ASYMPTOTIC FORMULA FOR COUNTING IN DETERMINISTIC AND RANDOM DYNAMICAL SYSTEMS

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The lattice point problem in dynamical systems investigates the distribution of certain objects with some length property in the space that the dynamics is defined. This problem in different contexts can be interpreted differently. In the context of symbolic dynamical systems, we are trying to investigate the growth of N(T), the number of finite words subject to a specific ergodic length T, as T tends to infinity. This problem has been investigated by Pollicott and Urbański to a great extent. We try to investigate it further, by relaxing a condition in the context of deterministic dynamical systems. Moreover, we investigate this problem in the context of random dynamical systems. The method for us is considering the Fourier-Stieltjes transform of N(T) and expressing it via a Poincaré series for which the spectral gap property of the transfer operator, enables us to apply some appropriate Tauberian theorems to understand asymptotic growth of N(T). For counting in the random dynamics, we use some results from probability theory.

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CHAPTER 1

INTRODUCTION

The lattice point (counting) problem in math has long history dating as far back as Gauss circle problem. Gauss tried to obtain an asymptotic formula for number of points in the plane with integer coordinates inside a circle of radius T as T grows. Later on Sierpinski, Walfisz, Iwaniec & Mozzochi [17], Huxley [15], Hardy [16, p. 372], Landau and Hafner [13] contributed to problems closely related to Gauss circle problem for purpose of obtaining better estimates.

The analogous problem in the context of hyperbolic spaces as well gained a lot of attention starting in 1942 with (unnoticed) work of Delsarte, where he considered the hyperbolic plane $\mathbb{H}^2$ and instead of $\mathbb{Z}^2$ he considered orbit of a point $z \in \mathbb{H}^2$ under the action of a Fuchsian group $G \subseteq \mathrm{PSL}(2, \mathbb{R})$. He obtained an asymptotic formula for number of $g \in G$ that move z at most by T as T grows. Here the distance is measured by a hyperbolic metric of constant negative curvature [7]. Independently, Huber published his result on this problem in 1956. His approach uses spectral decomposition of Laplacian operator, where G doesn't contain parabolic elements because he assumes the fundamental domain is compact [14]. In the same year, Selberg extended this decomposition for the case G contains parabolic elements where fundamental domain has finite area. He used the celebrated trace formula for this [37, p. 77]. This helped Patterson to approach the problem in generality providing some error term as well [30]. Along with these works, Margulis answered similar question in higher dimensional hyperbolic space in 1969 [24, p. 48]. Several others have contributed to this problem in different contexts including Sarnak [36], Lax & Phillips [23], Parry & Pollicott [29], Lalley [22], Mirzakhani [26] and etc.

Recently, Pollicott & Urbański jointly obtained an asymptotic formula in the context of conformal dynamical systems, see corollary 3.22 or [33, p. 39]. For this, it is enough to have graph directed Markov system in which our functions in the system are contractions and satisfy certain properties. The most important one is the conformal property which is angle preserving orientation preserving or orientation reversing. They use infinite theory of graph directed Markov system developed by Mauldin & Urbański [25] and complex transfer operator developed by Pollicott [32] to obtain an asymptotic formula for counting finite words in the shift space for which the corresponding composition function of the system has derivative at least e^{-T} as T grows. Further, they introduce slightly different system in which finitely many parabolic elements are allowed and they apply the aforementioned asymptotic formula for this system. These two kinds of conformal systems have many applications one of which is an asymptotic formula for the planer Apollonian circle packing problem. The circle packing was studied in 1970s by Boyd [3] and estimates of number of circles of radius at $1/T$ was obtained by him in 1980s [4]. This estimate had major improvement due to Kontorovich & Oh in 2011 [20] and Oh & Shah in 2012 [28]. The former article focuses on two cases: (a) number of circles of radius at least $1/T$ inside the biggest circle tangent to the three circles that generate the gasket, and (b) number of circles of radius at least $1/T$ between two parallel lines generating the gasket up to a period of the gasket. One year later, the latter article obtains a similar formula for case (c) number of circles of radius at least $1/T$ bounded in curvilinear triangle whose sides are parts of three circles tangent to each other . The method for Kontorovich-Oh-Shah is equidistribution of expanding closed horospheres on hyperbolic 3-manifolds $G\backslash\mathbb{H}^3$ where G is geometrically finite torsion-free discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Further they use Patterson-Sullivan theory of conformal density (measure) in which the Laplacian operator has simple isolated eigenvalue $-\delta_G(2-\delta_G)$ where δ_G is the Hausdorff dimension of the limit set under the assumption $\delta_G > 1$ [38, p. 195], [31, p. 272].

In Pollicott & Urbański's work the spectral theory is analyzed for the transfer operator instead where they assume their system has D-generic property which prevents the situation that the transfer operator admitting 1 as spectral value on the critical line $\mathrm{Re}(s) = \delta$ of the Poincaré series except at the exponent itself $s = \delta$. The other condition they impose on the system is strong regularity which can be perceived to be analogous to the assumption $\delta_G > 1$ mentioned above.

In this dissertation, we relax the D-generic assumption to see how Pollicott & Urbański's result changes, see theorem 3.19. We noticed that in this situation, we no longer obtain only one asymptotic formula. We may obtain continuum many relations. More precisely we can see that the ratio can converge to a full range of closed interval rather than just a point in Pollicott & Urbański's result, see example 3.25. However, we can obtain lower bound for the infimum and upper bound for the supremum. These bounds are shown to be sharp by an example, see example 3.25. We should mention that we only assume we are given a real-valued summable Hölder-type function on the shift space. We don't assume necessarily the function is induced by a conformal system. Theorem 3.19 is the main result of chapter 3. This involves spectral analysis of the transfer operator which we adapt from Pollicott & Urbański and a Tauberian theorem 2.42 of Graham & Vaaler. Further we investigate an asymptotic of the length for which the counting function is related to. Given $T > 0$ the maximum length contributing to the counting function is itself subject to an asymptotic formula, see proposition 3.23. The chapter 4 investigates counting problem for a random system. There might be different ways to define random systems. We believe our formulation of random systems is one natural way; however, the counting problem in this sense gets much harder than before. The main result of chapter 4 is corollary 4.10. This shows the difference of counting in deterministic and random settings. The main tool in this corollary is the law of iterated logarithm from probability theory. We further tried to find a formula for more general systems, but with some assumptions on the random factor. The main assumption restricts fluctuations of a random walk. Under this assumption we obtained some asymptotic formulas, see theorem 4.7 and theorem 4.5. Furthermore, we constructed an example in which the result suggested by the theorem 4.5 still holds under weaker condition, see example 4.12.

In chapter 2, we try to mention background materials for chapters 3 and 4. We first introduce the symbolic space. The notion of Hölder continuity for real or complex functions over the symbolic spaces comes after. Then we continue with some common dynamical notions such as pressure, Gibbs and equilibrium measure (state). Moreover, we define properties such as strong regularity and D-generic property. The latter property is of significant concern in this dissertation. Later, we introduce transfer operator and talk a bit about perturbation theory of analytic operator. This requires essential spectrum concept. We mention the notion of graph directed Markov systems. Furthermore, we introduce random dynamics in the context of graph directed Markov system. Finally, we briefly mention two Tauberian theorems before we finish the preliminaries chapter with some examples.

In chapter 3 our aim is obtaining a general asymptotic formula for counting in dynamical systems. There is no random dynamics flavors in this chapter. First we apply the analytic perturbation to the transfer operator to obtain a spectral representation of the transfer operator over its maximal eigenvalues. However, we need to show these eigenvalues are simple first. This requires introducing a weighted transfer operator. Next, we talk about the relation between a complex function (Poincaré series), and counting function. The idea is by taking Riemmann-Stieltjes integral against our target counting function, we obtain a Poincaré series. Furthermore, we bring some estimates to find some upper and lower bounds for counting finite periodic words in terms of our ordinary counting function. Then, we use the spectral representation to argue how Graham & Vaaler's Tauberian theorem is applicable to imply the main theorem 3.19 . We use this theorem to obtain Pollicott & Urbański's formula as a corollary. Finally, we talk about the asymptotic of the length that is conjectured to contribute the most to our counting function. We provide two asymptotic formulas for length. We finish by three examples to see how our estimates of bounds are sharp in the last section.

In chapter 4, we just focus on random systems. We try to investigate the counting problem for random graph directed Markov systems. The first section is devoted to formulating the problem first. Next, we investigate the problem for some class of random factor λ when they follow a periodic (or eventually periodic) pattern. In the next section, we loosen this pattern by some other conditions and we again obtain an exponential growth. Before finishing by examples we construct a system with non-exponential growth for counting. In this section we no longer use any Tauberian theorem and instead we compute our counting function directly.

We would like to mention that throughout this dissertation we try to stay loyal to the following conventions:

- e, ω_i, ρ_i : letter
- E : set of letters
- ω, τ, γ : finite words containing letters from E
- ρ, ρ' : infinite one-sided sequence containing letters from E
- $E_A^{\mathbb{N}}$: set of infinite one-sided sequences
- T : positive real number
- x, y : real number
- $s = x + iy$: complex number
- σ : shift map on $E_A^{\mathbb{N}}$
- m_x, μ, ν : measure
- f, g, h : real or complex functions on $E_A^{\mathbb{N}}$ or on $E_A^{\mathbb{N}} \times \Lambda$
- $\mathcal{B}$: Borel sets of $E_A^{\mathbb{N}}$
- $\mathbb{1}_B$: indicator function of a Borel set $B \in \mathcal{B}$
- $C^{0,\alpha}$: space of Hölder functions of exponent α
- $K, Q, c, c_1, C_1, C, D, c_\delta$: constants
- $\mathcal{L}, \mathcal{P}, \mathcal{Q}, \mathcal{D}, \mathcal{F}, \mathcal{E}$: operators
- $\mathcal{E}$: expected pressure (only in chapter 4)
- ξ : spectral value or eigenvalue
- Sp : spectrum
- Γ : interval of form (x_0, ∞)
- Γ^+ : some right half plane $\Gamma \times \mathbb{R}$
- $\mathcal{Z}$: countable (finite or infinite) set of real or complex numbers in the closed unit disk
- λ : infinite two-sided sequence containing letters from $\mathcal{Z}$
- $\mathcal{Z}^{\mathbb{Z}}$: set of infinite two-sided sequences
- $N_\rho(B, T), N_\rho(T), N_\rho^\lambda(T)$: counting function in T

- $\eta_\rho(B, s), \eta_\rho(s), \eta_\rho^\lambda(s)$: complex function in s
- $S_n f$: ergodic sum of f
- $s_{n,z}(\lambda)$: random walk on the random variables $\{\mathbb{1}_z\}_{z \in \mathcal{Z}}$
- X_v : a Euclidean domain
- t : a point of X_v
- ϕ_e : a function of a system on X_v
- ϕ'_e : derivative of ϕ_e
- ϕ_ω : composition of functions in order of the letters appearing in ω
- $\pi(\rho)$: limit point of $\rho \in E_A^{\mathbb{N}}$
- J : set of all the limit points
- δ : Hausdorff dimension of the limit set J
- $\mathrm{Int}(A)$: interior of the set A

CHAPTER 2

PRELIMINARIES

2.1. Shift Space

Let E be a countable (finite or infinite) set calling each of its elements a symbol, a letter or an alphabet. By $E^{\mathbb{N}}$ we mean the set of all infinite sequences of the form

$$e_1e_2e_3...e_n...,$$

where each e_i belongs to E. We usually represent the first n symbols of such a sequence, also called (finite) word or block, by ω throughout this work, i.e.

$$\omega = e_1e_2...e_n,$$

where we sometimes tend to identify ω_i with e_i and just have

$$\omega = \omega_1\omega_2...\omega_n.$$

When we write $|\omega| = n$ we just mean the word ω has n letters. By E^n we represent all the words of length n and by E^* we represent $\cup_{n=1}^{\infty} E^n$. As well we use the notation $|. \wedge .|$ to represent the number of common initial symbols in two sequences, i.e. for $\rho = e_1e_2...$ and $\rho' = e'_1e'_2...$ we have

$$|\rho \wedge \rho'| = m \Leftrightarrow e_1 = e'_1,\ e_2 = e'_2,\ ...,\ e_m = e'_m,\ e_{m+1} \neq e'_{m+1}.$$

One can as well introduce a metric by

$$d(\rho, \rho') = e^{-|\rho\wedge\rho'|}.$$

Further we set

$$d_\alpha = d^\alpha,\ \ 0 < \alpha < 1,$$

i.e. we have

$$d_\alpha(\rho, \rho') = e^{-\alpha|\rho\wedge\rho'|}.$$

Now we equip $E^{\mathbb{N}}$ with a metric space, which is called symbolic space. Note that the topology on $E^{\mathbb{N}}$ induce by this metric is the same as the Tychonoff topology where each E is equipped with ordinary discrete topology. This means for any α and β the topologies of d_α and d_β are the same, however the metrics are not equivalent for different α and β.
One can then see that the shift map $\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$ given by

$$\sigma(e_1e_2...) = e_2e_3...$$

is a continuous map.

Furthermore, we want to restrict ourselves to sequences that certain words are not appearing. We first introduce a map $A : E \times E \to \{0, 1\}$ (sometimes called incidence or transition matrix). We use $A_{ee'}$ notation instead of $A(e, e')$. A subshift of finite type consists of the sequences $e_1e_2e_3...$ in $E^{\mathbb{N}}$ such that

$$A_{e_1e_2} = 1,\ A_{e_2e_3} = 1,\ ...,\ A_{e_ne_{n+1}} = 1,\ ...\ .$$

Of course, if A only assumes the value 1, represented by $A = 1$, then this is just the space introduced earlier, that is why we sometimes call $(E^{\mathbb{N}}, \sigma)$ full shift space. Additionally, when $A_{e_1e_2} = 1$ we say e_1e_2 is A-admissible or just admissible. As well, by E_A^* we mean all admissible finite words of all lengths, by E_ρ^* we mean all $\omega \in E_A^*$ such that $\omega\rho$ is an admissible sequence, by E_ρ^n we mean all $\omega \in E_A^n$ such that $\omega\rho$ is an admissible sequence, by E_{per}^* we mean all $\omega \in E_A^*$ such that $\omega_n\omega_1$ is admissible and we say ω is periodic word, by $\bar{\omega}$ we mean the sequence $\omega\omega\omega...$ and by E_A^n we mean all admissible words of length n. Finally, for each finite word ω of length n we define the cylinder

$$[\omega] := \{\rho \in E_A^{\mathbb{N}} :\ \rho_1...\rho_n = \omega\}.$$

PROPOSITION 2.1. *For the subshift of finite type $E_A^{\mathbb{N}}$ the followings hold:*

a. *All the cylinders form a countable clopen basis.*

b. *Every open set can be written as countable union of mutually disjoint cylinders.*

c. *It is a Polish space.*

PROOF. (a). It is clear that for each positive integer n, we have countably many finite words of length n, therefore there are only countably many cylinders. Next we show each cylinder is a neighborhood in $E_A^{\mathbb{N}}$. Let ω be a finite word of length n, choose any fixed $\rho \in [\omega]$, we show $[\omega] = N(\rho, e^{-\alpha(n-1)})$. Note that ρ' is in $[\omega]$ iff $d_\alpha(\rho, \rho') < e^{-\alpha(n-1)}$ iff $|\rho \wedge \rho'| > n-1$ iff $|\rho \wedge \rho'| \geq n$ iff $\rho' \in [\omega]$. To see $[\omega]$ is closed, consider a sequence $\{\rho_{(i)}\}_i$ in $[\omega]$ converging to ρ. This means $|\rho_{(i)} \wedge \rho| \to \infty$ which clearly implies $\rho \in [\omega]$. Now for every open set V and every $\rho \in V$, note that there is $\epsilon > 0$ such that $\rho \in N(\rho, \epsilon) \subseteq V$. We choose n large enough such that $e^{-\alpha(n-1)} < \epsilon$, then obviously $[\rho_1\rho_2...\rho_n] = N(\rho, e^{-\alpha(n-1)}) \subset N(\rho, \epsilon) \subset V$.

(b). The fact that an open V can be written as countable union of cylinder is clear from part a. Then part b follows from the fact that for any two cylinders $[\omega]$ and $[\tau]$ that meet each other, we have either $[\omega] \subset [\tau]$ or $[\tau] \subset [\omega]$. To show this, assume ρ belongs to both of the cylinders $[\omega]$ and $[\tau]$. Further, assume $|\omega| \leq |\tau|$. Since $\rho \in [\tau]$, we should have $\rho = \tau\rho'$ for some $\rho' \in E_A^{\mathbb{N}}$, similarly $\rho \in [\omega]$ implies that $\rho = \omega\rho''$ for some $\rho'' E_A^{\mathbb{N}}$. Thus $\tau\rho' = \rho = \omega\rho''$ and since $|\omega| \leq |\tau|$ so $\tau = \omega\omega'$ for some finite word ω'. This implies $[\tau] \subseteq [\omega]$.

(c). Note that countable product of separable space is separable and countable product of complete metrizable space is complete metrizable. □

We would like to mention that we only work with probability measure over Borel sets all through this work.

DEFINITION 2.2. We call a subshift finitely irreducible if there exists a finite set Ω containing words such that for all $e, e' \in E$ there is $\omega \in \Omega$ such that $e\omega e'$ is admissible. As well subshift is called finitely primitive if it is finitely irreducible and all words in Ω are of fixed length.

Throughout this dissertation, we restrict ourselves to work with finitely irreducible subshifts.

REMARK 2.3. Note that this notion is just a generalization of irreducible matrix when E is countable. In fact, finitely irreducible condition guarantees that the shift map is topologically mixing, and finitely primitive guarantees that the shift map is topologically exact. If E is finite then A is irreducible iff σ is topologically mixing iff σ is transitive. Further, if E is finite then A is primitive iff σ is topologically exact. It is clear that if the shift space is finitely irreducible then the backward orbit of every element is dense, i.e.

$$\overline{\cup_{n=0}^{\infty}\sigma^{-n}(\rho)} = E_A^{\mathbb{N}}.$$

PROPOSITION 2.4. *If E is finite,*

$$\log r(A) = \lim_n \frac{1}{n} \log \# E_A^n,$$

where $r(A)$ is spectral radius of matrix A.

PROOF. We refer to theorem 3.2.22 [39]. □

2.2. Ergodicity

DEFINITION 2.5. For a measurable transformation $T : X \to X$ on a measure space $(X, \mathcal{B})$ we say a measure μ is $T-$invariant if for every $A \in \mathcal{B}$:

$$\mu(T^{-1}(A)) = \mu(A).$$

Further we say μ is ergodic if μ is $T-$invariant measure such that if $T^{-1}(A) = A$ then either $\mu(A) = 0$ or $\mu(A) = 1$.

We are now ready to express one of one of the main theorems in Ergodic Theory. Before that we need the following notion of Birkhoff sum for any real function $g : X \to \mathbb{R}$, we set

$$S_n g(x) := \sum_{i=0}^{n-1} g(T^i(x)).$$

THEOREM 2.6 (Birkhoff's Ergodic Theorem). *Let $T : X \to X$ be a map on probability space $(X, \mathcal{B}, \mu)$. If μ is $T-$invariant and ergodic measure, for any $\phi \in \mathcal{L}^1(X)$ we have*

$$\frac{1}{n} S_n \phi(x) \to \int_X \phi d\mu, \qquad a.e.\ x \in X$$

PROOF. We refer to corollary 8.2.14 in [39]. □

2.3. Hölder Continuity

Next we want to talk about the Hölder continuous maps. In Analysis textbooks [10, p. 52] we have different notions of Hölder continuity of exponent α for real or complex-valued functions on a Euclidean space D:

- Hölder at a point x_0: $\sup_{x\in U}\{|f(x)-f(x_0)|/|x-x_0|^\alpha\}$ is finite, where U is a neighborhood of x_0 in D.
- Hölder: $\sup_{x,y\in D}\{|f(x)-f(y)|/|x-y|^\alpha\}$ is finite.
- Locally Hölder: $\sup_{x,y\in K}\{|f(x)-f(y)|/|x-y|^\alpha\}$ is finite for every compact $K\subseteq D$.

We call each of the above suprema the Hölder coefficient. Of course D can be replaced with the metric space $E_A^{\mathbb{N}}$ to obtain similar notions on the shift space. We denote the set of complex-valued Hölder continuous functions of Hölder exponent α on $E_A^{\mathbb{N}}$ by $C^{0,\alpha}(E_A^{\mathbb{N}},\mathbb{C})$ or simply $C^{0,\alpha}$. We remind that the usual Hölder coefficient is defined by:

$$|g|_\alpha = \sup_{\rho,\rho'\in E_A^{\mathbb{N}}}\left\{\frac{|g(\rho)-g(\rho')|}{d_\alpha(\rho,\rho')}\right\}.$$

We would like to define another Hölder coefficient that is justified later. We set:

$$V_{\alpha,n}(f) := \sup\{|f(\rho_1)-f(\rho_2)|e^{\alpha(n-1)} : |\rho_1\wedge\rho_2|\geq n\geq 1\},$$

and

$$V_\alpha(f) := \sup_{n\geq 1} V_{\alpha,n}(f).$$

There is another notion of Hölder continuity useful for our purposes.

Definition 2.7. A complex-valued function f on $E_A^{\mathbb{N}}$ is called Hölder-type continuous with exponent $\alpha>0$ if $V_\alpha(f)<\infty$.

We define a norm on $C^{0,\alpha}(E_A^{\mathbb{N}},\mathbb{C})$ by

$$\|g\|_\alpha := \|g\|_\infty + V_\alpha(g). \tag{1}$$

We are ready to find relations between these different notions of Hölder continuity.

Proposition 2.8. *The followings hold:*

(a) *On* $E_A^{\mathbb{N}}$ *every complex-valued function is Hölder continuous iff it is Hölder-type continuous and bounded.*

(b) *The norm given above in 1 is equivalent to usual* $\|.\|_{C^{0,\alpha}} = \|.\|_\infty + |.|_\alpha$ *norm over* $C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$.

(c) $\left(C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C}), \|.\|_\alpha\right)$ *is Banach space.*

(d) *A Hölder-type continuous function is locally Hölder continuous and Hölder continuous at every point.*

Proof. a) Assume f is Hölder continuous function, then there is M such that

$$|f(\rho_1) - f(\rho_2)| \le M d(\rho_1, \rho_2)^\alpha = M e^{-\alpha|\rho_1 \wedge \rho_2|},$$

for every ρ_1 and ρ_2. Therefore

$$|f(\rho_1)| \le |f(\rho_1) - f(\rho_2)| + |f(\rho_2)| \le M + |f(\rho_2)|.$$

This gives boundedness of f. For Hölder-type, assuming $|\rho_1 \wedge \rho_2| \ge n$, it follows

$$|f(\rho_1) - f(\rho_2)| e^{\alpha(n-1)} \le M e^{-\alpha},$$

i.e. $V_\alpha(f) \le M e^{-\alpha}$.

For the converse, assuming that $|f| \le K$ for some constant K, and $|\rho_1 \wedge \rho_2| = n \ge 1$ we have

$$|f(\rho_1) - f(\rho_2)| e^{\alpha(n-1)} \le V_\alpha(f).$$

Therefore

$$|f(\rho_1) - f(\rho_2)| \le V_\alpha(f) e^{-\alpha(n-1)} = V_\alpha(f) e^\alpha d(\rho_1, \rho_2)^\alpha.$$

In case $|\rho_1 \wedge \rho_2| = 0$, we use boundedness of f to get

$$|f(\rho_1) - f(\rho_2)| \le 2K = 2K d(\rho_1, \rho_2)^\alpha.$$

Thus

$$|f(\rho_1) - f(\rho_2)| \le \max\{2K, V_\alpha(f) e^\alpha\} d(\rho_1, \rho_2)^\alpha,$$

for every ρ_1 and ρ_2.

b) From the proof above we realize that $V_\alpha(f) \le |f|_\alpha e^{-\alpha}$ which leaves

$$\|f\|_\alpha \le \|f\|_\infty + |f|_\alpha e^{-\alpha} \le \|f\|_\infty + |f|_\alpha.$$

Furthermore $|f|_\alpha \le \max\{2K, V_\alpha(f)e^\alpha\}$ gives us

$$\|f\|_\infty + |f|_\alpha \le 3\|f\|_\infty + V_\alpha(f)e^\alpha \le (3 + e^\alpha)\|f\|_\alpha.$$

c) This is a well-known fact, see for example [10, p. 73] for a Euclidean space.

d) This is easy to show. □

REMARK 2.9. We want to justify why we used the terminology Hölder-type:

- The Hölder-type continuous functions subject of study in this dissertation in the case of infinite alphabets are summable. This makes them unbounded and so they are not Hölder.
- Let $E = \mathbb{N}$. One can see that $f : E^{\mathbb{N}} \to \mathbb{R}$ defined by $f(kn_2n_3n_4...) = \ln 1/n_k^2$, is Hölder continuous at each point (consider $[kn_2n_3n_4...n_k]$) and locally Hölder continuous but is not Hölder-type continuous.
- Note that locally Hölder continuous on $E_A^{\mathbb{N}}$ wouldn't imply continuity necessarily, however Hölder continuity at a point clearly implies continuity.
- Regarding Hölder continuity at a point even if we were able to find a uniform bound for Hölder coefficients that worked for all the points it still doesn't imply Hölder-type continuity necessarily.
- Over shift space with finite alphabets Hölder continuity and Hölder-type continuity coincide.

Below we need to use sequence of finite words in the lemma. For that we use the notation $\omega_{(i)}$, to denote that it is not the i^{th} coordinate of ω which we represent it by ω_i.

LEMMA 2.10. *Let $\{\omega_{(i)}\}_{i\in I}$ be any collection of finite words with bounded length, i.e. there exists a positive integer k such that $|\omega_{(i)}| \le k$ for each i. If the cylinders $\{[\omega_{(i)}]\}_{i\in I}$ are* *mutually disjoint, then the indicator function of* $H := \cup_{i \in I}[\omega_{(i)}]$ *is Hölder continuous, i.e.* $\mathbb{1}_H \in C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$.

PROOF. We want to show there exists $M > 0$ such that

$$|\mathbb{1}_H(\rho) - \mathbb{1}_H(\rho')| \leq Md(\rho, \rho'),$$

for every $\rho, \rho' \in E_A^{\mathbb{N}}$. If $\rho, \rho' \in H$, there is nothing to prove as the left hand side is 0. Similarly if $\rho, \rho' \notin H$. If $\rho \in H$ and $\rho' \notin H$, then there is i such that $\rho \in [\omega_{(i)}]$. But $|\rho \wedge \rho'| < |\omega_{(i)}|$, otherwise $\rho' \in [\omega_{(i)}]$. Therefore

$$e^{-k} \leq e^{-|\omega_{(i)}|} \leq e^{-|\rho \wedge \rho'|} = d(\rho, \rho').$$

Thus if we just pick $M = e^k$, then for each ρ, ρ' we have

$$|\mathbb{1}_H(\rho) - \mathbb{1}_H(\rho')| \leq Md(\rho, \rho').$$

□

2.4. Basic Lemmas

LEMMA 2.11. *If* $f : E_A^{\mathbb{N}} \to \mathbb{C}$ *is Hölder-type continuous with* $V_\alpha(f) < \infty$ *then there exists* $K_f > 0$ *such that for any* $\omega \in E_A^n$ *and any* $\rho, \rho' \in E_A^{\mathbb{N}}$ *where* $\omega\rho, \omega\rho'$ *are admissible we have*

$$|S_n f(\omega\rho) - S_n f(\omega\rho')| \leq K_f d(\rho, \rho').$$

PROOF. We refer to [25, p. 26]. □

A sequence $\{a_n\}$ of real numbers is called subadditive if for every positive integer m, n:

$$a_{m+n} \leq a_m + a_n.$$

LEMMA 2.12 (Fekete's Lemma). *For every subadditive sequence* $\{a_n\}$*, the limit of the sequence* $\{\frac{a_n}{n}\}$ *exists and it is equal to* $\inf_n \left\{\frac{a_n}{n}\right\}$.

PROOF. We refer to [25, p. 5]. □

Lemma 2.13. *Let $f_i(T)$ be a collection of non-negative functions defined on $T > 0$. Then*

$$\sum_i \liminf_{T\to\infty} f_i(T) \leq \liminf_{T\to\infty} \sum_i f_i(T)$$

Proof. Of course if the collection is finite, this is clear. We show it for an infinite countable collection. As each f_i is non-negative so for each n

$$\sum_{i=1}^{n} f_i(T) \leq \sum_i f_i(T).$$

Taking liminf from both sides

$$\sum_{i=1}^{n} \liminf_{T\to\infty} f_i(T) \leq \liminf_{T\to\infty} \sum_i f_i(T).$$

This holds for each n, therefore we get the inequality. □

Unfortunately, analogous inequality for limsup doesn't hold even if $\sum_i f_i(T)$ is uniformly bounded above. Alternatively, we mention the following inequality.

Lemma 2.14. *For any two non-negative functions $f(T), g(T)$ defined on $T > 0$, we have*

$$\liminf_{T\to\infty} (f(T) + g(T)) \leq \liminf_{T\to\infty} f(T) + \limsup_{T\to\infty} g(T) \leq \limsup_{T\to\infty} (f(T) + g(T)).$$

Proof. Let $\underline{l} = \liminf_{T\to\infty} (f(T) + g(T))$, and $\overline{g} = \limsup_{T\to\infty} g(T)$. For $\epsilon > 0$ there is T_0 such that for $T > T_0$ we have

$$\underline{l} - \epsilon \leq f(T) + g(T) \leq f(T) + \overline{g} + \epsilon,$$

$$\underline{l} - \overline{g} - 2\epsilon \leq f(T),$$

which establishes the left inequality. Similar argument gives the right inequality. □

2.5. Transfer Operator

A real-valued function f on $E_A^{\mathbb{N}}$ is called summable if

$$\sum_{e\in E} \exp(\sup_{[e]} f) < \infty.$$

One purpose of this definition is to define an operator on the space of bounded complex-valued continuous functions on $E_A^{\mathbb{N}}$. Therefore we can extend this definition to complex-valued functions.

Definition 2.15. A complex-valued function f on $E_A^{\mathbb{N}}$ is called summable if

$$\sum_{e\in E} \exp(\sup_{[e]} \mathrm{Re}(f)) < \infty.$$

Definition 2.16. For a complex-valued Hölder-type summable function f we introduce Ruelle-Perron-Frobenius operator, also known as transfer operator

$$\mathcal{L}_f : C_b(E_A^{\mathbb{N}}, \mathbb{C}) \to C_b(E_A^{\mathbb{N}}, \mathbb{C})$$

$$\mathcal{L}_f(g)(\rho) = \sum_{e\in E_\rho} \exp\left(f(e\rho)\right) g(e\rho),$$

where the sum is taken over all $e \in E$ that $e\rho$ is admissible, i.e. $A_{e\rho_1} = 1$.

Remark 2.17. Here we would like to mention:

- If this f over shift with infinite letters is summable, then definition 2.15 yields that $\mathrm{Re}(f)$ should go to $-\infty$, i.e. f is unbounded. Therefore it is not Hölder continuous, see proposition 2.8.
- As well it is clear that when E is finite then every real-valued f is summable.
- Further one can see that this operator preserves $C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$.

Next we want to consider the adjoint operator $\mathcal{L}_f^*$ acting on $C_b(E_A^{\mathbb{N}}, \mathbb{C})^*$ which is the space of all regular bounded additive set functions [8, p. 262] (by additive set function we mean complex valued function g defined on the algebra, not necessarily σ-algebra, generated by the closed sets such that g is finitely additive, not necessarily countably additive). Below we mention a result which for case E finite is due to Ruelle [35] and for E infinite is due to Mauldin-Urbański [25, p. 50].

THEOREM 2.18. *If $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is real-valued summable and Hölder-type continuous function, then the adjoint operator $\mathcal{L}_f^*$ admits an eigenmeasure m with eigenvalue* $\exp(P(f))$.

This $P(f)$ is introduced below in definition 2.20.

2.6. Pressure and Equilibrium Measure

DEFINITION 2.19. A Gibbs state for a real-valued function f on $E_A^{\mathbb{N}}$ is a probability measure m on $E_A^{\mathbb{N}}$ for which there is $Q > 1$ and $P \in \mathbb{R}$ such that:

$$Q^{-1} \leq \frac{m([\omega])}{\exp\left(S_n f(\omega\rho) - Pn\right)} \leq Q, \quad \forall \omega \in E_A^n, \;\; \forall \omega\rho \text{ admissible.}$$

It is clear that a Gibbs state has full support, i.e.

$$\operatorname{supp}(m) = E_A^{\mathbb{N}}.$$

Another important fact is that once we get an eigenmeasure from theorem 2.18 it follows that it is actually a Gibbs state for f [25, p. 28]. Using this Gibbs state an invariant ergodic Gibbs state μ for f can be constructed as well [25, p. 14]. Furthermore, it is clear that if f is Hölder-type so is any constant multiple of f. However, the summable property of f doesn't necessarily carry on to any constant multiple of f. We set

$$\Gamma := \{x \in \mathbb{R} : xf \;\text{ summable}\}.$$

Clearly, if E is finite then $\Gamma = \mathbb{R}$ and if E is infinite then definition 2.15 tells us $x_1 \in \Gamma$ implies $x_2 \in \Gamma$ for any $x_2 > x_1$, i.e. Γ is half line. Therefore using the above explanation we obtain Gibbs state for xf $(x \in \Gamma)$ as well:

$$Q_x^{-1} \leq \frac{m_x([\omega])}{\exp\left(xS_n f(\omega\rho) - P(x)n\right)} \leq Q_x, \quad \forall \omega \in E_A^n, \;\; \forall \omega\rho \text{ admissible.} \tag{2}$$

DEFINITION 2.20. The topological pressure of a real-valued function f on $E_A^{\mathbb{N}}$ is defined by

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \ln\Big(\sum_{\omega\in E_A^n} \exp(\sup_{[\omega]} S_n f)\Big).$$

This limit exists by the Fekete's lemma 2.12.

DEFINITION 2.21. An invariant ergodic measure μ is called equilibrium state for a real-valued function f on $E_A^{\mathbb{N}}$ if it is a Gibbs state for f and it established the following equation:

$$P(f) = h_\mu(\sigma) + \int f \mathrm{d}\mu,$$

where h_μ is Kolmogorov entropy of the shift map σ. Note that in general under much weaker assumption for f we have the following equation known as variational principle:

$$P(f) = \sup\{h_\mu(\sigma) + \int f \mathrm{d}\mu\},$$

where the supremum is taken over invariant ergodic measures μ. Furthermore, we set

$$\chi_\mu = -\int f \mathrm{d}\mu,$$

and call it Lyapunov exponent.

One can see that P in definition 2.19 is actually the same as the topological pressure of f [25, p. 13]. This means

$$P(x) = P(xf), \quad x \in \Gamma.$$

We can actually show this function is strictly decreasing on Γ assuming some weak condition. This is well-known fact for function systems, but here we don't assume f is induced by a function system and so we prove it. First we need the following lemma.

LEMMA 2.22. *If μ is an invariant ergodic Gibbs measure then*

$$\lim_n \sup_{\omega \in E_A^n} \mu([\omega]) = 0.$$

PROOF. Let $b_n = \sup_{\omega \in E_A^n} \mu([\omega])$. Note that this supremum is attained so b_n is decreasing, therefore b_n is convergent to some b. Fix $0 < \epsilon < b$ and for each n define

$$F_n := \{\omega \in E_A^n : \epsilon \leq \mu([\omega])\}.$$

Clearly F_n is finite. If $\omega e \in F_{n+1}$ then $\epsilon \leq \mu([\omega e]) \leq \mu([\omega])$ which implies $\omega \in F_n$, i.e. each F_{n+1} extends some of F_n. If this extension process stops at moment m or in other words,

$F_m = \emptyset$ then $\mu([\omega]) < \epsilon$ for all $\omega \in E_A^m$, i.e. $b \leq b_m \leq \epsilon$. Therefore this process cannot stop and so we get at least one element $\rho = e_1e_2e_3... \in E_A^{\mathbb{N}}$ such that $\epsilon \leq \mu([e_1...e_n])$ for each n. This means $\epsilon \leq \mu(\{\rho\})$. We will show ρ is periodic and periodic orbit of ρ $O_+(\rho)$ has full measure which is a contradiction.

Let $A := \cup_{n\geq 0}\sigma^{-n}(\{\rho\})$. Clearly either $\sigma^{-1}(A) = A$ or $\sigma^{-1}(A) \cup \{\rho\} = A$. In the latter case $\mu(\sigma^{-1}(A)) + \mu(\{\rho\}) = \mu(A)$ which yields $\mu(\{\rho\}) = 0$ using invariant property of μ. In the former case ρ must be periodic with some period m. Since $\sigma^{-1}(A) = A$, ergoicity either yields $\mu(\{\rho\}) \leq \mu(A) = 0$, or otherwise $\mu(A) = 1$. Note that for each $i > 0$ we have $\sigma^{i-1}(\rho) \in \sigma^{-1}\left(\{\sigma^i(\rho)\}\right)$ so

$$\mu(\{\sigma^{i-1}(\rho)\}) \leq \mu\left(\sigma^{-1}(\{\sigma^i(\rho)\})\right) = \mu(\{\sigma^i(\rho)\}),$$

and since $\rho = \overline{e_1e_2...e_m}$ we have

$$\mu(\{\rho\}) \leq \mu(\{\sigma(\rho)\}) \leq ... \leq \mu(\{\sigma^{m-1}(\rho)\}) \leq \mu(\{\rho\}).$$

Therefore the inequalities in the above line are all equality. For each $n \geq 0$ we know $\sigma^{-n}(\{\rho\})$ meets $O_+(\rho)$ in exactly one point and since $\mu\left(\sigma^{-n}(\{\rho\})\right) = \mu(\{\rho\})$ thus the whole mass of $\sigma^{-n}(\{\rho\})$ is on $\sigma^{-n}(\{\rho\}) \cap O_+(\rho)$. Therefore

$$1 = \mu(A) = \mu(O_+(\rho)).$$

□

Proposition 2.23. *If $P(x_0) \leq 0$ for some x_0 then $P(x)$ is strictly decreasing on Γ.*

Proof. We start with the following estimate and we use 2 for it:

$$\exp\left(x_0 \sup_{[\omega]} S_n f - nP(x_0)\right) \leq Qm([\omega]).$$

Next we use the above lemma to find N such that for every $n \geq N$ and every $\omega \in E_A^n$:

$$\exp\left(x_0 \sup_{[\omega]} S_n f - nP(x_0)\right) \leq Q\mu([\omega]) \leq Q \sup_{\omega\in E_A^n} \mu([\omega]) \leq e^{-1}.$$

Then for all $k > 0$ and $\omega' \in E_A^{kN}$:

$$\exp\left(x_0 \sup_{[\omega']} S_{kN} f - kNP(x_0)\right) \leq \exp\left(x_0 k \sup_{[\omega]} S_N f - NkP(x_0)\right) \leq e^{-k}.$$

Consider $x_1 < x_2$ in Γ, we use the above estimate to find

$$\sum_{\omega' \in E_A^{kN}} \exp(x_2 \sup_{[\omega']} S_{nk} f) = \sum_{\omega' \in E_A^{kN}} \exp(x_1 \sup_{[\omega']} S_{nk} f) \exp\left((x_2 - x_1) \sup_{[\omega']} S_{nk} f\right)$$

$$= \sum_{\omega' \in E_A^{kN}} \exp(x_1 \sup_{[\omega']} S_{nk} f) \exp\left(\frac{x_2 - x_1}{x_0}\left(x_0 \sup_{[\omega']} S_{nk} f - kNP(x_0)\right)\right) \exp\left(\frac{x_2 - x_1}{x_0} kNP(x_0)\right)$$

$$\leq \sum_{\omega' \in E_A^{kN}} \exp(x_1 \sup_{[\omega']} S_{nk} f) \exp\left(-k\frac{x_2 - x_1}{x_0}\right) = \exp\left(-k\frac{x_2 - x_1}{x_0}\right) \sum_{\omega' \in E_A^{kN}} \exp(x_1 \sup_{[\omega']} S_{nk} f)$$

Now if we take log, divide by kN and let $k \to \infty$, we obtain

$$P(x_2) \leq -\frac{x_2 - x_1}{Nx_0} + P(x_1) < P(x_1).$$

□

Definition 2.24. A real-valued function $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is called regular if $P(x) = 0$ for some $x > 0$ and is called strongly regular if it is regular and $0 < P(x) < \infty$ for some $x > 0$.

Remark 2.25. It is worth to mention

- If $P(x)$ is strictly decreasing, it can have only one root say δ. Further, strong regularity means

$$\inf \Gamma < \delta.$$

- The above proposition can be proved under weaker assumption: $\inf_{x \in \Gamma} P(x) \leq 0$.

Proposition 2.26. *If f is strongly regular, the first derivative of P at δ is $P'(\delta) = -\chi_{\mu_\delta}$.*

Proof. We refer to proposition 2.6.13 in [25, p. 47]. □

2.7. Spectral Analysis

We start with considering family of functions $\{sf\}$ where s is usually a complex number in the right half plane $\Gamma^+ = \Gamma \times \mathbb{R}$. Then definitions 2.15 and 2.16 are applicable for such functions, however definitions 2.20 is not applicable as sf is not real anymore unless for real s. It is clear that when $s \in \Gamma^+$ then the series

$$\sum_{e \in E} \sup_{[e]} |\exp(sf)| = \sum_{e \in E} \exp(\mathrm{Re}(s) \sup_{[e]} f)$$

converges. Thus having a Hölder-type summable function, spectral theory of transfer operator on the right half plane Γ makes sense. Note that

- $\mathcal{L}_s := \mathcal{L}_{s\psi}$ is an operator on $C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ for any $s \in \Gamma^+$.
- The pressure function P is defined on Γ.

Another important property of the transfer operator to be discussed is D-generic property. This property prohibits the possibility of admitting specific eigenvalue. We adopt its definition from [33, p. 32]. Before mentioning the definition, we need an equivalency.

PROPOSITION 2.27. *The following conditions are equivalent:*

(i) $\exp(P(x) + ia)$ *is an eigenvalue of* $\mathcal{L}_{x+iy} : C_b(E_A^{\mathbb{N}}, \mathbb{C}) \to C_b(E_A^{\mathbb{N}}, \mathbb{C})$*, for some* $x \in \Gamma$.

(ii) $\exp(P(x) + ia)$ *is an eigenvalue of* $\mathcal{L}_{x+iy} : C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C}) \to C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$*, for all* $x \in \Gamma$.

PROOF. We refer to Proposition 2 in [32, p. 138] and Proposition 2.3.5 in [33, p. 32]. □

DEFINITION 2.28. We say a potential f is D-generic if either of the above statements (i) or (ii) from the above proposition fails for all non-zero y and $a = 0$. In other words, $\mathcal{L}_{x+iy} : C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C}) \to C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ doesn't admit $\exp(P(x))$ as eigenvalue if $y \neq 0$.
Further, we say the potential f is strongly D-generic, if either of the above statements (i) or (ii) from the above proposition fails for all non-zero y and all real a. In other words, $\mathcal{L}_{x+iy} : C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C}) \to C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ doesn't admit any eigenvalue of magnitude $\exp(P(x))$ for any $y \neq 0$.

One can obtain an alternative statement for D-generic and strongly D-generic properties.

PROPOSITION 2.29. *A potential f is D-generic iff the additive subgroup generated by the following set is not cyclic,*

$$\{S_{|\omega|}f(\bar{\omega}):\ \omega\in E^*_{per}\}.$$

And it is strongly D-generic iff the additive subgroup generated by the following set is not cyclic for any real β,

$$\{S_{|\omega|}f(\bar{\omega})-n\beta:\ \omega\in E^*_{per}\}.$$

Next we would like to bring some facts from spectral theory. We mostly refer to [8], [19], [5] or [1]. Assume $\mathfrak{B}$ is a Banach space, $\mathcal{L}$ a bounded operator on $\mathfrak{B}$. The spectrum of bounded operator $\mathcal{L}$, denoted by $\mathrm{Sp}(\mathcal{L})$, is defined to be all the complex numbers ζ such that the operator $(\mathcal{L}-\zeta I)$ is not bijective. Further the spectral radius of $\mathcal{L}$ is defined to be

$$r(\mathcal{L}):=\sup\{|\zeta|\ :\ \zeta\in\mathrm{Sp}(\mathcal{L})\}.$$

There is an alternative expression of spectral radius known as the Gelfand's formula:

$$r(\mathcal{L})=\lim_{n}\|\mathcal{L}^n\|^{\frac{1}{n}}.$$

Next we mention the essential spectrum definition. We notify that there are several other definitions of this concept in math community, however the radius of essential spectrum (defined below) remains the same for all the definitions. We adapt the following definition from [5, p. 107].

DEFINITION 2.30. The complex number ζ belongs to the essential spectrum of the operator $\mathcal{L}$, denoted by $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L})$, if at least one of the following condition holds:

(i) the operator $(\mathcal{L}-\zeta I)$ has a range which is not closed in $\mathfrak{B}$.

(ii) $\cup_{i\geq 0}\mathrm{Nul}(\mathcal{L}-\zeta I)$ is infinite dimensional.

(iii) the point ζ is a limit point of the spectrum of $\mathcal{L}$.

Furthermore, the essential spectral radius is

$$r_{\mathrm{ess}}(\mathcal{L}) := \sup\{|\zeta| \ : \ \zeta \in \mathrm{Sp}_{\mathrm{ess}}(\mathcal{L})\}.$$

Nussbaum showed the essential spectral radius as well follows a Gelfand type formula. Before bringing his formula we need to introduce a semi-norm. Consider, $\mathfrak{K}$ the ideal of all bounded compact operators on $\mathfrak{B}$, then

$$\|\mathcal{L}\|_{\mathfrak{K}} := \inf_{\mathcal{C}\in\mathfrak{K}} \|\mathcal{L} + \mathcal{C}\|,$$

defines a semi-norm on the space of bounded linear operators on $\mathfrak{B}$ [27, p. 474].

PROPOSITION 2.31. $r_{ess}(\mathcal{L}) = \lim_{n\to\infty} \|\mathcal{L}^n\|_{\mathfrak{K}}^{1/n}$.

PROOF. We refer to [27, p. 477]. □

Next we briefly talk about perturbation theory of linear operator. Our main sources are [19], [8] and [1]. It is now clear from definition 2.30 that for every r where $r_{\mathrm{ess}}(\mathcal{L}) < r \leq r(\mathcal{L})$ we should have only finitely many $\zeta \in \mathrm{Sp}(\mathcal{L})$ with $|\zeta| \geq r$, each of which are isolated eigenvalue with finite algebraic multiplicity. Kato calls these finite ζ's, finite system of eigenvalues [19, p. 181] or [1, p. 363]. This concept shows up in [8, p. 572] as spectral set. According to Schwartz-Dunford, spectral set is any clopen subset of the spectrum. The purpose is to obtain a perturbation theorem for a holomorphic family of operators $\mathcal{L}_s$ in complex variable s. The original idea of perturbation theory of self-adjoint operators over Hilbert space goes back to Schrödinger. The first major math result in this area was obtained by Rellich. Later on Sz. Nagy and Kato independently worked on this topic to generalize Rellich's result to a general closed operator over Banach space [18]. Many of these results can be found in [8, VII.6] or [19, Ch. VII] or [1, Ch. 10]. We first want to define holomorphic family of operators. Note that there are several definitions for this but all in the context of bounded operator-valued over a fixed Banach space coincide [1, 10.1], [1, 10.3].

DEFINITION 2.32. Let $(X, \|.\|)$ be Banach space, $\mathfrak{B}(X)$ be the space of all bounded linear operator on X, G a region in the complex plane and $s \mapsto \mathcal{L}_s$ a function from G into $\mathfrak{B}(X)$.

We say $\mathcal{L}_s$ is holomorphic in G if there exists an operator-valued function $s \mapsto \mathcal{L}'_s$ such that

$$\|\frac{\mathcal{L}_{s+h} - \mathcal{L}_s}{h} - \mathcal{L}'_s\| \to 0,$$

for all $s \in G$ and $h \to 0$.

We are ready to express one major result in perturbation theory of holomorphic family of bounded operators.

Theorem 2.33. *Let $\mathcal{L}_s$ be holomorphic family of bounded operators from a region G into $\mathfrak{B}(X)$. Let $s_0 \in G$ and $\xi_0, ...\xi_n$ be finite system of eigenvalues of $\mathcal{L}_{s_0}$, each of which with algebraic multiplicity 1. Then there is small enough neighborhood of s_0 such that $\mathcal{L}_s$ has the spectral representation*

$$\mathcal{L}_s = \sum_{i=1}^{n} \xi_i(s)\mathcal{P}_{i,s} + \mathcal{D}_s,$$

where each $\xi_i(s)$ is holomorphic function, $\mathcal{P}_i(s)$ is holomorphic operator-valued function and a projection, $\mathcal{D}(s)$ holomorphic operator-valued function and further

$$\xi_i(s_0) = \xi_i,$$

for each $i = 1, ..., n$.

In general if multiplicity of an eigenvalue is higher than 1 the eigenvalues may have algebraic singularities at s_0. The idea of the proof is first reducing it to the case where X is finite dimensional and then one can apply perturbation theory of holomorphic operators in finite dimension. For a detailed proof, first see theorem 1 in [1, p. 367], then theorem 1 in [1, p. 243], [1, p. 129] and [1, p. 131]. Another source of proof for the general form of the result is theorem 9 in [8, p. 587]. As well theorem 1.8 in [19, p. 370] provides a proof.
By projection in the above theorem we mean operator with property

$$\mathcal{P}_i^2 = \mathcal{P}_i.$$

Next, we would like to see how the above spectral representation of operators is related to spectral decomposition of operators. The following proposition is a consequence of the celebrated spectral mapping theorem [6, p. 209].

Proposition 2.34. *Suppose $\mathfrak{B}(X)$ is a Banach algebra of operators on the Banach space X. Let $\mathcal{L} \in \mathfrak{B}(X)$. Further, assume the spectrum of $\mathcal{L}$ can be written as*

$$Sp(\mathcal{L}) = F_1 \cup F_2,$$

for disjoint nonempty closed sets F_1, F_2. Then there is a nontrivial idempotent $\mathcal{E} \in \mathfrak{B}(X)$ such that

- *if $\mathcal{BL} = \mathcal{LB}$, then $\mathcal{BE} = \mathcal{EB}$.*
- *if $\mathcal{L}_1 = \mathcal{LE}$ and $\mathcal{L}_2 = \mathcal{L}(1 - \mathcal{E})$, then $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ and $\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2\mathcal{L}_1 = 0$.*
- *$Sp(\mathcal{L}_1) = F_1 \cup \{0\}$, $Sp(\mathcal{L}_2) = F_2 \cup \{0\}$.*

2.8. Graph Directed Markov System

Definition 2.35. We first consider directed multi-graph (V, E, i, t) and an incidence matrix $A : E \times E \to \{0, 1\}$, where V is the finite set of vertices, E is the countable (finite or infinite) set of directed edges and i, t (initial and tail) are functions

$$i, t : E \to V,$$

such that

$$A_{ab} = 1 \quad \Rightarrow \quad t(a) = i(b)$$

In addition, we have a finite family of Euclidean compact metric spaces $\{X_v\}_{v \in V}$ and countable family of contractions $\{\phi_e\}_{e \in E}$ and $\kappa \in (0, 1)$ such that

$$|\phi_e(t) - \phi_e(s)| \leq \kappa |t - s|,$$

for all $e \in E$ and $t, s \in X_{t(e)}$. Then

$$\mathcal{S} = \{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e \in E}$$

is called attracting graph directed Markov system.

We extend the functions $i, t : E \to V$ in a natural way to E_A^* as follows:

$$t(\omega) := t(\omega_n), \quad i(\omega) := i(\omega_1).$$

If $\omega \in E_A^n$ we define:

$$\phi_\omega = \phi_{\omega_1} \circ ... \circ \phi_{\omega_n} : X_{t(\omega)} \to X_{i(\omega)}.$$

Now for any $\rho \in E_A^{\mathbb{N}}$ the sets $\{\phi_{\rho_1\rho_2...\rho_n}(X_{t(\rho_n)})\}_{n\geq 1}$ form a descending sequence of non-empty compact sets and therefore $\cap_{n\geq 1}\phi_{\rho_1\rho_2...\rho_n}(X_{t(\rho_n)})$ is non-empty. Further since

$$\text{diam}(\phi_{\rho_1\rho_2...\rho_n}(X_{t(\rho_n)})) \leq \kappa^n \text{diam}(X_{t(\rho_n)}) \leq \kappa^n max\{\text{diam}(X_v)\}_{v\in V},$$

we find that this intersection is actually a singleton and we denote it by $\pi(\rho)$, in this way we have defined a map

$$\pi : E_A^{\mathbb{N}} \to \sqcup_{v\in V} X_v,$$

where $\sqcup_{v\in V} X_v$ is the disjoint union of the compact spaces $\{X_v\}_v$.

Definition 2.36. The set

$$J = \pi(E_A^{\mathbb{N}})$$

is called the limit set of system $\mathcal{S}$.

Definition 2.37. We call a graph directed Markov system conformal if the following conditions are satisfied for some $d \in \mathbb{N}$:

(a) For every $v \in V$, X_v is compact connected subset of $\mathbb{R}^d$ and $X_v = \overline{\text{Int}(X_v)}$.

(b) (Open Set Condition) For all different $e, e' \in E$,

$$\phi_e\left(\text{Int}(X_{t(e)}\right) \cap \phi_{e'}\left(\text{Int}(X_{t(e')}\right) = \emptyset.$$

(c) (Conformality) For every $v \in V$ there is an open connected W_v containing X_v. Further for each $e \in E$, ϕ_e extends to a C^1 conformal diffeomorphism from $W_{t(e)}$ into $W_{i(e)}$ with Lipschitz constant bounded by κ.

(d) (Bounded Distortion Property) There are two constants $L \geq 1$ and $\alpha > 0$ such that for every $e \in E$ and every $x, y \in X_{t(e)}$

$$\left|\frac{|\phi_e'(s)|}{|\phi_e'(t)|} - 1\right| \leq L\|s - t\|^\alpha,$$

where $|\phi_e'(t)|$ denotes the scaling factor of the derivative of ϕ_e' at t.

From now on we denote the conformal graph directed Markov system simply by CGDMS. Furthermore, we assign a real-valued function to a CGDMS:

$$f : E_A^{\mathbb{N}} \to \mathbb{R}, \quad f(\rho) = \log |\phi'_{\rho_1}(\pi(\sigma\rho))|,$$

and we call it the potential function.

2.9. Random Dynamics

We are now ready to define random graph directed Markov system. We want to adopt Roy-Urbański [34] definition here. We start with directed multi-graph (V, E, i, t), a mapping $A : E \times E \to \{0, 1\}$ and a family of compact metric spaces $\{X_v\}_v$ as in the previous section. Then we employ an invertible ergodic measure preserving map $T : (\Lambda, B, \nu) \to (\Lambda, \mathcal{F}, \nu)$ on a complete probability space (Λ, B, ν) and family of injective contractions $\{\phi_e^\lambda : X_{t(e)} \to X_{i(e)}\}_{e\in E,\lambda\in\Lambda}$ with Lipschitz constant at most $\kappa \in (0, 1)$. For each word ω we define

$$\phi_\omega^\lambda := \phi_{\omega_1}^\lambda \circ \phi_{\omega_2}^{T(\lambda)} \circ ... \circ \phi_{\omega_n}^{T^{n-1}(\lambda)}.$$

Note that for each $(\omega, \lambda) \in (E_A^*, \Lambda)$ the map $t \mapsto \phi_\omega^\lambda(t)$ is continuous, the map $\lambda \mapsto \phi_\omega^\lambda(t)$ is measurable for each $(\omega, t) \in (E_A^*, X_{t(\omega)})$ and $(t, \lambda) \mapsto \phi_\omega^\lambda(t)$ is jointly measurable for each word $\omega \in E_A^*$. Then for every $\lambda \in \Lambda$, $\rho \in E_A^{\mathbb{N}}$ similar to deterministic graph directed Markov system $\{\phi_{\rho_1...\rho_n}^\lambda(X_{t(\rho_n)})\}$ is a decreasing sequence of non-empty compact sets whose diameters are bounded by κ^n, so

$$\cap_{n\geq1}\phi_{\rho_1\rho_2...\rho_n}^\lambda(X_{t(\rho_n)})$$

is a singleton and we denote its only element by $\pi^\lambda(\rho)$. Therefore for each $\lambda \in \Lambda$ this defines a limit set

$$J^\lambda = \pi^\lambda(E_A^{\mathbb{N}}).$$

Definition 2.38. We call a system random CGDMS if the following conditions are satisfied for some $d \in \mathbb{N}$:

a) For every $v \in V$, X_v is compact connected subset of $\mathbb{R}^d$ and $X_v = \overline{\mathrm{Int}(X_v)}$.

b) (Open Set Condition) For almost every λ and all different $e, e' \in E$,

$$\phi_e^\lambda \left(\mathrm{Int}(X_{t(e)})\right) \cap \phi_{e'}^\lambda \left(\mathrm{Int}(X_{t(e')})\right) = \emptyset.$$

c) (Conformality) For every $v \in V$ there is open connected W_v containing X_v. Further for almost every λ and each $e \in E$, ϕ_e^λ extends to a C^1 conformal diffeomorphism from $W_{t(e)}$ into $W_{i(e)}$ with Lipschitz constant bounded by κ.

d) (Bounded Distortion Property) There are two constants $L \geq 1$ and $\alpha > 0$ such that for every $e \in E$ and every $s, t \in X_{t(e)}$

$$\left| \frac{|(\phi_e^\lambda)'(s)|}{|(\phi_e^\lambda)'(t)|} - 1 \right| \leq L \|s - t\|^\alpha,$$

for almost every λ, where $|(\phi_e^\lambda)'(t)|$ denotes the scaling factor of the derivative of $(\phi_e^\lambda)'$ at t.

For a random conformal graph directed Markov system we define a random potential by

$$f : (E_A^{\mathbb{N}}, \Lambda) \to \mathbb{R}, \quad f(\rho, \lambda) := \log \left| (\phi_{\rho_1}^\lambda)' \left(\pi^{T(\lambda)}(\sigma\rho)\right) \right|.$$

We know that essential supremum of a real-valued function f on a measure space (Λ, ν) is defined by

$$\operatorname*{ess\,sup}_\lambda f := \inf\{r : \ f(\lambda) \leq r \text{ for } \nu\text{-almost all } \lambda \in \Lambda\}.$$

Definition 2.39. A random potential f is called summable if

$$\sum_{e \in E} \exp \left(\operatorname*{ess\,sup}_\lambda f(e, \lambda) \right) < \infty.$$

For a real number x, then xf being summable means

$$\sum_{e \in E} \operatorname*{ess\,sup}_\lambda |(\phi_e^\lambda)'|^x < \infty.$$

We consider Γ the set of all such x and $\Gamma^+ = \Gamma \times \mathbb{R}$ as before. Then one can see that for $x \in \Gamma$ and almost every $\lambda \in \Lambda$, there exists a unique bounded measurable function

$\lambda \mapsto P^\lambda(x) := P^\lambda(xf)$, and a unique random probability measure $\{m_x^\lambda\}$ such that

$$(\mathcal{L}_x^\lambda)^* m_x^{T(\lambda)} = e^{P^\lambda(x)} m_x^\lambda,$$

for almost every $\lambda \in \Lambda$, see [34, p. 271]. That means $P^\lambda(x)$ and m_x are uniquely determined by

$$m_x^\lambda([e\omega]) = e^{-P^\lambda(x)} \int_{[\omega]} \left|(\phi_e^\lambda)' \left(\pi^{T(\lambda)}(\tau)\right)\right|^x \mathrm{d}m_x^{T(\lambda)}(\tau),$$

for almost every λ.

Remark 2.40. We would like to mention a few words about definition of random subshift of finite type. Bogenschutz defines it the way that each fiber is closed in a compact set [2, p. 420]. Roy-Urbański generalized it to a special case of subshift of finite type with infinite letters [34, p. 420]. Bogenschutz as well uses the infinite letter system $\mathbb{Z}_+$ but he makes the system compact (using one-point compactification $\bar{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{\infty\}$) so that his bundle random dynamical system theory (his PhD dissertation) applies and this doesn't include Roy-Urbański random system as special case simply because $\prod_{i=0}^\infty \mathbb{Z}_+$ is not a closed subset of $\prod_{i=0}^\infty \bar{\mathbb{Z}}_+$.

2.10. Tauberian Theorems

Before finishing the section, we mention two main Tauberian theorems needed later on. First Ikehara & Wiener's theorem [40, p. 127] and then Graham & Vaaler's theorem [12, p. 294] which is just a refinement of Ikehara-Wiener theorem. The motivation for Ikehara-Wiener theorem was to provide a simpler proof of the prime number theorem. We know that PNT was proved in the late 19^{th} century. However, Ikehara & Wiener used a theorem of Wiener to obtain the following result in the early 1930s that implies PNT [21, p. 127].

Theorem 2.41 (Ikehara-Wiener). *Let $\alpha(T)$ be monotone increasing function continuous from right such that*

$$\eta(s) = \int_0^\infty e^{-sT} d\alpha(T)$$

converges for $\mathrm{Re}(s) > \delta > 0$. *If*

$$\eta(s) - \frac{A}{s-\delta} = g(s)$$

has continuous extension to $\mathrm{Re}(s) = \delta$, *then*

$$e^{-\delta T}\alpha(T) \to \frac{A}{\delta}, \quad \textit{as } T \to \infty.$$

In 1980s, Graham & Vaaler on their journey to study extremal (minorant and majorant) function in Fourier analysis for some special classes of functions, obtained a refinement of Ikehara-Wiener theorem as a corollary. One may want to know that the early work in construction of extremal functions was done by Beurling and later on by Selberg (unpublished). For the proof of the following result see [12, p. 294].

THEOREM 2.42 (Graham-Vaaler). *Let* α *be a Borel measure on* $[0,\infty)$ *and that the Laplase-Stieltjes transform*

$$\eta(s) = \int_{0^-}^{\infty} e^{-sT}\, d\alpha(T), \qquad s = x + 2\pi i y,$$

exists for $\mathrm{Re}(s) > \delta$. *Suppose that for some number* $y_0 > 0$, *there is a constant* $A > 0$ *such that the analytic function* $\eta(s) - A/(s-\delta)$ *extends to a continuous function on the set* $\{\delta + 2\pi i y : |y| < y_0\}$. *Then*

$$\begin{aligned} Ay_0^{-1}\{\exp(\delta y_0^{-1}) - 1\}^{-1} &\le \liminf_{T\to\infty} e^{-\delta T}\alpha[0,T] \\ &\le \limsup_{T\to\infty} e^{-\delta T}\alpha[0,T] \\ &\le Ay_0^{-1}\{\exp(\delta y_0^{-1}) - 1\}^{-1}\exp(\delta y_0^{-1}). \end{aligned}$$

REMARK 2.43. It is worth noting that

- If $\eta(s) - A/(s-\delta)$ has continuous extension to the whole line $x = \delta$ then we may let $y_0 \to \infty$, this implies Ikehara-Wiener theorem 2.41.
- Graham & Vaaler or Korevaar [21, p. 30] assumed that A should be positive or non-negative. But since η is real non-negative on the real line, this assumption can be relaxed, i.e. A can be any complex number. Then one can see it has to be real non-negative. Furthermore it is clear that for us the measure α (possibly infinite

measure) is just taken to be the Borel measure generated by the right continuous, increasing function $N_\rho(B,T)$, see 16 and [9, Thm 1.16]. Moreover, Graham & Vaaler provide an example to show their bounds are both sharp.

2.11. Examples

EXAMPLE 2.44. Consider the iterated function system where in the multi-graph (V, E, i, t), V is Singleton $\{v\}$, E is finite $i = t$ are maps from E to the only element of V and the mapping $A : E \times E \to \{0, 1\}$ is just constant 1. This is an iterated function system. Then for the conformal graph directed Markov system we consider $X_v = [0, 1]$ and

$$\phi_e(t) = \alpha_e t + \beta_e,$$

where α_e, β_e are chosen appropriate enough from $(0, 1)$ so that we have all conditions for conformal graph directed Markov system satisfied, see definition 2.37. Then we know from below the definition 2.37 the potential is

$$f(\rho) = \log |(\phi_{\rho_1})' (\pi(\sigma\rho))| = \log \alpha_{\rho_1}$$

and

$$S_n f(\rho) = \sum_{e\in E} n_e \log \alpha_e,$$

where n_e is just number of letter e appearing in the word $\rho_1 ... \rho_n$. We can find the pressure:

$$P(x) = \lim_n \frac{1}{n} \log \sum_{|\omega|=n} \|\phi_\omega\|^x = \log(\sum_{e\in E} \alpha_e^x). \tag{3}$$

As well we know the following should hold

$$m_x([e\omega_1 ... \omega_n]) = \exp(-P(x)) \int_{[\omega_1 ... \omega_n]} |(\phi_e)' (\pi(\tau)) |^x dm_x(\tau), \tag{4}$$

for $e \in E$, Gibbs state m_x and pressure $P(x)$. This actually leaves

$$\frac{m_x([e\omega_1 ... \omega_n])}{m_t([\omega_1 ... \omega_n])} = \frac{\alpha_e^x}{\sum_{e\in E} \alpha_e^x}.$$

Example 2.45. Consider an iterated function system containing conformal maps

$$\phi_e(t) = \alpha t + \beta_e, \quad \alpha, \beta_e \in (0,1),$$

where β_e are appropriate enough for conformal conditions, see Definition 2.37, $E = \{0, 1, ..., k-1\}$, with some irreducible incidence matrix A. Then we know for this system we should have the potential

$$f(\omega) = \log |(\phi_{\omega_1})'(\pi(\sigma\omega)| = \log \alpha,$$

and a Gibbs state has the form

$$m_x([e\omega_1...\omega_n]) = \exp(-P(x)) \int_{[\omega_1...\omega_n]} |(\phi_e)'(\pi(\tau))|^x dm_x(\tau), \tag{5}$$

for appropriate $e \in E$, Gibbs state m_x and pressure $P(x)$. This actually yields

$$P(x) = \log (m_x([\omega_1...\omega_n])/m_x([e\omega_1...\omega_n])) + x \log \alpha.$$

Note that the first term of the above sum does not depend on t and it is actually equal to

$$\lim_n \log \#E_A^n/n = \log r(A),$$

where r(A) is the spectral radius of the incidence matrix A (see Proposition 2.4). We can see this simply by the pressure formula:

$$P(x) = \lim_n \frac{1}{n} \log \sum_{\omega \in E_A^n} \|\phi_\omega\|^x = \lim_n \frac{\log \#E_A^n}{n} + x \log \alpha = \log r(A) + x \log \alpha. \tag{6}$$

CHAPTER 3

COUNTING IN DETERMINISTIC DYNAMICS

In the following section we assume we have a summable strongly regular Hölder-type function (potential) $f : E_A^{\mathbb{N}} \to \mathbb{R}$ with $P(f) = P(1) = 0$.

3.1. Spectral Analysis of Transfer Operator

We recall $C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ is the Banach space of Hölder continuous complex-valued functions over $E_A^{\mathbb{N}}$, and $\mathfrak{B} := \mathfrak{B}\left(C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})\right)$ is Banach space of all bounded linear operators over $C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$. For every $s \in \Gamma^+$ it was stated in the previous section that $\mathcal{L}_s$ belongs to $\mathfrak{B}$. One major step is to establish holomorphy of operator $\mathcal{L}_s$.

Lemma 3.1. *For every $n \in \mathbb{N}$, the operator-valued function $s \mapsto \mathcal{L}_s^n$ is holomorphic on Γ^+.*

Proof. For each $\omega \in E_A^n$ one can consider the (idempotent) function $i_{[\omega]}$ in $C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ where it is defined to be 1 on ξ such that $\omega\xi$ is admissible and 0 otherwise. Then for each s in the right half plane Γ^+ and $g \in C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ we define $\mathcal{F}_{\omega,s}g$:

$$\mathcal{F}_{\omega,s}g(\rho) := i_{[\omega]}(\rho)\exp(sS_nf(\omega\rho))g(\omega\rho). \tag{7}$$

We want to show $\mathcal{F}_{\omega,s}$ is an operator on $C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$. First note that

$$\|\mathcal{F}_{\omega,s}g(\rho)\|_\infty \leq \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf)\|g\|_\infty.$$

To find Hölder coefficient of $\mathcal{F}_{\omega,s}g$ we let $|\rho \wedge \rho'| \geq k \geq 1$:

$$|\mathcal{F}_{\omega,s}g(\rho) - \mathcal{F}_{\omega,s}g(\rho')| \leq |\exp(sS_nf(\omega\rho))g(\omega\rho) - \exp(sS_nf(\omega\rho'))g(\omega\rho')|$$

$$= |\left(\exp(sS_nf(\omega\rho)) - \exp(sS_nf(\omega\rho'))\right)g(\omega\rho) + \exp(sS_nf(\omega\rho'))\left(g(\omega\rho) - g(\omega\rho')\right)|$$

$$\leq \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf).|s|.|S_nf(\omega\rho) - S_nf(\omega\rho')|.\|g\|_\infty$$
$$+ \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf)|g(\omega\rho) - g(\omega\rho')|.$$

By Lemma 2.11, we get

$$|\mathcal{F}_{\omega,s}g(\rho) - \mathcal{F}_{\omega,s}g(\rho')|\exp(\alpha k) \leq |\mathcal{F}_{\omega,s}g(\rho) - \mathcal{F}_{\omega,s}g(\rho')|\exp(\alpha|\rho \wedge \rho'|)$$

$$\leq \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf).|s|.K.\|g\|_\infty + \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf)V_\alpha(g)$$

$$\leq \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf)\|g\|_\alpha(1+|s|K),$$

where K depends only on f. Therefore we can write:

$$\|\mathcal{F}_{\omega,s}g\|_\alpha = \|\mathcal{F}_{\omega,s}g\|_\infty + V_\alpha(\mathcal{F}_{\omega,s}g)$$

$$\leq \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf)\|g\|_\infty + \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf)\|g\|_\alpha(1+|s|K)$$

$$\leq \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf)\|g\|_\alpha(2+|s|K),$$

so

$$\|\mathcal{F}_{\omega,s}\|_\alpha \leq \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf)(2+|s|K). \tag{8}$$

Next we want to show the map $s \mapsto \mathcal{F}_{\omega,s}$ is holomorphic on Γ^+. As expected derivative is

$$\mathcal{F}'_{\omega,s}g(\rho) = i_{[\omega]}(\rho).\exp(sS_nf(\omega\rho)).S_nf(\omega\rho).g(\omega\rho),$$

we first need to show this defines an operator on $C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ and then to check it is actually bounded. Note that $|S_nf|$ is bounded on $[\omega]$ by some C, see definition 2.19. If we review all the inequalities above and replace all the $g(\omega...)$ with $S_n(\psi)(\omega...)g(\omega...)$ we get:

$$\|\mathcal{F}'_{\omega,s}g\|_\alpha = \|\mathcal{F}'_{\omega,s}g\|_\infty + V_\alpha(\mathcal{F}'_{\omega,s}g)$$

$$\leq \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf).\|g\|_\infty.C + \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf).\|g\|_\alpha.(C+|s|KC+K)$$

$$\leq \exp(\mathrm{Re}(s)\sup_{[\omega]} S_nf)\|g\|_\alpha(2C+|s|KC+K).$$

Fix s_0 in Γ^+, we write:

$$\left(\mathcal{F}_{\omega,s} - \mathcal{F}_{\omega,s_0} - (s-s_0)\mathcal{F}'_{\omega,s_0}\right)g(\rho)$$

$$= i_{[\omega]}(\rho).\big(\exp(sS_nf(\omega\rho)) - \exp(s_0S_nf(\omega\rho))$$

$$-(s-s_0)\exp(s_0S_nf(\omega\rho))S_nf(\omega\rho)\big).g(\omega\rho),$$

therefore

$$s \mapsto \mathcal{F}_{\omega,s} \in \mathfrak{B}$$

is holomorphic iff

$$s \mapsto i_{[\omega]}(...) \exp(sS_n f(\omega...)) \in C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$$

is holomorphic. But

$$i_{[\omega]}(...) \exp(sS_n f(\omega...)) = i_{[\omega]}(...) \exp(s i_{[\omega]}(...) S_n f(\omega...))$$

and $i_{[\omega]}(...)S_n f(\omega...) \in C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$, thus since $i_{[\omega]}$ is a constant function of s problem boils down to holomorphy of the function $s \mapsto \exp(s\mathcal{T})$ for $\mathcal{T} \in C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$, and this is clearly holomorphic.

Thus the map $s \mapsto \mathcal{F}_{\omega,s}$ defines a holomorphic $\mathfrak{B}$-valued function on the right half plane Γ^+. Now because for $s \in \Gamma^+$, $\mathrm{Re}(s)f$ admits Gibbs state, $\mathfrak{B}$-valued function

$$s \mapsto \mathcal{L}_s^n = \sum_{\omega \in E_A^n} \mathcal{F}_{\omega,s}$$

converges and so is holomorphic on Γ^+. □

PROPOSITION 3.2. *The spectral radius of $\mathcal{L}_s$ is at most $e^{P(x)}$ and $r_{ess}(\mathcal{L}_s) < e^{P(x)}$.*

PROOF. For the case E is finite we just refer to [32, p. 140]. Assuming E is infinite, the former part is a straight-forward consequence of Ionescu Tulcea-Marinescu inequality (also known as Lasota-Yorke type inequality) shown in [25, p. 32]:

$$\|\mathcal{L}_s^n g\|_\alpha \le e^{nP(x)}(Qe^{-\alpha n}\|g\|_\alpha + C\|g\|_\infty), \tag{9}$$

which leaves

$$\|\mathcal{L}_s^n\|_\alpha \le e^{nP(x)}(Q + C). \tag{10}$$

First for every $\omega \in E_A^*$ choose $\hat{\omega} \in [\omega]$ arbitrarily. Then for every $n \ge 1$ consider the operator $\mathcal{E}_n$ on $C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ defined by:

$$\mathcal{E}_n(g) := \sum_{\omega \in E_A^n} g(\hat{\omega}) \mathbb{1}_{[\omega]}.$$

Therefore $\mathcal{E}_n g$ is constant on each cylinder $[\omega]$. It is clear that $\|\mathcal{E}_n g\|_\infty \le \|g\|_\infty$. We want to show $V_\alpha(\mathcal{E}_n g) \le V_\alpha(g)$. Remembering definition 2.7 if $m \ge n$ then clearly $V_{\alpha,m} = 0$, in case

$1 \leq m < n$ and $|\rho_1 \wedge \rho_2| \geq m$ there should be $\omega_1, \omega_2 \in E_A^n$ such that $\rho_1 \in [\omega_1]$ and $\rho_2 \in [\omega_2]$, therefore $|\hat{\omega}_1 \wedge \hat{\omega}_2| \geq m$ and

$$|\mathcal{E}_n g(\rho_1) - \mathcal{E}_n g(\rho_2)| e^{\alpha(m-1)} = |g(\hat{\omega}_1) - g(\hat{\omega}_2)| e^{\alpha(m-1)} \leq V_{\alpha,m}(g).$$

Thus we have:

$$\|\mathcal{E}_n g\|_\alpha \leq \|g\|_\alpha. \tag{11}$$

Next without loss of generality assume $E = \mathbb{N}$ and for each $N \geq 1$ define

$$E_A^n(N) := \{\omega \in E_A^n : \omega_1, \omega_2, ..., \omega_n \leq N\}$$

$$E_A^n(N+) := E_A^n \setminus E_A^n(N)$$

$$\mathcal{E}_{n,N} g := \sum_{\omega \in E_A^n(N)} g(\hat{\omega}) \mathbb{1}_{[\omega]}.$$

Note that n and N are independent. Moreover notice that this time since we have finite sum the operator $\mathcal{E}_{n,N}$ on $C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ is of finite rank and so compact. We use triangle inequality to write:

$$\|\mathcal{L}_s^n - \mathcal{L}_s^n \mathcal{E}_{n,N}\|_\alpha \leq \|(\mathcal{L}_s^n - \mathcal{L}_s^n \mathcal{E}_n) + (\mathcal{L}_s^n \mathcal{E}_n - \mathcal{L}_s^n \mathcal{E}_{n,N})\|_\alpha$$

$$\leq \|\mathcal{L}_s^n (\mathcal{I} - \mathcal{E}_n)\|_\alpha + \|\mathcal{L}_s^n (\mathcal{E}_n - \mathcal{E}_{n,N})\|_\alpha, \tag{12}$$

where $\mathcal{I}$ is just the identity operator. Note that 11 implies $\|g - \mathcal{E}_n g\|_\alpha \leq 2\|g\|_\alpha$. Furthermore, for any $\rho \in E_A^{\mathbb{N}}$ if set $\omega = \rho_1 ... \rho_n$ then we have $|\rho \wedge \hat{\omega}| \geq n$ and

$$|g(\rho) - \mathcal{E}_n g(\rho)| e^{\alpha(n-1)} = |g(\rho) - g(\hat{\omega})| e^{\alpha(n-1)} \leq V_{\alpha,n}(g) \leq V_\alpha(g).$$

Since ρ is arbitrarily, we obtain

$$\|g - \mathcal{E}_n g\|_\infty \leq V_\alpha(g) e^\alpha e^{-\alpha n} \leq \|g\|_\alpha e^\alpha e^{-\alpha n}.$$

Thus using two recent inequalities and 9 we find

$$\|\mathcal{L}_s^n (I - \mathcal{E}_n) g\|_\alpha \leq e^{nP(x)} (Q e^{-\alpha n} 2\|g\|_\alpha + C\|g\|_\alpha e^\alpha e^{-\alpha n})$$

$$\leq C_1 e^{nP(x)} \|g\|_\alpha e^{-\alpha n}, \tag{13}$$

for some constant $C_1 > 0$. Recalling $\mathcal{F}_{\omega,s}$ from the proof of previous lemma, we can write

$$\mathcal{F}_{\omega',s}(\mathcal{E}_n g - \mathcal{E}_{n,N} g) = \sum_{|\omega|=n} g(\hat{\omega})\mathcal{F}_{\omega',s}(\mathbb{1}_{[\omega]}) - \sum_{\omega \in E_A^n(N)} g(\hat{\omega})\mathcal{F}_{\omega',s}(\mathbb{1}_{[\omega]})$$

$$= \sum_{\omega \in E_A^n(N+)} g(\hat{\omega})\mathcal{F}_{\omega',s}(\mathbb{1}_{[\omega]}) = g(\hat{\omega'})\mathcal{F}_{\omega',s}(\mathbb{1}_{[\omega']}) \text{ or } 0,$$

depending on $\omega' \in E_A^n(N+)$ or not, so

$$\mathcal{L}_s^n(\mathcal{E}_n g - \mathcal{E}_{n,N} g) = \sum_{\omega' \in E_A^n} F_{\omega',s}(\mathcal{E}_n g - \mathcal{E}_{n,N} g) = \sum_{\omega \in E_A^n(N+)} g(\hat{\omega})\mathcal{F}_{\omega,s}(\mathbb{1}_{[\omega]}).$$

Then 8 leaves:

$$\|\mathcal{L}_s^n(\mathcal{E}_n g - \mathcal{E}_{n,N} g)\|_\alpha \leq \|g\|_\infty \sum_{\omega \in E_A^n(N+)} \|\mathcal{F}_{\omega,s}(\mathbb{1}_{[\omega]})\|_\alpha$$

$$\leq \|g\|_\infty (2 + |s|K) \sum_{\omega \in E_A^n(N+)} \exp(x \sup_{[\omega]} S_n f).$$

Now since A is finitely irreducible, there exists a finite set $\Omega \subseteq E_A^* = \cup_n E_A^n$ such that for every $e \in E$ and $\rho \in E_A^{\mathbb{N}}$, there is $\omega \in \Omega$ with $e\omega\rho$ being admissible. Thus there exists finite set $F \subseteq E_A^{\mathbb{N}}$ such that for every $e \in E$, there is $\tau \in F$ with $e\tau$ being admissible. For every $\omega \in E_A^*$ choose $\tau_\omega \in F$ with $\omega\tau_\omega$ admissible. Therefore using 2 we can continue

$$\leq \|g\|_\infty (2 + |s|K) Q^2 \sum_{\omega \in E_A^n(N+)} \exp(x S_n f(\omega\tau_\omega)).$$

Moreover if we consider

$$c_N := \sup_{j \geq N} \exp(\sup f[j]),$$

then the fact that f is summable implies that $c_N \to 0$. Now for each $\omega \in E_A^n(N+)$ there is $\omega_i > N$ so

$$\exp(S_n f(\omega\tau_\omega)) = \exp(S_{i-1} f(\omega\tau_\omega)) + f(\omega_i ... \omega_n \tau_\omega) + S_{n-i} f(\omega_{i+1} ... \omega_n \tau_\omega))$$

$$\leq Q.c_N.Q = Q^2 c_N.$$

Therefore for small enough $\epsilon > 0$ we have

$$\|\mathcal{L}_s^n(\mathcal{E}_n g - \mathcal{E}_{n,N} g)\|_\alpha$$

$$\le \|g\|_\infty (2+|s|K)Q^2 \sum_{\omega\in E_A^n(N+)} \exp(\epsilon S_n f(\omega\tau_\omega)) \exp((x-\epsilon)S_n f(\omega\tau_\omega))$$
$$\le \|g\|_\infty (2+|s|K)Q^2 Q^{2\epsilon} c_N^\epsilon \sum_{\omega\in E_A^n(N+)} \exp((x-\epsilon)S_n f(\omega\tau_\omega))$$
$$\le \|g\|_\infty (2+|s|K)Q^4 c_N^\epsilon \sum_{\tau\in F} \mathcal{L}_{x-\epsilon}^n(\mathbb{1})(\tau) \le \|g\|_\infty (2+|s|K)Q^4 c_N^\epsilon \#F \|\mathcal{L}_{x-\epsilon}^n\|_\alpha.$$

This together with 10 yields

$$\|\mathcal{L}_s^n(\mathcal{E}_n - \mathcal{E}_{n,N})\|_\alpha \le (2+|s|K)Q^4 c_N^\epsilon \#F \|\mathcal{L}_{x-\epsilon}^n\|_\alpha$$
$$\le (2+|s|K)Q^4 c_N^\epsilon \#F (Q+C) e^{nP(x-\epsilon)}.$$

For large enough N we get

$$\|\mathcal{L}_s^n(\mathcal{E}_n - \mathcal{E}_{n,N})\|_\alpha \le e^{nP(x-\epsilon)} e^{-\alpha n}.$$

Thus since P is strictly decreasing, the above inequality combined with 12 and 13 implies

$$\|\mathcal{L}_s^n - \mathcal{L}_s^n \mathcal{E}_{n,N}\|_\alpha \le C_1 e^{nP(x)} e^{-\alpha n} + e^{nP(x-\epsilon)} e^{-\alpha n} \le C_2 e^{nP(x-\epsilon)} e^{-\alpha n}.$$

Therefore we can estimate the essential spectral radius:

$$r_{\text{ess}}(\mathcal{L}_s) = \lim_n \|\mathcal{L}_s^n\|_{\mathfrak{K}}^{1/n} \le \limsup_n \|\mathcal{L}_s^n - \mathcal{L}_s^n \mathcal{E}_{n,N}\|_\alpha^{1/n} \le e^{P(x-\epsilon)} e^{-\alpha}.$$

Since ϵ was chosen small enough, this completes the proof. □

We want to introduce two operators closely related to the transfer operator. The first operator is $\mathcal{L}_0$. There is s hidden in the definition but we don't write that. It is defined by:

$$\mathcal{L}_0 := e^{-P(x)} \mathcal{L}_s,$$

and another operator is the weighted operator defined by:

$$\bar{\mathcal{L}}_s g := e^{-P(x)} \frac{1}{h_x} \mathcal{L}_s(g h_x),$$

where h_x is a fixed point of $\mathcal{L}_0$ obtained in [25, p. 34] as the (compactly) convergent point of the sequence $\{\frac{1}{n_k}\sum_{i=0}^{n_k-1} e^{-P(x)} \mathcal{L}_x^j(\mathbb{1})\}$. In other words h_x is actually an eigenfunction of $\mathcal{L}_x$ corresponding to the eigenvalue $e^{P(x)}$. Moreover, it is clear that $\int h_x \mathrm{d}m_x = 1$.

Lemma 3.3. *There is $c > 0$ such that $h_x > c$.*

PROOF. To show this we use theorem 2.3.5 from [25, p. 29]. Let $n_k - 1 = (M+1)t_k + r_k$ where $0 \le r_k \le M$ then

$$\frac{1}{n_k}\sum_{i=0}^{n_k-1}\mathcal{L}_0^j(\mathbb{1}) \ge \frac{1}{n_k}\sum_{i=1}^{(M+1)t_k}\mathcal{L}_0^j(\mathbb{1}) \ge \frac{1}{n_k}t_k R$$

which leaves $h_x \ge \frac{R}{M+1}$. □

LEMMA 3.4. *If $g \in C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ is non-negative then $\{\frac{1}{n}\sum_{j=1}^{n}\bar{\mathcal{L}}_x^j g\}$ has a converging subsequence with limit $\int g d\mu_x$, where μ_x is the equilibrium state of xf.*

PROOF. Observe that $\bar{\mathcal{L}}_x(\mathbb{1}) = \mathbb{1}$ and so $\bar{\mathcal{L}}_x^j(\mathbb{1}) = \mathbb{1}$ for each $j \ge 1$. Then one can start with $\|\bar{\mathcal{L}}_x^j g\|_\alpha \le \|g\|_\alpha$ and follow the same proof of theorem 2.4.3 [25, p. 34] to find that $\{\frac{1}{n}\sum_{j=1}^{n}\bar{\mathcal{L}}_x^j g\}$ has a converging subsequence with limit $g_1 \in C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$, where $\bar{\mathcal{L}}_x g_1 = g_1$. This leaves $g_1 h_x$ as a fixed point of $\mathcal{L}_0$. Since g is non-negative so is g_1 and $g_1 h_x$. Now theorem 2.4.7 [25, p. 39] tells us that

$$(\frac{g_1}{d}h_x m_x)\circ\sigma^{-1} = \frac{g_1}{d}h_x m_x, \quad d = \int g_1 h_x dm_x,$$

where m_x is eigenmeasure of $\mathcal{L}_x$. Therefore if one defines a measure by $\mu_1(A) = \frac{1}{d}\int_A g_1 h_x dm_x$, we find that

$$\mu_1(\sigma^{-1}(A)) = \frac{1}{d}\int_{\sigma^{-1}(A)} g_1 h_x dm_x$$

$$= \frac{1}{d}\int_A g_1\circ\sigma^{-1} h_x\circ\sigma^{-1} d(m\circ\sigma^{-1}) = \frac{1}{d}\int_A g_1 h_x dm_x = \mu_1(A).$$

That leaves an invariant absolutely continuous measure with respect to m_x. Then theorem 10.4.2 [39] implies that μ_1 must be μ_x, therefore the Randon-Nikodym derivative of μ_1 with respect to m is the same as that of μ with respect to m a.e. which means $g_1 = d$ a.e. and since g_1 is continuous so $g_1 = d = \int g_1 h_x dm_x = \int g_1 d\mu_x$ everywhere. Furthermore, it is not hard to see that $(\bar{\mathcal{L}}_x)^*(\mu_x) = \mu_x$ see theorem 2.4.4 [25, p. 36]. Since we had

$$\frac{1}{n}\sum_{j=1}^{n}\bar{\mathcal{L}}_x^j g \to g_1$$

on a sub-sequence, then

$$\int g \mathrm{d}\mu_x = \int \frac{1}{n} \sum_{j=1}^{n} \bar{\mathcal{L}}_x^j g \mathrm{d}\mu_x \to \int g_1 \mathrm{d}\mu_x,$$

i.e. $\int g \mathrm{d}\mu_x = \int g_1 \mathrm{d}\mu_x$. □

PROPOSITION 3.5. *The transfer operator $\mathcal{L}_s$ has at most finitely many eigenvalues of modules $e^{P(x)}$ all of which with multiplicity one.*

PROOF. Previous proposition implies there are at most finitely many spectral values of $\mathcal{L}_s$ with modulus $e^{P(x)}$ all are isolated eigenvalues with finite (algebraic) multiplicity, see definition 2.30. We would like to show first for each eigenvalues ξ with $|\xi| = e^{P(x)}$ the transfer operator $\mathcal{L}_s$ acts on $X := \mathcal{P}_{\xi,s}(C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})) = \cup_{m\geq 1} \ker(\mathcal{L}_s - \xi)^m$ diagonally. To see this we consider the Jordan normal form of $L := \mathcal{L}_s$ on finite dimensional space X, so there is an invertible transformation P, such that PLP^{-1} is the Jordan normal form of L. Consider a $k \times k$ Jordan block in matrix representation that has 1 above the diagonal. The n^{th} power of the block looks like

$$\begin{bmatrix} \xi^n & \binom{n}{1}\xi^{n-1} & \binom{n}{2}\xi^{n-2} & \dots & \binom{n}{k-1}\xi^{n-k+1} \\ & \xi^n & \binom{n}{1}\xi^{n-1} & \dots & \binom{n}{k-2}\xi^{n-k+2} \\ & & \xi^n & \dots & \binom{n}{k-3}\xi^{n-k+3} \\ & & & \ddots & \vdots \\ & & & & \xi^n \end{bmatrix}.$$

Then for $e = \begin{bmatrix} 0 & 0 & ... & 0 & 1 \end{bmatrix}^T$ we have

$$\begin{bmatrix} \xi^n & \binom{n}{1}\xi^{n-1} & \binom{n}{2}\xi^{n-2} & \dots & \binom{n}{k-1}\xi^{n-k+1} \\ & \xi^n & \binom{n}{1}\xi^{n-1} & \dots & \binom{n}{k-2}\xi^{n-k+2} \\ & & \xi^n & \dots & \binom{n}{k-3}\xi^{n-k+3} \\ & & & \ddots & \vdots \\ & & & & \xi^n \end{bmatrix} e = \begin{bmatrix} \binom{n}{k-1}\xi^{n-k+1} \\ \binom{n}{k-2}\xi^{n-k+2} \\ \binom{n}{k-3}\xi^{n-k+3} \\ \vdots \\ \xi^n \end{bmatrix}.$$

Notice that $\binom{n}{1}\xi^{n-1}$ is the $(k-1)^{th}$ coordinate of this vector. If we equip X with the norm

$$\|x\| = |x_1| + ... + |x_t|, \quad t = \dim X,$$

and if we view e and the the above vector in X, we will have:

$$\binom{n}{1}|\xi|^{n-1} \leq \|PL^nP^{-1}e\| \leq \|PL^nP^{-1}\| \leq \|P\|\|L^n\|\|P^{-1}\| \leq C_0|\xi|^n$$

for some constant C_0, where the last inequality holds by proposition 3.2 and because on finite dimensional space all the norms are equivalent. This is clearly a contradiction. Therefore there is no non-trivial Jordan block, i.e. L is diagonalizable. This implies

$$X = \ker(\mathcal{L}_s - \xi).$$

It is clear that if g is in $\ker(\mathcal{L}_s - \xi)$ then g/h_x is in $\ker(\bar{\mathcal{L}}_s - e^{-P(x)}\xi)$. Therefore to show each $\ker(\mathcal{L}_s - \xi)$ is one dimensional, it is enough to show $\ker(\bar{\mathcal{L}}_s - e^{-P(x)}\xi)$ is one dimensional. Let $g \in \ker(\bar{\mathcal{L}}_s - e^{-P(x)}\xi)$, for each n

$$|g| = |e^{-P(x)}\xi g| = |\bar{\mathcal{L}}_s^n g| \leq \bar{\mathcal{L}}_x^n |g|.$$

Therefore if we apply the above lemma to the function $|g|$ we obtain

$$|g| \leq \int |g| d\mu_x.$$

Continuity of g and the fact that $\text{supp}(\mu_x) = E_A^{\mathbb{N}}$ (see explanation below the definition 2.19) makes this inequality into equality, i.e. every eigenvector has constant modulus. It is not hard to see that

$$\bar{\mathcal{L}}_s^n g(\rho) =$$

$$\frac{e^{-nP(x)}}{h_x(\rho)} \sum_{\omega \in E_A^n} \exp(sS_nf(\omega\rho)) \frac{1}{h_x(\sigma^{n-1}\omega\rho)} \frac{1}{h_x(\sigma^{n-2}\omega\rho)} \cdots \frac{1}{h_x(\sigma\omega\rho)} h_x(\omega\rho)g(\omega\rho).$$

Moreover since $\bar{\mathcal{L}}_x(\mathbb{1}) = \mathbb{1}$ we get:

$$1 = \bar{\mathcal{L}}_x^n(\mathbb{1})(\rho) =$$

$$\sum_{\omega \in E_A^n} e^{-nP(x)} \frac{1}{h_x(\rho)} \exp(xS_nf(\omega\rho)) \frac{1}{h_x(\sigma^{n-1}\omega\rho)} \frac{1}{h_x(\sigma^{n-2}\omega\rho)} \cdots \frac{1}{h_x(\sigma\omega\rho)} h_x(\omega\rho).$$

Note that every term in this sum, say u_ω, is positive. Eventually we find:

$$e^{-nP(x)}\xi^n g(\rho) = \bar{\mathcal{L}}_s^n g(\rho) = \sum_{\omega \in E_A^n} u_\omega \exp(iyS_n f(\omega\rho)) g(\omega\rho).$$

Now note that $|\sum_j a_j| = \sum_j |a_j|$ implies all a_j are co-linear, this along with the fact that g has constant modulus we get

$$g(\omega\rho) = e^{-nP(x)}\xi^n \exp(-iyS_n f(\omega\rho)) g(\rho).$$

This means values of g on the dense set $\cup_n \sigma^{-n}(\rho)$ (see remark below the definition 2.2) is determined by $g(\rho)$, so g spans $\ker(\bar{\mathcal{L}}_s - e^{-P(x)}\xi)$ as long as g has at least one non-zero point. This shows ξ is simple eigenvalue and it finishes the proof. □

Thus everything is ready to obtain spectral representation of $\mathcal{L}_s$ corresponding to the eigenvalues $\xi_1, \xi_2, ..., \xi_p$ of modulus $e^{P(x)}$. We use the above proposition to see that for each $s = x + iy \in \Gamma^+$, $\mathcal{L}_s$ has only finitely many eigenvalues $\xi_1(s), ...\xi_n(s)$ of modulus $e^{P(x)}$ each of which isolated in the spectrum and actually they are all simple eigenvalues. Therefore we may use theorem 2.33 to obtain the following spectral representation of the transfer operator:

$$\mathcal{L}_s = \xi_1(s)\mathcal{P}_{1,s} + \xi_2(s)\mathcal{P}_{2,s} + ... + \xi_n(s)\mathcal{P}_{n,s} + \mathcal{D}_s,$$

where each $\mathcal{P}_{i,s}$ is projection. Note that in this equation the operators are analytic operators and eigenvalues are analytic functions. Further, composition of every two different operators on the right hand side vanishes by proposition 2.34. This yields

$$\mathcal{L}_s^m = \xi_1(s)^m\mathcal{P}_{1,s} + \xi_2(s)^m\mathcal{P}_{2,s} + ... + \xi_n(s)^m\mathcal{P}_{n,s} + \mathcal{D}_s^m. \tag{14}$$

Finally proposition 2.34 implies:

$$\mathrm{Sp}(\mathcal{L}_s) \cup \{0\} = \{\xi_1(s)\} \cup \{\xi_2(s)\} \cup ... \cup \{\xi_n(s)\} \cup \mathrm{Sp}(\mathcal{D}_s) \cup \{0\}.$$

We finish this section with the following lemma.

LEMMA 3.6. *For every s_0 on the line $x = 1$, there is a neighborhood U of s_0, $0 < \beta < 1$ and constant $C > 0$ such that for every positive integer m*

$$\|\mathcal{D}_s^m\|_\alpha \leq C\beta^m, \quad s \in U.$$

PROOF. The above spectral decomposition implies the spectral radius of $\mathcal{D}_s$ to be strictly less than that of $\mathcal{L}_s$. Furthermore, proposition 3.2 implies $r(\mathcal{L}_{s_0}) \leq e^{P(1)} = 1$ so for s_0 there is $0 < \beta < 1$ such that $r(\mathcal{D}_{s_0}) < \beta$. Thus there is constant C_1 and natural number q such that

$$\|\mathcal{D}_{s_0}^q\|_\alpha \leq C_1\beta^q \leq \frac{\beta}{2}.$$

Additionally, using continuity on a small enough ball U at s_0 we have

$$\|\mathcal{D}_s^q - \mathcal{D}_{s_0}^q\|_\alpha < \frac{\beta}{2}.$$

Combining these two recent inequalities yields $\|\mathcal{D}_s^q\|_\alpha \leq \beta$ on U. Furthermore, there is constant C_2 such that for each integer r with $0 \leq r < q$, we have $\|\mathcal{D}_s^r\|_\alpha \leq C_2$ on U. Since for each positive integer m we can write $m = lq + r$, we eventually get for some $C > 0$:

$$\|\mathcal{D}_s^m\|_\alpha \leq C(\beta^{1/q})^m \tag{15}$$

on U. □

3.2. Counting and Poincaré series

Given $\rho \in E_A^{\mathbb{N}}$ and $B \subseteq E_A^{\mathbb{N}}$, for every $T > 0$ we define several counting functions. (a) The central counting function for us is

$$N_\rho(B,T) := \#\{\omega \in \cup_{n=1}^\infty E_A^n \ : \ \omega\rho \text{ admissible}, \ \omega\rho \in B, \ S_{|\omega|}f(\omega\rho) \geq -T\}. \tag{16}$$

It is not so hard to see that this is a step function of T, continuous from right and increasing. In order to associate a complex function to this counting function we set $N_\rho(B,T) = 0$ for $T < 0$ and we consider the Laplace–Stieltjes transform of $T \mapsto N_\rho(B,T)$ which we call it Poincaré series:

$$\eta_\rho(B,s) := \int_0^\infty \exp(-sT)\mathrm{d}N_\rho(B,T).$$

We will talk about its convergence in the next proposition. Below we introduce other counting functions appropriate for our purposes.

(b) Let $H = \{\tau_{(i)}\}_{i\in I}$ be a countable (finite or infinite) collection of finite words of bounded length, i.e. there exists a positive integer k such that $|\tau_{(i)}| \leq k$ for each $i \in I$. Further, assume the cylinders $\{[\tau_{(i)}]\}_{i\in I}$ are mutually disjoint. We denote

$$[H] := \cup_{i\in I}[\tau_{(i)}],$$

then the corresponding Poincaré series is of the form

$$\eta_\rho([H], s) = \int_0^\infty \exp(-sT)\mathrm{d}N_\rho([H], T).$$

$$= \sum_{n=1}^{\infty} \exp(-sT_i)\left(N_\rho([H], T_i) - N_\rho([H], T_{i-1})\right)$$

where $T_1 < T_2 < T_3 < \ldots$ is the increasing sequence of discontinuities of $T \mapsto N_\rho([H], T)$. Eventually this sums up to

$$\eta_\rho([H], s) = \sum_{n=1}^{\infty} \sum_{\omega\rho\in[H]} \exp(sS_n f(\omega\rho)) = \sum_{n=1}^{\infty} \mathcal{L}_s^n(\mathbb{1}_{[H]})(\rho). \tag{17}$$

(c) If we require to count only words with certain initial blocks then we should define

$$N_\rho(H, T) := \#\{\omega \in \cup_{n=1}^{\infty} E_A^n : \tau \in H,\ \tau\omega\rho \ \text{ admissible },\ S_{|\tau\omega|}f(\tau\omega\rho) \geq -T\}.$$

Then similarly one can see that the corresponding Poincaré series has the form

$$\eta_\rho(H, s) = \sum_{n=1}^{\infty} \mathcal{L}_s^{k+n}(\mathbb{1}_{[H]})(\rho).$$

Therefore

$$\eta_\rho([H], s) = \eta_\rho(H, s) + \sum_{n=1}^{k} \mathcal{L}_s^n(\mathbb{1}_{[H]})(\rho). \tag{18}$$

(d) For any positive integer q we set

$$N_\rho([H], q, T) := \#\{\omega \in E_A^q : \omega\rho \ \text{ admissible},\ \omega\rho \in [H],\ S_{|\omega|}f(\omega\rho) \geq -T\},$$

then its Poincaré series would be

$$\eta_\rho([H], q, s) = \mathcal{L}_s^q(\mathbb{1}_{[H]})(\rho).$$

(e) Further we would like to deal with periodic words as well. For this purpose we define

$$N_{\text{per}}([H],T) := \#\{\omega \in \cup_{n=1}^{\infty} E_A^n \ : \ \omega \text{ periodic word}, \ \overline{\omega} \in [H], \ S_{|\omega|} f(\overline{\omega}) \geq -T\},$$

(f) And

$$N_{\text{per}}(H,T) := \#\{\omega \ : \tau \in H, \ \ \tau\omega \text{ periodic word}, \ S_{|\tau\omega|} f(\overline{\tau\omega}) \geq -T\}.$$

(g) Finally we introduce another counting function for any positive integer q:

$$N_{\text{per}}([H],q,T) := \#\{\omega : \omega \text{ periodic word of length } q, \ \overline{\omega} \in [H], \ S_{|\omega|} f(\overline{\omega}) \geq -T\}.$$

(h) If $H = E$ then obviously $[H]$ is the whole space $E_A^{\mathbb{N}}$. In this case, we drop the notation $E_A^{\mathbb{N}}$ in $N_\rho(E_A^{\mathbb{N}}, T)$ and we simply write $N_\rho(T)$, similarly $N_{\text{per}}(T)$.

Next we want to find some relations between these counting functions. Note that we do not introduce a Poincaré series for the periodic orbits, as it won't have an ordinary geometric series expression and therefore Tauberian theorems are not applicable, instead we use some approximations. Now for every finite word ω we pick (exactly) one $\omega^+ \in E_A^{\mathbb{N}}$ such that $\omega\omega^+$ is admissible. From now on in this section, we assume τ is a fix word of length $k \geq 0$. When $k = 0$ we mean there is no word involved.

Lemma 3.7. *Let q be a positive integer, and $\gamma \in E_A^q$ be any word of length q. Given any ω where $\tau\gamma\omega$ is admissible and it is further a period word, then we have*

$$|S_{|\tau\gamma\omega|} f(\overline{\tau\gamma\omega}) - S_{|\tau\gamma\omega|} f(\tau\gamma\omega\tau\gamma(\tau\gamma)^+)| \leq K e^{-(k+q)\alpha},$$

where K only depends on f.

Proof. It is enough to apply Lemma 2.11:

$$|S_{|\tau\gamma\omega|} f(\overline{\tau\gamma\omega}) - S_{|\tau\gamma\omega|} f(\tau\gamma\omega\tau\gamma(\tau\gamma)^+)| \leq K_f d(\overline{\tau\gamma\omega}, \tau\gamma(\tau\gamma)^+)^\alpha \leq K_f e^{-(k+q)\alpha}.$$

□

Lemma 3.8. *Let q be a positive integer, then the following inequalities hold:*

(i)

$$N_{per}([\tau], q, T) \leq N_{\tau\tau^+}([\tau], q, T+K),$$

(ii)

$$\sum_{\substack{\gamma \in E_A^q \\ \tau\gamma \in E_A^{k+q}}} N_{\tau\gamma(\tau\gamma)^+}(\tau\gamma, T - Ke^{-(k+q)\alpha}) \leq N_{per}(\tau, T),$$

(iii)

$$N_{per}([\tau], T) \leq \sum_{\substack{\gamma \in E_A^q \\ \tau\gamma \in E_A^{k+q}}} N_{\tau\gamma(\tau\gamma)^+}([\tau\gamma], T + Ke^{-(k+q)\alpha}),$$

(iv) *For* $i \geq k+q$

$$N_{\tau\gamma(\tau\gamma)^+}([\tau\gamma], i, T) \leq N_{\tau\tau^+}([\tau\gamma], i, T+K),$$

(v) *If* F *is any finite subset of* E_A^q *and* $F' = E_A^q \setminus F$, *then*

$$N_{per}([\tau], T) \leq \sum_{\substack{\gamma \in F \\ \tau\gamma \in E_A^{k+q}}} N_{\tau\gamma(\tau\gamma)^+}([\tau\gamma], T + Ke^{-(k+q)\alpha})$$

$$+ \sum_{\substack{\gamma \in F' \\ \tau\gamma \in E_A^{k+q}}} N_{\tau\tau^+}([\tau\gamma], T+2K) + \sum_{i=1}^{k+q-1} N_{\tau\tau^+}([\tau], i, T+K)$$

where K *only depends on* f.

PROOF. (i) Let ω be a finite word contributing to $N_{per}([\tau], q, T)$, then $|\omega| = q$. The fact that $\overline{\omega} \in [\tau]$ gives $\omega_1 = \tau_1$ Therefore since $\omega_q\omega_1$ is admissible, so is $\omega\tau\tau^+$. If $q \geq k = |\tau|$, since $\overline{\omega} \in [\tau]$ so is $\omega\tau\tau^+$, and if $q < k$, since $\overline{\omega} \in [\tau]$, we can write τ as m copies of ω and some remainders, i.e. $\tau = \omega^m\omega_1...\omega_r$. It is clear then the first k letters of $\omega^{m+1}\omega_1...\omega_r$ is again τ. Thus $\omega\tau\tau^+ \in [\tau]$. It remains to show $S_{|\omega|}\psi(\omega\tau\tau^+) \geq -T-K$. From our assumption $S_{|\omega|}\psi(\overline{\omega}) \geq -T$, we can apply Lemma 2.11 to see that

$$S_{|\omega|}\psi(\overline{\omega}) \leq S_{|\omega|}\psi(\omega\tau\tau^+) + K.$$

This finishes the proof for part (i).

(ii) Let γ be a word of length q with $\tau\gamma$ admissible. Let ω be a finite word contributing to $N_{\tau\gamma(\tau\gamma)^+}(\tau\gamma, T - Ke^{-(k+q)\alpha})$, we want to show $\gamma\omega$ contributes to $N_{per}(\tau, T)$. It is clear that, this way of contribution is injective, therefore that proves (ii). Since $\tau\gamma\omega\tau\gamma(\tau\gamma)^+$ is admissible, so is $\overline{\tau\gamma\omega}$. Furthermore, we know $S_{|\tau\gamma\omega|}\psi(\tau\gamma\omega\tau\gamma(\tau\gamma)^+) \geq -T + Ke^{-(k+q)\alpha}$. If we use the above lemma we find

$$-T + Ke^{-(k+q)\alpha} \leq S_{|\tau\gamma\omega|}\psi(\tau\gamma\omega\tau\gamma(\tau\gamma)^+) \leq S_{|\tau\gamma\omega|}\psi(\overline{\tau\gamma\omega}) + Ke^{-(k+q)\alpha},$$

which shows $-T \leq S_{|\tau\gamma\omega|}\psi(\overline{\tau\gamma\omega})$ as needed.

(iii). Let ω be a finite word contributing to $N_{per}([\tau], T)$ of length n. The fact that $\overline{\omega} \in [\tau]$ gives $\omega_1 = \tau_1$ Therefore since $\omega_n\omega_1$ is admissible, so is $\omega\tau$. Note that

$$[\tau] = \cup'[\tau\gamma],$$

where the union is over all γ with length q such that $\tau\gamma$ is admissible. Since $\overline{\omega} \in [\tau]$, there should be γ such that $\overline{\omega} \in [\tau\gamma]$. Since $\omega\tau$ is admissible, so is $\omega\tau\gamma(\tau\gamma)^+$. Next we want to show $\omega\tau\gamma(\tau\gamma)^+ \in [\tau\gamma]$. If we separate to two cases where $n \geq k+q$ and $n < k+q$, then in exactly similar manner as in part (i) we obtain this. It remains only to show $S_{|\omega|}\psi(\omega\tau\gamma(\tau\gamma)^+) \geq -T - Ke^{-(k+q)\alpha}$. We have already $S_{|\omega|}\psi(\overline{\omega}) \geq -T$, furthermore if we use lemma 2.11 we see that

$$|S_{|\omega|}\psi(\overline{\omega}) - S_{|\omega|}\psi(\omega\tau\gamma(\tau\gamma)^+)| \leq Kd(\overline{\omega}, \tau\gamma(\tau\gamma)^+) \leq Ke^{-(k+q)\alpha},$$

where the last inequality is due to $\overline{\omega} \in [\tau\gamma]$. Thus from this inequality, we obtain

$$-T - Ke^{-(k+q)\alpha} \leq S_{|\omega|}\psi(\omega\tau\gamma(\tau\gamma)^+).$$

This completes part (iii).

(iv) Take ω that contributes to $N_{\tau\gamma(\tau\gamma)^+}([\tau\gamma], t, T)$. Clearly, $\omega\tau\tau^+$ is admissible. Since $|\omega| \geq k+q$ then we have clearly $\omega\tau\tau^+ \in [\tau\gamma]$ as well. Further, note that

$$|S_{|\omega|}(\omega\tau\gamma(\tau\gamma)^+) - S_{|\omega|}(\omega\tau\tau^+)| \leq K.$$

(v) Take ω such that it contributes to $N_{per}([\tau], T)$. If its length is less than $k+q$, then we use part (i). This contributes to the third sum on the right hand side. If length of ω is at least $k+q$, then (iii) and (iv) tell us ω contributes to either of the first two sums on the right hand side. This finishes the proof of (v) and the lemma. $\square$

Below we want to prove item (v) from the above lemma without $[\tau]$. Let $\rho \in E_A^{\mathbb{N}}$, then due to our assumption that shift space is finitely irreducible, there exists a finite set consisting of finite words

$$\Omega = \{\tau_{(1)}, ..., \tau_{(r)}\}$$

such that for every finite word ω there exists $\tau_{(j)} \in \Omega$ with $\omega\tau_{(j)}\rho$ being admissible. Below we have a summation over all $\tau_{(j)}\rho$, while this might not be admissible for all $j = 1, ..., r$. Note that sum is only over those j where $\tau_{(j)}\rho$ is admissible. Note that Ω and r are independent of ρ.

LEMMA 3.9. *If F is any finite subset of E_A^q and $F' = E_A^q \setminus F$, for any $\rho \in E_A^{\mathbb{N}}$ we have*

$$N_{per}(T) \leq \sum_{\gamma \in F} N_{\gamma\gamma^+}([\gamma], T + Ke^{-q\alpha})$$

$$+ \sum_{\substack{j=1 \\ \tau_{(j)}\rho \in E_A^{\mathbb{N}}}}^{r} N_{\tau_{(j)}\rho}([F'], T + K) + \sum_{\substack{j=1 \\ \tau_{(j)}\rho \in E_A^{\mathbb{N}}}}^{r} \sum_{i=1}^{q-1} N_{\tau_{(j)}\rho}(i, T + K),$$

where K only depends on f.

PROOF. The proof is similar to item (v) in the above lemma. Let ω be a finite word contributing to $N_{\text{per}}(T)$, pick $\tau_{(j)} \in \Omega$ such that $\omega\tau_{(j)}\rho$ is admissible. If $|\omega| < q$, clearly ω is contributing to the third term on the right hand of the inequality. If $|\omega| \geq q$ and $\omega_1...\omega_q \in F$, we want to show ω contributes to $N_{\gamma\gamma^+}([\gamma], T + Ke^{-q\alpha})$ where $\gamma = \omega_1...\omega_q$. Since ω is periodic $\omega\omega_1$ is admissible, and so is $\omega\gamma\gamma^+$. It is clear that $\omega\gamma\gamma^+ \in [\gamma]$ as well. Further note that

$$|S_{|\omega|}f(\omega\gamma\gamma^+) - S_{|\omega|}f(\overline{\omega})| \leq Kd(\gamma\gamma^+, \overline{\omega}) \leq Ke^{-q\alpha}.$$

Finally, in case $|\omega| \geq q$ and $\omega_1...\omega_q \in F'$ we want to show ω is contributing to the second sum on the right hand side. This is similar to our previous case. $\square$

Moreover, we have the following two estimates for the eigenfunction h and the equilibrium state μ.

Lemma 3.10. *Let ω be a word of length n such that $\omega\rho$, $\omega\rho'$ are admissible, then we have*

$$1 - K_1 e^{-n\alpha} \leq \frac{h(\omega\rho)}{h(\omega\rho')} \leq 1 + K_1 e^{-n\alpha},$$

where K_1 only depends on h.

Proof. We know from [25, p. 34] that h is Hölder continuous, therefore there is a constant K_0, such that

$$|h(\omega\rho) - h(\omega\rho)| \leq K_0 d(\omega\rho, \omega\rho') \leq K_0 e^{-n\alpha}.$$

Dividing by $h(\omega\rho)$ and using lemma 3.3, we obtain

$$|\frac{h(\omega\rho)}{h(\omega\rho')} - 1| \leq \frac{K_0}{h(\omega\rho)} e^{-n\alpha} \leq K_1 e^{-n\alpha},$$

where $K_1 = K_0 \frac{M+1}{R}$. □

Lemma 3.11. *Let ω be a finite word of length n such that $\omega\rho$ is admissible, then*

$$(1 - K_1 e^{-n\alpha}) h(\omega\rho) m([\omega]) \leq \mu([\omega]) \leq (1 + K_1 e^{-n\alpha}) h(\omega\rho) m([\omega]),$$

where K_1 is a constant depending only on h.

Proof. We saw in the proof of the lemma 3.4 that $\mu(A) = \int_A h dm$. Therefore we have

$$\left(\inf_{[\omega]} h\right) m([\omega]) \leq \mu([\omega]) \leq \left(\sup_{[\omega]} h\right) m([\omega]).$$

Now we use the above lemma to see

$$(1 - K_1 e^{-n\alpha}) h(\omega\rho) \leq \inf_{[\omega]} h \leq \sup_{[\omega]} h \leq (1 + K_1 e^{-n\alpha}) h(\omega\rho).$$

This finishes the proof. □

Proposition 3.12. *The functions $\eta_\rho([H], s)$, $\eta_\rho(H, s)$ are holomorphic on* $\mathrm{Re}(s) > 1$, *and the function $\eta_\rho([H], q, s)$ is holomorphic on Γ^+.*

PROOF. Using the relation 18, if we show $\eta_\rho([H], s)$ is holomorphic then $\eta_\rho(H, s)$ will be holomorphic as well. In order to show $\eta_\rho([H], s)$ is holomorphic we need $|\mathcal{L}_s^n(\mathbb{1}_{[H]})|_\infty$:

$$|\mathcal{L}_s^n(\mathbb{1}_{[H]})|_\infty \leq |\mathcal{L}_s^n(\mathbb{1})|_\infty \leq \sum_{\omega \in E_A^n} \exp(\mathrm{Re}(s) \sup_{[\omega]} S_n f).$$

This reminds us of the pressure function. Using the fact that P is strictly decreasing on Γ from proposition 2.23, consider an arbitrary $s_0 = x_0 + iy_0$ with $x_0 > 1$, for any s with $x \geq x_0$ there is a negative r such that $P(x) < r < 0$, therefore there is N such that for $n > N$:

$$\frac{1}{n} \ln \big(\sum_{\omega \in E_A^n} \exp(x \sup_{[\omega]} S_n f \big) < r,$$

so

$$|\mathcal{L}_s^n(\mathbb{1}_{[H]})|_\infty \leq |\mathcal{L}_s^n(\mathbb{1})|_\infty \leq \sum_{\omega \in E_A^n} \exp(x \sup_{[\omega]} S_n f) < e^{rn}.$$

This shows $\eta_\rho([H], s)$ converges uniformly on compact sets, thus $\eta_\rho([H], s)$ as a sum of holomorphic functions is holomorphic on $\mathrm{Re}(s) > 1$.

The above expression of $\eta_\rho([H], q, s)$ shows it is holomorphic on Γ^+. □

PROPOSITION 3.13. *If $f : E_A^{\mathbb{N}} \to \mathbb{R}$ has D-generic property, then each $\eta_\rho([H], s)$ and $\eta_\rho(H, s)$ at each point of the critical line $\mathrm{Re}(s) = 1$ except $s = 1$ admits analytic continuation and at $s = 1$ admits a meromorphic extension with a simple pole and residue*

$$Res(\eta_\rho, 1) = \frac{h(\rho)}{\chi_\mu} m([H]).$$

If we lift the D-generic property, then there exists $y_1 > 0$ such that the above statement holds on the segment $\{1 + iy : |y| < y_1\}$ with the same residue at the simple pole $s = 1$. Furthermore, this y_1 doesn't depend on H or ρ.

PROOF. By reviewing equations 27 and 14, it is clear that we can write

$$\eta_\rho([H], s) = \sum_{k=1}^{\infty} \mathcal{L}_s^k(\mathbb{1}_{[H]})$$

$$= \sum_{k=1}^{\infty} \left(\xi_1(s)^k \mathcal{P}_{1,s}(\mathbb{1}_{[H]}) + \ldots + \xi_n(s)^k \mathcal{P}_{n,s}(\mathbb{1}_{[H]}) + \mathcal{D}_s^k(\mathbb{1}_{[H]}) \right).$$

Now we use proposition 2.23 to see $|\xi_i(s)| = e^{P(x)} < 1$ if $x > 1$. Therefore we can continue the above equation

$$= \xi_1(s)(1-\xi_1(s))^{-1}\mathcal{P}_{1,s}(\mathbb{1}_{[H]}) + ... + \xi_n(s)(1-\xi_n(s))^{-1}\mathcal{P}_{n,s}(\mathbb{1}_{[H]}) + \mathcal{Q}_s(\mathbb{1}_{[H]}),$$

where $\mathcal{Q}_s = \sum_{k=1}^{\infty} \mathcal{D}^k$ converges using lemma 3.6. This is a valid relation for the Poincaré series η_ρ on $x > 1$. We fix s_0 on the line $x = 1$, it is clear that $\mathcal{Q}_s(\mathbb{1}_{[H]})$ is a holomorphic function on the neighborhood U of s_0 obtained in lemma 3.6. Additionally all the projections $\mathcal{P}_{i,s}$ and function $\xi_i(s)$ are analytic as discussed just above the equation 14. Therefore the right hand side of the above equation is analytic on some neighborhood U_0 of s_0, as long as $\xi_i(s_0) \neq 1$. As we know for real $s = x + i0$, one of the eigenvalues of the transfer operator is $e^{P(x)}$ by theorem 2.18. We let $\xi_1(s)$ represent this eigenvalue, it is clear that $\xi_1(s)$ is not constant on any neighborhood of $s = 1$ as $|\xi_1(s)| = e^{P(x)}$ and P is strictly decreasing by proposition 2.23. Since $\xi_i(s)$ are isolated, simple eigenvalues and further analytic functions identity theorem from complex analysis guarantees existence of $y_1 > 0$ for which the equations $\xi_i(s) = 1$ on $\{1 + iy : |y| < y_1\}$ have solution only if $i = 1$ and $s = 1$. We deduce the right hand side of the equation above defines an analytic function on a neighborhood of $\{1+iy : 0 < |y| < y_1\}$. Note that $\xi_1(s)$ is simple eigenvalue, so near $s = 1$ we expect

$$(1 - \xi_1(s)) \sim s - 1.$$

In other words, we find that $\eta_\rho([H], s) - A/(s-1)$ admits analytic extension to the segment $\{1 + iy : |y| < y_1\}$, where

$$A = \lim_{s\to 1} \eta_\rho([H], s)(s-1) = \xi_1(1)\mathcal{P}_{1,1}(\mathbb{1}_{[H]}) \lim_{s\to 1} \frac{s-1}{1-\xi_1(s)}.$$

It is clear that using D-generic property y_1 can be taken to be ∞. Thus, it only remains to compute A. It is clear that $\xi_1 = \xi_1(1) = e^{P(1)} = 1$. To compute $\mathcal{P}_{1,1}(\mathbb{1}_{[H]})$ first note that $\mathcal{L}_0\mathcal{P}_{i,1} = \xi_i\mathcal{P}_{i,1}$ for each i, so

$$\int \mathcal{P}_{i,1}(g)dm = \int \mathcal{L}_0\mathcal{P}_{i,1}(g)dm = \xi_i \int \mathcal{P}_{i,1}(g)dm.$$

This gives $\int \mathcal{P}_{i,1}(g) = 0$ for every $g \in C^{0,\alpha}(E_A^{\mathbb{N}}, \mathbb{C})$ and $i \neq 1$. Therefore with respect to the measure m for each k:

$$\int g = \int \mathcal{L}_0^k(g) = \int \mathcal{P}_{1,1}(g) + \int \xi_2^k \mathcal{P}_{2,1}(g) + ... + \int \xi_n^k \mathcal{P}_{n,1}(g) + \int \mathcal{D}_1^k(g)$$

$$= \int \mathcal{P}_{1,1}(g) + \int \mathcal{D}_1^k(g),$$

now implementing the inequality obtained in lemma 3.6 would yield

$$\int g = \int \mathcal{P}_{1,1}(g).$$

This actually determines the action of $\mathcal{P}_{1,1}$ since if $P_{1,1}(g) = k_g h$ then $k_g = \int g$, i.e.

$$\mathcal{P}_{1,1}(g) = h \int g dm.$$

And lastly

$$\lim_{s \to 1} \frac{1 - \xi_1(s)}{s - 1} = \lim_{x \to 1} \frac{1 - e^{P(x)}}{x - 1} = -P'(1)e^{P(1)} = -\int f d\mu = \chi_\mu,$$

where the equality to the last follows from proposition 2.6.13 in [25, p. 47]. Thus we find that the residue is $h(\rho)m([H])/\chi_\mu$. □

3.3. Asymptotic Formula for Counting

In this section, we assume f is strongly regular, summable and Hölder-type continuous with $P(1) = P(f) = 0$. We keep this assumption to the end of proposition 3.18 and after that we consider general functions with $P(\delta) = P(\delta f) = 0$ for some $\delta > 0$. We want to find asymptotic formula for the counting functions presented in the previous section. We can provide formula for some estimate of lower bound and upper bound of all possible values. As well in this section by y_0 we mean

$$y_0 = \frac{y_1}{2\pi},$$

where y_1 was obtained in proposition 3.13. As mentioned in that proposition, this y_0 doesn't depend on H in $\eta_\rho([H], T)$. Further, we set

$$c_1 := y_0^{-1} \left(\exp(y_0^{-1}) - 1\right)^{-1}, \quad c_2 := y_0^{-1} \left(\exp(y_0^{-1}) - 1\right)^{-1} \exp(y_0^{-1}).$$

Proposition 3.14.

$$c_1 \frac{h(\rho)}{\chi_\mu} m([H]) \leq \liminf_{T\to\infty} \frac{N_\rho(H,T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N_\rho(H,T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\chi_\mu} m([H]),$$

and

$$c_1 \frac{h(\rho)}{\chi_\mu} m([H]) \leq \liminf_{T\to\infty} \frac{N_\rho([H],T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N_\rho([H],T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\chi_\mu} m([H]),$$

and for every positive integer q

$$\lim_{T\to\infty} \frac{N_\rho([H],q,T)}{\exp(T)} = 0.$$

Proof. The first two lines of inequalities follows from proposition 3.13 and applying Graham-Vaaler theorem 2.42. The last equality follows from proposition 3.12 and applying Ikehara-Wiener theorem 2.41. □

Proposition 3.15.

$$c_1 \frac{1}{\chi_\mu} \mu([\tau]) \leq \liminf_{T\to\infty} \frac{N_{per}(\tau,T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N_{per}(\tau,T)}{\exp(T)} \leq c_2 \frac{1}{\chi_\mu} \mu([\tau]),$$

and

$$c_1 \frac{1}{\chi_\mu} \mu([\tau]) \leq \liminf_{T\to\infty} \frac{N_{per}([\tau],T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N_{per}([\tau],T)}{\exp(T)} \leq c_2 \frac{1}{\chi_\mu} \mu([\tau]).$$

Proof. Let $\sum'$ represents the sum over all γ with length q such that $\tau\gamma$ is admissible. Then using part (ii) of lemma 3.8, lemma 2.13 and proposition 3.14 we can write:

$$\begin{aligned}
\liminf_{T\to\infty} \frac{N_{\mathrm{per}}(\tau,T)}{\exp(T)} &\geq \liminf_{T\to\infty} \sum{}' \frac{N_{\tau\gamma(\tau\gamma)^+}(\tau\gamma, T - Ke^{-(k+q)\alpha})}{\exp(T)} \\
&\geq \exp\left(-Ke^{-(k+q)\alpha}\right) \sum{}' \liminf_{T\to\infty} \frac{N_{\tau\gamma(\tau\gamma)^+}(\tau\gamma, T - Ke^{-(k+q)\alpha})}{\exp\left(T - Ke^{-(k+q)\alpha}\right)} \\
&= \exp\left(-Ke^{-(k+q)\alpha}\right) \sum{}' c_1 \frac{h(\tau\gamma(\tau\gamma)^+)}{\chi_\mu} m([\tau\gamma]).
\end{aligned}$$

We use lemma 3.11 at this step and continue:

$$\begin{aligned}
\liminf_{T\to\infty} \frac{N_{\mathrm{per}}(\tau,T)}{\exp(T)} &\geq c_1 \frac{\exp\left(-Ke^{-(k+q)\alpha}\right)}{\chi_\mu} \sum{}' (1 + K_1 e^{-(k+q)\alpha})^{-1} \mu([\tau\gamma]) \\
&= c_1 \frac{\exp\left(-Ke^{-(k+q)\alpha}\right)}{1 + K_1 e^{-(k+q)\alpha}} \frac{1}{\chi_\mu} \mu([\tau]).
\end{aligned}$$

Since q is arbitrary, by $q \to \infty$ we obtain

$$\liminf_{T\to\infty} \frac{N_{\text{per}}(\tau, T)}{\exp(T)} \geq c_1 \frac{1}{\chi_\mu} \mu([\tau])$$

If we show

$$\limsup_{T\to\infty} \frac{N_{\text{per}}([\tau], T)}{\exp(T)} \leq c_2 \frac{1}{\chi_\mu} \mu([\tau]),$$

we are done with the proof. We use lemma 3.8 part (v) for this and then we apply proposition 3.14 several times.

$$\limsup_{T\to\infty} \frac{N_{\text{per}}([\tau], T)}{\exp(T)} \leq \limsup_{T\to\infty} \sum_{\substack{\gamma\in F\\ \tau\gamma\in E_A^{\mathbb{N}}}} \frac{N_{\tau\gamma(\tau\gamma)^+}\left([\tau\gamma], T + Ke^{-(k+q)\alpha}\right)}{\exp(T)}$$

$$+ \limsup_{T\to\infty} \sum_{\substack{\gamma\in F'\\ \tau\gamma E_A^{\mathbb{N}}}} \frac{N_{\tau\tau^+}([\tau\gamma], T+2K)}{\exp(T)} + \limsup_{T\to\infty} \sum_{i=1}^{k+q-1} \frac{N_{\tau\tau^+}([\tau], i, T+K)}{\exp(T)}.$$

Now the first limsup easily passes through the finite sum and we use proposition 3.14 with $H = \tau\gamma$, for the second limsup note that

$$\sum_{\substack{\gamma\in F'\\ \tau\gamma E_A^{\mathbb{N}}}} N_{\tau\tau^+}([\tau\gamma], T+2K) = N_{\tau\tau^+}([\tau F'], T+2K),$$

therefore we apply proposition 3.14 with $H = \tau F'$ and the last limsup is clearly 0 using again proposition 3.14. Thus we get

$$\limsup_{T\to\infty} \frac{N_{\text{per}}([\tau], T)}{\exp(T)}$$

$$\leq \sum_{\substack{\gamma\in F\\ \tau\gamma E_A^{\mathbb{N}}}} \limsup_{T\to\infty} \frac{N_{\tau\gamma(\tau\gamma)^+}\left([\tau\gamma], T + Ke^{-(k+q)\alpha}\right)}{\exp\left(T + Ke^{-(k+q)\alpha}\right)} \exp\left(Ke^{-(k+q)\alpha}\right)$$

$$+ \limsup_{T\to\infty} \frac{N_{\tau\tau^+}([\tau F'], T+2K)}{\exp(T+2K)} \exp(2K)$$

$$= \exp\left(Ke^{-(k+q)\alpha}\right) \sum_{\substack{\gamma\in F\\ \tau\gamma E_A^{\mathbb{N}}}} c_2 \frac{h(\tau\gamma(\tau\gamma)^+)}{\chi_\mu} m([\tau\gamma]) + c_2 \frac{h(\tau\tau^+)}{\chi_\mu} m([\tau F']) \exp(2K).$$

Notice that since F was arbitrary for $\epsilon > 0$ we choose F such that

$$c_2 \frac{h(\tau\tau^+)}{\chi_\mu} m([\tau F']) \exp(2K) < \epsilon,$$

then we obtain

$$\limsup_{T\to\infty} \frac{N_{\rm per}([\tau],T)}{\exp(T)} \le \exp\left(Ke^{-(k+q)\alpha}\right) \sum_{\substack{\gamma\in F\\ \tau\gamma E_A^{\mathbb{N}}}} c_2 \frac{h(\tau\gamma(\tau\gamma)^+)}{\chi_\mu} m([\tau\gamma]) + \epsilon.$$

Now we apply left hand side of the lemma 3.11:

$$\limsup_{T\to\infty} \frac{N_{\rm per}([\tau],T)}{\exp(T)} \le \frac{\exp\left(Ke^{-(k+q)\alpha}\right)}{1-K_1 e^{-(k+q)\alpha}} \sum_{\substack{\gamma\in F\\ \tau\gamma E_A^{\mathbb{N}}}} c_2 \frac{1}{\chi_\mu} \mu([\tau\gamma]) + \epsilon.$$

Eventually we let $q\to\infty$ to get

$$\limsup_{T\to\infty} \frac{N_{\rm per}([\tau],T)}{\exp(T)} \le \sum_{\substack{\gamma\in F\\ \tau\gamma E_A^{\mathbb{N}}}} c_2 \frac{1}{\chi_\mu} \mu([\tau\gamma]) + \epsilon = c_2 \frac{1}{\chi_\mu} \mu([\tau F]) + \epsilon \le c_2 \frac{1}{\chi_\mu} \mu([\tau]) + \epsilon.$$

Since ϵ was arbitrary we have

$$\limsup_{T\to\infty} \frac{N_{\rm per}([\tau],T)}{\exp(T)} \le c_2 \frac{1}{\chi_\mu} \mu([\tau]).$$

□

PROPOSITION 3.16.

$$\limsup_{T\to\infty} \frac{N_{per}(T)}{\exp(T)} \le c_2 \frac{1}{\chi_\mu}.$$

PROOF. This proof is exactly similar to the proof of the previous proposition for limsup and implementing lemma 3.9. □

PROPOSITION 3.17. *For every open set $V \subseteq E_A^{\mathbb{N}}$ we have*

$$c_1 \frac{h(\rho)}{\chi_\mu} m(V) \le \liminf_{T\to\infty} \frac{N_\rho(V,T)}{\exp(T)} \le \limsup_{T\to\infty} \frac{N_\rho(V,T)}{\exp(T)} \le c_1 \frac{h(\rho)}{\chi_\mu} m(\overline{V}) + y_0^{-1} \frac{h(\rho)}{\chi_\mu},$$

and

$$c_1 \frac{1}{\chi_\mu} \mu(V) \le \liminf_{T\to\infty} \frac{N_{per}(V,T)}{\exp(T)} \le \limsup_{T\to\infty} \frac{N_{per}(V,T)}{\exp(T)} \le c_1 \frac{1}{\chi_\mu} \mu(\overline{V}) + y_0^{-1} \frac{1}{\chi_\mu}.$$

PROOF. We know from proposition 2.1 that V can be written as union of disjoint cylinders, so $V = \cup_i [\tau_{(i)}]$. Therefore using lemma 2.13 and proposition 3.14 with $H = \tau_{(i)}$ one can write

$$\liminf_{T\to\infty} \frac{N_\rho(V,T)}{\exp(T)} = \liminf_{T\to\infty} \sum_i \frac{N_\rho([\tau_{(i)}],T)}{\exp(T)} \ge \sum_i \liminf_{T\to\infty} \frac{N_\rho([\tau_{(i)}],T)}{\exp(T)}$$

$$\geq \sum_i c_1 \frac{h(\rho)}{\chi_\mu} m([\tau_{(i)}]) = c_1 \frac{h(\rho)}{\chi_\mu} m(V).$$

For the limsup we use lemma 2.14 and the above inequality for the open set $\overline{V}^c$ to find

$$c_1 \frac{h(\rho)}{\chi_\mu} m(\overline{V}^c) + \limsup_{T\to\infty} \frac{N_\rho(V,T)}{\exp(T)}$$

$$\leq \liminf_{T\to\infty} \frac{N_\rho(\overline{V}^c,T)}{\exp(T)} + \limsup_{T\to\infty} \frac{N_\rho(V,T)}{\exp(T)}$$

$$\leq \limsup_{T\to\infty} \frac{N_\rho(\overline{V}^c,T) + N_\rho(V,T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N_\rho(T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\chi_\mu},$$

where the last inequality holds if we apply proposition 3.14 for $H = E$ (all the alphabets). This yields

$$\limsup_{T\to\infty} \frac{N_\rho(V,T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\chi_\mu} - c_1 \frac{h(\rho)}{\chi_\mu} m(\overline{V}^c) = c_1 \frac{h(\rho)}{\chi_\mu} m(\overline{V}) + y_0^{-1} \frac{h(\rho)}{\chi_\mu}.$$

For counting periodic words, the idea is similar. Again we implement lemma 2.13 and this time proposition 3.15 to obtain:

$$\liminf_{T\to\infty} \frac{N_{\text{per}}(V,T)}{\exp(T)} = \liminf_{T\to\infty} \sum_i \frac{N_{\text{per}}([\tau_{(i)}],T)}{\exp(T)} \geq \sum_i \liminf_{T\to\infty} \frac{N_{\text{per}}([\tau_{(i)}],T)}{\exp(T)}$$

$$\geq \sum_i c_1 \frac{1}{\chi_\mu} \mu([\tau_{(i)}]) = c_1 \frac{1}{\chi_\mu} \mu(V).$$

Applying lemma 2.14 and the above inequality for the open set $\overline{V}^c$ gives us:

$$c_1 \frac{1}{\chi_\mu} \mu(\overline{V}^c) + \limsup_{T\to\infty} \frac{N_{\text{per}}(V,T)}{\exp(T)}$$

$$\leq \liminf_{T\to\infty} \frac{N_{\text{per}}(\overline{V}^c,T)}{\exp(T)} + \limsup_{T\to\infty} \frac{N_{\text{per}}(V,T)}{\exp(T)}$$

$$\leq \limsup_{T\to\infty} \frac{N_{\text{per}}(\overline{V}^c,T) + N_{\text{per}}(V,T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N_{\text{per}}(T)}{\exp(T)} \leq c_2 \frac{1}{\chi_\mu},$$

where the last inequality is due to the above proposition. This eventually gives

$$\limsup_{T\to\infty} \frac{N_{\text{per}}(V,T)}{\exp(T)} \leq c_2 \frac{1}{\chi_\mu} - c_1 \frac{1}{\chi_\mu} \mu(\overline{V}^c) = c_1 \frac{1}{\chi_\mu} \mu(\overline{V}) + y_0^{-1} \frac{1}{\chi_\mu}.$$

□

PROPOSITION 3.18. *For every Borel set $B \subseteq E_A^{\mathbb{N}}$ we have*

$$c_1 \frac{h(\rho)}{\chi_\mu} m(B^o) \leq \liminf_{T\to\infty} \frac{N_\rho(B,T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N_\rho(B,T)}{\exp(T)} \leq c_1 \frac{h(\rho)}{\chi_\mu} m(\overline{B}) + y_0^{-1} \frac{h(\rho)}{\chi_\mu},$$

and

$$c_1 \frac{1}{\chi_\mu} \mu(B^o) \leq \liminf_{T\to\infty} \frac{N_{per}(B,T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N_{per}(B,T)}{\exp(T)} \leq c_1 \frac{1}{\chi_\mu} \mu(\overline{B}) + y_0^{-1} \frac{1}{\chi_\mu}.$$

PROOF. We only prove the first line of inequalities. The other one is proved in a similar manner. We apply the above proposition to open set B^o:

$$c_1 \frac{h(\rho)}{\chi_\mu} m(B^o) \leq \liminf_{T\to\infty} \frac{N_\rho(B^o,T)}{\exp(T)} \leq \liminf_{T\to\infty} \frac{N_\rho(B,T)}{\exp(T)}.$$

For limsup we use lemma 2.14 and this inequality for $\overline{B}^c$:

$$c_1 \frac{h(\rho)}{\chi_\mu} m(\overline{B}^c) + \limsup_{T\to\infty} \frac{N_\rho(\overline{B},T)}{\exp(T)}$$

$$\leq \liminf_{T\to\infty} \frac{N_\rho(\overline{B}^c,T)}{\exp(T)} + \limsup_{T\to\infty} \frac{N_\rho(\overline{B},T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N_\rho(T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\chi_\mu},$$

where the last inequality holds if we apply proposition 3.14 for $H = E$ (all the alphabets). Thus

$$\limsup_{T\to\infty} \frac{N_\rho(B,T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N_\rho(\overline{B},T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\chi_\mu} - c_1 \frac{h(\rho)}{\chi_\mu} m(\overline{B}^c) = c_1 \frac{h(\rho)}{\chi_\mu} m(\overline{B}) + y_0^{-1} \frac{h(\rho)}{\chi_\mu}.$$

This finishes the proof. □

Note that so far we focused on the systems with $P(1) = 0$. We want to show that this is not restrictive and we can otherwise get corresponding counting formula as well. For a general Hölder-type function $f : E_A^{\mathbb{N}} \to \mathbb{R}$ we remember that $x \in \Gamma$ iff xf is summable. Assuming strong regularity we know there exists $\delta > 0$ such that $P(\delta) = 0$ and $\inf \Gamma < \delta$. Now if we consider a new function $g = \delta f$, first it is clear that g as well is strongly regular. Secondly, since $P(xg) = P(x\delta f)$ we have $P_g(1) = 0$. Therefore all the results obtained above is applicable for g. Additionally, note that $S_n g(\rho) = \delta S_n f(\rho)$, so we find that

$$N^g(\delta T) = N(T). \tag{19}$$

Moreover, it is clear

$$\mathcal{L}_{1g} = \mathcal{L}_{\delta f}.$$

Therefore if $\mathcal{L}_{\delta f} h_\delta = h_\delta$ then $\mathcal{L}_{1g} h_\delta = h_\delta$, similarly if $\mathcal{L}^*_{\delta f} m_\delta = m_\delta$ then $\mathcal{L}^*_{1g} m_\delta = m_\delta$. Additionally, if $\mathcal{L}_{sf}$ avoids $\exp(P(\delta f)) = 1$ as eigenvalue on

$$\{\delta + iy : 0 < |y| < y_0(f)\},$$

then $\mathcal{L}_{sg}$ does so on

$$\{1 + iy : 0 < |y| < \frac{y_0(f)}{\delta}\}.$$

This implies $y_0(g) = \frac{y_0(f)}{\delta}$ and so

$$c_1(g) = (\frac{y_0}{\delta})^{-1}\left(\exp((\frac{y_0}{\delta})^{-1}) - 1\right)^{-1}.$$

It is now enough to use proposition 3.18 for g with $m = m_\delta$, $\mu = \mu_\delta$ and $h = h_\delta$ to estimate:

$$c_1(g)\frac{h(\rho)}{\chi_\mu} m(B^o) \leq \liminf_{T\to\infty} \frac{N^g_\rho(B,T)}{\exp(T)} \leq \limsup_{T\to\infty} \frac{N^g_\rho(B,T)}{\exp(T)} \leq c_1(g)\frac{h(\rho)}{\chi_\mu} m(\overline{B}) + (\frac{y_0}{\delta})^{-1}\frac{h(\rho)}{\chi_\mu}.$$

Furthermore, note that

$$\chi_\mu = -\int g \ \mathrm{d}\mu_\delta = -\delta \int f \ \mathrm{d}\mu_\delta = \delta \chi_{\mu_\delta}.$$

Now we replace T with δT and use 19 to obtain the following estimate for f:

$$y_0^{-1}\left(\exp(\delta y_0^{-1}) - 1\right)^{-1}\frac{h_\delta(\rho)}{\chi_{\mu_\delta}} m_\delta(B^o) \leq \liminf_{T\to\infty} \frac{N_\rho(B,T)}{\exp(\delta T)}$$

$$\leq \limsup_{T\to\infty} \frac{N_\rho(B,T)}{\exp(\delta T)} \leq y_0^{-1}\left(\exp(\delta y_0^{-1}) - 1\right)^{-1}\frac{h_\delta(\rho)}{\chi_{\mu_\delta}} m_\delta(\overline{B}) + y_0^{-1}\frac{h_\delta(\rho)}{\chi_{\mu_\delta}}.$$

Similarly we can obtain a formula for $N_{\mathrm{per}}(B,T)$ which we omit its proof. We set

$$c_\delta := y_0^{-1}\left(\exp(\delta y_0^{-1}) - 1\right)^{-1} \tag{20}$$

and capture all the aforementioned arguments in the following theorem.

Theorem 3.19. *If $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is strongly regular Hölder-type function with $P(\delta f) = 0$, for every Borel set $B \subseteq E_A^{\mathbb{N}}$ and $\rho \in E_A^{\mathbb{N}}$ we have*

$$c_\delta \frac{h_\delta(\rho)}{\chi_{\mu_\delta}} m_\delta(B^o) \leq \liminf_{T\to\infty} \frac{N_\rho(B,T)}{\exp(\delta T)} \leq \limsup_{T\to\infty} \frac{N_\rho(B,T)}{\exp(\delta T)} \leq c_\delta \frac{h_\delta(\rho)}{\chi_{\mu_\delta}} m_\delta(\overline{B}) + y_0^{-1} \frac{h_\delta(\rho)}{\chi_{\mu_\delta}},$$

and

$$c_\delta \frac{1}{\chi_{\mu_\delta}} \mu(B^o) \leq \liminf_{T\to\infty} \frac{N_{per}(B,T)}{\exp(\delta T)} \leq \limsup_{T\to\infty} \frac{N_{per}(B,T)}{\exp(\delta T)} \leq c_\delta \frac{1}{\chi_{\mu_\delta}} \mu(\overline{B}) + y_0^{-1} \frac{1}{\chi_{\mu_\delta}}.$$

Remark 3.20. It is important to note that

- For N_ρ the eigenmeasure m and for N_{per} the equilibrium measure μ appears in the formula.
- The bounds are sharp as shown in example 3.25 below.
- The limit points of the ratio $\frac{N_\rho(B,T)}{\exp(\delta T)}$ can be a full closed interval, i.e.

$$\left\{ A \;:\; A = \lim_{n\to\infty} \frac{N_\rho(B,T_n)}{\exp(\delta T_n)},\;\; T_n \to \infty \;\text{ as }\; n \to \infty \right\} = [c, C],$$

for some $c, C > 0$. (see example 3.25)

Corollary 3.21. *If $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is strongly regular Hölder-type function with $P(\delta f) = 0$, for every Borel set $B \subseteq E_A^{\mathbb{N}}$ with boundary of measure 0 and $\rho \in E_A^{\mathbb{N}}$ we have*

$$c_\delta \frac{h_\delta(\rho)}{\chi_\mu} m(B) \leq \liminf_{T\to\infty} \frac{N_\rho(B,T)}{\exp(\delta T)} \leq \limsup_{T\to\infty} \frac{N_\rho(B,T)}{\exp(\delta T)} \leq c_\delta \frac{h_\delta(\rho)}{\chi_\mu} m(B) + y_0^{-1} \frac{h_\delta(\rho)}{\chi_\mu},$$

and

$$c_\delta \frac{1}{\chi_\mu} \mu(B) \leq \liminf_{T\to\infty} \frac{N_{per}(B,T)}{\exp(\delta T)} \leq \limsup_{T\to\infty} \frac{N_{per}(B,T)}{\exp(\delta T)} \leq c_\delta \frac{1}{\chi_\mu} \mu(B) + y_0^{-1} \frac{1}{\chi_\mu}.$$

Proof. We just need to apply the above theorem and noting that $m(\partial B) = 0$ implies $m(B) = m(\overline{B}) = m(B^o)$. □

Corollary 3.22 (Pollicott-Urbański). *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a strongly regular conformal graph directed Markov system with D-generic property. Let δ be the Hausdorff dimension of* *the limit set of $\mathcal{S}$, then for every Borel set $B \subseteq E_A^{\mathbb{N}}$ with boundary of measure* 0 *and $\rho \in E_A^{\mathbb{N}}$ we have*

$$\lim_{T\to\infty} \frac{N_\rho(B,T)}{\exp(\delta T)} = \frac{h_\delta(\rho)}{\delta \chi_{\mu_\delta}} m_\delta(B),$$

and

$$\lim_{T\to\infty} \frac{N_{per}(B,T)}{\exp(\delta T)} = \frac{1}{\delta \chi_{\mu_\delta}} \mu_\delta(B).$$

PROOF. It follows from the previous corollary. Note that when $\mathcal{S}$ is D-generic then we are allowed to let $y_0 \to \infty$ and this gives $c_\delta \to \frac{1}{\delta}$ from 20. □

3.4. Asymptotic Formula for Length

Before bringing some examples we would like to talk about counting with specified length. As indicated in the beginning of the previous section item (d) we had $N_\rho([H], q, T)$ which is counting number of words ω satisfying $S_{|\omega|} f(\omega\rho) \geq -T$ of length q. We addressed in proposition 3.14 that growth of this relative to $\exp(\delta T)$ tends to 0. Therefore if we would like to obtain fairly interesting growth we have to focus on some counting where q as well grows as T grows. We know $N_\rho(T) \sim C \exp(\delta T)$ but if we write

$$N_\rho(T) = \sum_{i=1}^{\infty} N_\rho(i,T),$$

first we should note that this sum is terminating at some point. More precisely, for ρ if we set

$$m(T) := \sup_{\omega \in E_\rho^*} \{|\omega| : S_{|\omega'|} f(\omega'\rho) \geq -T, \ \forall \omega' \in E_\rho^*, \ |\omega'| \leq |\omega|\}, \qquad b_n := \inf_{\omega \in E_\rho^n} S_n(\omega\rho),$$

$$M(T) := \sup_{\omega \in E_\rho^*} \{|\omega| \ : \ S_{|\omega|} f(\omega\rho) \geq -T\}, \qquad d_n := \sup_{\omega \in E_\rho^n} S_n(\omega\rho),$$

then $N_\rho(i,T) = 0$ for $i > M(T)$, therefore

$$N_\rho(T) = \sum_{i=1}^{M(T)} N_\rho(i,T).$$

The question we ask is which term of the above sum on the right hand side might have growth comparable to the left hand side, i.e. for which $i(T)$ the growth of $N_\rho(T)/N_\rho(i(T),T)$ is not too fast?! With the tools we have, we couldn't answer this question, however we have some words on that. First we prove the following.

PROPOSITION 3.23. *Both of the following limits exist:*

$$\lim_{T\to\infty}\frac{m(T)}{T}=r,\qquad \lim_{T\to\infty}\frac{M(T)}{T}=s.$$

PROOF. First we prove the latter one. We set $M := M(T)$, let ω be a finite word making the supremum possible in the definition of $M(T)$, then for any $\tau \in E_\rho^{M+1}$ we find

$$d_M \geq S_M f(\omega\rho) \geq -T > S_{M+1}(\tau\rho),$$

$$d_M \geq -T \geq d_{M+1},$$
$$\frac{d_M}{M} \geq \frac{-T}{M} \geq \frac{d_{M+1}}{M+1}\frac{M+1}{M}.$$

Therefore it is enough to show that d_n/n is convergent. To do so, we note that for arbitrary τ, γ with $|\tau| = m$, $|\gamma| = n$ where $\tau\gamma\rho$ is admissible, we can find $\omega \in \Omega$ such that $\tau\omega\rho$ is as well admissible by finitely irreducible definition 2.2. By lemma 2.11 we find:

$$\delta S_{m+n} f(\tau\gamma\rho) = \delta S_m f(\tau\gamma\rho) + \delta S_n f(\gamma\rho) \leq \delta S_m f(\tau\omega\rho) + \delta S_n f(\gamma\rho) + K_{\delta f}$$

$$= \delta S_{m+|\omega|} f(\tau\omega\rho) - \delta S_{|\omega|} f(\omega\rho) + \delta S_n f(\gamma\rho) + K_{\delta f}$$

$$= \delta S_{|\omega|} f(\tau\omega\rho) + \delta S_m f(\sigma^{|\omega|}(\tau\omega\rho)) - \delta S_{|\omega|} f(\omega\rho) + \delta S_n f(\gamma\rho) + K_{\delta f}$$

Now by 2 we know $\delta S_{|\omega|} f \leq \log Q_\delta$ and since Ω is finite, there is $C > 0$ such that

$$S_{m+n} f(\tau\gamma\rho) \leq S_m f(\sigma^{|\omega|}(\tau\omega\rho)) + S_n f(\gamma\rho) + C \leq d_m + d_n + C.$$

Thus we have $d_{m+n} \leq d_m + d_n + C$ and we can use Fekete's lemma 2.12 with $a_n = d_n + C$ to get convergence of d_n/n.

For the other one, note that if E is infinite then using 2 there are infinitely many n for which $b_n = -\infty$, therefore $m(T) = \sup \emptyset$ which we set it $-\infty$ and so $m(T)/T = -\infty$ for all $T > 0$. Let E be finite, for arbitrary τ, γ with $|\tau| = m$, $|\gamma| = n$ where $\tau\rho$ and $\gamma\rho$ are admissible there is $\omega \in \Omega$ such that $\tau\omega\gamma\rho$ is admissible as well. Therefore by lemma 2.11 we find:

$$\delta S_m f(\tau\rho) + \delta S_n f(\gamma\rho) \geq \delta S_m f(\tau\omega\gamma\rho) - K_{\delta f} + \delta S_n f(\gamma\rho)$$

$$= \delta S_{m+|\omega|} f(\tau\omega\gamma\rho) - \delta S_{|\omega|} f(\omega\gamma\rho) - K_{\delta f} + \delta S_n f(\gamma\rho)$$

$$= \delta S_{m+|\omega|+n} f(\tau\omega\gamma\rho) - \delta S_{|\omega|} f(\omega\gamma\rho) - K_{\delta f}$$

$$= \delta S_{|\omega|} f(\tau\omega\gamma\rho) + \delta S_{m+n} f(\sigma^{|\omega|}(\tau\omega\gamma\rho)) - \delta S_{|\omega|} f(\omega\gamma\rho) - K_{\delta f}.$$

Now for large m it is clear that by lemma 2.11 we have $\delta S_{|\omega|} f(\tau\omega\gamma\rho) \geq \delta S_{|\omega|} f(\tau\rho) - K_{\delta f}$, so again we use 2 and the fact that E and Ω are finite to obtain $C > 0$ such that:

$$\delta S_m f(\tau\rho) + \delta S_n f(\gamma\rho) \geq \delta S_{m+n} f(\sigma^{|\omega|}(\tau\omega\gamma\rho)) - C.$$

This gives $b_m + b_n \geq b_{m+n} - C$, and once again we use Fekete's lemma to find that b_n/n is convergent. Note that similar to above we can set $m := m(T)$ and let ω be a finite word making the supremum possible in the definition of $m(T)$, so:

$$\frac{b_m}{m} \geq \frac{-T}{m} \geq \frac{b_{m+1}}{m+1} \frac{m+1}{m}.$$

This finishes the proof. □

Note that $m(T)$ is the cutoff integer where before that the counting problem is just counting $\sum_{i=1}^{m(T)} \# E_A^i$, while after that not all words with generic length are included in $N_\rho(T)$. We continue this omitting process till we reach to $M(T)$ where no finite word of length bigger is counted anymore. Furthermore, it is obvious that $r \leq s$. We know equality and strict inequality are both possible, examples 3.25, 3.27 correspondingly. Our guess is the following

$$\frac{N_\rho(T)}{N_\rho(i(T), T)} = O(T) \iff \frac{i(T)}{T} \to \frac{1}{\chi_{\mu_\delta}},$$

where O is just the big O notation and $m(T) \leq i(T) \leq M(T)$. As stated, we couldn't show this by the tools we have. Note that this last assumption cannot be relaxed, for taking $i(T) = M(T) + 1$ in example 3.25 gives

$$\frac{N_\rho(T)}{1 + N_\rho(i(T), T)} = N_\rho(T) = O\left(\exp(\delta T)\right), \quad \frac{i(T)}{T} \to \frac{1}{\chi_{\mu_\delta}}.$$

In example 3.25 we have only one choice $i(T) = m(T) = M(T)$ and then $N_\rho(T)/N_\rho(i(T), T) = 1$. However, computations get much harder for example 3.27. Our computations using an asymptotic formula for partial sum of binomials [11, p. 492] suggest $N_\rho(T)/N_\rho(i(T), T) = O(T)$. In case, such a relation holds in general, it tells us that the main contributor to $N_\rho(T)$

is asymptotically $N_\rho(i(T), T)$. This is important because in some cases one needs to deal with words of specified length rather than any length when working with $N_\rho(T)$.

3.5. Examples

EXAMPLE 3.24. Recalling Example 2.45 from previous chapter. We should note that this system is not D-generic. Therefore we can use theorem 3.19. We know that for this system the transfer operator $\mathcal{L}_s$ for real $s = x$ due to Ruelle's theorem, see [32, p. 136], has only one eigenvalue of modulus $e^{P(x)}$ and this eigenvalue is $e^{P(x)}$, and since eigenvalue is analytic function then for any complex s, then eigenvalue is of the form

$$e^{\log r(A)+s\log\alpha},$$

which is 1 when

$$s = \frac{\log r(A)}{-\log\alpha} + \frac{2\pi ik}{-\log\alpha}, \quad k \in \mathbb{Z}.$$

Therefore $\delta = \log r(A)/-\log\alpha$ and $\eta_\rho - 1/(s-\delta)$ has continuous extension on the segment

$$\{s \in \mathbb{C} : s = \delta + 2\pi iy, |y| < -1/\log\alpha\}$$

of the critical line. Then theorem 3.19 for $y_0 = -1/\log\alpha$, $\delta = \log r(A)/-\log\alpha$ and $\chi_{\mu_\delta} = -\int \log\alpha d\mu_\delta = -\log\alpha$ gives us the following estimate:

$$\frac{h(\rho)}{r(A)-1} \leq \liminf_T \frac{N(T)}{\exp(\delta T)} \leq \limsup_T \frac{N(T)}{\exp(\delta T)} \leq \frac{h(\rho)r(A)}{r(A)-1}, \tag{21}$$

EXAMPLE 3.25. Recalling previous example if we consider the full shift for $n = 2$, i.e. the case $A = 1$, and $\alpha = \frac{1}{3}$ with maps

$$\phi_0(t) = \frac{1}{3}t, \quad \phi_1(t) = \frac{1}{3}t + \frac{2}{3}.$$

The limit set of this system is the Cantor set on unit interval. Therefore

$$f(\rho) = \log|\phi'_{\rho_1}(\pi(\sigma\rho))| = \log\frac{1}{3},$$

$$\mathcal{L}_s\mathbb{1}(\rho) = \exp(sf(0\rho)) + \exp(sf(1\rho)) = 2(\frac{1}{3^s})$$

$$h = \mathbb{1}$$

$$r(A) = 2.$$

$$y_0 = (\log 3)^{-1}$$

$$\delta = \log 2 / \log 3$$

Thus

$$1 \le \liminf_T \frac{N(T)}{\exp(\delta T)} \le \limsup_T \frac{N(T)}{\exp(\delta T)} \le 2. \tag{22}$$

This actually can be seen directly by computing $N(T) = 2^{\lfloor \frac{T}{\log 3} \rfloor + 1} - 2$, so

$$1 - \epsilon \le \frac{N(T)}{\exp(\delta T)} = 2(2^{\lfloor \frac{T}{\log 3} \rfloor - \frac{T}{\log 3}} - \frac{1}{2^{\frac{T}{\log 3}}}) \le 2 \tag{23}$$

for any $\epsilon > 0$ and T large enough, as we got in 22.

Remark 3.26. The above example establishes the fact that the lower bound and upper bound in theorem 3.19 are both sharp.

Example 3.27. Recalling example 2.44, consider the deterministic system with conformal maps of the unit interval

$$\phi_0(t) = \frac{1}{2}t + \frac{1}{20}, \quad \phi_1(t) = \frac{1}{3}t + \frac{1}{30}$$

on the full shift space $E^\infty = \{0,1\}^\infty$. Clearly, we have

$$f(\rho) = \log |\phi'_{\rho_1}(\sigma\rho)|,$$

$$S_n f(\omega\rho) = n_0 \log \frac{1}{2} + n_1 \log \frac{1}{3},$$

where $n_0 = n_0(\omega) = S_n \mathbb{1}_{[0]}(\omega\rho)$ and $n_1 = n_1(\omega) = S_n \mathbb{1}_{[1]}(\omega\rho)$. Basically n_0 is the number of 0s and n_1 is the number of 1s in $\omega \in E^n$. The pressure is calculated to be

$$\begin{aligned} P(x) = \lim_n \frac{1}{n} \log \sum_{|\omega|=n} \|\phi'_\omega\|^x &= \lim_n \frac{1}{n} \log \sum_{|\omega|=n} (\frac{1}{2^{n_0(\omega)}} \frac{1}{3^{n_1(\omega)}})^x \\ &= \lim_n \frac{1}{n} \log(\frac{1}{2^x} + \frac{1}{3^x})^n = \log(\frac{1}{2^x} + \frac{1}{3^x}) \end{aligned}$$

And a Gibbs state by 4 can be found first on $[\omega_1]$, then on $[\omega_1\omega_2]$ and so on:

$$m_x([\omega]) = \frac{(\frac{1}{2^x})^{n_0} (\frac{1}{3^x})^{n_1}}{(\frac{1}{2^x} + \frac{1}{3^x})^n}, \quad \omega \in E^n.$$

Note that, it defines a system with D-generic property. One way to see that the system is D-generic is by Proposition 2.29. Note that E^*_{per} is the set of periodic words of any length which is exactly E^*, since we work with the full shift. Therefore if the set

$$\{S_{|\omega|}f(\overline{\omega}) : \omega \in E^*_{\text{per}}\} = \{n_0 \log \frac{1}{2} + n_1 \log \frac{1}{3} : n_0 + n_1 = n \in \mathbb{N}\},$$

generates a cyclic additive group with a generator β, then there exist integers k, k' such that $k\beta = \log 1/2$ and $k'\beta = \log 1/3$. This yields $k/k' = \log 2/\log 3$ is rational. The other way to see that our system has D-generic property, is directly solving the following equation for the eigenvalue of maximal modulus of transfer operator:

$$1 = \xi(s) = \frac{1}{2^s} + \frac{1}{3^s}, \quad x = \delta$$

$$|\frac{1}{2^s} + \frac{1}{3^s}| = 1 = \frac{1}{2^\delta} + \frac{1}{3^\delta} = |\frac{1}{2^s}| + |\frac{1}{3^s}|,$$

so by properties of the triangle inequality there exists $b \geq 0$ such that

$$\frac{1}{2^s} = b\frac{1}{3^s} \;\Rightarrow\; b = \frac{3^s}{2^s} = \frac{3^\delta}{2^\delta} \exp(iy \log 3 - iy \log 2) \Rightarrow\; b = \frac{3^\delta}{2^\delta}, \quad y = \frac{2k\pi}{\log(3/2)}.$$

But,

$$1 = \frac{1}{2^s} + \frac{1}{3^s} = b\frac{1}{3^s} + \frac{1}{3^s} = b\frac{1}{3^\delta} \exp(-iy \log 3) + \frac{1}{3^\delta} \exp(-iy \log 3)$$
$$= \frac{1}{2^\delta} \exp(-iy \log 3) + \frac{1}{3^\delta} \exp(-iy \log 3) = \exp(-iy \log 3) \Rightarrow\; y = \frac{2k\pi}{\log 3},$$

i.e. y can only be 0. Now we are ready to apply corollary 3.22 to find

$$\frac{N_\rho(T)}{\exp(\delta T)} \to \frac{1}{\delta \chi_{\mu_\delta}}, \quad T \to \infty.$$

CHAPTER 4

COUNTING IN RANDOM DYNAMICS

4.1. Counting and Poincaré series

Theorems 3.21 and 3.22 in the previous chapter addressed asymptotic counting problem in deterministic systems. Later, M. Urbański wondered about an analogous result for random systems. This chapter is supposed to answer his question in some special cases. We only consider some special class of random CGDMS, with most attention toward random CIFS. The notion of random system is what we adopt from Roy and Urbański, see section 2.9 or [34]. For this purpose we need to specify our complete probability space with an ergodic invertible measure preserving transformation. We consider a countable (finite or infinite) set of complex numbers z within unit disk that are bounded away from 0:

$$\mathcal{Z} \subset \{z \in \mathbb{C} : 0 < \epsilon < |z| \leq 1\},$$

and then we set:

$$\Lambda := \mathcal{Z}^{\mathbb{Z}}.$$

From now on we represent an element of Λ by λ and of course λ_i is i^{th} coordinate of λ and $i \in \mathbb{Z}$. For the $\sigma-$Algebra $\mathcal{B}$ of measurable sets we just consider the Borel sets, for the ergodic invertible measure ν we consider a Bernoulli probability measure and for T the we consider the shift map on Λ. Therefore $(\Lambda, \mathcal{B}, \nu, T)$ is just two sided full shift space with an ergodic measure. Then we will have

- $\Lambda = \{\lambda = ...\lambda_{-(i-1)}...\lambda_{-2}\lambda_{-1}\lambda_0\lambda_1\lambda_2...\lambda_{i-1}... \ : \ \lambda_i \in \mathcal{Z}, \ i \in \mathbb{Z}\}$
- $[\lambda_{i_1} = z_1, \lambda_{i_2} = z_2, ..., \lambda_{i_k} = z_k] = \{\lambda \in \Lambda \ : \ \lambda_{i_1} = z_1, \lambda_{i_2} = z_2, ..., \ \lambda_{i_k} = z_k\}$
- $\sum_{z \in \mathcal{Z}} \nu([\lambda_0 = z]) = 1$
- $T(...\lambda_{-(i-1)}...\lambda_{-2}\lambda_{-1}\boldsymbol{\lambda_0}\lambda_1\lambda_2...\lambda_{i-1}...) = ...\lambda_{-(i-2)}...\lambda_{-1}\lambda_0\boldsymbol{\lambda_1}\lambda_2\lambda_3...\lambda_i...$
- $\nu(T^{-1}(B)) = \nu(B), \quad B \in \mathcal{B}.$

With this measurable system $(\Lambda, \mathcal{B}, \nu, T)$, we can introduce a random system for any deterministic CGDMS. This process can be defined in different ways. We explain one, which will be our main focus in this chapter. If $\{\phi_e\}_{e\in E}$ is a countable (finite or infinite) family of conformal contractions, then we define

$$\phi_e^\lambda := \lambda_0 \phi_e.$$

Note that we have to be careful with our definition so that we make sure it satisfy definition 2.38. For instant, we need to know that λ_0, chosen non-real, only makes sense when ϕ_e is a complex-valued function, or this λ_0 should be appropriate enough so that we make sure the image of ϕ_e^λ is still $X_{i(e)}$. We provide some examples to address these issues. As well note that this random CGDMS is summable (in the sense of definition 2.39) exactly for those x that the deterministic system is summable (in the sense of definition 2.15). Therefore without change we use the same notation Γ for all these x and Γ^+ for the right half-plane, see above the definition 2.20. Next, we want to compute pressure and transfer operator associated to this random CGDMS above. Referring to below the definition 2.38, we can obtain our random potential function

$$f(\rho,\lambda) = \log\left|(\phi_{\rho_1}^\lambda)'\left(\pi^{T(\lambda)}(\sigma\rho)\right)\right| = \log\left|(\phi_{\rho_1})'\left(\pi(\sigma\rho)\right)\lambda_0\right| = f(\rho) + \log|\lambda_0|,$$

where $f(\rho)$ is just the potential obtained from the deterministic system $\{\phi_e\}_{e\in E}$, see below the definition 2.37. Therefore we get

$$S_n f(\rho,\lambda) = f(\rho,\lambda) + f(\sigma\rho, T\lambda) + ... + f(\sigma^{n-1}\rho, T^{n-1}\lambda)$$

$$= f(\rho) + \log|\lambda_0| + f(\sigma\rho) + \log|\lambda_1| + f(\sigma^2\rho) + \log|\lambda_2| + ... + f(\sigma^{n-1}\rho) + \log|\lambda_{n-1}|$$

$$= S_n f(\rho) + \log|\lambda_0\lambda_1...\lambda_{n-1}|. \tag{24}$$

Also from the remark below definition 2.39 we obtain

$$P^\lambda(x) = P(x) + x\log|\lambda_0|,$$

for almost all λ. Therefore for the expected pressure we will have

$$\mathcal{E}P(x) = \int_\lambda P^\lambda(x)\mathrm{d}\nu = P(x) + \left(\sum_{z\in\mathcal{Z}} \nu([\lambda_0 = z])\log|z|\right)x \tag{25}$$

Note that if there is at least one $z \in \mathcal{Z}$ with $|z| < 1$, then the parenthesis above would be a negative value. Therefore the root of this expected pressure δ_Λ is not anymore the same as the root of the deterministic pressure δ. In fact:

$$\delta_\Lambda \leq \delta.$$

As well it is important to note that, this expected pressure is defined on Γ, so

$$\delta_\Lambda \in \Gamma.$$

Furthermore the random transfer operator is given by

$$\mathcal{L}_x^\lambda(g)(\rho) := \sum_{e \in E_\rho^1} \exp(xf(e\rho, \lambda))g(e\rho),$$

for a bounded continuous function g and $x \in \Gamma$. This leaves

$$(\mathcal{L}_x^\lambda)^n(\mathbb{1})(\rho) = \sum_{\omega \in E_\rho^n} \exp(xS_nf(\omega\rho, \lambda)) = \sum_{\omega \in E_\rho^n} \exp(xS_nf(\omega\rho))|\lambda_0\lambda_1...\lambda_{n-1}|^x$$

$$= |\lambda_0\lambda_1...\lambda_{n-1}|^x \mathcal{L}_x^n(\mathbb{1})(\rho).$$

It is important to note that this above expression is not n^{th} iteration of the operator $\mathcal{L}_x^\lambda$, simply because of the random variable λ that is involved. We want to investigate counting problem in random dynamics. Let $\rho \in E_A^{\mathbb{N}}$ and $T > 0$. Now we are ready to introduce the appropriate counting function:

$$N_\rho^\lambda(T) := \#\{\omega \in E_\rho^* : S_nf(\omega\rho, \lambda) \geq -T\}. \tag{26}$$

This define a Poincaré series

$$\eta_\rho^\lambda(s) := \int_0^\infty \exp(-sT)\mathrm{d}N_\rho^\lambda(T). \tag{27}$$

Alternatively, we can find another expression of η_ρ^λ in terms of the random transfer operator. In fact,

$$\eta_\rho^\lambda(s) = \sum_{n=1}^\infty \exp(-sT_i)\left(N_\rho^\lambda(T_i) - N_\rho^\lambda(T_{i-1})\right),$$

where $T_1 < T_2 < T_3 < \ldots$ is the increasing sequence of discontinuities of $N_\rho^\lambda(T)$. This sums to

$$\eta_\rho^\lambda(s) = \sum_{n=1}^{\infty} \sum_{\omega \in E_\rho^n} \exp(sS_n f(\omega\rho, \lambda))$$

$$= \sum_{n=1}^{\infty} \sum_{\omega \in E_\rho^n} \exp(sS_n f(\omega\rho, \lambda)) = \sum_{n=1}^{\infty} (\mathcal{L}_s^\lambda)^n(\mathbb{1})(\rho) = \sum_{n=1}^{\infty} |\lambda_0\lambda_1 \ldots \lambda_{n-1}|^s \mathcal{L}_s^n(\mathbb{1})(\rho).$$

Let δ^λ represent the critical line of this series. We focus on those λ that the series η_ρ^λ is convergent on $x > \delta^\lambda > \inf \Gamma$.

Remark 4.1. For example if E is finite all λ satisfy this.

This assumption enables us to obtain another expression of this Poincaré series. In fact by the spectral decomposition from 14 and noting that $\delta^\lambda \in \Gamma$ we can write

$$\eta_\rho^\lambda(s) = \sum_{k=1}^{\infty} |\lambda_0\lambda_1 \ldots \lambda_{k-1}|^s \mathcal{L}_s^k(\mathbb{1})(\rho)$$

$$= \sum_{k=1}^{\infty} |\lambda_0\lambda_1 \ldots \lambda_{k-1}|^s \left(\xi_1^k(s)\mathcal{P}_{1,s}(\mathbb{1}) + \xi_2^k(s)\mathcal{P}_{2,s}(\mathbb{1}) + \ldots + \xi_n^k(s)P_{n,s}(\mathbb{1}) + \mathcal{D}_s^k(\mathbb{1}) \right)$$

$$(28) \qquad = \left(\sum_{k=1}^{\infty} |\lambda_0\lambda_1 \ldots \lambda_{k-1}|^s \xi_1^k(s) \right) \mathcal{P}_{1,s}(\mathbb{1}) + \ldots + \left(\sum_{k=1}^{\infty} |\lambda_0\lambda_1 \ldots \lambda_{k-1}|^s \xi_n^k(s) \right) \mathcal{P}_{n,s}(\mathbb{1})$$

$$+ \sum_{k=1}^{\infty} |\lambda_0\lambda_1 \ldots \lambda_{k-1}|^s \mathcal{D}_s^k(\mathbb{1}).$$

It is important to note that analogue of the proposition 3.13 from the previous chapter is not anymore easy to construct for random systems as the transfer operator under consideration here is a random operator and the conventional notion of eigenvalue does not exist here for us in this chapter. Therefore to find an asymptotic formula for 26 we may apply Ikehara-Wiener theorem 2.41 or Garaham-Vaaler theorem 2.42 directly, depending on the behavior of this η_ρ^λ along the vertical line $x = \delta^\lambda$. This requires to understand behavior of each of parenthesis in 28 above, along $x = \delta^\lambda$.

4.2. Asymptotic Formula for D-generic Potential

In this section we assume the potential function of the deterministic system $\mathcal{S} = \{\phi_e\}_{e\in E}$ is D-generic, see definition 2.28.

Definition 4.2. We say $\lambda \in \Lambda$ is future periodic, if there exists $k \in \mathbb{N}$ such that $\sigma^k(\lambda_0\lambda_1...) = \lambda_0\lambda_1...$, and we say it is eventually future periodic, if for some $m \in \mathbb{N}$ we have $\sigma^k(\lambda_m\lambda_{m+1}...) = \lambda_m\lambda_{m+1}...$.

Theorem 4.3. *For each eventually future periodic λ, there exist $C, D > 0$, such that*

$$C \leq \liminf_{T\to\infty} \frac{N_\rho^\lambda(T)}{e^{\delta^\lambda T}} \leq \limsup_{T\to\infty} \frac{N_\rho^\lambda(T)}{e^{\delta^\lambda T}} \leq D.$$

Proof. We let

$$\lambda = ...\lambda_m\lambda_{m+1}\lambda_{m+2}...\lambda_{m+k-1}\lambda_m\lambda_{m+1}\lambda_{m+2}...\lambda_{m+k-1}...,$$

$$a_i(s) := |\lambda_m...\lambda_{m+k-1}|^s \xi_i(s)^k,$$

$$c_i(s) := |\lambda_0...\lambda_{m-1}|^s \left(|\lambda_m|^s\xi_i(s) + |\lambda_m\lambda_{m+1}|^s\xi_i(s)^2 + ... + |\lambda_m...\lambda_{m+k-1}|^s\xi_i(s)^k\right).$$

Then there exists a holomorphic function $b_i(s)$ such that the i^{th} parenthesis in 28 above, can be written as:

$$b_i(s) + a_i(s)c_i(s) + a_i(s)^2c_i(s) + a_i(s)^3c_i(s) + ... = b_i(s) + c_i(s)\frac{a_i(s)}{1 - a_i(s)}.$$

Now since $a_i(s)$ is holomorphic function, there should be $y_i > 0$ such that $a_i(s)$ omits 1 on $\{s = x + iy : -y_i < t < y_i,\ y \neq 0\}$, unless $\xi_i(s) = (\sqrt[k]{|\lambda_m...\lambda_{m+k-1}|})^{-s}$, which implies that $i = 1$. But in this case $\xi_1(s)$ meets 1 infinitely often which cannot happen as we assumed D-generic property. This means there is $y_0 > 0$ such that for each i, the function a_i omits 1 on $\{s = x + iy : -y_0 < t < y_0,\ t \neq 0\}$. Therefore the i^{th} parenthesis has continuous extension at least on this segment and so Graham-Vaaler theorem 2.42 is applicable. This finishes the proof. $\square$

Remark 4.4. Note that

- it is easy to see that δ^λ is the unique root of

$$P(x) + \frac{x}{k}\log|\lambda_m...\lambda_{m+k-1}| = 0.$$

- it is clear that for many future periodic λ, we have $\delta_\lambda \neq \delta_\Lambda$. Therefore $N_\rho^\lambda(T) \not\sim \exp(\delta_\Lambda T)$.

If further the potential of the deterministic system $\mathcal{S} = \{\phi_e\}_{e\in E}$ is assumed to be strongly D-generic (see definition 2.28), we can find an exact formula.

Theorem 4.5. *If the deterministic potential f is strongly D-generic, for each eventually future period λ, there is a constant C such that*

$$\lim_{T\to\infty} \frac{N_\rho^\lambda(T)}{e^{\delta^\lambda T}} = C.$$

Proof. Along the proof of previous theorem it is enough to notice that $a_i(s)$ can meet 1 only if $i = 1$ and s is real, which means all along the critical line of convergence (except at the real point) we have continuous extension and therefore Ikehara-Wiener theorem 2.41 is applicable in this case. □

4.3. Asymptotic Formula for Random Walk of Bounded Boundary

In this section we assume $\mathcal{Z}$ is finite, see the beginning of the this chapter. We adapt the notion of boundary from the theory of random walk to prove some results for our counting problem. For each positive integer n, $z \in \mathcal{Z}$ and $\lambda \in \Lambda$ we let $s_{n,z}(\lambda)$ denote the number of times that z appears in $\lambda_0...\lambda_{n-1}$. It is obvious that for each $\lambda \in \Lambda$ and $n \in \mathbb{N}$ we have

$$\sum_{z\in\mathcal{Z}} s_{n,z}(\lambda) = n.$$

Definition 4.6. We say $\lambda \in \Lambda$ is of bounded boundary if there are numbers p, q, l_z such that for all $n \in \mathbb{N}$:

$$nl_z + p \leq s_{n,z}(\lambda) \leq nl_z + q.$$

Note that in the above definition it is clear that l_z cannot be negative. This definition basically can be perceived as a condition to prevent fluctuation in a random walk. We remind that f is the potential function for the deterministic system $\mathcal{S} = \{\phi_e\}_{e\in E}$. We set

$$c := \sum_{z\in\mathcal{Z}} l_z \log|z|, \quad d := \sum_{z\in\mathcal{Z}} \log|z|.$$

THEOREM 4.7. *If $g = f + c$ inherits strong regularity from f, for λ of bounded boundary, there exist constants $C, D > 0$ such that*

$$C \leq \liminf_{T\to\infty} \frac{N_\rho^\lambda(T)}{e^{\delta^\lambda T}} \leq \limsup_{T\to\infty} \frac{N_\rho^\lambda(T)}{e^{\delta^\lambda T}} \leq D.$$

PROOF. By the bounded boundary definition we can find the following inequalities for the random ergodic sum:

$$S_n f(\omega\rho) + nc + q\sum_z \log|z| \leq S_n f(\omega\rho, \lambda) = S_n f(\omega\rho) + \log|\lambda_0 ... \lambda_{n-1}|$$
$$\leq S_n f(\omega\rho) + nc + p\sum_z \log|z|$$

Next we consider a new function defined by

$$g := f + c,$$

and its counting function as $N'_\rho(T)$. Then using the above inequalities we get

$$N'_\rho(T + qd) \leq N_\rho^\lambda(T) \leq N'_\rho(T + pd).$$

This yields

$$\exp(\delta^\lambda qd)\frac{N'_\rho(T+qd)}{\exp(\delta^\lambda(T+qd))} \leq \frac{N_\rho^\lambda(T)}{\exp(\delta^\lambda T)} \leq \frac{N'_\rho(T+pd)}{\exp(\delta^\lambda(T+pd))}\exp(\delta^\lambda pd).$$

Now it is enough $T \to \infty$ and use theorem 3.19. □

REMARK 4.8. First note that when E has finite number of alphabets then g inherits strong regularity from f, then the first assumption in this theorem is redundant. Furthermore, it is clear that every eventually periodic λ is of bounded boundary. Therefore this theorem implies theorem 4.3. However, it doesn't imply theorem 4.5. Finally we can find that δ^λ is the unique root of

$$P(x) + cx = 0.$$

Note that if f is just D-generic then the bounds obtained above may not necessarily be improved. However, by imposing strongly D-generic property on f we get a better estimate as

$$\lim_{T\to\infty} \frac{N'_\rho(T)}{\exp(\delta^\lambda T)},$$

exists. For this it is enough for us to notice

$$\mathcal{L}_{sg} = e^{sc}\mathcal{L}_{sf}$$

$$P(xg) = P(xf) + cx,$$

therefore in case $\mathcal{L}_{sg}$ admits $e^{P(x)+cx}$ as eigenvalue, then $\mathcal{L}_s$ must admit $e^{P(x)-ciy}$ as eigenvalue, but this cannot happen as $\mathcal{L}_s$ does not admit any eigenvalue of modulus $e^{P(x)}$.

4.4. Constructing System With non-Exponential Growth for Counting

Recalling example 2.45 from section 2.11, we can construct random IFS as follows. We let $\Lambda = \{z, w\}^{\mathbb{Z}}$ and consider a Bernoulli measure with $\nu([\lambda_0 = z]) = p$, $\nu([\lambda_0 = w]) = q$, where $0 < z < w < 1$ and $p + q = 1$. Therefore we can express our deterministic system as

$$\phi_i(t) = \alpha t + \alpha_i, \quad i = 0, 1$$

and our random system as

$$\phi_i^\lambda(t) := \lambda_0(\alpha t + \alpha_i), \quad i = 0, 1, \quad \lambda_0 = z, w.$$

Therefore from section 4.1, we can find the random potential:

$$f(\rho, \lambda) = \log |(\phi_{\rho_1}^\lambda)'(\pi^{T(\lambda)}(\sigma\rho))| = \log(\alpha\lambda_0),$$

the random pressure:

$$P^\lambda(x) = P(x) + x \log \lambda_0 = \log r(A) + x \log(\alpha\lambda_0),$$

and the expected pressure:

$$\mathcal{E}P(x) = \int_\Lambda P^\lambda(x)\mathrm{d}\lambda = \log r(A) + x\log\alpha + x(p\log z + q\log w).$$

Note that we consider a fixed Bernoulli measure on $E_A^{\mathbb{N}}$ for our random measure, see below the definition 2.39. Therefore the root of this expected pressure is

$$\delta_\Lambda = -\frac{\log r(A)}{\log\alpha + p\log z + q\log w}.$$

We mention below the inequality 21 from the example 2.45 again

$$C \le \liminf_T \frac{N(T)}{\exp(\delta T)} \le \limsup_T \frac{N(T)}{\exp(\delta T)} \le D, \tag{29}$$

for some $C, D > 0$ and we remind that $\delta = \log r(A)/-\log\alpha$.

THEOREM 4.9. *Given the random system above, for almost all λ we have*

$$\liminf_T \frac{N_\rho^\lambda(T)}{\exp(\delta_\Lambda T)} = 0, \quad \limsup_T \frac{N_\rho^\lambda(T)}{\exp(\delta_\Lambda T)} = \infty.$$

PROOF. Recalling $s_{n,z}(\lambda)$ from section 4.3, for an integer m one can find a sequence n_i such that $s_{n_i,z}(\lambda) \le pn_i + m < s_{n_i+1,z}(\lambda)$. Therefore we can give the following estimate:

$$n_i\log\alpha + (pn_i + m)\log z + (qn_i - m + 1)\log w \le S_{n_i}f(\omega\rho,\lambda) \tag{30}$$

$$\le n_i\log\alpha + (pn_i + m - 1)\log z + (qn_i - m)\log w.$$

Furthermore note that since $f(\rho,\lambda)$ and so $S_n(\rho,\lambda)$ are constant negative functions in ρ, then we find that for $T_i := -S_{n_i}f(\omega\rho,\lambda)$:

$$N_\rho^\lambda(T_i) = \#\{\omega \in E_\rho^* : S_{|\omega|}f(\omega\rho,\lambda) \ge -T_i\} = \#\{\omega \in E_\rho^* : |\omega| \le n_i\}$$

$$= \#\{\omega \in E_\rho^* : |\omega|\log\alpha \ge n_i\log\alpha\} = \#\{\omega \in E_\rho^* : S_{|\omega|}f(\omega\rho) \ge n_i\log\alpha\}$$

$$= N_\rho(-n_i\log\alpha).$$

Therefore the inequality 29 above for any small enough ϵ, there exists a large N such that for $i \ge N$:

$$(C-\epsilon)r(A)^{n_i} \le N_\rho^\lambda(T_i) \le (D+\epsilon)r(A)^{n_i}. \tag{31}$$

Additionally, 30 gives:

$$\exp\left(-\delta_\Lambda\left(n_i\log\alpha + (pn_i+m-1)\log z + (qn_i-m)\log w\right)\right)$$

$$\leq \exp(\delta_\Lambda T_i) = \exp(-\delta_\Lambda S_{n_i} f(\omega\rho,\lambda))$$

$$\leq \exp\left(-\delta_\Lambda\left(n_i\log\alpha + (pn_i+m)\log z + (qn_i-m+1)\log w\right)\right),$$

which can be rewritten as

$$r(A)^{n_i}\exp\left(\delta_\Lambda(m\log\frac{w}{z}+\log z)\right) \leq \exp(\delta_\Lambda T_i) \leq r(A)^{n_i}\exp\left(\delta_\Lambda(m\log\frac{w}{z}-\log w)\right).$$

This along with 31 yields:

$$(C-\epsilon)\exp\left(-\delta_\Lambda(m\log\frac{w}{z}-\log w)\right) \leq \frac{N_\rho^\lambda(T_i)}{\exp(\delta_\Lambda T_i)} \leq \exp\left(-\delta_\Lambda(m\log\frac{w}{z}+\log z)\right)(D-\epsilon).$$

By passing to a subsequence, we find:

$$(C-\epsilon)w^{\delta_\Lambda}(\frac{z}{w})^{\delta_\Lambda m} \leq \lim_i \frac{N_\rho^\lambda(T_i)}{\exp(\delta_\Lambda T_i)} \leq \frac{D-\epsilon}{z^{\delta_\Lambda}}(\frac{z}{w})^{\delta_\Lambda m}.$$

If $m \to \infty$:

$$\liminf_T \frac{N_\rho^\lambda(T)}{\exp(\delta_\Lambda T)} = 0,$$

and if $m \to -\infty$:

$$\limsup_T \frac{N_\rho^\lambda(T)}{\exp(\delta_\Lambda T)} = \infty.$$

□

Corollary 4.10. *Given the random system above and any $C, D, a > 0$, the set of all $\lambda \in \Lambda$ satisfying*

$$C\exp(aT) \leq N_\rho^\lambda(T) \leq D\exp(aT) \quad \textit{as} \;\; T\to\infty,$$

has ν measure 0.

4.5. Examples

EXAMPLE 4.11. In section 4.4, if we let for instance

$$\alpha = \frac{1}{3}, z = \frac{1}{5}, w = \frac{1}{7}.$$

Then it is not hard to see that we can actually get a stronger result than that of the theorem 4.9:

$$\left\{C : \lim_{n\to\infty} \frac{N_\rho^\lambda(T_n)}{\exp(\delta_\Lambda T_n)} = C\right\} = [0,\infty].$$

EXAMPLE 4.12. Recalling example 3.27 from section 3.5,

$$\phi_0(t) = \frac{1}{2}t + \beta_0,$$

$$\phi_1(t) = \frac{1}{3}t + \beta_1.$$

Note that the potential for this system is D-generic, but it is not strongly D-generic. We let

$$z = \frac{1}{5}, w = \frac{1}{7},$$

and we consider $\Lambda = \{z, w\}^{\mathbb{Z}}$. Then the random maps would be

$$\phi_0^\lambda(x) = \lambda_0(\frac{1}{2}x + \beta_0),$$

$$\phi_1^\lambda(x) = \lambda_0(\frac{1}{3}x + \beta_1).$$

Then the Poincaré series for future periodic λ can be simplified as (see proof of the theorem 4.3):

$$\eta_\rho^\lambda(s) = \left((\frac{1}{2^s} + \frac{1}{3^s})\lambda_0^s + (\frac{1}{2^s} + \frac{1}{3^s})^2\lambda_0^s\lambda_1^s + ... + (\frac{1}{2^s} + \frac{1}{3^s})^m\lambda_0^s...\lambda_{m-1}^s\right)\theta(s)^{-1}, \quad x > x_0$$

where

$$\theta(s) = 1 - (\frac{1}{2^s} + \frac{1}{3^s})^m\lambda_0^s...\lambda_{m-1}^s,$$

and x_0 is the unique real root of $\theta(s)$. Using the triangle inequality we can see x_0 is actually the only root of $\theta(s)$ along $x = x_0$, see example 3.27 for the similar argument. Note

$$\theta(s) = 1 - (\frac{1}{2^s} + \frac{1}{3^s})^m\lambda_0^s...\lambda_{m-1}^s = 1 - \left((\frac{\sqrt[m]{\lambda_0...\lambda_{m-1}}}{2})^s + (\frac{\sqrt[m]{\lambda_0...\lambda_{m-1}}}{3})^s\right)^m,$$

so by taking derivative

$$\theta'(s) = -m\left(\log\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2}(\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2})^s + \log\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{3}(\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{3})^s\right)$$

$$\times\left((\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2})^s + (\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{3})^s\right)^{m-1}$$

If θ has a root of order 2 or higher, then θ' as well vanishes at that root. Looking for such s we should have

$$\left((\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2})^s + (\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{3})^s\right)^m = 1, \tag{32}$$

$$\log\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2}(\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2})^s + \log\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{3}(\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{3})^s = 0.$$

The latter one leaves

$$(\frac{2}{3})^s = -\frac{\log \sqrt[m]{\lambda_0 \ldots \lambda_{m-1}} - \log 2}{\log \sqrt[m]{\lambda_0 \ldots \lambda_{m-1}} - \log 3}. \tag{33}$$

Let β represent the right hand side of the above equality. Note that since β is real, then the left hand side must be real either. This gives:

$$y = \frac{j\pi}{\log(2/3)}, \quad j \in \mathbb{Z}$$

Substituting 33 into 32, yields:

$$\left((\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2})^s + \beta(\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2})^s\right)^m = 1$$

$$\Longrightarrow (\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2})^{ms} = \frac{1}{(1+\beta)^m}.$$

Again the right hand of this is real, and so

$$y = l\pi(m\log\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2})^{-1}, \quad l \in \mathbb{Z}.$$

We found two expressions for y. This is possible only if $\log(\frac{2}{3})/\log(\frac{\sqrt[m]{\lambda_0 \ldots \lambda_{m-1}}}{2})$ is rational, which cannot happen. This shows all the roots of θ are of multiplicity one and so all the poles of the Poincaré series are simple. Therefore the Ikehara-Wiener theorem 2.41 is applicable. There exists $C > 0$,

$$\frac{N_\rho^\lambda(T)}{\exp(\delta_\Lambda T)} \to C.$$

It is clear that we can find a similar behavior for eventually future periodic λ. Furthermore, as we said the above system is not strongly D-generic, however the counting had similar result to that of strongly D-generic systems, see theorem 4.5.

Example 4.13. If we consider a Schottky group that generates the Apollonian gasket as our deterministic system, we know that with some modifications of the theorem 3.22 one can obtain an exponential counting growth formula for the number of circles of radius at least $1/T$ in the packing. However, investigating the counting problem for random Schottky group is not an easy question. But, it is good to notice that for $\mathbb{Z}$ as a subset of the unit circle, the answer would be exactly the same as that of the deterministic Schottky group due to the fact that the random ergodic sum is identical to the deterministic ergodic sum, see 4.1. This is actually expected, since each $z \in \mathcal{Z}$ may change each limit point, but it leaves the circles of inversions intact. In fact, these random factors play the role of rotations.

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