TRAVELING WAVE SOLUTIONS OF THE POROUS MEDIUM EQUATION

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We prove the existence of a one-parameter family of solutions of the porous medium equation, a nonlinear heat equation. In our work, with space dimension 3, the interface is a half line whose end point advances at constant speed. We prove, by using maximum principle, that the solutions are stable under a suitable class of perturbations. We discuss the relevance of our solutions, when restricted to two dimensions, to gravity driven flows of thin films. Here we extend the results of J. Iaia and S. Betelu in the paper "Solutions of the porous medium equation with degenerate interfaces" to a higher dimension.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Discussion of the Problem

Consider the porous medium equation(PME),

$$u_t = \Delta(u^m), \qquad m > 1, m \in \mathbb{R}. \tag{1}$$

where, $u = u(\mathbf{x}, t) > 0$, $\mathbf{x} \in \mathbb{R}^n$, n is the space dimension and $t \in \mathbb{R}$ the time.

This parabolic equation has been extensively studied in the literature due to its important physical applications. The function $u = u(\mathbf{x}, t)$ models the density of an ideal gas flowing in a homogeneous porous medium [1], [3] $n \leq 3$, the diffusion of strong thermal waves [2] $n \leq 3$, and the spreading of viscous gravity currents [6] with $n = 2$.

When $n = 2$ and $m = 4$ the porous medium equation models the spreading of a film of fluid on a horizontal surface under the action of gravity. In particular, if the free surface of the fluid is described by the graph $z = h(x, y, t)$, then it has been shown using *lubrication theory* [6] that h satisfies

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu} \nabla \cdot \left(h^3 \nabla h\right) = \frac{g}{12\nu} \Delta(h^4), \tag{2}$$

where g is the acceleration due to gravity and ν the kinematic viscosity of the fluid. After a change of scale Eq. (2) becomes Eq. (1) with $m = 4$.

To derive this equation, we neglect capillary, molecular and inertial forces and we assume the flow has a lateral extension which is much larger than the typical thickness. Several experimental studies (see for instance [8, 5] and references therein) show that this equation is a good approximation provided that the lateral extension of the spreading is much larger

Contents in section 1.1 and section 1.4 of this chapter have been reproduced, either in part or in full, from [7] J. Iaia and S. Betelu, European Journal of Applied Mathematics [doi:10.1017/S0956792512000423] with permission from Cambridge University Press.

than the capillary length $\sqrt{\gamma/\rho g}$, where γ is the surface tension, and provided that the fluid *wets* the surface i.e. the contact angle with the substrate at equilibrium is zero. In order to get a simpler, quadratic equation easier for analysis, we write the PME in terms of the *pressure* v where

$$v = \frac{m}{m-1}\, u^{m-1}.$$

Equation (1) then becomes

$$v_t = (m-1)v\Delta v + |\nabla v|^2. \tag{3}$$

1.2. Problem in Space Dimension $n = 2$

In the paper [7], it has been shown that in $n = 2$ dimensions, there are traveling wave solutions of (3) in the form

$$v(x, y, t) = r(x, y, t)F(\theta(x, y, t)), \tag{4}$$

where

$$r = \sqrt{(x-ct)^2 + y^2}, \quad c > 0,$$
$$\theta = \tan^{-1}\left(\frac{y}{x-ct}\right),$$

and

$$x - ct = r\cos\theta,$$

$$y = r\sin\theta.$$

Then the function $F(\theta)$ satisfies

$$F'^2 - cF'\sin\theta + F^2 + cF\cos\theta + (m-1)F(F'' + F) = 0. \tag{5}$$

In that paper it has also been shown that if $c > 0$ and

$$F(\pi) = a > 0, \quad F'(\pi) = 0,$$

then there is a positive solution of

$$F'^2 - cF'\sin\theta + F^2 + cF\cos\theta + (m-1)F(F'' + F) = 0 \quad \text{on} \quad (0, \pi)$$

such that

$$F(0) = 0, \quad F'(0) = 0$$

if and only if $c > a$.

In addition, when $c > a$ then F is 2π-periodic and there are solutions of (3) in the form (4).

1.3. Problem in Space Dimension $n = 3$

In this work, we attempt to prove a similar theorem in $n = 3$ dimensions. In particular, we attempt to find traveling wave solutions of (3) in $n = 3$ dimensions. We assume

$$v(x, y, z, t) = r(x, y, z, t)F(\phi(x, y, z, t)), \tag{6}$$

where

$$r = \sqrt{x^2 + y^2 + (z - ct)^2}, \quad c > 0, \tag{7}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \tag{8}$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z - ct}\right), \tag{9}$$

and:

$$x = r\cos\theta\sin\phi,$$

$$y = r\sin\theta\sin\phi,$$

$$z - ct = r\cos\phi.$$

We will then show that $F(\phi)$ satisfies

$$F'^2 - cF'\sin\phi + F^2 + cF\cos\phi + (m-1)F\left(F'' + \frac{\cos\phi}{\sin\phi}F' + 2F\right) = 0.$$

Problem in section 1.3 and its solution which is virtually the most part of my dissertation have been reproduced in elaborated form from my published article [9] L. Paudel and J. Iaia, Nonlinear Analysis(2012), Elsevier, doi:10.1016/j.na.2012.10.016 that I co-authored with my advisor J. Iaia.

Note that this equation has more terms than (5) and therefore requires a separate analysis than that contained in [7]. In this paper we will prove the following theorem:

Main Theorem: Let $c > 0$. Consider the differential equation

$$F'^2 - cF'\sin\phi + F^2 + cF\cos\phi + (m-1)F\left(F'' + \frac{\cos\phi}{\sin\phi}F' + 2F\right) = 0 \tag{10}$$

with

$$F(\pi) = a > 0, \quad F'(\pi) = 0. \tag{11}$$

There is a positive solution of (10)-(11) with

$$F(0) = 0, \quad F'(0) = 0$$

if and only if $c > a$.

In addition, when $c > a$ then F is 2π-periodic and there are solutions of (3) in the form of (6).

If $c = a$ then $F(\phi) = -a\cos(\phi)$ is an explicit solution of (10)-(11), and so there is not a positive solutions of (3) with $F(0) = 0$ and $F'(0) = 0$.

If $c < a$ and F is a solution of the equations (10)-(11), then F has a zero, z, with $\frac{\pi}{2} < z < \pi$ and

$$\lim_{\phi \to z^+} \frac{F(\phi)}{(\phi - z)^{\frac{m-1}{m}}} > 0.$$

1.4. Some Properties of the Porous Medium Equation Relevant to Our Work

Numerous interesting properties of the solutions of the PME have been studied, and they are nicely summarized in [11]. Some of those properties relevant to our work are summarized as follows:

- If the initial condition $u(\mathbf{x}, 0)$ is non-negative, then the solution remains non-negative for all times.

- The boundary of the set $S = \{\mathbf{x} : u(\mathbf{x}, t) = 0\}$ is called the *interface*. In the PME with $m > 1$ the interface can be static or move forward in the direction where $u = 0$, but never backwards.
- The function u is smooth in the regions where $u > 0$ and $t > 0$. At the interface, where $u = 0$, u is continuous but its derivative may be discontinuous.
- The solutions satisfy the maximum principle for parabolic equations: if $u_1(\mathbf{x}, 0) \leq u_2(\mathbf{x}, 0)$ then $u_1(\mathbf{x}, t) \leq u_2(\mathbf{x}, t)$ for $t > 0$.

In this paper, the interface is a half line that terminates abruptly, and since only the end point moves, we call this interface *degenerate*. As in all the applications of the porous medium equation, in this paper too, the propagating half line can be interpreted as a region *empty* of fluid.

CHAPTER 2

DERIVATION OF THE DIFFERENTIAL EQUATIONS

2.1. Differential Equation in Terms of the Pressure v

We assume

$$u = c\, v^k, \tag{12}$$

where, $c > 0, u > 0, v > 0, k \in \mathbb{R}$.

Then

$$u_t = ck\, v^{k-1} v_t. \tag{13}$$

Since

$$u^m = c^m\, v^{km},$$

differentiating with respect to 'x', we get

$$(u^m)_x = (c^m\, v^{km})_x = c^m\, km\, v^{km-1} v_x. \tag{14}$$

Again, differentiating with respect to 'x', we get

$$(u^m)_{xx} = (c^m\, km\, v^{km-1} v_x)_x = c^m\, km \left((km-1) v^{km-2} v_x^2 + v^{km-1} v_{xx} \right). \tag{15}$$

Proceeding similary,

$$(u^m)_{yy} = (c^m\, km\, v^{km-1} v_y)_y = c^m\, km \left((km-1) v^{km-2} v_y^2 + v^{km-1} v_{yy} \right). \tag{16}$$

$$(u^m)_{zz} = (c^m\, km\, v^{km-1} v_z)_z = c^m\, km \left((km-1) v^{km-2} v_z^2 + v^{km-1} v_{zz} \right). \tag{17}$$

Adding (15), (16) and (17), we get

$$\Delta(u^m) = c^m\, km \left((km-1) v^{km-2} |\nabla v|^2 + v^{km-1} \Delta v \right). \tag{18}$$

Using (13) and (18) in (1), we get

$$ck\, v^{k-1} v_t = c^m\, km \left((km-1) v^{km-2} |\nabla v|^2 + v^{km-1} \Delta v\right).$$

Thus

$$c\, v^{k-1} v_t = c^m\, m \left((km-1) v^{km-2} |\nabla v|^2 + v^{km-1} \Delta v\right).$$

Hence

$$\begin{aligned} v_t &= c^{m-1}\, m \left((km-1) v^{km-2-k+1} |\nabla v|^2 + v^{km-1-k+1} \Delta v\right) \\ &= m\, c^{m-1} \left((km-1) v^{km-k-1} |\nabla v|^2 + v^{km-k} \Delta v\right). \end{aligned}$$

Choosing k so that $km - k - 1 = 0$ i.e. $k = \frac{1}{m-1}$

then we get

$$\begin{aligned} v_t &= mc^{m-1} \left(\frac{|\nabla v|^2}{m-1} + v\, \Delta v\right) \\ &= \frac{mc^{m-1}}{(m-1)} \left((m-1) v \Delta v + |\nabla v|^2\right). \end{aligned}$$

Choosing c so that

$$\frac{m}{m-1} c^{m-1} = 1,$$

then

(19) $$v_t = (m-1)\, v\, \Delta v + |\nabla v|^2,$$

where

$$c^{m-1} = \frac{m-1}{m},$$

i.e.

$$c = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}},$$

and

$$u = c\, v^k$$

$$= \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} \quad v^{\frac{1}{m-1}},$$

equivalently,

$$v = \frac{m}{m-1}\; u^{m-1}.$$

2.2. Differential Equation in Terms of F

Using spherical coordinates, for $\vec{\mathbf{x}} = (x, y, z) \in \mathbb{R}^3$

$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z - ct = r\cos\phi,$

we have

$$r = \sqrt{x^2 + y^2 + (z-ct)^2}, \tag{20}$$

$$\tan\theta = \frac{y}{x},$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \tag{21}$$

$$\tan\phi = \frac{\sqrt{x^2+y^2}}{z-ct},$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z-ct}\right). \tag{22}$$

For a function $u(r, \theta, \phi)$, from [10] by Strauss,

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\phi\phi} + \frac{\cos\phi}{\sin\phi}\; u_\phi + \frac{1}{\sin^2\phi}\; u_{\theta\theta}\right) \tag{23}$$

Hence,

$$\Delta\phi = \frac{\cos\phi}{r^2\sin\phi}, \tag{24}$$

Let

$$v(x, y, z, t) = r(x, y, z, t)\;\; F(\phi(x, y, z, t)). \tag{25}$$

Then

$$v_t = r_t\; F(\phi) + r\; F'(\phi)\; \phi_t, \tag{26}$$

$$v_x = r_x \; F(\phi) + r \; F'(\phi) \; \phi_x, \tag{27}$$

$$v_y = r_y \; F(\phi) + r \; F'(\phi) \; \phi_y, \tag{28}$$

$$v_z = r_z \; F(\phi) + r \; F'(\phi) \; \phi_z, \tag{29}$$

Differentiating (27) with respect to ‘x’, we get

$$v_{xx} = r_{xx}F + r_x F'\phi_x + r_x F'\phi_x + rF''\phi_x^2 + rF'\phi_{xx}$$

Hence

$$v_{xx} = r_{xx}F + 2r_x F'\phi_x + rF''\phi_x^2 + rF'\phi_{xx}. \tag{30}$$

Similarly, differentiating (28) with respect to ‘y’ and (29) with respect to ‘z’ , we get

$$v_{yy} = r_{yy}F + 2r_x F'\phi_y + rF''\phi_y^2 + rF'\phi_{yy}. \tag{31}$$

$$v_{zz} = r_{zz}F + 2r_z F'\phi_z + rF''\phi_z^2 + rF'\phi_{zz}. \tag{32}$$

We have

$$r^2 = x^2 + y^2 + (z - ct)^2. \tag{33}$$

Differentiating (33) with respect to ‘t’, we get

$$2rr_t = 2(z - ct)(-c),$$

$$rr_t = -c(z - ct), \tag{34}$$

$$r_t = \frac{-c(z - ct)}{r} = -c\cos\phi. \tag{35}$$

Differentiating (33) with respect to ‘x’, we get

$$rr_x = x, \tag{36}$$

$$r_x = \frac{x}{r}. \tag{37}$$

Differentiating (33) with respect to 'y', we get

$$rr_y = y, \tag{38}$$

$$r_y = \frac{y}{r}. \tag{39}$$

Differentiating (33) with respect to 'z', we get

$$rr_z = z - ct, \tag{40}$$

$$r_z = \frac{z - ct}{r}. \tag{41}$$

Differentiating (36) with respect to x, we get

$$r_x^2 + rr_{xx} = 1. \tag{42}$$

Differentiating (38) with respect to y, we get

$$r_y^2 + rr_{yy} = 1. \tag{43}$$

Differentiating (40) with respect to z, we get

$$r_z^2 + rr_{zz} = 1. \tag{44}$$

Squaring and adding the equations (36), (38) and (40), we get

$$r^2(r_x^2 + r_y^2 + r_z^2) = x^2 + y^2 + (z - ct)^2 = r^2,$$

So

$$r_x^2 + r_y^2 + r_z^2 = 1,$$

i.e.

$$|\nabla r|^2 = 1. \tag{45}$$

Adding (42), (43) and (44), we get

$$|\nabla r|^2 + \Delta r\ r = 3.$$

Using (45), we get

$$\Delta r\, r = 2,$$

and

$$\Delta r = \frac{2}{r}. \tag{46}$$

We have

$$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z-ct}\right). \tag{47}$$

Differentiating (47) with respect to 't', we get

$$\begin{aligned}
\phi_t &= \frac{1}{1+\frac{x^2+y^2}{(z-ct)^2}}(-1)\frac{\sqrt{x^2+y^2}}{(z-ct)^2}(-c) \\
&= \frac{c\sqrt{x^2+y^2}}{x^2+y^2+(z-ct)^2} \\
&= \frac{c\sqrt{x^2+y^2}}{r^2}.
\end{aligned}$$

Thus

$$\phi_t = \frac{c\sqrt{x^2+y^2}}{r^2} = \frac{c\sin\phi}{r}. \tag{48}$$

Differentiating (47) with respect to 'x', we get

$$\begin{aligned}
\phi_x &= \frac{1}{1+\frac{x^2+y^2}{(z-ct)^2}}\frac{2x}{2\sqrt{x^2+y^2}(z-ct)} \\
&= \frac{x(z-ct)}{((z-ct)^2+x^2+y^2)\sqrt{x^2+y^2}} \\
&= \frac{x(z-ct)}{r^2\sqrt{x^2+y^2}}.
\end{aligned}$$

Hence

$$\phi_x = \frac{x(z-ct)}{r^2\sqrt{x^2+y^2}}. \tag{49}$$

Similarly,

$$\phi_y = \frac{y(z-ct)}{r^2\sqrt{x^2+y^2}}. \tag{50}$$

Differentiating (47) with respect to ‘z’, we get

$$\begin{aligned}
\phi_z &= \frac{1}{1+\frac{x^2+y^2}{(z-ct)^2}}(-1)\frac{\sqrt{x^2+y^2}}{(z-ct)^2} \\
&= \frac{-\sqrt{x^2+y^2}}{x^2+y^2+(z-ct)^2} \\
&= \frac{-\sqrt{x^2+y^2}}{r^2}.
\end{aligned}$$

Hence

$$\phi_z = \frac{-\sqrt{x^2+y^2}}{r^2}. \tag{51}$$

Then

$$\begin{aligned}
|\nabla\phi|^2 &= \phi_x^2+\phi_y^2+\phi_z^2 \\
&= \frac{x^2(z-ct)^2}{r^4(x^2+y^2)} + \frac{y^2(z-ct)^2}{r^4(x^2+y^2)} + \frac{(x^2+y^2)}{r^4} \\
&= \frac{(z-ct)^2}{r^4} + \frac{(x^2+y^2)}{r^4} \\
&= \frac{1}{r^2}.
\end{aligned}$$

So,

$$|\nabla\phi|^2 = \frac{1}{r^2}. \tag{52}$$

Since

$$\nabla r\cdot\nabla\phi = r_x\phi_x + r_y\phi_y + r_z\phi_z.$$

Using equations (36), (38), (40),(49),(50) and (51), we have

$$\begin{aligned}
\nabla r\cdot\nabla\phi &= \frac{x}{r}\,\frac{x(z-ct)}{r^2\sqrt{x^2+y^2}} + \frac{y}{r}\,\frac{y(z-ct)}{r^2\sqrt{x^2+y^2}} + \frac{(z-ct)}{r}\,\frac{(-1)\sqrt{x^2+y^2}}{r^2} \\
&= \frac{(x^2+y^2)(z-ct)}{r^3\sqrt{x^2+y^2}} - \frac{(z-ct)\sqrt{x^2+y^2}}{r^3} \\
&= \frac{(z-ct)\sqrt{x^2+y^2}}{r^3} - \frac{(z-ct)\sqrt{x^2+y^2}}{r^3} \\
&= 0.
\end{aligned}$$

So

$$\nabla r \cdot \nabla \phi = 0. \tag{53}$$

$$\begin{aligned}
|\nabla v|^2 &= v_x^2 + v_y^2 + v_z^2 \\
&= (r_x^2 + r_y^2 + r_z^2)F^2 + 2r(r_x\phi_x + r_y\phi_y + r_z\phi_z)FF' + (\phi_x^2 + \phi_y^2 + \phi_z^2)r^2F'^2 \\
&= |\nabla r|^2 F^2 + 2r(\nabla r \cdot \nabla \phi)FF' + |\nabla \phi|^2 r^2 F'^2 \\
&= F^2 + F'^2.
\end{aligned}$$

Hence

$$|\nabla v|^2 = F^2 + F'^2. \tag{54}$$

$$\begin{aligned}
\Delta v &= v_{xx} + v_{yy} + v_{zz} \\
&= (\Delta r)F + 2(\nabla r \cdot \nabla \phi)F' + |\nabla \phi|^2 rF'' + (\Delta \phi)rF' \\
&= \frac{2}{r}F + \frac{1}{r}F'' + \frac{\cos\phi}{r\sin\phi}F'.
\end{aligned}$$

Hence

$$\Delta v = \frac{2}{r}F + \frac{1}{r}F'' + \frac{\cos\phi}{r\sin\phi}F'. \tag{55}$$

Now, from the equations (25) and (55), we have

$$\begin{aligned}
v\ \Delta v &= rF\left(\frac{2}{r}F + \frac{1}{r}F'' + \frac{\cos\phi}{r\sin\phi}F'\right) \\
&= 2F^2 + FF'' + \frac{\cos\phi}{\sin\phi}\ FF'.
\end{aligned}$$

Thus

$$v\ \Delta v = 2F^2 + FF'' + \frac{\cos\phi}{\sin\phi}\ FF'. \tag{56}$$

From equation (26), we have

$$v_t = r_t\ F(\phi) + r\ F'(\phi)\ \phi_t.$$

Using (35) and (48), we get

$$
\begin{aligned}
v_t &= -c\cos\phi F + rF' \ \frac{c\sin\phi}{r} \\
&= -c\cos\phi\ F + c\sin\phi\ F'.
\end{aligned}
$$

Hence

$$v_t = -c\cos\phi\ F + c\sin\phi\ F'. \tag{57}$$

Using equations (54), (56) and (57) in

$$v_t = (m-1)\ v\ \Delta v + |\nabla v|^2.$$

we get

$$-c\cos\phi\ F + c\sin\phi\ F' = (m-1)\left(2F^2 + FF'' + \frac{\cos\phi}{\sin\phi}\ FF'\right) + F^2 + F'^2.$$

Finally,

$$F'^2 - c\sin\phi\ F' + F^2 + c\cos\phi\ F + (m-1)F\left(F'' + \frac{\cos\phi}{\sin\phi}\ F' \ + 2F\right) = 0. \tag{58}$$

CHAPTER 3

SOLUTIONS OF THE DIFFERENTIAL EQUATION

We consider the differential equation

$$F'^2 - c\sin\phi\, F' + F^2 + c\cos\phi\, F + (m-1)F\left(F'' + \frac{\cos\phi}{\sin\phi}\, F' \; + 2F\right) = 0. \tag{59}$$

By restricting ourselves to symmetric solutions about the $z-$ axis, we assign the boundary conditions as

$$F(\pi) = a, \tag{60}$$

$$F'(\pi) = 0. \tag{61}$$

3.1. Existence of the Solution F

Let $H = \left(\frac{m-1}{m}\right) F^{\frac{m}{m-1}}$, then the differential equation (59) is reduced to

$$\begin{aligned} H''\sin\phi + \; H'\cos\phi - \frac{c}{(m-1)}\left(\frac{m-1}{m}\right)^{\frac{m-1}{m}} H^{\frac{1}{m}-1}H'\sin^2\phi + \frac{m(2m-1)}{(m-1)^2}H\sin\phi \\ + \frac{c}{(m-1)}\left(\frac{m}{m-1}\right)^{\frac{1}{m}} H^{\frac{1}{m}}\cos\phi\sin\phi = 0 \end{aligned} \tag{62}$$

and satisfies

$$H(\pi) = \left(\frac{m-1}{m}\right) a^{\frac{m}{m-1}} \tag{63}$$

$$H'(\pi) = 0. \tag{64}$$

Rewriting the equation (62),

$$(H'\sin\phi)' - \frac{c}{(m-1)}\left(\frac{m-1}{m}\right)^{\frac{m-1}{m}} \left(mH^{\frac{1}{m}}\right)'\sin^2\phi + \frac{m(2m-1)}{(m-1)^2}H\sin\phi$$

$$+\frac{c}{(m-1)}\left(\frac{m}{m-1}\right)^{\frac{1}{m}} H^{\frac{1}{m}} \cos\phi \sin\phi = 0$$

and integrating over ϕ to π, we get

$$-H' \sin\phi + \frac{cm}{m-1}\left(\frac{m-1}{m}\right)^{\frac{m-1}{m}} H^{\frac{1}{m}} \sin^2\phi + \frac{m(2m-1)}{(m-1)^2}\int_\phi^\pi H \sin\theta \, d\theta$$

$$+\frac{c}{(m-1)}\left[2m\left(\frac{m-1}{m}\right)^{\frac{m-1}{m}} + \left(\frac{m}{m-1}\right)^{\frac{1}{m}}\right]\int_\phi^\pi H^{\frac{1}{2}} \sin\theta\cos\theta \, d\theta = 0.$$

Dividing by $\sin\phi$ and integrating once again over ϕ to π, we get

$$H(\pi) - H(\phi) = \int_\phi^\pi \frac{cm}{m-1}\left(\frac{m-1}{m}\right)^{\frac{m-1}{m}} H^{\frac{1}{m}} \sin\theta \, d\theta + \int_\phi^\pi \left(\frac{m(2m-1)}{(m-1)^2 \sin t}\int_t^\pi H \sin\theta \, d\theta\right) dt$$

$$+\int_\phi^\pi \left[\frac{c}{(m-1)\sin t}\left(2m\left(\frac{m-1}{m}\right)^{\frac{m-1}{m}} + \left(\frac{m}{m-1}\right)^{\frac{1}{m}}\right)\int_t^\pi H^{\frac{1}{2}} \sin\theta\cos\theta \, d\theta\right] dt.$$

So,

(65)

$$H(\phi) = H(\pi) - \int_\phi^\pi P H^{\frac{1}{m}} \sin\theta \, d\theta - \int_\phi^\pi \left(\frac{Q}{\sin t}\int_t^\pi H \sin\theta \, d\theta\right) dt - \int_\phi^\pi \left(\frac{R}{\sin t}\int_t^\pi H^{\frac{1}{2}} \sin\theta\cos\theta \, d\theta\right) dt.$$

where,

$P = \frac{cm}{(m-1)}\left(\frac{m-1}{m}\right)^{\frac{m-1}{m}}$,

$Q = \frac{m(2m-1)}{(m-1)^2}$, and

$R = \frac{c}{(m-1)}\left[2m\left(\frac{m-1}{m}\right)^{\frac{m-1}{m}} + \left(\frac{m}{m-1}\right)^{\frac{1}{m}}\right]$.

From (65), it can be shown that the map T defined by $T(H) =$ right hand side of (65) is a contraction mapping , provided ϕ is sufficiently close to π. Hence by the contraction mapping principle, T has a fixed point H such that $T(H) = H$ for ϕ sufficiently close to π. Hence the solution H of equations (62)-(64) exists in the neighborhood of $\phi = \pi$. This proves the existence of the solution F of (59)-(61) in the neighborhood of $\phi = \pi$.

3.2. Preliminary Observation

Let us suppose

$$W = F'^2 - c\sin\phi\, F' + F^2 + c\cos\phi\, F. \tag{66}$$

Then

$$\begin{aligned} W' &= 2\, F'F'' - c\sin\phi\, F'' - c\cos\phi\, F' + 2FF' + c\cos\phi\, F' - c\sin\phi F \\ &= 2F'(F'' + F) - c\sin\phi(F'' + F) \\ &= (2F' - c\sin\phi)(F'' + F). \end{aligned}$$

Hence

$$W' = (2F' - c\sin\phi)(F'' + F). \tag{67}$$

From equations (59) and (66), we have

$$W + (m-1)F\left(F'' + F + \frac{\cos\phi\, F' + \sin\phi\, F}{\sin\phi}\right) = 0. \tag{68}$$

So

$$F'' + F = -\frac{W}{(m-1)F} - \frac{\cos\phi}{\sin\phi}\, F' - F.$$

Then

$$W' = (2F' - c\sin\phi)\left(-\frac{W}{(m-1)F} - \frac{\cos\phi}{\sin\phi}\, F' - F\right).$$

$$W' + \frac{(2F' - c\sin\phi)}{(m-1)F} W = (2F' - c\sin\phi)\left(-\frac{\cos\phi}{\sin\phi}\, F' - F\right). \tag{69}$$

Note that by using (60) and (61) in (66), we get

$$W(\pi) = a^2 - ac \tag{70}$$

Taking the limit as $\phi \to \pi^-$ in equation (59), we get

$$a^2 - ac + (m-1)\, a\left(F''(\pi) + \lim_{\phi\to\pi^-} \frac{\cos\phi}{\sin\phi} F'(\phi) + 2F(\pi)\right) = 0.$$

Using L'Hospital's rule to calculate, we get

$$a^2 - ac + (m-1)\;2\;\;a\,(F''(\pi) + F(\pi)) = 0.$$

$$(a-c) + 2(m-1)\,(F''(\pi) + F(\pi)) = 0.$$

And thus

$$F''(\pi) + F(\pi) = \frac{c-a}{2(m-1)}. \tag{71}$$

Hence,

$$F''(\pi) + F(\pi) < 0 \qquad \text{for } a > c, \tag{72}$$

$$F''(\pi) + F(\pi) = 0 \qquad \text{for } a = c, \tag{73}$$

and

$$F''(\pi) + F(\pi) > 0 \qquad \text{for } a < c. \tag{74}$$

If we let

$$D = \cos\phi\; F' + \sin\phi\; F, \tag{75}$$

Then

$$D' = (F'' + F)\cos\phi. \tag{76}$$

From (82)

$$W + (m-1)F\left(F'' + F + \frac{\cos\phi\; F' + \sin\phi\; F}{\sin\phi}\right) = 0.$$

Using (75) and (76), we get

$$W + (m-1)F\left(\frac{D'}{\cos\phi} + \frac{D}{\sin\phi}\right) = 0.$$

Therefore

$$W + (m-1)\frac{F}{\cos\phi}D' + (m-1)\frac{F}{\sin\phi}D = 0. \tag{77}$$

From equation (67)

$$W' = (2F' - c\sin\phi)(F'' + F).$$

Using (76), we get

$$W' = (2F' - c\sin\phi)\frac{D'}{\cos\phi}. \tag{78}$$

LEMMA 3.1. *The function F is symmetric about the line $\phi = \pi$, i.e. $F(\pi+\phi) = F(\pi-\phi)$.*

PROOF. Let $F(\pi+\phi) = G_1(\phi)$ and $F(\pi-\phi) = G_2(\phi)$, then

$$G_1'(\phi) = F'(\pi+\phi), \quad G_2'(\phi) = -F'(\pi-\phi),$$

$$G_1''(\phi) = F''(\pi+\phi), \quad G_2''(\phi) = F''(\pi-\phi).$$

So

$$G_1(0) = a = G_2(0) \quad \text{and} \quad G_1'(0) = 0 = G_2'(0). \tag{79}$$

Replacing ϕ by $\pi+\phi$ in equation (59), we get

$$G_1'^2 - c\sin(\pi+\phi)\,G_1' + G_1^2 + c\cos(\pi+\phi)\,G_1 + (m-1)G_1\left(G_1'' + \frac{\cos(\pi+\phi)}{\sin(\pi+\phi)}\,G_1' + 2G_1\right) = 0.$$

Thus

$$G_1'^2 + c\sin\phi\,G_1' + G_1^2 - c\cos\phi\,G_1 + (m-1)G_1\left(G_1'' + \frac{\cos\phi}{\sin\phi}\,G_1' + 2G_1\right) = 0. \tag{80}$$

Replacing ϕ by $\pi-\phi$ in equation (59), we get

$$G_2'^2 + c\sin(\pi-\phi)\,G_2' + G_2^2 + c\cos(\pi-\phi)\,G_2 + (m-1)G_2\left(G_2'' - \frac{\cos(\pi-\phi)}{\sin(\pi-\phi)}\,G_2' + 2G_2\right) = 0.$$

Thus

$$G_2'^2 + c\sin\phi\,G_2' + G_2^2 - c\cos\phi\,G_2 + (m-1)G_2\left(G_2'' + \frac{\cos\phi}{\sin\phi}\,G_2' + 2G_2\right) = 0. \tag{81}$$

From equations (80) and (81), we see that G_1 and G_2 satisfy the same differential equation.

From (79), they also satisfy the same initial conditions. Hence by the uniqueness theorem [4],

$$G_1(\phi) = G_2(\phi).$$

So

$$F(\pi + \phi) = F(\pi - \phi).$$

This completes the proof of lemma 3.1. $\square$

CHAPTER 4

THE CASE $\quad 0 < c = a$

Lemma 4.1. *For $c = a$, $F = -a\cos\phi$ is a solution for the system of equations (59)-(61).*

Proof. Let

$$W = F'^2 - c\sin\phi\, F' + F^2 + c\cos\phi\, F.$$

Then equation (59) is reduced to

$$W + (m-1)F\left(F'' + \frac{\cos\phi}{\sin\phi}\, F' + 2F\right) = 0.$$

$$\frac{W}{(m-1)F} + F'' + \frac{\cos\phi}{\sin\phi}\, F' + 2F = 0. \tag{82}$$

Consider the equation

$$F'' + \frac{\cos\phi}{\sin\phi}\, F' \ + 2F = 0. \tag{83}$$

Note that for

$$F = \cos\phi,$$

$$F' = -\sin\phi,$$

$$F'' = -\cos\phi,$$

Using in equation (83), we get

$$-\cos\phi + \frac{\cos\phi}{\sin\phi}\,(-\sin\phi) + 2\cos\phi = 0.$$

Then, $F = \cos\phi$ is a solution for equation (83). Besides, equation (83) is a linear homogeneous differential equation. So, $F = k\cos\phi$ is a solution of equation (83) for any k.

Also, consider the equation

$$W = F'^2 - c\sin\phi\, F' + F^2 + c\cos\phi\, F = 0. \tag{84}$$

Let

$$F = -a\cos\phi,$$

then

$$F' = a\sin\phi,$$

$$F'' = a\cos\phi.$$

Since $c = a$,

$$\begin{aligned} W &= a^2\sin^2\phi - c\sin\phi\, a\sin\phi + a^2\cos^2\phi - ac\cos^2\phi \\ &= (a^2 - ac)\sin^2\phi + (a^2 - ac)\cos^2\phi \\ &= (a^2 - ac) \\ &= 0 \end{aligned}$$

Combining this with the fact that $F = -a\cos\phi$ satisfies equation (83), we see that $F = -a\cos\phi$ satisfies the equation (82). Hence, whenever $c = a$, $F = -a\cos\phi$ is a solution for the system of equations (59)-(61). This completes the proof of lemma 4.1. $\square$

CHAPTER 5

THE CASE $\quad 0 < c < a$

Lemma 5.1. *Let $0 < c < a$. Then F has a zero on $(\frac{\pi}{2}, \pi)$.*

Proof. By (60) we have $F(\pi) = a > 0$. Contrary to the lemma let us assume that $F > 0$ on $(\frac{\pi}{2}, \pi)$. Then from (60)-(61) and (75)-(76) we have

$$D(\pi) = 0, \quad D\left(\frac{\pi}{2}\right) = F\left(\frac{\pi}{2}\right) \geq 0,$$

$$D'(\pi) = [F''(\pi) + F(\pi)]\cos\pi = (-1)[F''(\pi) + F(\pi)].$$

Using this and (71) we see

$$D'(\pi) > 0.$$

Therefore D is negative to the immediate left of π and $D(\frac{\pi}{2}) \geq 0$ so it follows that D has a negative local minimum on $(\frac{\pi}{2}, \pi)$. So there is a $p \in (\frac{\pi}{2}, \pi)$ such that $D'(p) = 0$, $D' > 0$ on $(p, \pi]$ and $D(p) < 0$. Then it follows that

$$D(\phi) < 0 \text{ on } [p, \pi). \tag{85}$$

In addition, it follows from calculus that

$$D''(p) \geq 0. \tag{86}$$

Since $D'(p) = 0$ and $p \in (\frac{\pi}{2}, \pi)$, then from (78) it follows that $W'(p) = 0$. Next, we differentiate (77) and see that we obtain

$$\frac{W'}{m-1} + \left(\frac{F}{\cos\phi}\right)' D' + \left(\frac{F}{\cos\phi}\right) D'' + \left(\frac{F}{\sin\phi}\right)' D + \left(\frac{F}{\sin\phi}\right) D' = 0. \tag{87}$$

Evaluating at $\phi = p$ (where $W'(p) = D'(p) = 0$), this reduces to

$$\frac{F(p)}{\cos p} D''(p) + \left(\frac{F'(p)\sin p - F(p)\cos p}{\sin^2 p}\right) D(p) = 0.$$

Since $p \in \left(\frac{\pi}{2}, \pi\right)$, then $\cos p < 0$ and along with (86), we obtain

$$\left(\frac{F'(p)\sin p - F(p)\cos p}{\sin^2 p}\right) D(p) \geq 0.$$

Since $D(p) < 0$, we then have

$$F'(p)\sin p - F(p)\cos p \leq 0.$$

Thus since $p \in (\frac{\pi}{2}, \pi)$ we have

$$F'(p)\sin p \leq F(p)\cos p < 0.$$

Hence

$$F'(p) < 0. \tag{88}$$

Also from (60)-(61) and (71) we see that F has a local maximum at π. Then for some $\epsilon > 0$

$$F' > 0 \tag{89}$$

on $(\pi - \epsilon, \pi)$. From (88) we see that F is decreasing near p and from (89) we see that F is increasing to the immediate left of π. Therefore there exists $q \in (p, \pi)$ where F has a local minimum and thus $F'(q) = 0$. Since $q \in (p, \pi)$, using (75) and (85) we see that

$$0 > D(q) = F'(q)\cos q + F(q)\sin q = F(q)\sin q$$

which contradicts the assumption $F > 0$. This completes the proof of lemma 5.1. $\square$

Due to the preceding lemma, we will now assume that there is a z with $\frac{\pi}{2} < z < \pi$ such that $F(z) = 0$ and

$$F > 0 \text{ on } (z, \pi]. \tag{90}$$

Lemma 5.2. *Let $0 < c < a$. Then $D' > 0$ on $(z, \pi]$.*

Proof. We assume by the way of contradiction that there exists $w \in (z, \pi] \subset (\frac{\pi}{2}, \pi]$ such that $D'(w) \leq 0$. Now recall from the beginning of the proof of lemma 5.1 that $D'(\pi) > 0$

and $D(\pi) = 0$. Hence there exists some $p \in [w, \pi)$ such that $D'(p) = 0$ and $D' > 0$ on $(p, \pi]$. Then

$$D''(p) \geq 0 \;\text{ and }\; D(p) < 0. \tag{91}$$

Also, from (78) we have

$$W'(p) = 0.$$

Using these facts in (87), we obtain

$$\frac{F(p)}{\cos p} D''(p) + \left(\frac{F'(p)\sin p - F(p)\cos p}{\sin^2 p} \right) D(p) = 0. \tag{92}$$

Since $p \in (z, \pi) \subset (\frac{\pi}{2}, \pi)$,

$$\cos p < 0 \;\text{ and }\; F(p) > 0,$$

and thus from (91) we obtain

$$(F'(p)\sin p - F(p)\cos p)\, D(p) \geq 0.$$

Since $D(p) < 0$ (by 91) we then have

$$F'(p)\sin p - F(p)\cos p \leq 0$$

and so

$$F'(p)\sin p \leq F(p)\cos p < 0.$$

Thus:

$$F'(p) < 0.$$

So F is decreasing near p and since F has a local maximum at π, F is increasing to the immediate left of π. Thus, there exists $q \in (p, \pi)$ such that $F'(q) = 0$. Then from (75) we have

$$D(q) = F(q)\sin q > 0. \tag{93}$$

Now from (91) and the comments immediately preceding this lemma, $D(p) < 0, D(\pi) = 0$, and $D' > 0$ on $(p, \pi]$. Thus $D(q) < 0$ which is a contradiction to (93). Thus $D' > 0$ on $(z, \pi]$. This completes the proof of lemma 5.2. $\square$

Note: It follows from lemma 5.2 that $D' = (F''+F)(\cos\phi) > 0$ on (z,π). Thus $F''+F < 0$ on (z,π). Since $F > 0$ on (z,π) then it follows that $F'' < 0$ on $(z,\pi]$. Thus, F' is strictly decreasing on (z,π). Since $F'(\pi) = 0$ this implies $F' > 0$ on (z,π). Also, since F' is strictly decreasing on (z,π), we see that $\lim\limits_{\phi\to z^+} F'(\phi)$ exists (but may be infinite). Thus $\lim\limits_{\phi\to z^+} F'(\phi) = A$ where $0 < A \leq \infty$.

Lemma 5.3. *Let* $0 < c < a$. *Then* $\lim\limits_{\phi\to z^+} F' = \infty$.

Proof. We assume by the way of contradiction that $\lim\limits_{\phi\to z^+} F'(\phi) = A$, where $0 < A < \infty$. Now let

$$E = \frac{1}{2}\left(F'^2 + F^2\right).$$

From the note after lemma 5.2 we have

$$F'' + F < 0 \text{ and } F' > 0 \text{ on } (z,\pi).$$

Thus we see that

$$E' = \frac{1}{2}\left(F'^2 + F^2\right)' = F'\left(F'' + F\right) < 0 \text{ on } (z,\pi). \tag{94}$$

Therefore E is decreasing on (z,π) and thus:

$$\frac{a^2}{2} = E(\pi) \leq \lim_{\phi\to z^+} E(\phi) = \frac{A^2}{2}.$$

Therefore

$$a \leq A,$$

and since $0 < c < a$, we see that

$$0 < c < a \leq A. \tag{95}$$

Then using (95) and the fact that $|\sin(z)| \leq 1$, we see that

$$\lim_{\phi\to z^+} W(\phi) = \lim_{\phi\to z^+}[F'^2 - cF'\sin\phi + F^2 + cF\cos\phi] = A^2 - cA\sin z = A(A - c\sin z) > 0. \tag{96}$$

Next, dividing (69) by W and taking the limit as $\phi \to z^+$, we see (using $z \in (\frac{\pi}{2}, \pi)$ and (95)) that

$$\lim_{\phi \to z^+} \left[\frac{W'}{W} + \frac{2}{m-1} \frac{F'}{F} - \frac{c}{m-1} \frac{\sin\phi}{F} \right] = -\frac{\cos z}{\sin z} \left(\frac{2A - c\sin z}{A - c\sin z} \right) > 0. \tag{97}$$

Next we let

$$G = W F^{\frac{2}{m-1}} \exp\left(\frac{c}{m-1} \int_\phi^\pi \frac{\sin t}{F(t)}\, dt \right). \tag{98}$$

Taking logs and differentiating we obtain

$$\frac{G'}{G} = (\ln G)' = \frac{W'}{W} + \frac{2}{m-1} \frac{F'}{F} - \frac{c}{m-1} \frac{\sin\phi}{F}$$

Hence from (97),

$$\lim_{\phi \to z^+} \frac{G'}{G} = -\frac{\cos z}{\sin z} \left(\frac{2A - c\sin z}{A - c\sin z} \right) \equiv B > 0. \tag{99}$$

From (90) and (96) observe that $G > 0$ to the immediate right of z. So, from (99), $G' > 0$ near $\phi = z$. Thus, G is increasing near $\phi = z$ and therefore

$$\lim_{\phi \to z^+} G = L \geq 0. \tag{100}$$

Now from (99) we have

$$\frac{G'}{G} \leq 2B \quad \text{on} \quad (z, z+\epsilon).$$

Thus integrating on $(\phi, \phi_1) \subset (z, z+\epsilon)$ we obtain

$$G(\phi_1) e^{2B(\phi - \phi_1)} \leq G(\phi)$$

and therefore

$$0 < G(\phi_1) e^{2B(z - \phi_1)} \leq \lim_{\phi \to z^+} G(\phi) = L.$$

Thus

$$\lim_{\phi \to z^+} G = L > 0. \tag{101}$$

We already know from (96) that $\lim_{\phi\to z^+} W(\phi) = A(A - c\sin z) > 0$. So, from (98) and (101) we see

$$\lim_{\phi\to z^+}\left[F^{\frac{2}{m-1}}(\phi)\exp\left(\frac{c}{m-1}\int_\phi^\pi \frac{\sin t}{F(t)}\,dt\right)\right] \text{ is positive and finite.}$$

Taking natural logs implies

$$\lim_{\phi\to z^+}\left[2\ln F + c\int_\phi^\pi \frac{\sin t}{F(t)}\,dt\right] \text{ exists and is finite.}$$

Rewriting, we see that

$$\lim_{\phi\to z^+} 2\ln F\left[1 + \frac{c\int_\phi^\pi \frac{\sin t}{F(t)}\,dt}{2\ln F}\right] \text{ exists and is finite.} \tag{102}$$

Since $\lim_{\phi\to z^+} F'(\phi) = A$ it follows from L'Hopital's rule that

$$\lim_{\phi\to z^+}\frac{F(\phi)}{\phi - z} = \lim_{\phi\to z^+} F'(\phi) = A.$$

Using this and the fact that $\frac{\pi}{2} < z < \pi$ we see that the integral in (102) becomes unbounded as $\phi \to z^+$. Hence applying L'Hopital's rule we see that

$$\lim_{\phi\to z^+}\frac{c\int_\phi^\pi \frac{\sin t}{F(t)}\,dt}{2\ln F} = -\lim_{\phi\to z^+}\frac{\frac{c\sin\phi}{F}}{\frac{2F'}{F}} = -\frac{c\sin z}{2A}. \tag{103}$$

Now, from (95) we have $A \geq a > c > \frac{c}{2}\sin z$ and so

$$1 + \lim_{\phi\to z^+}\frac{c\int_\phi^\pi \frac{\sin t}{F(t)}\,dt}{2\ln F} = 1 - \frac{c\sin z}{2A} > 0. \tag{104}$$

However, since

$$\lim_{\phi\to z^+} 2\ln F = -\infty \tag{105}$$

we see that (104) and (105) contradict (102). Therefore it must be the case that $A = \infty$. This completes the proof of lemma 5.3. $\square$

Lemma 5.4. *Let $0 < c < a$. Then*

$$\lim_{\phi\to z^+}\frac{FF''}{F'^2} = -\frac{1}{m-1}.$$

Proof. Dividing (59) by F'^2 gives

$$1 - \frac{c\sin\phi}{F'} + \frac{(2m-1)F^2}{F'^2} + \frac{cF\cos\phi}{F'^2} + \frac{(m-1)FF''}{F'^2} + \frac{(m-1)F\cos\phi}{F'\sin\phi} = 0. \tag{106}$$

Since $F(z) = \lim_{\phi\to z^+} F(\phi) = 0$ and $\lim_{\phi\to z^+} F'(\phi) = \infty$ (by lemma 5.3), it follows then that

$$\lim_{\phi\to z^+} \frac{c\sin\phi}{F'} = \lim_{\phi\to z^+} \frac{(2m-1)F^2}{F'^2} = \lim_{\phi\to z^+} \frac{cF\cos\phi}{F'^2} = \frac{(m-1)F\cos\phi}{F'\sin\phi} = 0. \tag{107}$$

Taking the limit as $\phi \to z^+$ in (106) and using (107) we see that

$$1 + \lim_{\phi\to z^+} \frac{(m-1)FF''}{F'^2} = 0.$$

Thus,

$$\lim_{\phi\to z^+} \frac{FF''}{F'^2} = -\frac{1}{m-1}.$$

This completes the proof of lemma 5.4. □

Lemma 5.5. *Let* $0 < c < a$. *Then*

$$\lim_{\phi\to z^+} \frac{(\phi - z)F'}{F} = \frac{m-1}{m}.$$

Proof. From lemma 5.4, it follows that given $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\frac{-1}{m-1} - \epsilon < \frac{FF''}{F'^2} < \frac{-1}{m-1} + \epsilon \text{ for } z < \phi < z + \delta.$$

Since $\frac{FF''}{F'^2} = 1 - \left(\frac{F}{F'}\right)'$, we may rewrite the above as

$$\frac{m}{m-1} - \epsilon < \left(\frac{F}{F'}\right)' < \frac{m}{m-1} + \epsilon \text{ for } z < \phi < z + \delta.$$

Integrating on (ϕ_1, ϕ) gives

$$\left(\frac{m}{m-1} - \epsilon\right)(\phi - \phi_1) < \frac{F}{F'} - \frac{F(\phi_1)}{F'(\phi_1)} < \left(\frac{m}{m-1} + \epsilon\right)(\phi - \phi_1) \text{ for } z < \phi_1 < \phi < z + \delta.$$

Using the facts $\lim_{\phi_1\to z^+} F(\phi_1) = F(z) = 0$ and $\lim_{\phi\to z^+} F'(\phi) = \infty$ (by lemma 5.3), we see when taking the limit as $\phi_1 \to z^+$ that

$$\left(\frac{m}{m-1} - \epsilon\right)(\phi - z) \le \frac{F}{F'} \le \left(\frac{m}{m-1} + \epsilon\right)(\phi - z) \text{ for } z < \phi < z + \delta.$$

Thus

$$\frac{m}{m-1} - \epsilon \leq \frac{F}{(\phi - z)F'} \leq \frac{m}{m-1} + \epsilon \text{ for } z < \phi < z + \delta.$$

Therefore

$$\lim_{\phi \to z^+} \frac{F}{(\phi - z)F'} = \frac{m}{m-1}.$$

Thus

$$\lim_{\phi \to z^+} \frac{(\phi - z)F'}{F} = \frac{m-1}{m}.$$

This completes the proof of lemma 5.5. $\square$

LEMMA 5.6. *Let* $0 < c < a$. *Then*

$$\lim_{\phi \to z^+} F'F^{\frac{1}{m-1}} \text{ exists and is positive.}$$

PROOF. Dividing (69) by W and then dividing the numerator and denominator of the right hand side of this expression by F'^2 and rewriting we see that

$$\frac{W'}{W} + \frac{2}{m-1}\frac{F'}{F} - \frac{c}{m-1}\frac{\sin\phi}{F} = -\frac{(2 - \frac{c\sin\phi)}{F'})\left(\frac{\cos\phi}{\sin\phi} + \frac{F}{F'}\right)}{1 - \frac{c\sin(\phi)}{F'} + \frac{F^2}{F'^2} - \frac{cF\cos(\phi)}{F'^2}}.$$

Then using lemma 5.2, we have

$$\lim_{\phi \to z^+} \frac{G'}{G} = \lim_{\phi \to z^+} [\frac{W'}{W} + \frac{2}{m-1}\frac{F'}{F} - \frac{c}{m-1}\frac{\sin\phi}{F}] = \frac{2\cos(z)}{\sin(z)} \equiv -C < 0.$$

Therefore G is decreasing near z and so for some L with $0 < L \leq \infty$

$$\lim_{\phi \to z^+} G = L.$$

Then integrating on $(\phi, \phi_1) \subset (z, z+\epsilon)$ we have

$$\ln\left(\frac{G(\phi_1)}{G(\phi)}\right) = \int_\phi^{\phi_1} \frac{G'}{G}\,dt > \int_\phi^{\phi_1} -2C\,dt = -2C(\phi_1 - \phi).$$

Thus

$$L = \lim_{\phi \to z^+} G(\phi) \leq G(\phi_1)e^{2C(\phi_1 - z)}.$$

Thus the limit of G is a positive, finite number.

Next, choosing $0 < \epsilon < \frac{1}{m}$ then by lemma 5.5 we have for some $\delta > 0$ that:

$$\frac{(\phi - z)F'}{F} < 1 - \frac{1}{m} + \epsilon \ \text{ for } \ z < \phi < z + \delta.$$

Dividing by $\phi - z$, and integrating on $(\phi, z + \delta)$ where $z < \phi < z + \delta$ gives

$$\ln\left(\frac{F(z+\delta)}{F(\phi)}\right) \leq \left(1 - \frac{1}{m} + \epsilon\right) \ln\left(\frac{\delta}{\phi - z}\right).$$

Therefore

$$\frac{F(z+\delta)}{F} \leq \frac{\delta^{1-\frac{1}{m}+\epsilon}}{(\phi - z)^{1-\frac{1}{m}+\epsilon}} \text{ for } z < \phi < z + \delta.$$

Thus, we have

$$\frac{1}{F} \leq \frac{C(\delta)}{(\phi - z)^{1-\frac{1}{m}+\epsilon}} \text{ for } z < \phi < z + \delta,$$

where $C(\delta) = \frac{\delta^{1-\frac{1}{m}+\epsilon}}{F(z+\delta)}$. Since $0 < \epsilon < \frac{1}{m}$ we see that

$$\int_z^\phi \frac{1}{F(t)} dt \leq \int_z^\phi \frac{C(\delta)}{(t-z)^{1-\frac{1}{m}+\epsilon}} dt = \frac{C(\delta)}{\frac{1}{m} - \epsilon} (\phi - z)^{\frac{1}{m}-\epsilon} < \infty \text{ on } (z, z+\delta).$$

Hence $\frac{1}{F}$ is integrable on $(z, z+\delta)$ for some $\delta > 0$. Also, on $[z+\delta, \pi]$ we know that F is bounded from below by a strictly positive constant and therefore we see that $\frac{1}{F}$ is integrable on $(z, \pi]$. Also, since $(\int_\phi^\pi \frac{\sin t}{F(t)}\, dt)' = -\frac{\sin\phi}{F(\phi)} < 0$ on (z, π) we see that

$$\lim_{\phi \to z^+} \exp\left(\frac{c}{m-1} \int_\phi^\pi \frac{\sin t}{F(t)}\, dt\right) \text{ exists and is finite and positive.} \tag{108}$$

Since $a > c > 0$ we see from (108) and (98) that the limit of G is positive and finite so that

$$\lim_{\phi \to z^+} W(\phi) F^{\frac{2}{m-1}}(\phi) \text{ exists and is positive.}$$

Rewriting this using (66) we see therefore that there exists $T > 0$ such that

$$\lim_{\phi \to z^+} F'^2 F^{\frac{2}{m-1}} \left[1 - \frac{c \sin\phi}{F'} + \frac{F^2}{F'^2} + \frac{cF\cos\phi}{F'^2}\right] = T > 0. \tag{109}$$

Again, since $F(z) = \lim_{\phi \to z^+} F = 0$ and $\lim_{\phi \to z^+} F' = \infty$ (by lemma 5.3) it follows from (107) and (109) that

$$\lim_{\phi \to z^+} F'^2 F^{\frac{2}{m-1}} = T > 0. \tag{110}$$

From the note between lemma 5.2 and lemma 5.3 we know that $F' > 0$ on (z, π) and therefore it follows from (110) that

$$\lim_{\phi \to z^+} F' F^{\frac{1}{m-1}} = \sqrt{T} > 0. \tag{111}$$

This completes the proof of the lemma 5.6. $\square$

LEMMA 5.7. *Let $0 < c < a$. Then*

$$\lim_{\phi \to z^+} \frac{F}{(\phi - z)^{\frac{m-1}{m}}} = \left(\frac{m\sqrt{T}}{m-1} \right)^{\frac{m-1}{m}}.$$

PROOF. Letting,

$$K = F^{\frac{m}{m-1}},$$

then we see that

$$K' = \frac{m}{m-1} F^{\frac{1}{m-1}} F',$$

and so by (111)

$$\lim_{\phi \to z^+} K' = \frac{m\sqrt{T}}{m-1}.$$

Thus from L'Hopital's rule we see that

$$\lim_{\phi \to z^+} \frac{F^{\frac{m}{m-1}}}{\phi - z} = \lim_{\phi \to z^+} \frac{K}{\phi - z} = \lim_{\phi \to z^+} K' = \frac{m\sqrt{T}}{m-1} > 0.$$

This completes the proof of lemma 5.7 and the proof of the main theorem in the case $0 < c < a$. $\square$

Note: We have assumed throughout chapter 5 that F is defined and positive on (z, π). However, if F is defined and positive only on (x_0, π) where $z < x_0 < \pi$, it is easy to show that D does not have a negative local minimum on (x_0, π). A similar proof as in Lemma 5.2 shows $D' > 0$ on (x_0, π).

If $x_0 \leq \frac{\pi}{2}$, we have shown earlier in the chapter that F has a zero on $\left(\frac{\pi}{2}, \pi\right)$ which contradicts that $F > 0$ on (x_0, π).

If $x_0 > \frac{\pi}{2}$, then by lemma 5.2 $F'' + F < 0$ on (x_0, π). Also $F > 0$ on (x_0, π). So, $F'' < 0$ on (x_0, π). Hence F' is decreasing on (x_0, π). So, $\lim_{\phi \to x_0^+} F'(\phi)$ exists (possibly ∞). Since

$F'(\pi) = 0$ and $F'' < 0$ on (x_0, π) then $F'(\phi) \geq 0$ on (x_0, π) and thus $\lim_{\phi \to x_0^+} F(\phi)$ exists.

Next we show that if $\lim_{\phi \to x_0^+} F(\phi) \neq 0$ then $\lim_{\phi \to x_0^+} F'(\phi) \neq \infty$.

Suppose $\lim_{\phi \to x_0^+} F(\phi) = L > 0$ and $\lim_{\phi \to x_0^+} F'(\phi) = \infty$. We have from (10),

$$(m-1)FF'' + F'^2 = F' \left(c \sin\phi - \frac{(m-1)\cos\phi}{\sin\phi} F \right) - c\ \cos\phi\ F - 2(m-1)F^2. \tag{112}$$

Since $\left(c \sin\phi - \frac{(m-1)\cos\phi}{\sin\phi} F \right)$ is positive, the right hand side of equation (112) converges to ∞ as ϕ approaches to x_0^+. Then,

$$\begin{aligned} \left(F' F^{\frac{1}{m-1}} \right)' &= F'' F^{\frac{1}{m-1}} + \frac{1}{m-1} F^{\frac{1}{m}-1} F'^2 \\ &= \frac{1}{m} F^{\frac{1}{m-1}-1} \left((m-1)FF'' + F'^2 \right). \end{aligned}$$

which converges to ∞ by equation(112). This contradicts the lemma 5.6 that $\lim_{\phi \to x_0^+} \left(F' F^{\frac{1}{m-1}} \right)$ exists and is finite. Thus if $\lim_{\phi \to x_0^+} F(\phi) > 0$ then $\lim_{\phi \to x_0^+} F'(\phi)$ is finite. Then we can find an interval $(x_0 - \epsilon, \pi)$, for some $\epsilon > 0$, where F is defined contradicting the maximality of the interval (x_0, π). So $x_0 = z$ and F is defined on all of (z, π).

CHAPTER 6

THE CASE $c > a > 0$

We first observe that if $a = \frac{c}{m}$ then $F = \frac{c}{2m}(1 - \cos\phi)$ is a positive solution on $(0, \pi)$ of (58) such that

$$F(\pi) = a = \frac{c}{m}, F'(\pi) = 0,$$

and

$$F(0) = 0, F'(0) = 0.$$

We will now show that a similar result is true for all other values of a where $0 < a < c$.

Lemma 6.1. *Let $c > a > 0$. Then*

$$D' < 0 \ \ on \ \ \left(\frac{\pi}{2}, \pi\right) \quad and \quad F'' + F > 0 \ \ on \ \ \left(\frac{\pi}{2}, \pi\right).$$

Proof. We have from (71)-(76) that $D(\pi) = 0$ and $D'(\pi) < 0$. Thus D is decreasing near π and if the lemma were not true then there would exist a $q \in (\frac{\pi}{2}, \pi)$ with

$$D'(q) = 0, \ \ D''(q) \leq 0, \ \ D' < 0 \ \text{on} \ (q, \pi), \ \ \text{and} \ \ D > 0 \ \text{on} \ [q, \pi). \tag{113}$$

Thus from (75)

$$F' \cos\phi + F \sin\phi > 0 \quad \text{on} \quad [q, \pi),$$

and therefore we see that

$$\left(\frac{F}{\cos\phi}\right)' > 0 \quad \text{on} \quad [q, \pi).$$

Integrating on (ϕ, π) gives

$$\frac{a}{\cos\pi} - \frac{F}{\cos\phi} > 0 \quad \text{on} \quad [q, \pi),$$

and thus

$$F > -a\cos\phi > 0 \quad \text{on} \ [q, \pi). \tag{114}$$

Next, from (77) we have

$$W + (m-1)\frac{F}{\cos\phi}D' + (m-1)\frac{F}{\sin\phi}D = 0,$$

and therefore from (113) and (114)

$$W(q) = -(m-1)\frac{F(q)}{\sin q}D(q) < 0. \tag{115}$$

From (113) we have $D'(q) = 0$ and so by (78) $W'(q) = 0$. Also, from (113)-(114) and the fact that $q \in \left(\frac{\pi}{2}, \pi\right)$, we see

$$\frac{F(q)}{\cos q}D''(q) \geq 0. \tag{116}$$

Using these facts in (87), we then see

$$\left(\frac{F'(q)\sin(q) - F(q)\cos(q)}{\sin^2(q)}\right) D(q) \leq 0. \tag{117}$$

Since $D(q) > 0$ then (117) implies

$$F'(q)\sin q - F(q)\cos q \leq 0. \tag{118}$$

In addition, from (115)

$$F'^2(q) - cF'(q)\sin q + F^2(q) + cF(q)\cos q = W(q) < 0.$$

Then, using (118), we obtain

$$0 \leq F'^2(q) + F^2(q) < cF'(q)\sin q - cF(q)\cos q \leq 0. \tag{119}$$

This forces $F(q) = F'(q) = 0$ which implies $D(q) = 0$ but this contradicts (113). This completes the proof of the first part of the lemma. The second part of the lemma follows from (76) and the fact that $\phi \in (\frac{\pi}{2}, \pi)$. This completes the proof of lemma 6.1. □

Lemma 6.2. *Let $c > a > 0$. Then $F > 0$ on $(\frac{\pi}{2}, \pi]$.*

Proof. By the previous lemma, $D' < 0$ on $\left(\frac{\pi}{2}, \pi\right]$. Also since $D(\pi) = 0$ we then have that $D > 0$ on $\left(\frac{\pi}{2}, \pi\right)$. Thus from (75)

$$F'\cos\phi + F\sin\phi > 0 \text{ on } \left(\frac{\pi}{2}, \pi\right),$$

and therefore

$$\left(\frac{F}{\cos\phi}\right)' > 0 \ \text{ on } \ \left(\frac{\pi}{2}, \pi\right).$$

Integrating over (ϕ, π), we get

$$-a \geq \frac{F}{\cos\phi} \ \text{ for } \ \phi \in \left(\frac{\pi}{2}, \pi\right),$$

and thus

$$F \geq -a\cos\phi > 0 \ \text{ for } \ \phi \in \left(\frac{\pi}{2}, \pi\right).$$

As $F(\pi) = a > 0$, we see that

$$F > 0 \ \text{ on } \ \left(\frac{\pi}{2}, \pi\right].$$

This completes the proof of lemma 6.2. □

LEMMA 6.3. *Let $c > a > 0$. Then $W < 0$ on $(\frac{\pi}{2}, \pi]$.*

PROOF. From lemmas 6.1 and 6.2 we have

$$F > 0 \text{ and } D' < 0 \ \text{ on } \ \left(\frac{\pi}{2}, \pi\right].$$

Thus

$$\frac{F}{\cos\phi} D' > 0 \ \text{ on } \ \left(\frac{\pi}{2}, \pi\right]. \tag{120}$$

Also since:

$$D(\pi) = 0 \ \text{ and } \ D' < 0 \ \text{ on } \ \left(\frac{\pi}{2}, \pi\right],$$

then

$$D \geq 0 \ \text{ on } \ \left(\frac{\pi}{2}, \pi\right].$$

Thus

$$\frac{F}{\sin\phi} D \geq 0 \ \text{ on } \ \left(\frac{\pi}{2}, \pi\right]. \tag{121}$$

Hence using (120) and (121) in (77), we have

$$W < 0 \ \text{ on } \ \left(\frac{\pi}{2}, \pi\right].$$

This completes the proof of the lemma. □

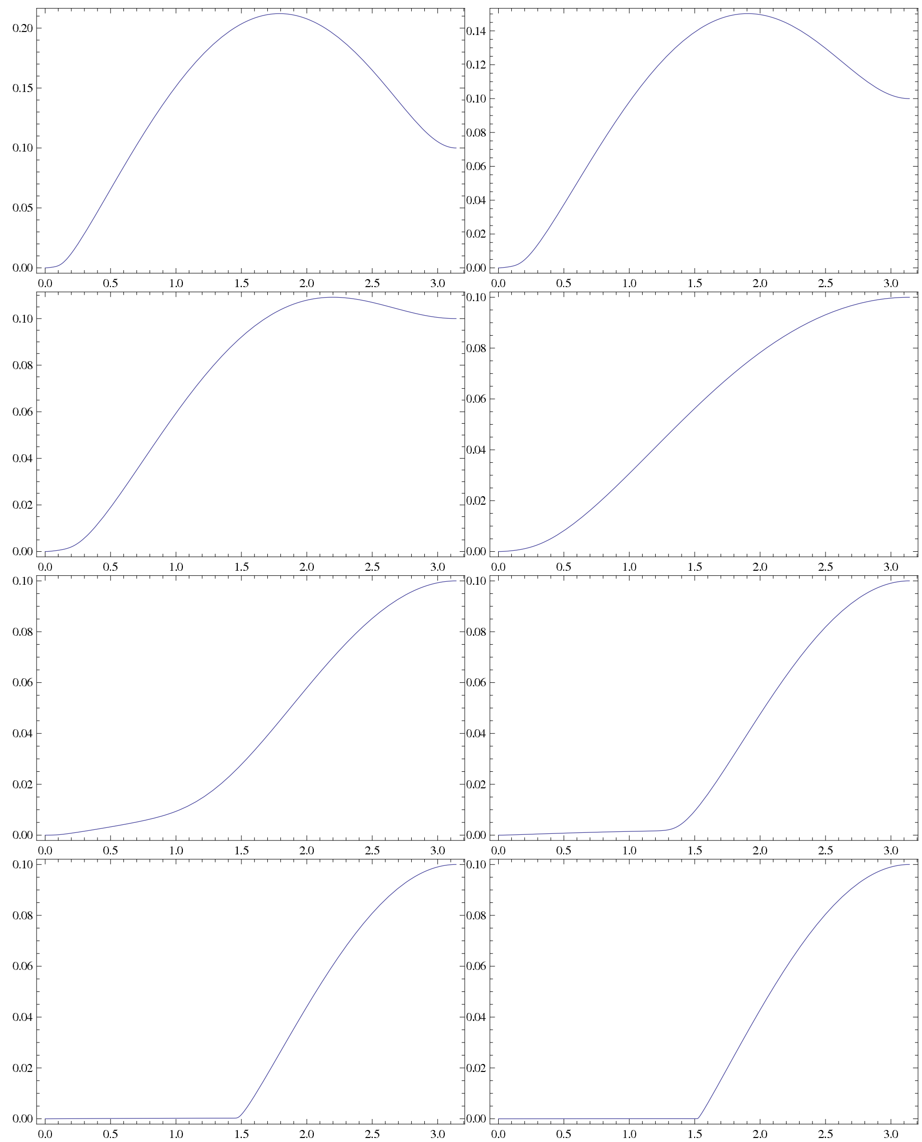

FIGURE 6.1. The function $F(\phi)$ for $a = 0.1$, $m = 4$ and for dimensionless speed values $c = 4.0, 2.0, 1.0, 0.50, 0.25, 0.15, 0.12$ and 0.11 (in lexicographical order)

Note: Let $c > a > 0$. Since $D(\pi) = 0$ and $D' < 0$ on $(\frac{\pi}{2}, \pi)$, it follows from (75) that $F(\frac{\pi}{2}) = D(\frac{\pi}{2}) > 0$.

LEMMA 6.4. *Let $c > a > 0$. Then $F'\left(\frac{\pi}{2}\right) > 0$.*

PROOF. From lemma 6.2, we have

$$F > 0 \quad \text{on} \ \ (\frac{\pi}{2}, \pi].$$

From lemma 6.3, we have $W < 0$ on $(\frac{\pi}{2}, \pi]$. Then by continuity,

$$W\left(\frac{\pi}{2}\right) \le 0,$$

and so from (66) we have

$$F'^2\left(\frac{\pi}{2}\right) + F^2\left(\frac{\pi}{2}\right) \le c\, F'\left(\frac{\pi}{2}\right). \tag{122}$$

By the note before this lemma, $F\left(\frac{\pi}{2}\right) > 0$. Then from (122) we see that

$$F'\left(\frac{\pi}{2}\right) > 0.$$

This completes the proof of the lemma. □

LEMMA 6.5. *Let $c > a > 0$. Then $F''(\frac{\pi}{2}) + F(\frac{\pi}{2}) > 0$.*

PROOF. From lemma 6.1, we have

$$F'' + F > 0 \quad \text{on} \quad \left(\frac{\pi}{2}, \pi\right].$$

And so by continuity

$$F'' + F \ge 0 \quad \text{on} \quad \left[\frac{\pi}{2}, \pi\right].$$

Now let us assume by way of contradiction that

$$(F'' + F)\left(\frac{\pi}{2}\right) = 0. \tag{123}$$

Since $F'' + F > 0$ on $(\frac{\pi}{2}, \pi]$ it follows from (123) that

$$(F'' + F)'\left(\frac{\pi}{2}\right) \ge 0. \tag{124}$$

From equations (76)-(77), we have

$$W + (m-1)F(F'' + F) + (m-1)\frac{F}{\sin\phi}D = 0. \tag{125}$$

Differentiating (125) we obtain

$$\frac{W'}{m-1} + F'(F''+F) + F(F''+F)' + \left(\frac{F}{\sin\phi}\right)' D + \left(\frac{F}{\sin\phi}\right) D' = 0 \tag{126}$$

Also note from (67), (76), and (123) that

$$W'(\frac{\pi}{2}) = D'(\frac{\pi}{2}) = 0.$$

Therefore using this and (124) in (126), we obtain

$$F'(\frac{\pi}{2})D(\frac{\pi}{2}) \le 0. \tag{127}$$

From the note before lemma 6.4, we have

$$F\left(\frac{\pi}{2}\right) = D\left(\frac{\pi}{2}\right) > 0 \tag{128}$$

and thus by (127)

$$F'\left(\frac{\pi}{2}\right) \le 0. \tag{129}$$

Evaluating (66) and (125) at $\phi = \frac{\pi}{2}$ and using (123), (128), and (129) gives

$$0 < F'^2\left(\frac{\pi}{2}\right) - cF'\left(\frac{\pi}{2}\right) + F^2\left(\frac{\pi}{2}\right) = W(\frac{\pi}{2}) = -(m-1)F^2\left(\frac{\pi}{2}\right) < 0,$$

which is impossible. Thus the lemma must hold. This completes the proof of lemma 6.5 $\square$

Lemma 6.6. *Let $c > a > 0$. Then D has a local maximum at $\frac{\pi}{2}$.*

Proof. From (76) we have

$$D'(\frac{\pi}{2}) = (F''+F)(\frac{\pi}{2})\cos(\frac{\pi}{2}) = 0.$$

Differentiating (76) we have

$$D'' = (F'''+F')\cos\phi - (F''+F)\sin\phi.$$

Thus by lemma 6.5 we have

$$D''\left(\frac{\pi}{2}\right) = -(F''+F)\left(\frac{\pi}{2}\right) < 0.$$

Hence D has a local maximum at $\frac{\pi}{2}$. $\square$

Lemma 6.7. *Let $c > a > 0$. If $b \geq 0$ and $F > 0$ on $\left(b, \frac{\pi}{2}\right)$, then $F' \geq 0$ on $\left(b, \frac{\pi}{2}\right)$.*

Proof. Contrary to the lemma, let us assume

$$F'(p) < 0 \quad \text{for some } \ p \in \left(b, \frac{\pi}{2}\right).$$

By lemma 6.4,

$$F'\left(\frac{\pi}{2}\right) > 0.$$

Thus F is decreasing near p, increasing near $\frac{\pi}{2}$, and so there exists a local minimum $q \in \left(p, \frac{\pi}{2}\right)$ such that

$$F'(q) = 0 \quad \text{and} \quad F''(q) \geq 0. \tag{130}$$

From equations (59) and (66) we have

$$W + (m-1)F\left(F'' + \frac{\cos\phi}{\sin\phi}\, F' + 2F\right) = 0. \tag{131}$$

At $\phi = q$, we have

$$F'(q) = 0, \quad F(q)F''(q) \geq 0, \quad \text{and} \quad F^2(q) > 0.$$

Using this in (131) we obtain

$$W(q) < 0.$$

On the other hand, from (66) and (130) we have

$$W(q) = F^2(q) + cF(q)\cos q > 0$$

which is a contradiction. This completes the proof of the lemma. $\square$

Lemma 6.8. *Let $c > a > 0$. If $b \geq 0$ and $F > 0$ on $\left(b, \frac{\pi}{2}\right)$, then D does not have a positive local minimum on $\left(b, \frac{\pi}{2}\right)$.*

Proof. On the contrary, let us assume that D has a positive local minimum on $\left(b, \frac{\pi}{2}\right)$. Then there is a $p \in \left(b, \frac{\pi}{2}\right)$ with the following properties

$$D(p) > 0, \quad D'(p) = 0, \quad D'(\phi) > 0 \ \text{ on } \ \left(p, \frac{\pi}{2}\right), \quad \text{and} \ \ D''(p) \geq 0. \tag{132}$$

From (78), we see that

$$W'(p) = 0.$$

Using (132) and (87) at $\phi = p$, we see

$$\left(\frac{F'(p)\sin(p) - F(p)\cos(p)}{\sin^2 p}\right) D(p) \leq 0.$$

Since $D(p) > 0$ we then have

$$F'(p)\sin p - F(p)\cos p \leq 0. \tag{133}$$

Also note from (77) and (132) that

$$W(p) = -(m-1)\frac{F(p)}{\sin p} D(p) < 0.$$

Using this in (66) we see

$$W(p) = F'^2(p) - cF'(p)\sin p + F^2(p) + cF(p)\cos p < 0.$$

Thus using (133) and rewriting this implies

$$0 < F'^2(p) + F^2(p) \leq c\,[F'(p)\sin p - F(p)\cos p] \leq 0 \tag{134}$$

which is impossible. Hence the lemma must be true. This completes the proof of the lemma. $\square$

Lemma 6.9. *Let $c > a > 0$. If there exists $z \in \left[0, \frac{\pi}{2}\right)$ such that $F(z) = 0$ and $F > 0$ on $\left(z, \frac{\pi}{2}\right)$ then*

$$D' \geq 0 \;\; \textit{and} \;\;\; F'' + F \geq 0 \;\; \textit{on} \left(z, \frac{\pi}{2}\right).$$

Proof. By lemma 6.8, we see that D does not have a positive local minimum on $\left(z, \frac{\pi}{2}\right)$. Since $F > 0$ on $\left(z, \frac{\pi}{2}\right)$, by lemma 6.5 we have $F' \geq 0$ on $\left(z, \frac{\pi}{2}\right)$. This implies

$$D = F'\cos\phi \; + F\sin\phi > 0 \text{ on } (z, \frac{\pi}{2}). \tag{135}$$

By lemma 6.6, D has a local maximum at $\frac{\pi}{2}$. Therefore if there were a $q \in (z, \frac{\pi}{2})$ with $D'(q) < 0$ then these facts along with (135) imply that D has a positive local minimum in

$(z, \frac{\pi}{2})$ contradicting lemma 6.8.

Hence, we see that

$$D' \geq 0 \ \text{ on} \left(z, \frac{\pi}{2}\right).$$

This proves the first part of the lemma. The second part of the lemma follows from (76) and the fact that $\cos\phi > 0$ on $\left(z, \frac{\pi}{2}\right) \subset \left(0, \frac{\pi}{2}\right)$. This completes the proof of the lemma. $\square$

LEMMA 6.10. *Let $c > a > 0$. Then F has a zero on $\left[0, \frac{\pi}{2}\right)$.*

PROOF. Contrary to the lemma, let us assume

$$F > 0 \quad \text{on} \quad \left[0, \frac{\pi}{2}\right).$$

By lemma 6.7,

$$F' \geq 0 \quad \text{on} \quad \left(0, \frac{\pi}{2}\right),$$

and by lemma 6.9,

$$F'' + F \geq 0 \ \text{ on} \left(0, \frac{\pi}{2}\right).$$

Therefore

$$F\left(F'' + \frac{\cos\phi}{\sin\phi}\, F' \ + 2F\right) > 0 \ \text{ on} \left(0, \frac{\pi}{2}\right). \tag{136}$$

Now, from (59) and (66) we have

$$W + (m-1)F\left(F'' + \frac{\cos\phi}{\sin\phi}\, F' \ + 2F\right) = 0,$$

and therefore by (136)

$$W = -(m-1)F\left(F'' + \frac{\cos\phi}{\sin\phi}\, F' \ + 2F\right) < 0 \ \text{ on} \left(0, \frac{\pi}{2}\right).$$

By continuity

$$W = F'^2 - cF'\sin\phi + F^2 + cF\cos\phi \leq 0 \ \text{ on} \left[0, \frac{\pi}{2}\right).$$

Evaluating at $\phi = 0$, we obtain

$$F'^2(0) + F^2(0) + cF(0) \leq 0$$

which is impossible since we are assuming $F(0) > 0$. Thus the lemma must be true. This completes the proof of the lemma. $\square$

So now we assume that there exists $z \in [0, \frac{\pi}{2})$ such that $F(z) = 0$ and $F > 0$ on $(z, \frac{\pi}{2})$.

LEMMA 6.11. *Let $c > a > 0$. If $z \in \left[0, \frac{\pi}{2}\right)$ such that $F(z) = 0$ and $F > 0$ on $\left(z, \frac{\pi}{2}\right)$ then $W < 0$ on $\left(z, \frac{\pi}{2}\right)$.*

PROOF. By assumption

$$F > 0 \quad \text{on} \quad \left(z, \frac{\pi}{2}\right),$$

and by lemma 6.7 we see that

$$F' \geq 0 \quad \text{on} \quad \left(z, \frac{\pi}{2}\right),$$

and by lemma 6.9 we see that

$$F'' + F \geq 0 \ \text{ on} \left(z, \frac{\pi}{2}\right).$$

So

$$F\left(F'' + \frac{\cos\phi}{\sin\phi}\, F' \ + 2F\right) > 0 \ \text{ on} \left(z, \frac{\pi}{2}\right).$$

Using this inequality along with (59) and (66), we see

$$W = -(m-1)F\left(F'' + \frac{\cos\phi}{\sin\phi}\, F' \ + 2F\right) < 0 \ \text{ on} \left(z, \frac{\pi}{2}\right).$$

This completes the proof of the lemma. $\square$

LEMMA 6.12. *Let $c > a > 0$. If z is a zero of F on $\left[0, \frac{\pi}{2}\right)$ and $F > 0$ on $\left(z, \frac{\pi}{2}\right)$ then $0 < F' < c\ \sin\phi$ on $\left(z, \frac{\pi}{2}\right)$.*

PROOF. We have

$$F > 0 \quad \text{on} \quad \left(z, \frac{\pi}{2}\right).$$

By lemma 6.7

$$F' \geq 0 \quad \text{on} \quad \left(z, \frac{\pi}{2}\right),$$

and by lemma 6.11

$$W = F'^2 - cF'\sin\phi + F^2 + cF\cos\phi < 0 \quad \text{on} \quad \left(z, \frac{\pi}{2}\right).$$

Thus

$$F'^2 - cF' \sin\phi < -F^2 - cF\cos\phi < 0 \quad \text{on} \quad \left(z, \frac{\pi}{2}\right).$$

This yields

$$F'\left(F' - c\sin\phi\right) < 0 \text{ on } (z, \frac{\pi}{2}). \tag{137}$$

Since $F' \geq 0$ then (137) implies

$$F' > 0 \quad \text{and} \quad F' - c\ \sin\phi < 0.$$

Thus we conclude that

$$0 < F' < c\ \sin\phi \quad \text{on} \quad \left(z, \frac{\pi}{2}\right).$$

This completes the proof of the lemma. □

Lemma 6.13. *Let $c > a > 0$. Then there exists an A with $0 \leq A \leq c\ \sin z$ such that $\lim\limits_{\phi\to z^+} F'(\phi) = A$.*

Proof. Let

$$E = \frac{1}{2}\left[F'^2 + F^2\right].$$

Then

$$E' = F'\left[F'' + F\right].$$

By lemma 6.7, we know that

$$F' \geq 0 \ \text{ on} \left(z, \frac{\pi}{2}\right).$$

Also, by lemma 6.9, we know that

$$F'' + F \geq 0 \ \text{ on} \left(z, \frac{\pi}{2}\right)$$

and thus

$$E' \geq 0 \ \text{ on} \left(z, \frac{\pi}{2}\right).$$

Thus $\lim_{\phi\to z^+} E$ exists. As $\lim_{\phi\to z^+} F = 0$, we see that $\lim_{\phi\to z^+} F'^2$ exists. Again from lemma 6.7 we have that $F' \geq 0$ on $\left(z, \frac{\pi}{2}\right)$ and so we see that there exists an A such that

$$\lim_{\phi\to z^+} F' = A \geq 0.$$

We also know by lemma 6.11 that

$$W = F'^2 - cF'\sin\phi + F^2 + cF\cos\phi < 0 \text{ on } \left(z, \frac{\pi}{2}\right).$$

Taking the limit as $\phi \to z^+$, we obtain

$$A^2 - cA\sin z = A(A - c\sin z) \leq 0.$$

Hence

$$0 \leq A \leq c\ \sin z.$$

This completes the proof of the lemma. $\square$

Notes: By lemma 6.3, lemma 6.11, and continuity we have

$$W = F'^2 - cF'\sin\phi + F^2 + cF\cos\phi \leq 0 \quad \text{on} \quad (z, \pi].$$

Therefore completing the square, we have

$$(F' - \frac{c}{2}\sin\phi)^2 + (F + \frac{c}{2}\cos\phi)^2 \leq \frac{c^2}{2},$$

and therefore

$$|F| \leq |F + \frac{c}{2}\cos\phi| + |\frac{c}{2}\cos\phi| \leq \frac{c}{2} + \frac{c}{2} = c \tag{138}$$

and

$$|F'| \leq |F' - \frac{c}{2}\sin\phi| + |\frac{c}{2}\sin\phi| \leq \frac{c}{2} + \frac{c}{2} = c. \tag{139}$$

Hence F and F' are uniformly bounded on $(z, \pi]$.

LEMMA 6.14. $\lim_{\phi\to z^+} F' = 0.$

Proof. From lemma 6.13 we know that $\lim_{\phi\to z^+} F' = A$ where $0 \le A \le c\sin z$. Note that if $z = 0$ then $A = 0$ and the lemma is proved. So now let us suppose $z > 0$ and also that $0 < A < c\sin z$.

Multiplying (59) by $\frac{F'}{F}$ and rewriting gives:

$$\frac{F'}{F}[F'^2 - cF'\sin\phi] + (2m-1)FF' + cF'\cos\phi + (m-1)F'F'' + \frac{(m-1)\cos\phi}{\sin\phi}F'^2 = 0.$$

Integrating this on (ϕ, ϕ_2) where $z < \phi < z + \epsilon = \phi_2$ and $\epsilon > 0$ gives

$$-\int_\phi^{\phi_2} \frac{F'}{F}[F'^2 - cF'\sin t]\,dt = \frac{2m-1}{2}[F^2(\phi_2) - F^2] + \frac{m-1}{2}[F'(\phi_2)^2 - F'^2] \tag{140}$$

$$+c[F(\phi_2)\cos\phi_2 - F\cos\phi] + c\int_\phi^{\phi_2} F\sin t\,dt + (m-1)\int_\phi^{\phi_2}\frac{\cos t}{\sin t}F'^2\,dt.$$

From (138) and (139) we have that F and F' are bounded and thus we see that the right-hand side of (140) is bounded. Thus we see there exists an $M > 0$ such that

$$\left|-\int_\phi^{z_2}\frac{F'}{F}[F'^2 - cF'\sin t]\,dt\right| \le M. \tag{141}$$

Also, by assumption we have $0 < A < c\sin(z)$ and therefore it follows from (66) that

(142)

$$\lim_{\phi\to z^+} W(\phi) = \lim_{\phi\to z^+}\left[F'^2 - cF'\sin\phi + F^2 + cF\cos\phi\right] = \lim_{\phi\to z^+} F'[F' - c\sin\phi] = A[A - c\sin z].$$

Thus if ϕ is sufficiently close to z and $\phi > z$ then we see that

$$-[F'^2 - cF'\sin\phi] \ge \frac{A|A - c\sin z|}{2} > 0.$$

However using (141) we see that for ϵ sufficiently small we have

(143)

$$M \ge -\int_\phi^{\phi_2}\frac{F'}{F}[F'^2 - cF'\sin t]\,dt \ge \frac{A|A - c\sin z|}{2}\int_\phi^{\phi_2}\frac{F'}{F} = \frac{A|A - c\sin z|}{2}\ln\left[\frac{F(\phi_2)}{F(\phi)}\right].$$

Since we have assumed $0 < A < c\sin z$, we see that the right-hand side of (143) goes to $+\infty$ as $\phi \to z^+$ since $\lim_{\phi\to z^+} F(\phi) = F(z) = 0$ whereas the left-hand side of (143) is bounded. This is a contradiction and thus we see that either $A = 0$ or $A = c\sin z$.

So now we suppose $z > 0$ and $A = c \sin z > 0$. Then we see from (142) that $\lim_{\phi \to z^+} W(\phi) = 0$. Also

$$\lim_{\phi \to z^+} [2F' - c\sin(\phi)] = 2A - c\sin(z) = 2c\sin(z) - c\sin(z) = c\sin(z) > 0. \tag{144}$$

And from (67)

$$W' = (2F' - c\sin\phi)(F'' + F).$$

By lemma 6.9 and (144), we see that $W' \geq 0$ on $(z, z+\epsilon)$ and since $\lim_{\phi \to z^+} W(\phi) = 0$, we see that W is nonnegative to the immediate right of z contradicting lemma 6.11. Thus $A = 0$ and this completes the proof of the lemma. □

Lemma 6.15. $z = 0$.

Proof. By lemma 6.10, we have $z \in \left[0, \frac{\pi}{2}\right)$. So, $z \geq 0$. Let us assume that $z > 0$. By lemma 6.11, $W < 0$ on $\left(z, \frac{\pi}{2}\right)$. Then

$$-cF'\sin\phi \ + cF\cos\phi \leq F'^2 - cF'\sin\phi \ + F^2 + cF\cos\phi < 0 \ \text{ on } \left(z, \frac{\pi}{2}\right). \tag{145}$$

Since $F > 0$ then by lemma 6.12, $F' > 0$ on $\left(z, \frac{\pi}{2}\right)$, and so by (145)

$$0 < \frac{F}{F'} \leq \frac{\sin\phi}{\cos\phi} \quad \text{on} \quad \left(z, \frac{\pi}{2}\right),$$

and so $\frac{F}{F'}$ is bounded near z. Then there exists some finite positive number M such that

$$\frac{F}{F'} \leq M \quad \text{on} \ \ (z, z+\epsilon), \tag{146}$$

where $\epsilon > 0$ and $z + \epsilon < \frac{\pi}{2}$.

Since $F' > 0$ on $(z, z+\epsilon)$ we may divide (59) by F' and after rewriting we obtain

$$F' + (2m-1)\frac{F}{F'}F + c\frac{F}{F'}\cos\phi + (m-1)\frac{F}{F'}F'' + (m-1)\frac{\cos\phi}{\sin\phi}F = c\sin\phi.$$

Taking the limit as $\phi \to z^+$ and using (146) and lemma 6.14 we obtain

$$\lim_{\phi \to z^+}\left(c\frac{F}{F'}\cos\phi + (m-1)\frac{FF''}{F'}\right) = c\sin z > 0.$$

Thus there is an $\epsilon_1 > 0$ with $0 < \epsilon_1 < \epsilon$ such that

$$c\frac{F}{F'}\cos\phi + (m-1)\frac{FF''}{F'} > \frac{c}{2}\sin z \quad \text{on} \ \ (z, z+\epsilon_1).$$

Thus

$$(m-1)\frac{FF''}{F'} > \frac{c}{2}\sin z - c\frac{F}{F'}\cos\phi \quad \text{on} \ \ (z, z+\epsilon_1).$$

Therefore

$$F'' > \frac{c\ \sin z}{2(m-1)}\frac{F'}{F} - \frac{c}{m-1}\cos\phi \quad \text{for } z < \phi < z+\epsilon_1. \tag{147}$$

Now we let $\delta > 0$ and integrate (147) on $(z+\delta, \phi)$ where $z+\delta < \phi < z+\epsilon_1$ and we obtain

$$F' - F'(z+\delta) \geq \frac{c\sin z}{2(m-1)}\ln\left[\frac{F}{F(z+\delta)}\right] - \frac{c}{m-1}\left[\sin\phi - \sin(z+\delta)\right]. \tag{148}$$

Now as $\delta \to 0$ the left hand side of (148) goes to $F'(\phi)$ (by lemma 6.14) while the right hand side goes to $+\infty$ (since $\lim\limits_{\phi\to z^+} F(\phi) = F(z) = 0$) yielding a contradiction. Thus we see that $z = 0$. This completes the proof of the lemma. □

It then follows from lemma 6.14 that

$$\lim_{\phi\to 0^+} F'(\phi) = 0.$$

We also saw earlier in lemma 3.1 that $F(\pi - \phi) = F(\pi + \phi)$ and therefore we see that F is defined on $[0, 2\pi]$. And also we know from lemma 6.15 that $F(0) = F(2\pi) = 0$. Thus we can extend F to be 2π-periodic on all of $\mathbf{R}$ and so F satisfies (59) on all of $\mathbf{R}$. Thus it follows that:

Lemma 6.16. *F is $2\pi-$periodic on $\mathbf{R}$.*

This completes the proof of the main theorem in the case $c > a > 0$.

Note: Throughout chapter 6 we assumed that F is defined on $(0,\pi)$. However, if we suppose F is defined only on (x_0, π) for some $x_0 > 0$. Then by the similar argument as in the note after lemma 6.13, we can show that F and F' are uniformly bounded on (x_0,π) by c. Then from equation (10), it follows that $|F''| \leq M$ on $(x_0, \pi-\epsilon)$ for some $\epsilon > 0$ and some finite $M > 0$. So, F, F' and F'' are bounded on $(x_0, \pi-\epsilon)$. So $\lim_{\phi\to x_0^+} F(\phi)$ and

$\lim_{\phi \to x_0^+} F'(\phi)$ exist. Then we can extend F to a larger interval, contradicting maximality of (x_0, π). This gives $x_0 \leq 0$. Hence F exists on all of $(0, \pi)$.

CHAPTER 7

FINAL COMMENTS

One can also look for solutions of (3) in spherical coordinates in $\mathbf{R}^n$ where $n \geq 3$. Suppose

$$v(x_1, x_2, \ldots, x_n, t) = r(x_1, x_2, \ldots, x_n, t) F(\phi(x_1, x_2, \ldots, x_n, t))$$

where

$$r(x_1, x_2, \ldots, x_n, t) = \sqrt{x_1^2 + \cdots + x_{n-1}^2 + (x_n - ct)^2},$$

and

$$\phi(x_1, x_2, \ldots, x_n, t) = \tan^{-1}\left(\frac{\sqrt{x_1^2 + \cdots + x_{n-1}^2}}{x_n - ct}\right).$$

One can then show that F will satisfy

$$F'^2 - cF' \sin\phi + F^2 + cF\cos\phi + (m-1)F\left(F'' + (n-2)\frac{\cos\phi}{\sin\phi}F' + (n-1)F\right) = 0.$$

Note that this reduces in the $n = 2$ and $n = 3$ cases to the equations obtained in the introduction. We conjecture that a theorem similar to the Main Theorem is true in this case as well.

In ([7]), the authors considered the behavior of solutions when $a \to \infty$ and also when $a \to 0^+$. Preliminary investigations indicate that a result similar to the result in ([7]) is also true. In fact, we conjecture that if we denote the solution of (59)-(61) as F_a then

$$\lim_{a\to\infty}\left(\frac{F_a}{a}\right) = H$$

where H satisfies

$$H'^2 + H^2 + (m-1)H\left(H'' + \frac{\cos\phi}{\sin\phi}H' + 2H\right) = 0,$$

and

$$\lim_{a\to 0^+}\left(\frac{F_a}{\max F_a}\right) = C\sin(\phi)$$

for some $C > 0$.

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