x|) = f(\alpha) \).

Now letting \( x = \tau \) (i.e. the \( x \) in the statement of the theorem) gives us our result. \( \square \)

**Lemma 8.** If \( f \in [\omega_1]^{\omega_1} \), then \( \{ \alpha : \sup_{\beta < \alpha} f(\beta) = \alpha \} \) is a club.

**Proof.** First note that, since \( f(\alpha) < f(\beta) \leftrightarrow \alpha < \beta \), if \( f(\alpha) \neq \alpha \), then \( f(\beta) \neq \beta \) for all \( \beta > \alpha \). If \( f(\alpha) = \alpha \) for all \( \alpha \), then we are done. Assume not. Let \( F = \{ \alpha < \omega_1 : \sup_{\beta < \alpha} f(\beta) = \alpha \} \).

To show that \( F \) is non-empty and unbounded, let \( a_0 \) be some ordinal arbitrarily high such that \( f(a_0) \neq a_0 \). Define recursively \( a_{n+1} = f(a_n) \) and let \( \alpha = \sup_{n < \omega} a_n \). Note that \( a_n < a_{n+1} < \alpha \).

Let \( \beta \leq \alpha \) and \( n < \omega \). \( \sup_{\beta < \alpha} f(a_n) = \sup_{n < \omega} a_{n+1} = \alpha \). So, \( \alpha \in F \). Now let \( a_0 < a_1 < ... \) all be members of \( F \) and let \( \alpha = \sup_{n < \omega} a_n \). Since \( a_{n+1} \in F, f(a_n) \leq a_{n+1} \). \( \sup_{\beta < \alpha} f(\beta) \geq \alpha, \alpha \in F \). Therefore, \( F \) is closed. \( \square \)

**Theorem 9.** (AD) If \( L = \{ \alpha < \omega_1 : \alpha \text{ is a limit ordinal} \} \) and \( f \in [L]^{\omega_1} \) is such that \( \forall^* \alpha f(\alpha) > \alpha \), then \( f \) is almost everywhere one of the two types.
Proof. Note first that, via the previous lemma, since $f$ is not almost everywhere equal to the identity function it must be almost everywhere discontinuous. Let $T$ be the Kunen tree. We fix some canonical procedure $\langle, \rangle$ such that $\langle, \rangle : \alpha^{<\omega} \to \alpha$ is a bijection for all $\alpha$. Let $x \in \omega^\omega$ be such that $\forall \alpha f(\alpha) < |T_x \restriction \alpha|$. Therefore, $\forall \beta < f(\alpha) \exists u \in T_x \downarrow \alpha \, |u|_{T_x \restriction \alpha} = \beta$. Let $\lambda_\alpha = \lambda$ be the least $\gamma \leq \alpha$ such that $\sup\{|u|_{T_x \restriction \alpha} : \langle u \rangle \leq \gamma\} = f(\alpha)$.

First suppose that $\forall^* \alpha \lambda_\alpha = \alpha$. $f$ has uniform cofinality $\alpha$ almost everywhere via $g(\alpha, \beta) = |u|_{T_x \restriction \alpha}$ where $\langle u \rangle = \beta$.

Now suppose that $\forall^* \alpha \lambda_\alpha < \alpha$. The map $\alpha \mapsto \lambda_\alpha$ is then a pressing down function, so $\exists \gamma < \omega_1 \forall^* \alpha \lambda_\alpha = \gamma$. $\gamma$ has countable cofinality, so fix some $\langle \gamma_n \rangle_{n<\omega}$ cofinal sequence in $\gamma$ such that $\sup\{|u_n|_{T_x \restriction \alpha} : n \in \omega\} = f(\alpha)$, where $\langle u_n \rangle = \gamma_n$. $f$ has uniform cofinality $\omega$ almost everywhere via $g(\alpha, n) = |u_n|_{T_x \restriction \alpha}$. \hfill $\Box$

Lemma 10. (AD) If we define $\gamma_f$ as the ordinal in $[\omega_1]^{\omega_1}/\mu_\omega$ representing $f \in [\omega_1]^{\omega_1}$, then $\gamma_f$ is a limit ordinal whenever $f$ is one of the two types.

Proof. Let $f$ be, without loss of generality, of the correct type and let $g$ be the function that witnesses its uniform cofinality. Suppose $h \in [\omega_1]^{\omega_1}$ is such that $\gamma_h < \gamma_f$. Then $\forall^* \alpha h(\alpha) < f(\alpha)$. Define $h' \in [\omega_1]^{\omega_1}$ by $h'(\alpha) = g(\alpha, N)$, where $N$ is the least integer such that $g(\alpha, N) > h(\alpha)$. $N$ must exist by the definition of the uniform cofinality (in actuality $\forall^* \alpha N$ exists, which is all we really care about). Now we have $\forall^* \alpha h(\alpha) < h'(\alpha) < f(\alpha)$, so $\gamma_h < \gamma_h' < \gamma_f$, which shows that $\gamma_f$ cannot be a successor ordinal. A similar but simpler argument shows that $\gamma_f \neq 0$. \hfill $\Box$

Lemma 11. (AD) Given any partition $h : [\omega_1]^{\omega_1} \to \{0, 1\}$ there is a club $F \subseteq \omega_1$ and an $i \in \{0, 1\}$ such that $\forall f \in [F]^{\omega_1}_c \, h(f) = i$. This is also written as $\forall^* f \, f \in C_k$. The result would still hold if we were to replace "the correct type" with "the other type".

Definition 14. Let $\kappa$ be a cardinal and $\Gamma$ a pointclass. Define the Laczkovich-Komjath property by $LK(\kappa, \Gamma)$ iff whenever $\langle A_\alpha \rangle_{\alpha<\kappa}$ is a sequence of sets in $\Gamma$ such that $\limsup_{\alpha<\kappa} A_{f(\alpha)}$ is uncountable whenever $f \in [\kappa]^{<\kappa}$ is cofinal, there exists a particular $f \in [\kappa]^{<\kappa}$
cofinal such that $\bigcap_{\alpha<\kappa} A_{f(\alpha)}$ has a perfect subset. That is, Laczkovich proved $LK(\omega, B)$ and Komjath proved $LK(\omega, \Sigma^1_1)$, where $B$ is the pointclass of Borel sets.

We finally come to the main result.

**Theorem 12.** $(AD + V=L(\mathbb{R})) \forall \kappa LK(\kappa, \mathcal{P}(\omega))$

Note: As we show in chapter 3, if we only assume AD we still obtain an impressive result. That is, our main theorem will still hold whenever $\kappa$ has the strong partition property.
CHAPTER 2
THE COUNTABLE CASE

For the most part, we are following Komjath’s original proof for the analytic case. The only problem is what we are calling the Selection Lemma (see below). In Komjath’s situation, the Selection Lemma (he did not use this name) was quite easy to prove. In our situation, it takes a bit more effort.

**Definition 15.** If $f, g \in [\omega]^{\omega}$, define $f \leq g$ iff $\text{ran}(f) \subseteq \text{ran}(g)$. Let $f \leq^* g$ iff $\exists n \in \omega \ f\upharpoonright [n, \infty) \leq g$. When viewing elements of $[\omega]^{\omega}$ as subsets of $\omega$, we sometimes use the notation $G \leq H$ to mean $G \in [H]^{\omega}$.

**Definition 16.** Until we complete the proof of the Selection Lemma, let $\varepsilon$ be the Ellentuck topology on $[\omega]^{\omega}$.

**Lemma 13.** Let $A \subseteq [\omega]^{\omega}$ be $\varepsilon$-non-meager and $\varepsilon$-Baire. $A$ has non-empty $\varepsilon$-interior.

**Proof.** $A$ must be $\varepsilon$-comeager in some $\varepsilon$-basic open set, $[t, Z]$. Since $A$ is completely Ramsey, for some $Y \in [Z]^{\omega}$ either $[t, Y] \subseteq A$ or $[t, Y] \cap A = \emptyset$. But $[t, Z] \setminus A$ is $\varepsilon$-meager and $[t, Y] \subseteq [t, Z]$, so $[t, Y] \subseteq A$. □

**Theorem 14 (Well-Ordered Additivity of Ellentuck-Meager Sets).** ($AD+V=L(\mathbb{R})$) Let $\langle A_\alpha \rangle_{\alpha < \kappa}$ be a well-ordered sequence of subsets of $[\omega]^{\omega}$. If $A_\alpha$ is $\varepsilon$-meager for every $\alpha$, then $\bigcup_{\alpha < \kappa} A_\alpha$ is also $\varepsilon$-meager.

**Proof.** $^2$ Suppose not and let $\kappa$ be the least such cardinal. By the $\sigma$-additivity of category we must have $\kappa \geq \omega_1$. $\kappa$ must be regular. If not, then let $\lambda = \text{cof}(\kappa) < \kappa$ and let $\langle t_\beta \rangle_{\beta < \lambda}$ be a cofinal sequence in $\kappa$. Let $C_\beta = \bigcup_{\alpha < t_\beta} A_\alpha$. By the minimality of $\kappa$, $C_\beta$ must be $\varepsilon$-meager. But then, also by the minimality of $\kappa$, $\bigcup_{\alpha < \kappa} A_\alpha = \bigcup_{\beta < \lambda} C_\beta$ must be $\varepsilon$-meager, which is $^2$Thanks goes to [12] for the inspiration for this proof. They were able to prove a similar assertion arguing from AC, interestingly.
a contradiction.

Without loss of generality we may assume that the \( A_\alpha \)'s are disjoint. \( \kappa < \Theta \), so there must exist some \( S^* \subseteq \mathbb{R} \) and some prewellordering \( \prec^* \) on \( S^* \) with rank \( \kappa \) that has the \( \Sigma_1^1 \)-boundedness property. Let \( R^* = \{(x,y)|x \in A_{|y|}\} \subseteq [\omega]^\omega \times S^* \), where \( |y| \) is \( y \)'s rank in \( \prec^* \).

Therefore, the triplet \( \langle S^*, \prec^*, R^* \rangle \) can be considered a counterexample to the theorem. Now define the ternary relation \( \mathcal{F}(S, \prec, R) \leftrightarrow (\langle S, \prec, R \rangle \) is a counterexample to the theorem and \( \prec \) has the \( \Sigma_1^1 \)-boundedness property). That is, \( \mathcal{F}(S, \prec, R) \) iff

1. \( S \subseteq \mathbb{R} \)
2. \( \prec \) is a prewellordering on \( S \) with the \( \Sigma_1^1 \)-boundedness property.
3. \( R \subseteq [\omega]^\omega \times \mathbb{R} \)
4. \( \forall y \in S \ \exists X \in [\omega]^\omega \ R(X, y) \) (Non-triviality of sections)
5. \( \forall X \in [\omega]^\omega, y, y' \in S \ [R(X, y) \rightarrow (R(X, y') \leftrightarrow |y| = |y'|)] \) (Invariance and disjointness of sections)
6. \( R \) has meager sections on the left coordinate but the union of those sections is non-meager.

It can be checked that \( \mathcal{F} \) is a projective relation. By the existence of \( \langle S^*, \prec^*, R^* \rangle \), \( \mathcal{F} \) is non-empty. Therefore, \( \mathcal{F} \) must be satisfied by some \( \Delta_1^2 \) triplet. Let \( \langle S, \prec, R \rangle \) be such a triplet. Since \( R \) is at worst \( \Delta_1^2 \), it must have a uniformizing function, \( F \).

Enumerate the left-coordinate sections of \( R \) by \( \langle D_\alpha \rangle_{\alpha < \tau} \). By possibly switching to a subsequence of the \( D_\alpha \)'s, we assume without loss of generality that \( \tau \) is minimal. That is, given any \( \lambda < \tau \) any sub-union of the \( D_\alpha \)'s of length \( \lambda \) is meager.

By the previous lemma \( \bigcup_{\alpha < \tau} D_\alpha \) must contain some \([t, W] \). Let \( \langle U_n \rangle_{n \in \omega} \) be an enumeration of a basis for \( \mathbb{R} \) (with regards to the standard topology) with \( U_0 = \mathbb{R} \). We now recursively define a sequence \( \langle X_n \rangle_{n \in \omega} \) in \([\omega]^\omega \) in the following way: Let \( X_0 = W \). Recursively assume that for each \( k \leq n \)

1. \( X_{k+1} \in [X_k]^\omega \cap [t, \omega] \)
(2) For every \( s \subseteq \{ \min(X_j \setminus t) : j \leq k \} \) either \([t \cup s, X_k] \subseteq F^{-1}[U_k] \) or \([t \cup s, X_k] \cap F^{-1}[U_k] = \emptyset \).

By \( 2^{n+1} \) applications of the completely Ramsey property, we now choose an \( X_{n+1} \in [t, X_n \setminus \{ \min(X_n \setminus t) \}] \) that satisfies the recursive requirements. Now let \( Y = t \cup \{ \min(X_n \setminus t) : n \in \omega \} \). It follows that \( F|\![t, Y] \) is continuous. It is important to note that this is continuity in the topology that \([t, Y]\) inherits from the Cantor space, not the Ellentuck topology. Let \( K \) be the image of \([t, Y]\) under \( F \). \([t, Y]\) is Borel (once again, in the inherited topology), so \( K \) must be analytic. But since \( \prec \) has the \( \Sigma^1_1 \)-boundedness property, \( K \) must be \( \prec \)-bounded. By the minimality of \( \tau \), \( \bigcup_{y \in K} D|y \) must be \( \varepsilon \)-meager, where \( |y| \) is \( y \)'s rank in \( \prec \). But that set contains \([t, Y]\), which is a contradiction since no non-empty \( \varepsilon \)-open set can be \( \varepsilon \)-meager.

This gives us the following lemma.

**Lemma 15 (Selection Lemma).** \((AD+V=L(\mathbb{R}))\) Let \( \langle A_n \rangle_{n \in \omega} \) be a sequence of sets of reals and \( V = \bigcup_{\alpha < \kappa} V_\alpha \) a set of reals with \( \kappa < \Theta \). If \( V \cap \limsup_{n \in \omega} A_{f(n)} \) is uncountable for every \( f \in [\omega]^\omega \), then for some \( \alpha < \kappa \) and \( H \in [\omega]^\omega \) \( V_\alpha \cap \limsup_{n \in \omega} A_{f(n)} \) is uncountable for every \( f \in [H]^\omega \).

**Proof.** Let \( R(\alpha, f) \leftrightarrow V_\alpha \cap \limsup_{n \in \omega} A_{f(n)} \) is uncountable. Since \( \limsup_{n \in \omega} A_{f(n)} \) only depends on a tail end of \( f \), the sets \( R_\alpha = \{ f \in [\omega]^\omega : R(\alpha, f) \} \) are \( E_0 \)-invariant. Consider some \( f \in [\omega]^\omega \). By assumption \( \bigcup_{\alpha < \kappa} V_\alpha \cap \limsup_{n \in \omega} A_{f(n)} = \bigcup_{\alpha < \kappa} (V_\alpha \cap \limsup_{n \in \omega} A_{f(n)}) \) is uncountable. Therefore, there must be some \( \alpha < \kappa \) such that \( R(\alpha, f) \). This means that \( \bigcup_{\alpha < \kappa} R_\alpha = [\omega]^\omega \). Since \( [\omega]^\omega \) is \( \varepsilon \)-non-meager, by the previous theorem \( R_\alpha \) must be \( \varepsilon \)-non-meager for some \( \alpha \). \( R_\alpha \) must contain some \([t, H]\). \( R_\alpha \) is also \( E_0 \)-invariant, so in fact \([H]^\omega = [\emptyset, H] \subseteq R_\alpha \). □

The main theorem will follow as a corollary of the following theorem.

**Theorem 16 (Penultimate Claim).** \((AD+V=L(\mathbb{R}))\) \( \forall \kappa \ LK(\omega, S_\kappa) \), where \( S_\kappa = \{ A \subseteq \omega^\omega : A \text{ is } \kappa\text{-Suslin} \} \).
Also assume without loss of generality that the $A\kappa$ mentioned will not necessarily be the $\kappa$ of the Penultimate Claim.

Until we prove the Penultimate Claim we will consider $\langle A_n \rangle_{n \in \omega}$ to be fixed. We will also assume without loss of generality that the $A_n$'s are subsets of $\omega^\omega$.

**Definition 17.** Let $V$ be a set of reals and $f \in [\omega]^\omega$. We say $f$ is **good** for $V$ if $V \cap \limsup_{n \in \omega} A_{f(n)}$ is uncountable for every $g \leq f$. Obviously, $\leq$ can be replaced with $\leq^*$.

**Lemma 17.** (AD+$V=L(\mathbb{R})$) Let $F, A \subseteq \omega^\omega, f \in [\omega]^\omega$ with $F$ closed. If $f$ is good for $F \cap A$, then for some $x \in F$ and $f' \leq f$ $f'$ is good for $F \cap A \cap U$ for every open neighborhood $U$ of $x$.

**Proof.** Since $\omega^\omega$ is a separable metric space, $F$ can be written as $\bigcup_{n \in \omega} F_{(n)}$, where each $F_{(n)}$ is closed and $\text{diam}(F_{(n)}) < 1$. By the Selection Lemma, there is an $n_0 \in \omega$ and $f_0 \leq f$ such that $f_0$ is good for $F_{(n_0)} \cap A$. We now write $F_{(n_0)} = \bigcup_{n \in \omega} F_{(n_0, n)}$, where each $F_{(n_0, n)}$ is closed and $\text{diam}(F_{(n_0, n)}) < \frac{1}{2}$. Once again, by the Selection Lemma there is an $n_1 \in \omega$ and $f_1 \leq f_0$ such that $f_1$ is good for $F_{(n_0, n_1)} \cap A$. We recursively extend this procedure to produce $n_0, n_1, n_2, \ldots \in \omega$ and $f_0 \geq f_1 \geq f_2 \geq \ldots$. By the completeness of $\omega^\omega$, let $x$ be the unique element of $\bigcap_{k \in \omega} F_{(n_0, n_1, \ldots, n_k)}$.

Let $a_0$ be the least element of $f_0$. Recursively, let $a_{n+1}$ be the least element of $f_{n+1}$ that is greater than $a_n$. Let $f' = \{a_n : n \in \omega\}$. We claim that $f'$ works. Let $U$ be any open neighborhood of $x$. By the decreasing diameters, $F_{(n_0, \ldots, n_k)} \subseteq U$ for some $k$. Since $f_k$ is good for $F_{(n_0, \ldots, n_k)} \cap A$, it is also good for $F \cap A \cap U$. $f' \leq^* f_k$, so $f'$ is good for $F \cap A \cap U$. \qed

**Definition 18.** Let $P, Q \subseteq \omega^\omega$ be closed and non-empty. $\delta^*(P, Q) = \min\{\rho(x, y) : x \in P, y \in Q\}$, where $\rho$ is the standard metric on $\omega^\omega$.

**Corollary 1.** (AD+$V=L(\mathbb{R})$) Let $F, A \subseteq \omega^\omega, f \in [\omega]^\omega$, and $\varepsilon > 0$ with $F$ closed (we are no longer using $\varepsilon$ for the Ellentuck topology). If $f$ is good for $F \cap A$, then there is an $f' \leq f$ and perfect sets $F_0, F_1 \subseteq F$ such that $\text{diam}(F_i) < \varepsilon$, $\delta^*(F_0, F_1) > 0$ and $f'$ is good for both $F_0 \cap A$ and $F_1 \cap A$.  

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Proof. By the previous lemma, there is some \( x \in F \) and \( f_1 \leq f \) such that \( f_1 \) is good for \( F \cap A \cap U \) for every open neighborhood \( U \) of \( x \). For each \( n \in \omega \) let \( Q_n = \{ y \in F : \rho(x, y) \geq \frac{\varepsilon}{2n+1} \} \). Since \( F = \{ x \} \cup \bigcup_{n \in \omega} Q_n \) and no element of \([\omega]^\omega\) can be good for a singleton, the Selection Lemma says that there is a \( k \in \omega \) and an \( f_2 \leq f_1 \) such that \( f_2 \) is good for \( F \cap A \cap U \) for every open neighborhood \( U \) of \( x \). For each \( n \in \omega \) let \( Q_n = \{ y \in F : \rho(x, y) \geq \frac{\varepsilon}{2n+2} \} \). Since \( F = \{ x \} \cup \bigcup_{n \in \omega} Q_n \) and no element of \([\omega]^\omega\) can be good for a singleton, the Selection Lemma says that there is a \( k \in \omega \) and an \( f_2 \leq f_1 \) such that \( f_2 \) is good for \( Q_k \).

Write \( Q_k \) as \( Q_k = \bigcup_{n \in \omega} P_n \), where each \( P_n \) is closed and \( \text{diam}(P_n) < \varepsilon \). Once again, we use the Selection Lemma to get an \( n \in \omega \) and an \( f' \leq f_2 \) such that \( f' \) is good for \( P_n \). Now we let \( F_0 = \{ y \in F : \rho(x, y) \leq \frac{\varepsilon}{2n+2} \} \) and \( F_1 = P_n \). Note finally that \( F_i \) must be uncountable and closed. Under AD every set has the perfect set property, so without loss of generality we may reduce \( F_i \) to its non-empty perfect part. Since we are only removing countably many elements the goodness of \( f' \) has not been affected. \qed

Now,

Proof. (Penultimate Claim) Since \( A_n \) is \( \kappa \)-Suslin we can write it as \( A_n = \bigcup_{z \in \kappa^n} \bigcap_{k \in \omega} F^z|k \), where \( F^z|k \) is a closed set with \( \text{diam}(F^z|k) < \frac{1}{2n} \). This requires Countable Choice to pick out the Suslin representations for each \( A_n \). Given a \( t \in \kappa^{<\omega} \) define \( A^t_n = \bigcup_{z \in \kappa^n, t \subseteq z} \bigcap_{k \in \omega} F^z|k \).

Without loss of generality assume that \( A_0 = \omega^\omega \).

We now recursively define for each \( n \in \omega \) the following objects:

1. \( a_n \in \omega \)
2. A perfect set \( P_s \) for each \( s \in 2^n \)
3. A sequence \( t(k, s) \in \kappa^n \) for each \( k \leq n \) and \( s \in 2^n \)
4. \( H_n \in [\omega]^\omega \)

They will be defined so as to satisfy the following conditions:

i. \( a_n < a_{n+1} \in H_n \)
ii. \( H_{n+1} \in [H_n]^\omega \)
iii. If \( s \in 2^n \), then \( P_{st(0)} \cdot P_{st(1)} \subset P_s \) and \( \delta^s(P_{st(0)}, P_{st(1)}) > 0 \).
iv. If \( s \in 2^n \), then \( \text{diam}(P_s) < \frac{1}{2n} \).
(v) If \( s \in 2^n \), then \( H_n \) is good for \( P_s \cap \bigcap_{k=0}^n A^{t(k,s)}_{a_k} \).
(vi) If \( s \in 2^n \), then \( P_s \subseteq \bigcap_{k=0}^{n} F_{a_k}^{t(k,s)} \).

(vii) If \( s \in 2^n, k \leq n \), and \( i \in \{0, 1\} \), then \( t(k,s) \subseteq t(k,s \ast \langle i \rangle) \).

The construction goes as follows: Let \( a_0 = 0, P_0 = \omega^\omega \), and \( H_0 = \omega \). Obviously, \( t(0,\emptyset) = \emptyset \).

Now assume that all the objects have been chosen up to stage \( n \). We first claim that there exists \( b \in H_n, G \leq H_n \) such that \( b > a_n \) and, for every \( s \in 2^n \), \( G \) is good for \( P_s \cap \bigcap_{k=0}^{n} A_{a_k}^{t(k,s)} \cap A_b \).

Suppose not. Then we can recursively construct \( b_0 < b_1 < \ldots \) and \( H_n = G_0 \geq G_1 \geq \ldots \) such that \( b_n \in G_n \) and for each \( m \in \omega \) there is an \( s_m \in 2^n \) such that

\[
P_{s_m} \cap \bigcap_{k=0}^{n} A_{a_k}^{t(k,s_m)} \cap A_m \cap \lim sup_{t \in G_{m+1}} A_t
\]

is countable. Since \( s_m \) can only take at most \( 2^n \) different values there must be a particular \( s \in 2^n \) such that \( S = \{b_m : s_m = s\} \) is infinite. Therefore,

\[
P_{s_m} \cap \bigcap_{k=0}^{n} A_{a_k}^{t(k,s_m)} \cap \bigcup_{m \in S} A_m \cap \lim sup_{t \in S} A_t
\]

is countable, which is impossible by (v) and the fact that \( S \leq H_n \). Therefore, \( b \) and \( G \) must exist.

Let \( a_{n+1} = b \). By \( 2^n \) applications of Corollary 1 we get, for each \( s \in 2^n \), perfect sets \( P'_{s \ast \langle 0 \rangle}, P'_{s \ast \langle 1 \rangle} \subseteq P_s \) and \( G' \leq G \) such that \( \text{diam}(P'_{s \ast \langle i \rangle}) < \frac{1}{2^{n+1}}, \delta^*(P'_{s \ast \langle 0 \rangle}, P'_{s \ast \langle 1 \rangle}) > 0 \), and for every \( s \in 2^n \) \( G' \) is good for \( P'_{s \ast \langle i \rangle} \cap \bigcap_{k=0}^{n} A_{a_k}^{t(k,s)} \cap A_{a_{n+1}} \).

Now,

\[
\bigcap_{k=0}^{n} A_{a_n}^{t(k,s)} \cap A_{a_{n+1}} \subseteq \bigcup_{z_0} \bigcup_{z_1} \ldots \bigcup_{z_{n+1}} \bigcap_{k=0}^{n+1} A_{z_k}
\]

where \( z_0, \ldots, z_{n+1} \in \kappa^{n+1} \) and \( t(k,s) \subseteq z_k \) for each \( k \leq n \). That union has length \( \kappa \), so by \( 2^{n+1} \) applications of the Selection Lemma, we can select a particular \( \langle z_0, \ldots, z_{n+1} \rangle \) and a \( H_{n+1} \leq G' \) such that \( H_{n+1} \) is good for \( P'_{s \ast \langle i \rangle} \cap \bigcap_{k=0}^{n+1} A_{a_k}^{z_k} \). Now, for each \( k \leq n+1 \) let \( t(k,s \ast \langle i \rangle) \) be the corresponding \( z_k \).

Finally, let \( Q_{s \ast \langle i \rangle} = P'_{s \ast \langle i \rangle} \cap \bigcap_{k=0}^{n+1} F_{a_k}^{t(k,s \ast \langle i \rangle)} \). Since \( H_{n+1} \) is good for it, \( Q_{s \ast \langle i \rangle} \) must be uncountable. It is also closed, so we let \( P_{s \ast \langle i \rangle} \) be its perfect part. This finishes the construction.
Now we can finish the proof. Let $P = \bigcap_{n \in \omega} \bigcup_{s \in 2^n} P_s$. $P$ is perfect. Let $x \in P$. For each $n \in \omega$ there is a unique $s_n \in 2^n$ with $x \in P_{s_n}$. Not only this, but $s_0 \subseteq s_1 \subseteq \ldots$. By (vi) and (vii), if $n \geq m$, then $x \in F_{a_m}^{t(m, s_n)}$ and $t(m, s_m) \subseteq t(m, s_{m+1}) \subseteq \ldots$. Therefore, $x \in A_{a_m}^{t(m, s_m)} \subseteq A_{a_m}$ and so $P \subseteq \bigcap_{n \in \omega} A_{a_n}$. □

We finally complete our goal.

**Theorem 18.** $(AD + V = L(\mathbb{R})) \text{ LK}(\omega, \mathcal{P}(\omega))$

**Proof.** Suppose not. Define a unary relation $\mathcal{F}$ by $\mathcal{F}(\langle B_n \rangle_{n \in \omega})$ iff $\langle B_n \rangle_{n \in \omega}$ is a sequence of subsets of $\omega^\omega$ that is a counterexample to the Main Theorem. $\mathcal{F}$ is a projective relation and, by assumption, non-empty. Therefore, we can select a particular $\Delta^2_1$ sequence $\langle C_n \rangle_{n \in \omega}$ that is a counterexample. The $C_n$’s must be uniformly $\kappa$-Suslin for some $\kappa < \Theta$ and the Penultimate Claim yields a contradiction. □
CHAPTER 3
THE GENERAL CASE

In this chapter we prove the general case. That is, we consider uncountable sequences of sets of reals. It is a bit striking that the countable case takes considerably more effort than the, supposedly, more complicated uncountable case.

**Theorem 19.** (AD) If $\kappa$ has the strong partition property, then $LK(\kappa, P(\omega^\omega))$.

**Proof.** Suppose not. If we define $C_f = \bigcap_{\alpha < \kappa} A_f(\alpha)$, then $\forall f \in [\kappa]^\kappa C_f$ is countable. For each $f$ and $\beta < \kappa$ let $f_\beta$ be the function that enumerates $\{f(\alpha) : f(\alpha) > f(\beta)\}$ and let $f_{-1} = f$.

Note that if $\alpha < \alpha'$, then $C_f(\alpha) \subseteq C_f(\alpha')$. Since, under AD, there cannot be a strictly increasing uncountable union of countable sets of reals, $\forall f \exists \alpha \forall \alpha' > \alpha C_{f_{\alpha'}} = C_{f_\alpha}$. Let $\alpha_f$ be the least such $\alpha$. Define a partition on $[\kappa]^\kappa$ by $C_0 = \{f : \alpha_f = -1\}$ and $C_1 = \{f : \alpha_f > -1\}$.

Suppose first that $C_1$ has a homogeneous set, $H$. Let $f = H_{\alpha_f}$. Since $f \in [H]^\kappa$, $C_{f_0} \neq C_f$.

But $C_{f_0} = C_{H_{\alpha_{f+1}}} = C_{H_{\alpha_f}} = C_f$, a contradiction.

Therefore, $C_0$ has a homogenous set, $H$. Suppose that $x \in \limsup_{\alpha \in H} A_\alpha$. By a simple induction, it follows from the choice of $H$ that, given any $f, f' \in [H]^\kappa$, if $f$ and $f'$ are $E_0$-equivalent, then $C_f = C_{f'}$. From there we get $\forall f \in [H]^\kappa \forall \beta C_f \subseteq \bigcap_{\alpha < f(\beta), \alpha \in H} A_\alpha$. Now $x \in C_f$ for some $f \in [H]^\kappa$, $\forall \beta x \in \bigcap_{\alpha < f(\beta), \alpha \in H} A_\alpha$, which means that $x \in C_H$. Therefore, $\limsup_{\alpha \in H} A_\alpha \subseteq C_H$. But $\limsup_{\alpha \in H} A_\alpha$ is uncountable, whereas $C_H$ is countable, a contradiction. $\square$

**Theorem 20.** (AD + $V=L(\mathbb{R})$) $\forall \kappa > \omega LK(\kappa, P(\omega^\omega))$

**Proof.** We may first assume that $\kappa$ is regular. If not, then we switch to some regular cofinal subsequence and start again. Let $P = \limsup_{\alpha \in H} A_\alpha$. By assumption, $P$ is uncountable. For each $x \in P$ define $g_x : \kappa \to \kappa$ by $g_x(\alpha) = \beta$ such that $x \in A_\beta$. $\mu_\omega$ is a normal measure, so we let $\lambda = [\kappa]^\kappa / \mu_\omega$ and $\gamma_f$ be ordinal in the ultrapower representing $f \in [\kappa]^\kappa$.

For each $\beta < \lambda$, let $G_{\beta} = \{x \in P : \gamma_f = \beta\}$. Under AD we cannot have an uncountable, well-ordered union of distinct countable sets, so there must be some $\beta < \lambda$ such that $G_\beta$ is
uncountable. Let $P'$ be some perfect subset of $G_\beta$ and pick some $g$ such that $\gamma_g = \beta$. That means for each $x \in P' g_x$ will equal $g$ on some measure 1 set (with regards to $\mu_\omega$, of course). This gives us the statement

$$\forall x \in P' \ \forall^* \alpha \ x \in A_{g(\alpha)}$$

The goal is to switch the quantifiers by switching to an uncountable $P'' \subseteq P'$. Note that a subset of $\kappa$ will have measure 1 if it has an unbounded subset that is closed under limits of length $\omega$. Every function $f : \kappa \rightarrow \kappa$ such that $\forall \alpha \ f(\alpha) > \alpha$ defines an $\omega$-closed set. That is, the closed set is $\{ \alpha < \kappa : \alpha$ has countable cofinality and $\forall \beta < \alpha \ f(\beta) < \alpha \}$. We will now show that each $x \in P'$ has an $\omega$-closed set that works for (4) (that is, $x \in A_{g(\alpha)}$ for each $\alpha$ in that set) such that the increasing function that defines the closed set can actually be coded into a real. First consider some $x \in P'$, some $\omega$-closed set $C$ that works for (4), and some prewellordering of $S \subseteq \mathbb{R}$ of length $\kappa$ with the $\Sigma^1_1$-boundedness property (the prewellordering is, of course, chosen independently from $x$). Now we consider a game $G(C)$ where players I and II play reals $y_1, y_2 \in \mathbb{R}$ against each other and II wins iff $y_1 \in S \rightarrow (y_2 \in S \ \wedge \ |y_1| < |y_2| \in C$. We claim that II has a winning strategy in $G(C)$. Suppose not and let $\sigma$ be a winning strategy for I. $\sigma[\mathbb{R}] \subseteq S$ is $\Sigma^1_1$ and hence bounded by some $\alpha < \kappa$. II can then defeat I’s use of $\sigma$ by playing any $y_2$ with $\alpha \leq |y_2| \in C$.

Now define the relation $R(x, \tau) \leftrightarrow$

$$x \in P' \ \wedge \ \tau \text{ is a winning strategy for II in } G(C) \text{ for some } C \text{ that works for (4)}$$

Since $P'$ has a perfect subset, we can use AD to uniformize $R$ continuously on a comeager subset. That is, there is some uncountable $Q \subseteq P'$ and a continuous function $x \mapsto \tau_x$ on $Q$ such that $R(x, \tau_x)$. Let $P'' \subseteq Q$ be a perfect set. For each $x \in P''$, let $C_x$ be the $\omega$-closed set defined by the map $h_x(\alpha) = \inf \{ \tau_x(y) : |y| = \alpha \}$. Now we will define a master $\omega$-closed set that will allow us to switch the quantifiers in (4). For each $y \in S$ the map $x \mapsto \tau_x(y)$ is continuous and so as before must be bounded. So, define $h(\alpha) = \sup_{x \in P''} h_x(\alpha)$ and let $C$ be the $\omega$-closed set defined by $h$. Let $x \in P''$ and $\alpha \in C$. By the definition of $C$, $\alpha$ has
countable cofinality and $\forall \beta < \alpha \ h(\beta) < \alpha$. Since $h_x(\beta) \leq h(\beta)$, we have $\alpha \in C_x$. Therefore,

$C \subseteq \bigcap_{x \in P''} C_x$.

Finally we have switched the quantifiers in (4) as desired. That is,

\begin{equation}
\forall^* \alpha \ \forall x \in P'' \ x \in A_{g(\alpha)}
\end{equation}

where the "$\forall^* \alpha$" means "$\forall \alpha \in C$". We now have an uncountable $P'' \subseteq \bigcap_{\alpha \in C} A_{g(\alpha)}$, which completes the proof. \qed
CHAPTER 4
GENERALIZING THE SELECTION LEMMA

In this chapter, we will develop some generalizations of the Selection Lemma that was used in chapter 2. It was originally thought that the generalization of the main theorem to uncountably long sequences of sets of reals would follow a similar line of argumentation to the countable case. Even though the proof ended up going in a completely different direction, the following chapter is an interesting discussion on its own. We will be assuming AD for this chapter.

**Definition 19.** For this chapter we will adopt the following notation: Given any ordinal \( \beta < \omega_2 \), there is some equivalence class of functions on \( \omega_1 \) that \( \beta \) represents in the ultrapower. Let \( \pi(\beta) \) be a function in that equivalence class. There is an abuse here, since there is no way to actually pick out a representative function. However, all functions in the equivalence class will agree almost everywhere and so we will use this notation when the ambiguity won’t cause any problems.

**Definition 20.** Given a limit ordinal \( \lambda \) define temporarily \( C_\lambda = \{ f : \xi \to \lambda : \xi \leq \lambda \wedge f \text{ is increasing and unbounded} \} \). Given two infinite cardinals \( \lambda, \kappa < \theta \) define Sel(\( \lambda, \kappa \)) iff given any sequence of sets of reals \( \langle A_\alpha \rangle_{\alpha < \lambda} \) and a set of reals \( V = \bigcup_{\beta < \kappa} V_\beta \), if \( V \cap \limsup_{\alpha \in \text{dom}(f)} A_{f(\alpha)} \) is uncountable for every \( f \in C_\lambda \), then for some \( g \in C_\lambda \) and \( \beta < \kappa \) \( V_\beta \cap \limsup_{\alpha \in \text{dom}(f)} A_{(g \circ f)(\alpha)} \) is uncountable for every \( f \in C_{\text{dom}(g)} \). This is an unpleasantly complicated definition, but note that it is simply a generalization of the property seen in the second chapter’s Selection Lemma. That is, the Selection Lemma simply states that Sel(\( \omega, \kappa \)) holds for every \( \kappa < \theta \).

**Lemma 21.**

(i) Sel(cof(\( \lambda \)), \( \kappa \)) \( \rightarrow \) Sel(\( \lambda, \kappa \))

(ii) If \( \kappa \) is singular, then \( (\forall \xi < \kappa \text{ Sel}(\lambda, \xi)) \rightarrow \text{Sel}(\lambda, \kappa) \).

**Proof.**

(i) Let \( \lambda \) be a cardinal and let \( \xi = \text{cof}(\lambda) \). Fix a sequence of reals \( \langle A_\alpha \rangle_{\alpha < \lambda} \) that satisfies the premise of Sel(\( \lambda, \kappa \)). Let \( h : \xi \to \lambda \) be a cofinal function. Since
ξ is regular and \( \langle A_{h(a)} \rangle_{a < \xi} \) satisfies the premise of Sel(ξ, κ), there must be some \( f \in [\xi]^\xi \) and \( \beta < \kappa \) granted by the conclusion of Sel(ξ, κ). \( h \circ f \) and \( \beta \) now satisfy the conclusion of Sel(λ, κ).

(ii) Let \( ξ = \text{cof}(κ) \) and let \( \langle t_β \rangle_{β < ξ} \) be some cofinal sequence. For each \( β < ξ \) let \( W_β = \bigcup_{τ < t_β} V_τ \). \( \bigcup_{β < ξ} W_β = \bigcup_{β < κ} V_β \) and Sel(λ, ξ), so there exists some \( f \in C_λ \) and some \( β < ξ \) that satisfy the conclusion of Sel(λ, ξ) (using \( W_β \) instead of \( V_β \)). Let \( β' = |β| \). \( β' < κ \), so Sel(λ, β') holds. Therefore, there exists some \( g \in C_{\text{dom}(f)} \) and some \( τ < β \) such that \( f \circ g \) and \( τ \) satisfy the conclusion of Sel(λ, β') (using \( V_τ \), \( τ < β \)). But \( f \circ g \) and \( τ \) also satisfy the conclusion of Sel(λ, κ).

□

Therefore, we will now restrict our attention to regular λ’s and also realize that, given a λ, the first κ for which Sel(λ, κ) can fail must be regular. This gives us a much cleaner definition of Sel that allows us to forget about the definition of \( C_λ \).

**Theorem 22.** Sel(ω₁, ω)

**Proof.** Suppose not and let \( \langle \langle A_a \rangle_{a < ω_1}, \langle V_n \rangle_{n < ω} \rangle \) be a counterexample. We will first prove that there is a club \( F' \subseteq ω_1 \) and an \( n_1 < ω \) such that for every \( f \in [F]^{ω_1}_{c} V_{n_1} \cap \limsup_{a < ω_1} A_{f(a)} \) is uncountable. Suppose not. Let \( C_n = \{ f \in [ω_1]^{ω_1}_{c} : V_n \cap \limsup_{a < ω_1} A_{f(a)} \text{ is countable} \} \). \( C_n \) and \( [ω_1]^{ω_1}_{c} \setminus C_n \) obviously partition \( [ω_1]^{ω_1}_{c} \). Since we are supposing our claim is false, \( [ω_1]^{ω_1}_{c} \setminus C_n \) cannot have a homogeneous club. Therefore, \( C_n \) must have a homogeneous club \( F_n \). We can select an \( F_n \) for each \( n \) using countable choice. Let \( F = \bigcap_{n < ω} F_n \). \( F \) is a club, so \( \forall f \in [F]^{ω_1}_{c} \left( V_n \cap \limsup_{a < ω_1} A_{f(a)} \text{ is countable} \right) \). But \( \bigcup_{n < ω} \left( V_n \cap \limsup_{a < ω_1} A_{f(a)} \right) = V \cap \limsup_{a < ω_1} A_{f(a)} \), which is uncountable. Since the countable union of countable sets is countable, we have a contradiction. This proves our claim.

By a similar argument, we construct a club \( F \subseteq F' \) and an \( n_2 < ω \) such that for every \( f \in [F]^{ω_1}_{c} V_{n_2} \cap \limsup_{a < ω_1} A_{f(a)} \) is uncountable. Since our theorem fails, there must be
some $f \in [F]^{\omega_1}$ such that $(V_{n_1} \cup V_{n_2}) \cap \lim sup_{\alpha<\omega_1} A_{f(\alpha)}$ is countable. By possibly thinning $f$ out, we may assume that $f = f' \circ g$ for some $f' \in [F]^{\omega_1}$ of one of the types. Without loss of generality, we assume that $f' \in [F]^{\omega_1}$. Let $G \subseteq \omega_1$ be some club such that $f(\alpha) = f'(\alpha)$ for every $\alpha \in G$. Let $k \in [\omega_1]^{\omega_1}$ enumerate $G$ and let $h = f' \circ g$. Since $f'$ is in $[F]^{\omega_1}$, so is $h$. But $V_{n_1} \cap \lim sup_{\alpha<\omega_1} A_{h(\alpha)} = V_{n_1} \cap \lim sup_{\alpha<\omega_1} A_{f'(\alpha)} = V_{n_1} \cap \lim sup_{\alpha<\omega_1} A_{f(\alpha)}$, which is a contradiction since $V_{n_1} \cap \lim sup_{\alpha<\omega_1} A_{h(\alpha)}$ must be uncountable.

\textbf{Theorem 23.} \textit{Sel}(\omega_1, \omega_1)

\textbf{Proof.} Consider an appropriate $(\langle A_\alpha \rangle_{\alpha<\omega_1}, \langle V_\beta \rangle_{\beta<\omega_1})$. For each $f \in [\omega_1]^{\omega_1}$ let $\beta_f$ be the least ordinal such that $V_{\beta_f} \cap \lim sup_{\alpha<\omega_1} A_{f(\alpha)}$ is uncountable. Let $C_0 = \{ f \in [\omega_1]^{\omega_1} : \beta_f < f(0) \}$ and $C_1 = \{ f \in [\omega_1]^{\omega_1} : \beta_f \geq f(0) \}$. Suppose that $C_1$ has a homogeneous set $G$. Let $G' = \{ \alpha \in G : \alpha > \beta_G \}$. $G'E_0G$, so $\beta_{G'} = \beta_G$. But $\beta_{G'} < G'(0)$, a contradiction. Therefore, let $H$ be a homogeneous set for $C_0$. Let $f \in [H]^{\omega_1}$. Let $f' = f \cup \{ H(0) \}$. $\beta_f = \beta_f$, so $\beta_f < H(0)$. Since $H(0)$ is a countable ordinal, the pair $(\langle A_\alpha \rangle_{\alpha \in H}, \langle V_\beta \beta < H(0) \rangle)$ satisfies the conditions of $\textit{Sel}(\omega_1, \omega)$, which we know to be true.

\textbf{Theorem 24.} \textit{Sel}(\omega_1, \omega_2)

\textbf{Proof.} We will assume without loss of generality that the $V_\beta$'s are increasing and continuous at limit stages. By the same argument given in the proof of $\textit{Sel}(\omega_1, \omega)$, we will first limit ourselves to only those functions of the correct type. For each $f \in [\omega_1]^{\omega_1}$ let $\beta_f$ be as before.

\textbf{Claim 1.} Each $\beta_f$ is a successor ordinal.

\textbf{Proof.} Suppose $\beta_f$ is a limit ordinal. $V_{\beta_f} = \bigcup_{\beta<\beta_f} V_\beta$. First suppose that $\beta_f$ has countable cofinality, say with some cofinal sequence $(\gamma_n)$. $V_{\beta_f} \cap \lim sup_{\alpha<\omega_1} A_{f(\alpha)} = \bigcup_{n<\omega} (V_{\gamma_n} \cap \lim sup_{\alpha<\omega_1} A_{f(\alpha)})$, which is a countable union of countable sets, violating the definition of $\beta_f$. Therefore, $\beta_f$ has uncountable cofinality. Let

\begin{equation}
C = \{ \beta < \beta_f : (V_\beta \setminus \bigcup_{\gamma<\beta} V_{\gamma}) \cap \lim sup_{\alpha<\omega_1} A_{f(\alpha)} \neq \emptyset \}
\end{equation}
By the definition of $\beta_f$, $C$ must be cofinal in $\beta_f$ and therefore uncountable. But then $\langle (V_\beta \setminus \bigcup_{\gamma<\beta} V_\gamma) \cap \limsup_{\alpha<\omega_1} A_{f(\alpha)} \rangle_{\beta \in C}$ is an uncountable sequence of distinct non-empty countable sets, which is impossible under AD. \qed

Since $\gamma_f$ must be a limit ordinal when $f$ is of the correct type, $\beta_f \neq \gamma_f'$ for every $f, f'$.

**Claim 2.** $\forall f \beta_f < \gamma_f$

**Proof.** Suppose not. Since $\beta_f \neq \gamma_f$ for every $f$, $\forall f \gamma_f < \beta_f$. By the definition of $\beta_f$, $V_{\beta_f} \cap \limsup_{\alpha<\omega_1} A_{f(\alpha)}$ is countable for all $f$'s of the correct type within some club $F$. That means that, given any $f \in [F]^{\omega_1}$, all but countably many $x \in V_{\gamma_f}$ are only in countably many $A_{f(\alpha)}$. That countable set of reals will most likely vary between $f$'s, even those $f$'s that agree with each other almost everywhere. Nevertheless, we have the following:

**Sub-Claim.** $\forall f \in [F]^{\omega_1} V_{\beta_f} \cap \limsup_{\alpha \in F} A_\alpha$ is countable.

**Proof.** Suppose not and fix a counterexample, $f$. For each $x \in V_{\gamma_f} \cap \limsup_{\alpha \in F} A_\alpha$ let $g_x$ enumerate $\{ \alpha \in F : x \in A_\alpha \}$. By possibly thinning out $g_x$ we may assume that $\gamma_{g_x} > \gamma_f$ (note that $g_x$ may not be of the correct type, which is okay). Since the well-ordered union of countable sets is countable, there must be some $g \in [F]^{\omega_1}$ (once again, not necessarily of the correct type) such that $\{ x \in V_{\gamma_f} \cap \limsup_{\alpha \in F} A_\alpha : \gamma_{g_x} = \gamma_g \}$ is uncountable. Fix such a $g$ and call that uncountable set $W$. Since $\gamma_g > \gamma_f$, $\forall^* \alpha f(\alpha) < g(\alpha)$. Define recursively a function $f'$ in the following way:

(i) $f'(0) = f(0)$

(ii) If $\alpha$ is an "even" ordinal with $f'(\alpha) = f(\alpha')$, then $f'(\alpha + 1) = g(\alpha')$.

(iii) If $\alpha$ is an "odd" ordinal, then $f'(\alpha + 1)$ is the least value of $f$ greater than $f'(\alpha)$.

(iv) If $\lambda$ is a limit ordinal, then $f'(\lambda)$ is the least value of $f$ greater than $\sup_{\alpha<\lambda} f'(\alpha)$.

Given the fact $\forall^* \lambda \sup_{\alpha<\lambda} f'(\alpha) = \lambda = \sup_{\alpha<\lambda} f(\alpha)$, we have $\forall^* \alpha f(\alpha) = f'(\alpha)$. Furthermore, since $\forall^* \alpha (f(\alpha) \text{ is in the range of } f')$, $\forall^* \alpha (g(\alpha) \text{ is in the range of } f')$. Now, given any $x \in W$, 22
\( \forall \alpha \, g_x(\alpha) = g(\alpha) \), which means that \( \forall \alpha \) \((g_x(\alpha)\) is in the range of \( f' \)). But then
\[
(8) \quad x \in V_{\gamma_f} \cap \limsup_{\alpha<\omega_1} A_{f'(\alpha)} = V_{\gamma_f} \cap \limsup_{\alpha<\omega_1} A_{f'(\alpha)}
\]
which means that \( W \subseteq V_{\gamma_f} \cap \limsup_{\alpha<\omega_1} A_{f'(\alpha)} \), contradicting the fact that the latter set is countable. \( \square \)

Let \( C_f = V_{\gamma_f} \cap \limsup_{\alpha \in F} A_\alpha \). The assignment \( f \mapsto C_f \) is \( \mu_\omega \) invariant, so, for some particular countable set \( C \), \( \forall^* f \) (relative to \( F \)) \( C_f = C \). By removing those countably many reals, we may assume that \( C = \emptyset \). Call the club that witnesses this fact \( F' \) (so, \( F' \subseteq F \)). In other words, \( \forall f \in [F']_{\omega_1} \forall x \in V_{\gamma_f} \, x \not\in \limsup_{\alpha \in F'} A_\alpha \). Since every \( x \in V \) is eventually in some \( V_{\gamma_f} \), we have \( \forall x \in V \, x \not\in \limsup_{\alpha \in F'} A_\alpha \), which violates the assumption of the theorem. \( \square \)

Consider pairs \( \langle h, f \rangle \) of functions of the correct type such that \( \sup_{\beta<\alpha} f(\beta) < h(\alpha) < f(\alpha) \).

Claim 3. \( \forall^* \langle h, f \rangle \beta_f < \gamma_h \)

Proof. Suppose not. Then \( \forall^* \langle h, f \rangle \beta_f > \gamma_h \). So,
\[
(9) \quad \forall^* \langle h, f \rangle \forall \alpha \, h(\alpha) = \pi(\gamma_h)(\alpha) < \pi(\beta_f)(\alpha)
\]
Let \( f \) be a function of the correct type and let \( g : \omega_1 \times \omega \to \omega_1 \) be a function that witnesses the fact that \( f \) has uniform cofinality \( \omega \). By the definition of \( g \),
\[
(10) \quad \forall^* \alpha \exists n < \omega \, \pi(\beta_f)(\alpha) < g(\alpha, n) > \sup_{\beta<\alpha} f(\beta)
\]
\( \mu_\omega \) is countably complete, so
\[
(11) \quad \exists N < \omega \forall^* \alpha \, \pi(\beta_f)(\alpha) < g(\alpha, N) > \sup_{\beta<\alpha} f(\beta)
\]
Define \( h \) by \( h(\alpha) = g(\alpha, N) \). \( \langle h, f \rangle \) is therefore one of the pairs we are considering. Therefore,
\[
(12) \quad \forall^* \alpha \, h(\alpha) < \pi(\beta_f)(\alpha) < g(\alpha, N) = h(\alpha)
\]
which is a contradiction. \( \square \)
Let $F \subseteq \omega_1$ be some club such that $\forall \langle h, f \rangle \in [F]^{\omega_1}_c \beta_f < \gamma_h$. By renaming we may assume that $F = \omega_1$. Consider some $f \in [\omega_1]^{\omega_1}_c$. Define $h_f$ by $h_f(\alpha) = \sup_{\beta < \alpha} f(\beta) + 1$. $\langle h_f, f \rangle$ is one of the pairs we are considering, so $\beta_f < \gamma_{h_f}$. But $\forall f \forall^* \alpha h_f(\alpha) = \alpha + 1$. Therefore, if we define $h$ by $h(\alpha) = \alpha + 1$, then $\beta_f < \gamma_h$ for every $f$. But this says that the $\beta_f$'s are bounded and therefore there are only $\omega_1$ of them. Since we know that Sel($\omega_1, \omega_1$) is true, we are done. This concludes the consideration of functions of the correct type.

We will now consider functions of the other type. The argument actually follows exactly the same as before, with one exception: In the last claim the function $h$ must be defined differently. The argument goes as follows: Fix a function $f$ of the other type and let $g$ be a function that witnesses the fact that $f$ has uniform cofinality $\alpha$. Since $\beta_f < \gamma_f$,

$$\forall^* \alpha \exists \beta < \alpha \pi(\beta_f)(\alpha) < g(\alpha, \beta) > \sup_{\gamma < \alpha} f(\gamma)$$

Define a function $k : \omega_1 \to \omega_1$ by $k(\alpha) = \text{the least } \beta < \alpha \text{ such that } \pi(\beta_f)(\alpha) < g(\alpha, \beta) > \sup_{\gamma < \alpha} f(\gamma)$ (to be fair, this definition won’t make sense for all $\alpha$, but it will for almost all of them). Since $\mu_\omega$ is a normal measure, $\exists \gamma < \omega_1 \forall^* \alpha k(\alpha) = \gamma$. We then define $h$ by $h(\alpha) = g(\alpha, \gamma)$ and the rest of the proof continues as before.

**Corollary 2.** Sel($\omega_1, \omega_\omega$)

**Proof.** All the cardinals from $\omega_3$ until $\omega_\omega$ are singular.


