DESCRIPTIVE SET THEORY AND MEASURE THEORY IN LOCALLY COMPACT AND NON-LOCALLY COMPACT GROUPS

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In this thesis we study descriptive-set-theoretic and measure-theoretic properties of Polish groups, with a thematic emphasis on the contrast between groups which are locally compact and those which are not. The work is divided into three major sections. In the first, working jointly with Robert Kallman, we resolve a conjecture of Gleason regarding the Polish topologization of abstract groups of homeomorphisms. We show that Gleason's conjecture is false, and its conclusion is only true when the hypotheses are considerably strengthened. Along the way we discover a new automatic continuity result for a class of functions which behave like but are distinct from functions of Baire class 1. In the second section we consider the descriptive complexity of those subsets of the permutation group S_∞ which arise naturally from the classical Levy-Steinitz series rearrangement theorem. We show that for any conditionally convergent series of vectors in Euclidean space, the sets of permutations which make the series diverge, and diverge properly, are $\mathbf{\Sigma}^0_3$-complete. In the last section we study the phenomenon of Haar null sets a la Christensen, and the closely related notion of openly Haar null sets. We identify and correct a minor error in the proof of Mycielski that a countable union of Haar null sets in a Polish group is Haar null. We show the openly Haar null ideal may be distinct from the Haar null ideal, which resolves an uncertainty of Solecki. We show that compact sets are always Haar null in S_∞ and in any countable product of locally compact non-compact groups, which extends the domain of a result of Dougherty. We show that any countable product of locally compact non-compact groups decomposes into the disjoint union of a meager set and a Haar null set, which gives a partial positive answer to a question of Darji. We display a translation property in the homeomorphism group $\mathrm{Homeo}^{+}[0,1]$ which is impossible in any non-trivial locally compact group. Other related results are peppered throughout.

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CONTENTS

CHAPTER 1

INTRODUCTION

In this work we investigate several topics which lay at the crossroads between the fields of descriptive set theory, measure theory, and topological group theory. Especially we are interested in the contrast between the structural properties of the so-called “large” Polish groups, i.e., the non-locally compact, separable, completely metrizable groups, with their “small” locally compact counterparts. The characteristics of locally compact groups were thoroughly explored in the mid-twentieth century by von Neumann, Pontryagin, Mackey, Weil, Gleason, Montgomery, Zippin, Yamabe, and others in the search for a solution to Hilbert’s fifth problem. Among their famous and deep results are that every locally compact group carries translation-invariant Haar measures, and that every locally compact group which is connected is the projective limit of a sequence of Lie groups.

In recent decades, however, researchers in descriptive set theory, functional analysis, and dynamics have provided a mounting body of evidence that depicts the non-locally compact groups as wildly varying in nature. For instance, among the large groups there exist extremely amenable groups, i.e. groups whose every continuous action on a compact space admits a fixed point, lying in stark contrast to the locally compact groups, which always admit free group actions. Many non-locally compact groups have also been shown to have a constructible universal minimal flow, whereas each non-compact locally compact group is known to have a somewhat pathological, non-metrizable universal minimal flow. Non-locally compact groups also do not admit reasonable translation-invariant or even quasi-invariant measures, so developing an adequate measure theory is much more difficult in the large group setting. We aim in this thesis to contribute to our collective understanding of these less well-behaved groups, especially by studying their descriptive-set-theoretic and measure-theoretic properties.

In Chapter 2 we recall the major definitions and facts that we need for the later results. The results of Chapter 2 are all well-known, except for the characterization of the point-stability groups of S_∞ presented in Section 2.2.3, which is apparently a new observation. We also provide details on some of the results mentioned in the previous two paragraphs regarding the difference between locally compact and non-locally compact groups. After these introductory facts are established, the remainder of the thesis may be organized into three major topics which may be read and understood largely independently of one another. The first topic is the assignment of Polish topologies to abstract (un-topologized) symmetry groups, and embodies Chapter 3. Chapter 4 examines the Borel complexity of some natural subsets of the permutation group S_∞, a non-locally compact Polish group when endowed with its topology of pointwise convergence. Chapter 5 deals primarily with the σ-ideal of Haar null sets a la Christensen, which represents a generalization of the ideal of Haar measure zero sets to the non-locally compact Polish group setting. We also study the closely related ideal of openly Haar null sets. The theorems in these latter three chapters are new unless otherwise stated.

1.1. Polish Topologization of Abstract Symmetry Groups

In Chapter 3, in joint work with Robert R. Kallman, we address an old problem of Gleason on the topologization of abstract symmetry groups. We ask, following Gleason, if G is an abstract group of homeomorphisms of a topological space M, under what circumstances can G be given a topology such that the pair (G, M) is a topological transformation group? That is, when can G be given a topological group topology such that the mapping $(g, m) \mapsto g(m)$, $G \times M \to M$, is continuous? If G acts by homeomorphisms on a topological space M, define a *frame* for the action to be any finite n-tuple $(m_1, ..., m_n) \in M^n$ for which the action of G restricted to the orbit $G \cdot (m_1, ..., m_n)$ is a free action. Gleason [22] posed the following very general conjecture on topologizing symmetry groups:

Gleason's Conjecture. Suppose that G is an abstract group acting by homeomorphisms on a Polish topological space M, and suppose a frame $m = (m_1, ..., m_n)$ exists for the action of G on M. Further suppose that for every $q \in M$, the orbit $G \cdot (m, q)$ is an analytic set in M^{n+1}. Then G may be endowed with a natural topology τ for which (G, τ) forms a Polish topological group, and G acts as a topological transformation group on M.

Remarkably, Gleason proved his own conjecture in the special case where $n = 1$. If $n = 1$ then the action of G on M admits a unary frame, i.e. G acts simply transitively on M. In this case G may be topologized in such a way that it is homeomorphic to M. The "analytic orbit" condition above is a bit puzzling, as it is now a well-known application of the Lusin-Souslin theorem that if G and M are in fact Polish, then every such G-orbit is actually a Borel set and not just analytic. Since Gleason's paper is quite old, it seems possible that he was unaware of this at the time. In the modern context, we reformulate the analytic orbit condition to the following: for every $q \in X$, the G-orbit of (m, q) is Borel in M^{n+1}. *A priori* this condition is stronger and so the reformulated conjecture is weaker.

We show that the stronger conjecture, and hence Gleason's original conjecture, is false for all $n \geq 2$. Our constructed counterexample group G is a certain K_σ subgroup of the classical $ax + b$ group, which acts on the Polish space $\mathbb{R}$ by affine transformations, and whose action admits frames of size n for all $n \geq 2$. The orbits of the group action are each K_σ, hence Borel, but the group cannot admit a Polish topology by an automatic continuity theorem of Dudley [13]. So the conjecture fails.

Now suppose the hypotheses of Gleason's conjecture are substantially strengthened as follows: for each $q \in M$, the orbit $G \cdot (m, q)$ is Borel, and in addition the orbit $G \cdot m$ is a G_δ set. In this case we are able to show that the conclusion of Gleason's conjecture holds. Unfortunately the result is not sharp, as we provide an example of a Polish group G acting transitively by homeomorphisms on a Polish space M, with an n-ary frame for each $n \geq 2$, but with the property that $G \cdot m$ is not G_δ for any frame $m \in M^n$. So a complete characterization of which actions (G, M) admit a desirable topologization will require some additional hypotheses.

During our investigation of Gleason's conjecture we discover an apparently new automatic continuity result. Recall that if X is a topological space and $A \subseteq X$ has the Baire property, then $D(A)$ is the smallest closed set in X for which $A \backslash D(A)$ is meager in X. Suppose that Y is a metric space and $\varphi : X \to Y$ is a Baire-measurable function with the property that $\varphi^{-1}(V) \subseteq D(\varphi^{-1}(V))$ for every open set $V \subseteq Y$. Then the set of points of continuity of φ is comeager in X.

1.2. Borel Complexity of Series Rearrangements in S_∞

Let X be a Polish space, and let $\mathcal{B}(X)$ be the family of Borel sets in X, which is the smallest family of subsets of X which includes the open sets and which is closed under countable unions and complements. The Borel sets may be stratified by their relative definable complexity into a Borel hierarchy indexed by the countable ordinals. We denote the open sets by $\mathbf{\Sigma}_1^0$ and the closed sets by $\mathbf{\Pi}_1^0$. The class $\mathbf{\Sigma}_2^0$ consists of all countable unions of closed sets, while $\mathbf{\Pi}_2^0$ is comprised of countable intersections of open sets. $\mathbf{\Sigma}_3^0$ is the class of all countable unions of $\mathbf{\Pi}_2^0$ sets, and so on.

If $\mathbf{\Gamma}$ is the first level where a set A appears in the Borel hierarchy, then we will say that A is $\mathbf{\Gamma}$-*complete.* It is an empirical phenomenon that a great bulk of those Borel sets which present themselves in the everyday study of mathematics will fall into the very bottom few levels of the Borel hierarchy. Thus there has been some industry for descriptive set theorists in finding "natural" examples of Borel sets which are "more complex than usual." Here we will take *more complex than usual* to mean at least on the third level of the Borel hierarchy. *Natural* here can only be defined sociologically rather than mathematically, but we take it to mean that the set in question appears commonly in everyday mathematical practice, and is not constructed in an *ad hoc* way or for a contrived purpose.

In Chapter 4 we seek to compute the exact descriptive complexity of some subsets of the group S_∞, which arise naturally from the following theorem of classical real analysis:

Levy-Steinitz Theorem. Let $\sum_{k=0}^{\infty} v_k$ be a conditionally convergent series of vectors in $\mathbb{R}^d$. Then there exists a non-trivial affine subspace $A \subseteq \mathbb{R}^d$ (that is, a space of the form $A = v + M$ where $v \in \mathbb{R}^d$ and $M \subseteq \mathbb{R}^d$ is a linear subspace with $\dim M \geq 1$) such that whenever $a \in A$, there is $\pi \in S_\infty$ with $\sum_{k=0}^{\infty} v_{\pi(k)} = a$.

Say that a series of d-dimensional vectors *diverges properly* if the series diverges, but does not diverge to ∞, where ∞ is the point at infinity in the one-point compactification of

$\mathbb{R}^d$. Fix any conditionally convergent series $\sum_{k=0}^{\infty} v_k$. Let $\mathcal{D}$ denote the set of all permutations $\pi \in S_\infty$ for which $\sum_{k=0}^{\infty} v_{\pi(k)}$ diverges, and let $\mathcal{DP}$ be the set of all $\pi \in S_\infty$ for which $\sum_{k=0}^{\infty} v_{\pi(k)}$ diverges properly. We show that both $\mathcal{D}$ and $\mathcal{DP}$ are $\mathbf{\Sigma}_3^0$-complete in S_∞.

The proof is primarily geometric in nature, and relies on the existence of a particular bounded rearrangement constant, now called the Steinitz constant, which was discovered and employed by Steinitz himself in his original proof of the Levy-Steinitz Theorem.

1.3. Haar Null Sets and Openly Haar Null Sets

In Chapter 5 we investigate the general theory of Haar null sets as defined by Christensen, as well as the newer notion of openly Haar null sets as defined briefly by Solecki. It is a famous result of Haar/Weil/Cartan that every locally compact topological group admits a left Haar measure, i.e. a regular Borel measure which is invariant under left translations. Of course every locally compact group also admits a right Haar measure, which is defined analogously. The left and right Haar measures need not agree, but they are always absolutely continuous with respect to one another, i.e. they define the same σ-ideal of measure zero sets. In this way the algebraic and topological structure of each locally compact group uniquely defines a measure-theoretic σ-ideal which is invariant under left, right, and two-sided translations: the Haar measure zero sets.

Unfortunately the situation is quite different for non-locally compact groups. Every locally compact Polish group is σ-compact, hence "small." Conversely, if a Polish group is non-locally compact, then its compact subsets are meager, and hence the Baire category theorem implies that it cannot be σ-compact; so the non-locally compact groups are "large." This turns out to be a fundamental problem with respect to regular measures: it is well-known that if a Polish group G admits a regular σ-finite non-zero Borel measure whose family of measure zero sets is invariant under translations, then G must be σ-compact, and hence locally compact. In other words if G is not locally compact, then no single measure on G can generate a translation-invariant zero-ideal of subsets of G.

In the absence of a single canonical measure generating a translation-invariant ideal on a non-locally compact group, Christensen [7] has defined the following alternative notion. If G is a Polish group and $A \subseteq G$ is universally measurable, we say A is a Haar null set (or according to some authors a shy set) if there exists a Borel probability measure μ on G with the property that $\mu(gAh) = 0$ for every $g, h \in G$. Denote the class of subsets of G which are contained in a Haar null set by $\mathcal{HN}(G)$ or just $\mathcal{HN}$. It can be shown that $\mathcal{HN}(G)$ forms a σ-ideal in any Polish group which is invariant under translations, and that $\mathcal{HN}(G)$ is exactly the class of Haar measure zero sets whenever G is locally compact. So the notion is a novel generalization of the Haar zero ideal to the non-locally compact setting.

The fact that $\mathcal{HN}(G)$ is closed under countable unions is non-trivial. The most commonly cited source for this fact is the 1992 paper of Mycielski [40], which establishes many of the fundamental facts about Haar null sets. We have identified a small but significant error in the proof presented in [40], which we describe and correct in Section 5.2.

A closely related family of sets was defined briefly by Solecki [51]. If $A \subseteq G$ is universally measurable, then A is called openly Haar null if there exists a Borel probability measure μ on G such that for every $\epsilon > 0$, there is an open superset $U \supseteq A$ with the property that $\mu(gUh) < \epsilon$ for every $g, h \in G$. Denote the class of openly Haar null sets in G by $\mathcal{OHN}(G)$, or just $\mathcal{OHN}$. Then $\mathcal{OHN}(G)$ forms a sub-σ-ideal of $\mathcal{HN}(G)$.

The family $\mathcal{OHN}$ has the following interesting application. If G is a group with the property that every singleton subset of G is openly Haar null, then G admits a decomposition $G = A \cup B$, where A is a comeager Haar null set and B is meager and co-Haar null. This gives a large group analogue to the classical dichotomy theorem regarding meager/Haar measure zero sets in locally compact groups (See Theorem 2.30). Of course in any uncountable group the singletons are Haar null, but it turns out they need not be openly Haar null in every group. We show that in many groups (both locally compact and non-locally compact) there are Haar null sets which are not openly Haar null, and so the containment of $\mathcal{OHN}$ in $\mathcal{HN}$ is sometimes proper. This resolves an uncertainty of Solecki. In fact we exhibit groups G (both locally compact and non-locally compact) where $\mathcal{OHN}(G) = \{\emptyset\}$ is the trivial σ-ideal.

We are able to get positive results in some other groups. If $G = \prod_{k \in \omega} G_k$ is a countably infinite product of locally compact, non-compact groups G_k, we show that every compact subset of G is openly Haar null. Thus the compact sets are small in both the topological and measure-theoretic senses. On the other hand, it follows that G decomposes into the union of a Haar null set and a meager set, so the topological and measure-theoretic ideals always differ drastically in such groups. This gives a partial positive answer to a question of Darji. In S_∞ we show that the compact subsets are Haar null, which extends the domain of a result of Dougherty [11] for groups which admit a two-sided-invariant metric.

Using a dynamical argument, we show that the group $\mathrm{Homeo}^+[0,1]$ of increasing homeomorphisms of the interval has the following curious property: if K is a compact subset and U is a nonempty open subset, then there exist some $g, h \in \mathrm{Homeo}^+[0,1]$ for which $gKh \subseteq U$. An immediate corollary is that there are no non-empty openly Haar null sets in $\mathrm{Homeo}^+[0,1]$. We observe that such a property cannot hold in any non-locally compact group, and hence it may be fairly called a "large group property."

CHAPTER 2

PRELIMINARIES

The content of this chapter consists primarily of well-known definitions and facts from descriptive set theory, measure theory, and topological group theory, which are given excellent and detailed treatments in many standard references (see for instance [8], [17], [18], [28], [33], [35], [36], and [41]). For this reason we aim not to give a comprehensive development of the ideas described, but rather to develop just enough technology to facilitate the proofs of the later chapters. With this in mind, we avoid giving definitions and facts in their fullest generality, and instead restrict our attention mainly to the specific setting of Polish spaces and Polish groups. We assume the reader already has familiarity with the basics of sets, groups, and topological spaces.

2.1. Definitions and Facts

In the next two subsections we briefly recall what major facts we need to know from descriptive set theory, topological group theory, and measure theory.

2.1.1. Descriptive Set Theory and Baire Category

Definition 2.1. Let X be a set. A family $\mathcal{A}$ of subsets of X is called a *σ-algebra* of subsets of X if $\emptyset, X \in \mathcal{A}$ and it is

(1) closed under countable unions, i.e. if $(A_i)_{i\in\omega}$ is a sequence of subsets in $\mathcal{A}$, then $\bigcup_{i\in\omega} A_i \in \mathcal{A}$, and
(2) closed under complementation, i.e. if $A \in \mathcal{A}$ then $X \backslash A \in \mathcal{A}$.

Any σ-algebra of subsets of X is also closed under countable intersections. Since any intersection of σ-algebras is again a σ-algebra, for any family $\mathcal{F}$ of subsets of X there is a smallest σ-algebra containing $\mathcal{F}$ as a subset. This is the σ-algebra *generated* by $\mathcal{F}$, denoted $\sigma(\mathcal{F})$.

Definition 2.2. A nonempty family $\mathcal{I}$ of subsets of X is called a *σ-ideal* of subsets of X if $X \notin \mathcal{I}$ and it is

(1) closed under countable unions, i.e. if $(A_i)_{i\in\omega}$ is a sequence of subsets in $\mathcal{I}$, then $\bigcup_{i\in\omega} A_i \in \mathcal{I}$, and

(2) closed under taking subsets, i.e. if $A \subseteq B$, $B \in \mathcal{I}$, then $A \in \mathcal{I}$.

Intuitively, we think of the members of a σ-ideal as the "small subsets" of X.

Definition 2.3. If X is a topological space and τ the topology on X, then the family $\mathcal{B}(X) = \sigma(\tau)$ is the family of *Borel* subsets of X. If Y is another topological space and $f : X \to Y$ is a function, then f is called *Borel-measurable* if $f^{-1}(V) \in \mathcal{B}(X)$ for every open set $V \subseteq Y$. If f is Borel-measurable and also has a Borel-measurable inverse f^{-1}, we say that f is a *Borel isomorphism.*

Definition 2.4. A topological space X is called *Polish* if it is completely metrizable and separable.

Polish spaces are the natural setting for descriptive set theory. The following facts are well-known and may be used to build many new Polish spaces from old ones.

Theorem 2.5. *(1) The disjoint union of a finite or infinite sequence of Polish spaces is Polish.*

(2) The product of a finite or infinite sequence of Polish spaces is Polish.

(3) (Alexandrov-Mazurkiewicz) If X is Polish and $Y \subseteq X$, then Y is Polish if and only if Y is G_δ.

(4) (Sierpinski) If X is Polish and $f : X \to Y$ is a continuous open surjection, then Y is Polish.

Definition 2.6. Let X be a Polish space. A subset $A \subseteq X$ is called *analytic* if there is a Polish space Z and a continuous function $f : Z \to X$ for which $f(Z) = A$. Denote the family of analytic subsets of X by $\mathbf{\Sigma}^1_1(X)$ or just $\mathbf{\Sigma}^1_1$.

Theorem 2.7 ([33] Theorem 14.12). *Let X, Y be Polish spaces and $f : X \to Y$ a function. The following statements are equivalent:*

(1) f is Borel-measurable;

(2) The graph of f is a Borel subset of $X \times Y$;

(3) The graph of f is an analytic subset of $X \times Y$.

THEOREM 2.8 ([33] Lusin-Souslin's Theorem 15.1 and Corollary 15.2). *Let X, Y be Polish spaces and $f : X \to Y$ a continuous function. If $A \subseteq X$ is Borel and $f \restriction A$ is injective, then $f(A)$ is Borel in Y, and $f \restriction A : A \to f(A)$ is a Borel isomorphism.*

The following theorem is fundamental in descriptive set theory and asserts that the continuum hypothesis is "true for analytic sets." We use this fact to construct our main counterexample in Chapter 3.

THEOREM 2.9 ([33] Exercise 14.13). *Let X be a Polish space and $A \in \mathbf{\Sigma}_1^1(X)$. Then either A is countable or A contains a compact perfect set.*

DEFINITION 2.10. Let X be a Polish space. A subset $F \subseteq X$ is said to be *nowhere dense* if $\operatorname{Int}_X \operatorname{cl}_X F = \emptyset$. A subset $A \subseteq X$ is said to be *meager* or *first category* if A is contained in any countable union of nowhere dense sets. Intuitively, we think of the meager sets as "small in the topological sense." A subset $B \subseteq X$ is called *comeager* or *residual* if $X \backslash B$ is meager. Equivalently, B is comeager if B contains a dense G_δ subset of X.

A set $A \subseteq X$ is said to have the *Baire property* if there is an open set $U \subseteq X$ such that $A \Delta U$ is meager. Denote the class of all sets with the Baire property by $\mathcal{BP}(X)$. If Y is another topological space and $f : X \to Y$ is a function, then f is called $\mathcal{BP}(X)$-measurable if $f^{-1}(V) \in \mathcal{BP}(X)$ for every open $V \subseteq Y$.

The next fact motivates some of the above definitions.

THEOREM 2.11 ([33] Proposition 8.22). *Let X be a Polish space. The collection $\mathcal{BP}(X)$ forms a σ-algebra of subsets of X, and the collection of all meager subsets of X forms a σ-ideal of subsets of X.*

THEOREM 2.12 ([33] Theorem 13.7 and Corollary 29.14). *Every Borel set is analytic, and* *every analytic set has the Baire property. If a function is continuous then it is Borel-measurable, and if a function is Borel-measurable then it is $\mathcal{BP}(X)$-measurable.*

2.1.2. Topological Groups and Measures

DEFINITION 2.13. A *topological group* is a group G endowed with a topology for which the group multiplication map $\cdot : G \times G \to G$ and the group inversion map $^{-1} : G \to G$ are both continuous. Of course it follows that for each fixed $g \in G$, the map $x \mapsto gx$, $G \to G$ is a continuous bijection with a continuous inverse given by $x \mapsto g^{-1}x$, $G \to G$, and hence a homeomorphism. The inversion map $g \mapsto g^{-1}$, $G \to G$ is also a homeomorphism since it is its own inverse. For any $g, h \in G$ and any subset $A \subseteq G$, we can define the following sets:

$$gA = \{ga : a \in A\}$$
$$Ah = \{ah : a \in A\}$$
$$gAh = \{gah : a \in A\}$$

The sets above are called a left translation of A, a right translation of A, and a two-sided translation of A respectively.

Note that since a left, right, or two-sided translation is a homeomorphism, the family $\mathcal{B}(G)$ of Borel subsets of G, the family $\mathbf{\Sigma}^1_1(G)$ of analytic subsets of G, the family $\mathcal{BP}(G)$ of subsets of G with the Baire property, and the family of meager subsets of G are all invariant under such translations.

If G, H are topological groups and there exists a group isomorphism $\phi : G \to H$ which is both continuous and has a continuous inverse, then G and H are called *topologically isomorphic.*

A metric d on G is called *left-invariant* if $d(x, y) = d(gx, gy)$ for every $g, x, y \in G$. A metric d is called *two-sided-invariant* if $d(x, y) = d(gxh, gyh)$ for every $g, h, x, y \in G$. A group G is called *TSI* if G admits a two-sided-invariant metric.

THEOREM 2.14. *Let G and H be topological groups, and $f : G \to H$ a group homomorphism. Then f is continuous if and only if f is continuous at the identity element $e \in G$.*

Proof. If f is continuous then f is continuous at e. Conversely, suppose f is continuous at e. Let $g \in G$ be arbitrary and let $V \subseteq Y$ be an open neighborhood of $f(g)$. Then $[f(g)]^{-1}V$ is an open neighborhood of $f(e)$, and so there is an open set $U \subseteq G$ containing e for which $f(U) \subseteq [f(g)]^{-1}(V)$. Then gU is an open neighborhood of g and $f(gU) = f(g)f(U) \subseteq f(g)[f(g)]^{-1}V = v$, so f is continuous at g and hence everywhere. $\square$

The following theorem is fundamental for the theory of topological groups and the reader may consult [18] Theorem 2.2.1 for a proof.

Theorem 2.15 (Birkhoff-Kakutani). *Let G be a Hausdorff topological group. Then G is metrizable if and only if G admits a countable basis of open sets at the identity. If G is metrizable then G admits a left-invariant metric.*

Definition 2.16. A topological group is *Polish* if its underlying topology is Polish.

Theorem 2.17.
1. *([18] Proposition 2.2.3) A product of a finite or infinite sequence of Polish groups is a Polish group.*
2. *([31] Proposition 4.2) If G, H are Polish groups and $\theta : H \to \mathrm{Aut}\, G$ is a group homomorphism for which $(g, h) \mapsto [\theta(h)](g)$, $G \times H \to G$ is continuous, then the semidirect product $G \rtimes_\theta H$ is a Polish group.*
3. *([18] Proposition 2.2.1) If H is a subgroup of a Polish group G, then H is Polish if and only if H is G_δ if and only if H is closed.*
4. *([18] Proposition 2.2.10) If H is a closed normal subgroup of a Polish group G, then G/H is a Polish group.*

Theorem 2.18 (Montgomery [39]). *Let G be a group endowed with a completely metrizable topology. If the mappings $x \mapsto gx$, $G \to G$ and $x \mapsto xh$, $G \to G$ are continuous for every $g, h \in G$, then G is a completely metrizable topological group.*

Definition 2.19. Let G be a Polish group and H a closed subgroup of G. A *coset selector* for G/H is a function $\phi : G/H \to G$ such that $\phi(gH) \in gH$ for every $g \in G$.

THEOREM 2.20 ([33] Theorem 12.17). *Let G be a Polish group and H a closed subgroup of G. Then there exists a Borel-measurable coset selector for G/H.*

DEFINITION 2.21. A topological group is called *locally compact* if it admits a base of topology consisting of compact sets.

THEOREM 2.22. *Every second countable locally compact Hausdorff topological group is Polish.*

DEFINITION 2.23. Let X be a set, and $\mathcal{A}$ a σ-algebra of subsets of X. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a (countably additive) *measure* on $\mathcal{A}$ if $\mu(\emptyset) = 0$ and $\mu\left(\bigcup_{i\in\omega} A_i\right) = \sum_{i\in\omega} \mu(A_i)$ whenever $(A_i)_{i\in\omega}$ is a sequence of pairwise disjoint Borel subsets of X. The measure μ is called *finite* if $\mu(X) < \infty$. The measure μ is called a *probability measure* if $\mu(X) = 1$.

If X is a topological space and $\mathcal{A} = \mathcal{B}(X)$ is the family of Borel subsets of X, then μ is called a *Borel measure* on X.

A Borel measure μ on X is called *regular* if the following three properties hold.

(1) $\mu(K) < \infty$ whenever $K \subseteq X$ is compact and Borel;
(2) $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ is compact and Borel}\}$ for each open set $U \subseteq X$; and
(3) $\mu(A) = \inf\{\mu(V) : V \supseteq A, V \text{ is open}\}$ for each Borel set $A \subseteq X$.

LEMMA 2.24 ([8] Proposition 8.1.10). *Every finite Borel measure on a Polish space is regular.*

DEFINITION 2.25. Let X be a set, $\mathcal{A}$ a σ-algebra of subsets of X, and μ a finite measure on $\mathcal{A}$. The *completion* $\mathcal{A}_\mu$ of $\mathcal{A}$ with respect to μ is the set

$$\mathcal{A}_\mu = \{A \subseteq X : \exists\ E, F \in \mathcal{A}, E \subseteq A \subseteq F, \mu(F \backslash E) = 0\}.$$

It is easy to check that the completion $\mathcal{A}_\mu$ is a σ-algebra of subsets of X. The measure μ may be naturally extended to a measure on $\mathcal{A}_\mu$ by requiring that $\mu(A) = \mu(E) = \mu(F)$ for every such triple (A, E, F) as above.

Now suppose X is a topological space. A set A is called *universally measurable* if A is in the completion of $\mathcal{B}(X)$ with respect to every finite Borel measure μ. Since the class of universally measurable sets is an intersection of σ-algebras (across all possible μ) it is again a σ-algebra of subsets of X.

THEOREM 2.26 ([8] Corollary 8.4.3). *Every analytic subset of a Polish space is universally measurable.*

DEFINITION 2.27. If G is a topological group, then a Borel probability measure μ on G is called a *left Haar measure* on G if it is both regular and invariant under left translations, i.e. $\mu(gA) = \mu(A)$ for every $g \in G$ and Borel set $A \subseteq G$. Likewise, a *right Haar measure* on G is a regular Borel probability measure which is invariant under right translations.

The following theorem was proven in the special case of second countable groups by Haar in 1933 [25], and then in total generality by Weil in 1940 [57], using the axiom of choice. Cartan [6] later furnished an independent proof which avoids the use of choice.

THEOREM 2.28 (Haar/Weil/Cartan). *Let G be a locally compact topological group. Then there exists a left Haar measure on G. Moreover this measure is unique up to scalar multiplication, that is, if μ and ν are both left Haar measures on G then $\mu = c\nu$ for some constant $c \in \mathbb{R}$.*

Of course an immediate corollary is that every locally compact group admits a right Haar measure which is also unique up to scalar multiplication. In general the left and right Haar measures on a group need not be the same, but the following is true.

THEOREM 2.29 ([8] Corollary 9.3.7). *Let G be a topological group. Let μ_L be any left Haar measure on G and μ_R be any right Haar measure on G. Then μ_L and μ_R are absolutely continuous with respect to one another, i.e., for every Borel set $A \subseteq G$, we have $\mu_L(A) = 0$ if and only if $\mu_R(A) = 0$.*

It is easy to see that for any Borel measure μ on a topological space X, the family of measure zero sets $\mathcal{I} = \{A \subseteq X : \exists B \supseteq A, B \in \mathcal{B}(X), \mu(B) = 0\}$ forms a σ-ideal. Then the previous two theorems imply that each locally compact group admits a canonical family of measures (left and right Haar measures) which all generate the same σ-ideal of measure zero sets in G. Since it is generated by left and right Haar measures, this σ-ideal is invariant under left, right, and therefore two-sided translations. So the locally compact groups admit a canonical measure-theoretic analogue to the topological translation-invariant σ-ideal of meager sets.

On the other hand, the two σ-ideals always differ substantially, as the next theorem indicates.

THEOREM 2.30 ([41] Theorem 16.5). *Let G be an uncountable locally compact topological group. Then G may be written as a disjoint union $G = A \cup B$, where A is a comeager Haar measure zero set and B is a meager set of full Haar measure.*

DEFINITION 2.31. Let G be a locally compact topological group and let μ be a left Haar measure on G. For each $g \in G$ define a new measure μ_g on X by the rule $\mu_g(A) = \mu(Ag)$, for each $A \in \mathcal{B}(G)$. It is easy to check that μ_g is another left Haar measure, and hence it is a positive scalar multiple of μ; say $\mu_g = \Delta(g)\mu$ for some $\Delta(g) \in \mathbb{R}^+$. The function $\Delta : G \to \mathbb{R}^+$ defined this way is called the *modular function* of G.

The definition of Δ does not depend on the choice of left Haar measure μ. Δ is a continuous homomorphism from G into $\mathbb{R}$.

If Δ is the constant function $\Delta = 1$, then G is called *unimodular*. Evidently a group is unimodular if and only if its left and right Haar measures coincide.

2.2. Some Groups of Interest

In this section we seek to clarify our domain of discourse by mentioning many examples of groups both large and small, and developing some facts about those most important to our present purposes. First, we provide lists of some famous locally compact and non-locally compact (Polish) groups.

Examples of Locally Compact Polish Groups.

- The additive group $\mathbb{Z}$ with the discrete topology.
- More generally, any countable group G with the discrete topology.
- The additive groups $\mathbb{R}$ and $\mathbb{C}$ with the usual Euclidean topology.
- The multiplicative group $\mathbb{R}^+$ with the usual Euclidean topology.
- The circle group $\mathbb{T}$.
- The $ax + b$ group.
- The matrix groups $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $O(n, \mathbb{C})$, $SO(n, \mathbb{C})$, etc.
- More generally, any finite-dimensional Lie group.
- Any finite product of locally compact Polish groups.

Examples of Non-Locally Compact Polish Groups.

- Any infinite-dimensional separable Banach space with its norm topology.
- The unitary group $U(\mathcal{H})$ of an infinite-dimensional separable Hilbert space, with the strong operator topology.
- Any countably infinite product of non-compact locally compact second countable groups.
- Many continuous function spaces, e.g. $C([0, 1])$, $C(2^\omega, \mathbb{R})$, etc., with the compact-open topology.
- Many homeomorphism groups, e.g. $\mathrm{Homeo}[0, 1]$, $\mathrm{Homeo}\,\mathbb{R}$, $\mathrm{Homeo}\,2^\omega$, $\mathrm{Homeo}\,\mathbb{T}$, etc., with the compact-open topology.
- S_∞, the group of permutations of a countable set, with the topology of pointwise convergence.
- Aut $\mathbb{Q}$, the group of order-preserving self-bijections of the rationals, and Aut R, the group of edge-relation-preserving self-bijections of the random graph, with the topology of pointwise convergence.
- $\mathrm{Iso}\,\mathbb{U}$, the group of isometries of the universal Urysohn space, with the compact-open

topology.

- Aut (X, μ), the group of measure-preserving transformations of a standard Lebesgue measure space, with the weak topology.

In the next three subsections we develop just a few of these groups in greater detail. The structure of the $ax + b$ group and its subgroups prove useful in Chapters 3 and 5, while the properties of S_∞ and $\mathrm{Homeo}^+[0,1]$ are especially relevant in Chapters 4 and 5.

2.2.1. The Homeomorphism Groups $\mathrm{Homeo}^+\mathbb{R}$ and $\mathrm{Homeo}^+[0,1]$

Let $\mathrm{Homeo}^+\mathbb{R}$ denote the group of all increasing self-homeomorphisms of $\mathbb{R}$, with function composition as group operation. Equip $\mathrm{Homeo}^+\mathbb{R}$ with the compact-open topology, which has for a topological subbase all sets of the form

$$U_{K,V} = \{f \in \mathbb{R} : f(K) \subseteq V\},$$

where $K \subseteq \mathbb{R}$ is a compact set and $U \subseteq \mathbb{R}$ is open. Then $\mathrm{Homeo}^+\mathbb{R}$ forms a Polish group.

Similarly, let $\mathrm{Homeo}^+[0,1]$ be the group of all increasing self-homeomorphisms of the interval $[0,1]$, with function composition as group operation, and equipped with the compact-open topology. Then $\mathrm{Homeo}^+[0,1]$ is also a Polish group, and we have:

THEOREM 2.32. $\mathrm{Homeo}^+\mathbb{R}$ *is topologically isomorphic to* $\mathrm{Homeo}^+[0,1]$.

PROOF. Fix a homeomorphism $\phi : (0,1) \to \mathbb{R}$. Define a map $\psi : \mathrm{Homeo}^+\mathbb{R} \to \mathrm{Homeo}^+[0,1]$ by

$$[\psi(f)](x) = \begin{cases} \phi^{-1} \circ f \circ \phi(x) & : \quad x \in (0,1) \\ x & : \quad x = 0, 1 \end{cases}.$$

Since every increasing self-homeomorphism of $[0,1]$ fixes endpoints, it is easy to check that ψ is a group isomorphism. To see that ψ is continuous, let $f \in \mathrm{Homeo}^+[0,1]$ and let $W \subseteq \mathrm{Homeo}^+[0,1]$ be basic open about $\psi(f)$. Then

$$W = \{g \in \mathrm{Homeo}^+[0,1] : g(K_1) \subseteq V_1, ..., g(K_n) \subseteq V_n\}$$

for some compact sets $K_1, ..., K_n$ and open sets $V_1, ..., V_n$ in $[0, 1]$. Define $U \subseteq \text{Homeo}^+ \mathbb{R}$ by

$$U = \{h \in \text{Homeo}^+ \mathbb{R} : h(\phi(K_1)) \subseteq \phi(V_1), ..., h(\phi(K_n)) \subseteq \phi(V_n)\}.$$

So U is basic open in $\text{Homeo}^+ \mathbb{R}$, and since $\psi(f) \in W$ we have $f \in U$. Clearly $\psi(U) \subseteq W$, so ψ is continuous at f and hence continuous everywhere. A similar argument shows ψ^{-1} is continuous, so ψ is a topological isomorphism. $\square$

DEFINITION 2.33. Define a metric ρ on $\text{Homeo}^+[0, 1]$ by $\rho(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$, the *uniform metric*. The uniform metric ρ generates a topological group topology on $\text{Homeo}^+[0, 1]$.

We desire the following characterization of the compact-open topology, which we use in Chapter 5.

THEOREM 2.34. *Let τ_1 be the compact-open topology on $\text{Homeo}^+[0, 1]$, and let τ_2 be the topology generated by the uniform metric ρ. Then $\tau_1 = \tau_2$.*

PROOF. Let $B_\rho(\text{id}, \epsilon)$ be the basic open ϵ-ball about the identity $\text{id} : [0, 1] \to [0, 1]$ in $\text{Homeo}^+[0, 1]$. Let d be the standard Euclidean metric on $[0, 1]$. Let $\{B_d(x_k, \frac{\epsilon}{4})\}_{1 \le k \le n}$ be a finite covering of $[0, 1]$ by $\frac{\epsilon}{4}$-balls. Define a set $U \subseteq \text{Homeo}^+[0, 1]$ by

$$U = \{f \in \text{Homeo}^+[0, 1] : f(\text{cl}(B_d(x_k, \tfrac{\epsilon}{4})) \subseteq B_d(x_k, \tfrac{\epsilon}{2}), 1 \le k \le n\}.$$

U is a τ_1-open neighborhood of id. Let $f \in U$ and $x \in [0, 1]$. We have $x \in B_d(x_k, \frac{\epsilon}{4})$ for some x_k, and hence $f(x) \in B_d(x_k, \frac{\epsilon}{2})$. Then $d(f(x), x) \le d(f(x), x_k) + d(x, x_k) < \frac{\epsilon}{4} + \frac{\epsilon}{2} = \frac{3\epsilon}{4}$. Since x was arbitrary, $\rho(f, \text{id}) \le \frac{3\epsilon}{4} < \epsilon$, and hence $f \in B_\rho(\text{id}, \epsilon)$. So $U \subseteq B_\rho(\text{id}, \epsilon)$. Since τ_1 and τ_2 are group topologies, this implies that $\tau_1 \subseteq \tau_2$.

On the other hand, suppose $K_1, ..., K_n \subseteq [0, 1]$ are compact and $V_1, ..., V_n \subseteq [0, 1]$ are open, and $K_k \subseteq V_k$ for each $k \in \{1, ..., n\}$. Consider the τ_1-basic open neighborhood of the identity given by

$$U = \{f \in \text{Homeo}^+[0, 1] : f(K_k) \subseteq V_k, 1 \le k \le n\}.$$

Let $\epsilon = \min\limits_{1\leq k\leq n} d(K_k, [0,1]\backslash V_k)$, and consider the τ_2-open neighborhood of identity $B_\rho(\mathrm{id}, \epsilon) \subseteq \mathrm{Homeo}^+[0,1]$. Let $f \in B_\rho(\mathrm{id}, \epsilon)$ and let $x \in K_k$ for $k \in \{1, ..., n\}$. If $f(x) \notin V_k$, then we would have $f(x) \in [0,1]\backslash V_k$ and hence $d(K_k, [0,1]\backslash V_k) \leq d(x, f(x)) < \epsilon \leq d(K_k, [0,1])$, a contradiction. So $f(x) \in V_k$ and $K_k \subseteq V_k$ for each k. This shows $f \in U$, and therefore $B_\rho(\mathrm{id}, \epsilon) \subseteq U$. Again since τ_1 and τ_2 are group topologies, we have shown $\tau_2 \subseteq \tau_1$. So $\tau_1 = \tau_2$.

□

2.2.2. The $ax + b$ Group

There are two natural ways to realize the $ax + b$ group.

(1) Let G be the group of all *affine transformations* of $\mathbb{R}$, i.e. the group of all functions $f \in \mathrm{Homeo}^+ \mathbb{R}$ of the form $f(x) = ax + b$ for some $a, b \in \mathbb{R}$, with function composition as the group operation. Endow G with the subspace topology inherited from $\mathrm{Homeo}^+ \mathbb{R}$ with its compact-open topology, and G becomes a Polish group.
(2) Let G' be the natural semidirect product group $G' = \mathbb{R}^+ \ltimes \mathbb{R} = \mathbb{R}^+ \ltimes_\phi \mathbb{R}$, where $\phi : \mathbb{R} \to \mathrm{Aut}\ \mathbb{R}$ is given by $[\phi(c)](x) = cx$ for each $x \in \mathbb{R}$, for each $c \in \mathbb{R}$. The mapping $(a, b) \mapsto ab$, $\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous and hence G' is a Polish group by Theorem 2.17 (2). As a topological space $G' = \mathbb{R}^+ \times \mathbb{R}$ (so G' is locally compact), and the group operation on G' is given by $(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1)$ for each $(a_1, b_1), (a_2, b_2) \in G'$.

Theorem 2.35. *G and G', as defined above, are topologically isomorphic.*

Proof. Define $\phi : G' \to G$ by $[\phi(a, b)](x) = ax + b$ for $x \in \mathbb{R}$, for each $(a, b) \in G'$. Again it is easy to check that ϕ is a group isomorphism, so we need only verify that ϕ and ϕ^{-1} are both continuous at the identity $(1, 0) \in G'$.

Let $U \subseteq G$ be a basic open neighborhood of $\phi(1, 0) = \mathrm{id}_\mathbb{R} : \mathbb{R} \to \mathbb{R}$ in G, so

$$U = \{f \in G : f(K_k) \subseteq V_k, 1 \leq k \leq n\}$$

for some compact sets $K_1, ..., K_n \subseteq \mathbb{R}$ and some open sets $V_1, ..., V_n \subseteq \mathbb{R}$ with $K_k \subseteq V_k$ $(1 \leq k \leq n)$. Let d be the standard Euclidean metric on $\mathbb{R}$, and let $\delta = \min\limits_{1\leq k\leq n} d(K_n, \mathbb{R}\backslash V_n\}$. Choose an M so large that $\bigcup\limits_{k=1}^{n} K_k \subseteq (-M, M)$. Define an open neighborhood W of identity in G by

$$W = (1 - \tfrac{\delta}{2M}, 1 + \tfrac{\delta}{2M}) \times (-\tfrac{\delta}{2}, \tfrac{\delta}{2}).$$

Suppose $(a, b) \in W$ and let $f(x) = [\phi(a, b)](x) = ax + b$. Let $x \in K_k$ be arbitrary. If $0 \leq x < M$, we have $f(x) - x = ax + b - x = (a-1)x + b < \frac{\delta}{2M} \cdot x + \frac{\delta}{2} < \frac{\delta}{2M} \cdot M + \frac{\delta}{2} = \delta$, and $f(x) - x = ax + b - x = (a-1)x + b > -\frac{\delta}{2M} \cdot x - \frac{\delta}{2} > -\frac{\delta}{2M} \cdot M - \frac{\delta}{2} = -\delta$. On the other hand if $-M < x < 0$, then we have $f(x) - x = (a-1)x + b < -\frac{\delta}{2M} \cdot x + \frac{\delta}{2} < -\frac{\delta}{2M} \cdot (-M) + \frac{\delta}{2} = \delta$, and $f(x) - x = (a-1)x + b > \frac{\delta}{2M} \cdot x - \frac{\delta}{2} > \frac{\delta}{2M} \cdot (-M) - \frac{\delta}{2} = -\delta$. So $f(x) - x \in (-\delta, \delta)$, i.e. $d(x, f(x)) < \delta$. If we had $f(x) \in \mathbb{R}\backslash V_k$, then we would have $d(K_k, \mathbb{R}\backslash V_k) \leq d(x, f(x)) < \delta \leq d(K_k, \mathbb{R}\backslash V_k)$, a contradiction. So $f(x) \in V_k$ and hence $f(K_k) \subseteq V_k$ for each k. This shows $\phi(W) \subseteq U$ and hence ϕ is continuous at the identity.

To see ϕ^{-1} is also continuous, let $U = (1-\epsilon, 1+\epsilon) \times (-\delta, \delta)$ be a basic open neighborhood of identity $(1, 0) = \phi^{-1}(\mathrm{id}_{\mathbb{R}})$. Without loss of generality shrink δ so that $\delta < \epsilon$. Let $K_1 = \{0\}$ and $K_2 = \{1\}$, so K_1 and K_2 are compact subsets of $\mathbb{R}$. Let $V_1 = (-\delta, \delta)$ and $V_2 = (1 - (\epsilon - \delta), 1 + (\epsilon - \delta))$, so V_1, V_2 are open intervals in $\mathbb{R}$. Define a basic open set of identity $W \subseteq G$ by

$$W = \{f \in G : f(K_1) \subseteq V_1, f(K_2) \subseteq V_2\}.$$

Let $f \in W$, and write $f(x) = ax + b$ for some $a \in \mathbb{R}^+$, $b \in \mathbb{R}$. We have $b = f(0) \in (-\delta, \delta)$. Then since $f(1) = a + b \in (1 - (\epsilon - \delta), 1 + (\epsilon - \delta))$, we have $a = (a+b) - b < 1 + \epsilon - \delta + \delta = 1 + \epsilon$, and $a = (a+b) - b > 1 - \epsilon + \delta - \delta = 1 - \epsilon$. So $a \in (1-\epsilon, 1+\epsilon)$ and hence $(a, b) = \phi^{-1}(f) \in W$. This shows $\phi^{-1}(W) \subseteq U$ and ϕ^{-1} is continuous at the identity. So ϕ and ϕ^{-1} are both continuous everywhere and ϕ is a homeomorphism. $\square$

2.2.3. The Permutation Group S_∞ and Its Closed Subgroups

Let S_∞ be the group of all permutations $\pi : \omega \to \omega$, with function composition as group operation. For a finite sequence of integers $s = (s(0), ..., s(n-1)) \in \omega^{<\omega}$, let $\mathrm{lh}(s) = n$ denote the length of the sequence s. Endow S_∞ with the topology generated by basic open sets $\{N_s\}_{s\in\omega^{<\omega}}$ of the form

$$N_s = \{\pi \in S_\infty : \pi \upharpoonright \mathrm{lh}(s) = s\}.$$

Then S_∞ becomes a Polish group. The set of all finitely-supported permutations is dense in S_∞. S_∞ famously has the following automatic continuity property.

Theorem 2.36 (Kallman [32]). *Let G be a Polish group and let $\phi : G \to S_\infty$ be an abstract group isomorphism. Then ϕ is a topological isomorphism.*

Corollary 2.37 (Kallman [32]). *The standard topology on S_∞ is the unique topology which makes S_∞ into a Polish group.*

The previous corollary confirms our intuition that the topology of S_∞ is closely tied to its combinatorial structure. For instance, it is helpful to view S_∞ topologically as a subspace of the Baire space ω^ω. The following definitions and lemmas will prove to be of use in Chapter 5.

Definition 2.38. A *tree* on ω is a subset $T \subseteq \omega^{<\omega}$ which is closed under initial segments, i.e. if $t \in T$ then $t \upharpoonright n \in T$ for all integers $n \leq \mathrm{lh}(t)$. The elements of T are called *nodes*. If $s, t \in T$ with $\mathrm{lh}(t) = \mathrm{lh}(s) + 1$ and $t \upharpoonright \mathrm{lh}(s) = s$, then t is called a *child* of s and s is called a *parent* of t. A tree is called *pruned* if every node in T has a child. A tree is called *finitely branching* if each node in T has only finitely many children. A *branch* of T is an infinite sequence $x \in \omega^\omega$ such that $x \upharpoonright n \in T$ for every integer n. Denote the set of all branches through T by $[T]$.

Lemma 2.39. *Let $F \subseteq \omega^\omega$. Then F is closed if and only if $F = [T]$ for some pruned tree T on ω. F is compact if and only if the tree T is finitely branching.*

Proof. Suppose F is closed. Define $T = \{\pi \upharpoonright n : \pi \in F, n \in \omega\}$. T is a pruned tree on ω. If $\pi \in F$, then clearly $\pi \in [T]$ by the definition of T, so $F \subseteq [T]$. Conversely, suppose $\pi \in [T]$. Then for each $n \in \omega$, $\pi \upharpoonright n \in T$, so there exists a permutation $\pi_n \in F$ such that $\pi_n \upharpoonright n = \pi \upharpoonright n$. Then the sequence $(\pi_n)_{n\in\omega}$ of elements of F converges to $\pi \in \omega^\omega$, and hence $\pi \in F$ since F is closed. So $F = [T]$.

Suppose T is not finitely branching. Then there is a node $s \in T$ with infinitely many children $t_1, t_2, ...$ Then $\{N_{t_1}, N_{t_1}, ...\} \cup \{\omega^\omega \backslash \bigcup_{i\in\omega} N_{t_i}\}$ forms an open cover of F by pairwise disjoint sets. For each t_i there is $\pi_i \in F$ with $\pi_i \in N_{t_i}$ by the definition of T; so this cover admits no finite subcover. Therefore F is not compact.

On the other hand, suppose T is finitely branching, but for the sake of contradiction that F is not compact. Then F has an open cover $\{U_i\}_{i\in\omega}$ which cannot be finitely subcovered. Since T is finitely branching, there are only finitely many $t_1, ..., t_n \in T$ of length 1. Since $\{N_{t_k}\}_{1\leq k\leq n}$ is an open cover of $[T] = F$, there must be one particular t_k such that N_{t_k} cannot be covered by only finitely many elements of $\{U_i\}_{i\in\omega}$ (since otherwise $F = [T]$ would be finitely subcoverable). Set $x(0) = t_k(0)$ for this t_k.

Now suppose by way of induction that we have constructed a finite sequence $(x(0), ..., x(n-1)) \in T$ with the property that $N_{(x(0),...,x(n-1))}$ cannot be covered by only finitely many elements of $\{U_i\}_{i\in\omega}$. Since T is finitely branching, this node has only finitely many children $t_1, ..., t_k$ in T. Again one of these nodes t_k must have the property that N_{t_k} cannot be covered by finitely many elements of $\{U_i\}_{i\in\omega}$, since otherwise $N_{(x(0),...,x(n-1))}$ would be finitely coverable by elements of $\{U_i\}_{i\in\omega}$, which it is not. Set $x(n) = t_k(n)$ and continue the induction.

In this way we construct a sequence $x = (x(0), x(1), ...) \in \omega^\omega$ with the property that $N_{x\upharpoonright n}$ is not finitely subcoverable by elements of $\{U_i\}_{i\in\omega}$, for any $n \in \omega$. But $x \in U_i$ for some $i \in \omega$, and since U_i is open there is some basic neighborhood $N_{x\upharpoonright n}$ with $x \in N_{x\upharpoonright n} \subseteq U_i$. So this basic neighborhood was finitely coverable after all, a contradiction. Therefore F is compact and the theorem is proved. □

Corollary 2.40. *Let $F \subseteq S_\infty$. Then F is closed if and only if $F = [T] \cap S_\infty$ for some pruned tree T on ω. F is compact if and only if T is finitely branching and $[T] \subseteq S_\infty$.*

Proof. The first statement above is just the definition of the subspace topology. If F is compact in S_∞, then F is compact in ω^ω and hence $F = [T]$ for a finitely branching tree T. Then $[T] = F \subseteq S_\infty$. □

Corollary 2.41. *S_∞ is not locally compact.*

Proof. Any basic clopen neighborhood $N_s \subseteq S_\infty$, $s \in \omega^{<\omega}$ is equal to $[T] \cap S_\infty$ for some infinitely branching tree T, so no such neighborhood has compact closure. □

Next we mention and make use of a powerful structural theorem of Dixon, Neumann, and Thomas. First let us set up a little notation.

Definition 2.42. For $F \subseteq \omega$ let $S_{(F)} = \{\pi \in S_\infty : \pi(n) = n \text{ for all } n \in F\}$ and $S_{\{F\}} = \{\pi \in S_\infty : \pi(n) \in F \text{ for all } n \in F\}$. Let S_f be the set of all finitely-supported permuations.

Theorem 2.43 (Dixon-Neumann-Thomas [10]). *Let H be a subgroup of S_∞. If H is of countable index in S_∞, then there exists a finite set $F \subseteq \omega$ such that*

$$S_{(F)} \leq H \leq S_{\{F\}}.$$

The above may be applied to prove the following theorem, which gives a purely algebraic characterization of the stability groups for the natural action of S_∞ on ω.

Proposition 2.44. *If X is a set and $x \in X$, then $H = S_x$ has exactly two double cosets in $S(X)$ and the only subgroup of H that is normal in $S(X)$ is $\{e\}$. Conversely, if X is countable and H is a subgroup of $S(X)$ with exactly two double cosets and such that the only subgroup of H that is normal in $S(X)$ is $\{e\}$, then there is a unique $x \in X$ such that $H = S_x$.*

Proposition 2.44 should be compared with the results of William G. Wright [59], who gave a similar complete characterization in the class of closed subgroups of the point stabilizers of the homeomorphism groups of connected manifolds. It would be interesting if the closed subgroup hypothesis in Wright's results can be dispensed with.

Before turning to the proof, we need the following lemma.

LEMMA 2.45. *Let $H \subseteq S_\infty$ be a subgroup with exactly two double cosets and such that the only normal subgroup of H that is normal in S_∞ is $\{e\}$. Then there is a finite subset $F \subseteq \omega$ such that $S_{(F)} \subseteq H \subseteq S_{\{F\}}$. In particular S_∞/H is countable and H is open.*

PROOF. $S_f \nsubseteq H$ since S_f is normal in S_∞. Hence, there is some transposition $(a, b) \notin H$. Therefore the double coset not equal to H must be $H(a, b)H$. Let $B = \{a, b\}$. $S_{(B)}$ is open in S_∞, $S_\infty/S_{(B)}$ is countable and therefore $H/(H \cap S_{(B)})$ is countable. Let $\{h_\ell\}_{\ell \geq 1} \subseteq H$ be distinct coset representatives for $H/(H \cap S_{(B)})$. Therefore $H(a, b)H = \cup_{\ell \geq 1} h_\ell (H \cap S_{(B)})(a, b)H = \cup_{\ell \geq 1} h_\ell (a, b)(H \cap S_{(B)})H = \cup_{\ell \geq 1} h_\ell (a, b)H$. The lemma now follows from Theorem 2.43. $\square$

PROOF OF PROPOSITION 2.44. If $x \in X$ and $H = S_x$, then H has exactly two orbits on X since S_x fixes x and acts transitively on $X - \{x\}$. Since there is an equivariant bijection between $S(X)/S_x$ and X as $S(X)$-spaces, there are exactly two H double cosets in $S(X)$. If N is a normal subgroup of H, then $N \subseteq \cap_{a \in S(X)} a S_x a^{-1} = \cap_{a \in S(X)} S_{ax} = \{e\}$.

To prove the converse, we may assume that $X = \omega$ and $S(X) = S_\infty$. Suppose H is a subgroup of S_∞ that has exactly two double cosets and has the property that the only subgroup of H that is normal in S_∞ is $\{e\}$. Therefore there is $F \subseteq \omega$ with $|F| < +\infty$ such that $S_{(F)} \subseteq H \subseteq S_{\{F\}}$ by Lemma 2.45. We will be done if $|F| = 1$. Suppose that $|F| > 1$. This will lead to a contradiction if $S_{\{F\}}$ and therefore H has more than two double cosets. To see this, choose distinct elements a, $b \in F$ and distinct elements x, $y \in \omega - F$. The two finite permutations $(a, x)(b, y)$ and (a, x) are in distinct $S_{\{F\}}$ double cosets, for suppose that π, $\rho \in S_{\{F\}}$ and $(a, x)(b, y) = \pi(a, x)\rho$. If $\rho(a) = a$, then $\rho(b) \in F - \{a\}$ and $(\pi(a, x)\rho)(b) \in F$ but $((a, x)(b, y))(b) = y \notin F$, a contradiction. If $\rho(a) \in F - \{a\}$, then $(\pi(a, x)\rho)(a) \in F$ but $((a, x)(b, y))(a) = x \notin F$, again a contradiction. Thus $|F| = 1$ and Proposition 2.44 is proved. $\square$

As a corollary we have the well-known result of Schreier and Ulam [49] that every automorphism of S_∞ is inner.

COROLLARY 2.46. *Let X and Y be countably infinite sets and let $\varphi : S(X) \to S(Y)$ be an* *algebraic isomorphism. Then there is a bijection $\psi : X \to Y$ such that $\varphi(\pi) = \psi\pi\psi^{-1}$ for every $\pi \in S(X)$.*

PROOF. If $x \in X$, then S_x has exactly two double cosets and the only subgroup of it normal in $S(X)$ is $\{e\}$. Since $\varphi(S_x)$ has exactly the same algebraic properties, there is a unique $y \in Y$ such that $\varphi(S_x) = S_y$. Let $\psi(x) = y$. ψ is one-to-one and ψ is onto by applying the same reasoning using φ^{-1}, so $\psi : X \mapsto Y$ is a bijection. $S_{\psi(\pi(x))} = \varphi(S_{\pi(x)}) = \varphi(\pi S_x \pi^{-1}) = \varphi(\pi)\varphi(S_x)\varphi(\pi^{-1}) = \varphi(\pi)S_{\psi(x)}\varphi(\pi^{-1}) = S_{\varphi(\pi)(\psi(x))}$ and therefore $\psi\pi = \varphi(\pi)\psi$. □

S_∞ interests many mathematicians because it and and its closed subgroups play the role of the automorphism groups of countable structures. The remainder of this section will rigorize the preceding comment. We generally follow the presentation and definitions given in Section 4.5 of [44].

DEFINITION 2.47. A *signature* L is a countable family of symbols R_i, $i \in I$, which represent relations, and symbols f_j, $j \in J$, which represent functions. To each symbol we associate a particular *arity*, i.e. a positive integer $n(i)$ for relation symbols, or a nonnegative integer $m(j)$ for function symbols.

A *structure* $\mathfrak{A}$ in the signature L is a set A together with a family of relations $R_i^{\mathfrak{A}} \subseteq A^{n(i)}$, one for each $i \in I$, and a family of functions $f_j^{\mathfrak{A}} : A^{m(j)} \to A$, one for each $j \in J$. The set A is called the *universe* of $\mathfrak{A}$ and the relations $R_i^{\mathfrak{A}}$ and functions $f_j^{\mathfrak{A}}$ are called *interpretations* of the relation and function symbols in L. Note that relation symbols of arity 1 may be interpreted as constants, or distinguished subsets, of A. If L has no function symbols, then we call $\mathfrak{A}$ a *relational structure*.

EXAMPLE 2.48. (1) If $L = \emptyset$, then the structures in L are just sets.

(2) Suppose $L = \{<\}$, where $<$ is a relation symbol of arity 2. Then linearly ordered sets are examples of structures in L. So are partially ordered sets.

(3) Suppose L again contains just one relation symbol of arity 2. A graph (V, E) is a structure in L, where the universe is the set V of vertices, and the relation symbol

is the irreflexive, symmetric relation E of edges. If E is instead irreflexive and antisymmetric, then the directed graph (V, E) is also a structure in L.

(4) Let $S \subseteq [0, \infty)$ be a countable set. Let $L = \{D_s : s \in S\}$, where each D_s is a binary relation symbol. An *S-valued metric space* (X, d) is a structure in L, where we understand that $D_s^X(x, y) \leftrightarrow d(x, y) = s$ for all $x, y \in X$. For instance, a discrete space is a $\{0, 1\}$-valued metric space, and $\mathbb{Q}$ is a $\mathbb{Q} \cap [0, \infty)$-valued metric space.

(5) Let $\mathbb{F}$ be a countable field. A vector space V over F can be regarded as a structure in a signature which includes a constant symbol 0, a binary function symbol $+$, which represents addition in V, and a unary function symbol for each $\lambda \in \mathbb{F}$, which represents multiplication by the scalar λ.

(6) A Boolean algebra is a structure in a signature which includes binary function symbols $\vee$ and $\wedge$, a unary function symbol $\neg$, and constant symbols 0 and 1.

Definition 2.49. Given two structures $\mathfrak{A}$ and $\mathfrak{B}$ in the same signature L, with universes A and B respectively, a map $\pi : A \to B$ is called a *homomorphism* if for every $R_i \in L$, and every $n(i)$-tuple $(a_1, ..., a_{n(i)})$, we have

$$(a_1, ..., a_{n(i)}) \in R_i^{\mathfrak{A}} \leftrightarrow (\pi(a_1), ..., \pi(a_{n(i)})) \in R_i^{\mathfrak{B}}$$

and for every $f_j \in L$, and every $m(j)$-tuple $(a_1, ..., a_{m(j)})$, we have

$$f_j^{\mathfrak{B}}(\pi(a_1), ..., \pi(a_{m(j)})) = \pi(f_j^{\mathfrak{A}}(a_1, ..., a_{m(j)})).$$

If π is also an injection, we call π a *monomorphism* or an *embedding*, and we say $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, denoted $\mathfrak{A} \leq \mathfrak{B}$. We also write $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ if we wish to draw attention to the fact that π is our distinguished embedding. If $A \subseteq B$ and $\mathrm{id}_A : \mathfrak{A} \hookrightarrow \mathfrak{B}$, then we write $\mathfrak{A} \sqsubseteq \mathfrak{B}$ (this notation may not be standard). If π is bijective then we call π an *isomorphism.*

A structure $\mathfrak{F}$ is called *ultrahomogenous* if every isomorphism between finite substructures $\mathfrak{A} \sqsubseteq \mathfrak{F}$ and $\mathfrak{B} \sqsubseteq \mathfrak{F}$ extends to an automorphism of $\mathfrak{F}$.

A structure $\mathfrak{F}$ is called *locally finite* if every finite subset $D \subseteq F$ is contained in some finite substructure $\mathfrak{D} \sqsubseteq \mathfrak{F}$.

Remark 2.50. All relational structures are locally finite. Every example listed above is locally finite, except for the vector space V over $\mathbb{F}$, which is locally finite if and only if $\mathbb{F}$ is a finite field.

Definition 2.51. If $\mathfrak{A}$ is a structure, then the *age* of $\mathfrak{A}$, denoted $\mathrm{Age}(\mathfrak{A})$, is the class of all finite structures which are isomorphic to a substructure of $\mathfrak{A}$.

The following two theorems are achieved using a "back-and-forth" or "shuttle" argument.

Theorem 2.52. *Let $\mathfrak{F}$ be a countable locally finite structure. Then the following are equivalent:*

(1) *$\mathfrak{F}$ is ultrahomogeneous.*
(2) *$\mathfrak{F}$ has the finite extension property, i.e., whenever $\mathfrak{A}, \mathfrak{B} \in \mathrm{Age}(\mathfrak{F})$, and $\mathfrak{A} \sqsubseteq \mathfrak{B}$, then every embedding of $\mathfrak{A}$ into $\mathfrak{F}$ extends to an embedding of $\mathfrak{B}$ into $\mathfrak{F}$.*

Theorem 2.53. *Every two countable ultrahomogenous structures in the same language, having the same age, are isomorphic.*

Definition 2.54. A structure $\mathfrak{F}$ is called a *Fraïssé structure* if it is countably infinite, locally finite, and ultrahomogeneous.

Theorem 2.55 (Fraïssé [16]). *A nonempty class $\mathcal{C}$ of finite structures in a signature L is the age of a Fraïssé structure if and only if it satisfies the following conditions:*

(1) *$\mathcal{C}$ is closed under isomorphisms;*
(2) *$\mathcal{C}$ is closed under taking substructures;*
(3) *$\mathcal{C}$ contains structures of arbitrarily high finite cardinality;*
(4) *$\mathcal{C}$ satisfies the joint embedding property, i.e., whenever $\mathfrak{A}, \mathfrak{B} \leq \mathcal{C}$, then there is $\mathfrak{D} \in \mathcal{C}$ containing both $\mathfrak{A}$ and $\mathfrak{B}$ as substructures; and*
(5) *$\mathcal{C}$ satisfies the amalgamation property, i.e. whenever $f_1 : \mathfrak{A} \hookrightarrow \mathfrak{B}_1$ and $f_2 : \mathfrak{A} \hookrightarrow \mathfrak{B}_2$ are monomorphisms of structures in $\mathcal{C}$, then there is $\mathfrak{D} \in \mathcal{C}$ and embeddings g_1 :*

$\mathfrak{B}_1 \hookrightarrow \mathfrak{D}$ *and* $g_2 : \mathfrak{B}_2 \hookrightarrow \mathfrak{D}$ *such that* $g_1 \circ f_1 = g_2 \circ f_2$.

If this is the case then $\mathcal{C}$ *is the age of a unique up to isomorphism Fraïssé structure* $\mathfrak{F} = \mathrm{Flim}(\mathcal{C})$*, called the Fraïssé limit of* $\mathcal{C}$*.*

Fraïssé's theorem gives us a class of "universal countable objects" which have a large degree of symmetry and therefore a rich automorphism group:

EXAMPLE 2.56.

(1) The class of all finite sets is a Fraïssé class, whose Fraïssé limit is ω.
(2) The class of all finite linear orders is a Fraïssé class, whose Fraïssé limit is order isomorphic to the rationals $\mathbb{Q}$.
(3) The class of all finite graphs is a Fraïssé class, whose limit is the random graph R.
(4) The finite dimensional vector spaces over a finite field $\mathbb{F}$ form a Fraïssé class, whose limit is the infinite dimensional vector space V over $\mathbb{F}$.
(5) Finite Boolean algebras form a Fraïssé class, and their limit is the countable atomless Boolean algebra B_∞, which can be realized as the space of all clopen subsets of the Cantor space 2^ω.
(6) The finite $\mathbb{Q}$-valued metric spaces form a Fraïssé class, and their limit is the *rational Urysohn space* $\mathbb{U}_\mathbb{Q}$.

DEFINITION 2.57. Let $\mathfrak{F}$ be a structure with countable universe F. Let $\mathrm{Aut}\,\mathfrak{F}$ denote the group of all automorphisms of $\mathfrak{F}$, endowed with the topology of pointwise convergence on F, where F is viewed as a discrete space. This topology makes $\mathrm{Aut}\,\mathfrak{F}$ into a Polish group.

THEOREM 2.58. *Let* $\mathfrak{F}$ *be a structure with universe* ω*. Then* $\mathrm{Aut}\,\mathfrak{F}$ *is a closed subgroup of* S_∞*.*

PROOF. $\mathrm{Aut}\,\mathfrak{F}$ is obviously a subgroup, so it suffices to show that $\mathrm{Aut}\,\mathfrak{F}$ is closed in S_∞. We will show that the complement $X = S_\infty - \mathrm{Aut}\,\mathfrak{F}$ is open. Let L be the signature for which $\mathfrak{F}$ is a structure. If $\pi \in X$, then either there exists an n-ary relation symbol $R \in L$ and a tuple $(x_1, ..., x_n) \in \omega^n$ such that

$$(x_1,...,x_n) \in R^{\mathfrak{F}} \text{ and } (\pi(x_1),...,\pi(x_n)) \notin R^{\mathfrak{F}}, \text{ or } (x_1,...,x_n) \notin R^{\mathfrak{F}} \text{ and } (\pi(x_1),...,\pi(x_n)) \in R^{\mathfrak{F}},$$

or else there exists an n-ary function symbol $f \in L$ and a tuple $(x_1,...,x_n) \in \omega^n$ such that

$$f^{\mathfrak{F}}(\pi(x_1),...,\pi(x_n)) \neq \pi(f^{\mathfrak{F}}(x_1,...,x_n)).$$

In either case, the open set $U = \{\sigma \in S_\infty : \sigma(x_1) = \pi(x_1),...,\sigma(x_n) = \pi(x_n)\}$ is a neighborhood of π which is disjoint from Aut $\mathfrak{F}$. □

Definition 2.59. Let G be a subgroup of S_∞. We associate a structure $\mathfrak{F}_G$ to G as follows: For each $n \in \omega$, let ω^n/G denote the set of all distinct G-orbits in ω^n. Let L_G be a signature consisting of relation symbols $R_{n,o}$, where $n \in \omega$ and $o \in \omega^n/G$, where the arity of each $R_{n,o}$ is n. The universe of the structure $\mathfrak{F}_G$ is ω, and we define each $R_{n,o}^{\mathfrak{F}_G}$ by the rule

$$(x_1,...,x_n) \in R_{n,o}^{\mathfrak{F}_G} \leftrightarrow (x_1,...,x_n) \in o.$$

In other words, we set $R_{n,o}^{\mathfrak{F}_G} = o$. We call $\mathfrak{F}_G$ the *canonical structure* associated to G.

Remark 2.60. It is obvious that every $g \in G$ will induce an automorphism of $\mathfrak{F}_G$. Thus we have $G \leq$ Aut $\mathfrak{F}$.

Theorem 2.61. Aut $\mathfrak{F}_G$ *is the closure of* G *in* S_∞.

Proof. Since we already know Aut $\mathfrak{F}_G$ is closed, it suffices to show that G is dense in Aut $\mathfrak{F}_G$. So let $\pi \in$ Aut $\mathfrak{F}_G$, let $x = (x_1,...,x_n) \in \omega^n$, and consider the basic open neighborhood U of π defined by

$$U = \{\sigma \in S_\infty : \sigma(x_i) = \pi(x_i), i = 1,...,n\}.$$

Let $o \in \omega^n/G$ be the unique orbit for which $x \in o$. Then since $\pi \in \mathrm{Aut}\ \mathfrak{F}_G$, we have $\pi x \in o$. But by definition we have $o = G \cdot x$, and hence there exists a $g \in G$ for which $gx = \pi x$. Hence $g \in U$, and the proof is finished. $\square$

PROPOSITION 2.62. *$\mathfrak{F}_G$ is a Fraïssé structure.*

PROOF. $\mathfrak{F}_G$ is obviously countably infinite, and since it is a relational structure, it is also locally finite. So we need only verify that $\mathfrak{F}_G$ is ultrahomogeneous.

So suppose $\mathfrak{A}, \mathfrak{B} \sqsubseteq \mathfrak{F}_G$ are finite and $p : \mathfrak{A} \hookrightarrow \mathfrak{B}$ is an isomorphism. Let $A = \{x_1, ..., x_n\} \subseteq \omega$ and $B = \{y_1, ..., y_n\} \subseteq \omega$ be the universes of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, enumerated in such a way that $p(x_i) = y_i$ for all $i = 1, .., n$. Let $o \in \omega^n/G$ be arbitrary. Since we have $R^{\mathfrak{A}}_{n,o} = R^{\mathfrak{F}_G}_{n,o} \cap A^n = o \cap A^n$ and $R^{\mathfrak{B}}_{n,o} = R^{\mathfrak{F}_G}_{n,o} \cap B^n = o \cap B^n$, it follows that

$$\begin{aligned}
(x_1, ..., x_n) \in o &\leftrightarrow (x_1, ..., x_n) \in o \cap A^n \\
&\leftrightarrow (x_1, ..., x_n) \in R^{\mathfrak{A}}_{o,n} \\
&\leftrightarrow (y_1, ..., y_n) \in R^{\mathfrak{B}}_{o,n} \\
&\leftrightarrow (y_1, ..., y_n) \in o \cap B^n \\
&\leftrightarrow (y_1, ..., y_n) \in o.
\end{aligned}$$

So $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ belong to the same G-orbit, i.e. there exists a $g \in G$ with $(g(x_1), ..., g(x_n)) = (y_1, ..., y_n) = (p(x_1), ..., p(x_n))$. So $g \in \mathrm{Aut}\ \mathfrak{F}_G$ and g extends p; thus $\mathfrak{F}_G$ is ultrahomogeneous. $\square$

COROLLARY 2.63. *The automorphism groups of Fraïssé structures on ω are exactly the closed subgroups of S_∞.*

Corollary 2.64. *Every automorphism group of a Fraïssé structure is topologically isomorphic to a closed subgroup of S_∞. In particular,* Aut $\mathbb{Q}$*,* Aut R*,* Aut $\mathbb{F}^\infty$*,* Aut B_∞*, and* Aut $\mathbb{U}_\mathbb{Q}$ *may all be regarded as closed subgroups of S_∞.*

2.3. Locally Compact vs. Non-Locally Compact Groups

In this section we seek to highlight the contrast between the deep and well-developed theory of locally compact groups, and the somewhat more mysterious theory of non-locally compact groups. We open with what are arguably the most significant results in the history of the theory of topological groups, which most mathematicians agree comprise the solution to Hilbert's fifth problem, i.e. the question of what are the minimal hypotheses one can put on a topological group G to conclude that G is a Lie group.

Theorem 2.65 (Gleason-Montgomery-Zippin). *Let G be a topological group. Then G is locally Euclidean if and only if G is a finite-dimensional Lie group.*

Definition 2.66. A group G is said to have the *no-small-subgroup property* if there exists an open neighborhood of identity $U \subseteq G$ which does not contain any non-trivial subgroup of G.

Theorem 2.67 (Yamabe). *Any locally compact connected topological group G is the inverse limit of a sequence of finite-dimensional Lie groups. If G is locally compact and has the no-small-subgroup property, then G is a finite-dimensional Lie group.*

Of course, every finite-dimensional Lie group is locally Euclidean and hence locally compact. These theorems imply that the locally compact groups are "not far" from Lie groups, which a priori have considerably more structure. So we might expect the behavior of locally compact groups to be more homogeneous than their non-locally compact counterparts.

Let us consider the dynamical properties of topological groups.

Definition 2.68. A *topological transformation group* is a triple (G, X, ϕ) where G is a topological group, X is a topological space, and ϕ is an abstract homomorphism from G

into the homeomorphism group of X with the property that the mapping $(g, x) \mapsto [\phi(g)](x)$, $G \times X \to X$ is continuous. The point $[\phi(g)](x)$ is usually abbreviated $g \cdot x$ or gx.

The space X above is typically called a *G-space*, and we say *G acts on X*. If X is compact, X is called a *G-flow*.

If Y is another G-space, and $f : X \to Y$ is a function, we say that f is *G-equivariant* if $f(gx) = gf(x)$ for every $g \in G$, $x \in X$.

If $x \in X$ the set $G \cdot x = \{gx : g \in G\}$ is called the *G-orbit* of x. The action of G on X is called *transitive* if $G \cdot x = X$ for any point $x \in X$ (and hence all points $x \in X$). If for every $x \in X$ and every non-identity element $g \in G$, we have $gx \neq x$, then the action of G on X is called *free*. If the action is both transitive and free, we call the action *simply transitive*.

Notice that if G is a compact topological group, then G may act on itself by left translation, and hence G admits a free action on a compact space, namely itself. This fact was extended to include locally compact spaces by Veech in 1977.

THEOREM 2.69 (Veech [54]). *Every locally compact group admits a free action on a compact space.*

Let us define a property which, according to the theorem above, is impossible for locally compact groups.

DEFINITION 2.70. A group G is called *extremely amenable* if every action of G on a compact space X has a common fixed point, i.e. a point x for which $g \cdot x = x$ for every x.

The idea of extreme amenability first appeared in the mid-1960's as a property of topological semigroups rather than groups. It was known even before Veech's theorem that no locally compact group could be extremely amenable, and for a while it was an open question whether extremely amenable groups existed at all. The first example was given by Herer and Christensen in 1975 [27] in an *ad hoc* construction. A few years later it was shown by Gromov and Milman that the unitary group $U(\ell^2)$ is extremely amenable. In the last few decades there have been a flurry of papers proving that various non-locally compact groups are extremely amenable:

THEOREM 2.71. *The following groups are extremely amenable.*

1. *(Gromov and Milman* [24]*) The unitary group $U(\mathcal{H})$ of a separable infinite-dimensional Hilbert space.*
2. *(Pestov* [42]*)* $\mathrm{Homeo}^+[0,1]$ *and* $\mathrm{Homeo}^+\mathbb{R}$*.*
3. *(Pestov* [42]*)* $\mathrm{Aut}\ \mathbb{Q}$*.*
4. *(Pestov* [43]*)* $\mathrm{Iso}\,\mathbb{U}$*.*
5. *(Kechris, Pestov, and Todorcevic* [34]*) The automorphism group* $\mathrm{Aut}\ \mathfrak{F}$ *for many Fraïssé structures* $\mathfrak{F}$*.*
6. *(Giordano and Pestov* [19]*)* $\mathrm{Aut}\ (X, \mu)$*.*

DEFINITION 2.72. Let G be a topological group and X a compact G-flow. X is called *minimal* if $\mathrm{cl}_X(G \cdot x) = X$ for every $x \in X$. X is called a *universal minimal flow* if X is minimal and for every other minimal G-flow Y there is a continuous G-equivariant surjection $\pi : X \to Y$.

It is known that each group G admits a universal minimal flow X, and moreover that this flow is unique up to topological conjugacy. For locally compact groups which are not compact, the universal minimal flow can be quite complicated or pathological. At least we have the following.

THEOREM 2.73. *Let G be a locally compact non-compact topological group and let X be a universal minimal G-flow. Then X is not metrizable.*

Conversely, for extremely amenable non-locally compact groups the universal minimal flow is the simplest compact set possible: a singleton set. In addition, for many non-locally compact groups which are not extremely amenable, the universal minimal flow has actually been explicitly computed, and turns out to be metrizable. Just a few examples are listed below.

THEOREM 2.74. *(1) (Pestov* [42]*) The universal minimal flow of* $\mathrm{Homeo}^+\mathbb{T}$ *is* $\mathbb{T}$*, with* *the evaluation action.*

(2) *(Glasner and Weiss* [20]*) The universal minimal flow of S_∞ is the space LO of all linear orders of ω, viewed topologically as a closed subset of $2^{\omega\times\omega}$, with the natural action.*

(3) *(Glasner and Weiss* [21]*) The universal minimal flow of* Homeo 2^ω *is the space X of all maximal chains of compact subsets of 2^ω.*

For our present purposes, we are mostly concerned with the measure-theoretic gap between locally compact and non-locally compact groups. Recall that every locally compact group G admits both left and right Haar measures, and that these measures together generate a translation-invariant measure zero σ-ideal in G. The following argument, attributed to Weil but presented here as in [46], indicates that no such σ-ideal can be generated by any single (reasonable) measure μ if G is not locally compact.

Theorem 2.75. *Suppose μ is a non-zero σ-finite regular measure μ on G whose zero sets comprise a translation-invariant σ-ideal. Then G must be locally compact.*

Proof. (*Weil*) Since μ is regular, there is a σ-compact set F for which F has full μ-measure in G. Let $g \in G$ be arbitrary; since the zero-sets are preserved under translations, $\mu(gF) > 0$. But then $gF \cap F \neq \emptyset$ and hence $g \in FF^{-1}$. So $G = FF^{-1}$ is σ-compact, and hence locally compact. □

In Chapter 5 we study the phenomenon of Haar null sets, defined by Christensen [7] to address the aforementioned shortcoming of non-locally compact groups. Under this lens we see that different classes of groups may exhibit very different measure-theoretic properties.

CHAPTER 3

POLISH TOPOLOGIZATION OF ABSTRACT SYMMETRY GROUPS

3.1. Introduction

Felix Klein emphasized the intrinsic connection between symmetry groups and geometries in his Erlangen Program. Perhaps motivated by Klein, Gleason ([22]) posed a very general conjecture on topologizing symmetry groups that he regarded as fundamental for a general study of geometries. Gleason in fact proved his conjecture in a very special case. In this chapter, working jointly with Robert R. Kallman, we show that Gleason's general conjecture is false as originally stated and that it is true only under very strong hypotheses. Along the way new general results in descriptive set theory are proved about a class of functions that behave like but are distinct from functions of Baire class 1.

Paraphrasing Gleason, we ask: if G is an abstract group of homeomorphisms of a topological space M, under what circumstances can G be given a topology such that the pair (G, M) is a topological transformation group? That is, when can G be given a (reasonable) topological group topology such that the mapping $(g, m) \mapsto g(m)$, $G \times M \mapsto M$, is continuous? Gleason gives a plausibility argument relating this question to very general geometries. Define a (topological) geometry as a topological space in which certain lines (subsets homeomorphic to $\mathbb{R}$) are distinguished. Let G be the group of automorphisms of a geometry M, i.e., the group of homeomorphisms of M that induce a permutation of the lines of M. It is reasonable to assume that M is homogeneous, i.e., that G acts transitively on M. It is also reasonable to assume some local uniqueness for lines which in turn implies that G is not "too big." Gleason's hope was that every topological geometry of finite dimension that satisfies some weak geometrical axioms must be the homogeneous space of some Lie group. In this context Gleason defined a frame for the action of G on M to be an element $(m_1, \ldots, m_n) \in M^n$ such that the mapping $g \mapsto (g(m_1), \ldots, g(m_n))$, $G \to M^n$ is an injection. In this paper such a frame will be called a finite frame and n will be called the size of the finite frame. It is convenient to define a countably infinite frame to be an element

$(m_1, m_2, \dots) \in M^{\mathbb{N}}$ such that the mapping $g \to (g(m_1), g(m_2), \dots)$, $G \mapsto M^{\mathbb{N}}$ is an injection. Guided by this very general geometric model, Gleason considered the following axioms:

Axiom 1.

M is a Polish space and G is a group of homeomorphisms of M that acts transitively on M;

Axiom 2(a).

There is a finite frame $(m_1, \dots, m_n) \in M^n$ for the action of G on M such that

$$\{(g(m_1), \dots, g(m_n), g(q)) \ : \ g \in G\} \subseteq M^{n+1}$$

is an analytic set for each $q \in M$.

Gleason's Conjecture. If G and M satisfy Axiom 1 and Axiom 2(a), then G can be given a Polish group topology such that the pair (G, M) is a topological transformation group.

The assumption in Axiom 1 that M is a Polish space and that G is a group of homeomorphisms of G is a very mild condition. The assumption that G acts transitively on M is a very restrictive and powerful assumption. The existence of a finite frame for the action of G on M in Axiom 2(a) corresponds to the assumption that lines are locally unique and that G is not too big. The analyticity condition in Axiom 2(a) is a smoothness assumption and is somewhat problematic in that it does not correspond to any obvious geometric assumption. However, Gleason pointed out that some assumption like Axiom 2(a) is needed. Specifically, if $M = \mathbb{C}^2 - \{(0,0)\}$ and G is the group of nonzero quaternion matrices, then there is even a simply transitive action of G by homeomorphisms on M that violates Axiom 2(a) and such that the pair (G, M) is not a topological transformation group.

Gleason ([22]) proved his conjecture in the very special case of size one frames, i.e., Gleason proved his conjecture if G acts simply transitively on M. However, in this case notice that the analytic set $\{(g(m), g(q)) \ : \ g \in G\} \subseteq M^2$ is the graph of a function on M for each $q \in M$. Hence, by Theorem 2.7 $\{(g(m), g(q)) \ : \ g \in G\}$ is actually a Borel set and not merely an analytic set. On the other hand, if Gleason's conjecture is true, then the mapping $g \mapsto (g(m_1), \ldots, g(m_n), g(q)))$, $G \to M^{n+1}$ is a continuous injection for each $q \in M$. Hence, Lusin-Souslin's theorem implies that $\{(g(m_1), \ldots, g(m_n), g(q))) \ : \ g \in G\} \subseteq M^{n+1}$ is in fact a Borel set. This suggests that Axiom $2(a)$ should be replaced by the stronger

Axiom $2(b)$.
There is a finite frame $(m_1, \ldots, m_n) \in M^n$ for the action of G on M such that

$$\{(g(m_1), \ldots, g(m_n), g(q)) \ : \ g \in G\} \subseteq M^{n+1}$$

is a Borel set for each $q \in M$.

However, Axiom 2(b) still will not be sufficient to guarantee that the conclusion of Gleason's conjecture is true. This situation is further clarified in Section 3.2, where it is shown that there is a G and an M that satisfy Axiom 1 and Axiom $2(a)$ such that $\{(g(m_1), \ldots, g(m_n), g(q)) \ : \ g \in G\}$ is a K_σ for every $q \in M$, but there is no way to make G into a Polish group let alone have (G, M) be a Polish transformation group. Notice that in this counterexample the G-orbit of any frame is also a K_σ since the continuous image of any K_σ is again a K_σ. This example suggest a further strengthening of Axiom $2(b)$.

Axiom $2(c)$.
There is a finite frame $F = (m_1, \ldots, m_n) \in M^n$ for the action of G on M such that the G-orbit of the frame

$$\{(g(m_1), \ldots, g(m_n)) \ : \ g \in G\} \subseteq M^n$$

is a G_δ in M^n and

$$\{(g(m_1), \ldots, g(m_n), g(q)) \ : \ g \in G\} \subseteq M^{n+1}$$

is a Borel subset of M^{n+1} for each $q \in M$.

Axiom 2(c) is consistent with the size one frame case proved by Gleason since in that simply transitive case the G-orbit of a frame, i.e., a single point, is M itself, trivially a G_δ subset of M. We also consider two weaker variants of this axiom. To motivate the next axiom note that there exist transitive Polish transformation groups (G, M) with no finite frame. For example, take M to be the unit sphere of a separable infinite dimensional complex Hilbert space $\mathcal{H}$ and let G be the full unitary group of $\mathcal{H}$. Any orthonormal basis is then a frame for the action of G on M.

Axiom 2(d).

There is a countably infinite frame $F = (m_1, m_2, \dots) \in M^{\mathbb{N}}$ for G such that the G-orbit of the frame

$$\{(g(m_1), g(m_2), \dots) \;:\; g \in G\} \subseteq M^{\mathbb{N}}$$

is a G_δ-subset of $M^{\mathbb{N}}$ and

$$\{(g(m_1), g(m_2) \dots, g(q)) \;:\; g \in G\} \subseteq M^{\mathbb{N}} \times M$$

is a Borel set for each $q \in M$.

and

Axiom 2(e).

There is a dense sequence $\{m_\ell\}_{\ell \geq 1}$ in M such that the G-orbit

$$\{(g(m_1), g(m_2), \dots) \;:\; g \in G\} \subseteq M^{\mathbb{N}}$$

is a G_δ-subset of $M^{\mathbb{N}}$ and

$$\{(g(m_1), g(m_2) \dots, g(q)) \;:\; g \in G\} \subseteq M^{\mathbb{N}} \times M$$

is a Borel set for each $q \in M$.

Of course if $\{m_\ell\}_{\ell \geq 1}$ is dense in M, then $(m_1, m_2, \dots)$ is a countably infinite frame for G. It will be proved that if there is a frame that satisfies Axiom $2(c)$ then there is a frame that satisfies Axiom $2(e)$ and therefore a frame that satisfies $2(d)$. The purpose of this chapter is to prove that the conclusion of Gleason's conjecture is indeed true if Axiom 1 and Axiom $2(c)$ or Axiom $2(d)$ or Axiom $2(e)$ hold.

An example is given in Section 3.2 plausibly showing that the G_δ condition in the axioms is needed. A new general result in descriptive set theory is given in Section 3.3. Gleason's results for the simply transitive case (frames of size one) are recalled and generalized for the convenience of the reader in Section 3.4. The relations among Axiom $2(c)$, Axiom $2(d)$ and Axiom $2(e)$ are proved in Section 3.5 together with the proof of the fact that Axiom 1 and Axiom $2(e)$ imply that G can be made into a Polish group so that (G, M) is a topological transformation group. An application of this general result is given in Section 3.6.

3.2. A Counterexample to Gleason's Conjecture

The purpose of this section is to show that Axiom 1 and Axiom $2(b)$ can hold even though the conclusion to Gleason's conjecture is false.

LEMMA 3.1. *Let $K \subseteq \mathbb{R}$ be an uncountable compact set whose elements are linearly independent over $\mathbb{Q}$. Such a K exists. Let H be the additive subgroup of $(\mathbb{R}, +)$ algebraically generated by K. Then H is σ-compact and there is no algebraic isomorphism of H with any Polish group.*

PROOF. Von Neumann [55] proved that there is a injection $f : (0, +\infty) \to \mathbb{R}$ whose range consists of numbers that are algebraically independent over $\mathbb{Q}$. A simple inspection of von Neumann's construction shows that f is a Borel mapping. Thus the range of f is an uncountable Borel set by Lusin-Souslin's theorem and therefore contains a compact perfect set by Theorem 2.9. Thus such a K exists. It is simple to check that H is σ-compact. As an abstract group $H = \oplus_{x \in K} \mathbb{Z}x$. Suppose that G is a Polish group and $\varphi : G \mapsto H$ is an algebraic isomorphism. Then Lemma 2 and Theorem 1 of Dudley [13] imply that φ is continuous if H is given the discrete topology. In particular $e_G = \varphi^{-1}(e_H)$ is open and therefore G is a discrete Polish group. This implies that G is countable, a contradiction, since G is algebraically isomorphic to H, an uncountable group. $\square$

PROPOSITION 3.2. *There exist a σ-compact subgroup G of a Polish group K and a Polish space M such that (G, M) is a transitive topological transformation group with a frame of every size $n \geq 2$ such that there is no algebraic isomorphism of G with any Polish group.*

PROOF. Notice that under the assumptions of the proposition if there is a frame of size n for the action of G on M, then there is a frame of size $n+1$ for the action of G on M by merely adding any element of M as the $n+1$-st entry to the original frame. It therefore suffices to prove the proposition to show the existence of a frame of size 2.

Let A be the exponentiation of the additive subgroup H of the reals given in Lemma 3.1, so that A is a subgroup of the multiplicative group of positive reals. There is no algebraic isomorphism of A with any Polish group since A is algebraically and topologically isomorphic to H. Let B be the additive group of the reals and let $G = B \rtimes A$ be the natural semidirect product. G is a σ-compact subgroup of the classical $ax+b$ group, a Polish group. If X is the real numbers, then (G, X) is a transitive topological transformation group and $(1, -1) \in X^2$ is a frame for (G, X). Suppose that L is a Polish group and $\varphi : L \mapsto G$ is an algebraic isomorphism. It is simple to check that A is maximal abelian in G. $A_L = \varphi^{-1}(A)$ is maximal abelian in L and therefore is closed in L and is itself a Polish group. This is a contradiction since $\varphi \restriction A_L : A_L \to A$ is an algebraic isomorphism of A with a Polish group A_L. $\square$

Though unrelated to the other results of this section, it should be noted that the elements of a frame for a transitive group action are not at all analogous to a basis for a vector space, even after extraneous elements of the frame are omitted. This is the case even for finite groups. For example, let G be the symmetric group on a set of size six $X = \{1, 2, 3, 4, 5, 6\}$. If $\emptyset \neq S \subseteq X$, let $G_S = \{g \in G \ : \ g(s) = s \text{ for all } s \in S\}$ and let $M = G/G_{\{1,2,3\}}$, a transitive G-space. Choose g_1, g_2 and $g_3 \in G$ such that $g_1 G_{\{1,2,3\}} g_1^{-1} = G_{\{4,5,6\}}$, $g_2 G_{\{1,2,3\}} g_2^{-1} = G_{\{1,2,4\}}$ and $g_3 G_{\{1,2,3\}} g_3^{-1} = G_{\{1,2,5\}}$. Then $(G_{\{1,2,3\}}, g_1 G_{\{1,2,3\}})$ and $(G_{\{1,2,3\}}, g_2 G_{\{1,2,3\}}, g_3 G_{\{1,2,3\}})$ are two frames for the action of G on M of different sizes that cannot be reduced in size by omitting judiciously chosen elements.

3.3. An Automatic Continuity Result

We start with a trivial observation. If X and Y are Polish spaces, $\varphi : X \to Y$ is continuous, $V \subseteq Y$ is open, $U \subseteq X$ is open, and $U \cap \varphi^{-1}(V) \neq \emptyset$, then $U \cap \varphi^{-1}(V)$ is a nonempty open subset of X and therefore is nonmeager in X. What about the converse? That is, suppose that X and Y are Polish spaces and $\varphi : X \to Y$ satisfies the property that if $V \subseteq Y$ is open and if $U \subseteq X$ is open and if $U \cap \varphi^{-1}(V) \neq \emptyset$, then $U \cap \varphi^{-1}(V)$ is nonmeager in X. Does this imply that φ satisfies some sort of nontrivial continuity property? In general, the answer is no. For example, let $X = \mathbb{R}$, $Y = \{0, 1\}$, $B \subseteq \mathbb{R}$ a Bernstein set ([41], pp. 32 – 33) and let $\varphi = \chi_B$, the characteristic function of B. The construction of $B = \chi_B^{-1}(1)$ shows that $B^c = \chi_B^{-1}(0)$ is also a Bernstein set, i.e., neither B nor B^c contains a compact perfect set by Theorem 2.9. If $A \subseteq \mathbb{R}$ is an uncountable analytic set, then $\varphi \restriction A$ cannot be continuous. This follows since A contains a compact perfect set K. If $x \in K \cap B$, then every relative neighborhood of x in K contains a compact perfect set and therefore contains of point of B^c, showing that $\chi_B \restriction K$ cannot be continuous. So some a priori weak smoothness assumption is needed on φ to in order to conclude that φ has some sort of reasonable continuity property.

Before proceeding further, we set up some notation and recall some very general results about Baire category. Let X be a topological space, $A \subseteq X$, and $M(A)$ the union of all open sets $V \subseteq X$ such that $V \cap A$ is meager in X. Then $M(A)$ is open in X and $A \cap \mathrm{cl}_X(M(A))$ is meager in X ([35], p. 201). Define $D(A) = M(A)^c$, a closed subset of X, and define $ID(A) = \mathrm{Int}(D(A))$, an open subset of X. $D(A)$ is the set of points in X at which A is not locally of the first category in X and $ID(A)$ is the interior of the set of points in X at which A is not locally of the first category in X. The following lemma consists of well-known results, most of which can be extracted from Kuratowski [36].

Lemma 3.3. *Let X be a topological space and A, B, A_n, $A_\iota \subseteq X$. Then:*
(1) if $A \subseteq B$, then $M(B) \subseteq M(A)$ and therefore $D(A) \subseteq D(B)$;

(2) $M(A \cup B) = M(A) \cap M(B)$ *and therefore* $D(A \cup B) = D(A) \cup D(B)$*;*

(3) $\mathrm{cl}_X(A)^c \subseteq M(A)$ *and therefore* $D(A) \subseteq \mathrm{cl}_X(A)$*;*

(4) $ID(A) = X - \mathrm{cl}_X(M(A))$*;*

(5) if $U \subseteq X$ *is open, then* $D(U) - U$ *is meager;*

(6) $D(A) = \emptyset$ *if and only if* A *is meager;*

(7) $A - D(A)$ *is meager and* $D(A - D(A)) = \emptyset$*;*

(8) $A - ID(A)$ *is meager and* $D(A - ID(A)) = \emptyset$*;*

(9) $D(A) - D(B) \subseteq D(A - B)$*;*

(10) $D(\cap_\iota A_\iota) \subseteq \cap_\iota D(A_\iota)$*;*

(11) $\cup_\iota D(A_\iota) \subseteq D(\cup_\iota A_\iota)$*;*

(12) if $U \subseteq X$ *is open, then* $U \cap D(A) = U \cap D(U \cap A)$*;*

(13) $D(D(A)) = D(A)$*;*

(14) $D(A) = \mathrm{cl}_X(ID(A))$*;*

(15) $ID(A) = \mathrm{Int}(\mathrm{cl}_X(ID(A)))$*;*

(16) $ID(A) = \emptyset$ *if and only if* A *is meager;*

(17) $D(\cup_{n\geq 1} A_n) - \cup_{n \geq 1} D(A_n)$ *is nowhere dense;*

(18) if A *is nonmeager, then* $A \cap ID(A)$ *is nonempty;*

(19) if $A \subseteq U$*, where* U *is open and* A *is nonmeager, then* $U \cap ID(A) \neq \emptyset$*;*

LEMMA 3.4. *Let* X *be a topological space and let* $A \subseteq X$ *be any set. Then* $A \subseteq D(A)$ *if and only if the following property holds: whenever* $U \subseteq X$ *is open and* $U \cap A \neq \emptyset$*, then* $U \cap A$ *is nonmeager in* X*.*

PROOF. Suppose $A \subseteq D(A)$ and let $U \subseteq X$ be open such that $U \cap A \neq \emptyset$. Let $x \in U \cap A$. Then $x \in D(A)$ and hence $x \notin M(A)$. Since U is an open neighborhood of x, it follows that $U \cap A$ is nonmeager.

Conversely, suppose whenever $U \subseteq X$ is open and $U \cap A \neq \emptyset$, then $U \cap A$ is nonmeager. Let $x \in A$ be arbitrary. If U is any open neighborhood of x, we have that $x \in U \cap A \neq \emptyset$, and hence $U \cap A$ is nonmeager by hypothesis. This implies $x \notin M(A)$, i.e., $x \in D(A)$. So $A \subseteq D(A)$. $\square$

COROLLARY 3.5. *Let X and Y be topological spaces and let $\varphi : X \to Y$. Then the following are equivalent:*

(1) $\varphi^{-1}(V) \subseteq D(\varphi^{-1}(V))$ for every open $V \subseteq Y$;

(2) if $V \subseteq Y$ is open, $U \subseteq X$ is open and $U \cap \varphi^{-1}(V) \neq \emptyset$, then $U \cap \varphi^{-1}(V)$ is nonmeager in X.

Recall that if X be a topological space, then a set $A \subseteq X$ is said to be a set with the Baire property in X if there exists an open set $U \subseteq X$ such that $A \bigtriangleup U$ is meager in X. Let $\mathcal{BP}(X)$ be the collection subsets of X with the Baire property. $\mathcal{BP}(X)$ is the smallest σ-algebra of subsets of X generated by the open sets and the first category sets and therefore contains the Borel subsets of X. It is a nontrivial fact that if X is a Polish space then $\mathcal{BP}(X)$ contains the analytic subsets of X. Again, the following lemma consists of well-known facts that can be gleaned from various places in Kuratowski [36].

LEMMA 3.6. *Let X be a topological space and let $A \subseteq X$. Then the following statements are equivalent:*

(0) $A \in \mathcal{BP}(X)$;

(1) $A = G \cup M$, where G is a G_δ and M is meager in X;

(2) $A = F - M$, where F is an F_σ and M is meager in X;

(3) $A = (U - B) \cup C$, where U is open and B and C are meager in X;

(4) $A = (F - B) \cup C$, where F is closed and B and C are meager in X;

(5) there is a set M meager in X such that $A - M$ is both open and closed relative to M^c;

(6) $D(A) \cap D(A^c)$ is nowhere dense in X and therefore every nonempty open set contains a point at which either A or A^c is of the first category in X;

(7) $D(A) - A$ is meager in X;

(8) $A \bigtriangleup D(A)$ is meager in X;

(9) $A \bigtriangleup ID(A)$ is meager in X.

Corollary 3.7 (Gleason). *Let X be a Baire space and let A, $B \in \mathcal{BP}(X)$. Then $ID(A) \cap ID(B) \neq \emptyset \implies A \cap B \neq \emptyset$.*

Proof. Suppose that $A \cap B = \emptyset$ and that the open set $ID(A) \cap ID(B) \neq \emptyset$. But then an elementary computation shows that $\emptyset \neq ID(A) \cap ID(B) \subseteq ((ID(A) - A) \cup A) \cap ((ID(B) - B) \cup B) \subseteq (ID(A) - A) \cup (ID(B) - B)$ is meager by Lemma 3.6 (9). But $ID(A) \cap ID(B)$ is not meager since X is a Baire space, a contradiction. Hence, $A \cap B \neq \emptyset$. □

Proposition 3.8. *Let X be a topological space, let (Y, d) be a metric space and let $\varphi : X \to Y$ be a function that satisfies:*

(1) $\varphi^{-1}(B) \in \mathcal{BP}(X)$ for every ball $B \subseteq Y$; and

(2) $\varphi^{-1}(V) \subseteq D(\varphi^{-1}(V))$ if $V \subseteq Y$ is open.

Then the set of points of continuity of φ is comeager in X.

Proof. Fix $n \geq 1$. We will show that there is an open dense set U_n in X for which φ has oscillation less than or equal to $\frac{1}{n}$ at each point in U_n.

For each $x \in X$, let $V_x \subseteq Y$ be the open ball of d-radius $\frac{1}{2n}$ about $\varphi(x)$. Let $U_x = ID(\varphi^{-1}(V_x))$, and set $U_n = \bigcup_{x \in X} U_x$. U_n is open in X and we claim that U_n is dense in Y. To see this, let $W \subseteq X$ be any nonempty open set and choose $x \in W$. Then $x \in \varphi^{-1}(V_x) \subseteq D(\varphi^{-1}(V_x))$, so $W \cap D(\varphi^{-1}(V_x))$ is nonempty and relatively open in $D(\varphi^{-1}(V_x))$. Since U_x is dense in $D(\varphi^{-1}(V_x))$ by Lemma 3.3 (14), it follows that $W \cap U_x$ is nonempty. Thus $W \cap U_n$ is nonempty and U_n is dense as required.

Next we show that φ has oscillation less than or equal to $\frac{1}{n}$ at w for every $w \in U_n$. To accomplish this, we will first show that if $V \subseteq Y$ is open and $x \in ID(\varphi^{-1}(V))$, then $\varphi(x) \in \mathrm{cl}_Y(V)$. If not then there is some open neighborhood V' of $\varphi(x)$ which misses V. Since $x \in \varphi^{-1}(V') \subseteq D(\varphi^{-1}(V'))$, we have $ID(\varphi^{-1}(V)) \cap D(\varphi^{-1}(V')) \neq \emptyset$. But $ID(\varphi^{-1}(V'))$ is dense in $D(\varphi^{-1}(V'))$ by Lemma 3.3 (14), so in fact $ID(\varphi^{-1}(V)) \cap ID(\varphi^{-1}(V')) \neq \emptyset$. Since $\varphi^{-1}(V)$ and $\varphi^{-1}(V')$ have the Baire property, it follows from Corollary 3.7 that $\varphi^{-1}(V) \cap \varphi^{-1}(V') \neq \emptyset$. This contradicts our assumption that V misses V', so $\varphi(x) \in \mathrm{cl}_Y(V)$.

Now suppose $w \in U_n$. Let $z \in X$ be such that $w \in U_z$. It follows from the above paragraph that for every $x \in U_z$ we have $\varphi(x) \in \mathrm{cl}_Y(V_z)$, where V_z is the ball of radius $\frac{1}{2n}$ about $\varphi(z)$. So $d(\varphi(w), \varphi(x)) \leq d(\varphi(w), \varphi(z)) + d(\varphi(x), \varphi(z)) \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$. Since w was arbitrary, φ has oscillation less than or equal to $\frac{1}{n}$ at every point in U_n.

Set $U = \bigcap_{n \geq 1} U_n$. Then U is a countable intersection of dense open sets in X and φ has oscillation 0 at every point in U. So φ is continuous on a comeager set. $\square$

Though Gleason did not formulate Proposition 3.8, a glance at [22] shows that he had most of the technology in hand to prove it. A word of caution perhaps is in order for readers of [22]: the notation $D(A)$ used here is consistent with that defined in Kuratowski [36], whereas Gleason's notation of $D(A)$ coincides with the $ID(A)$ used here in spite of the fact that he refers to Kuratowski [36] for the properties of his $D(A)$.

The conclusion of Theorem 3.8 is reminiscent of a property of Baire class 1 functions (a theorem of Baire, [33], Theorem 24.14). However, in general, there is no connection between the functions that satisfy the hypotheses of Theorem 3.8 and functions of Baire class 1. For example, the function $\varphi = \delta_0$ is a Baire class 1 function, but $\varphi^{-1}((1/2, 3/2)) = \{0\}$ is certainly meager in $\mathbb{R}$ and therefore does not satisfy the hypotheses of Theorem 3.8. On the other hand, let $B = \{(x, y) \ : \ y > 0\} \cup \mathbb{Q}$ and let $\varphi = \chi_B$, the characteristic function of B. Then φ is Borel measurable and hence is in $\mathcal{BP}(\mathbb{R}^2)$ and if $U \subseteq \mathbb{R}$ is open, then $\varphi^{-1}(U)$ is either empty or contains a nonempty open subset of $\mathbb{R}^2$ and therefore is second category. But φ is not a Baire class 1 function since $\varphi^{-1}((-1/2, 1/2)) \cap \mathbb{R}$ is the set of irrational numbers, which is not an F_σ.

3.4. A Strengthening of Gleason's Results

Most, but not all, of the results given in this section are due to Gleason in less general form. They are given here because of their importance in what follows and for the convenience of the reader since Gleason's paper is somewhat obscure. The statements of the results are more general than Gleason's statements and the proofs are somewhat different.

We start with a result on descriptive set theory. It illustrates the power of a transitive group assumption.

Theorem 3.9 (Gleason). *Let X be a Polish space, Y a separable metric space, G a group that acts as a group of homeomorphisms on X and Y and that is transitive on X, and let $\varphi : X \to Y$ be $\mathcal{BP}(X)$-measurable and G-equivariant. Then φ is continuous.*

Proof. Let $U \subseteq X$ and $V \subseteq Y$ be open and satisfy $U \cap \varphi^{-1}(V) \neq \emptyset$. Let $x \in Uf \cap \varphi^{-1}(V)$ so $(x, \varphi(x)) \in U \times V$. If $x' \in X$ choose $g \in G$ such that $g(x) = x'$. Then $(x', \varphi(x')) = (g(x), \varphi(g(x)) = (g(x), g(\varphi(x))) \in g(U) \times g(V)$. Therefore graph$(\varphi) \subseteq \cup_{g \in G}(g(U) \times g(V))$. Since graph$(\varphi)) \subseteq X \times Y$ is separable metrizable and therefore Lindelöf there exists a sequence $\{g_n\}_{n \geq 1} \subseteq G$ such that graph$(\varphi) \subseteq \cup_{n \geq 1}(g_n(U) \times g_n(V))$. Let $X_n = \{x \in X \mid (x, \varphi(x)) \in g_n(U) \times g_n(V)\} = g_n(U \cap \varphi^{-1}(V))$. Since $\cup_{n \geq 1} X_n = X$, some X_n is second category and therefore $U \cap \varphi^{-1}(V)$ is nonmeager in X. Corollary 3.5 plus Theorem 3.8 imply that the set of points of continuity of φ in X is comeager in X and therefore nonempty. Since G is transitive and φ is G-equivariant, φ is continuous everywhere. □

Gleason did not point out the following corollary (c.f. [15]).

Corollary 3.10. *Let X be a Polish space, $x \in X$, and G a Polish group that acts as an abstract transitive group of homeomorphisms of X such that $g \mapsto g(x)$, $G \to X$, is continuous. Let $G_x = \{x \in X \ : \ g \cdot x = x\}$, a closed subgroup of G. Then the natural G-equivariant mapping $\varphi : gG_x \mapsto g(x)$, $G/G_x \to X$, is a homeomorphism and (G, X) is a topological transformation group that is naturally homeomorphic to the topological transformation group $(G, G/G_x)$.*

Proof. The quotient space G/G_x is a Polish space by a theorem of Hausdorff [26] and the natural G-equivariant mapping $\varphi : gG_x \to g(x)$, $G/G_x \mapsto X$ is a continuous bijection. G acts transitively on both X and G/G_x and $\varphi^{-1} : X \to G/G_x$ is a G-equivariant Borel mapping by Lusin-Souslin's theorem. Proposition 3.9 now implies that φ^{-1} is continuous and therefore φ is a homeomorphism. □

Corollary 3.11 (Gleason). *Let G be an abstract group and also a Polish space such that, for each fixed $g \in G$, the mapping $h \mapsto gh$, $G \to G$, is continuous and for each fixed $h \in G$,* *the mapping $g \mapsto gh$, $G \to G$, is $\mathcal{BP}(G)$-measurable. Then G is a Polish group.*

Proof. Fix $h_0 \in G$ and let $\varphi(g) = gh_0$, a $\mathcal{BP}(G)$-measurable mapping. Left translations are continuous by assumption and therefore homeomorphisms since they are invertible. Left translations also act transitively on G. Proposition 3.9 implies that φ is continuous since $k\varphi(g) = \varphi(kg)$ for all k, $g \in G$. Therefore both left and right translations of G are continuous. The corollary now follows from Theorem 2.18. $\square$

Corollary 3.12. *Let G be an abstract group and also a Polish space such that, for each fixed $g \in G$, the mapping $h \mapsto hg$, $G \to G$, is continuous and for each fixed $h \in G$, the mapping $g \mapsto hg$, $G \to G$, is $\mathcal{BP}(G)$-measurable. Then G is a Polish group.*

Proof. Let G^* be the group whose underlying set is G with the topology of G but with the multiplication $a * b = ba$. Then G^* is a Polish group and this corollary follows by applying Corollary 3.12 to G^*. $\square$

Corollary 3.13 (Gleason). *Let G and M satisfy Axiom 1 and Axiom 2(a) for a frame of size one. Then the conclusion to Gleason's Conjecture is true.*

Proof. G acts simply transitively on M since $n = 1$. Fix $m_0 \in M$ and topologize G by requiring that the bijection $g \mapsto g(m_0)$, $G \to M$ be a homeomorphism. G is then an abstract group and a Polish space. $h_n \to h$ if and only if $h_n(m_0) \to h(m_0)$ which implies that $gh_n(m_0) = g(h_n(m_0)) \to g(h(m_0)) = gh(m_0)$ which in turn implies that $gh_n \to gh$ for each $g \in G$, i.e., the mapping $h \mapsto gh$, $G \to G$, is continuous for each $g \in G$.

On the other hand, fix $h \in G$. The graph of the mapping $g \mapsto gh$, $G \to G$, is homeomorphic to $\{(g(m_0), g(h(m_0))) \ : \ g \in G\}$, an analytic set. From this it easily follows that the mapping $g \mapsto gh$, $G \to G$, is $\mathcal{BP}(G)$-measurable. The present corollary now follows from Corollary 3.12. $\square$

3.5. The General Case

We first start with some basic properties of frames.

LEMMA 3.14. *Let M be a Polish space, G an abstract group of homeomorphisms of M, I and J nonempty finite or countably infinite disjoint index sets, $(p_i)_{i\in I}$ a frame for G acting on M and $(q_j)_{j\in J}$ a tuple of elements of M. Let $A = \{(g(p_i))_{i\in I} \ : \ g \in G\}$, $C(r) = \{(g(p_i))_{i\in I} \oplus (g(r)) \ : \ g \in G\}$ ($r \in M$) and $B = \{(g(p_i))_{i\in I} \oplus (g(q_j))_{j\in J} \ : \ g \in G\}$ and suppose that $C(r)$ is an analytic set for every $r \in M$. Then A is analytic set and B is an analytic set Borel isomorphic to A. If A is a Borel set, then B is a Borel set. If A is a G_δ-set, then B is a G_δ-set homeomorphic to A.*

PROOF. A is an analytic set since it is the continuous image of any $C(r)$. If $j_0 \in J$ then $C_{j_0} = \{(g(p_i))_{i\in I} \oplus (g(q_{j_0})) \ : \ g \in G\} \oplus \prod_{j\in J-\{j_0\}} M$ is an analytic subset of $M^{I\cup J}$ and therefore $B = \cap_{j\in J} C_{q_j}$ is an analytic set since the intersection of a sequence of analytic sets is analytic. The natural projection of B onto A is a continuous bijection since $(p_i)_{i\in I}$ is a frame for G acting on M and therefore is a Borel isomorphism by [38], Theorem 4.2. Let $\varphi : A \to B$ be given by $\varphi : (g(p_i))_{i\in I} \mapsto (g(p_i))_{i\in I} \oplus (g(q_j))_{j\in J}$, the inverse of projection of B onto A and therefore a Borel mapping. We are now done in the analytic and Borel cases. Finally, suppose A is a G_δ. Then A is a Polish space, B is a separable metric space and φ is a G-equivariant mapping. Proposition 3.9 implies that φ is continuous. Since we have already noted that the natural projection of B onto A, viz. φ^{-1}, is continuous, we have that φ is a homeomorphism. Hence, B is homeomorphic to a G_δ and therefore itself is a G_δ. □

COROLLARY 3.15. *If Axiom 2(c) holds, then Axiom 2(d) holds, and if Axiom 2(d) holds, then Axiom 2(e) holds.*

PROOF. Any augmentation of a finite or countably infinite frame for the action of G on M which has a G_δ-orbit by a finite or countably infinite number of elements of M is again a frame for the action of G on M with a G_δ-orbit by Lemma 3.14. □

PROPOSITION 3.16. *Let G and M satisfy Axiom 1 and Axiom 2(c) or Axiom 2(d) or Axiom 2(e). Then the conclusion of Gleason's Conjecture is true.*

PROOF. Corollary 3.15 implies that it suffices to give the proof under the assumption that

Axiom $2(e)$ holds.

Suppose that $\{m_\ell\}_{\ell\geq 1}$ is a dense sequence in M such that $\mathcal{O} = \{(g(m_1), g(m_2), \dots) : g \in G\} \subseteq M^{\mathbb{N}}$ is a G_δ subset of $M^{\mathbb{N}}$ and that $\{(g(m_1), g(m_2), \dots, g(q)) : g \in G\} \subseteq M^{\mathbb{N}} \times M$ is a Borel subset of $M^{\mathbb{N}} \times M$ set for each $q \in M$. The natural diagonal action of G on $M^{\mathbb{N}}$ is an abstract group of homeomorphisms of $M^{\mathbb{N}}$. If $F = (m_1, m_2, \dots) \in M^{\mathbb{N}}$ is the frame and $Q = (q_1, q_2, \dots) \in \mathcal{O}$, then, with obvious notation, the set $\{(g(F), g(Q)) : g \in G\} \subseteq \mathcal{O} \times \mathcal{O}$ is a Borel set by Lemma 3.14. $\mathcal{O}$ is a Polish space since it is a G_δ-subset of a Polish space and G acts as a simply transitive group of homeomorphisms of $\mathcal{O}$. Then the pair G and $\mathcal{O}$ satisfy Axiom 1 and Axiom $2(a)$. Therefore the pair $(G, \mathcal{O})$ can be made into a Polish topological transformation group by Corollary 3.13.

If $x \in M$ choose $h \in G$ such that $x = h(m_1)$. Now the mapping $g \mapsto gh \mapsto gh(F) \mapsto gh(m_1) = g(x)$ is continuous for every $x \in M$ and G-equivariant. Corollary 3.10 now implies that the pair (G, M) is a transitive Polish topological transformation group. $\square$

The next result implies that the topology on G determined by Theorem 3.16 is unique.

Proposition 3.17. *Let M be a Polish space and let G be an abstract group of homeomorphisms of M. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two Polish group topologies on G such that both $((G, \mathcal{T}_1), M)$ and $((G, \mathcal{T}_2), M)$ are topological transformation groups, then $\mathcal{T}_1 = \mathcal{T}_2$.*

Proof. Let F be a frame whose coordinates are dense in M. The mapping $g \mapsto g \cdot F$, $G \to G \cdot F \subset M^{\mathbb{N}}$ is a bijection. Hence, the mappings $g \mapsto g \cdot F$, $(G, \mathcal{T}_\ell) \to G \cdot F$ $(\ell = 1, 2)$ are Borel isomorphisms since they are continuous. Therefore, the group isomorphism $g \mapsto g$, $(G, \mathcal{T}_1) \to (G, \mathcal{T}_2)$ is a Borel mapping and therefore a topological isomorphism. $\square$

The counterexample given in Section 3.2 strongly suggests that the sufficient G_δ G-orbit of a frame condition in Proposition 3.16 cannot be omitted. Unfortunately this condition is not necessary, as the following proposition demonstrates.

Proposition 3.18. *There exist a Polish group G and a Polish space X such that (G, X) is a transitive topological transformation group with a frame such that the following propery* *holds: the orbit $G \cdot x$ is not a G_δ set in X^n for every frame $x \in X^n$ ($n \in \mathbb{N} \cup \{\infty\}$).*

PROOF. Let X be the reals, let A be the multiplicative group of positive rationals with the discrete topology, let B be the additive group of the reals and let $G = B \rtimes A$, the natural semidirect product of B and A. (G, X) is a transitive topological transformation group and $(1, -1)$ is a frame for G in X^2. Let $n \in \mathbb{N}$ and let $x \in X^n$ be a frame for G. Then $x \neq 0 \in X^n$. Suppose that $G \cdot x$ is a G_δ subset of X^n. Then $G \cdot x$ is a Polish space and the mapping $g \mapsto gx$ is a homeomorphism by Corollary 3.10. In follows that $A \cdot x$ is a G_δ in X^n since A closed in G implies $A \cdot x$ is closed in $G \cdot x$. But $A \cdot x = \{qx \mid q \in \mathbb{Q},\, q > 0\}$ is therefore a G_δ in $\{qx \ : \ q \in \mathbb{R},\, q > 0\}$, a contradiction since the positive rationals are not a G_δ-subset of the positive reals. □

Recall the following theorem of Becker-Kechris.

THEOREM 3.19 ([4] Theorem 5.1.5). *Let G be a Polish group, let X be a Polish G-space, and let $E \subseteq X$ be a G-invariant Borel set. There exists a Polish topology finer than the original topology of X (and thus having the same Borel structure) in which E is now open and the action of G on X is still continuous.*

The following proposition shows in a rather strong manner that the assumption that G is a Polish group cannot be omitted from the Becker-Kechris Theorem 3.19.

PROPOSITION 3.20. *There exists a separable metrizable topological group G, a Polish G-space X and a G-invariant K_σ-subset $E \subseteq X$ such that there is no finer Polish topology on X which makes E a G_δ and such that the action of G on E is still continuous.*

PROOF. Let G be as in Proposition 3.2, let $M = \mathbb{R}$ and let $X = M^2$. G is a separable metrizable K_σ group and the pair (G, M) is a topological transformation group. The orbit of any frame for G in M^n and therefore in X^n is a K_σ. $(1, -1) \in X$ is a frame for G, $E = G \cdot (1, -1) \subseteq X$ is a K_σ, G acts simply transitively on E and $G \cdot (x, q) \subseteq X^2$ is K_σ for every x, $q \in X$. Suppose that there is a Polish topology on X that makes E into a G_δ and such that the action of G on X is still continuous. This new topology and the original topology on X generate the same Borel sets and therefore $G \cdot (x, q)$ is still a Borel set for every x, $q \in X$. Then the hypotheses of Corollary 3.13 are satisfied and the abstract group G can be made into a Polish group such that the pair (G, X) is a Polish topological transformation group. But G cannot be given any Polish group topology by Proposition 3.2, a contradiction. $\square$

3.6. A Corollary on Lie Groups and Manifolds

The following corollary is obviously motivated by Gleason [22], Corollary 2, who gave a terse indication of its proof. Perhaps this is the result desired by Gleason.

Corollary 3.21. *Let G and M satisfy Axiom 1 and Axiom 2(c). In addition assume that M is of finite dimension and the G-orbit $G \cdot F$ is locally connected at one point. Then (G, M) can be made into a Polish topological transformation group such that G is a Lie group and M is a manifold homeomorphic with a quotient group of G.*

Proof. (G, M) can be made into a Polish topological transformation group by Theorem 3.16. M^n is of finite dimension ([30], Theorem III 4, The Product Theorem, p. 33) and therefore the G-orbit $G \cdot F \subseteq M^n$ is of finite dimension ([30], Theorem III 1, p. 26). Since the G-orbit is locally connected at one point, it is locally connected since G acts transitively on the orbit. Therefore G is a finite-dimensional locally connected Polish group since it is homeomorphic to the G-orbit $G \cdot F$. If U is a connected open subset of G, then U is a connected, locally connected complete metrizable space and therefore U is arcwise connected ([28], Theorem 3-17). Hence, G is a finite dimensional locally arcwise connected Polish group and therefore is a Polish Lie group ([23], Theorem 7.2). Finally, if $x \in M$ and G_x is the G-stability group at x, then G_x is a closed subgroup of G, G/G_x is a manifold ([56], Theorem 3.58) and M is homeomorphic to G/G_x by Corollary 3.10. $\square$

As a final comment, Gleason proved the following corollary.

Corollary 3.22 (Gleason [22], Corollary 3). *Let G be a topological group acting continuously and effectively on a complete separable metric space M. Let T be an analytic subgroup* *of G which is simply transitive on M. Then T is closed.*

As Gleason notes in his proof, we have $G = G_x T$, where G_x is the stability group at $x \in M$, a closed subgroup of G, and $G_x \cap T = \{e\}$. Therefore Proposition 5 and Corollary 6 of [2] provide more general results, at least in the case for which (G, M) is a Polish transformation group. The proofs of this proposition and corollary appear to have nothing in common with the techniques employed by Gleason.

CHAPTER 4

BOREL COMPLEXITY OF SERIES REARRANGEMENTS IN S_∞

4.1. Introduction

The goal of this chapter is to establish the exact descriptive complexity of some interesting subsets of the Polish group S_∞ of permutations of ω, endowed with the topology of pointwise convergence on ω, considered as a discrete set. Our methods involve a blending of the techniques of classical real analysis and geometry, with the descriptive set theoretic notion of continuous reducibility between Polish spaces.

First we recall Bernhard Riemann's celebrated rearrangement theorem of 1876 [45], now a staple of every graduate course in real analysis, which states the following remarkable fact (presented here as in [48]): given a conditionally convergent series of real numbers $\sum_{k=0}^{\infty} a_k$, and two extended real numbers $\alpha, \beta \in [-\infty, \infty]$ with $\alpha \leq \beta$, it is possible to find an infinite permutation $\pi \in S_\infty$ for which $\liminf_{n\to\infty} \sum_{k=0}^{n} a_{\pi(k)} = \alpha$ and $\limsup_{n\to\infty} \sum_{k=0}^{n} a_{\pi(k)} = \beta$. In other words, by varying one's choice of α and β, it is possible to rearrange the terms of a conditionally convergent infinite series so that the partial sums converge to any particular real number, or diverge to plus or minus infinity, or even diverge properly.

Almost as famous as Riemann's original theorem is the following d-dimensional analogue:

Levy-Steinitz Theorem. Let $\sum_{k=0}^{\infty} v_k$ be a conditionally convergent series of vectors in $\mathbb{R}^d$. Then there exists a non-trivial affine subspace $A(v_k) \subseteq \mathbb{R}^d$ (that is, a space of the form $A(v_k) = w + M$ where $w \in \mathbb{R}^d$ and $M \subseteq \mathbb{R}^d$ is a linear subspace with $\dim M \geq 1$) such that whenever $a \in A(v_k)$, there is $\pi \in S_\infty$ with $\sum_{k=0}^{\infty} v_{\pi(k)} = a$.

The statement above implies that the set of all possible sums of rearrangements of a conditionally convergent series of d-dimensional vectors is at least as rich as in the 1-dimensional case. An incomplete proof was first given by Levy in 1905 [37], and the complete proof was furnished by Steinitz in 1913 [53]. Steinitz's proof relied on a particular geometric constant for Euclidean spaces which is now referred to as the *Steinitz constant*. The proof is nontrivial, and an excellent concise version of it may be found in the paper [47] by P. Rosenthal. Our proofs also rely heavily on the existence of a Steinitz constant.

The Levy-Steinitz theorem gives rise to a natural partition of S_∞, into the set $\mathcal{D}$ of all permutations π for which $\sum_{k=0}^{\infty} v_{\pi(k)}$ diverges (either properly or to ∞, where ∞ denotes the point at infinity in the one-point compactification of $\mathbb{R}^d$), and the complement set $S_\infty \backslash \mathcal{D}$ of permutations π for which $\sum_{k=0}^{\infty} v_{\pi(k)}$ converges to some vector in $\mathbb{R}^d$. Both $\mathcal{D}$ and its complement are interesting nontrivial sets. For instance, it is easy to observe, as we do briefly in Section 4, that both $\mathcal{D}$ and $S_\infty \backslash \mathcal{D}$ are uncountable and dense in S_∞, and also that $\mathcal{D}$ is a comeager set.

We wish to examine these collections from the vantage point of descriptive set theory, or, loosely speaking, the study of the definable subsets of Polish spaces. *Definable* here may refer to Borel, analytic, projective, or any other class of "well-behaved" sets, which are typically closed under continuous preimages. Of course the Borel sets may be stratified by their relative complexity into a *Borel hierarchy* indexed by the countable ordinals, whose exact definition we recall for the reader in Section 2. It is an empirical phenomenon that a great bulk of those Borel sets which present themselves in the everyday study of mathematics will fall into the very bottom few levels of the Borel hierarchy. Thus there has been some industry for descriptive set theorists in finding "natural" examples of Borel sets which are "more complex" than usual. For some instances of such sets, the reader may consult the well-known references [3] and Sections 23, 27, 33, and 37 of [33], or the paper [1], which produces many examples in the field of ordinary differential equations.

Our objective here will be to establish the exact descriptive complexity of our set $\mathcal{D}$ and its complement. In classical terminology, we show that $\mathcal{D}$ is $G_{\delta\sigma}$ but not $F_{\sigma\delta}$ (and hence not F_σ, G_δ, open, nor closed). Using the more modern notation, we prove:

Proposition 4.1. *Let* $\sum_{k=0}^{\infty} v_k$ *be any conditionally convergent series of vectors in* $\mathbb{R}^d$*, and let*

$\mathcal{D} \subseteq S_\infty$ be the set of all permutations π for which $\sum_{k=0}^{\infty} v_{\pi(k)}$ diverges. Then $\mathcal{D}$ is $\mathbf{\Sigma}^0_3$-complete.

Of course, it follows immediately that $S_\infty \backslash \mathcal{D}$ is $\mathbf{\Pi}^0_3$-complete. Now, for $\pi \in S_\infty$, say that the rearrangement $\sum_{k=0}^{\infty} v_{\pi(k)}$ *diverges properly* if the series diverges, but does not diverge to infinity. Our methods also give the following result:

PROPOSITION 4.2. *Let $\sum_{k=0}^{\infty} v_k$ be any conditionally convergent series of vectors in $\mathbb{R}^d$, and let $\mathcal{DP} \subseteq S_\infty$ be the set of all permutations π for which $\sum_{k=0}^{\infty} v_{\pi(k)}$ diverges properly. Then $\mathcal{DP}$ is $\mathbf{\Sigma}^0_3$-complete.*

It follows that the set $S_\infty \backslash \mathcal{DP}$ of series rearrangements which either converge to a vector in $\mathbb{R}^d$, or which diverge to ∞, is also a $\mathbf{\Pi}^0_3$-complete set in S_∞. Notice that, remarkably, none of the above statements depend on the nature of the particular conditionally convergent series $\sum_{k=0}^{\infty} v_k$ that we choose! Thus, for each conditionally convergent series, we exhibit a naturally defined subset of S_∞ which lies no lower on the Borel hierarchy than the third level.

4.2. The Borel Hierarchy

First we recall the definition of the Borel hierarchy. Given a Polish space X, we let $\mathbf{\Sigma}^0_1(X)$ be the family of all open subsets of X, and $\mathbf{\Pi}^0_1(X)$ the family of all closed subsets of X. We set $\mathbf{\Delta}^0_1(X) = \mathbf{\Sigma}^0_1(X) \cap \mathbf{\Pi}^0_1(X)$, so $\Delta^0_1(X)$ consists of the clopen sets in X. The rest of the levels of the hierarchy are defined recursively as follows: Suppose for some countable ordinal β, we have defined the classes $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{\Pi}^0_\alpha(X)$ for all $\alpha < \beta$. Then we set

$$\mathbf{\Sigma}^0_\beta(X) = \{\textstyle\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_{\alpha_n} \text{ for some } \alpha_n < \beta\},$$
$$\mathbf{\Pi}^0_\beta(X) = \{A^c : A \in \mathbf{\Sigma}^0_\beta(X)\}, \text{ and}$$
$$\mathbf{\Delta}^0_\beta(X) = \mathbf{\Sigma}^0_\beta(X) \cap \mathbf{\Pi}^0_\alpha(X).$$

It is well known that $\mathbf{\Delta}^0_\beta(X) \subseteq \mathbf{\Sigma}^0_\beta(X), \mathbf{\Pi}^0_\beta(X) \subseteq \mathbf{\Delta}^0_{\beta+1}$ for each β, and that the inclusions are all proper.

Let X, Y be Polish spaces and $A \subseteq X$, $B \subseteq Y$. If there exists a continuous function $f : X \to Y$ such that $f^{-1}(B) = A$, then we say that A is *Wadge reducible* or *continuously reducible* to B, and we write $A \leq_W B$. Intuitively, we think that A is "no more complex" than B.

Let $\mathbf{\Gamma}$ be any of the pointclasses $\mathbf{\Sigma}^0_\beta$, $\mathbf{\Pi}^0_\beta$, or $\mathbf{\Delta}^0_\beta$. A standard inductive argument through the hierarchy shows that $\mathbf{\Gamma}$ is closed under continuous preimages, i.e., whenever X and Y are Polish, $A \subseteq X$, $B \in \mathbf{\Gamma}(Y)$, and A is continuously reducible to B, then we have $A \in \mathbf{\Gamma}(X)$.

The above comment provides a useful tool for determining the complexity of a set. We say that a subset B of a Polish space Y is $\mathbf{\Gamma}$-*hard* if for every Polish space X and every $A \in \mathbf{\Gamma}(X)$ we have $A \leq_W B$. It follows from the above comments that if B is $\mathbf{\Gamma}$-hard, then $\mathbf{\Gamma}$ is a lower bound for the descriptive complexity of B. If in addition we have $B \in \mathbf{\Gamma}(Y)$, then we say that B is $\mathbf{\Gamma}$-*complete*, and we have determined its exact complexity in the Borel hierarchy.

The most common method for showing that a set B is $\mathbf{\Gamma}$-hard is to find a set A which is already known to be $\mathbf{\Gamma}$-complete, and prove that $A \leq_W B$ by constructing an explicit continuous reduction. This is the method of our proof in Section 4, and we make use of the following subset $\mathcal{C}$ of the Baire space ω^ω:

$$\mathcal{C} = \{x \in \omega^\omega : \lim_{n\to\infty} x(n) = \infty\}.$$

Exercise 23.2 of [33] asks the reader to show that $\mathcal{C}$ is in fact $\mathbf{\Pi}^0_3$-complete. It necessarily follows that the complement $\omega^\omega \backslash \mathcal{C}$ is $\mathbf{\Sigma}^0_3$-complete. Our proof continuously reduces this complement $\omega^\omega \backslash \mathcal{C}$, simultaneously, to both $\mathcal{D}$ and $\mathcal{DP}$, and thus establishes the $\mathbf{\Sigma}^0_3$-hardness of the latter two sets.

We regard each nonnegative integer n as a von Neumann ordinal, i.e. we think of each n as the set $\{0, ..., n-1\}$. If a function $\pi : n \to \omega$ is injective, then we call π a *finite partial* *permutation*. We use the notation $\mathrm{dom}(\pi)$ to refer to the map's domain $n = \{0, ..., n-1\}$ and $\mathrm{ran}(\pi)$ to refer to its range $\{\pi(0), ..., \pi(n-1)\}$. If $\sigma \in S_\infty$ or if σ is a finite partial permutation, then we say σ *extends* π if $\sigma \restriction \mathrm{dom}(\pi) = \pi$.

4.3. The Bounded Walk Lemma

In this section we develop the main technical lemma on which our proof is built. An intuitive explanation for the Bounded Walk lemma is as follows: Consider a conditionally convergent series as an abstract infinite collection $(v_k)_{k\in\omega}$ of vectors in $\mathbb{R}^d$ from which we may build finite paths. Let α and β be two points in $\mathbb{R}^d$. Suppose we have already chosen some finite subcollection $\{v_{\pi(0)}, v_{\pi(1)}, ..., v_{\pi(J)}\}$, $J \in \omega$, of vectors from $(v_k)_{k\in\omega}$ whose sum $\displaystyle\sum_{k=0}^{J} v_{\pi(k)}$ (here visualized as a path of vectors laid end-to-end) is very close to α. We wish to extend the path we have already built by choosing finitely many more vectors, from among those we have not already chosen, in such a way that the extended path will terminate very close to β, i.e. we wish to "walk from α to β."

We will show that if all the remaining vectors to choose from are sufficiently small (say less than $\frac{1}{3C_d}||\beta - \alpha||$ where C_d is some constant to be determined later), then it is possible to build a finite path which (1) extends the path we have already walked; (2) uses up all except arbitrarily small remaining vectors; (3) takes us arbitrarily close to β; and (4) does not wander arbitrarily far from the straight-line path connecting α and β. In addition we may (5) use up any particular vector we wish. (Note that conditions (1) and (2) allow us to repeat this "bounded walk" process between as many points as we like, as often as we like.)

Now we aim to establish such a lemma. Before we do so, we first recall the following classical result as stated in [47], which is attributed to Steinitz, and which asserts the existence of a very useful "bounded rearrangement constant" C_d in Euclidean space, now referred to as the Steinitz constant:

LEMMA 4.3 (Polygonal Confinement Theorem). *Let $d \geq 1$ be any integer. Then there exists a constant C_d which satisfies the following statement: Whenever $v_0, v_1, ..., v_m$ are vectors in*

$\mathbb{R}^d$ *which sum to* 0 *and satisfy* $||v_i|| \leq 1$ *for each* $i \leq m$*, then there is a finite permutation* $P \in S_m$ *with the property that*

$$\left|\left|v_0 + \sum_{i=1}^{j} v_{P(i)}\right|\right| \leq C_d$$

for every j*.*

The Polygonal Confinement Theorem is the basis for the remaining lemmas in this section.

LEMMA 4.4. *Let* $\alpha, v_1, ..., v_m$ *be vectors in* $\mathbb{R}^d$ *which sum to* $\beta \in \mathbb{R}^d$*, and let* C_d *be as in the Polygonal Confinement Theorem. Further suppose we have* $||v_i|| \leq \frac{1}{C_d}||\beta - \alpha||$ *for each* $i \leq m$*. Then there is a finite permutation* $P \in S_m$ *with the property that*

$$\left|\left|\sum_{i=1}^{j} v_{P(i)}\right|\right| \leq 2||\beta - \alpha||$$

for every j*.*

PROOF. Without loss of generality we may assume $\alpha = 0$, for if not, replace α with 0 and β with $\beta - \alpha$. We may also without loss of generality take $||\beta|| = C_d$, for if not, replace β with $\beta \cdot \dfrac{C_d}{||\beta||}$ and v_i with $v_i \cdot \dfrac{C_d}{||\beta||}$. In this case we have $||v_i|| \leq 1$ for each i.

Now let s be an integer sufficiently large so that $\frac{||\beta||}{s} \leq 1$, and set $v_{m+1} = v_{m+2} = ... = v_{m+s} = -\beta/s$. Then $\alpha, v_1, ..., v_m, v_{m+1}, ..., v_{m+s}$ are a collection of vectors which satisfy the hypotheses of the Polygonal Confinement Theorem, and hence there exists a permutation $P' \in S_{m+s}$ for which

$$\left|\left|\alpha + \sum_{i=1}^{j'} v_{P'(i)}\right|\right| = \left|\left|\sum_{i=1}^{j'} v_{P'(i)}\right|\right| \leq C_d$$

for every $j' \leq m + s$. Let $P \in S_m$ be the unique permutation which arranges $1, ..., m$ in the same order as P'.

Now let $j \leq m$ be arbitrary. Let $j' \geq j$ be the least integer for which $\{P(1), ..., P(j)\} \subseteq \{P'(1), ..., P'(j')\}$. Note that since P and P' arrange $1, ..., j$ in the same order, then for any

$i \le j'$, we must have either $(P')^{-1}(i) \in \{1, ..., j\}$ or $(P')^{-1}(i) \in \{m+1, ..., m+s\}$. Let $I = \{i \le j' : P^{-1}(i) \in \{m+1, ..., m+s\}\}$. Then we have:

$$\begin{aligned}
\left|\left|\sum_{i=1}^{j} v_{P(i)}\right|\right| &= \left|\left|\sum_{i=1}^{j'} v_{P'(i)} - \sum_{i\in I} v_i\right|\right| \\
&\le \left|\left|\sum_{i=1}^{j'} v_{P'(j)}\right|\right| + \sum_{i\in I} ||v_i|| \\
&\le C_d + \sum_{i\in I} \frac{||\beta||}{s} \\
&\le ||\beta|| + s \cdot \frac{||\beta||}{s} \\
&= 2||\beta||
\end{aligned}$$

as required. $\square$

Lemma 4.5. *Let σ be any finite partial permutation, and let $\sum_{k=0}^{\infty} v_k$ be a series of vectors. If $\pi \in S_\infty$ is any permutation for which $\sum_{k=0}^{\infty} v_{\pi(k)}$ converges, then there exists another permutation $\pi' \in S_\infty$ for which*

(1) π' extends σ, and

(2) $\sum_{k=0}^{\infty} v_{\pi'(k)} = \sum_{k=0}^{\infty} v_{\pi(k)}$.

Proof. This may be accomplished by simply finding a finitely supported permutation $\tau \in S_\infty$ for which $\tau \circ \pi \restriction \mathrm{dom}(\sigma) = \sigma$, and setting $\pi' = \tau \circ \pi$. $\square$

For the remainder of the chapter, given a conditionally convergent series $\sum_{k=0}^{\infty} v_k$ in $\mathbb{R}^d$, let $A(v_k) \subseteq \mathbb{R}^d$ denote the affine subspace promised in the statement of the Levy-Steinitz Theorem.

Lemma 4.6 (Bounded Walk Lemma). *Let* $\sum_{k=0}^{\infty} v_k$ *be a conditionally convergent series of vectors in* $\mathbb{R}^d$. *Let* $\epsilon > 0$ *and* $n \in \omega$ *be arbitrary. Let* $\alpha \in \mathbb{R}^d$, $\beta \in A(v_k)$, *and let* C_d *be as in the Polygonal Confinement Theorem. Suppose* π *is a finite partial permutation with* $\mathrm{dom}(\pi) = J+1 \in \omega$, *for which* $\left|\left|\alpha - \sum_{k=0}^{J} v_{\pi(k)}\right|\right| < \frac{1}{3}||\beta-\alpha||$, *and such that* $||v_k|| \leq \frac{1}{3C_d}||\beta-\alpha||$ *whenever* $k \notin \mathrm{ran}(\pi)$.

Then there exists a finite partial permutation σ *with* $\mathrm{dom}(\sigma) = I+1 \in \omega$ *which satisfies the following properties:*

(1) σ *extends* π;

(2) $||v_k|| \leq \frac{1}{C_d} \cdot \epsilon$ *whenever* $k \notin \mathrm{ran}(\sigma)$;

(3) $\left|\left|\beta - \sum_{k=0}^{I} v_{\sigma(k)}\right|\right| < \epsilon$;

(4) $\left|\left|\sum_{k=J+1}^{i} v_{\sigma(k)}\right|\right| \leq 6||\beta-\alpha||$ *whenever* $J+1 \leq i \leq I$; *and*

(5) $n \in \mathrm{ran}(\sigma)$.

Proof. Since $\beta \in A(v_k)$, there is $\tau \in S_\infty$ for which $\sum_{k=0}^{\infty} v_{\tau(k)} = \beta$. By applying Lemma 4.5, we may assume without loss of generality that τ extends π. Choose $I \in \omega$ to be so large that $\tau^{-1}(n) \leq I$, $\left|\left|\beta - \sum_{k=0}^{I} v_{\tau(k)}\right|\right| < \min(\epsilon, \frac{1}{3}||\beta-\alpha||)$, and that $||v_{\tau(k)}|| \leq \frac{1}{C_d} \cdot \epsilon$ for all $k > I$.

Set $\alpha_1 = \sum_{k=0}^{J} v_{\tau(k)} = \sum_{k=0}^{J} v_{\pi(k)}$ and set $\beta_1 = \sum_{k=0}^{I} v_{\tau(k)}$. Now notice that the images $\tau(J+1), ..., \tau(I)$ do not lie in the range of π, since τ is a bijection extending π and $\mathrm{dom}(\pi) = \{0, ..., J\}$. It follows that for each $i \in \{J+1, ..., I\}$ we have

$$\begin{aligned}
\frac{1}{C_d}||\beta_1-\alpha_1|| &= \frac{1}{C_d}||\beta-\alpha-(\beta-\beta_1)+(\alpha-\alpha_1)|| \\
&\geq \frac{1}{C_d}||||\beta-\alpha||-||\beta-\beta_1||-||\alpha-\alpha_1|||| \\
&\geq \frac{1}{C_d}||||\beta-\alpha||-\frac{1}{3}||\beta-\alpha||-\frac{1}{3}||\beta-\alpha|||| \\
&= \frac{1}{3C_d}||\beta-\alpha|| \\
&\geq ||v_i||.
\end{aligned}$$

Hence we may apply Lemma 4.4 to find a bijection $P : \{\tau(J+1), ..., \tau(I)\} \to \{\tau(J+1), ..., \tau(I)\}$ which satisfies

$$\left|\left|\sum_{k=J+1}^{i} v_{P(\tau(k))}\right|\right| \leq 2||\beta_1-\alpha_1||$$

whenever $J+1 \leq i \leq I$. If we define $\sigma : I+1 \to \omega$ by $\sigma(k) = \tau(k)$ for $k \leq J$ and $\sigma(k) = P(\tau(k))$ for $J < k \leq I$, then σ is a finite partial permutation with domain $I+1$ which clearly satisfies (1) above.

Note that if $k \notin \mathrm{ran}(\sigma)$, then $k \notin \{\sigma(0), ..., \sigma(J), \sigma(J+1), ..., \sigma(I)\} = \{\tau(0), ..., \tau(J), P(\tau(J+1)), ..., P(\tau(I)\} = \{\tau(0), ..., \tau(I)\}$. So $\tau^{-1}(k) > I$, and hence by our choice of I, we have $||v_k|| = ||v_{\tau(\tau^{-1}(k))}|| \leq \frac{1}{C_d} \cdot \epsilon$. Thus (2) is also satisfied.

Since P is a bijection, we have

$$\begin{aligned}
\left|\left|\beta-\sum_{k=0}^{I} v_{\sigma(k)}\right|\right| &= \left|\left|\beta-\left(\sum_{k=0}^{J} v_{\tau(k)}+\sum_{k=J+1}^{I} v_{P(\tau(k)}\right)\right|\right| \\
&= \left|\left|\beta-\sum_{k=0}^{I} v_{\tau(k)}\right|\right| < \epsilon,
\end{aligned}$$

so (3) is satisfied.

Lastly note that $||\beta_1-\alpha_1|| \leq ||\beta-\beta_1||+||\beta-\alpha||+||\alpha-\alpha_1|| \leq 3||\beta-\alpha||$. Hence, we have

$$\begin{aligned} \left|\left|\sum_{k=J+1}^{i} v_{\sigma(k)}\right|\right| &= \left|\left|\sum_{k=J+1}^{i} v_{P(\tau(k))}\right|\right| \\ &\leq 2||\beta_1 - \alpha_1|| \\ &\leq 6||\beta - \alpha|| \end{aligned}$$

whenever $J+1 \leq i \leq I$. So (4) is also satisfied. (5) holds simply because $\tau^{-1}(n) \leq I = \mathrm{dom}(\tau)$, and hence $n \in \mathrm{ran}(\tau) = \mathrm{ran}(\sigma)$. So the lemma is proved. $\square$

REMARK 4.7. For simplicity, from now on when we apply the Bounded Walk lemma, we will say that we use it to *walk from α to β*, where α and β are as in the statement of the theorem.

4.4. Proof of Propositions 4.1 and 4.2

First let us make a few introductory observations. We describe our sets of interest using logical notation, with the assumption in place that all quantified variables range over ω. Notice that by Cauchy's criterion for convergence, the following equivalence holds:

$$\pi \in \mathcal{D} \;\leftrightarrow\; \exists m\, \forall n\, \exists i\, \exists j\ \left[i, j \geq n \wedge \left|\left|\sum_{k=i}^{j} v_{\pi(k)}\right|\right| \geq \tfrac{1}{m}\right].$$

Since the latter predicate is an open condition in S_∞, a count of quantifiers verifies that $\mathcal{D}$ indeed lies in $\mathbf{\Sigma}_3^0(S_\infty)$. Now fix any $m \geq 1$, and consider the set $\mathcal{D}_m$ defined by the following rule:

$$\pi \in \mathcal{D}_m \;\leftrightarrow\; \forall n\, \exists i\, \exists j\ \left[i, j \geq n \wedge \left|\left|\sum_{k=i}^{j} v_{\pi(k)}\right|\right| \geq \tfrac{1}{m}\right].$$

Then $\mathcal{D}_m$ is a nonempty $\mathbf{\Pi}_2^0$ (G_δ)-subset of $\mathcal{D}$ which is invariant under multiplication by finitely supported permutations, and hence dense in S_∞. This shows that $\mathcal{D}$ is a comeager set. The complement $S_\infty \backslash \mathcal{D}$ is also nonempty and invariant under finitely supported permutations, and hence dense as well.

Define a map $\phi : 2^\omega \to S_\infty$ by the rule

$$[\phi(x)](2k) = 2k \text{ and } [\phi(x)](2k+1) = 2k+1 \text{ if } x(k) = 0; \text{ and}$$
$$[\phi(x)](2k) = 2k+1 \text{ and } [\phi(x)](2k+1) = 2k \text{ if } x(k) = 1;$$

for $x \in 2^\omega$ and $k \in \omega$. Let $T = \operatorname{ran}(\phi) \subseteq S_\infty$. The map ϕ is injective and hence T is uncountable. The permutations in T act only by transposing consecutive integers, and as a consequence it is simple to check that both $\mathcal{D}$ and $S_\infty \backslash \mathcal{D}$ are invariant under multiplication on the left by elements of T. Thus both sets $\mathcal{D}$ and $S_\infty \backslash \mathcal{D}$ are uncountable dense, i.e. in some sense they are "large" nontrivial sets in S_∞, as promised in the introduction. The reader may consult [5] for similar observations about some other sets in S_∞ which are closely related to our $\mathcal{D}$ and $\mathcal{DP}$.

Next define a set $\mathcal{I} \subseteq S_\infty$ by the rule

$$\pi \in \mathcal{I} \;\leftrightarrow\; \exists m\ \forall n\ \exists i\ \left[i \geq n \wedge \left\| \sum_{k=0}^{i} v_{\pi(k)} \right\| \leq m \right].$$

Then $\mathcal{I}$ is a $\mathbf{\Sigma}^0_3$ set, which consists exactly of those permutations whose corresponding series rearrangements $\sum_{k=0}^{\infty} v_{\pi(k)}$ do not diverge to infinity. Since $\mathcal{DP} = \mathcal{D} \cap \mathcal{I}$, so too we have $\mathcal{DP} \in \mathbf{\Sigma}^0_3(S_\infty)$.

Proof of Propositions 4.1 and 4.2. In light of our comments above, it suffices to show that $\mathcal{D}$ and $\mathcal{DP}$ are $\mathbf{\Sigma}^0_3$-hard. Recall that the set $\mathcal{C} = \{x \in \omega^\omega : \lim_{n\to\infty} x(n) = \infty\}$ is known to be $\mathbf{\Pi}^0_3$-complete, and the complement $\omega^\omega \backslash \mathcal{C}$ is $\mathbf{\Sigma}^0_3$-complete. We will build a function $f : \omega^\omega \to S_\infty$ that will be a continuous reduction from $\omega^\omega \backslash \mathcal{C}$ to both $\mathcal{D}$ and $\mathcal{DP}$ simultaneously. That is, both of the following will hold:

$$x \in \omega^\omega \backslash \mathcal{C} \leftrightarrow f(x) \in \mathcal{D},$$
$$x \in \omega^\omega \backslash \mathcal{C} \leftrightarrow f(x) \in \mathcal{DP}.$$

Fix an arbitrary $x \in \omega^\omega$. Let $\boldsymbol{v} = \sum_{k=0}^{\infty} v_k$. We will recursively construct a sequence of integers $(J_n)_{n\geq -1}$ and a sequence of finite partial permutations $(\pi_n)_{n\geq -1}$, each with domain $\{0, ..., J_n\}$, which satisfy the following seven conditions whenever $n \geq 0$:

(I) π_n extends π_{n-1};

(II) $n \in \{\pi_n(0), ..., \pi_n(J_n)\}$;

(III) the definitions of $\pi_n(J_{n-1}+1)$, ..., $\pi_n(J_n)$ depend only on the values of $x(n)$ and $x(n+1)$;

(IV) $\left|\left|\boldsymbol{v} - \sum_{k=0}^{J_n} v_{\pi_n(k)}\right|\right| < \dfrac{1}{x(n+1)+1}$;

(V) $\left|\left|\sum_{k=J_{n-1}+1}^{j} v_{\pi_n(k)}\right|\right| \leq 36 \cdot \dfrac{1}{x(n)+1}$ for every $j \in \{J_{n-1}+1, ..., J_n\}$;

(VI) there exist $i, j \in \{J_{n-1}, ..., J_n\}$ for which $\left|\left|\sum_{k=i}^{j} v_{\pi_n(k)}\right|\right| > \frac{1}{x(n)+1}$; and

(VII) $||v_k|| \leq \frac{1}{C_d} \cdot \frac{1}{x(n+1)+1}$ for all $k \notin \mathrm{ran}(\pi_n)$.

After this construction is finished, we will let π be the unique permutation which extends all the π_n's, and set $f(x) = \pi$. Conditions (I) and (II) will guarantee that π is indeed a permutation, while (III) will guarantee that the map f is continuous. Conditions (IV) and (V) will ensure that if $x \in \mathcal{C}$, then $\sum_{k=0}^{\infty} v_{\pi(k)}$ will converge to $\boldsymbol{v}$, while condition (VI) will guarantee that if $x \notin \mathcal{C}$, then $\sum_{k=0}^{\infty} v_{\pi(k)}$ will diverge properly. (Condition (VII) is just a technical requirement to facilitate our recursive definition.)

We will now proceed with our construction. Here for the sake of convenience our base case will be $n = -1$. Let $J_{-1} \geq 0$ be so large that $||\boldsymbol{v} - \sum_{k=0}^{J_{-1}} v_k|| < \frac{1}{x(0)+1}$, and that $||v_k|| \leq \frac{1}{C_d} \cdot \frac{1}{x(0)+1}$ for all $k > J_{-1}$. Let $\pi_{-1} : J_0 + 1 \to J_0 + 1$ be the identity permutation. Note that π_{-1} and J_{-1} trivially satisfy (IV) and (VII) above; this will be enough to facilitate our induction.

Now we assume that J_i and π_i are defined for all $i < n$, and satisfy at least (IV) and (VII), and we proceed with the inductive step of defining J_n and π_n. As we go we will verify that J_n and π_n really do satisfy all of conditions (I)-(VII).

By the Levy-Steinitz theorem, there is a non-trivial affine subspace $A(v_k) \subseteq \mathbb{R}^d$ of points β such that some rearrangement of $\sum_{k=0}^{\infty} v_k$ converges to β. In particular, there is at least a line of such points. So we may choose some $\beta \in A(v_k)$ for which $||\beta - \boldsymbol{v}|| = 3 \cdot \frac{1}{x(n)+1}$. Now we will define π_n and J_n by applying the Bounded Walk lemma twice: first, we will use the lemma to "walk out" to a point near β, and then we will use the lemma to "walk back in" to a point near $\boldsymbol{v}$.

To "walk out": observe that by condition (VII) of our inductive hypothesis, we have $||v_k|| \leq \frac{1}{C_d} \cdot \frac{1}{x(n)+1} \leq \frac{1}{3C_d}||\beta - \boldsymbol{v}||$ for all $k \notin \operatorname{ran}(\pi_{n-1})$, and by condition (IV) of our inductive hypothesis, we have $\left|\left|\boldsymbol{v} - \sum_{k=0}^{J_{n-1}} v_{\pi_{n-1}(k)}\right|\right| < \frac{1}{x(n)+1} = \frac{1}{3}||\beta - \boldsymbol{v}||$. So we may apply the Bounded Walk lemma to walk from $\boldsymbol{v}$ to β, extending the finite partial permutation π_{n-1} and with $\epsilon = \frac{1}{x(n)+1}$. Thus we obtain an index $I > J_{n-1}$ and a finite partial permutation $\sigma : I + 1 \to I + 1$ which satisfies properties (1)–(5) of the lemma. In particular, condition (3) ensures that we have

$$\begin{aligned}
\left|\left|\sum_{k=J_{n-1}+1}^{I} v_{\sigma(k)}\right|\right| &= \left|\left|\beta - \boldsymbol{v} - \beta + \sum_{k=0}^{I} v_{\sigma(k)} + \boldsymbol{v} - \sum_{k=0}^{J_{n-1}} v_{\sigma(k)}\right|\right| \\
&\geq ||\beta - \boldsymbol{v}|| - \left|\left|\beta - \sum_{k=0}^{I} v_{\sigma(k)}\right|\right| - \left|\left|\boldsymbol{v} - \sum_{k=0}^{J_{n-1}} v_{\sigma(k)}\right|\right| \\
&> 3 \cdot \frac{1}{x(n)+1} - \frac{1}{x(n)+1} - \frac{1}{x(n)+1} \\
&= \frac{1}{x(n)+1}
\end{aligned}$$

while condition (4) guarantees that $\left|\left|\sum_{k=J_{n-1}+1}^{i} v_{\sigma(k)}\right|\right| \leq 6\,||\beta - \boldsymbol{v}|| = 18 \cdot \frac{1}{x(n)+1}$ whenever $J + 1 \leq i \leq I$.

Next we "walk back in." By property (2) in our previous application of the Bounded Walk lemma, we have $||v_k|| \leq \frac{1}{C_d} \cdot \frac{1}{x(n)+1} = \frac{1}{3C_d}||\boldsymbol{v}-\beta||$ whenever $k \notin \operatorname{ran}(\sigma)$, and by property (3) we have $\left|\left|\beta - \sum_{k=0}^{I} v_{\sigma(k)}\right|\right| < \frac{1}{x(n)+1} = \frac{1}{3}||\boldsymbol{v}-\beta||$. So we may apply the Bounded Walk lemma to walk from β to $\boldsymbol{v}$, extending the finite partial permutation σ, with $\epsilon = \frac{1}{x(n+1)+1}$. Then we obtain an index $J_n > I$ and a finite partial permutation $\pi_n : J_n + 1 \to J_n + 1$ which again satisfies properties (1)–(5). By the previous inequality, and applying condition (4) for π_n, we see that

$$
\begin{aligned}
\left|\left|\sum_{k=J_{n-1}+1}^{i} v_{\pi_n(k)}\right|\right| &\leq \left|\left|\sum_{k=J_{n-1}+1}^{I} v_{\pi_n(k)}\right|\right| + \left|\left|\sum_{k=I+1}^{i} v_{\pi_n(k)}\right|\right| \\
&\leq \left|\left|\sum_{k=J_{n-1}+1}^{I} v_{\sigma(k)}\right|\right| + 6\,||\beta - \boldsymbol{v}|| \\
&\leq 18 \cdot \frac{1}{x(n)+1} + 18 \cdot \frac{1}{x(n)+1} \\
&= 36 \cdot \frac{1}{x(n)+1}
\end{aligned}
$$

whenever $I + 1 \leq i \leq J_n$. Thus we have shown that (V) holds for π_n. (I), (III), and (IV) obviously hold from our definition of π_n, and (II) holds if we utilize condition (5) in either of our two applications of the Bounded Walk lemma to ensure that $n \in \operatorname{ran}(\pi_n)$. We have shown that (VI) holds if we take $i = J_{n-1}$ and $j = I$, and (VII) follows from condition (2) in our second application of the Bounded Walk lemma. So our construction is complete and we may let $\pi \in S_\infty$ be the unique permutation which extends all of the π_n's.

Define the map $f : \omega^\omega \to S_\infty$ by $f(x) = \pi$, where π is as we have constructed above. The function f, as a map between the Polish space ω^ω and its Polish subspace S_∞, is continuous by condition (III). We claim that f is in fact the continuous reduction we desire.

To see this, suppose $x \in \mathcal{C}$, so $\lim_{n\to\infty} x(n) = \infty$ and hence $\lim_{n\to\infty} \frac{1}{x(n)+1} = 0$. For any $i \in \omega$, let n_i be the greatest integer for which $J_{n_i-1} < i \leq J_{n_i}$. Then by (IV) and (V) we have

$$\begin{aligned}\left\|\boldsymbol{v}-\sum_{k=0}^{i} v_{\pi(k)}\right\| &\leq \left\|\boldsymbol{v}-\sum_{k=0}^{J_{n-1}} v_{\pi_{n-1}(k)}\right\| + \left\|\sum_{k=J_{n-1}+1}^{i} v_{\pi_n(k)}\right\| \\ &\leq \frac{1}{x(n_i)+1} + 36\cdot\frac{1}{x(n_i)+1} \\ &= 37\cdot\frac{1}{x(n_i)+1}.\end{aligned}$$

Now taking the limit as $i \to \infty$ (and as $n_i \to \infty$) we see that $\sum_{k=0}^{\infty} v_{\pi(k)}$ converges to $\boldsymbol{v}$. Hence $f(x) \in S_\infty \backslash \mathcal{D}$ and $f(x) \in S_\infty \backslash \mathcal{DP}$.

On the other hand, suppose $x \in \omega^\omega \backslash \mathcal{C}$. Then the sequence $(x(n))$ is cofinally bounded, i.e. there is an $M < \infty$ such that $x(n) < M$ infinitely often. Hence $\frac{1}{x(n)+1} > \frac{1}{M+1}$ infinitely often. It follows from (VI) that there are infinitely many blocks $i, ..., j$ of integers for which $\left\|\sum_{k=i}^{j} v_{\pi(k)}\right\| > \frac{1}{x(n)+1} > \frac{1}{M+1}$, and hence $\sum_{k=0}^{\infty} v_{\pi(k)}$ diverges by the Cauchy criterion. In addition, we have already demonstrated that for any n_i depending on i as above, we have

$$\left\|\boldsymbol{v}-\sum_{k=0}^{i} v_{\pi(k)}\right\| \leq 37\cdot\tfrac{1}{x(n_i)+1} \leq 37.$$

This implies that all partial sums of the rearranged series are bounded, and so the series must diverge properly. Thus in this case we have $f(x) \in \mathcal{D}$ and $f(x) \in \mathcal{DP}$. So f is the reduction we seek, and $\mathcal{D}$ and $\mathcal{DP}$ are $\mathbf{\Sigma}_3^0$-complete. $\square$

CHAPTER 5

HAAR NULL SETS AND OPENLY HAAR NULL SETS

5.1. Introduction

Let G be a Polish group. When should a subset of G be considered "small"? If we wish to define a suitable family $\mathcal{F}$ of subsets of G which are to be considered small, then, emulating Hunt, Sauer, and Yorke in [29], we ask that $\mathcal{F}$ satisfy the following collection of test properties:

(I) If $A \in \mathcal{F}$ and $B \subseteq A$, then $B \in \mathcal{F}$, i.e. a subset of a small set is small.

(II) If $(A_n)_{n\in\omega}$ is a sequence of sets, where $A_n \in \mathcal{F}$, then $\bigcup_{n\in\omega} A_n \in \mathcal{F}$, i.e. a countable union of small sets is small.

(III) If $A \in \mathcal{F}$, then $G \backslash A$ is dense in G, i.e. a large set is dense.

(IV) If $A \in \mathcal{F}$, then $gAh \in \mathcal{F}$, i.e., a translate of a small set is small.

Conditions (I), (II), and (III) above will guarantee that $\mathcal{F}$ is a σ-ideal of subsets of G, while (IV) demands that $\mathcal{F}$ in addition be translation-invariant. Every group admits a natural translation-invariant σ-ideal: the family of meager subsets of G, which are defined purely topologically.

The astounding classical Theorem 2.28 of Haar/Weil/Cartan asserts that every locally compact group G carries a left Haar measure μ, that is, a regular Borel measure which is invariant under left translations. Moreover this measure is unique up to scalar multiplication, so any other left-translation-invariant regular Borel measure on G is simply a multiple of μ. Of course G also admits a unique-up-to-scalars right Haar measure as well. The left and right Haar measures need not in general be the same, but they do share the same family of measure zero sets in G. This family comprises a translation-invariant σ-ideal. Thus the algebraic and topological structure of G uniquely determine a measure-theoretic analogue to the meager sets: the family of Haar measure zero sets.

Now we pose the question for groups G which are not locally compact. Can we find a suitable measure-theoretic smallness notion in G, which satisfies properties (I) through (IV)? If μ is any nonzero Borel measure on G, then the collection of μ-measure zero sets certainly satisfies conditions (I)-(III), so the difficult property to achieve is (IV). Unfortunately, if G is not locally compact, then its topological nature precludes us from finding a single measure on G which induces a translation-invariant zero-ideal.

A Polish locally compact group is "small" in the (unrelated) sense that it is σ-compact. On the other hand if G is not locally compact, then the compact sets are meager in G, and hence the Baire category theorem implies that G cannot be written as a countable union of compact sets, so G is "large." The following argument, attributed to Weil but presented here as in [47], indicates that this is precisely the distinction which prevents us from finding a measure μ on a large group G which meets our desires.

THEOREM 5.1. *Suppose μ is a non-zero σ-finite regular measure μ on G whose zero sets comprise a translation-invariant σ-ideal. Then G must be locally compact.*

PROOF. (*Weil*) Since μ is regular, there is a σ-compact set F for which F has full μ-measure in G. Let $g \in G$ be arbitrary; since the zero-sets are preserved under translations, $\mu(gF) > 0$. But then $gF \cap F \neq \emptyset$ and hence $g \in FF^{-1}$. So $G = FF^{-1}$ is σ-compact, and hence locally compact. $\square$

In the absence of a single canonical measure generating a translation-invariant σ-ideal in a non-locally compact group, Christensen has defined the following measure-theoretic smallness notion as a potential substitute.

DEFINITION 5.2 (Christensen 1972 [7]). Let G be a Polish group and let A be a universally measurable subset of G. Say that A is *Haar null* if there exists a Borel probability measure μ on G (not unique) such that $\mu(gAh) = 0$ for all $g, h \in G$.

If we extend the definition of Haar null sets to include all subsets of such a universally measurable set, then the resulting family, which we may call $\mathcal{HN}(G)$ or just $\mathcal{HN}$ if no confusion may arise, constitutes a translation-invariant σ-ideal in any Polish group G. Moreover, if G is locally compact, then $\mathcal{HN}(G)$ is exactly the Haar measure zero sets, so the notion is a true generalization of this well-studied phenomenon.

Notice that in case G is an abelian group, we may write that a universally measurable subset A in G is Haar null if there is an appropriate measure μ such that $\mu(g + A) = 0$ for all $g \in G$ (where we denote the group operation additively instead of multiplicatively). This is how Christensen first formulated the notion in [7], and also Hunt, Sauer, and Yorke, who rediscovered it in [29]. For abelian groups the situation seems greatly simplified, and most positive results in the literature have come in regard to such groups. The definition of Haar null sets was extended to the non-abelian case by Tøpsoe and Hoffman-Jørgenson in [14] and, independently, by Mycielski in [40]. Haar null sets are frequently called *shy* sets in the literature. Following Hunt, Sauer, and Yorke, we will call a set $A \subseteq G$ *prevalent* if $G \backslash A$ is Haar null.

The phenomenon of Haar null sets is well-understood in abelian groups, even in non-locally compact ones. In large non-abelian groups, however, the nature of the ideal remains somewhat mysterious. Any mathematician who has studied Baire category or Haar measure may easily call to mind a slew of natural questions to ask about the family of Haar null sets in this general setting. Here I will mention just a few that we will try to address:

Questions.

(1) Is every compact set in a non-locally compact group Haar null?

(2) May every uncountable group be written as the disjoint union of a Haar null set and a meager set? (Darji 2011 [9])

(3) Is every Haar null set contained in a Haar null G_δ set? (Mycielski 1994 [40])

Question (2) is motivated by the classical fact that every uncountable Polish locally compact group may be written as the disjoint union of a Haar measure zero set and a meager set (see Theorem 2.30). Note that since countable dense sets are easily Haar null in an uncountable group, a positive answer to (3) would imply a positive answer to (2).

Solecki has also defined the following closely related family of sets in [51]:

Definition 5.3 (Solecki 2001 [51]). Let G be a Polish group and let A be a universally measurable subset of G. Say that A is *openly Haar null* if there exists a Borel probability measure μ on G such that for all $\epsilon > 0$, there is an open set U containing A with $\mu(gUh) < \epsilon$ for all $g, h \in G$.

The openly Haar null sets form a translation-invariant σ-ideal which is contained in the family of Haar null sets. In abelian locally compact groups, the Haar measures guarantee that every Haar null set is openly Haar null, so the ideals coincide in what is perhaps the simplest class of groups under our consideration. Can this fact be true in general?

Questions. (4) Is every Haar null set also openly Haar null?

In this chapter we show that the answer to (4) is no in general, and that there exist groups both locally compact and non-locally compact where the ideals differ. We show that the answer to (1) is yes in the permutation group S_∞ and in any countable product group $G = \prod_{i\in\omega} G_i$ where the groups G_i are locally compact non-compact Polish groups, which extends the domain of a result of Dougherty for TSI groups (Proposition 12 in [11]). In fact we show that the compact sets are openly Haar null in the latter group, and hence each σ-compact set is contained in a Haar null G_δ. An immediate consequence is that such groups always decompose into the disjoint union of a Haar null set and a meager set, which provides a partial positive answer to (2). In start contrast, however, we use a dynamical argument to show that even singleton sets are not openly Haar null in the homeomorphism group $\text{Homeo}^+[0,1]$.

5.2. Known Facts About Haar Null Sets

The purpose of this section is to establish the fundamental facts which justify our interest in the family of Haar null sets $\mathcal{HN}(G)$, namely that: (1) $\mathcal{HN}(G)$ is an ideal in any metric group G; (2) $\mathcal{HN}(G)$ is a σ-ideal if G is Polish; and (3) $\mathcal{HN}(G)$ is the class of Haar measure zero sets if G is locally compact.

First we recall the definition of convolution of measures.

Definition 5.4. Let G be a group and μ and ν two Borel probability measures on G. Define the *convolution* $\mu * \nu$ of μ and ν by the rule

$$\mu * \nu(A) = \mu \times \nu(\{(x, y) \in G^2 : xy \in A\})$$

for every Borel set $A \subseteq G$.

It is not hard to check that $\mu * \nu$ is a countably additive Borel probability measure on G. We also wish to define a notion of an infinite convolution of measures.

Definition 5.5. Let G be a group and $(\mu_i)_{i\in\omega}$ a sequence of Borel probability measures on G. Then we define their convolution $\mathop{\Huge *}_{i=0}^{\infty} \mu_i$ by the rule

$$\mathop{\Huge *}_{i=0}^{\infty} \mu_i(A) = \left(\prod_{i=0}^{\infty} \mu_i\right) (\{(x_i) \in G^\omega : x_0 x_1 ... \text{ converges and } x_0 x_1 ... \in A\})$$

for every Borel set $A \subseteq G$.

It is once again not hard to check that $\mathop{\Huge *}_{i=0}^{\infty} \mu_i$ is a countably additive Borel measure. The potential difficulty with this notion is that $\mathop{\Huge *}_{i=0}^{\infty} \mu_i$ need not in general be a probability measure nor even non-zero. For instance, if G is the multiplicative group $\mathbb{R}^+$ of positive reals and μ_i is Lebesgue measure restricted to $\mathbb{R}^+$, then $\left(\prod_{i=0}^{\infty} \mu_i\right)$-almost every infinite product diverges and hence $\mathop{\Huge *}_{i=0}^{\infty} \mu_i(G) = 0$.

Now we are ready to establish that finite unions of Haar null sets are Haar null.

Lemma 5.6. *Let G be a metric group. Let $A \subseteq G$ be a Haar null set as witnessed by a probability measure μ. Let ν be an arbitrary probability measure. Then $\mu * \nu(gAh) = \nu * \mu(gAh) = 0$ for every $g, h \in G$.*

PROOF. Let $g, h \in G$ be arbitrary. Note that for any fixed $y \in G$ we have $\{(x,y) \in G^2 : xy \in gAh\} = gAhy^{-1}$, and for any fixed $x \in G$ we have $\{(x,y) \in G^2 : xy \in gAh\} = x^{-1}gAh$. Hence by Fubini's theorem, we have

$$\mu * \nu(gAh) = \mu \times \nu(\{(x,y) \in G^2 : xy \in gAh\}) = \int_G \mu(gAhy^{-1}) d\nu(y)$$

and

$$\nu * \mu(gAh) = \nu \times \mu(\{(x,y) \in G^2 : xy \in gAh\}) = \int_G \mu(x^{-1}gAh) d\nu(y).$$

Since both integrands above are constantly 0, we have $\mu * \nu(gAh) = \nu * \mu(gAh) = 0$. □

COROLLARY 5.7. *Let G be a metric group. Then $\mathcal{HN}(G)$ is closed under finite unions.*

PROOF. If A and B are Haar null sets in G as witnessed by μ and ν respectively, then the convolution $\mu * \nu$ gives $\mu * \nu(g(A \cup B)h) \leq \mu * \nu(gAh) + \mu * \nu(gBh) = 0$ for every $g, h \in G$ by Lemma 5.6. □

Now we wish to establish that if G is in addition separable, that is, if G is a Polish group, then $\mathcal{HN}(G)$ is actually closed under countable unions as well as finite ones. This result is not new. Christensen gives a proof in the case where G is abelian in [7], and Hunt, Sauer, and Yorke prove it for topological vector spaces in [29]. Proofs of this fact for general Polish groups appear in the papers [14] by Topsøe and Hoffman-Jørgenson and [40] by Mycielski. We wish to point out that in the latter paper, there appears to be an error in the proof of this fact (Theorem 3 in [40]). The error can be corrected to yield the same conclusion, but it is substantive enough to warrant a change in the proof's main construction. For the sake of clarity we seek to describe and correct the mistake here, especially because [40] is so frequently cited as a fundamental paper in the general theory of Haar null sets.

We paraphrase the argument of [40] as follows: Let G be a Polish group with compatible metric d. Let $(A_i)_{i\in\omega}$ be a sequence of Haar null sets in G. For each i, find a measure μ_i

which witnesses that A_i is a Haar null set, and which is supported on a compact set K_i containing the identity e with d-diameter less than 2^{-i} (Lemma 5.9 below guarantees that this is possible). Then $\prod_{i=0}^{\infty} \mu_i$-almost every sequence $(g_i)_{i\in\omega}$ has $g_i \in K_i$ and hence $d(e, g_i) < 2^{-i}$ for every i. Then the sequence of partial products $(g_0...g_i)_{i\in\omega}$ forms a Cauchy sequence, in which case the infinite product $g_0g_1...$ converges. Then the infinite convolution $\mathop{\ast}_{i=0}^{\infty} \mu_i$ gives measure 1 to G, and hence becomes a probability measure witnessing that $\bigcup_{i=0}^{\infty} A_i$ is Haar null.

The problem with the above argument lies in the conclusion that if $d(e, g_i) < 2^{-i}$ for every i, then $d(g_0...g_{i-1}, g_0...g_{i-1}g_i) < 2^{-i}$ for every i, and hence the sequence of partial products $(g_0...g_i)$ forms a Cauchy sequence. This conclusion would be true if the metric d in question were both complete and left-invariant. (Groups which admit such a metric are known as *CLI groups*.) Although every Polish group admits a complete metric by definition and a left-invariant metric by the Birkhoff-Kakutani Theorem 2.15, there are many well-known groups which do not admit a simultaneously complete and left-invariant metric. If the metric d is left-invariant then the sequence of partial products $(g_0...g_i)_{i\in\omega}$ will indeed be Cauchy, but need not converge; and if d is complete but not left-invariant then there is no reason to think that the partial products $(g_0, ..., g_i)$ form a Cauchy sequence at all.

For a simple concrete example, take $G = S_\infty$, which admits no compatible complete left-invariant metric. Let d be any metric generating the topology on S_∞ (complete or left-invariant or otherwise). Let $(\delta_i)_{i\in\omega}$ be a sequence of positive reals converging to 0 (for instance we could take $\delta_i = 2^{-i}$ as in the above comments). For each $i \in \omega$, let k_i be so large that the basic open set $U_i = \{\pi \in S_\infty : \pi \restriction (k_i - 1) = e \restriction (k_i - 1)\}$ is a subset of the open ball $B_d(e, \delta_i)$. Assume without loss of generality that the sequence (k_i) is increasing. Now for each $i \in \omega$ let g_i be the transposition (k_i, k_{i+1}). By definition $g_i \in U_i \subseteq B_d(e, \delta_i)$ for each i, so $d(e, g_i) < \delta_i$. But $g_0g_1...g_i$ is the cyclic permutation $(k_0, k_1, ..., k_i, k_{i+1})$, and hence $(g_0g_1...g_i)^{-1}(k_0) = k_{i+1}$ for each i. Since (δ_i) converges to 0, (k_{i+1}) diverges to infinity and hence the sequence of partial products $(g_0g_1...g_i)_{i\in\omega}$ diverges.

Such an example of a divergent product in fact exists for every sequence (δ_i) in every group which is not CLI, as the next proposition establishes.

PROPOSITION 5.8. *Let G be a topological group. Then the following are equivalent.*

(1) There exists a metric d compatible with the topology of G, and a sequence $(\delta_i)_{i\in\omega}$, with the property that if $(x_i) \in G^\omega$ is a sequence such that $d(e, x_i) < \delta_i$ for every $i \in \omega$, then the infinite product $x_0x_1...$ converges.

(2) G is a CLI group.

PROOF. From our previous comments, (2) implies (1) by taking d to be a complete left-invariant metric and $(\delta_i) = (2^{-i})$. To see that (1) implies (2), suppose that G has such a metric d and such a sequence (δ_i). By the Birkhoff-Kakutani Theorem 2.15, there is also a left-invariant metric ρ compatible with the topology of G. We will show ρ is complete.

Let $(y_i)_{i\in\omega}$ be a ρ-Cauchy sequence. For each i let $\epsilon_i > 0$ be so small that $B_\rho(e, \epsilon_i) \subseteq B_d(e, \delta_i)$. Since (y_i) is ρ-Cauchy, we may pass to a subsequence (y_{i_k}) with the property that $\rho(y_{i_k}, y_{i_{k+1}}) < \epsilon_k$ for each $k \in \omega$. Set $x_k = y_{i_k}^{-1}y_{i_{k+1}}$ for each $k \in \omega$. Then by the left-invariance of ρ, we have $\rho(e, x_k) = \rho(e, y_{i_k}^{-1}y_{i_{k+1}}) = \rho(y_{i_k}, y_{i_{k+1}}) < \epsilon_k$ for each k, and hence $d(e, x_k) < \delta_k$ for k by our choice of ϵ_k. Thus the infinite product $x_0x_1...$ converges by our hypothesis. But then $y_0x_0x_1... = \lim_{k\to\infty} y_0x_0x_1...x_{k-1} = \lim_{k\to\infty} y_{i_k}$, so (y_{i_k}) converges and hence (y_i) converges. This shows ρ is a complete left-invariant metric on G. □

We should comment that the original proof given by Topsøe and Hoffman-Jørgenson in [14] that a countable union of Haar null sets is Haar null in any Polish group is to the best of our knowledge sound. Their proof invokes the topological and algebraic properties of the space $\mathcal{M}(G)$ of Borel probability measures on G endowed with the topology of weak convergence. This topology may be metrized by the Levy-Prokhorov metric, which is complete and generates a separable topology if G is Polish. Moreover the convolution operation $*$ on $\mathcal{M}(G)$ is associative with identity element δ_e, the Dirac measure at $e \in G$, as well as separately continuous with respect to the topology on $\mathcal{M}(G)$. So $\mathcal{M}(G)$ forms a Polish topologized semigroup. Topsøe and Hoffman-Jørgenson then note that given a sequence (A_i) of Haar null sets, it is possible to choose a sequence of witnessing measures (μ_i) which converge to δ_e so quickly that the finite convolutions $(\mu_0 * ... * \mu_i)_{i\in\omega}$ form a Cauchy sequence in $\mathcal{M}(G)$; then the limit of this sequence is the desired probability measure $\mu = \mathop{\huge *}_{i=0}^{\infty} \mu_i$, for which $\mu\left(g\left(\bigcup_{i=0}^{\infty} A_i\right)h\right) = 0$ for all $g, h \in G$.

The proof outlined above is "technological" in the sense that it relies upon the Polish semigroup structure of $\mathcal{M}(G)$. Given that the definition of Haar null sets is so elementary, we believe there is merit in presenting a direct proof that $\mathcal{HN}(G)$ is closed under countable unions when G is Polish, which does not rely upon the metrizability of $\mathcal{M}(G)$. We give such an argument in Proposition 5.11, which is modeled after Mycielski's argument in [40], but with the necessary repair provided by Lemma 5.10.

Lemma 5.9. *Let G be a Polish group and $A \subseteq G$ a universally measurable set. Let $\delta > 0$ be arbitrary. Then A is Haar null if and only if there exists a compactly supported Borel probability measure μ on G such that $\mu(gAh) = 0$ for every $g, h \in G$, $e \in \operatorname{supp}\mu$, and $\operatorname{diam}\operatorname{supp}\mu < \delta$.*

Proof. If such a compactly supported measure exists, then A is Haar null. Conversely, suppose A is Haar null as witnessed by a measure Borel probability measure μ on G. Let $x \in \operatorname{supp}\mu$ be arbitrary. Define a Borel probability measure μ' on G by setting $\mu'(A) = \mu(x^{-1}A)$ for each Borel set $A \subseteq G$. Then $\mu'(gAh) = \mu(x^{-1}gAh) = 0$ for every $g, h \in G$ and $e \in \operatorname{supp}\mu'$.

Let U be the open ball about e of radius $\frac{\delta}{3}$. Since $e \in \operatorname{supp} U$, $\mu'(U) > 0$. Since G is Polish, μ' is regular by Lemma 2.24. Hence there exists a compact subset $K \subseteq U$ with $\mu(K) > 0$. Define a Borel measure ν on G by setting $\nu(A) = \dfrac{\mu'(A \cap K)}{\mu'(K)}$ for every Borel set A. Then ν is a probability measure for which $\nu(gAh) = 0$ for every $g, h \in G$, $e \in \operatorname{supp}\nu$, and $\operatorname{diam}\operatorname{supp}\nu = \operatorname{diam} K \leq \operatorname{diam} U < \delta$.

$\square$

Lemma 5.10. *Let (G, d) be a metric group and let $\beta > 0$ be arbitrary. For each $g \in G$ define*

$$\alpha_g = \sup\{\alpha : gB_d(e, \alpha) \subseteq B_d(g, \beta)\}.$$

If K is any compact subset of G, then $\inf\{\alpha_g : g \in K\} > 0$.

PROOF. Note that since multiplication on the left by g is continuous, $\alpha_g > 0$ for every $g \in G$. So if $K \subseteq G$ is compact, we have $\inf\{\alpha_g : g \in K\} \geq 0$. Thus we need only show that the infimum is not exactly 0.

Suppose toward a contradiction that $\inf\{\alpha_g : g \in K\} = 0$. Then there exists a sequence $(g_n)_{n\in\omega}$ of points in K for which $(\alpha_{g_n})_{n\in\omega}$ converges to 0. Since K is compact, we may assume by passing to a subsequence that (g_n) converges to some point $g \in K$.

Now for each n, we have $gB_d(e, \alpha_{g_n} + \frac{1}{n}) \not\subseteq B_d(g_n, \beta)$ since $\alpha_{g_n} + \frac{1}{n} > \alpha_{g_n}$. So we may find some $x_n \in B_d(e, \alpha_{g_n} + \frac{1}{n})$ for which $g_n x_n \notin B_d(g_n, \beta)$. So $d(g_n, g_n x_n) \geq \beta$ for all n. Note that the sequence (x_n) converges to e since $(\alpha_{g_n} + \frac{1}{n})$ converges to 0.

Choose N so large that for all $n \geq N$, we have $d(g, g_n) < \frac{\beta}{2}$. Then for all $n \geq N$, we have $d(g_n, g_n x_n) \leq d(g, g_n) + d(g, g_n x_n)$ and hence $d(g, g_n x_n) \geq d(g_n, g_n x_n) - d(g, g_n) > \beta - \frac{\beta}{2} = \frac{\beta}{2}$. This means that the sequence $(g_n x_n)$ does not converge to $g = ge$, despite the fact that $g_n \to g$ and $x_n \to e$, contradicting the continuity of group multiplication in G. $\square$

PROPOSITION 5.11. *Let G be a Polish group and $(A_i)_{i\in\omega}$ a sequence of Haar null sets in G. Then $\bigcup_{i\in\omega} A_i$ is a Haar null set.*

PROOF. Let d be a compatible complete metric for G. We will construct a sequence of Borel probability measures $(\mu_i)_{i\in\omega}$ such that

(1) $\mu_i(gA_ih) = 0$ for every $g, h \in G$ and $i \in \omega$;

(2) $K_i = \operatorname{supp}(\mu_i)$ is compact for each $i \in \omega$; and

(3) if $x \in K_0K_1...K_{i-1}$ and $x_i \in K_i$, then $d(x, xx_i) < 2^{-i}$ for each $i \geq 1$.

To start the construction, by Lemma 5.9 let μ_0 be an arbitrary compactly-supported measure with $\mu(gA_0h) = 0$ for all $g, h \in G$, and let $K_0 = \operatorname{supp}\mu_0$. Now by way of induction suppose that we have found $(\mu_0, ..., \mu_{i-1})$ and $(K_0, ..., K_{i-1})$ with our desired three properties above. For each $g \in G$, define

$$\alpha_g = \sup\{\alpha : gB_d(e, \alpha) \subseteq B_d(g, 2^{-i})\}.$$

Since $K_0K_1...K_{i-1}$ is compact, by Lemma 5.10 we may find some fixed $\delta_i > 0$ such that $\delta_i < \alpha_x$ for all $x \in K_0K_1...K_{i-1}$. By Lemma 5.9, there is a probability measure μ_i on G for which $\mu_i(gA_ih) = 0$ for all $g, h \in G$, and $K_i = \operatorname{supp} \mu_i$ is compact with $e \in K_i$ and $\operatorname{diam} K_i < \delta_i$. Note that if $x \in K_0K_1...K_{i-1}$ and $x \in K_{i+1}$, then $d(e, x_i) < \delta_i$, and hence $xx_i \in xB_d(e, \delta_i) \subseteq xB_d(e, \delta_x) \subseteq B_d(x, 2^{-i})$; so $d(x, xx_i) < 2^{-i}$. This completes the construction.

Finally, define $\mu = \mathop{\ast}\limits_{k=0}^{\infty} \mu_k$ as in Definition 5.5. Then μ is a measure on G. We wish to check that μ is a probability measure, i.e. that $\mu(G) = 1$. To that end, notice that $\left(\prod_{i=0}^{\infty} \mu_k\right)$-almost every sequence $(g_0, g_1, ...) \in G^\omega$ is an element of $\prod_{k=0}^{\infty} K_i$ since $K_i = \operatorname{supp} \mu_i$ for each i. Thus for a full-measure set of sequence (g_i) we have that each partial product $g_0g_1...g_i$ lies in $K_0K_1...K_i$, whence if $n < m$ we have

$$\begin{aligned} d(g_0g_1...g_n, g_0g_1...g_m) &\leq d(g_0g_1...g_n, g_0g_1...g_{n+1}) + ... + d(g_0g_1...g_{m-1}, g_0g_1...g_m) \\ &< 2^{-(n+1)} + ... + 2^{-m} \\ &< 2^{-n} \end{aligned}$$

by our construction of the sets (K_i). Thus, for $\left(\prod_{i=0}^{\infty} \mu_k\right)$-almost every sequence $(g_i)_{i\in\omega}$, the partial products $(g_0g_1...g_i)$ form a Cauchy sequence and hence the infinite product $g_0g_1...$ converges since d is a complete metric. It follows that $\mu(G) = \left(\prod_{i=0}^{\infty} \mu_k\right)(\{(g_i) \in G^\omega : g_0g_1... \text{ converges}\}) = 1$ and μ is a probability measure.

For any $i \in \omega$ and any $g, h \in G$, we have

$$\mu(gA_ih) = \left(\mathop{\ast}\limits_{j=0}^{i-1} \mu_j\right) * \mu_i * \left(\mathop{\ast}\limits_{k=i+1}^{\infty} \mu_k\right)(gA_ih) = 0$$

by Lemma 5.6. Therefore for any $g, h \in G$ we have $\mu\left(g\left(\bigcup_{i\in\omega} A_i\right)h\right) = \mu\left(\bigcup_{i\in\omega} gA_ih\right) \leq \sum_{i=0}^{\infty} \mu(gA_ih) = 0$ and thus $\bigcup_{i\in\omega} A_i$ is a Haar null set. $\square$

Corollary 5.12. *Let G be any Polish group. Then $\mathcal{HN}(G)$ is a σ-ideal of subsets of G.*

The following corollary follows not from the statements but from the proofs of Corollary 5.7, Lemma 5.9 and Proposition 5.11. We sketch the necessary arguments.

Corollary 5.13. *Let G be a metric group.*

(1) If A and B are openly Haar null subsets of G, then $A \cup B$ is openly Haar null.

(2) If G is Polish and $(A_i)_{i \in \omega}$ is a sequence of openly Haar null subsets of G, then $\bigcup_{i=0}^{\infty} A_i$ is openly Haar null. Hence $\mathcal{OHN}(G)$ is a σ-ideal of subsets of G.

Proof. *(Proof of (1).)* Let A and B be openly Haar null sets as witnessed by μ and ν respectively. Let $\epsilon > 0$. Let $U \supseteq A$ and $V \supseteq B$ be open sets for which $\mu(gUh) < \frac{\epsilon}{2}$ and $\nu(gVh) < \frac{\epsilon}{2}$ for every $g, h \in G$. By Fubini's theorem again, for every $g, h \in G$ we have

$$\mu * \nu(gUh) = \int_G \mu(gUhy^{-1}) d\nu(y) < \int_G \tfrac{\epsilon}{2} d\nu(y) = \tfrac{\epsilon}{2}$$

and

$$\mu * \nu(gVh) = \int_G \nu(x^{-1}gVh) d\mu(x) < \int_G \tfrac{\epsilon}{2} d\nu(y) = \tfrac{\epsilon}{2}.$$

Then $U \cup V$ is open about $A \cup B$ and $\mu * \nu(g(U \cup V)h) = \mu * \nu(gUh \cup gVh) \leq \mu * \nu(gUh) + \mu * \nu(gVh) < \epsilon$ for every $g, h \in G$. Hence $A \cup B$ is openly Haar null.

(Proof of (2).) Let $(A_i)_{i \in \omega}$ be a sequence of openly Haar null sets in G and let $\epsilon > 0$. By an argument exactly analogous to the proof of Lemma 5.9, for any $\delta < 0$ it is possible to find for each i a measure μ_i witnessing that A_i is openly Haar null, which is supported on a compact set K_i containing the identity e and for which $\operatorname{diam} K_i < \delta$ for a suitable complete metric. Then by the same inductive construction employed in the proof of Proposition 5.11, we may build a probability measure μ of the form

$$\mu = \mathop{\Large *}_{i=0}^{\infty} \mu_i.$$

For each i find an open set $U_i \supseteq A_i$ such that $\mu_i(gU_ih) < \epsilon 2^{-(i+1)}$. Since $\mu = \left(\mathop{\ast}\limits_{j=0}^{i-1} \mu_j\right) \ast \mu_i \ast \left(\mathop{\ast}\limits_{k=i+1}^{\infty} \mu_k\right)$, by Fubini's theorem we have

$$\mu(gA_ih) = \int_G \int_G \mu_i(x^{-1}gA_ihy^{-1})d\left(\mathop{\ast}\limits_{j=0}^{i-1} \mu_j\right)(x)d\left(\mathop{\ast}\limits_{k=i+1}^{\infty} \mu_k\right)(y).$$

Since the integrand above is bounded above by $\epsilon 2^{-(i+1)}$ we have $\mu(gA_ih) < \epsilon 2^{-(i+1)}$. Hence for any $g, h \in G$ we have $\mu\left(g\left(\bigcup_{i=0}^{\infty} A_i\right)h\right) \leq \sum_{i=0}^{\infty} \mu(gA_ih) < \sum_{i=0}^{\infty} \epsilon 2^{-(i+1)} = \epsilon$. So $\bigcup_{i=0}^{\infty} A_i$ is openly Haar null. □

The following theorem establishes that $\mathcal{HN}(G)$ is the Haar measure zero sets when G is locally compact.

Theorem 5.14 (Mycielski [40]). *Let G be a Polish locally compact group and $A \subseteq G$ a universally measurable set. Then the following are equivalent.*

(1) A is a Haar null set.

(2) A is a Haar measure zero set.

(3) There exists a Borel probability measure μ on G such that $\mu(gA) = 0$ for every $g \in G$.

Part (3) in the theorem above may motivate the following question: May the two-sided translations involved in the definition of Haar null sets be dispensed with? That is, do we get a reasonable or useful σ-ideal if we consider the family of *left Haar null sets*, i.e. the class of all universally measurable sets $A \subseteq G$ such that there is a Borel probability measure μ on G, with $\mu(gA) = 0$ for every $g \in g$? If G is locally compact or abelian, then evidently the left Haar null sets are exactly the Haar null sets. Solecki has shown, however, that there is a severe difference for an important class of non-locally compact non-abelian groups.

Definition 5.15. A Polish group G is said to *have a free subgroup at the identity* if it has a non-discrete free subgroup whose all finitely-generated subgroups are discrete.

Solecki shows that the class of Polish groups which have a free subgroup at the identity includes: S_∞, Aut $\mathbb{Q}$, Homeo 2^ω, Homeo$[0,1]^n$ ($n \in \mathbb{N}$), and many others. For these groups, the family of left Haar null sets badly fails to comprise a σ-ideal, as the next theorem shows.

THEOREM 5.16 (Solecki [52]). *Let G be a Polish group which has a free subgroup at the identity. Then there exists a Borel set $B \subseteq G$ such that B is left Haar null and $G = B \cup Bg$ for some $g \in G$.*

Since a right translation of a left Haar null set is left Haar null, the left Haar null sets are not even closed under finite unions, let alone countably infinite ones.

Now we turn to what is known about questions (1)-(4) above. The most general known result regarding the Haar-null-ness of compacta belongs to Dougherty. Recall that a *TSI* group is a group which admits a two-sided invariant metric.

THEOREM 5.17 (Dougherty [11]). *Let G be a non-locally compact TSI Polish group. Then the compact subsets of G are Haar null.*

We are also interested in extending the following classical theorem to the non-locally compact setting.

THEOREM 5.18 ([41] Theorem 16.5). *Let G be an uncountable locally compact topological group. Then G may be written as a disjoint union $G = A \cup B$, where A is a comeager Haar measure zero set and B is a meager set of full Haar measure.*

Mycielski and Dougherty have made the following classification in S_∞.

THEOREM 5.19 (Dougherty and Mycielski [12]). *Let $A \subseteq S_\infty$ be the set of all permutations which have infinitely many infinite cycles and only finitely many finite cycles. Then A is prevalent in S_∞.*

On the other hand, it is easy to prove the following.

PROPOSITION 5.20. *Let $B \subseteq S_\infty$ be the set of all permutations which have only finite cycles. Then B is comeager in S_∞.*

PROOF. Using the logical notation and allowing the variables m, n to range over ω, we write

$$\pi \in B \leftrightarrow \forall m\ \exists n\ [\pi^n(m) = m].$$

This displays B as a G_δ set, and since B contains the finitely-supported permutations, B is dense. $\square$

So Dougherty and Mycielski have shown by explicit construction that S_∞ decomposes into the union of a Haar null comeager set and a prevalent meager set. As far as we are aware, this is the only previously existing example of this phenomenon in the literature.

Lastly we turn to questions (3) and (4). Note that every openly Haar null set is contained in a Haar null G_δ set. So a positive answer to (4) would imply a positive answer to (3) (and in turn a positive answer to (2)). The following result shows that it is consistent with ZFC that the answer to (3) (and hence (4)) is no.

Theorem 5.21 (Dougherty [11]). *Assume the continuum hypothesis. Let G be a non-locally compact TSI Polish group. Then there exists a Haar null set $S \subseteq G$ such that if $B \subseteq G$ is a Borel set with $S \subseteq B$, then B is prevalent.*

5.3. Haar Null Sets in Products and Inverse Limits of Locally Compact Groups

Definition 5.22. Let G be a topological group, μ a Borel measure on G, and $A \subseteq G$ a Borel set. We will say that μ is *openly (δ-)bounded* at A if there exists some real number δ with $0 < \delta < \mu(G)$, and some open set $U \supseteq A$, such that $\mu(gUh) < \delta$ for all $g, h \in G$.

Obviously if μ is openly δ-bounded at A then $\mu(gAh) < \delta$ for all $g, h \in G$ as well.

For any unimodular group G, the two-sided Haar measures on G are all openly bounded at every Borel set. But in general the left and right Haar measures on a non-unimodular group cannot be openly bounded at any Borel set A, for every open set in a non-unimodular group has two-sided translates of unbounded measure.

Lemma 5.23. *Let G be a non-unimodular locally compact topological group. Let $N = \ker(\Delta)$ denote the kernel of the modular function Δ of G, so G/N is also a locally compact group, and let $\pi : G \to G/N$ be the canonical projection. Let $0 < \delta < 1$. If A is any subset of G for* *which $\pi(A)$ has finite Haar measure in G/N, then there exists a probability measure μ on G such that μ is openly δ-bounded at A.*

Proof. Since $\Delta : G \to \mathbb{R}^+$ is a continuous group homomorphism, the map $\Delta^* : G/N \to \mathbb{R}^+$ defined by $\Delta^*(gN) = \Delta(g)$ is a well-defined, continuous, injective group homomorphism from G/N into $\mathbb{R}$. In particular, G/N is an abelian locally compact group, and hence unimodular. So G/N admits two-sided Haar measures. Let λ be any two-sided Haar measure on G/N for which $\lambda(\pi(A)) \leq \frac{\delta}{2}$. Observe that G/N is not compact; for if it were, then the image $\Delta^*(G/N)$ would be compact in $\mathbb{R}$, but unbounded in $\mathbb{R}$, a contradiction. So $\lambda(G/N) = \infty$.

Now let $\phi : G/N \to G$ be a Borel selector for the cosets of H, i.e. let ϕ be a Borel measurable function which satisfies $\phi(gH) \in gH$ for each $g \in G$ (See Theorem 2.20). Note that ϕ is a right inverse for the projection map π, i.e. $\pi \circ \phi$ is the identity on G/N.

Now let ν be the measure on G defined by $\nu(X) = \lambda(\phi^{-1}(X))$ for every Borel set $X \subseteq G$. Since $\lambda(G/N) = \infty$, we have $\nu(G) = \infty$; so find any compact set K for which $\nu(K) = 1$, and set $\mu = \nu \restriction K$. We claim that μ is our desired probability measure.

To see this, note that since $\delta > \frac{\delta}{2} = \lambda(\pi(A))$, the regularity of λ implies that there is some open set $V \subseteq G/N$ with $V \supseteq \pi(A)$ and $\lambda(V) < \delta$. Set $U = \pi^{-1}(V)$, so U is an open superset of A in G. Then

$$\begin{aligned}
\mu(gUh) &\leq \nu(gUh) \\
&\leq \lambda(\pi(gUh)) \\
&= \lambda(\pi(g)V\pi(h)) \\
&= \lambda(V) \\
&< \delta
\end{aligned}$$

as required. $\square$

Corollary 5.24. *Let G be a non-compact, locally compact topological group. Let $0 < \delta < 1$. If K is a compact subset of G, then there exists a regular probability measure μ on G such that μ is openly δ-bounded at K.*

Proof. If G is unimodular, then find the unique two-sided Haar measure λ on G for which $\lambda(K) < \delta$, and set $\mu = \lambda \upharpoonright F$, where F is any compact set with $\lambda(F) = 1$. The translation-invariance of λ implies the open δ-boundedness of μ at K. Otherwise if G is not unimodular, then let N be the kernel of the modular function Δ of G and let $\pi : G \to G/N$ be the canonical projection. Since $\pi(K)$ is a compact subset of G/N, it has finite Haar measure in G/N, and now Lemma 5.23 applies to finish the proof. $\square$

Proposition 5.25. *Let $(G_n)_{n\in\omega}$ be a sequence of topological groups such that infinitely many of the G_n are locally compact but not compact. Let $G = \prod_{n\in\omega} G_n$. Suppose $A \subseteq G$ has the property that for infinitely many n, G_n is locally compact non-compact and $\pi_n(A)$ is precompact in G_n. Then A is openly Haar null in G.*

Proof. For each n, let A_n denote the topological closure of $\pi_n(A)$ in G_n. By hypothesis there is some subsequence $(n_k)_{k\in\omega}$ of ω for which A_{n_k} is compact and G_{n_k} is locally compact but not compact, for all k. For each such coordinate n_k, use Corollary 5.24 to find a probability measure μ_{n_k} on G_{n_k} which is openly $\frac{1}{2}$-bounded at A_{n_k}. For every coordinate n which does not appear in the sequence $(n_k)_{k\in\omega}$, let μ_n be an arbitrary probability measure, and set $\mu = \prod_{n\in\omega} \mu_n$.

Let $\epsilon > 0$. Choose N large enough so that $2^{-(N+1)} < \epsilon$. For each $0 \leq k \leq N$, use the open $\frac{1}{2}$-boundedness of μ_{n_k} to find an open set U_{n_k} containing A_{n_k}, such that $\mu_{n_k}(g_{n_k} U_{n_k} h_{n_k}) < \frac{1}{2}$, for all $g_{n_k}, h_{n_k} \in G_{n_k}$. For all coordinates n which are not in $\{n_0, ..., n_N\}$, set $U_n = G_n$, and set $U = \prod_{n\in\omega} U_n$. So U is an open subset of G containing A, and for all $g, h \in G$ we have

$$\mu(gUh) = \prod_{n\in\omega} \mu_n(\pi_n(g) U_n \pi_n(h))$$

$$
\begin{aligned}
&= \left[\prod_{k=0}^{N} \mu_{n_k}(\pi_{n_k}(g)U_{n_k}\pi_{n_k}(h))\right] \cdot \left[\prod_{n \notin \{n_0,\ldots,n_N\}} \mu_n(G_n)\right] \\
&< \prod_{k=0}^{N} \frac{1}{2} \\
&= 2^{-(N+1)} \\
&< \epsilon.
\end{aligned}
$$

So A is openly Haar null. $\square$

Corollary 5.26. *If $G = \prod_{n \in \omega} G_n$ where infinitely many of the groups G_n are locally compact but not compact, then the compact subsets of G are openly Haar null.*

Since products of locally compact groups need not in general admit two-sided invariant metrics, Corollary 5.26 significantly extends the domain of Theorem 5.17 of Dougherty. We can also pass the conclusion of Theorem 5.17 between groups and their subgroups of small index, via the next proposition.

Proposition 5.27. *Let G be a Polish group. Let H be an open subgroup of G. Then the compact sets in H are Haar null in H if and only if the compact sets in G are Haar null in G.*

Proof. First suppose the compact sets in H are Haar null in H. Since H is open, it is also closed in G and hence Polish. Since $[G : H]$ is countable, we may enumerate all the right cosets of H by $Hg_0, Hg_1, \ldots$ for some $g_0, g_1, \ldots \in G$ and all the left cosets of H by $h_0H, h_1H, \ldots$ for some $h_0, h_1, \ldots \in G$.

Let $K \subseteq G$ be compact. Set $F = \bigcup_{i,j \in \omega} g_iKh_j \cap H$. F is a σ-compact subset of H and hence a Haar null set in H by our hypothesis. So find a probability measure μ, which has support in H, and for which $\mu(k_0Fk_1) = 0$ for all $k_0, k_1 \in H$.

Then μ is also a measure on G, and we claim that μ also witnesses that K is Haar null in G. For if $g, h \in G$ are arbitrary, we may find some g_i and some h_j for which $g \in Hg_i$ and $h \in h_j H$. Then there exist $k_0, k_1 \in H$ for which $g = k_0 g_i$ and $h = h_j k_1$, and hence we have

$$\begin{aligned}
gKh \cap H &= k_0 g_i K h_j k_1 \cap H \\
&= k_0[g_i K h_j \cap k_0^{-1} H k_1^{-1}]k_1 \\
&= k_0[g_i K h_j \cap H]k_1 \\
&\subseteq k_0 F k_1.
\end{aligned}$$

Since the support of μ is a subset of H, it follows that we have $\mu(gKh) = \mu(gKh \cap H) \leq \mu(k_0 F k_1) = 0$. So K is indeed Haar null in G.

Conversely, suppose the compact sets in G are Haar null in G, and let $K \subseteq H \subseteq G$ be compact. Since K is Haar null in G, let μ be a measure on G witnessing it. Since μ is countably additive and there are only countably many cosets Hg_i in G, there must be some particular $i \in \omega$ for which $\mu(Hg_i) > 0$; fix this i. Let $c = 1/\mu(Hg_i)$ and define ν on H by $\nu(A) = c\mu(Ag_i)$. Then ν is a probability measure on H, and for all $g, h \in H$ we have $\nu(gKh) = c\mu(gKhg_i) = 0$, so K is Haar null in H. $\square$

The next corollary gives the first positive answer to Question (2) of Darji for a large class of Polish groups.

Corollary 5.28. *Let* $G = \prod_{n \in \omega} G_n$ *where infinitely many of the groups* G_n *are locally compact but not compact. If* G *is Polish, then* G *may be represented as the disjoint union* $G = A \cup B$, *where* A *is a comeager Haar null set and* B *is a meager prevalent set.*

Proof. Let D be a countable dense set in G. Since D is σ-compact, D is an openly Haar null set by Corollary 5.26. It follows that D is contained in a Haar null G_δ set A. Since A is dense, A is comeager, and hence its complement $B = G \backslash A$ is meager as well as prevalent. $\square$

The next result is a slight modification of the forward direction of Proposition 8 in [11], and is achieved by essentially the same proof.

Lemma 5.29. *Let G and H be Polish groups and $\phi : G \to H$ a continuous surjective homomorphism, and $A \subseteq H$ a universally measurable set. If A is openly Haar null in H, then $\phi^{-1}(A)$ is openly Haar null in G.*

Proof. First suppose A is openly Haar null in H, as witnessed by the probability measure ν on H. Define a map $\theta : G/\ker(\phi) \to H$ by $\theta(g\ker(\phi)) = \phi(g)$. The map θ is a well-defined continuous bijection and hence a Borel isomorphism by Lusin-Souslin's theorem. By Theorem 2.20, let $\psi : G/\ker(\phi) \to G$ be a Borel-measurable coset selector. Set $\kappa = \psi \circ \theta^{-1} : H \to G$. Then κ is Borel-measurable and $\kappa(h) \in \phi^{-1}(h)$ for every $h \in H$; in other words $\phi \circ \kappa : H \to H$ is the identity map.

Define μ on G by $\mu(B) = \nu(\kappa^{-1}(B))$ for all Borel sets $B \subseteq G$, so μ is a probability measure. Let $\epsilon > 0$ and let $V \subseteq H$ be an open superset of A, for which $\nu(h_1 V h_2) < \epsilon$ for all $h_1, h_2 \in H$. Set $U = \phi^{-1}(V)$, so U is open in G and $\phi^{-1}(A) \subseteq U$. Then for any $g_1, g_2 \in G$ we have $\mu(g_1 U g_2) = \nu(\kappa^{-1}(g_1 U g_2)) \leq \nu(\phi(g_1 U g_2)) = \nu(\phi(g_1)\phi(U)\phi(g_2)) = \nu(\phi(g_1)V\phi(g_2)) < \epsilon$. So $\phi^{-1}(A)$ is openly Haar null in H. □

Corollary 5.30. *Let G be Polish group. If there exists a continuous surjective homomorphism $\phi : G \to H$ where H is an uncountable locally compact unimodular Polish group, then the singleton sets in G are openly Haar null, and hence G may be written as the disjoint union of a Haar null set and a meager set.*

Proof. If $g \in G$, then the singleton set $\{\psi(g)\}$ is openly Haar null in H as witnessed by any two-sided Haar measure on H. So by our lemma above, $\{g\} \subseteq \psi^{-1}(\{\psi(g)\})$ is a subset of an openly Haar null set and hence openly Haar null. The decomposition now follows exactly as in the proof of Corollary 5.28. □

For instance, if $G = \varprojlim G_n$ is an inverse limit of Polish groups G_n, and at least one G_n is an uncountable locally compact unimodular group, then G admits such a decomposition.

5.4. Groups Where $\mathcal{OHN} \neq \mathcal{HN}$

The next few propositions establish the existence of several groups, both locally compact and non-locally compact, where the family of openly Haar null sets is a proper subcollection of the family of Haar null sets. This provides a negative answer to question (4) and resolves an uncertainty of Solecki in [51].

PROPOSITION 5.31. *Let G be the $ax + b$ group as defined in Subsection 2.2.2. Then there exist closed subsets of G which are Haar null but not openly Haar null.*

PROOF. For instance, let $F = \mathbb{R}^+ \times \{0\} \subseteq G$. F is obviously Haar measure zero, i.e. Haar null in G. But we will show that F is not openly Haar null.

To that end, let μ be an arbitrary compactly-supported measure on G; say that the support of μ is a subset of the closed bounded box $[i, j] \times [-k, k]$, where $i, j \in \mathbb{R}^+$ and $k \in \mathbb{R}$. Let U be an arbitrary open set containing F. Since U may be written as a countable union of basic open boxes, and the line segment $[i, j] \times \{0\}$ is a compact subset of F, by passing to a finite subcover and taking appropriate minimums, we may find some $\epsilon > 0$ for which

$$[i, j] \times \{0\} \subseteq (i - \epsilon, j + \epsilon) \times (-\epsilon, \epsilon) \subseteq U.$$

Set $V = (i - \epsilon, j + \epsilon) \times (-\epsilon, \epsilon)$. Now let $N \in \mathbb{R}^+$ be so large that $N\epsilon > k$. Let us compute membership in the set $(\frac{1}{N}, 0) \cdot V \cdot (N, 0)$:

$$\begin{aligned}
(a, b) \in (N, 0) \cdot V \cdot \left(\frac{1}{N}, 0\right) &\leftrightarrow \left(\frac{1}{N}, 0\right) \cdot (a, b) \cdot (N, 0) \in V \\
&\leftrightarrow \left(\frac{1}{N}, 0\right) \cdot (aN, b) \in V \\
&\leftrightarrow \left(a, \frac{b}{N}\right) \in V \\
&\leftrightarrow a \in (i - \epsilon, j + \epsilon) \wedge \frac{b}{N} \in (-\epsilon, \epsilon) \\
&\leftrightarrow a \in (i - \epsilon, j + \epsilon) \wedge b \in (-N\epsilon, N\epsilon) \\
&\leftrightarrow (a, b) \in (i - \epsilon, j + \epsilon) \times (-N\epsilon, N\epsilon).
\end{aligned}$$

So by the equivalence above, we have $[i,j] \times [-k,k] \subseteq (i-\epsilon, i+\epsilon) \times (-N\epsilon, N\epsilon) \subseteq (N,0) \cdot V \cdot (\frac{1}{N}, 0)$. Since the support of μ is contained in $[i,j] \times [-k,k]$, we have $\mu((N,0) \cdot U \cdot (\frac{1}{N}, 0)) \geq \mu((N,0) \cdot V \cdot (\frac{1}{N}, 0)) = 1$. So U has a two-sided translate of full measure. Since μ and U were taken arbitrarily, F cannot possibly be openly Haar null. $\square$

So $\mathcal{OHN}(G) \subsetneq \mathcal{HN}(G)$ in G the $ax+b$ group. However, there are certainly many openly Haar null sets in G, as the next theorem shows.

PROPOSITION 5.32. *Let G be a non-unimodular locally compact topological group. Let $N = \ker(\Delta)$ denote the kernel of the modular function Δ of G, so G/N is also a locally compact group, and let $\pi : G \to G/N$ be the canonical projection. If A is any subset of G for which $\pi(A)$ is Haar measure zero in G/N, then A is openly Haar null in G.*

PROOF. This proof is almost identically the proof of Lemma 5.23, so we will move through the details a bit more quickly. Since G/N is abelian, it admits a two-sided Haar measure λ. By Theorem 2.20, let $\phi : G/N \to G$ be a Borel selector for the cosets of N and define a measure ν on G by the rule $\nu(A) = \lambda(\phi^{-1}(A))$. Let K be any subset of G with $\nu(K) = 1$ and let $\mu = \nu \restriction K$, so μ is a probability measure on G. For any $\epsilon > 0$, use the regularity of λ to find $V \subseteq G/N$ such that V contains $\pi(A)$ and $\lambda(V) < \epsilon$, and let $U = \pi^{-1}(V)$. Then U is open in G, U contains A, and for any $g,h \in G$ we have $\mu(gUh) \leq \nu(gUh) = \lambda(\pi(g)V\pi(h)) = \lambda(V) < \epsilon$. Hence A is openly Haar null as required. $\square$

Now we will mention that the openly Haar null sets may differ from the Haar null sets in non-locally compact groups as well as locally compact ones, by looking at an "infinite-dimensional $ax+b$ group." The construction is somewhat *ad hoc* and was actually the first example discovered by the author of a group where $\mathcal{OHN} \neq \mathcal{HN}$. The proof of Proposition 5.34 is in spirit extremely similar to that of Proposition 5.31.

DEFINITION 5.33. Let $\phi : \mathbb{R}^+ \to \mathrm{Aut}(\mathbb{R}^\omega)$ be defined by the rule $[\phi(a)](b) = a \cdot b = (ab(0), ab(1), ...)$ for every $a \in \mathbb{R}^+$ and $b = (b(n))_{n\in\omega} \in \mathbb{R}^\omega$. Define G to be the semidirect product group

$$G = \mathbb{R}^+ \ltimes_\phi \mathbb{R}^\omega.$$

Stated otherwise, G is the group of all tuples $(a, b) = (a, b(0), b(1), ...) \in \mathbb{R}^+ \times \mathbb{R}^\omega$, with the group multiplication defined by

$$(a, b) \cdot (c, d) = (ac, a \cdot d + b) = (ac, ad(0) + b(0), ad(1) + b(1), ...).$$

Theorem 2.17 (2) implies that G is a Polish group. An easy computation shows that in general, if $(a, b) \in G$, then $(a, b)^{-1} = (\frac{1}{a}, \frac{1}{a} \cdot (-b))$.

PROPOSITION 5.34. *Let G be as in Definition 5.33. Then there exist closed subsets of G which are Haar null but not openly Haar null.*

PROOF. For instance, set

$$F = \mathbb{R}^+ \times [-1, 1]^\omega.$$

First let us show that F is a Haar null set in G. We must find a measure μ on G which is transverse to F. Let ν be an arbitrary probability measure on $\mathbb{R}^+$. For each $n \in \omega$, let μ_n be the uniform probability measure on the interval $[-n, n]$. Define μ on G by $\mu = \nu \times \prod_{n\in\omega} \mu_n$.

Now let (a, b) and (c, d) be arbitrary elements of G. We compute the members of the set $(a, b)^{-1} \cdot F \cdot (c, d)^{-1}$ below:

$$\begin{aligned}
(e, f) \in (a, b)^{-1} \cdot F \cdot (c, d)^{-1} &\leftrightarrow (a, b) \cdot (e, f) \cdot (c, d) \in F \\
&\leftrightarrow (aec, a \cdot f + ae \cdot d + b) \in F \\
&\leftrightarrow \forall n \in \omega \ \ (af(n) + aed(n) + b(n) \in [-1, 1]) \\
&\leftrightarrow \forall n \in \omega \ \ \left(f(n) \in \left[\frac{-1 - aed(n) - b(n)}{a}, \frac{1 - aed(n) - b(n)}{a} \right] \right).
\end{aligned}$$

Let $I_n = \left[\frac{-1-aed(n)-b(n)}{a}, \frac{1-aed(n)-b(n)}{a}\right]$. Then the above computation reveals that, setwise, we have

$$(a,b)^{-1} \cdot F \cdot (c,d)^{-1} = \mathbb{R}^+ \times \prod_{n \in \omega} I_n.$$

Thus we have $\mu((a,b)^{-1} \cdot F \cdot (c,d)^{-1}) = \nu(\mathbb{R}^+) \cdot \prod_{n\in\omega} \mu_n(I_n)$. But the length of each interval I_n is just $\frac{2}{a}$, and our measures μ_n are uniformly distributed on $[-n,n]$ for each n. It follows that for all $n > \frac{4}{a}$, we have $\mu_n(I_n) \leq \frac{2/a}{n} < \frac{1}{2}$. Hence $\mu((a,b)^{-1} \cdot F \cdot (c,d)^{-1}) = 0$, and F is a Haar null set in G.

However, F cannot be openly Haar null. For let μ be an arbitrary probability measure with compact support on G and let $U \subseteq G$ be an open set which contains F. Let $\psi : G \to \mathbb{R}^+$ be the obvious projection map, and for each $n \in \omega$ let $\pi_n : G \to \mathbb{R}$ be the n-th projection map. Let $L = \psi(\operatorname{supp}(\mu))$ and let $K_n = \pi_n(\operatorname{supp}(\mu))$, so $L \times \prod_{n\in\omega} K_n$ is compact in G and $\operatorname{supp}(\mu) \subseteq L \times \prod_{n\in\omega} K_n$.

Now since $L \times [0,1]^\omega$ is compact by Tychonoff's theorem, we may find finitely many basic open sets $U_1, ..., U_p$ such that $L \times [0,1]^\omega \subseteq \bigcup_{i=1}^{p} U_i \subseteq U$. We may write each U_i, $1 \leq i \leq p$, as a product

$$U_i = V_i \times \prod_{j=0}^{m_i} U_i^j \times \prod_{k>m_i} \mathbb{R},$$

where m_i is an integer depending on i, V_i is open in $\mathbb{R}^+$, and U_i^j is open in $\mathbb{R}$ for each $0 \leq j \leq m_i$.

Let now $M = \max\{m_i : 1 \leq i \leq p\}$. Notice that each K_j, $0 \leq j \leq M$, is a compact and hence bounded subset of the real line. Hence we may choose some N so large that

$$L \times \prod_{j=0}^{M} K_j \times \prod_{k>M} \mathbb{R} \subseteq L \times \prod_{j=0}^{M} [-N,N] \times \prod_{k>M} \mathbb{R}.$$

We claim that the set $(N,0) \cdot \left[\bigcup_{i=1}^{p} U_i\right] \cdot (\frac{1}{N}, 0)$ contains $L \times \prod_{n\in\omega} K_n$, and hence contains $\operatorname{supp}(\mu)$ and has full μ-measure.

Equivalently, we may show that $(\frac{1}{N},0)\cdot\left[L\times\prod_{n\in\omega}K_n\right]\cdot(N,0)\subseteq\bigcup_{i=1}^{p}U_i$. To see this, take an arbitrary $(e,f)\in(\frac{1}{N},0)\cdot\left[L\times\prod_{n\in\omega}K_n\right]\cdot(N,0)$. Then the following implications hold:

$$
\begin{aligned}
(e,f)\in\left(\frac{1}{N},0\right)\cdot\left[L\times\prod_{n\in\omega}K_n\right]\cdot(N,0) &\to (N,0)\cdot(e,f)\cdot\left(\frac{1}{N},0\right)\in L\times\prod_{n\in\omega}K_n\\
&\leftrightarrow (e,N\cdot f)\in L\times\prod_{n\in\omega}K_n\\
&\to (e,N\cdot f)\in L\times\prod_{j=0}^{M}K_j\times\prod_{k>M}\mathbb{R}\\
&\to (e,N\cdot f)\in L\times\prod_{j=0}^{M}[-N,N]\times\prod_{k>M}\mathbb{R}\\
&\leftrightarrow e\in L\wedge Nf(0)\in[-N,N]\wedge...\wedge Nf(M)\in[-N,N]\\
&\leftrightarrow e\in L\wedge f(0)\in[-1,1]\wedge...\wedge f(M)\in[-1,1]\\
&\leftrightarrow (e,f)\in L\times\prod_{j=0}^{M}[-1,1]\times\prod_{k>M}\mathbb{R}.
\end{aligned}
$$

Now define a point $f'\in\mathbb{R}^\omega$ by $f'(j)=f(j)$ for all $0\le j\le M$ and $f(k)=0$ for all $k>M$. It follows from the last statement above that $(e,f')\in L\times[-1,1]^\omega$ and hence $(e,f')\in U_i$ for some $i\in\{1,...,p\}$. This implies $f(j)=f'(j)\in U_i^j$ for all $j\in\{1,...,M\}\supseteq\{1,...,m_i\}$. So in fact we also have $(e,f)\in U_i$ and the claim is proved.

It follows that the two-sided translate $(N,0)\cdot U\cdot(\frac{1}{N},0)$ contains the support of μ and hence $\mu((N,0)\cdot U\cdot(\frac{1}{N},0))=1$. Since μ and U were chosen arbitrarily, we see that our set F cannot be openly Haar null. $\square$

Remark 5.35. Let $\mathbb{Q}_d^+$ denote the multiplicative group of positive rationals with the discrete topology. Let $\phi:\mathbb{Q}_d^+\to\operatorname{Aut}\mathbb{R}$ be the natural mapping defined by $[\phi(q)](x)=qx$ for every

$q \in \mathbb{Q}_d^+$ and $x \in \mathbb{R}$, and let $\phi_\omega : \mathbb{Q}_d^+ \to \mathrm{Aut}\ \mathbb{R}^\omega$ be the analogous natural mapping.

Define two semidirect product groups by $G_1 = \mathbb{Q}_d^+ \ltimes_\phi \mathbb{R}$ and $G_2 = \mathbb{Q}^+ \ltimes_{\phi_\omega} \mathbb{R}^\omega$. Then G_1 is a locally compact Polish group and G_2 is a non-locally compact Polish group. We may think informally of G_1 as a "$qx + b$ group" where q is rational, and G_2 as the infinite-dimensional version of G_1. By arguments very similar to those in the proofs of Propositions 5.31 and 5.34, it may be shown that G_1 and G_2 also have no nonempty openly Haar null sets. That is, $\mathcal{OHN}(G_1) = \mathcal{OHN}(G_2) = \{\emptyset\}$. So the family $\mathcal{OHN}$ may degenerate in a multitude of Polish groups, even locally compact ones.

5.5. Haar Null Sets in S_∞

In this section we define a useful class of measures in S_∞, and use such a measure to give a sufficient criterion for a set to be Haar null in this group. A corollary is that every compact set is Haar null, which extends the result of Theorem 5.17 to another group which does not admit a two-sided invariant metric.

Definition 5.36. Let S_∞ denote the group of permutations of ω, endowed with the topology of pointwise convergence.

For each $s \in \omega^{<\omega}$, let $N_s = \{\pi \in S_\infty : \pi \restriction \mathrm{lh}(s) = s\} \subseteq S_\infty$, so the family of all such N_s is the standard basis for the topology of S_∞.

Suppose K is some compact subset of S_∞, so by Corollary 2.40 K is the set of branches of some finitely branching tree $T \subseteq \omega^{<\omega}$, i.e. $K = [T]$. For any $\sigma, \tau \in S_\infty$, let $\sigma T \tau$ denote the unique pruned tree for which $\sigma K \tau = [\sigma T \tau]$. For every $s \in \omega^{<\omega}$ and tree $S \subseteq \omega^{<\omega}$, let $\mathcal{C}(S, s)$ denote the set

$$\mathcal{C}(S, s) = \{t \in S : \mathrm{lh}(t) = \mathrm{lh}(s) + 1 \text{ and } t \restriction \mathrm{lh}(s) = s\},$$

so $\mathcal{C}(S, s)$ is the set of all children of s which lie in S. This set is finite if $[S]$ is compact.

We will say that μ is a *uniform probability measure on $M = [S]$* if μ is a probability measure on S_∞ which satisfies the following properties:

(1) $\operatorname{supp}(\mu) = M$, and

(2) $\mu(N_s) = \displaystyle\prod_{k=0}^{\mathrm{lh}(s)-1} \frac{1}{|\mathcal{C}(S, s \restriction k)|}$ whenever $s \in S$.

Lemma 5.37. *Every compact set $M \subseteq S_\infty$ admits a unique uniform probability measure μ on M.*

Proof. This is a routine application of Kolmogorov's extension theorem. $\square$

Definition 5.38. Let $\mathrm{Im}\ (T, j) \subseteq \omega$ denote the set

$$\mathrm{Im}\ (T, j) = \{n \in \omega : \exists \pi \in [T]\ (\pi(j) = n)\},$$

so $\mathrm{Im}\ (T, j)$ is a set of all integers which appear at the j-th level of T. Let $\mathcal{N}(T, j) \subseteq \omega^{<\omega}$ denote the set

$$\mathcal{N}(T, j) = \{s \in T : \mathrm{lh}(s) = j + 1\},$$

so $\mathcal{N}(T, j)$ is the set of all nodes of T at the j-th level. It is clear that $|\mathrm{Im}\ (T, j)| \leq |\mathcal{N}(T, j)|$, and that if $\sigma, \tau \in S_\infty$, then $|\mathrm{Im}\ (\sigma T \tau, \tau^{-1}(j))| = |\mathrm{Im}\ (T, j)|$.

Suppose $\{I, ..., J\}$ are consecutive integers (with $I < J$) and $s \in \omega^{<\omega}$ with $\mathrm{lh}(s) > J$. We will say that s is a *cycle* on $\{I, ..., J\}$ if whenever $i \in \{I, ..., J\}$ we have

$$s(i) = [(i - I + n) \mod (J - I)] + I$$

for some integer n. Note that if s is a cycle on $\{I, ..., J\}$ then s is a cyclic permutation of $\{I, ..., J\}$. If $0 = I_0 < I_1 < ... < I_k$ and $s \in \omega^{<\omega}$ with $\mathrm{lh}(s) = I_k$, then we will say that s *consists of cycles* on $I_0, ..., I_k$ if s is a cycle on $\{I_i, ..., I_{i+1} - 1\}$ for every $i \in \{0, ..., k - 1\}$.

Note that if s consists of cycles on $I_0, ..., I_k$ then s is a bijective function from $I_k \to I_k$. Moreover, there are exactly $\displaystyle\prod_{i=1}^{k}(I_i - I_{i-1})$ many distinct $s \in \omega^{<\omega}$ which consist of cycles on $I_0, ..., I_k$.

Proposition 5.39. *Let F be a subset of S_∞, and let $T \subseteq \omega^{<\omega}$ be pruned tree on ω for which $F = [T] \cap S_\infty$. If Im $(T, j) = F \cdot j$ is finite for infinitely many $j \in \omega$, then F is a Haar null* *set.*

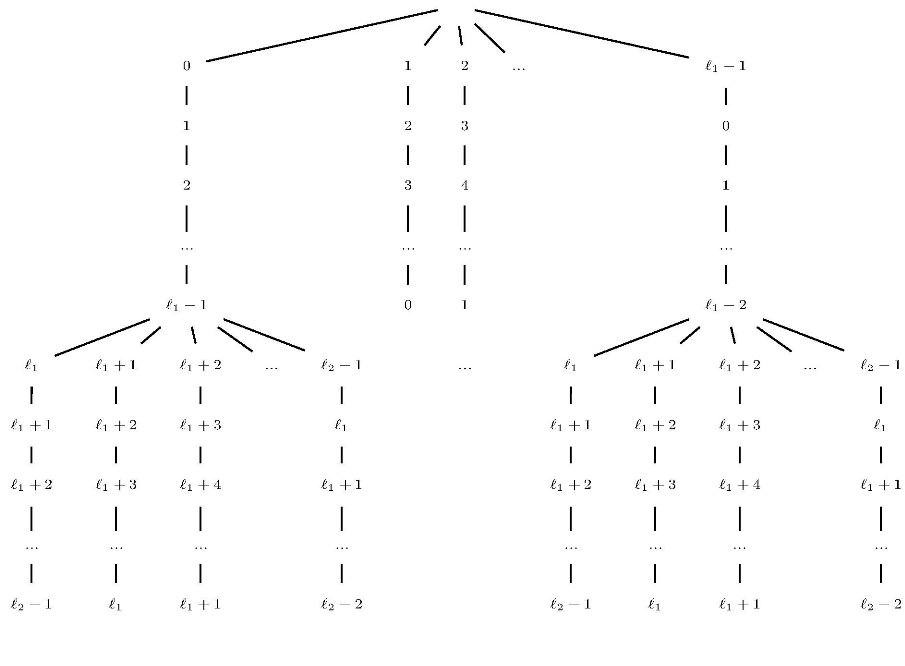
Proof. We recursively define increasing sequences $(\ell_k)_{k\in\omega}$ and $(n_k)_{k\in\omega}$ of integers as follows: Let $\ell_0 = 0$. Let n_0 be the least integer for which $|\mathrm{Im}\ (T, j)| = n_0$, for some $j \in \omega$.

Now fix k and suppose ℓ_i and n_i are defined for all $i < k$. Set $\ell_k = 2 \cdot \sum_{i=0}^{k-1} n_i$. Finally, let n_k be large enough so that there are at least $\ell_k + 1$ distinct integer levels j for which $|\mathrm{Im}\ (T, j)| \leq n_k$. This completes our recursive definition.

Now we will define a new tree S on $\omega^{<\omega}$ whose branches will form a subset of S_∞ and which splits "at least twice as much" as any translate of T at infinitely many levels. Define S as follows:

$$
\begin{aligned}
s \in S \leftrightarrow \exists k \in \omega\ \exists t \in \omega^{<\omega}\ & (\mathrm{lh}(t) = \ell_k)\wedge \\
& (t \restriction \mathrm{lh}(s) = s)\wedge \\
& (t \text{ consists of cycles on } \ell_0, ..., \ell_k).
\end{aligned}
$$

The first few levels of the tree S are pictured below:



Now set $M = [S]$. If $\pi \in M$, then clearly $\pi \upharpoonright \ell_k$ consists of cycles on $\ell_0, ..., \ell_k$ for every k, and hence π is really a permutation. Thus $M \subseteq S_\infty$. Then let μ be the uniform probability measure generated by M.

We claim that μ will witness the Haar null-ness of K. In order to show this, we first wish to make the following five technical claims about S and μ.

(1) For any $j \in \omega$ with $\ell_k \leq j < \ell_{k+1}$, we have

$$|\text{Im } (S, j)| = |\text{Im } (S, \ell_k)| = 2n_k,$$

and

$$|\mathcal{N}(S, j)| = |\mathcal{N}(S, \ell_k)| = \prod_{i=0}^{k} |\text{Im } (S, \ell_i)| = \prod_{i=0}^{k} (2n_i).$$

(2) For any $j \in \omega$ with $\ell_k \leq j < \ell_{k+1}$, if $s \in \mathcal{N}(S, j)$, then

$$\mu(N_s) = \frac{1}{|\mathcal{N}(S, j)|} = \prod_{i=0}^{k} \frac{1}{2n_i}.$$

(3) For any $j \in \omega$ with $\ell_k \leq j < \ell_{k+1}$, and for any tree U on $\omega^{<\omega}$ with $[U] \subseteq S_\infty$, we have

$$|\mathcal{N}(U \cap S, j)| \leq |\mathcal{N}(U \cap S, \ell_{k-1})| \cdot |\text{Im } (U, j)|$$

(Here we take $|\mathcal{N}(U \cap S, \ell_{k-1})|$ to be 1 if $k = 0$.)

(4) For any $j, j' \in \omega$ with $j \leq j'$ and any tree U on $\omega^{<\omega}$ with $[U] \subseteq S_\infty$, we have

$$\frac{|\mathcal{N}(U \cap S, j')|}{|\mathcal{N}(S, j')|} \leq \frac{|\mathcal{N}(U \cap S, j)|}{|\mathcal{N}(S, j)|}.$$

(5) For any $j \in \omega$ and any tree U on $\omega^{<\omega}$ with $[U] \subseteq S_\infty$, we have

$$\mu([U]) \leq \frac{|\mathcal{N}(U \cap S, j)|}{|\mathcal{N}(S, j)|}.$$

Proof of (1). Notice that s appears on the j-th level of T if and only if s is extended by some $\pi \in [S]$ for which $\pi \restriction \ell_{k+1}$ is a cycle on $\{\ell_k, ..., \ell_{k+1}-1\}$. Hence $s(j) = \pi(j) \in \{\ell_k, ..., \ell_{k+1}-1\}$, and we have $|\text{Im } (S, j)| \leq \ell_{k+1} - \ell_k = 2n_k$. On the other hand if m is one of the $2n_k$ integers in the set $\{\ell_k, ..., \ell_{k+1} - 1\}$ then it is clear that $\pi(j) = m$ for some $\pi \in [S]$. So in fact we have $|\text{Im } (S, j)| = 2n_k$ for all j satisfying $\ell_k \leq j < \ell_{k+1}$.

We will show the second statement of the claim by induction on k. We have shown the base case $k = 0$ in the above paragraph. So suppose that, for some $k \geq 1$, we have $|\mathcal{N}(S, j')| = |\mathcal{N}(S, \ell_k)| = \prod_{i=0}^{k} |\text{Im } (S, \ell_k)| = \cdot \prod_{i=0}^{k} (2n_i)$ whenever j satisfies $\ell_{k-1} \leq j' < \ell_k$. Now suppose we have $j \in \omega$ with $\ell_k \leq j < \ell_{k+1}$. If $s \in S$ has $\text{lh}(s) = \ell_{k+1}$, then s extends $s \restriction \ell_k$, which is one of the nodes in $\mathcal{N}(S, \ell_k - 1)$. Also, s is a cycle on $\{\ell_k, ..., \ell_{k+1} - 1\}$ by our definition of S. Since there are exactly $\ell_{k+1} - \ell_k = 2n_k$ many ways to build such a cycle extending any node in $\mathcal{N}(S, \ell_k - 1)$, and each of these cycles takes a distinct value at j, we must have $|\mathcal{N}(S, j)| = |\mathcal{N}(S, \ell_k - 1)| \cdot 2n_k = |\mathcal{N}(S, \ell_k)| \cdot |\text{Im } (S, \ell_k)|$. Now our inductive hypothesis proves the claim.

Proof of (2). We again proceed by induction on k. If $k = 0$, simply observe that $\mathcal{N}(S, \ell_k) = \mathcal{N}(S, 0) = \mathcal{C}(S, \emptyset)$, and that by our construction we have $|\mathcal{C}(S, s \restriction i)| = 1$ for every $1 \leq i < j$. Then by claim (1) and the fact that μ is a uniform measure on $[U]$, we get

$$\mu(N_s) = \prod_{i=0}^{j-1} \frac{1}{|\mathcal{C}(S, s \restriction i)|}$$

$$= \frac{1}{|\mathcal{C}(S,\emptyset)|} \cdot \prod_{i=1}^{j-1} \frac{1}{|\mathcal{C}(S, s \restriction i)|}$$
$$= \frac{1}{|\mathcal{N}(S,0)|}$$
$$= \frac{1}{|\mathcal{N}(S,j)|},$$

as claimed.

Now by way of induction, suppose the claim holds for all integers below ℓ_k, and let j satisfy $\ell_k \leq j < \ell_{k+1}$. Notice that since $s \restriction \ell_k$ lies in $\mathcal{N}(S, \ell_k - 1)$, then as we pointed out in our proof of claim (1), $s \restriction \ell_k$ has exactly one child in S for every integer in Im $(S, \ell_k) = \{\ell_k, ..., \ell_{k+1} - 1\}$. So $|\mathcal{C}(S, s \restriction \ell_k)| = |\text{Im } (S, \ell_k)|$. On the other hand if $\ell_k < i < j$, then $s \restriction i$ has only one child in S, i.e. $|\mathcal{C}(S, s \restriction i)| = 1$. Then applying our inductive hypothesis, claim (1), and the fact that μ is a uniform measure on S, we get

$$\mu(N_s) = \prod_{i=0}^{j-1} \frac{1}{|\mathcal{C}(S, s \restriction i)|}$$
$$= \prod_{i=0}^{\ell_k - 1} \frac{1}{|\mathcal{C}(S, s \restriction i)|} \cdot \frac{1}{|\mathcal{C}(S, s \restriction \ell_k)|} \cdot \prod_{i=\ell_k+1}^{j-1} \frac{1}{|\mathcal{C}(S, s \restriction i)|}$$
$$= \frac{1}{|\mathcal{N}(S, \ell_k - 1)|} \cdot \frac{1}{|\text{Im } (S, \ell_k)|}$$
$$= \frac{1}{|\mathcal{N}(S, \ell_k)|}$$
$$= \frac{1}{|\mathcal{N}(S,j)|}.$$

This completes the induction and proves the claim.

Proof of (3). If $k = 0$ then $|\mathcal{N}(U \cap S, j)| = |\text{Im } (U \cap S, j)| \leq |\text{Im } (U, j)|$. Otherwise suppose $k \geq 1$ and $s \in U \cap S$ with $\text{lh}(s) = j + 1$. Then $s \restriction \ell_{k-1} \in \mathcal{N}(U \cap S, \ell_{k-1})$, and by definition the number of possible values for $s(j)$ is bounded above by $|\text{Im } (U, j)|$. Thus the number of ways to get such an s is no more than $|\mathcal{N}(U \cap S, \ell_{k-1})| \cdot |\text{Im }(U, j)|$, i.e. we must have $|\mathcal{N}(U \cap S, j)| \leq |\mathcal{N}(U \cap S, \ell_{k-1})| \cdot |\text{Im }(U, j)|$.

Proof of (4). Let k, k' be the unique integers for which $\ell_k \leq j < \ell_{k+1}$ and $\ell_{k'} \leq j' < \ell_{k'+1}$. Since $j \leq j'$, we have $k \leq k'$. We will prove the claim by induction on k'. Suppose $k = k'$. Notice that in this case every node s in $\mathcal{N}(U \cap S, j')$ restricts to a unique node in $\mathcal{N}(U \cap S, j)$, and hence we have $|\mathcal{N}(U \cap S, j')| \leq |\mathcal{N}(U \cap S, j)|$. Thus we have

$$\frac{|\mathcal{N}(U \cap S, j')|}{|\mathcal{N}(S, j')|} \leq \frac{|\mathcal{N}(U \cap S, j)|}{|\mathcal{N}(S, \ell_k)|} = \frac{|\mathcal{N}(U \cap S, j)|}{|\mathcal{N}(S, j)|}.$$

Then suppose the claim holds for some $k' - 1$ and consider k'. Then by the base case above, the inductive hypothesis, and claim (1), we obtain the following:

$$\begin{aligned}
\frac{|\mathcal{N}(U \cap S, j')|}{|\mathcal{N}(S, j')|} &\leq \frac{|\mathcal{N}(U \cap S, \ell_{k'})|}{|\mathcal{N}(S, \ell_{k'})|} \\
&\leq \frac{|\mathcal{N}(U \cap S, \ell_{k'-1})| \cdot |\text{Im }(U \cap S, \ell_{k'})|}{|\mathcal{N}(S, \ell_{k'-1})| \cdot |\text{Im }(S, \ell_{k'})|} \\
&\leq \frac{|\mathcal{N}(U \cap S, \ell_{k'-1})|}{|\mathcal{N}(S, \ell_{k'-1})|} \cdot 1 \\
&\leq \frac{|\mathcal{N}(U \cap S, j)|}{|\mathcal{N}(S, j)|}.
\end{aligned}$$

This completes the induction and proves the claim.

Proof of (5). Observe that the collection $\{N_s : s \in \mathcal{N}(U \cap S, j)\}$ is an open cover of $[U \cap S]$. Hence, using claim (2) and the fact that $\text{supp}(\mu) = M = [S]$, we have

$$\begin{aligned}
\mu([U]) &= \mu([U] \cap [S]) \\
&= \mu([U \cap S]) \\
&\leq \sum_{s \in \mathcal{N}(U \cap S, j)} \mu(N_s)
\end{aligned}$$

$$= \sum_{s \in \mathcal{N}(U \cap S, j)} \frac{1}{|\mathcal{N}(S,j)|}$$
$$= \frac{|\mathcal{N}(U \cap S, j)|}{|\mathcal{N}(S,j)|},$$

as desired. This last claim gives us an easy computational method for estimating the measures of sets given by trees on $\omega^{<\omega}$.

Now we are ready to show that μ gives measure 0 to every two-sided translate of F.

To see this, first fix any $\sigma, \tau \in S_\infty$, and consider the tree $\sigma T \tau$. We claim that there exists a sequence $(j_k)_{k\in\omega}$ of integers such that $(\tau^{-1}(j_k))_{k\in\omega}$ is increasing, and such that

$$\frac{|\mathcal{N}(\sigma T\tau \cap S, \tau^{-1}(j_k))|}{|\mathcal{N}(S, \tau^{-1}(j_k))|} \leq \frac{1}{2^k}$$

for each k. We will construct this sequence recursively.

To begin, by our choice of n_0, there exists some $j_0 \in \omega$ for which $|\mathrm{Im}\ (T, j_0)| = |\mathrm{Im}\ (\sigma T\tau, \tau^{-1}(j_0))| = n_0$. Let k_0 be the unique integer for which $\ell_{k_0} \leq \tau^{-1}(j_0) < \ell_{k_0+1}$. If $k_0 = 0$, then by claims (1) and (3) above we have

$$\frac{|\mathcal{N}(\sigma T\tau \cap S, \tau^{-1}(j_0))|}{|\mathcal{N}(S, \tau^{-1}(j_0))|} \leq \frac{|\mathrm{Im}\ (\sigma T\tau, \tau^{-1}(j_0))|}{|\mathcal{N}(S, \ell_0)|}$$
$$\leq \frac{n_0}{2n_0}$$
$$= \frac{1}{2}.$$

Otherwise if $k_0 > 1$, then again by claims (1) and (3), we have

$$\frac{|\mathcal{N}(\sigma T\tau \cap S, \tau^{-1}(j_0))|}{|\mathcal{N}(S, \tau^{-1}(j_0))|} \leq \frac{|\mathcal{N}(\sigma T\tau \cap S, \ell_{k_0-1})| \cdot |\mathrm{Im}\ (\sigma T\tau, \tau^{-1}(j_0))|}{\prod_{i=0}^{k_0}(2n_i)}$$

$$\leq \frac{|\mathcal{N}(S, \ell_{k_0-1})|}{|\mathcal{N}(S, \ell_{k_0-1})|} \cdot \frac{n_0}{2n_{k_0}}$$
$$\leq \frac{n_0}{2n_0}$$
$$= \frac{1}{2}.$$

Now suppose we have constructed the necessary j_{k-1} for some $k \geq 1$; we will construct j_k. First let k_0 be the unique integer which puts $\ell_{k_0} \leq \tau^{-1}(j_{k-1}) < \ell_{k_0+1}$. By our choice of n_{k_0+1}, there are at least $\ell_{k_0+1} + 1$ distinct integers j for which $|\text{Im }(\sigma T \tau, \tau^{-1}(j)| = |\text{Im }(T, j)| \leq n_{k_0+1}$. Hence at least one of these integers, which we will now call j_k, is such that $\tau^{-1}(j_k) \geq \ell_{k_0+1}$ (since τ^{-1} is an injection!). Let k_1 be the unique integer for which $\ell_{k_1} \leq \tau^{-1}(j_k) < \ell_{k_1+1}$, so $\ell_{k_1-1} \geq \ell_{k_0} > \tau^{-1}(j_{k-1})$. Now by claims (1), (3), and (4), and our inductive hypothesis, we get

$$\frac{|\mathcal{N}(\sigma T \tau \cap S, \tau^{-1}(j_k))|}{|\mathcal{N}(S, \tau^{-1}(j_k))|} \leq \frac{|\mathcal{N}(\sigma T \tau \cap S, \ell_{k_1-1})| \cdot |\text{Im }(\sigma T \tau, \tau^{-1}(j_k))|}{\prod_{i=0}^{k_1}(2n_i)}$$
$$= \frac{|\mathcal{N}(\sigma T \tau \cap S, \ell_{k_1-1})|}{|\mathcal{N}(S, \ell_{k_1-1})|} \cdot \frac{n_{k_0}}{2n_{k_1}}$$
$$\leq \frac{|\mathcal{N}(\sigma T \tau \cap S, \tau^{-1}(j_{k-1}))|}{|\mathcal{N}(S, \tau^{-1}(j_{k-1}))|} \cdot \frac{n_{k_0}}{2n_{k_0}}$$
$$\leq \frac{1}{2^{k-1}} \cdot \frac{1}{2}$$
$$= \frac{1}{2^k}.$$

This completes the construction. It now follows from claim (5) that

$$\mu(\sigma F \tau) = \mu(\sigma F \tau \cap M)$$

$$
\begin{aligned}
&\leq \mu([\sigma T\tau \cap S]) \\
&\leq \frac{|\mathcal{N}(\sigma T\tau \cap S, \tau^{-1}(j_k))|}{|\mathcal{N}(S, \tau^{-1}(j_k))|} \\
&\leq \frac{1}{2^k}
\end{aligned}
$$

for every integer k. So $\mu(\sigma F\tau) = 0$, and F is Haar null. $\square$

COROLLARY 5.40. *Every compact subset of S_∞ is Haar null.*

PROOF. Every compact subset K of S_∞ is the set of branches through some finitely branching tree $T \subseteq \omega^{<\omega}$ by Corollary 2.40. So Im (T, j) is finite for every $j \in \omega$, and Proposition 5.39 applies. $\square$

5.6. A Large Group Property in $\mathrm{Homeo}^+[0,1]$

In this section we observe a somewhat bizarre property of the homeomorphism group $\mathrm{Homeo}^+[0,1]$ which cannot exist in any (non-trivial) locally compact group. The definition given below is certainly not standard.

DEFINITION 5.41. Let G be topological group. We will say G is *amorphous* if it has the following property: whenever $K \subseteq G$ is compact and $U \subseteq G$ is open and nonempty, there exist $g, h \in G$ such that $gKh \subseteq U$.

LEMMA 5.42. *Let G be a Hausdorff locally compact group consisting of at least two points, and let μ be a left Haar measure on G. Then there is a compact set $K \subseteq G$ and a nonempty open set $U \subseteq G$ such that $\mu(K) > \mu(U)$.*

PROOF. Since G is Hausdorff and has at least two points, let U, V be disjoint nonempty open sets in G with compact closure. Set $K = \mathrm{cl}_G(U) \cup \mathrm{cl}_G(V)$. Since V is nonempty open, $\mu(V) > 0$, and hence we have $\mu(U) < \mu(U) + \mu(V) \leq \mu(U \cup V) \leq \mu(\mathrm{cl}_G(U) \cup \mathrm{cl}_G(V)) = \mu(K)$. $\square$

Proposition 5.43. *Let G be a Hausdorff locally compact topological group consisting of at least two points. Then G is not amorphous.*

Proof. First suppose G is unimodular, and let μ be a two-sided Haar measure on G. Use 5.42 to find a compact set $K \subseteq G$ and a nonempty open set $U \subseteq G$ for which $\mu(K) > \mu(U)$. If there were $g, h \in G$ for which $gKh \subseteq U$, we would have $\mu(K) = \mu(gKh) \leq \mu(U)$, a contradition; so no such g and h exist.

If G is not unimodular, then let Δ be the modular function of G and let $N = \ker \Delta$. G/N is a Hausdorff locally compact group, and it must contain at least two points, for if it were a singleton then G would have been unimodular in the first place. G/N is group isomorphic to a subgroup of $\mathbb{R}^+$ and hence abelian, so let μ be a two-sided Haar measure on G/N. Use Lemma 5.42 to find a compact subset $L \subseteq G/N$ and a nonempty open set $V \subseteq G/N$ such that $\mu(L) > \mu(V)$.

Let $\pi : G \to G/N$ be the natural projection. Let $K \subseteq G$ be a compact set for which $\pi(K) = L$ (see [58] Exercise 18E (4)) and let $U = \pi^{-1}(V) \subseteq G$, so U is open and nonempty. If there were $g, h \in G$ for which $gKh \subseteq U$, then we would have $\pi(g)L\pi(h) = \pi(g)\pi(K)\pi(h) = \pi(gKh) \subseteq \pi(U) = V$, and hence $\mu(L) = \mu(\pi(g)L\pi(h)) \leq \mu(V)$, again a contradiction. So no such g and h exist. $\square$

The following lemma is presented here as in [50]. We make use of it here to prove Proposition 5.45. A corollary of Proposition 5.45 is that $\mathrm{Homeo}^+[0,1]$ contains no non-empty openly Haar null sets- so $\mathcal{OHN}(\mathrm{Homeo}^+[0,1])$ is the trivial σ-ideal.

Lemma 5.44 (Shi/Thomson). *A set $K \subseteq \mathrm{Homeo}^+[0,1]$ is compact iff K is closed, equicontinuous, and for every closed nonempty set $K_0 \subseteq K$, the functions $f_1(x) = \inf\{k(x) : k \in K_0\}$ and $f_2(x) = \sup\{k(x) : k \in K_0\}$, $x \in [0,1]$, are homeomorphisms of $[0,1]$.*

Proposition 5.45. *Let $K \subseteq \mathrm{Homeo}^+[0,1]$ be compact, and let U be the ϵ-ball about the identity in $\mathrm{Homeo}^+[0,1]$ (with the uniform metric), for some $\epsilon > 0$. Then there exist some $g, h \in \mathrm{Homeo}^+[0,1]$ for which $gKh \subseteq U$.*

Proof. Define $f_1(x) = \inf\{k(x) : k \in K\}$ and $j(x) = \sup\{k(x) : k \in K\}$ for $x \in [0,1]$; by Lemma 5.44, $f_1, j \in \mathrm{Homeo}^+[0,1]$. Clearly we have $f_1(x) \leq k(x) \leq j(x)$ for all $x \in [0,1]$ and all $k \in K$. Set $f_2(x) = \sqrt{j(x)}$, so $f_2 \in \mathrm{Homeo}^+[0,1]$ and $f_2(x) > j(x)$ for all $x \in (0,1)$. Thus we have $f_1(x) \leq k(x) \leq f_2(x)$ for all $x \in [0,1]$, and $f_1(x) < f_2(x)$ for all $x \in (0,1)$.

Let $x_0 \in (0,1)$ be arbitrary, and for each $n \in \mathbb{Z}$, set $x_n = (f_1^{-1} f_2)^n(x_0)$. Set $y_0 = f_2(x)$, and for each $n \in \mathbb{Z}$, set $y_n = (f_2 f_1^{-1})^n(y_0)$. Thus we obtain two bi-infinite sequences $(x_n)_{n\in\mathbb{Z}}$ and $(y_n)_{n\in\mathbb{Z}}$. Notice that $y_n = f_2(x_n)$ and $f_1(x_n) = y_{n-1}$ for all $n \in \mathbb{Z}$.

Notice also that for all $n \in \mathbb{Z}$, we have $f_1(x_n) < f_2(x_n)$ and hence $x_n < f_1^{-1} f_2(x_n) = x_{n+1}$, so the sequence $(x_n)_{n=0}^{\infty}$ is increasing and bounded above by 1, and the sequence $(x_{-n})_{n=0}^{\infty}$ is decreasing and bounded below by 0. In addition, the inverses f_1^{-1} and f_2^{-1} are homeomorphisms which satisfy $f_2^{-1} < f_1^{-1}$ on $(0,1)$, so for each $n \in \mathbb{Z}$ we have $f_2^{-1}(y_n) < f_1^{-1}(y_n)$ and hence $y_n < f_2 f_1^{-1}(y_n) = y_{n+1}$. So $(y_n)_{n=0}^{\infty}$ is increasing and bounded by 1, and $(y_{-n})_{n=0}^{\infty}$ is decreasing and bounded by 0. So all four sequences converge to some respective limits.

We claim that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$ and $\lim_{n\to-\infty} x_n = \lim_{n\to-\infty} y_n = 0$. To see this for the first sequence, suppose for a contradiction that $\lim_{n\to\infty} x_n = L$ for some $L < 1$. Then by the continuity of $f_1^{-1} f_2$, we have $f_1^{-1} f_2(L) = \lim_{n\to\infty} f_1^{-1} f_2(x_n) = \lim_{n\to\infty} x_{n+1} = L$, i.e. $f_1(L) = f_2(L)$. This contradicts the fact that $f_1 < f_2$ on $(0,1)$. So we must have $\lim_{n\to\infty} x_n = 1$. Analogous arguments, repeated three times, will show that the other sequences also converge to the endpoints of the interval in the way we intend.

Now let $(z_n)_{n\in\mathbb{Z}}$ be an arbitrary bi-infinite sequence in $[0,1]$ which satisfies the following properties:

(1) $z_{n+1} > z_n$ for all $n \in \mathbb{Z}$;
(2) $z_{n+1} - z_n < \dfrac{\epsilon}{4}$ for all $n \in \mathbb{Z}$;
(3) $\lim_{n\to\infty} z_n = 1$; and
(4) $\lim_{n\to-\infty} z_n = 0$.

Now, using the fact that each of our bi-infinite sequences $(x_n), (y_n), (z_n)$ accumulates only at the endpoints of $[0,1]$, we let h be any increasing homeomorphism of the interval for which

$$h([z_{2n-1}, z_{2n+1}]) = [x_n, x_{n+1}] \text{ for all } n \in \mathbb{Z},$$

and we let g be any homeomorphism for which

$$g([y_n, y_{n+1}]) = [z_{2n}, z_{2n+2}] \text{ for all } n \in \mathbb{Z}.$$

(For instance, we could take both g and h to be piece-wise linear.)

Let $k \in K$ be arbitrary. We claim that $gkh \in U$. Let $x \in (0,1)$. Find the unique $n \in \mathbb{Z}$ for which $x \in [z_{2n-1}, z_{2n+1}]$. Then $h(x) \in [x_n, x_{n+1}]$, and since $f_1 \leq k \leq f_2$, and all functions involved are increasing, we have

$$\begin{aligned} y_{n-1} &= f_1(x_n) \\ &\leq f_1(h(x)) \\ &\leq k(h(x)) \\ &\leq f_2(h(x)) \\ &\leq f_2(x_{n+1}) \\ &= y_{n+1}. \end{aligned}$$

So $kh(x) \in [y_{n-1}, y_{n+1}]$ and hence $gkh(x) \in [z_{2n-2}, z_{2n+2}]$. Since we also have $x \in [z_{2n-1}, z_{2n+1}] \subseteq [z_{2n-2}, z_{2n+2}]$, and the latter set has diameter $< \epsilon$, we have $|gkh(x) - x| < \epsilon$. Since x was taken arbitrarily, we have shown $gkh \in U$ and hence $gKh \subseteq U$. $\square$

Corollary 5.46. $\mathrm{Homeo}^+[0,1]$ *is amorphous.*

Proof. Let $K \subseteq \mathrm{Homeo}^+[0,1]$ be compact and $U \subseteq \mathrm{Homeo}^+[0,1]$ open nonempty. Let $u \in U$. Then $u^{-1}U$ is a neighborhood of identity and hence contains an ϵ-ball about the identity in the uniform metric by Theorem 2.34. By Proposition 5.45 there are g, h for which $gKh \subseteq u^{-1}U$, whence $ugKh \subseteq U$. This proves the corollary. $\square$

COROLLARY 5.47. *There are no nonempty openly Haar null sets in* $\mathrm{Homeo}^+[0,1]$.

PROOF. Let A be any nonempty set in $\mathrm{Homeo}^+[0,1]$. Let μ be any measure on G which is supported on a compact set $K \subseteq G$, and let $V \supseteq A$ be open. Let $v \in V$, so $v^{-1}V$ is a nonempty neighborhood of the identity. Then there exist homeomorphisms g and h such that $gKh \subseteq v^{-1}V$, and hence $(vg)^{-1}Vh^{-1} \supseteq K$. Thus $\mu((vg)^{-1}Vh^{-1}) = 1$, and V has a two-sided translate with full measure. Since V was arbitrary, A cannot be openly Haar null. $\square$

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