TOPOLOGICAL CONJUGACY RELATION ON THE SPACE OF TOEPLITZ SUBSHIFTS

Ping Yu

Dissertation Prepared for the Degree of

DOCTOR OF PHILOSOPHY

UNIVERSITY OF NORTH TEXAS

August 2022

APPROVED:

Steve Jackson, Major Professor
Su Gao, Committee Member
Lior Fishman, Committee Member
Ralf Schmidt, Chair of the Department of Mathematics
John Quintanilla, Interim Dean of the College of Science
Victor Prybutok, Dean of the Toulouse Graduate School

Yu, Ping. *Topological Conjugacy Relation on the Space of Toeplitz Subshifts*. Doctor of Philosophy (Mathematics), August 2022, 62 pp., 34 numbered references.

We proved that the topological conjugacy relation on T_1, a subclass of Toeplitz subshifts, is hyperfinite, extending Kaya's result that the topological conjugate relation of Toeplitz subshifts with growing blocks is hyperfinite. A close concept about the topological conjugacy is the flip conjugacy, which has been broadly studied in terms of the topological full groups. Particularly, we provided an equivalent characterization on Toeplitz subshifts with single hole structure to be flip invariant.

ACKNOWLEDGMENTS

I would like to thank the University of North Texas Department of Mathematics and the Toulouse Graduate School for support, quality education and opportunity on teaching math courses which help me become better and more confident.

I thank my original advisor Su Gao who poses the problem on topological conjugacy relation which led to my dissertation and my current advisor Steve Jackson who takes over and help me graduate. Thanks to both of them for many stimulating meetings, a lot of patience and respect, generosity on sharing me a lot of references and brilliant ideas and spending time on reading the initial draft.

I thank our graduate students' advisor Olav Richter for advising me graduate too and all other staffs in our department who help me learn more and move on.

I am especially grateful to my husband Taylor Jones who makes my life brighter and happier.

TABLE OF CONTENTS

Page

ACKNOWLEDGEMENTS iv

CHAPTER 1 INTRODUCTION 1

1.1. Background 1

1.2. Borel Reduction and Countable Borel Equivalence Relations 4

1.2.1. Borel Equivalence Relations and Borel Reduction 4

1.2.2. Countable Borel Equivalent Relations 5

1.3. Main Results and Remaining Problems 8

1.4. Organization of the Dissertation 10

CHAPTER 2 PRELIMINARIES 12

2.1. Basic Conceptions on Toeplitz Sequences and Toeplitz Subshifts 12

2.2. A Criterion for Toeplitz Subshifts to Be Topologically Conjugate 14

2.3. The Borelness of the Space of Toeplitz Subshift 18

2.3.1. The Effros Borel Space of $F(X)$ 19

2.3.2. The Space of Toeplitz Subshift is Standard Borel 19

CHAPTER 3 VARIOUS SUBSPACES OF TOEPLITZ SUBSHIFTS 22

3.1. Toeplitz Subshifts Having Separated Holes or Weakly Separated Holes 22

3.2. Toeplitz Subshifts Having Growing Blocks or Unbounded Block Gaps 24

3.3. Examples 29

CHAPTER 4 THE COMPLEXITY OF TOPOLOGICAL CONJUGACY OF TOEPLITZ SUBSHIFTS 39

4.1. Topological Conjugacy on Toeplitz Subshifts is a Countable Borel Equivalence Relation 39

4.2. The Topological Conjugacy Relation on Toeplitz Subshifts is Nonsmooth 39

4.3. The Topological Conjugacy on $\mathcal{T}_1$ is Hyperfinite 43

CHAPTER 5 THE INVERSE PROBLEM ON TOEPLITZ SUBSHIFTS 53

5.1. The Inverse Problem and an Application of Curtis-Hedlund-Lyndon Theorem 53

5.2. A Characterization of Inverse Problem on Teoplitz Subshifts Having a Single Hole Structure 54

REFERENCES 60

CHAPTER 1

INTRODUCTION

1.1. Background

Borel reduction introduced by Friedman and Stanley to compare the relative complexity of different Borel equivalence relations is now a rich and well-developed theory, see [10] and [21]. Let E, F be Borel equivalences on standard Borel spaces X, Y, E is Borel reducible to F if there exists a Borel function such that any two elements are E-equivalent if and only if their images are F-equivalent. If E is Borel reducible to F, then E is considered to be at most as complex as F, since every inquiry about E can be transferred and answered in a Borel way by an inquiry about F.

The equivalence relation we are interested in is the topological conjugacy relation on subshifts. Fix a finite alphabet $\mathfrak{n}$, the left shift map σ on $\mathfrak{n}^{\mathbb{Z}}$ is defined by $\sigma(x)(n) = x(n+1)$ for all $x \in \mathfrak{n}^{\mathbb{Z}}$, and $n \in \mathbb{Z}$. Two subshifts are topologically conjugate if there exists a homeomorphism commuting with the left-shift map. Namely, two subshifts $\mathcal{O}, \mathcal{O}'$ are topologically conjugate if there is a homeomorphism $f : \mathcal{O} \to \mathcal{O}'$ such that $f(\sigma(x)) = \sigma(f(x))$, for all $x \in \mathcal{O}$. It turns out that the topological conjugacy relation of subshifts is a countable Borel equivalence relation.

The complexity problems of topological conjugacy relations on various subclasses of subshifts over a finite alphabet have been studied extensively in the context of Borel reducibility. Clemens [3] proved that the topological conjugacy relation on the space of subshifts is a universal countable Borel equivalence relation. Gao, Jackson and Seward [12] generalized Clemens' result to G-subshifts when G is not locally finite, while it is Borel bireducible with E_0, the eventual agreement relation on the Cantor space $2^{\mathbb{N}}$, when G is locally finite. Dominik Kwietniak announced that, at the 8th Visegrad Conference on Dynamical Systems, the topological conjugacy on subshifts with specification is a universal countable Borel equivalence relation. Gao, Jackson and Seward [12] also proved that the topological conjugacy on minimal subshifts is not smooth, and they asked the question of determining its Borel complexity, which generated a lot of studies on the subclasses of minimal subshifts. For example, Gao and Hill [11] proved that the topological conjugacy relation on the space of minimal rank-1 subshifts is Borel bireducible with E_0. Thomas [32] showed that the topological conjugacy on Toeplitz subshifts, a subclass of minimal subshifts, is not smooth. Later, Sabok and Tsankov [27] proved that the topological conjugacy on Toeplitz subshifts with separated holes is 1-amenable, and the topological conjugacy on Toeplitz G-subshifts is not hyperfinite when G is residually finite and non-amenable. They also posed the question of determining the complexity of the topological conjugacy on Toeplitz subshifts. Kaya [18], [19] proved that the topological conjugacy on Toeplitz subshifts with growing blocks(w.r.t. the natural factorization) is hyperfinite, giving a partial affirmative answer to Sabok and Tsankov's question.

However, the class of Toeplitz subshifts with growing blocks is not an invariant set under the topological cojugacy, and we will construct examples to explain this. In this paper, we consider various subclasses of Toeplitz subshifts which are invariant under the topological conjugacy, for example, Toeplitz subshifts with weakly separated holes, Toeplitz subshifts with unbounded block gaps, and generalize Kaya's result to $\mathcal{T}_1$, a subset of Toeplitz subshifts having unbounded block gaps.

A close concept of the topological conjugacy relation is the flip conjugacy relation. Two subshifts $\mathcal{O}, \mathcal{O}'$ are flip conjugate if there is a homeomorphism $f : \mathcal{O} \to \mathcal{O}'$ such that $f(\sigma(x)) = \sigma(f(x))$ or $f(\sigma(x)) = \sigma^{-1}(f(x))$, for all $x \in \mathcal{O}$. The flip conjugacy relation on the space of subshifts is also a countable Borel equivalence relation. Since Clemens in [3] proved that the topological conjugacy relation on the space of subshifts over any finite alphabet is a countable universal equivalence relation, then it follows that the flip conjugacy relation is Borel reducible to the topological conjugacy relation on the space of subshifts. However, we don't know whether the flip conjugacy relation on the space of Toeplitz subshifts(minimal subshifts) is Borel reducible to the topological conjugacy relation on the same space respectively or not.

There are some nice properties and characterizations on flip conjugacy in dynamics which might help us determine the complexity of the flip conjugacy relation, or even the topological conjugacy relation on the Toeplitz subshifts.

Recall that the topological full group of a Cantor minimal system (X, T), denoted by $[[T]]$, is the group of all homeomorphisms $f : X \to X$ such that there is a clopen partition of $X = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ and integers $l_1, l_2, \ldots, l_n$ such that $f \restriction A_i = T^{l_i} \restriction A_i$ for $1 \leq i \leq n$. The following theorem combines the work of Giordano–Putnam–Skau [13] and Bezuglyi–Medynets [2], and characterizes the flip conjugacy relation in the group setting.

Theorem 1.1. *Assume (X, T), (Y, S) are Cantor minimal systems, the following statements are equivalent:*

(1) *(X, T) and (Y, S) are flip conjugate.*

(2) *The topological full groups $[[T]]$ and $[[S]]$ are isomorphic as abstract groups.*

(3) *The commutator subgroups of the topological full groups $[[T]]$ and $[[S]]$ are isomorphic as abstract groups.*

The topological full groups of Toeplitz subshifts have a lot of nice properties. Matui in [24], and Juschenko and N. Monod in [17] find the following nice group properties of Cantor minimal systems.

Theorem 1.2. *Assume (X, T) is a Cantor minimal system, then we have:*

(1) *$[[T]]$ is amenable.*

(2) *The commutator subgroup of $[[T]]$ is an infinite simple group.*

(3) *The commutator subgroup of $[[T]]$ is finitely generated if and only if (X, T) is topologically conjugate to a minimal subshift over finite alphabet.*

We know that there are many Toeplitz subshifts $(\mathcal{O}, \sigma)$, and $(\mathcal{O}, \sigma^{-1})$ are not topologically conjugate(such examples are in Chapter 5). It seems that a Toeplitz subshift $(\mathcal{O}, \sigma)$ to be topologically conjugate to $(\mathcal{O}, \sigma^{-1})$ must have some nice property. In this dissertation, we also work on the problem that under what condition a Toeplitz subshift $(\mathcal{O}, \sigma)$ is topologically conjugate to $(\mathcal{O}, \sigma^{-1})$, and we call it the inverse problem. We will give an equivalent characterization of Toeplitz subshift having a single hole structure over $\{0, 1\}$ on this problem.

1.2. Borel Reduction and Countable Borel Equivalence Relations

1.2.1. Borel Equivalence Relations and Borel Reduction

Recall that X is a Polish space if it is separable and complete metrizable. Let X be a set, $\mathcal{S}$ be a σ-algebra on X. $(X, \mathcal{S})$ is called a standard Borel space if there exists a Polish topology on X, and the σ-algebra generated by the open sets on X is $\mathcal{S}$. By the technique of finding a finer topology, for every Borel set in a Polich space, we can turn it into a clopen set, see theorem 13.1 in [20]. It follows that if $(X, \mathcal{S})$ is a standard Borel space, and $B \in \mathcal{S}$, then $(B, \{A \cap B : A \in \mathcal{S}\})$ is also a standard Borel set.

If $f : (X, \mathcal{S}_1) \to (Y, \mathcal{S}_2)$ is a function between two standard Borel spaces, f is called Borel if $f^{-1}(A) \in \mathcal{S}_1$ for any $A \in \mathcal{S}_2$. It is well-known that any two uncountable standard Borel spaces are Borel isomorphic.

Definition 1.3. An equivalence relation $E \subseteq X \times X$ on a standard Borel space X is called a Borel equivalence relation if E is a Borel subset of $X \times X$. Let E, F be Borel equivalence relations on standard Borel spaces X, Y respectively. We say E is Borel reducible to F, denoted by $E \leq_B F$, if there exists a Borel function $f : X \to Y$ such that

$$x_1 E x_2 \iff f(x_1) F f(x_2), \forall x_1, x_2 \in X.$$

If such a Borel reduction function is a continuous injection, we denote it by $E \sqsubseteq_C F$, and if E is Borel reducible to F, but F is not Borel reducible to E, we denote it by $E <_B F$.

We say a Borel equivalence relation is smooth if it is Borel reducible to the identity relation, $=$, on the Cantor space $2^{\mathbb{N}}$ or equivalently by the Borel isomorphism theorem, if it is reducible to the identity relation on some uncountable Polish space X. The smooth equivalence relations are also called concretely classifiable since any such Borel reduction provides a Borel procedure which classifies the object to a concrete invariant. Any closed equivalence relation on Polish space is smooth, see proposition 5.4.7 in [10].

The eventual agreement relation E_0 is the equivalence relation defined on 2^ω by

$$x E_0 y \iff \exists m \forall n \geq m x(n) = y(n).$$

For a Borel equivalence relation E on a standard Borel space X, a measure μ on X is E-nonatomic if for any E-equivalent class A, $\mu(A) = 0$, and it is E-ergodic if for any E invariant Borel set $A \subseteq X$, $\mu(A)$ is either 0 or 1. For any Borel equivalence relation E, if there is an E-nonatomic and E-ergodic measure, then E is not smooth. Note that the product mesure on 2^ω is E_0-nonatomic and E_0-ergodic, so E_0 is not smooth(see proposition 6.1.7 in [10]). In fact, E_0 is the first non-smooth Borel equivalence relation in terms of Borel reduction, which follows from the following remarkable dichotomy theorem in [14].

Theorem 1.4 (Harrington–Kechris–Louveau). *Let E be a Borel equivalence relation on a Polish space X. Then either E is smooth or else $E_0 \sqsubseteq_C E$.*

1.2.2. Countable Borel Equivalent Relations

A Borel equivalence relation E is countable if every E-equivalent class is countable. There is a vast amount of research in countable Borel equivalence relations and the study of these relations have become intertwined with the study of countable group theory and ergodic theory. A lot of interesting problems are unsolved. For readers who want to study more on this topic, we refer to [21] and [16].

Recall that a Borel action of a topological group G on a topological space X is a Borel map $a : G \times X \to X$ such that $a(1_G, x) = x$, and $a(g, a(h, x)) = a(gh, x)$. And when the action a is understood, we write $g \cdot x$ for $a(g, x)$. The orbit equivalence relation, denoted by E_G^X, is defined by $x E_G^X y \iff \exists g(g \cdot x = y)$. It is clear that every orbit equivalence relation generated by a Borel action of a countable group is a countable Borel equivalence relation. An interesting fact noticed by Feldman and Moore in [7] reveals that every countable Borel equivalence relation turns out to be an orbit equivalence relation generated by some countable group.

Theorem 1.5 (Feldman and Moore). *Let E be a countable Borel equivalence relation on a standard Borel space X. Then there is a countable group G and a Borel action of G on X such that $E = E_G^X$.*

In classifying a countable Borel equivalence relation, usually we are interested in whether it is smooth, or hyperfinite, or countable universal.

DEFINITION 1.6. A Borel relation E on a standard Borel space X is hyperfinite if there is a sequence of finite Borel equivalence relations E_n, $n \in \mathbb{N}^+$ on X such that $E_n \subsetneq E_{n+1}$ for all $n \in \mathbb{N}^+$ and $E = \bigcup_{n\in\mathbb{N}^+} E_n$.

Every hyperfinite equivalence relation is a countable Borel equivalence relation. The equivalence relation E_0 is hyperfinite since $E_0 = \bigcup_{n\geq 1} E_n$, where each $E_n \subsetneq 2^\omega \times 2^\omega$ defined by

$$xE_n y \iff \forall m \geq n(x(n) = y(n)),$$

is a finite Borel equivalence relation.

It is clear that if E is a hyperfinite Borel equivalence relation, F is a countable Borel equivalence relation, and $F \leq_B E$, then F is hyperfinite. In fact, such closure property for hyperfiniteness still holds if F is weakly Borel reducible to E.

DEFINITION 1.7. Let E, F be countable Borel equivalence relations. A weak Borel reduction of E to F is a countable-to-1 Borel homomorphism f from E to F(for any $x_1Ex_2 \Rightarrow f(x_1)Ff(x_2)$). We denote this by $f : E \leq_B^w F$. If such an f exists, we say that E is weakly Borel reducible to F, in symbols $E \leq_B^w F$.

THEOREM 1.8 (Dougherty–Jackson–Kechris [4]). *If E, F are countable Borel equivalence relations, F is hyperfinite and $E \leq_B^w F$, then E is hyperfinite.*

One famous problem about hyperfinite equivalence relation is that assume $E = \bigcup_{n\in\mathbb{N}} E_n$, where each E_n is a hyperfinite equivalence relation, and $E_n \subsetneq E_{n+1}$, then is E hyperfinite?

DEFINITION 1.9. A Borel relation E on a standard Borel space X is hypersmooth if there is a sequence of smooth Borel equivalence relations E_n, $n \in \mathbb{N}^+$ on X such that $E_n \subsetneq E_{n+1}$ for all $n \in \mathbb{N}^+$ and $E = \bigcup_{n\in\mathbb{N}^+} E_n$.

A hypersmooth equivalence relation may not be countable. For example, the equivalence relation E_1 defined on $(2^{\mathbb{N}})^{\mathbb{N}}$ by

$$xE_1 y \iff \exists m \forall n \geq m(x(n) = y(n))$$

is a hypersmooth equivalence relation but not countable. We will see every hypersmooth countable Borel equivalence relation turns out to be hyperfinite. For a countable Borel equivalence relation, there are many equivalent statements about the hyperfiniteness from the following remarkable theorem based on earlier work by Weiss and by Slaman–Steel, and appeared in this form in Reference [4].

THEOREM 1.10 (Dougherty–Jackson–Kechris). *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*

(1) *E is hyperfinite.*

(2) *E is hypersmooth.*

(3) *$E \leq_B E_0$.*

(4) *$E \sqsubseteq_B E_0$.*

(5) *$E = \bigcup_{n\in\mathbb{N}^+} E_n$, where E_n are finite Borel equivalence relations, $E_n \subseteq E_{n+1}$, and every E_n-equivalence class has at most n elements.*

(6) *There exists a Borel action of $\mathbb{Z}$ on X such that $E = E_{\mathbb{Z}}^X$.*

(7) *There is a Borel assignment $C \to <_C$ associating with each E-equivalence class C a linear order $<_C$ on C so that there is an order-preserving map from $(C, <_C)$ into $(Z, <)$. Here $C \to <_C$ is Borel when the relation $R(x, y, z) \iff y <_{[x]_E} z$ is Borel.*

A remarkable theorem that the orbit equivalence relation generated by any countable abelian group is hyperfinite is first obtained by Gao and Jackson. Recall that a group is locally nilpotent if every finitely generated subgroup is nilpotent. Schnneider and Seward in [28] improve this result to the orbit equivalence relation generated by any countable locally nilpotent group.

THEOREM 1.11. *The orbit equivalence relation generated by a countable Abelian group is hyperfinite.*

DEFINITION 1.12. For any class $\mathcal{C}$ of equivalence relations, a universal equivalence relation for $\mathcal{C}$ is some $F \in \mathcal{C}$ such that for any $E \in \mathcal{C}$, $E \leq_B F$.

A universal equivalence relation in $\mathcal{C}$ is regarded as the most complicated one in $\mathcal{C}$. For all the Borel equivalence relations, there is no universal one since if $E \subseteq X \times X$ is a Borel equivalence relation. Consider the Friedman–Stanley jump $E^+ \subseteq X^{\mathbb{N}} \times X^{\mathbb{N}}$ defined as $xE^+y \iff \forall n \exists m (x(n) E y(m)) \wedge \forall m \exists n (x(n) E y(m))$. Friedman and Stanley in [8] proved that $E <_B E^+$. However, for the class of countable Borel equivalence relations, there are some well-known universal ones.

Example 1.13. Let $\mathbb{F}_2$ be a free group with two generators. Let E_∞ denote the orbit equivalence relation induced by the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$, where the shift action is defined by $g \cdot x(h) = x(g^{-1}h)$, for all $g, h \in \mathbb{F}_2$, $x \in 2^{\mathbb{F}_2}$. Doughterty–Jackson–Kechris in [4] proved that E_∞ is a universal countable Borel equivalence relation.

Example 1.14. Gao in [9] proved that the orbit equivalence relation of the canonical action of $SL(2, \mathbb{Z})$ on the subsets of $\mathbb{Z}^2$ is also a universal countable Borel equivalence relation.

Example 1.15. Clemens in [3] proved that the topological conjugacy relation on the space of subshifts over finite alphabet is a countable universal equivalence relation.

However, there are many countable Borel equivalence relations which are non-smooth, non-hyperfinite, and non-(countable universal). In fact, there is a rich structure of countable Borel equivalence relations E such that $E_0 <_B E <_B E_\infty$, which are called intermediate. All known proofs of the existence of intermediate countable Borel equivalence relations use measure theoretic methods of ergodic theory. References on such intermediate equivalence relations are [1], [21], [31], [26], and [15].

Theorem 1.16 (Adams–Kechris [1]). *There exist uncountably many free countable Borel equivalence relations up to Borel bireducibility.*

1.3. Main Results and Remaining Problems

Theorem 1.17. *The topological conjugacy relation on $\mathcal{T}_1$ is Borel bireducible with E_0.*

However, we don't know the complexity of topological conjugacy on Toeplitz subshifts with unbounded block gaps. Also, there exists a Toeplitz subshift not having unbounded block gaps. The questions of determining the complexities of the topological conjugacy on minimal subshifts and that on Toeplitz subshifts are still open.

Problem 1.18. Is the topological conjugacy relation hyperfinite on the space of Toeplitz subshifts with unbounded block gaps? Is it hyperfinite on the space of Toeplitz subshifts?

Recently, Julien Melleray in [25] proved that the orbit equivalence of Toeplitz subshifts is a universal equivalence relation for all orbit equivalence relations induced by Borel actions of Polish groups, which might imply that the topological conjugacy relation of Toeplitz subshifts could be kind of complicated too.

Problem 1.19. Is the flip conjugacy relation Borel reducible to the topological conjugacy relation on the space of Toeplitz(minimal) subshifts?

It is trivial that, on any subclass of the space of subshifs, the topological conjugacy relation is weakly Borel reducible to the flip conjugacy relation since the identity map witnesses such a reduction. By theorem 1.8, if the flip conjugacy relation is hyperfinite, then the flip conjugacy relation and the topological conjugacy relation are Borel bireducible to each other since each of them are hyperfinite and nonsmooth.

We say a Toeplitz subshift (X, σ) is flip invariant if (X, σ) is topologically conjugate to (X, σ^{-1}). We will provide an equivalent characterization of flip invariant property on Toeplitz subshifts having a single hole structure.

Theorem 1.20. *A Toeplitz subshift over* $\{0, 1\}$ *with a single hole structure is flip invariant if and only if it has a nice symmetric filling property.*

Problem 1.21. Characterize the inverse problem on the space of Toeplitz subshifts.

By a combined theorem 1.1 of Giordano–Putnam–Skau and Bezuglyi–Medynets, we know there is a bridge between the flip conjugacy of Toeplitz subshifts and the group properties of their topological full groups. And theorem 1.2 claims that the commutator subgroup of the topological full group of a Toeplitz subshift is a finitely generated simple amenable group, a natural question is that what kind of groups can be realised in this way.

Problem 1.22. For what kind of finitely generated simple amenable group G, can G be realized as the commutator subgroup of the topological full group of a Toeplitz subshift?

1.4. Organization of the Dissertation

The rest of the thesis is organized as follows.

In Chapter 2, we will introduce some basic concepts and facts about Toeplitz sequences(subshifts), like period structures, p-skeletons, natural partitions of a Toeplitz subshift, a criterion for Toeplitz subshifts to be topologically conjugate, and the Borelness of the space of Toeplitz subshifts, etc. Related references in this part are [5], [34] and [18].

In Chapter 3, we will discuss various subclasses of Toeplitz subshifts, for example, Toeplitz subshifts having separated holes, weakly separated holes, growing blocks and unbounded block gaps. We will study the closure properties of those subclasses under the topological conjugacy relation and whether they are independent from the choice of their period structures. We give examples to explain the proper inclusion of those subclasses, the existence of Toelitz subshifts not having unbounded block gaps, and the class of Toeplitz subshifts having growing blocks is not closed under topological conjugacy, etc.

In Chapter 4, we discuss the complexity of the topological conjugacy relation on Toeplitz subshifts in terms of Borel reduction. we give a more elementary proof of the fact that the topological conjugacy relation on the space of Toeplitz subshifts is not smooth, which was first proved by Simon Thomas in [32]. We follow Kaya's idea and generalize the result that the topological conjugacy on $\mathcal{T}_1$ is hyperfinite.

In chapter 5, we work on the problem under what condition, a Toeplitz subshift $(\mathcal{O}, \sigma)$ is topologically conjugate to $(\mathcal{O}, \sigma^{-1})$, which we call the inverse problem. Reapplying the criterion of Downarowicz and Kwiatkowski on the topological conjugacy on Toeplitz subshifts, we characterize the inverse problem on Toeplitz subshifts having a single-hole structure by a nice symmetric property on each of such Toeplitz subshifts itself.

CHAPTER 2

PRELIMINARIES

2.1. Basic Conceptions on Toeplitz Sequences and Toeplitz Subshifts

Fix a finite alphabet $\mathfrak{n}$, $|\mathfrak{n}| \geq 2$, assume $* \notin \mathfrak{n}$. Let $d_{\mathfrak{n}}$ be a metric on $\mathfrak{n}$. A natural metric on the product space $\mathfrak{n}^{\mathbb{Z}}$ is

$$d(x, y) := \Sigma_{n=-\infty}^{\infty} 2^{-|n|} d_{\mathfrak{n}}(x(n), y(n)).$$

Consider the full shift space $(\mathfrak{n}^{\mathbb{Z}}, \sigma)$, where $\sigma : \mathfrak{n}^{\mathbb{Z}} \to \mathfrak{n}^{\mathbb{Z}}$ is the left shift map defined as $(\sigma(\alpha))(i) = \alpha(i+1)$, for all $i \in \mathbb{Z}$, $\alpha \in \mathfrak{n}^{\mathbb{Z}}$. $\alpha \in \mathfrak{n}^{\mathbb{Z}}$ is called a Toeplitz sequence if for all $i \in \mathbb{Z}$, there exists $j \in \mathbb{Z}$ such that $\alpha(i + kj) = \alpha(i)$ for all $k \in \mathbb{Z}$. This means the set of Toeplitz sequences are those where every subblock recurs periodically. Period sequences are Toeplitz, but they are too simple and there are only countably many periodic sequences, so we ignore them and from now on, when we talk about Toeplitz sequences, we mean non-periodic Toeplitz sequences. A subshift is a closed set in $\mathfrak{n}^{\mathbb{Z}}$ which is also invariant under σ and σ^{-1}. A subshift is called a Toeplitz subshift if it is the closure of the σ-orbit of some Toeplitz sequence. Recall that a subshift on $\mathfrak{n}^{\mathbb{Z}}$ is minimal if it is a closed, σ and σ^{-1} invariant, and does not contain any nonempty proper subshift. Therefore, every Toeplitz subshift is minimal. Throughout the whole paper, when we talk about Toeplitz sequences(subshifts), we always consider they are elements(subsets) of $\mathfrak{n}^{\mathbb{Z}}$.

For each sequence $\alpha \in \mathfrak{n}^{\mathbb{Z}}$ and $p \in \mathbb{N}^+$, we associate them with the following notions and notations:

(1) The p-periodic part of α:

$$Per_p(\alpha) := \{i \in \mathbb{Z} : \alpha(i) = \alpha(i + kp) \text{ for all } k \in \mathbb{Z}\}.$$

Every integer not in the p-periodic part of α is called a p-hole of α,and we denote the set of p-holes of α by $Hole_p(\alpha)$.

(2) The p-skeleton of α is the sequence, denoted by $Skel_p(\alpha)$, defined as, for each $i \in \mathbb{Z}$,

$$Skel_p(\alpha)(i) = \begin{cases} \alpha(i), & \text{if } i \in Per_p(\alpha), \\ *, & \text{otherwise.} \end{cases}$$

(3) The set of p-symbols of α is

$$W_p(\alpha) = \{\alpha[kp,(k+1)p) : k \in \mathbb{Z}\}.$$

A period structure of a Toeplitz sequence α is a sequence $(p_t)_{t\in\mathbb{N}}$ such that $p_t|p_{t+1}$ for all $t \in \mathbb{N}$, $Per_{p_t}(\alpha) \neq \emptyset$, and $\bigcup_{t\in\mathbb{N}} Per_{p_t}(\alpha) = \mathbb{Z}$. Note that $\alpha = \lim_{t\to\infty} Skel_{p_t}(\alpha)$, and every $Skel_{p_t}(\alpha)$ is a periodic sequence in $(\mathfrak{n} \cup \{*\})^{\mathbb{Z}}$ with a period p_t, this implies we can construct every Toeplitz sequence in a periodic manner inductively.

Assume α is a Toeplitz sequence, $(p_t)_{t\in\mathbb{N}}$ is a period structure of α, and $\mathcal{O}$ is the Toeplitz subshift containing α. For each t, $0 \leq k < p_t$, we define

$$A(\alpha, p_t, k) := \{\sigma^i(\alpha) : i \equiv k(mod\ p_t)\},$$

$$Parts(\mathcal{O}, p_t) := \{\overline{A(\alpha, p_t, k)} : 0 \leq k < p_t\}.$$

If $W \in Parts(\mathcal{O}, p_t)$, it is clear that all the sequences in W have the same p_t skeleton, and we define

$$Skel_{p_t}(W) = Skel_{p_t}(\alpha),$$

where α is any Toeplitz sequence in W. Williams in [34] proved that $Parts(\mathcal{O}, p_t) = \{\overline{A(\alpha, p_t, k)} : 0 \leq k < p_t\}$ is a partition of $\mathcal{O}$. It is clear the collection $\{\overline{A(\alpha, p_t, k)} : 0 \leq k < p_t, t \in \mathbb{N}\}$ is also a basic clopen base of $\mathcal{O}$ equipped with the subspace topology.

Lemma 2.1. *Assume α is a Toeplitz sequence and $(p_t)_{t\in\mathbb{N}}$ is a period structure of α. Then $\beta \in \overline{A(\alpha, p_t, i)}$ if and only if there is a sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that $\beta = \lim_{k\to\infty} \sigma^{n_k p_t + i}(\alpha)$.*

Proof. $\Leftarrow$. It is obvious since σ is continuous and $\overline{A(\alpha, p_t, i)}$ is closed.

$\Rightarrow$. Assume $\beta \in \overline{A(\alpha, p_t, i)}$. Let $\beta_k \in A(\alpha, p_t, i)$ be such that $\beta = \lim_{k\to\infty} \beta_k$. For each k, there exists $n_k \in \mathbb{Z}$ such that $\beta_k = \sigma^{n_k p_t + i}(\alpha)$, then $\beta = \lim_{k\to\infty} \sigma^{n_k p_t + i}(\alpha)$. □

Theorem 2.2 (S. Williams). *Assume $\mathcal{O}$ is a Toeplitz subshift, $(p_t)_{t\in\mathbb{N}}$ is a period structure of $\mathcal{O}$, and α is a Toeplitz sequence in $\mathcal{O}$. For each t, we have*

(1) $\sigma(\overline{A(\alpha,p_t,i)} = \overline{A(\alpha,p_t,i+1)}$ *for* $i\in\mathbb{Z}$;

(2) $\overline{A(\alpha,p_t,i)} \supseteq \overline{A(\alpha,p_{t'},j)}$ *for* $i\in\mathbb{Z}$, *for* $t'\geq t$, *and* $i\equiv j(mod\ p_t)$;

(3) $\{\overline{A(\alpha,p_t,i)}:0\leq i<p_t\}$ *is a clopen partition of* $\mathcal{O}$.

Proof. $(1),(2)$ are trivial and we only provide a proof of (3).

First, to show $\mathcal{O}=\bigcup_{0\leq i<p_t}\overline{A(\alpha,p_t,i)}$. We only need to show $\mathcal{O}\subseteq\bigcup_{0\leq i<p_t}\overline{A(\alpha,p_t,i)}$. If $\beta\in\mathcal{O}$, then there exists a sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that $\beta=\lim_{k\to\infty}\sigma^{n_k}(\alpha)$. Let $n'_k\in[0,p_t)$ be such that $n'_k\equiv n_k(mod\ p_t)$. Then there is a subsequence $\{n'_{k_m}\}_{m\in\mathbb{N}}$ of $\{n'_k\}_{k\in\mathbb{N}}$ which converges to a number in $[0,p_t)$, say $j:=\lim_{m\to\infty}n'_{k_m}$, then $\beta\in\overline{A(\alpha,p_t,j)}$.

Now ony need to show if $\overline{A(\alpha,p_t,i)}\cap\overline{A(\alpha,p_t,j)}\neq\emptyset$, then $\overline{A(\alpha,p_t,i)}=\overline{A(\alpha,p_t,j)}$. Assume $\beta\in\overline{A(\alpha,p_t,i)}\cap\overline{A(\alpha,p_t,j)}$. By symmetry we only need to show $\overline{A(\alpha,p_t,i)}\subseteq\overline{A(\alpha,p_t,j)}$. For any $\gamma\in\overline{A(\alpha,p_t,i)}$, let $\{n_k\}_{k\in\mathbb{N}}$, $\{n'_k\}_{k\in\mathbb{N}}$, $\{m_k\}_{k\in\mathbb{N}}\subseteq\mathbb{Z}$ be such that $\beta=\lim_{k\to\infty}\sigma^{n_kp_t+i}(\alpha)$, $\beta=\lim_{k\to\infty}\sigma^{n'_kp_t+j}(\alpha)$, and $\gamma=\lim_{k\to\infty}\sigma^{m_kp_t+i}(\alpha)$. Then we have,

$$\begin{aligned}\gamma&=\lim_{k\to\infty}\sigma^{m_kp_t+i}(\sigma^{-n_kp_t-i}(\beta))\\&=\lim_{k\to\infty}\sigma^{m_kp_t+i}(\sigma^{-n_kp_t-i}(\sigma^{n'_kp_t+j}(\alpha)))\\&=\lim_{k\to\infty}\sigma^{(m_k-n_k+n'_k)p_t+j}(\alpha)\end{aligned}$$

So, $\gamma\in\overline{A(\alpha,p_t,j)}$. Therefore, $\{\overline{A(\alpha,p_t,i)}:0\leq i<p_t\}$ is a clopen partition of $\mathcal{O}$. □

2.2. A Criterion for Toeplitz Subshifts to Be Topologically Conjugate

We say two subshifts $\mathcal{O}$ and $\mathcal{O}'$ are topologically conjugate if there exists a homeomorphism $\pi:\mathcal{O}\to\mathcal{O}'$, and $\pi\circ\sigma=\sigma\circ\pi$, and π is called a topological conjugacy between $\mathcal{O}$ and $\mathcal{O}'$.

The following fundamental theorem connects the homomorphisms between subshifts and block codes, one recourse is [22] theorem 6.2.9. Block code plays a significant role in topological conjugacy between subshifts when it comes to determine the complexity of topological conjugacy or homomorphism on various subshift spaces.

Theorem 2.3 (Curtis–Hedlund–Lyndon). *Let $\pi : \mathcal{O} \to \mathcal{O}'$ be a one-to-one and onto map between subshifts $\mathcal{O}$, $\mathcal{O}'$. Then π is a topological conjugacy between $\mathcal{O}$ and $\mathcal{O}'$ if and only if there exist $i \in \mathbb{N}$ and a map $C : \mathfrak{n}^{2i+1} \to \mathfrak{n}$, such that*

$$\pi(\alpha)(k) = C(\alpha[k-i, k+i]), \textit{ for all } k \in \mathbb{Z}, \alpha \in \mathcal{O}.$$

Proof. $\Leftarrow$. Assume there exist $i \in \mathbb{N}$ and $C : \mathfrak{n}^{2i+1} \to \mathfrak{n}$, such that

$$\pi(\alpha)(k) = C(\alpha[k-i, k+i]), \text{ for all } k \in \mathbb{Z}, \alpha \in \mathcal{O}.$$

It is easy to check that π is a continuous function. Since π is also one-to-one and onto, $\mathcal{O}$ and $\mathcal{O}'$ are compact, then π is a homeomorphism. To show that π is a topological conjugacy, we only need to show that π commutes with σ. For any $\alpha \in \mathcal{O}$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\sigma(\pi(\alpha))(k) &= \pi(\alpha)(k+1) \\
&= C(\alpha[k+1-i, k+1+i]) \\
&= C(\sigma(\alpha)[k-i, k+i]) \\
&= \pi(\sigma(\alpha))(k).
\end{aligned}$$

$\Rightarrow$. Assume $\pi : \mathcal{O} \to \mathcal{O}'$ is a topological conjugacy. And assume toward a contradiction that for each $i \in \mathbb{N}$, the set

$$A_i := \{\alpha \in \mathcal{O} : \exists \beta \in \mathcal{O}(\alpha[-i, i] = \beta[-i, i], \text{ and } \pi(\alpha)(0) \neq \pi(\beta)(0))\}$$

is nonempty.

We claim that for each i, A_i is closed. Suppose $(\alpha_n)_{n\in\mathbb{N}}$ is a convergent sequence in A_i, and $\alpha = \lim_{n\to\infty} \alpha_n$. Let n_0 be such that for all $n \geq n_0$, $\alpha_n[-i, i] = \alpha_{n_0}[-i, i]$, and $\pi(\alpha_n)(0) = \pi(\alpha_{n_0})(0)$. Since $\alpha_{n_0} \in A_i$, there is $\beta \in \mathcal{O}$ such that $\alpha_{n_0}[-i, i] = \beta[-i, i]$, and $\pi(\alpha)(0) \neq \pi(\beta)(0)$. Then $\alpha[-i, i] = \beta[-i, i]$, and $\pi(\alpha)(0) \neq \pi(\beta)(0)$, namely, $\alpha \in A_i$.

Therefore, $(A_i)_{i\in\mathbb{N}}$ is a sequence of nonempty closed decreasing sets in a compact space $\mathcal{O}$, so $\cap_i A_i \neq \emptyset$. Let $\alpha \in \cap_i A_i$, let $\beta_i \in \mathcal{O}$ be such that $\alpha[-i, i] = \beta_i[-i, i]$, and $\pi(\alpha)(0) \neq$

$\pi(\beta_i)(0)$. Then $\alpha = \lim_{i\to\infty} \beta_i$. Since π is continuous, $\lim_{i\to\infty} \pi(\beta_i) = \pi(\alpha)$. Hence, for all large enough i, $\pi(\beta_i)(0) = \pi(\alpha)(0)$, a contradiction. □

In theorem 2.3, we say C is a block code inducing π, and i is the length of C. We define the length of π, denoted by $|\pi|$, as follows:

$$|\pi| = \max\{\min\{|C| : C \text{ induces } \pi\}, \min\{|C| : C \text{ induces } \pi^{-1}\}\}.$$

Applying Curtis–Hedlund–Lyndon's theorem, Downarowicz and Kwiatkowski [5] found a criterion for Toeplitz subshifts to be topologically conjugate.

THEOREM 2.4 (Downarowicz–Kwiatkowski). *Assume Toeplitz subshifts $\mathcal{O}$ and $\mathcal{O}'$ have period structures $(p_t)_{t\in\mathbb{N}}$ and $(q_t)_{t\in\mathbb{N}}$ respectively. Then $\mathcal{O}$ and $\mathcal{O}'$ are topologically conjugate if and only if there exist Toeplitz sequences $\alpha \in \mathcal{O}$, $\beta \in \mathcal{O}'$, some $t \in \mathbb{N}$, and $\phi \in Sym(\mathfrak{n}^{p_t})$ such that*

$$\beta[kp_t, (k+1)p_t) = \phi(\alpha[kp_t, (k+1)p_t)), \textit{ for all } k \in \mathbb{Z}.$$

PROOF. $\Leftarrow$. Assume ϕ is a bijection such that for all $k \in \mathbb{Z}$, $\beta[kp_t, (k+1)p_t) = \phi(\alpha[kp_t, (k+1)p_t))$. Let $\hat{\phi}$ be a homeomorphism induced by ϕ on $\mathfrak{n}^{\mathbb{Z}}$ defined by

$$\hat{\phi}(x)[kp_t, (k+1)p_t) = \phi(x[kp_t, (k+1)p_t)),$$

for all $k \in \mathbb{Z}$ and $x \in \mathfrak{n}^{\mathbb{Z}}$.

Consider a map $\pi : \mathcal{O} \to \mathcal{O}'$ defined as

$$\pi(x) = \sigma^i(\hat{\phi}(\sigma^{-i}(x))) \text{ if } x \in \overline{A(\alpha, p_t, i)}.$$

It is easy to check that π is a conjugacy between $\mathcal{O}$ and $\mathcal{O}'$.

$\Rightarrow$. Assume $\pi : \mathcal{O} \to \mathcal{O}'$ is a topological conjugacy and $\alpha \in \mathcal{O}$ is a Toeplitz sequence. Let t be large enough such that $[-|\pi|, |\pi|] \subseteq Per_{p_t}(\alpha), Per_{p_t}(\pi(\alpha))$. Define $\phi : W_{p_t}(\alpha) \to W_{p_t}(\pi(\alpha))$ as follows:

$$\phi(\alpha[kp_t, (k+1)p_t) = \pi(\alpha)[kp_t, (k+1)p_t).$$

First, to show ϕ is a well-defined function. Let C be a block code inducing π and $|C| \leq |\pi|$. If $\alpha[k_1 p_t, (k_1+1)p_t) = \alpha[k_2 p_t, (k_2+1)p_t)$, then

$$\alpha[k_1 p_t - |C|, (k_1+1)p_t + |C|) = \alpha[k_2 p_t - |C|, (k_2+1)p_t + |C|).$$

For all $0 \leq i < p_t$, we have

$$\begin{aligned}\pi(\alpha)(k_1 p_t + i) &= C(\alpha[k_1 + i - |C|, k_1 p_t + i + |C|]) \\ &= C(\alpha[k_2 + i - |C|, k_2 p_t + i + |C|]) \\ &= \pi(\alpha)(k_2 p_t + i)\end{aligned}$$

So, ϕ is a function and obviously, it is onto. So $|W_{p_t}(\alpha)| \leq |W_{p_t}(\pi(\alpha))|$. Symmetrically, we can show that $|W_{p_t}(\alpha)| \geq |W_{p_t}(\pi(\alpha))|$. So $|W_{p_t}(\alpha)| = |W_{p_t}(\pi(\alpha))| < \infty$. Hence ϕ is a bijection, which can be extended to an element in $Sym(\mathfrak{n}^{p_t})$. □

COROLLARY 2.5. *Toeplitz subshifts $\mathcal{O}$ and $\mathcal{O}'$ are topologically conjugate if and only if there exist Toeplitz sequences $\alpha \in \mathcal{O}$, $\beta \in \mathcal{O}'$, some $m \in \mathbb{N}$, and $\phi \in Sym(\mathfrak{n}^m)$ such that*

$$\beta[km, (k+1)m) = \phi(\alpha[km, (k+1)m)), \text{ for all } k \in \mathbb{Z}.$$

REMARK 2.6. Kaya noticed the following fact. Assume α, β, $\mathcal{O}$, $\mathcal{O}'$, $(p_t)_{t\in\mathbb{N}}$, and ϕ are as stated in theorem 2.4. Let π be as defined in the proof, namely

$$\pi(x) = \sigma^i \hat{\phi}(\sigma^{-i}(x)), \text{ if } x \in \overline{A(\alpha, p_t, i)}.$$

Then π is a topological conjugacy which maps the partition $Parts(\mathcal{O}, p_t)$ onto $Parts(\mathcal{O}', p_t)$. However, for some $0 < i < p_t$, it may happen that we cannot find a $\psi \in Sym(\mathfrak{n}^{p_t})$ such that $\hat{\psi}(\overline{A(\alpha, p_t, i)}) = \overline{A(\beta, p_t, i)}$. Therefore, $\mathcal{O}$ and $\mathcal{O}'$ are topologically conjugate if and only if some elements of the partition $Parts(\mathcal{O}, p_t)$ are mapped onto some elements of $Parts(\mathcal{O}', p_t)$ under the natural group action of $Sym(\mathfrak{n}^{p_t})$ for large enough p_t.

For a Toeplitz sequence α, a positive integer p is called an essential period of α if p is a period of α, and $Per_q(\alpha) \neq Per_p(\alpha)$ whenever $0 < q < p$.

We call a period structure $(p_t)_{t\in\mathbb{N}}$ a good period structure of a Toeplitz sequence α if

(1) $(p_t)_{t\in\mathbb{N}}$ is a period structure of α,

(2) For any prime number p,

(a) If there exist n, and t such that $p^n|p_t$, let t_0 be the least such number, and for all t, $p^{n+1}\nmid p_t$. Define

$$p'_t = \begin{cases} p_t, & \text{if } t < t_0, \\ \frac{p_t}{p}, & \text{otherwise.} \end{cases}$$

Then $(p'_t)_{t\in\mathbb{N}}$ is not a period structure of α, which means, in this situation, there is some $k\in\mathbb{Z}$ such that $k\in\cap_{t\in\mathbb{N}}Hole_{p'_t}(\alpha)$.

(b) If for all n, there exists t such that $p^n|p_t$. Let q_t be the greatest factor of p_t such that $p\nmid q_t$. Then for all $n\in\mathbb{N}$, $(p^n\times q_t)_{t\in\mathbb{N}}$ is not a period structure of α.

We call $(p_t)_{t\in\mathbb{N}}$ a good period structure of a Toeplitz subshift $\mathcal{O}$ if it is a good period structure of a Toeplitz sequence $\alpha\in\mathcal{O}$.

For each $i\in\mathbb{N}^+$, let r_i be the i-th prime number. If $(p_t)_{t\in\mathbb{N}}$ is a good period structure of α, let

$$\dot{p}_t = \Pi_{0\le i\le t+1} r_i^{min\{k_i,t+1\}}, \text{ where } k_i = sup\{j\in\mathbb{N} : \text{ there exists } t\in\mathbb{N}, r_i^j|p_t\}.$$

The sequence we get from $(\dot{p}_t)_{t\in\mathbb{N}}$ by deleting all 1's and the repeated terms is called the natural factorization of $(p_t)_{t\in\mathbb{N}}$, and it is also called the natural factorization of α.

Similarly, We call $(p_t)_{t\in\mathbb{N}}$ the natural factorization of a Toeplitz subshift $\mathcal{O}$ if it is the natural factorization of a Toeplitz sequence $\alpha\in\mathcal{O}$.

2.3. The Borelness of the Space of Toeplitz Subshift

In this section, we will show that the space of Toeplitz subshifts as a subspace of $F(\mathfrak{n}^{\mathbb{Z}})$ is a standard Borel space, observed at the beginning by Sabok and Tsankov [27] by applying odometers in Theorem 5.1 in [6]. Here, we will reprove it without using odometers.

2.3.1. The Effros Borel Space of $F(X)$

Recall that for a Polish space(separable, completely metrizable space) X, the collection of closed subset, denoted by $F(X)$, endowed with the σ-algebra generated by the sets of the form $\{F \in F(X) : F \bigcap U \neq \emptyset\}$, where U varies over open subsets of X, is called the Effros Borel space of $F(X)$. It is well-known that the Effros Borel space of $F(X)$ is a standard Borel space, namely, the σ-algebra here can be generated by a Polish topology on $F(X)$. Particularly, when X is also compact, the Effros Borel space of $F(X)$ admits a compatible topology that is compact metrizable. More details are available in [20] section 4.F and 12.

2.3.2. The Space of Toeplitz Subshift is Standard Borel

First, let's see a non-Toeplitz sequence in a Toeplitz subshift through the following example, the trick in which shows that the set of non-Toeplitz sequences in a Toeplitz subshift is a nonempty F_σ set.

Example 2.7. We start with the two-sided unfilled sequence say $\cdots * * * \ldots$.

At step $2n+1$, fill the leftmost unfilled position of $\alpha_{2n}[k2^{2n+1}, (k+1)2^{2n+1})$ by 0 for all $k \in \mathbb{Z}$, and denote the sequence we obtained by α_{2n+1}.

At step $2n+2$, fill the rightmost unfilled position of $\alpha_{2n+1}[k2^{2n+2}, (k+1)2^{2n+2})$ by 1 for all $k \in \mathbb{Z}$, and denote the sequence we obtained by α_{2n+2}.

The diagram below shows our first three constructions:

$$step1 : ...0 * 0 * 0 * 0 * 0 * 0 * 0 * 0 * ...$$

$$step2 : ...0 * 010 * 010 * 010 * 01...$$

$$step3 : ...00010 * 0100010 * 01...$$

Let x be the Toeplitz sequence we constructed as above, and X be the Toeplitz subshift containing x. For each $k \in \mathbb{N}$, let

$$y_k = \begin{cases} \sigma^{-\frac{2^{k+1}-1}{3}}(x), & \text{if } k \text{ odd,} \\ \sigma^{-\frac{2^k-1}{3}}(x), & \text{otherwise.} \end{cases}$$

Note that for each n, if n is odd, then $\frac{2^{n+1}-1}{3}$ is a 2^n-hole of x, and if n is even, then $\frac{2^n-1}{3}$ is a 2^n-hole of x. Therefore, 0 is always a 2^k-hole of y_k. Since X is compact, let $\{y_{k_l}\}_{l\in\mathbb{N}}$ be a convergent subsequence of $\{y_k\}_{k\in\mathbb{N}}$. Then $\lim_{l\to\infty} y_{k_l}$ is not a Toeplitz sequence since its 0 coordinate won't repeat with any period 2^n.

LEMMA 2.8. *If $\mathcal{O}$ is a Toeplitz subshift, then the set of Toeplitz sequences in $\mathcal{O}$ is a proper dense G_δ subset.*

PROOF. Fix a period structure of $\mathcal{O}$, say $(p_t)_{t\in\mathbb{N}}$, and let $x \in \mathcal{O}$ be a Toeplitz sequence. Then $\{\sigma^n(x) : n \in \mathbb{Z}\}$ is a dense subset of Toeplitz sequences in $\mathcal{O}$. For each $t \in \mathbb{N}$ and $n \in \mathbb{Z}$, define

$$B_{t,n} := \bigcup_{\substack{0\le i<p_t \\ Skel_{p_t}(\sigma^i(x))(n)=*}} \overline{A(\alpha, p_t, i)}.$$

Fix n, $\{B_{t,n}\}_{t\in\mathbb{N}}$ is a sequence of decreasing compact sets in $\mathcal{O}$, therefore, $\bigcap_{t\in\mathbb{N}} B_{t,n}$ is a nonempty closed set. Let $Toep(\mathcal{O})$ be the set of Toeplitz sequences in $\mathcal{O}$, then

$$\begin{aligned} x \notin Toep(\mathcal{O}) &\iff \exists n \in \mathbb{Z}(x(n) \text{ won't repeat with any } p_t) \\ &\iff \exists n \in \mathbb{Z} \forall t \in \mathbb{N}(x \in B_{t,n}) \end{aligned}$$

Hence, $Toep(\mathcal{O}) = \bigcap_{n\in\mathbb{Z}} \bigcup_{t\in\mathbb{N}} (\mathcal{O} \setminus B_{t,n})$ is a proper dense G_δ subset. □

To show that the space of Toeplitz subshifts as a subspace of the Effros Borel space of $F(\mathfrak{n}^{\mathbb{Z}})$ is Borel, we need the following well-known facts in descriptive set theory, references are section 5.8 in [30], section 16.A and 18.B in [20] or [23].

DEFINITION 2.9. Let X, Y be standard Borel spaces. A function $\mathcal{I} : X \to \mathcal{P}(\mathcal{P}(Y))$ is called Borel on Borel if for every Borel set $B \subset X \times Y$, the set $\{x \in X : B_x \in \mathcal{I}(x)\}$ is Borel, where $\mathcal{P}(Y)$ is the family of all subset on Y and $B_x := \{y : (x, y) \in B\}$.

Theorem 2.10. *Let X, Y be Polish spaces and $g : X \to F(Y)$ be a Borel function. The map $\mathcal{I} : X \to \mathcal{P}(\mathcal{P}(Y))$ defined by:*

$$\mathcal{I}(x) = \{I \subseteq Y : I \text{ is meager in } g(x)\}$$

is Borel on Borel.

Theorem 2.11. *Let X, Y be Polish spaces. Assume $\mathcal{I} : X \to \mathcal{P}(\mathcal{P}(Y))$ is a Borel on Borel map assigning each $x \in X$ a σ-ideal $\mathcal{I}(x)$ of subset of Y. If $B \subseteq X \times Y$ is Borel and for $x \in Proj_X(B) := \{x \in X : \exists y((x, y) \in B)\}$, $B_x \notin \mathcal{I}(x)$, then $Proj_X(B)$ is Borel and there is a Borel uniformization for B, namely, a Borel function $f : Proj_X(B) \to Y$, such that $(x, f(x)) \in B$ for $x \in Proj_X(B)$.*

Theorem 2.12 (M. Sabok and T. Tsankov). *The space of Toeplitz subshifts as a subspace of the Effros Borel space of $F(\mathfrak{n}^{\mathbb{Z}})$ is Borel. Moreover, there is a Borel function picking a Toeplitz sequence from each Toeplitz subshift containing it.*

Proof. Fix a countable base $\{U_n\}_{n\in\mathbb{N}}$ of X. Consider the set $B \subseteq F(X) \times X$ defined as

$$B := \{(\mathcal{O}, x) : x \in \mathfrak{n}^{\mathbb{Z}} \text{ is a Toeplitz sequence and } \mathcal{O} = \overline{\{\sigma^n(x) : n \in \mathbb{Z}\}}\}.$$

We have

$$(\mathcal{O}, x) \in B \iff x \in \mathfrak{n}^{\mathbb{Z}} \wedge \forall m \in \mathbb{Z} \exists l \in \mathbb{N} \forall k \in \mathbb{Z}(x(m) = x(m + kl)) \wedge$$

$$\forall n \in \mathbb{N}(U_n \cap \mathcal{O} \neq \emptyset \Rightarrow \exists i \in \mathbb{Z}(\sigma^i(x) \in U_n)).$$

So, B is Borel. It is clear that $Proj_{F(X)}(B)$ is the set of Toeplitz subshifts on $\mathfrak{n}^{\mathbb{Z}}$.

Let $g : F(X) \to F(X)$ be the identity map. Let $\mathcal{I} : F(X) \to \mathcal{P}(\mathcal{P}(X))$ be the Borel on Borel map defined as

$$\mathcal{I}(\mathcal{O}) = \{I \subseteq X : I \text{ is meager in } \mathcal{O}\}.$$

By theorem 2.10, $Proj_{F(X)}(B)$ is Borel.

By lemma 2.8 we know that for $\mathcal{O} \in Proj_{F(X)}(B)$, $B_{\mathcal{O}} \notin \mathcal{I}(\mathcal{O})$, applying theorem 2.11, the Borel uniformization of B picks a Toeplitz sequence in each Toeplitz subshift containing it. □

CHAPTER 3

VARIOUS SUBSPACES OF TOEPLITZ SUBSHIFTS

3.1. Toeplitz Subshifts Having Separated Holes or Weakly Separated Holes

DEFINITION 3.1. A Toeplitz sequence α is said to have weakly separated holes w.r.t. its period structure $(p_t)_{t\in\mathbb{N}}$ if there exists $M > 0$ such that

$$\lim_{t\to\infty} \min\{|i-j| > M : i, j \text{ are } p_t \text{ holes of } \alpha\} = \infty.$$

A Toeplitz subshift $\mathcal{O}$ is said to have weakly separated holes w.r.t. its period structure $(p_t)_{t\in\mathbb{N}}$ if there exists a (equivalently, every) Toeplitz sequence $\alpha \in \mathcal{O}$ having weakly separated holes w.r.t. $(p_t)_{t\in\mathbb{N}}$.

Recall that for a Toeplitz sequence α having a period structure $(p_t)_{t\in\mathbb{N}}$, if $\lim_{t\to\infty} \min\{|i-j| : i, j \text{ are distinct } p_t \text{ holes of } \alpha\} = \infty$, then we say α has separated holes w.r.t. $(p_t)_{t\in\mathbb{N}}$. We say $i < j$ are consecutive p_t holes of α if i, j are p_t holes of α, and for any $k \in (i, j)$, k is not a p_t hole of α. If $\lim_{t\to\infty} \min\{|i-j| > 1 : i, j \text{ are consecutive } p_t \text{ holes of } \alpha\} = \infty$, then we say that α has growing blocks w.r.t. $(p_t)_{t\in\mathbb{N}}$.

A Toeplitz subshift is said to have separated holes(growing blocks) w.r.t. $(p_t)_{t\in\mathbb{N}}$ if there exists a Toeplitz sequence $\alpha \in \mathcal{O}$ having separated holes(growing blocks) w.r.t. $(p_t)_{t\in\mathbb{N}}$. Obviously, the class of Toeplitz sequences (subshifts) having growing blocks is contained in the class of Toeplitz sequences (subshifts) having weakly separated holes.

Kaya gives an example in [18] showing that a Toeplitz subshift having growing blocks depends on the choice of its periods structures. However, he points out that whether or not a Toeplitz subshift has separated holes is independent of the choice of its period structures. It turns out that having or not having weakly separated holes is also independent of the choice of the period structures.

PROPOSITION 3.2. *Let $(p_t)_{t\in\mathbb{N}}$, $(q_t)_{t\in\mathbb{N}}$ be period structures of a Toeplitz subshift $\mathcal{O}$. Then $\mathcal{O}$ has weakly separated holes w.r.t. $(p_t)_{t\in\mathbb{N}}$ if and only if $\mathcal{O}$ has weakly separated holes w.r.t*

$(q_t)_{t\in\mathbb{N}}$.

Proof. Assume $\alpha \in \mathcal{O}$ is a Toeplitz sequence having weakly separated holes w.r.t. $(p_t)_{t\in\mathbb{N}}$. Let $M > 0$ be such that

$$\lim_{t\to\infty} \min\{|i-j| > M : i, j \text{ are } p_t \text{ holes of } \alpha\} = \infty.$$

For any $N > 0$, there exists t_0 such that

$$\min\{|i-j| > M : i, j \text{ are } p_{t_0} \text{ holes of } \alpha\} > N.$$

Since $(p_t)_{t\in\mathbb{N}}$ and $(q_t)_{t\in\mathbb{N}}$ are period structures for $\mathcal{O}$, there exist t_1, r_1, $r_2 \in \mathbb{N}$ such that $p_{t_0} = r_1 \cdot r_2$, and for any $t > t_1$, $r_1 | q_t$ and $(r_2, \frac{q_t}{r_1}) = 1$.

Claim:$Per_{p_{t_0}}(\alpha) = Per_{r_1}(\alpha)$.

Proof. It is obvious that $Per_{r_1}(\alpha) \subseteq Per_{p_{t_0}}(\alpha)$ since $r_1|p_{t_0}$. We only need to show that $Hole_{r_1}(\alpha) \subseteq Hole_{p_{t_0}}(\alpha)$. For any $i \in Hole_{r_1}(\alpha)$, let $k_0 \in \mathbb{Z}$ satisfy $\alpha(i) \neq \alpha(i + k_0 r_1)$. Let $t > t_1$ be such that $i + k_0 r_1 \in Per_{q_t}(\alpha)$ since $(q_t)_{t\in\mathbb{N}}$ is a period structure of α. Since $(r_2, \frac{q_t}{r_1}) = 1$, there exist $a, b \in \mathbb{Z}$ such that

$$ar_2 + b\frac{q_t}{r_1} = 1.$$

For all $k \in \mathbb{Z}$, we have

$$\begin{aligned}\alpha(i + k_0 r_1) &= \alpha(i + k_0 r_1 + k q_t)\\ &= \alpha(i + k_0 r_1 + k\frac{r_1 - ar_1 r_2}{b})\\ &= \alpha(i + (k_0 + \frac{k}{b})r_1 - \frac{ka}{b}p_{t_0})\end{aligned}$$

Consider $k = -bk_0$, we have

$$\alpha(i + k_0 r_1) = \alpha(i + ak_0 p_{t_0}).$$

Hence $\alpha(i) \neq \alpha(i + ak_0 p_{t_0})$, and $i \in Hole_{p_{t_0}}(\alpha)$. □

Therefore, $Per_{p_{t_0}}(\alpha) = Per_{r_1}(\alpha) \subseteq Per_{q_t}(\alpha)$, which implies that

$$\min\{|i-j| > M : i,j \text{ are } q_t \text{ holes of } \alpha\} \geq \min\{|i-j| > M : i,j \text{ are } p_{t_0} \text{ holes of } \alpha\}$$
$$\geq N.$$

Hence, $\mathcal{O}$ has weakly separated holes w.r.t. $(q_t)_{t\in\mathbb{N}}$.

The other direction can be proved with a symmetrical argument. □

3.2. Toeplitz Subshifts Having Growing Blocks or Unbounded Block Gaps

Given a Toeplitz subshift $\mathcal{O}$, pick a Toeplitz sequence $\alpha \in \mathcal{O}$, let $(p_t)_{t\in\mathbb{N}}$ be a period structure, for any $t \in \mathbb{N}$, we denote $L_{p_t,0}(\mathcal{O}) = 0$. Assume $L_{p_t,n}(\mathcal{O})$ has been defined. If there exist consecutive p_t holes of α, say i, j, such that $|i-j| > L_{p_t,n}(\mathcal{O})$, define

$$L_{p_t,n+1}(\mathcal{O}) = \min\{|i-j| > L_{p_t,n}(\mathcal{O}) : i,j \text{ are consecutive } p_t \text{ holes of } \alpha\}.$$

Otherwise, let $L_{p_t,n+1}(\mathcal{O}) = L_{p_t,n}(\mathcal{O})$.

For any $n, t \in \mathbb{N}$, $L_{p_t,n}$ is well-defined since it is independent of the choice of Toeplitz sequence in $\mathcal{O}$. It is easy to check, for any $n, t \in \mathbb{N}$, $L_{p_t,n}$ is Borel on the set of Toeplitz subshifts. Denote $L_{p_t}(\mathcal{O}) = \max\{L_{p_t,n+1}(\mathcal{O}) - L_{p_t,n}(\mathcal{O}) : n \in \mathbb{N}\}$.

Definition 3.3. A Toeplitz subshift $\mathcal{O}$ is said to have unbounded block gaps with its period structure $(p_t)_{t\in\mathbb{N}}$, if $\lim\limits_{t\to\infty} L_{p_t}(\mathcal{O}) = \infty$.

A Toeplitz sequence α is said to have unbounded block gaps with its period structure $(p_t)_{t\in\mathbb{N}}$ if the Toeplitz subshift containing α does.

Proposition 3.4. *The class of Toeplitz subshifts having weakly separated holes is contained in the class of Toeplitz subshifts having unbounded block gaps.*

Proof. Let α be a Toeplitz sequence with a period structure $(p_t)_{t\in\mathbb{N}}$. Assume α has weakly separated holes. Let M be such that

$$\lim_{t\to\infty} \min\{|i-j| > M : i,j \text{ are } p_t \text{ holes of } \alpha\} = \infty.$$

For any $N > 0$, there exists t_0 such that for all $t > t_0$,

$$\min\{|i-j| > M : i, j \text{ are } p_t \text{ holes of } \alpha\} > N + M.$$

Let $t_1 > t_0$ be such that $[0, M] \subseteq Per_{p_{t_1}}(\alpha)$. For all $t > t_1$, $[0, M] \subseteq Per_{p_{t_1}}(\alpha) \subseteq Per_{p_t}(\alpha)$, so there exists s_t such that $L_{p_t,s_t}(\mathcal{O}) \leq M$, and $L_{p_t,s_t+1}(\mathcal{O}) > M$. For all $t > t_1$, we have

$$\begin{aligned} L_{p_t}(\mathcal{O}) &\geq L_{p_t,s_t+1}(\mathcal{O}) - L_{p_t,s_t}(\mathcal{O}) \\ &\geq \min\{|i-j| > M : i, j \text{ are } p_t \text{ holes of } \alpha\} - M \\ &> N. \end{aligned}$$

Therefore, $\mathcal{O}$ has unbounded block gaps. □

LEMMA 3.5. *Assume $\mathcal{O}, \mathcal{O}'$ are Toeplitz subshifts having a same period structure $(p_t)_{t\in\mathbb{N}}$, $\pi : \mathcal{O} \to \mathcal{O}'$ is a topological conjugacy, $\alpha \in \mathcal{O}$ is a Toeplitz sequence, and $\beta = \pi(\alpha)$. If i, j are consecutive p_t holes of α, and $j - i > 2|\pi|$, then there exist $i' \in [i - |\pi|, i + |\pi|]$, $j' \in [j - |\pi|, j + |\pi|]$, and $i' < j'$ are consecutive p_t holes of β.*

Particularly, for any $n \in \mathbb{N}$, if $L_{p_t,n}(\mathcal{O}) > 2|\pi|$, there exists $n' > 0$ such that $|L_{p_t,n'}(\mathcal{O}') - L_{p_t,n}(\mathcal{O})| \leq 2|\pi|$.

PROOF. Let $i < j$ be consecutive p_t holes of α, such that $j - i > 2|\pi|$. Let C be a block code inducing π, and $|C| \leq |\pi|$. For any $n \in (i + |\pi|, j - |\pi|)$, we have

$$\begin{aligned} \pi(\alpha)(n) &= C((\alpha[n - |C|, n + |C|]) \\ &= C((\alpha[n - |C| + kp_t, n + |C| + kp_t]), \forall k \in \mathbb{Z} \\ &= \pi(\alpha)(n + kp_t), \forall k \in \mathbb{Z}. \end{aligned}$$

So, $(i + |\pi|, j - |\pi|) \subseteq Per_{p_t}(\pi(\alpha))$.

Let D be a block code inducing π^{-1}, and $|D| \leq |\pi|$. Since i is a p_t hole of α, there exists $k_1 \in \mathbb{Z}$ such that $\alpha(i) \neq \alpha(i + k_1 p_t)$. Note that

$$\alpha(i) = \pi^{-1}(\beta)(i)$$

$$= D(\beta[i - |D|, i + |D|]).$$

$$\alpha(i + k_1 p_t) = \pi^{-1}(\beta)(i + k_1 p_t)$$
$$= D(\beta[i - |D| + k_1 p_t, i + |D| + k_1 p_t]).$$

So, $\beta[i-|D|, i+|D|] \neq \beta[i-|D|+k_1 p_t, i+|D|+k_1 p_t]$. Since $|D| \leq |\pi|$, we have $\beta[i-|\pi|, i+|\pi|] \neq \beta[i - |\pi| + k_1 p_t, i + |\pi| + k_1 p_t]$. This implies that there exists a p_t hole of β in the interval $[i - |\pi|, i + |\pi|]$. Similarly, since j is a p_t hole of α, there exists a p_t hole of β in the interval $[j - |\pi|, j + |\pi|]$.

Let i' be the largest p_t hole of β in the interval $[i - |\pi|, i + |\pi|]$, and j'_1 be the smallest p_t hole of β in the interval $[j - |\pi|, j + |\pi|]$, we have $j - i + 2|\pi| \geq j' - i' \geq j - i - 2|\pi| > 0$, and i', j' are consecutive p_t holes of β. □

We will see that the class of Toeplitz subshifts with growing blocks w.r.t. some particular period structure is not invariant under the topological conjugacy raletion. While, the set of Toeplitz subshifts having weakly separated holes is invariant under the topologically conjugate ralation.

Proposition 3.6. *If $\mathcal{O}$ and $\mathcal{O}'$ are topologically conjugate Toeplitz subshifts, then $\mathcal{O}$ has weakly separated holes if and only if $\mathcal{O}'$ does.*

Proof. Assume $\mathcal{O}$ has weakly separated holes, $(p_t)_{t\in\mathbb{N}}$ be a period structure of $\mathcal{O}$, and $\pi : \mathcal{O} \to \mathcal{O}'$ is a topological conjugacy. Pick a Toeplitz sequence $\alpha \in \mathcal{O}$. Let $M > 0$ be such that

$$\lim_{t\to\infty} \min\{|i - j| > M : i, j \text{ are } p_t \text{ holes of } \alpha\} = \infty.$$

For any $N > 0$, there exists t_N such that for all $t \geq t_N$,

$$\min\{|i - j| > M : i, j \text{ are } p_t \text{ holes of } \alpha\} > N + 2|\pi|.$$

For any $t \geq t_N$, to show

$$\min\{|i - j| > M + 2|\pi| : i, j \text{ are } p_t \text{ holes of } \pi(\alpha)\} > N.$$

Assume i, j are p_t holes of $\pi(\alpha)$, and $|i-j| > M + 2|\pi|$. By lemma 3.5, there exist $i' \in [i - |\pi|, i + |\pi|]$, $j' \in [j - |\pi|, j + |\pi|]$, and i', j' are p_t holes of α. Then $|i' - j'| \geq |i - j| - 2|\pi| > M$, and hence,

$$|i' - j'| \geq \min\{|i - j| > M : i, j \text{ are } p_t \text{ holes of } \alpha\} > N + 2|\pi|.$$

Therefore, $|i - j| \geq |i' - j'| - 2|\pi| > N$, which means $\mathcal{O}'$ has weakly separated holes.

The proof of the other direction is symmetrical. □

The class of Toeplitz subshifts having unbounded block gaps is also invariant under the topological conjugacy raletion.

LEMMA 3.7. *Assume $\mathcal{O}$ is a Toeplitz subshift having unbounded block gaps w.r.t. $(p_t)_{t\in\mathbb{N}}$, but not having weakly separated holes. Then there exists T such that for all $t > T$, we have $L_{p_t}(\mathcal{O}) = max\{L_{p_t,n+1}(\mathcal{O}) - L_{p_t,n}(\mathcal{O}) : n \in \mathbb{N}^+\}$.*

PROOF. Assume $\mathcal{O}$ is a Toeplitz subshift satisfying the hypothesis. Assume toward a contradiction that there is a subsequence $(t_k)_{k\in\mathbb{N}}$ such that $L_{p_{t_k}}(\mathcal{O}) = L_{p_{t_k},1}(\mathcal{O})$. Then $\lim_{k\to\infty} L_{p_{t_k},1}(\mathcal{O}) = \infty$, which implies that $\mathcal{O}$ has separated holes. Hence, $\mathcal{O}$ has weakly separated holes, a contradiction. □

PROPOSITION 3.8. *If $\mathcal{O}$ and $\mathcal{O}'$ are topologically conjugate Toeplitz subshifts with period struture $(p_t)_{t\in\mathbb{N}}$, then $\mathcal{O}$ has unbounded block gaps w.r.t. $(p_t)_{t\in\mathbb{N}}$ if and only if $\mathcal{O}'$ does.*

PROOF. By symmetry, we only need to show one direction.

Since having weakly separated holes is invariant under topological conjugacy, we can assume that $\mathcal{O}$ has unbounded block gaps w.r.t. $(p_t)_{t\in\mathbb{N}}$, but doesn't have weakly separated holes. By lemma 3.7, there exists T_0 such that for all $t > T_0$,

$$L_{p_t}(\mathcal{O}) = max\{L_{p_t,n+1}(\mathcal{O}) - L_{p_t,n}(\mathcal{O}) : n \in \mathbb{N}^+\}.$$

For any $N > 0$, let $T_1 > T_0$ be such that for all $t > T_1$,

$$L_{p_t}(\mathcal{O}) > N + 4|\pi|.$$

For each $t > T_1$, let $m_t \in \mathbb{N}^+$ be such that

$$L_{p_t}(\mathcal{O}) = L_{p_t,m_t+1}(\mathcal{O}) - L_{p_t,m_t}(\mathcal{O}).$$

By the lemma 3.5, we can let m'_t be the minimal integer such that

$$|L_{p_t,m_t+1}(\mathcal{O}) - L_{p_t,m'_t+1}(\mathcal{O}')| \leq 2|\pi|.$$

Since $L_{p_t,m'_t}(\mathcal{O}') < L_{p_t,m'_t+1}(\mathcal{O}')$, and by the minimality of m'_t, we know that

$$L_{p_t,m'_t}(\mathcal{O}') < L_{p_t,m_t+1}(\mathcal{O}) - 2|\pi|.$$

Claim: For each $t > n_1$, $L_{p_t,m'_t}(\mathcal{O}') \leq L_{p_t,m_t}(\mathcal{O}) + 2|\pi|$.

PROOF. Assume toward a contradiction that

$$L_{p_t,m'_t}(\mathcal{O}') > L_{p_t,m_t}(\mathcal{O}) + 2|\pi|.$$

By the lemma 3.5, there exists m''_t such that

$$|L_{p_t,m'_t}(\mathcal{O}') - L_{p_t,m''_t}(\mathcal{O})| \leq 2|\pi|.$$

Then,

$$L_{p_t,m''_t}(\mathcal{O}) \geq L_{p_t,m'_t}(\mathcal{O}') - 2|\pi| > L_{p_t,m_t}(\mathcal{O}),$$

$$L_{p_t,m''_t}(\mathcal{O}) \leq L_{p_t,m'_t}(\mathcal{O}') + 2|\pi| < L_{p_t,m_t+1}(\mathcal{O}).$$

Hence, $L_{p_t,m_t}(\mathcal{O}) < L_{p_t,m''_t}(\mathcal{O}) < L_{p_t,m_t+1}(\mathcal{O})$, contradicting the definition of $L_{p_t,m_t+1}(\mathcal{O})$.

□

Therefore, for all $t > n_1$, we have

$$\begin{aligned} L_{p_t}(\mathcal{O}) &\geq L_{p_t,m'_t+1}(\mathcal{O}') - L_{p_t,m'_t}(\mathcal{O}') \\ &\geq L_{p_t,m_t+1}(\mathcal{O}) - L_{p_t,m_t}(\mathcal{O}) - 4|\pi| \\ &> N, \end{aligned}$$

which means $\mathcal{O}'$ has unbounded block gaps w.r.t. $(p_t)_{t\in\mathbb{N}}$. □

Example 3.9. We present an example to show that there exists a Toeplitz sequence not having weakly separated holes. First, let $w_0 = 0***$, $p_0 = 4$, and α_0 be the sequence defined by $\alpha[4k, 4k+4) = w_0$, for all $k \in \mathbb{Z}$. Assume w_n, p_n, α_n have been defined, and w_n has at least 3 unfilled positions. In w_n, let i_1 be the smallest unfilled position, i_2 be the second largest unfilled position, and i_3 be the largest unfilled position. For $0 \leq j < p_n$, define:

$$w_{n,1}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = i_2, \\ 0, & \text{otherwise.} \end{cases}$$

$$w_{n,2}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq * \text{ or } j = i_2, \\ 1, & \text{otherwise.} \end{cases}$$

$$w_{n,3}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = i_3, \\ 0, & \text{otherwise.} \end{cases}$$

$$w_{n,4}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = i_3, \\ 1, & \text{otherwise.} \end{cases}$$

$$w_{n,5}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = i_1, \\ 0, & \text{otherwise.} \end{cases}$$

Define $w_{n+1} = w_{n,1}w_{n,2}w_{n,3}w_{n,4}w_nw_{n,5}$, $p_{n+1} = |w_{n+1}|$, and α_{n+1} is the sequence defined by $\alpha_{n+1}[kp_{n+1}, (k+1)p_{k+1}) = w_{n+1}$ for all $k \in \mathbb{Z}$. In w_{n+1}, there are at least 3 positions unfilled(in fact, at least 8 unfilled positions). Let $\alpha = \lim_{n\to\infty} \alpha_n$, we know α is a Toeplitz sequence since every the position has been filled in some periodic manner. It is clear $(p_n)_{n\in\mathbb{N}}$ is a period structure of α. We can check every unfilled position in α_n is a p_n hole of α, and α does not have weakly separated holes.

Example 3.10 (Kaya). This example shows that having growing blocks depends on the choice of the period structures.

Consider a period structure $(2^t \cdot 5)_{t\in\mathbb{N}}$, we define a Toeplitz sequences by the following inductive process:

For $t = 0$, we start with the 5-skeleton consisting of the repeated blocks $0 * * * 0$.

When t is even, along each interval $[k2^t5, (k+1)2^t5)$, we fill the first two holes with 0, fill the first two holes in $[k2^t5 + 2^{t-1}5, (k+1)2^t5)$ with 1, and fill the rest three holes in $[k2^t5 + 2^{t-1}5, (k+1)2^t5)$ by 101.

When t is odd, along each interval $[k2^t5, (k+1)2^t5)$, we fill the third hole with 1.

The following diagram shows the first four constructions:

$$st.0: \ \ldots 0 * * * 00 * * * 00 * * * 00 * * * 00 * * * 00 * * * 00 * * * 00 * * * 0 \ldots$$

$$st.1: \ \ldots 0 * 1 * 00 * * * 00 * 1 * 00 * * * 00 * 1 * 00 * * * 00 * 1 * 00 * * * 0 \ldots$$

$$st.2: \ \ldots 001000 * * * 00111001010001000 * * * 00111001010 \ldots$$

$$st.3. \ \ldots 001000 * 1 * 00111001010001000 * * * 00111001010 \ldots$$

Eventually, all the positions get filled periodically, and we get a Toeplitz sequence, denote by α. It is easy to check that α doesn' have growing blocks with respect to $(2^t5)_{t\in\mathbb{N}}$. However, α does have growing block with respect to $(4^t5)_{t\in\mathbb{N}}$.

Definition 3.11. Assume $w \in \{0,1,*\}^p \backslash \{0,1\}^p$, define $\alpha \in \{0,1,*\}^{\mathbb{Z}}$ by $\alpha[kp,(k+1)p) = w$, $k \in \mathbb{Z}$. If $\alpha(n) = *$, we call n an unfilled position of α.

For $M > 0$, we say (α, p) has the property $< M$ if for any unfilled positions $i_1 < j_1$ of α, if $j_1 - i_1 > min\{j - i : i < j$ are unfilled positions of $\alpha\}$, then there exist unfilled positions $i_2 < j_2$, such that $0 < (j_1 - i_1) - (j_2 - i_2) < M$.

Proposition 3.12. *Having unbounded block gaps depends on the choice of period structures.*

Proof. We construct such a Toeplitz sequence α and its period structures by induction.

For $n = 1$, define $w_1 = 0 * *$, $p_1 = 3$, and define α_1 by $\alpha_1[3i, 3(i+1)) = w_1$, for all $i \in \mathbb{Z}$.

Assume w_n is defined, let $p_n = |w_n|$, and define α_n as $\alpha_n[ip_n, (i+1)p_n) = w_n$, for all $i \in \mathbb{Z}$.

Let k be the number of unfilled positions of w_n, let $m_1 < m_2 < \cdots < m_{3k}$ be all the unfilled positions of $w_n w_n w_n$. Denote

$$A_n := \{(m_i, m_j) : 0 < m_j - m_i < p_n + m_k - m_1, 1 \leq i \leq k, i < j \leq 3k\}.$$

Define a partial order $\leq_n$ on A_n as follows:

$$\begin{aligned}(m_i, m_j) \leq_n (m_{i'}, m_{j'}) \longleftrightarrow & (m_j - m_i > m_{j'} - m_{i'}) \text{ or} \\ & (m_j - m_i = m_{j'} - m_{i'} \text{ and } m_i \leq m_{i'}).\end{aligned}$$

For each $0 \leq j < |w_n|$, define:

$$w_{n,-1}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = m_1, \\ 0, & \text{otherwise.} \end{cases}$$

$$w_{n,1}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = m_k, \\ 0, & \text{otherwise.} \end{cases}$$

$$w_{n,2}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = m_k, \\ 1, & \text{otherwise.} \end{cases}$$

$$w_{n,3}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = m_{k-1}, \\ 0, & \text{otherwise.} \end{cases}$$

$$w_{n,4}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = m_{k-1}, \\ 1, & \text{otherwise.} \end{cases}$$

Let $|A_n|$ be the cardinal of A_n. For each $0 < l \leq |A_n|$, let (l_1, l_2) be the l-th largest $\leq_n$ element, for each $0 \leq j < 3|w_n|$, define

$$w_{n,4+l}(j) = \begin{cases} 0, & \text{if } l_1 < j < l_2, \text{ and } \alpha_n(j) = *, \\ \alpha_n(j), & \text{otherwise.} \end{cases}$$

If n is odd, define

$$w_{n+1} = w_{n,1}w_{n,2}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,3}w_{n,4}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,-1}.$$

If n is even, define

$$\begin{aligned} w_{n+1} = & w_{n,1}w_{n,2}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,3}w_{n,4}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,5}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,6} \\ & \underbrace{w_n \dots w_n}_{7 \text{ times}} \cdots w_{n,3+|A_n|}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,4+|A_n|}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,-1}. \end{aligned}$$

Let $p_{n+1} = |w_{n+1}|$. Define α_{n+1} as $\alpha_{n+1}[ip_{n+1}, (i+1)p_{n+1}) = w_{n+1}$, $i \in \mathbb{Z}$.

Let $\alpha = \lim_{n\to\infty} \alpha_n$, then α is a Toeplitz sequence since all the positions are filled in a periodic manner. It is clear that both $(p_n)_{n\in 2\mathbb{N}^+}$ and $(p_n)_{n\in 2\mathbb{N}+1}$ are period structures of α. We can check that every unfilled position of α_n is a p_n hole of α inductively. Let $\mathcal{O}$ be the Toeplitz subshift containing α.

Claim 1: α has unbounded block gaps w.r.t. $(p_n)_{n\in 2\mathbb{N}^+}$.

Proof. When n is odd, the largest difference between two consecutive unfilled positions of α_{n+1} is $p_n + m_k - m_1$, the second largest difference between two consecutive unfilled positions of α_{n+1} is p_n. Note that $0 < m_1 < p_{n-1}$, $p_n - p_{n-1} < m_k < p_n$, and $p_n \geq 19p_{n-1}$, we have

$$L_{p_{n+1}}(\mathcal{O}) \geq p_n + m_k - m_1 - p_n = m_k - m_1 > p_n - 2p_{n-1} \geq 17p_{n-1}.$$

Hence, $\lim_{n\to\infty} L_{2n}(\mathcal{O}) = \infty$, and α has unbounded block gaps w.r.t. $(p_n)_{n\in 2\mathbb{N}^+}$. $\square$

Claim 2: For each $n \in \mathbb{N}^+$, (α_n, p_n) has the property < 3.

Proof. (By induction) For $n = 1$, we know the unfilled positions of α_1 are $\{1 + 3k, 2 + 3k : k \in \mathbb{Z}\}$. Assume $i < j$ are unfilled positions of α_1 and $j - i > 1$.

Case 1. $i \equiv 1$ (mode 3), then $i+1 \equiv 2$ (mode 3), and α_1 is unfilled at $i+1$. Let $i' = i+1$, $j' = j$, then $j' - i' = j - i - 1 > 0$, and $(j-i) - (j'-i') = 1 \in (0,3)$.

Case 2. $i \equiv 2$ (mode 3) and $j \equiv 1$ (mode 3). Then $i - 1 \equiv 1$ (mode 3), $j - 2 \equiv 2$ (mode 3), and α_1 is unfilled at $i-1$ and $j-2$. Let $i' = i-1$, $j' = j-2$, then $j' - i' = j - i - 1 > 0$, and $(j-i) - (j'-i') = 1 \in (0,3)$.

Case 3. $i \equiv 2$ (mode 3) and $j \equiv 2$ (mode 3). Then $j - 1 \equiv 1$ (mode 3), and α_1 is unfilled at $j-1$. Let $i' = i$, $j' = j-1$, then $j' - i' = j - i - 1 > 0$, and $(j-i) - (j'-i') = 1 \in (0,3)$.

Hence, (α_1, p_1) has the property < 3.

Assume the statement in the claim holds for n.

To show (α_{n+1}, p_{n+1}) has the property < 3.

Assume $i < j$ are unfilled positions of α_1 and $j - i > 1$. Then, i, j are unfilled positions of α_n, apply the induction hypothesis, there exist $i' < j'$, unfilled positions of α_n such that $0 < (j-i) - (j'-i') < 3$.

If i' and j' are unfilled positions of α_{n+1}, done.

Now assume i' is filled of α_{n+1}. Then $i' + 3p_n$, $i' + 4p_n$, $i' + 5p_n$, $i' + 6p_n$ are all unfilled positions of α_{n+1}. If $j' + 3p_n$ is unfilled of α_{n+1}, then $i' + 3p_n < j' + 3p_n$ are unfilled positions of α_{n+1} such that

$$0 < (j-i) - ((j' + 3p_n) - (i' + 3p_n)) = (j-i) - (j'-i') < 3.$$

If $j' + 3p_n$ is filled of α_{n+1}, then $j' + 6p_n$ is unfilled of α_{n+1}. Then $i' + 6p_n < j' + 6p_n$ are unfilled positions of α_{n+1} such that

$$0 < (j-i) - ((j' + 6p_n) - (i' + 6p_n)) = (j-i) - (j'-i') < 3.$$

□

Claim 3: α does not have unbounded block gaps w.r.t. $(p_n)_{n \in 2\mathbb{N}+1}$.

Proof. If n is even, the largest difference between two consecutive unfilled positions of α_{n+1} is $p_n + m_k - m_1$. For any consecutive unfilled positions $i < j$ of α_{n+1}, i, j are unfilled positions of α_n. Since (α_n, p_n) has the property < 3 by the claim 2 above, if $j - i > 1$, there exist unfilled positions $i' < j'$ of α_n such that $0 < (j-i) - (j'-i') < 3$. Let $i'' \in [0, p_n)$ be such that $i'' \equiv i' \pmod{p_n}$, let $j'' = j' - i' + i''$, then i'' and $j'' \equiv j' \pmod{p_n}$ are unfilled of α_n, note that $0 < j'' - i'' = j' - i' < p_n + m_k - m_1$, then $j'' \in [0, 3p_n)$. Therefore, $(i'', j'') \in A_n$, say $(i'', j''))$ is the l-th largest $\leq_n$ element. By our construction of $w_{n,4+l}$ and α_{n+1}, we know $(17 + 8(l-1))p_n + i''$ and $(17 + 8(l-1))p_n + j''$ are consecutive unfilled positions of α_{n+1}, and

$$
\begin{aligned}
&(j-i) - (((17 + 8(l-1))p_n + j'') - ((17 + 8(l-1))p_n + i'')) \\
&= (j-i) - (j'' - i'') \\
&= (j-i) - (j' - i') \in (0, 3).
\end{aligned}
$$

So, $L_{p_{n+1}}(\mathcal{O}) < 3$ for all even number n. Hence, α does not have unbounded bolck gaps w.r.t. $(p_n)_{n \in 2\mathbb{N}+1}$. □

□

Theorem 3.13 (Dirichlet, 1837). *If $a, b \in \mathbb{N}$ are coprime, then there are infinitely many prime numbers p such that $p \equiv a (mod\ b)$.*

An elementary proof of the Dirichlet theorem was given by Atle Selberg in [29].

Theorem 3.14. *There exists a Toeplitz sequence not having unbounded block gaps.*

Proof. For $n = 1$, define $w_1 = 0 * *$, $p_1 = q_1 = 3$, and define α_1 by $\alpha_1[3i, 3(i+1)) = w_1$, for all $i \in \mathbb{Z}$.

Assume w_n has been defined, let $p_n = |w_n|$, $q_n = \frac{p_n}{p_{n-1}}$ if $n > 1$, and define α_n as $\alpha_n[ip_n, (i+1)p_n) = w_n$, $i \in \mathbb{Z}$. Let k be the number of unfilled positions of w_n, let $m_1 < m_2 < \cdots < m_{3k}$ be all the unfilled positions of $w_n w_n w_n$. Define A_n, the partial order $\leq_n$ on A_n, words $w_{n,-1}, w_{n,1}, w_{n,2}, \ldots, w_{n,4+|A_n|}$, and sequence α_n in the same way as in proposition 3.12.

Define

$$w_{n+1} = w_{n,1}w_{n,2}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,3}w_{n,4}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,5}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,6}\underbrace{w_n \dots w_n}_{7 \text{ times}}$$

$$\cdots w_{n,3+|A_n|}\underbrace{w_n \dots w_n}_{7 \text{ times}} w_{n,4+|A_n|}\underbrace{w_n \dots w_n}_{l \text{ times}} w_{n,-1},$$

where l is the least integer such that $l \geq 7$, $q_{n+1} \equiv 1(mod\ p_n)$, and $q_{n+1} = \frac{p_{n+1}}{p_n}$ is prime. Such l must exist because of the Dirichlet's Theorem.

Define α_{n+1} as $\alpha_{n+1}[ip_{n+1}, (i+1)p_{n+1}) = w_{n+1}$, $i \in \mathbb{Z}$. Let $\alpha = \lim_{n\to\infty} \alpha_n$, it is clear that α is a Toeplitz sequence, and from the proof in proposition 3.12, we know that α does not have unbounded block gaps w.r.t. $(q_1q_2 \dots q_n)_{n\in\mathbb{N}^+}$.

Claim: For any integer $t \geq 1$, for any finitely many integers $a_1, a_2, \dots, a_m$, if $t+1 < a_1 < a_2 < \dots < a_m$, then $Per_{p_tq_{a_1}q_{a_2}\cdots q_{a_m}}(\alpha) = Per_{p_t}(\alpha)$.

Proof. We only need to prove that for $m \geq 1$, $Per_{p_tq_{a_1}q_{a_2}\cdots q_{a_m}}(\alpha) \subseteq Per_{p_t}(\alpha)$. Assume toward a contradiction that there exists an $i \in \mathbb{Z}$ such that $i \in Per_{p_tq_{a_1}q_{a_2}\cdots q_{a_m}}(\alpha) \setminus Per_{p_t}(\alpha)$. Since $q_{n+1} \equiv 1(mod\ p_n)$ for all $n \in \mathbb{N}^+$, there exists $k' \in \mathbb{Z}$ such that $p_tq_{a_1}q_{a_2}\dots q_{a_m} = p_t + k'p_{t+1}$. Then for all $k \in \mathbb{Z}$, we have

$$\begin{aligned}\alpha_{t+1}(i + kp_tq_{a_1}q_{a_2}\dots q_{a_m}) &= \alpha_{t+1}(i + kp_t + kk'p_{t+1}) \\ &= \alpha_{t+1}(i + kp_t)\end{aligned}$$

Since $i \notin Per_{p_t}(\alpha)$, we have $\alpha_t(i + kp_t) = *$ for all $k \in \mathbb{Z}$. Let $0 \leq i' < p_t$ and $k'' \in \mathbb{Z}$ be such that $i = i' + k''p_t$. Now consider the filling process from step t to $t+1$ by our inductive construction.

Case 1. $\alpha_{t+1}(i') = *$.

By the definition of α_{t+1}, we have $\alpha_{t+1}(i' + 9p_t) = 0$, and $\alpha_{t+1}(i' + 10p_t) = 1$. Then $0 = \alpha_{t+1}(i+(9-k'')p_t) = \alpha_{t+1}(i+(9-k'')p_tq_{a_1}q_{a_2}\dots q_{a_m})$, so $\alpha(i+(9-k'')p_tq_{a_1}q_{a_2}\dots q_{a_m}) = 0$. Similarly, we have $\alpha(i + (10-k'')p_tq_{a_1}q_{a_2}\dots q_{a_m}) = 1$. Since $i \in Per_{p_tq_{a_1}q_{a_2}\cdots q_{a_m}}(\alpha)$, we have $\alpha(i + (9-k'')p_tq_{a_1}q_{a_2}\dots q_{a_m}) = \alpha(i + (10-k'')p_tq_{a_1}q_{a_2}\dots q_{a_m})$, then $0 = 1$, a contradiction.

Case 2. $\alpha_{t+1}(i') \neq *$.

In this case, we have $\alpha_{t+1}(i') = 0$, and $\alpha_{t+1}(i' + p_t) = 1$. Then $0 = \alpha_{t+1}(i - k''p_t) = \alpha_{t+1}(i - k''p_t q_{a_1} q_{a_2} \dots q_{a_m})$, so $\alpha(i - k''p_t q_{a_1} q_{a_2} \dots q_{a_m}) = 0$. Similarly, we can get $\alpha(i + (1 - k'')p_t q_{a_1} q_{a_2} \dots q_{a_m}) = 1$. Since $\alpha(i - k''p_t q_{a_1} q_{a_2} \dots q_{a_m}) = \alpha(i + (1 - k'')p_t q_{a_1} q_{a_2} \dots q_{a_m})$, which implies $0 = 1$, a contradiction.

Hence, such i cannot exist, and $Per_{p_t q_{a_1} q_{a_2} \cdots q_{a_m}}(\alpha) = Per_{p_t}(\alpha)$. □

Now assume $(r_n)_{n \in \mathbb{N}^+}$ is a period structure for α, then for all large enough n, there exists an integer $t \geq 1$, and finitely many integers $a_1, a_2, \dots, a_m$ with $t+1 < a_1 < a_2 < \cdots < a_m$, such that $r_n = p_t$ or $r_n = p_t q_{a_1} q_{a_2} \dots q_{a_m}$. By the claim above and the fact that α does not have unbounded block gaps w.r.t. $(p_n)_{n \in \mathbb{N}^+}$, it follows straightforward that α does not have unbounded block gaps w.r.t. $(r_n)_{n \in \mathbb{N}^+}$. □

Remark. If $1 < a_1 < a_2 < \cdots < a_m$, then $q_{a_1} q_{a_2} \dots q_{a_m} \equiv 1 (mod\ q_1)$. Using the same idea as the proof in the previous claim, we can show that $Per_{q_{a_1} q_{a_2} \cdots q_{a_m}}(\alpha) = \emptyset$.

THEOREM 3.15. *The class of Toeplitz subshifts with growing blocks is not invariant under topological conjugacy.*

PROOF. We only need to show there exist two topologically conjugate Toeplitz subshifts, one has growing blocks w.r.t. any of its period structure, while the other doesn't have growing blocks w.r.t. any of its period structures.

For n=1, let $w_1 = 0 * 0 * 0$, $p_1 = q_1 = 5$, and define α_1 by $\alpha_1[5i, 5(i+1)) = w_1$, for all $i \in \mathbb{Z}$.

Assume w_n has been defined, let $p_n = |w_n|$, $q_n = \frac{p_n}{p_{n-1}}$ if $n > 1$, and define α_n as $\alpha_n[ip_n, (i+1)p_n) = w_n$, $i \in \mathbb{Z}$. Assume w_n satisfies the following requirements:

(1) if $n > 1$, then q_n is prime and $q_n \equiv 1 (mod\ p_n)$;

(2) there exists a unique k such that $w_n[5k, 5(k+1)) = w_1$, denote this unique number by k_0.

Let k_1 be the largest unfilled position of w_n, let k_2 be the smallest unfilled position of w_n. Define words $w_{n,1}$, $w_{n,2}$, $w_{n,3}$, $w_{n,4}$ and $w_{n,5}$ as follows:

$$w_{n,1}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = k_1, \\ 0, & \text{otherwise.} \end{cases}$$

$$w_{n,2}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = k_1, \\ 1, & \text{otherwise.} \end{cases}$$

$$w_{n,3}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = k_2, \\ 0, & \text{otherwise.} \end{cases}$$

$$w_{n,4}(j) = \begin{cases} w_n(j), & \text{if } w_n(j) \neq *, \text{ or } j = k_2, \\ 1, & \text{otherwise.} \end{cases}$$

$$w_{n,5}(j) = \begin{cases} 0, & \text{if } w_n(j) = *, \text{ and } j \in [0, 5k_0)) \cup [5(k_0+1), p_n), \\ w_n(j), & \text{otherwise.} \end{cases}$$

Define

$$w_{n+1} = w_{n,1}w_{n,2}w_{n,3}w_{n,4}w_{n,5}\underbrace{w_3 \ldots w_3}_{l \text{ times}},$$

where l is the least integer such that $l \geq 1$, $q_{n+1} := \frac{p_{n+1}}{p_n}$ is prime, and $q_{n+1} \equiv 1(mod\ p_n)$, here $p_{n+1} := |w_{n+1}|$. Again, l must exist due to Dirichlet's Theorem. Define α_{n+1} as $\alpha_{n+1}[ip_{n+1}, (i+1)p_{n+1}) = w_{n+1}$, $i \in \mathbb{Z}$.

Let $\alpha = \lim_{n\to\infty} \alpha_n$, it is clear that α is a Toeplitz sequence not having growing blocks w.r.t.$(p_n)_{n\in\mathbb{N}^+}$.

Repeat the construction above with the initial word $w'_1 = 0 * *00$, we will obtain a Toeplitz sequence β. It is easy to check β has growing blocks w.r.t. $(p_n)_{n\in\mathbb{N}^+}$.

Carrying out the idea of the proof in theorem 3.14, we can show that α doesn't have growing blocks w.r.t. any period structure of it, while β has growing blocks w.r.t. every period structure of it.

Define a map $\Phi : \{0 * 0 * 0 : \text{ each } * \text{ is 0 or 1 }\} \to \{0 * *00 : \text{ each } * \text{ is 0 or 1 }\}$ as follows:

$$\Phi(00000) = 00000,\ \ \Phi(00010) = 00100,$$

$$\Phi(01000) = 01000,\ \ \Phi(01010) = 01100.$$

Then Φ satisfies $\beta[5i, 5(i+1)) = \Phi(\alpha[5i, 5(i+1)))$, for all $i \in \mathbb{Z}$. By theorem 2.4, we know that $\overline{Orb(\alpha)}$ and $\overline{Orb(\beta)}$ are topologically conjugate.

$\square$

CHAPTER 4

THE COMPLEXITY OF TOPOLOGICAL CONJUGACY OF TOEPLITZ SUBSHIFTS

4.1. Topological Conjugacy on Toeplitz Subshifts is a Countable Borel Equivalence Relation

PROPOSITION 4.1. *The topological conjugacy relation of Toeplitz subshifts over any finite alphabet* $\mathfrak{n}$ *is a countable Borel equivalence relation.*

PROOF. Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable topological base on $\mathfrak{n}^{\mathbb{Z}}$. Let E denote the topological conjugacy relation of Toeplitz subshifts over $\mathfrak{n}$. By theorem 2.12, there is a Borel function f defined on the space of Toeplitz subshifts such that for each Toeplitz subshift $\mathcal{O}$, $f(\mathcal{O})$ is a Toeplitz sequence in $\mathcal{O}$. By corollary 2.5, we have

$$\begin{aligned}(\mathcal{O},\mathcal{O}')\in E &\iff \exists m\exists\phi\in Sym(\mathfrak{n}^m)\hat{\phi}(f(\mathcal{O}))\in\mathcal{O}'\\ &\iff \exists m\exists\phi\in Sym(\mathfrak{n}^m)\forall n[\hat{\phi}(f(\mathcal{O}))\in U_n\Rightarrow U_n\cap\mathcal{O}'\neq\emptyset]\end{aligned}$$

So E is Borel and since $\underset{m\in\mathbb{N}}{\cup} Sym(\mathfrak{n}^m)$ is countable, it follows that E is a countable Borel equivalence relation. □

REMARK 4.2. In fact, Clemens in [3] showed that the topological conjugacy on the space of subshifts over finite alphabet is a countable Borle equivalence relation.

4.2. The Topological Conjugacy Relation on Toeplitz Subshifts is Nonsmooth

Thomas [32] proved that the topological conjugacy on Toeplitz subshifts is not smooth via a weakly Borel reduction argument. Here, we provide a slightly different but more direct proof in this section.

For every non-eventually constant number $x\in 2^{\mathbb{N}}$, we construct the following sequences inductively.

At step 0, let α_0 be the completely unfilled sequence marked as $\alpha_0(k)=*$ for all $k\in\mathbb{Z}$.

At step $2n+1$, fill the leftmost unfilled position of $\alpha_{2n}[k2^{2n+1},(k+1)2^{2n+1})$ by $x(2n)$ for all $k\in\mathbb{Z}$, and denote the sequence we obtained by α_{2n+1}.

At step $2n+2$, fill the rightmost unfilled position of $\alpha_{2n+1}[k2^{2n+2},(k+1)2^{2n+2})$ by $x(2n+1)$ for all $k \in \mathbb{Z}$, and denote the sequence we obtained by α_{2n+2}.

The diagram below shows our first three constructions:

$$step1 : ...x(0) * x(0) * x(0) * x(0) * x(0) * x(0) * x(0) * x(0) * x(0)...$$

$$step2 : ...x(0) * x(0)x(1)x(0) * x(0)x(1)x(0) * x(0)x(1)x(0) * x(0)x(1)...$$

$$step3 : ...x(0)x(2)x(0)x(1)x(0) * x(0)x(1)x(0)x(2)x(0)x(1)x(0) * x(0)x(1)...$$

We define the corresponding Toeplitz sequence $\widetilde{x}$ to be the limit of the sequences we construct above. Obviously, for each $n \in \mathbb{N}$, $\widetilde{x}$ has exactly one 2^n-hole in $[0,2^n)$ if and only if x is not eventually constant.

Consider a function $g : \mathbb{N}^+ \to \mathbb{N}$ defined as:

$$g(n) = \begin{cases} \frac{2^n-2}{6}, & \text{if } n \text{ is odd,} \\ \frac{5\cdot 2^n-2}{6}, & \text{otherwise.} \end{cases}$$

It is clear that at each step $n \geq 1$, the position we filled in the interval $[0,2^n)$ is exactly $g(n)$.

Consider a function $h : \mathbb{N}^+ \to \mathbb{N}^+$ defined by

$$h(n) = \begin{cases} \frac{2^{n+1}-1}{3}, & \text{if } n \text{ is odd,} \\ \frac{2^n-1}{3}, & \text{otherwise.} \end{cases}$$

It is straightforward that for each non-eventually constant element $x \in 2^{\mathbb{N}}$, for each $n \in \mathbb{N}^+$, and for each $0 \leq i < 2^n$, we have

$$Skel_{2^n}(\tilde{x}, i) = * \iff i = h(n).$$

LEMMA 4.3. *Given $\omega \in 2^{\mathbb{Z}}$ and a not eventually constant element $x \in 2^{\omega}$. If there exist an m_0 and a bijection $\Pi : W_{2^{m_0}}(\tilde{x}) \mapsto W_{2^{m_0}}(\omega)$ such that for all $k \in \mathbb{Z}$, $\omega[k2^{m_0},(k+1)2^{m_0}) = \Pi(\tilde{x}[k2^{m_0},(k+1)2^{m_0}))$, then*

(1) *ω is a Toplitz sequence;*

(2) *If* $Hole_{2^{m_0}}(\widetilde{x}) = Hole_{2^{m_0}}(\omega)$*, then for all* $m \geq m_0$*, we have* $Hole_{2^m}(\widetilde{x}) = Hole_{2^m}(\omega)$*.*

PROOF. (1) For any $i \in \mathbb{Z}$, let $k_0 \in \mathbb{Z}$ be such that $i \in [k_0 2^{m_0}, (k_0+1)2^{m_0})$. Since $\tilde{x}$ is a Toeplitz sequence, there exists $m_1 \in \mathbb{N}$ such that for all $k \in \mathbb{Z}$, $\tilde{x}[k_0 2^{m_0}, (k_0+1)2^{m_0}) = \tilde{x}[k_0 2^{m_0} + k2^{m_1}, (k_0+1)2^{m_0} + k2^{m_1})$. Then, for all $k \in \mathbb{Z}$,

$$\begin{aligned}\omega[k_0 2^{m_0}, (k_0+1)2^{m_0}) &= \Pi(\tilde{x}[k_0 2^{m_0}, (k_0+1)2^{m_0}))\\ &= \Pi(\tilde{x}[k_0 2^{m_0} + k2^{m_1}, (k_0+1)2^{m_0} + k2^{m_1}))\\ &= \omega[k_0 2^{m_0} + k2^{m_1}, (k_0+1)2^{m_0} + k2^{m_1})\end{aligned}$$

So, $\omega(i) = \omega(i + k2^{m_1})$ for all $k \in \mathbb{Z}$, which implies that ω is a Toeplitz sequence.

(2)Assume that $Hole_{2^{m_0}}(\widetilde{x}) = Hole_{2^{m_0}}(\omega)$. By a symmetric argument, we only need to prove $Hole_{2^m}(\widetilde{x}) \subseteq Hole_{2^m}(\omega)$ for all $m \geq m_0$.

For each $k \in \mathbb{Z}$, let l be a 2^m-hole of $\widetilde{x}$ in $[k2^m, (k+1)2^m)$, $i \in \mathbb{N}$ be such that $l \in [i \cdot 2^{m_0}, (i+1) \cdot 2^{m_0})$, and k_0 be such that $\widetilde{x}(l) \neq \widetilde{x}(l + k_0 \cdot 2^m)$. Then $\widetilde{x}[i \cdot 2^{m_0}, (i+1) \cdot 2^{m_0}) \neq \widetilde{x}[i \cdot 2^{m_0} + k_0 \cdot 2^m, (i+1) \cdot 2^{m_0} + k_0 \cdot 2^m)$. Therefore,

$$\begin{aligned}\omega[i \cdot 2^{m_0}, (i+1) \cdot 2^{m_0}) &= \Pi(\tilde{x}[i \cdot 2^{m_0}, (i+1) \cdot 2^{m_0})\\ &\neq \Pi(\tilde{x}[[i \cdot 2^{m_0} + k_0 \cdot 2^m, (i+1) \cdot 2^{m_0} + k_0 \cdot 2^m))\\ &= \omega[[i \cdot 2^{m_0} + k_0 \cdot 2^m, (i+1) \cdot 2^{m_0} + k_0 \cdot 2^m)).\end{aligned}$$

Since $Hole_{2^m}(\widetilde{x}) \subseteq Hole_{2^{m_0}}(\widetilde{x}) = Hole_{2^{m_0}}(\omega)$, and $\widetilde{x}$ has a unique 2^{m_0}-hole in each interval of length 2^{m_0}, then it must be $\omega(l) \neq \omega(k_0 \cdot 2^m + l)$, namely, $l \in Hole_{2^m}(\omega)$. □

We define an equivalence relation E_0^* on $2^{\mathbb{N}}$ as xE_0^*y if and only if xE_0y or $(1-x)E_0y$, where $1-x$ is the sequence defined as $(1-x)(n) = 1 - x(n)$ for all $n \in \mathbb{N}$.

Let $X = \{x \in 2^{\mathbb{N}} : x \text{ is non-eventually constant}\}$, $Y = \{\overline{Orb(\tilde{x})} : x \in X\}$.

THEOREM 4.4. *Let* $x, y \in Y$*. Then* $\overline{Orb(\tilde{x})}$ *and* $\overline{Orb(\tilde{y})}$ *are topologically conjugate if and only if* xE_0^*y*.*

PROOF. The sufficiency is trivial, and we only need to prove the necessity.

Let π be a Topological conjugacy from $\overline{Orb(\widetilde{x})}$ to $\overline{Orb(\widetilde{y})}$, let $\omega = \pi(\tilde{x})$. Then there exist m_0 and a bijetion $\Pi : W_{2^{m_0}}(\tilde{x}) \mapsto W_{2^{m_0}}(\omega)$ such that for all $k \in \mathbb{Z}$, $\Pi(\tilde{x}[k2^{m_0}, (k+1)2^{m_0})) = \omega[k2^{m_0}, (k+1)2^{m_0})$.

Note that $W_{2^{m_0}}(\tilde{x}) = \{a * b : * \text{ is } 0 \text{ or } 1\}$, $W_{2^{m_0}}(\widetilde{y}) = \{c * d : * \text{ is } 0 \text{ or } 1\}$, where a, b, c, d are determined by $x[0, m_0 - 1]$ and $y[0, m_0 - 1]$. Let $k_0 \in [0, 2^{m_0})$ be such that $\omega \in \overline{A(\tilde{y}, 2^{m_0}, k_0)}$, then $Skel_{2^{m_0}}(\omega) = \sigma^{k_0} Skel_{2^{m_0}}(\widetilde{y})$. Define $\Pi' : W_{2^{m_0}}(\omega) \to W_{2^{m_0}}(\sigma^{-k_0}(\omega))$ as

$$\Pi'(\omega[k2^{m_0}, (k+1)2^{m_0})) = \sigma^{-k_0}(\omega)[k2^{m_0}, (k+1)2^{m_0}), \forall k \in \mathbb{Z}.$$

It is clear that Π' is a bijection.

Define $\Pi'' : W_{2^{m_0}}(\tilde{x}) \to W_{2^{m_0}}(\sigma^{-k_0}(\omega))$ as

$$\Pi''(\tilde{x}[k2^{m_0}, (k+1)2^{m_0})) = \sigma^{-k_0}(\omega)[k2^{m_0}, (k+1)2^{m_0}), \forall k \in \mathbb{Z}.$$

Then $\Pi'' = \Pi' \circ \Pi$ is also a bijection.

Replace ω by $\sigma^{-k_0}(\omega)$, we can assume $W_{2^{m_0}}(\omega) = \{c * d : * \text{ is } 0 \text{ or } 1\}$.

Case 1: $\Pi(a0b) = c0d$, and $\Pi(a1b) = c1d$.

For all $m \geq m_0$, by lemma 4.3 we know the unique 2^m-hole of ω in $[0, 2^m)$ is the same as the hole of $\widetilde{x}$, which is the same as that of $\tilde{y}$. By lemma 2.3 in [?], we know that $\{\overline{A(\tilde{y}, 2^m, k)} : 0 \leq k < 2^m\}$ is a partition of $\overline{Orb(\tilde{y})}$, so $\omega \in \overline{A(\tilde{y}, 2^m, 0)}$ for all $m \geq m_0$. Then $\omega \in \bigcap_{m \geq m_0} \overline{A(\tilde{y}, 2^m, 0)} = \{\tilde{y}\}$. So, $\omega = \tilde{y}$, $x(m) = y(m)$ for $m \geq m_0$, which implies xE_0y.

Case 2. $\Pi(a0b) = c1d$, and $\Pi(a1b) = c0d$.

In this case, we can show $(1 - x)E_0y$ with a similar argument.

Hence, xE_0^*y. □

COROLLARY 4.5. *The topological conjugacy on Y is Borel bireducible with E_0.*

PROOF. It is clear that E_0^* is a hyperfinite equivalence relation, hence Borel bireducible to E_0. We only need to show that the topological conjugacy on Y is Borel bireducible with E_0^* on X.

Assume x, y are not eventually constant, and $\overline{Orb(\tilde{x})} = \overline{Orb(\tilde{y})}$, then we have $\tilde{y} \in \bigcap_{n\in\mathbb{N}^+} \overline{A(\tilde{x}, 2^n, 0)} = \{\tilde{x}\}$. Therefore, $x = y$.

It is easy to check the map $f : X \to Y$ defined as

$$f(x) = \overline{Orb(\tilde{x})},$$

is a Borel reduction from E_0^* to the topologically conjugate relation on Y by theorem 4.4.

For the other direction, recall that h and g are functions coding the filled positions and unfilled positions respectively. Define $f_1 : Y \to 2^{\mathbb{N}}$ by

$$f_1(\mathcal{O}) = \bigcap_{n\in\mathbb{N}} A_n,$$

where each $A_n \in Parts(\mathcal{O}, 2^n)$, $Skel_{2^n}(A_n)(h(n)) = *$.

Define $f_2 : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$f_2(x)(n) = x(g(n)), n \in \mathbb{N}.$$

It is obvious that both f_1 and f_2 are Borel functions, and $f_2 \circ f_1$ maps $\mathcal{O} \in Y$ to a sequence $x \in X$ such that $\mathcal{O} = \overline{Orb(\tilde{x})}$. Again, by theorem 4.4, $f_2 \circ f_1$ witnesses that the topological conjugacy on Y is Borel reducible to E_0^* on X.

□

4.3. The Topological Conjugacy on $\mathcal{T}_1$ is Hyperfinite

It is well-known that the class of Toeplitz subshifts on $\mathfrak{n}^{\mathbb{Z}}$ is a standard Borel space. It turns out that the class of Toeplitz subshifts on $\mathfrak{n}^{\mathbb{Z}}$ having unbounded block gaps is also a standard Borel space.

PROPOSITION 4.6. *The class of Toeplitz subshifts over $\mathfrak{n}^{\mathbb{Z}}$ having unbounded block gaps is Borel.*

PROOF. For each Toeplitz subshift $\mathcal{O}$, let $\tau(\mathcal{O}) = (p_t)_{t\in\mathbb{N}}$ be the natural factorization of $\mathcal{O}$. For each $t \in \mathbb{N}^+$, let $X(\mathcal{O}, p_t)$ be the set defined as

$$X(\mathcal{O}, p_t) = \{q : p_t | q, q | \tau(\mathcal{O})_n \text{ for some n, and } L_q(\mathcal{O}) > L_{p_t}(\mathcal{O})\}.$$

Define a function $\phi : \mathcal{T} \to \mathbb{N}^{\mathbb{N}}$ inductively as follows,

$$\phi(\mathcal{O})_0 = \tau(\mathcal{O})_0,$$

$$\phi(\mathcal{O})_{t+1} = \begin{cases} \min X(\mathcal{O}, \phi(\mathcal{O})_t), & \text{if } X(\mathcal{O}, \phi(\mathcal{O})_t) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that ϕ is Borel, and $\mathcal{O}$ has unbounded block gaps if and only if $\phi(\mathcal{O})$ is not eventually zeroes. Hence, the class of Toeplitz subshifts having unbounded block gaps is Borel.

□

DEFINITION 4.7. Define the set $\mathcal{T}_1$ by $\mathcal{O} \in \mathcal{T}_1$ if and only if $\mathcal{O}$ is a Toeplitz subshift, $(p_t)_{t\in\mathbb{N}}$ is the natural factorization of $\mathcal{O}$, and there exists $M \in \mathbb{N}$ such that $\lim\limits_{t\to\infty} L_{p_t,s_t+1}(\mathcal{O}) - L_{p_t,s_t}(\mathcal{O}) = \infty$, where $(s_t)_{t\in\mathbb{N}}$ is the sequence

(1) for all $t \in \mathbb{N}$, s_t is the largest integer such that $L_{p_t,s_t+1}(\mathcal{O}) \neq L_{p_t,s_t}(\mathcal{O})$, and $s_t \neq 0$ implies for any $0 \leq i < s_t$, $L_{p_t,i+1}(\mathcal{O}) - L_{p_t,i}(\mathcal{O}) \leq M$, or
(2) for all $t \in \mathbb{N}$, s_t is the smallest integer such that for any $i > s_t$, $L_{p_t,i+1}(\mathcal{O}) - L_{p_t,i}(\mathcal{O}) \leq M$.

We say such a pair M, $(s_t)_{t\in\mathbb{N}}$ witnesses $\mathcal{O} \in \mathcal{T}_1$. Moreover, if $\mathcal{O} \in \mathcal{T}_1$, M, $(s_t)_{t\in\mathbb{N}}$ witnesses $\mathcal{O} \in \mathcal{T}_1$, and if M, $(s_t)_{t\in\mathbb{N}}$ satisfying (1), we say $\mathcal{O} \in \mathcal{T}_1$ is of the first case, and $M, (s_t)_{t\in\mathbb{N}}$ witnesses $\mathcal{O} \in \mathcal{T}_1$ of the first case. Otherwise, $\mathcal{O} \in \mathcal{T}_1$ is of the second case.

It is easy to check that $\mathcal{T}_1$ is a Borel set. If $\mathcal{O} \in \mathcal{T}_1$ is of the first case, we can associate a unique pair $M, (s_t)_{t\in\mathbb{N}}$ in a Borel way, such that $M, (s_t)_{t\in\mathbb{N}}$ witnesses $\mathcal{O} \in \mathcal{T}_1$ of the first case, and if $M' < M$, $(s'_t)_{t\in\mathbb{N}}$ satisfies (1), then $M', (s'_t)_{t\in\mathbb{N}}$ doesn't witness $\mathcal{O} \in \mathcal{T}_1$. If $\mathcal{O} \in \mathcal{T}_1$ is not of the first case, we associate a unique pair $M, (s_t)_{t\in\mathbb{N}}$ in a Borel way too, where $M, (s_t)_{t\in\mathbb{N}}$ witnesses $\mathcal{O} \in \mathcal{T}_1$, and for any $M' < M$, $(s'_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, $M', (s'_t)_{t\in\mathbb{N}}$ doesn't witness $\mathcal{O} \in \mathcal{T}_1$.

If $\mathcal{O}$ has unbounded block gaps w.r.t. $(p_t)_{t\in\mathbb{N}}$. We call $\alpha \in \mathbb{N}^{\mathbb{N}}$ is the E_0-smallest(E_0-largest) real that witnesses $\mathcal{O}$ having unbounded block gaps w.r.t. $(p_t)_{t\in\mathbb{N}}$ if it satisfies the following conditions:

(1) $\lim_{t\to\infty} L_{p_t,\alpha(t)+1}(\mathcal{O}) - L_{p_t,\alpha(t)}(\mathcal{O}) = \infty$;

(2) $\forall\beta \in \mathbb{N}^{\mathbb{N}}$, if $\exists^\infty n,\ \beta(n) < \alpha(n)(\exists^\infty n,\ \beta(n) \geq \alpha(n)$respectively), then $\lim_{t\to\infty} L_{p_t,\beta(t)+1}(\mathcal{O}) - L_{p_t,\beta(t)}(\mathcal{O}) \neq \infty$.

Proposition 4.8. *Assume $\mathcal{O}$ has unbounded block gaps w.r.t. $(p_t)_{t\in\mathbb{N}}$. $\alpha \in \mathbb{N}^{\mathbb{N}}$ is the E_0-smallest(E_0-largest) real that witnesses $\mathcal{O}$ having unbounded block gaps w.r.t. $(p_t)_{t\in\mathbb{N}}$ if and only if there exists $M \in \mathbb{N}$ and $(s_t)_{t\in\mathbb{N}}$ such that*

(1) $\alpha E_0 (s_t)_{t\in\mathbb{N}}$,

(2) $\lim_{t\to\infty} L_{p_t,s_t+1}(\mathcal{O}) - L_{p_t,s_t}(\mathcal{O}) = \infty$,

(3) $\forall t \in \mathbb{N}, \forall 0 \leq i < s_t$ *($\forall t \in \mathbb{N}, \forall i \geq s_t$ respectively), we have* $L_{p_t,i+1}(\mathcal{O}) - L_{p_t,i}(\mathcal{O}) \leq M$.

Proof. $\Longleftarrow$. It is obvious.

$\Longrightarrow$. Assume α is the E_0-smallest real that witnesses $\mathcal{O}$ having unbounded w.r.t $(p_t)_{t\in\mathbb{N}}$. Note that if $\mathcal{O}$ has separated holes, then $M = 0$, $s_t = 0$ for all $t \in \mathbb{N}$ satisfy the requirements. Now assume $\mathcal{O}$ doesn't have separated holes, then $\alpha(n) \neq 0$ for all but finitely many n. Let $n_0 \in \mathbb{N}$ be the first number such that $\alpha(n) > 0$ for all $n \geq n_0$.

Assume toward a contradiction that there doesn't exist M and $(s_t)_{t\in\mathbb{N}}$ satisfying the three requirements listed in the proposition. With this assumption, we can construct a real number inductively which will destroy the E_0-smallestness of α as follows.

Let $m_0 = \max\{1 + L_{p_t}(\mathcal{O}) : t \leq n_0\}$, then there exist $t_0 > n_0$ and $0 \leq i_0 < s_{t_0}$, such that $L_{p_{t_0},i_0+1}(\mathcal{O}) - L_{p_{t_0},i_0}(\mathcal{O}) > m_0$. Define

$$\alpha^0(t) = \begin{cases} i_0, & \text{if } t = t_0, \\ \alpha(t). & \text{otherwise.} \end{cases}$$

Assume we have defined m_k, t_k, i_k, α^k such that $\alpha E_0 \alpha^k$.

Let $m_{k+1} = \max\{1 + L_{p_t}(\mathcal{O}) : t \le t_k\} > m_k$. Then there exist $t_{k+1} > t_k$ and $0 \le i_{k+1} < \alpha^k(t_{k+1})$ such that $L_{p_{t_{k+1}},i_{k+1}+1}(\mathcal{O}) - L_{p_{t_{k+1}},i_{k+1}}(\mathcal{O}) > m_{k+1}$. Define

$$\alpha^k(t) = \begin{cases} i_{k+1}, & \text{if } t = t_{k+1}, \\ \alpha^k(t). & \text{otherwise.} \end{cases}$$

Let $\beta = \lim_{k\to\infty} \alpha^k$. It is clear that $\beta(n) < \alpha(n)$ for infinitely many n, and $\lim_{t\to\infty} L_{p_t,\beta(t)+1}(\mathcal{O}) - L_{p_t,\beta(t)}(\mathcal{O}) = \infty$. A contradiction! □

For each $\mathcal{O} \in \mathcal{T}_1$, let $(p_t)_{t\in\mathbb{N}}$ be its natural factorization, and $M, (s_t)_{t\in\mathbb{N}}$ be the pair we associate with $\mathcal{O}$ as described above. For each $t \in \mathbb{N}$, we denote $Parts_1(\mathcal{O}, p_t) = \{W \in Parts(\mathcal{O}, p_t) : Skel_{p_t}(W)(0) \ne *,\ Skel_{p_t}(W)(-1) = *,$ and $length(W) > L_{p_t,s_t}(\mathcal{O})\}$, where $length(W)$ is the minimal natural number such that $Skel_{p_t}(W)(length(W)) = *$.

Lemma 4.9. *Assume $\mathcal{O}, \mathcal{O}'$ are Toeplitz subshifts, and $\pi : \mathcal{O} \to \mathcal{O}'$ is a topological conjugacy. If there exists a pair, $M, (s_t)_{t\in\mathbb{N}}$, witnessing $\mathcal{O} \in \mathcal{T}_1$, then there exists $M', (s'_t)_{t\in\mathbb{N}}$ witnessing $\mathcal{O}' \in \mathcal{T}_1$, $M' \le 6|\pi| + M$, and $|L_{p_t,s_t+1}(\mathcal{O}) - L_{p_t,s'_t+1}(\mathcal{O}')| \le 2|\pi|$ for all large enough t.*

Proof. We only need to show it is true for $M, (s_t)_{t\in\mathbb{N}}$ witnessing $\mathcal{O} \in \mathcal{T}_1$ of the first case, the other case can be proved with a similar argument.

Assume $\pi : \mathcal{O} \to \mathcal{O}'$ is a topological conjugacy, and $M, (s_t)_{t\in\mathbb{N}}$ witnesses $\mathcal{O} \in \mathcal{T}_1$ of the first case. Let n_0 be such that for all $t > n_0$, we have

$$L_{p_t,s_t+1}(\mathcal{O}) - L_{p_t,s_t}(\mathcal{O}) > 12|\pi| + M.$$

From now, we always assume $t > n_0$. Let

$$A_t := \{m \in \mathbb{N} : L_{p_t,s_t+1}(\mathcal{O}) - 2|\pi| \le L_{p_t,m+1}(\mathcal{O}') \le L_{p_t,s_t+1}(\mathcal{O}) + 2|\pi|\}.$$

By lemma 3.5, we know $A_t \ne \emptyset$. Let $s'_t = \min A_t$.

Case 1. $L_{p_t,s_t}(\mathcal{O}) \le 2|\pi|$.

In this case, we have $0 \le L_{p_t,s'_t}(\mathcal{O}') \le 4|\pi|$. Since otherwise, by the choice of s'_t, we know that

$$4|\pi| < L_{p_t,s'_t}(\mathcal{O}') < L_{p_t,s_t+1}(\mathcal{O}) - 2|\pi|.$$

By lemma 3.5, there exists n such that

$$|L_{p_t,n}(\mathcal{O}) - L_{p_t,s'_t}(\mathcal{O}')| \leq 2|\pi|.$$

Then, we can get the following inequalities

$$2|\pi| < L_{p_t,s'_t}(\mathcal{O}') - 2|\pi| \leq L_{p_t,n}(\mathcal{O}) \leq L_{p_t,s'_t}(\mathcal{O}') + 2|\pi| < L_{p_t,s_t+1}(\mathcal{O}).$$

This contradicts the assumption $L_{p_t,s_t}(\mathcal{O}) \leq 2|\pi|$. Therefore,

$$\begin{aligned} L_{p_t,s'_t+1}(\mathcal{O}') - L_{p_t,s'_t}(\mathcal{O}') &\geq L_{p_t,s_t+1}(\mathcal{O}) - 6|\pi| \\ &> L_{p_t,s_t}(\mathcal{O}) + 6|\pi| + M \\ &\geq 6|\pi| + M. \end{aligned}$$

And for any m, if $0 \leq m < s'_t$, we have

$$L_{p_t,m+1}(\mathcal{O}') - L_{p_t,m}(\mathcal{O}') \leq L_{p_t,m+1}(\mathcal{O}') \leq L_{p_t,s'_t}(\mathcal{O}') \leq 4|\pi|.$$

Case 2. $L_{p_t,s_t}(\mathcal{O}) > 2|\pi|$.

In this case, we have

$$L_{p_t,s'_t}(\mathcal{O}') \leq L_{p_t,s_t}(\mathcal{O}) + 2|\pi|.$$

Since otherwise,

$$L_{p_t,s'_t}(\mathcal{O}') > L_{p_t,s_t}(\mathcal{O}) + 2|\pi|.$$

Apply lemma 3.5 agian, there exists $n \in \mathbb{N}^+$ such that

$$|L_{p_t,n}(\mathcal{O}) - L_{p_t,s'_t}(\mathcal{O}')| \leq 2|\pi|.$$

Then, we have the following inequalities

$$\begin{aligned} L_{p_t,s_t}(\mathcal{O}) &< L_{p_t,s'_t}(\mathcal{O}') - 2|\pi| \\ &\leq L_{p_t,n}(\mathcal{O}) \\ &\leq L_{p_t,s'_t}(\mathcal{O}') + 2|\pi| \\ &< L_{p_t,s_t+1}(\mathcal{O}). \end{aligned}$$

Hence, $s_t < n < s_t + 1$, a contradiction.

By a similar argument, we have

$$L_{p_t,s_t}(\mathcal{O}) - 2|\pi| \leq L_{p_t,s'_t}(\mathcal{O}').$$

Therefore,

$$L_{p_t,s'_t+1}(\mathcal{O}') - L_{p_t,s'_t}(\mathcal{O}') \geq L_{p_t,s_t+1}(\mathcal{O}) - L_{p_t,s_t}(\mathcal{O}) - 4|\pi| > 8|\pi| + M.$$

Claim: In case 2, for any m, if $0 \leq m < s'_t$, we have

$$L_{p_t,m+1}(\mathcal{O}') - L_{p_t,m}(\mathcal{O}') \leq 6|\pi| + M.$$

PROOF. Assume toward a contradiction that there is $0 \leq m < s'_t$ such that

$$L_{p_t,m+1}(\mathcal{O}') - L_{p_t,m}(\mathcal{O}') > 6|\pi| + M.$$

Let $B_t := \{n \in \mathbb{N} : L_{p_t,m+1}(\mathcal{O}') - 2|\pi| \leq L_{p_t,n+1}(\mathcal{O}) \leq L_{p_t,m+1}(\mathcal{O}') + 2|\pi|\}$. By lemma 3.5, we know $B_t \neq \emptyset$, and let $l = \min B_t$.

Subcase 2.1. $L_{p_t,m}(\mathcal{O}') \leq 2|\pi|$.

Using a similar proof in the case 1 can show that $L_{p_t,l}(\mathcal{O}) \leq 4|\pi|$. Then,

$$L_{p_t,l+1}(\mathcal{O}) - L_{p_t,l}(\mathcal{O}) \geq L_{p_t,m+1}(\mathcal{O}') - 6|\pi| > L_{p_t,m}(\mathcal{O}') + M \geq M.$$

This contradicts the assumption that $M, (s_t)_{t\in\mathbb{N}}$ witnesses $\mathcal{O} \in \mathcal{T}_1$ of the first case.

Subcase 2.2. $L_{p_t,m}(\mathcal{O}') > 2|\pi|$.

In this subcase, we can verify that

$$L_{p_t,m}(\mathcal{O}') - 2|\pi| \leq L_{p_t,l}(\mathcal{O}) \leq L_{p_t,m}(\mathcal{O}') + 2|\pi|.$$

Then, we can obtain the following inequalities,

$$\begin{aligned} L_{p_t,l+1}(\mathcal{O}) - L_{p_t,l}(\mathcal{O}) &\geq L_{p_t,m+1}(\mathcal{O}') - L_{p_t,m}(\mathcal{O}') - 4|\pi| \\ &> 2|\pi| + M \\ &\geq M. \end{aligned}$$

This contradicts the assumption that $M, (s_t)_{t\in\mathbb{N}}$ witnesses $\mathcal{O} \in \mathcal{T}_1$ of the first case. $\square$

Let $M' = M + 6|\pi|$. For $t \leq t_o$, let s'_t be the largest integer such that

$$L_{p_t, s'_t+1}(\mathcal{O}') \neq L_{p_t, s_t}(\mathcal{O}'),$$

and $s'_t \neq 0$ implies that for any $0 \leq i < s'_t$,

$$L_{p_t, i+1}(\mathcal{O}) - L_{p_t, i}(\mathcal{O}) \leq M.$$

Then $M', (s'_t)_{t \in \mathbb{N}}$ witnesses $\mathcal{O}' \in \mathcal{T}_1$ of the first case. Moreover, $M' \leq 6|\pi| + M$, and $|L_{p_t, s_t+1}(\mathcal{O}) - L_{p_t, s'_t+1}(\mathcal{O}')| \leq 2|\pi|$ for all $t > n_0$. □

We introduce the following notions from Kaya [18].

For each $\phi \in Sym(\mathfrak{n}^p)$, define $\widehat{\phi}$ to be the homeomorphism on $\mathfrak{n}^{\mathbb{Z}}$ as follows:

$$\widehat{\phi}(\alpha)[kp, (k+1)p) = \phi(\alpha[kp, (k+1)p)),\ \forall \alpha \in \mathfrak{n}^{\mathbb{Z}}, k \in \mathbb{Z}.$$

For each $p \in \mathbb{N}^+$, consider the Borel action of the symmetric group $Sym(\mathfrak{n}^p)$ on the hyperspace $K(\mathfrak{n}^{\mathbb{Z}})$:

$$\phi \cdot W \to \widehat{\phi}(W),\ \forall \phi \in Sym(\mathfrak{n}^p),\ W \in K(\mathfrak{n}^{\mathbb{Z}}).$$

For each $W \in K(\mathfrak{n}^{\mathbb{Z}})$, denote $[W]_p = \{\widehat{\phi}(W) : \phi \in Sym(\mathfrak{n}^p)\}$.

Let $Fin(K(\mathfrak{n}^{\mathbb{Z}})) := \{F \subseteq K(\mathfrak{n}^{\mathbb{Z}}) : F \text{ is nonempty and finite}\}$. $Fin(K(\mathfrak{n}^{\mathbb{Z}}))$ is a Borel space in the polish space $K(\mathfrak{n}^{\mathbb{Z}})$. The following equivalence relation is hypersmooth as pointed out by Kaya.

Definition 4.10. Define an equivalence relation $\sim$ on $\mathbb{N}^{\mathbb{N}} \times (Fin(K(\mathfrak{n}^{\mathbb{Z}})))^{\mathbb{N}}$ by

$$\begin{aligned}(r, (F_i)_{i \in \mathbb{N}}) \sim (r', (F'_i)_{i \in \mathbb{N}}) \iff & (r = r') \wedge \exists i \forall j \geq i \\ & \{[W]_{r_i} : W \in F_i\} = \{[W']_{r_i} : W' \in F'_i\}.\end{aligned}$$

Proposition 4.11. *The Borel equivalence relation $\sim$ defined on $\mathbb{N}^{\mathbb{N}} \times (Fin(K(\mathfrak{n}^{\mathbb{Z}})))^{\mathbb{N}}$ is hypersmooth.*

Proof. For each $i \in \mathbb{N}$, consider a relation $E^{(i)}$ on $\mathbb{N}^{\mathbb{N}} \times (Fin(K(\mathfrak{n}^{\mathbb{Z}})))$ defined by

$$(r, F_i) E^{(i)} (r', F'_i) \iff (r = r') \wedge \{[W]_{r_i} : W \in F_i\} = \{[W']_{r_i} : W' \in F'_i\}.$$

It is clear that each $E^{(i)}$ is a finite Borel equivalence relation, hence smooth. Therefore, the relation E on $(\mathbb{N}^{\mathbb{N}} \times Fin(K(\mathfrak{n}^{\mathbb{Z}})))^{\mathbb{N}}$ defined by

$$(r, F_i)_{i\in\mathbb{N}} E (r', F'_i)_{i\in\mathbb{N}} \iff \exists j \forall i (r, F_i) E^{(i)} (r', F'_i).$$

Then E is a hypersmooth equivalence relation.

Note that the function $f : \mathbb{N}^{\mathbb{N}} \times (Fin(K(\mathfrak{n}^{\mathbb{Z}})))^{\mathbb{N}} \to (\mathbb{N}^{\mathbb{N}} \times Fin(K(\mathfrak{n}^{\mathbb{Z}})))^{\mathbb{N}}$ defined by

$$f((r, (F_i)_{i\in\mathbb{N}})) = (r, F_i)_{i\in\mathbb{N}}$$

is a Borel reduction from $\sim$ to E. Hence $\sim$ is hypersmooth. □

THEOREM 4.12. *The topological conjugate relation on $\mathcal{T}_1$ is hyperfinite.*

PROOF. We only need to show that the topological conjugate relation on $\mathcal{T}_1$ is Borel reducible to $\sim$.

Define $f : \mathcal{T}_1 \to \mathbb{N}^{\mathbb{N}} \times (Fin(K(\mathfrak{n}^{\mathbb{Z}})))^{\mathbb{N}}$ by

$$f(\mathcal{O}) = (\tau(\mathcal{O}), \chi(\mathcal{O})),$$

where $\tau(\mathcal{O})$ is the natural factorization of $\mathcal{O}$, and the sequence $\chi(\mathcal{O})$ is defined as, for each t, if $Parts_1(\mathcal{O}, \tau(\mathcal{O})_t) \neq \emptyset$, then

$$\chi(\mathcal{O})_t = \{\sigma^{\lfloor j/2 \rfloor}[W] : W \in Parts_1(\mathcal{O}, \tau(\mathcal{O})_t), \text{ and } length(W) = j\},$$

otherwise, $\chi(\mathcal{O})_t = \{\mathfrak{n}^{\mathbb{Z}}\}$.

It is easy to check that f is Borel. We only need to show that f is a reduction.

We only need to consider Toeplitz subshifts of the first case since the other case can be proved with a similar argument.

Assume $\mathcal{O}, \mathcal{O}' \in \mathcal{T}_1$ are of the first case, and $\pi : \mathcal{O} \to \mathcal{O}'$ is a topological conjugacy. Then $\mathcal{O}, \mathcal{O}'$ have the same natural factorization, denoted by $(p_t)_{t\in\mathbb{N}}$. Let n_0 be such that $Parts_1(\mathcal{O}, p_{n_0}) \neq \emptyset$. For all $t \geq n_0$, $Parts_1(\mathcal{O}, p_t) \neq \emptyset$.

Let $M, (s_t)_{t\in\mathbb{N}}$, and $M', s'_t)_{t\in\mathbb{N}}$ be the associated pairs such that M, $(s_t)_{t\in\mathbb{N}}$, $M', (s'_t)_{t\in\mathbb{N}}$ witness $\mathcal{O}, \mathcal{O}' \in \mathcal{T}_1$ of the first case respectively. Since

$$\lim_{t\to\infty} L_{p_t,s_t+1}(\mathcal{O}) - L_{p_t,s_t}(\mathcal{O}) = \infty,$$

$$\lim_{t\to\infty} L_{p_t,s'_t+1}(\mathcal{O}') - L_{p_t,s'_t}(\mathcal{O}') = \infty,$$

and for all large enough t (by lemma 4.9),

$$|L_{p_t,s_t+1}(\mathcal{O}) - L_{p_t,s'_t+1}(\mathcal{O}')| \le 2|\pi|,$$

there exists $n_1 \ge n_0$ such that for all $t \ge n_1$, we have the following inequalities,

$$L_{p_t,s_t+1}(\mathcal{O}) - L_{p_t,s_t}(\mathcal{O}) > 4|\pi|,$$

$$L_{p_t,s'_t+1}(\mathcal{O}') - L_{p_t,s'_t}(\mathcal{O}') > 4|\pi|,$$

$$|L_{p_t,s_t+1}(\mathcal{O}) - L_{p_t,s'_t+1}(\mathcal{O}')| \le 2|\pi|.$$

We claim that for all $t \ge n_1$,

$$\{[W]_{p_t} : W \in \chi(\mathcal{O})_t\} = \{[W]_{p_t} : W \in \chi(\mathcal{O}')_t\}.$$

Pick $W \in \chi(\mathcal{O})_t$, then W is of the form $\sigma^{\lfloor j/2\rfloor}[Z]$ for some $Z \in Parts_1(\mathcal{O}, p_t)$ with $length(Z) = j$. Choose a Toeplitz sequence $\alpha \in W$ and set $\beta = \pi(\alpha)$. We know that $-1 - \lfloor j/2\rfloor$ and $j - \lfloor j/2\rfloor$ are consecutive p_t holes of α. By lemma 3.5, there exist $m_1 \in [-1 - \lfloor j/2\rfloor - |\pi|, -1 - \lfloor j/2\rfloor + |\pi|]$ and $m_2 \in [j - \lfloor j/2\rfloor - |\pi|, j - \lfloor j/2\rfloor + |\pi|]$, where m_1, m_2 are consecutive p_t holes of β. Then,

$$\begin{aligned} m_2 - m_1 &\ge j - \lfloor j/2\rfloor - |\pi| - (-1 - \lfloor j/2\rfloor + |\pi|) \\ &= j + 1 - 2|\pi| \\ &\ge L_{p_t,s_t+1}(\mathcal{O}) + 1 - 2|\pi| \\ &\ge L_{p_t,s'_t+1}(\mathcal{O}') + 1 - 4|\pi| \\ &> L_{p_t,s'_t}(\mathcal{O}') + 1. \end{aligned}$$

So, $m_2 - m_1 \ge L_{p_t,s'_t+1}(\mathcal{O}')$. Let $j' = \lceil (m_1 + m_2)/2 \rceil$, then $\sigma^{j'}[\pi[W]] \in \chi(O')_t$.

By the choice of t, we know that $[-2|\pi|, 2|\pi|] \subseteq Per_{p_t}(\alpha), Per_{p_t}(\sigma^{j'}(\beta))$. Note that

$$\begin{aligned} j' &= \lceil (m_1 + m_2)/2 \rceil \\ &\leq \lceil (-1 - \lfloor j/2 \rfloor + |\pi| + j - \lfloor j/2 \rfloor + |\pi|)/2 \rceil \\ &= \lceil (j-1)/2 - \lfloor j/2 \rfloor + |\pi| \rceil \\ &= \lceil (j-1)/2 \rceil - \lfloor j/2 \rfloor + |\pi| \\ &= |\pi|. \end{aligned}$$

Therefore, the topological conjugacy $\sigma^{j'} \circ \pi$ maps α to $\sigma^{j'}(\beta)$, and $|\sigma^{j'} \circ \pi| \leq 2|\pi|$. By theorem 2.4, there exists a $\phi \in Sym(\mathfrak{n}^{p_t})$ such that for all $k \in \mathbb{Z}$,

$$\phi(\alpha[kp_t, (k+1)p_t)) = \sigma^{j'}(\beta)[kp_t, (k+1)p_t)).$$

Then it easily follows from the proof of Lemma 5.5.2 that the induced homeomorphism $\hat{\phi}$ bijectively maps W onto $\sigma^{j'}[\pi[W]]$. Therefore, W and $\sigma^{j'}[\pi[W]]$ are D_{p_t}-equivalent, which shows that

$$\{[W]_{D_{p_t}} : W \in \chi(\mathcal{O})_t\} \subseteq \{[W]_{D_{p_t}} : W \in \chi(\mathcal{O}')_t\}.$$

By symmetry, we can show that

$$\{[W]_{D_{p_t}} : W \in \chi(\mathcal{O})_t\} = \{[W]_{D_{p_t}} : W \in \chi(\mathcal{O}')_t\}.$$

Hence, $f(\mathcal{O}) \sim f(\mathcal{O}')$ whenever $\mathcal{O}$ and $\mathcal{O}'$ are topologically conjugate.

Now pick $\mathcal{O}, \mathcal{O}' \in \mathcal{T}_1$, and assume that $f(\mathcal{O}) \sim f(\mathcal{O}')$, then

$$\tau(\mathcal{O}) = \tau(\mathcal{O}').$$

And for some sufficiently large t, there exists $W \in Parts_1(\mathcal{O}, \tau(\mathcal{O})_t)$ which is bijectively mapped onto some $W' \in Parts_1(\mathcal{O}', \tau(\mathcal{O})_t)$ via a homeomorphism $\hat{\phi}$ induced by a permutation $\phi \in Sym(\mathfrak{n}^{\tau(\mathcal{O})_t})$. It follows from theorem 2.4 that $\mathcal{O}$ and $\mathcal{O}'$ are topologically conjugate. □

CHAPTER 5

THE INVERSE PROBLEM ON TOEPLITZ SUBSHIFTS

5.1. The Inverse Problem and an Application of Curtis-Hedlund-Lyndon Theorem

Given a sequence $x \in 2^{\mathbb{Z}}$, denote $x^{\perp}$ as the sequence defined as:

$$x^{\perp}(k) = x(-k+1), \text{ for all } k \in \mathbb{Z}.$$

For a subshift X, we denote $X^{\perp} := \{x^{\perp} : x \in X\}$.

LEMMA 5.1. *For any subshift X, (X, σ^{-1}) and $(X^{\perp}, \sigma)$ are topologically conjugate.*

PROOF. Let $f : X \to X^{\perp}$ be the function $f(x) = x^{\perp}$ for all $x \in X$. It is obvious that f is a continuous bijection and $f^{-1}(x)(-k+1) = x(k)$ for all $k \in \mathbb{Z}$ and $x \in X$.

For all $k \in \mathbb{Z}$ and $x \in X$, we have

$$\begin{aligned} f(\sigma^{-1}(f^{-1}(x)))(k) &= \sigma^{-1}(f^{-1}(x))(-k+1) \\ &= f^{-1}(x)(-k) \\ &= x(k+1) \\ &= \sigma(x)(k) \end{aligned}$$

Therefore, f is a topological conjugacy between (X, σ^{-1}) and $(X^{\perp}, \sigma)$. □

DEFINITION 5.2. We say a subshift (X, σ) is flip invariant if (X, σ) and (X, σ^{-1}) are topologically conjugate.

COROLLARY 5.3. *A subshift (X, σ) is flip invariant if and only if (X, σ) and $(X^{\perp}, \sigma)$ are topologically conjugate. Particularly, By the criterion of topological conjugacy of Toeplitz subshifts, we have a Toeplitz subshift (X, σ) is flip invariant if and only if there exist m and $\phi \in sym(\mathfrak{n}^m)$ such that $\hat{\phi}(X) \cap X^{\perp} \neq \emptyset$.*

5.2. A Characterization of Inverse Problem on Teoplitz Subshifts Having a Single Hole Structure

Definition 5.4. Given two words a, b, if $w = a * b \dots a * b$ is n copy of $a * b$, $m \in [0, n-1]$, we fill w by $x(0), x(1), \dots x(m-1), m(m+1), \dots, x(n-1)$ as follows:

$$ax(0)bax(1)b \dots ax(m-1)ba * bax(m+1)b \dots ax(n-1)b.$$

We say such a filling is of first symmetry if

(1) for all $i, j \in [0, n-1] \setminus \{m\}$, if $(i+j)/2 = m$, then $x(i) = x(j)$, and
(2) if $m < n/2$, then $x(2m+1)x(2m+2) \dots x(n-1) = x(n-1)x(n-2) \dots x(2m+1)$; if $m \geq n/2$, then $x(0)x(1) \dots x(2m-n) = x(2m-n)x(2m-n-1) \dots x(1)x(0)$.

We say such a filling is of second symmetry if

(1) n is odd, and
(2) for all $i, j \in [0, n-1] \setminus \{m\}$, if $(i+j)/2 = m$, then $x(i) = 1 - x(j)$, and
(3) if $m < n/2$, then $x(2m+1)x(2m+2) \dots x(n-1) = (1-x(n-1))(1-x(n-2)) \dots (1-x(2m+1))$; if $m > n/2$, then $x(0)x(1) \dots x(2m-n) = (1-x(2m-n))(1-x(2m-n-1) \dots (1-x(0))$.

Example 5.5. Consider $w = a * ba * ba * ba * b$, where a, b are given words, for $m = 2$, we fill w by $x(0) = 0, x(1) = 1, x(3) = 1$. This is a first symmetric filling.

Consider $w = a * ba * ba * ba * ba * ba * ba * b$, where a, b are given words, for $m = 2$, we fill w by $x(0) = 0, x(1) = 0, x(3) = 1, x(4) = 1, x(5) = 0, x(6) = 1$. This is a second symmetric filling.

Lemma 5.6. *Assume* $w = \underbrace{a * b \dots a * b}_{n \text{ copies of } a*b}$, $v = \underbrace{c * d \dots c * d}_{n \text{ copies of } c*d}$, *and* $ba = dc := e$. *For any* $m \in [0, n-1]$, $x(0), x(1), \dots, x(m-1), x(m+1), \dots, x(n-1)$, *let* $y(i) = 1 - x(i)$ *for each* $i \in [0, n-1] \setminus \{m\}$, *and let*

$$w_1 = ax(0)bax(1)b \dots ax(m-1)ba * bax(m+1)b \dots ax(n-1)b,$$

$$v_1 = cx(n-1)dcx(n-2)d \dots cx(m+1)dc * dcx(m-1)d \dots cx(0)d,$$

$$v_2 = cy(n-1)dcy(n-2)d \dots cy(m+1)dc * dcy(m-1)d \dots cy(0)d.$$

(1) *If $x(0), x(1), \dots, x(m-1), x(m+1), \dots, x(n-1)$ is a first symmetric filling of w, then the words between two consecutive $*$ in w_1w_1 and v_1v_1 are the same;*

(2) *If $x(0), x(1), \dots, x(m-1), x(m+1), \dots, x(n-1)$ is a second symmetric filling of w, then the two words between two consecutive $*$ in w_1w_1 and v_2v_2 are the same.*

PROOF. It is trivial and we omit the proof here. □

DEFINITION 5.7. Assume x is a Toeplitz sequence in a minimal subshift X, $(p_t)_{t\in\mathbb{N}}$ is a period structure for x, and for each t, $Skel_{p_t}(x)[0, p_t) = a_t * b_t$ for some words a_t and b_t over $\{0,1\}$. We say that x and X have the nice symmetric filling property if there exists t_0 such that

(1) for all $t \geq t_0$, there exist m_t, $z_t(0), z_t(1), \dots, z_t(m_t - 1), z_t(m_t + 1), \dots, z_t(\frac{p_{t+1}}{p_t} - 1)$ such that $Skel_{p_{t+1}}(x)[0, p_{t+1}) = a_t z_t(0) b_t \dots a_t z_t(m_t - 1) b_t a_t * b_t a_t z_t(m_t + 1) b_t \dots a_t z_t(\frac{p_{t+1}}{p_t} - 1) b_t$ is of the first symmetric filling, or

(2) for all $t \geq t_0$, there exist m_t, $z_t(0), z_t(1), \dots, z_t(m_t - 1), z_t(m_t + 1), \dots, z_t(\frac{p_{t+1}}{p_t} - 1)$ such that $Skel_{p_{t+1}}(x)[0, p_{t+1}) = a_t z_t(0) b_t \dots a_t z_t(m_t - 1) b_t a_t * b_t a_t z_t(m_t + 1) b_t \dots a_t z_t(\frac{p_{t+1}}{p_t} - 1) b_t$ is of the second symmetric filling.

In the situation (1), we say x has the first symmetric filling w.r.t. $(p_t)_{t\in\mathbb{N}}$ for $t \geq t_0$. Similarly, in case (2), we say x has the second symmetric filling w.r.t. $(p_t)_{t\in\mathbb{N}}$ for $t \geq t_0$.

LEMMA 5.8. *Assume that x is a Toeplitz sequene with a period structure $(p_t)_{t\in\mathbb{N}}$, $Skel_{p_t}(x)[0, p_t)$ has exactly one unfilled position for each $t \in \mathbb{N}$, and X is the Toeplitz subshift containing x.*

(1) *If there exists t_0 such that for all $t \geq t_0$, x has the first symmetric filling w.r.t. $(p_t)_{t\in\mathbb{N}}$, then (X, σ) is flip invariant.*

(2) *Similarly, if there exists t_0 such that for all $t \geq t_0$, x has the second symmetric filling w.r.t. $(p_t)_{t\in\mathbb{N}}$, then (X, σ) is flip invariant.*

PROOF. We only provide a proof for (1), and (2) can be proved in a similar way.

Let t_0 be such that for all $t \geq t_0$, x has the first symmetric filling. Let $m \in [0, p_{t_0})$ be the hole of $Skel_{p_{t_0}}(x)$. Let a, b, c, d be such that

$$Skel_{p_{t_0}}(x)[0, p_{t_0}) = a * b,$$

$$Skel_{p_{t_0}}(x^{\perp})[0, p_{t_0}) = c * d.$$

Fix $\phi \in Sym(2^{p_{t_0}})$ with $\phi(a0b) = c0d$, and $\phi(a1b) = c1d$.

Claim. Assume $ba = dc$, then $\hat{\phi}(x) \in \overline{Orb(x^{\perp})}$.

PROOF. We only need to show that for all $t \geq t_0$, the blocks between 2 consecutive p_t holes of $Skel_{p_t}(\hat{\phi}(x))$ and of $Skel_{p_{t_0}}(x^{\perp})$ are the same. This can be proved inductively on $t \geq t_0$.

For $t = t_0$, it follows directly from lemma 5.6.

Assume that $t \geq t_0$, and the blocks between 2 consecutive p_t holes of $Skel_{p_t}(\hat{\phi}(x))$ and of $Skel_{p_{t_0}}(x^{\perp})$ are the same. That is, let u_1, u_2, v_1, v_2 be such that

$$Skel_{p_t}(\hat{\phi}(x))[0, p_t) = u_1 a * b u_2,$$

$$Skel_{p_t}(x^{\perp})[0, p_t) = v_1 c * d v_2,$$

then $bu_2u_1a = dv_2v_1c$.

Since x has the first symmetric filling for $t \geq t_0$, $\phi(a0b) = c0d$, $\phi(a1b) = c1d$, applying lemma 5.6, we have the blocks between 2 consecutive p_{t+1} holes of $Skel_{p_{t+1}}(\hat{\phi}(x))$ and of $Skel_{p_{t_0}}(x^{\perp})$ are the same. □

If $ba \neq dc$, let e_1 be the word of $|a|$ many 1, e_2 be the word of $|b|$ many 1. Fix $\psi \in Sym(2^{p_{t_0}})$ with $\psi(a0b) = e_1 0 e_2$, $\psi(a1b) = e_1 1 e_2$. Let Y be the Toeplitz subshift containing $\hat{\psi}(x)$. The proof in the claim shows that (Y, σ) is flip invariant. Since (X, σ) and (Y, σ) are topologically conjugate, then (X, σ) is flip invariant.

□

LEMMA 5.9. *Assume x is a Toeplitz sequence with a period structure $(p_t)_{t \in \mathbb{N}}$, $Skel_{p_t}(x)[0, p_t)$ has exactly one hole, and X is a Toeplitz subshift containing x. If x doesn't have the nice filling property w.r.t. $(p_t)_{t \in \mathbb{N}}$, then (X, σ) is not flip invariant.*

Proof. Assume toward a contradiction that there exists $\pi \in Sym(2^{p_{t_0}})$ such that $y := \hat{\pi}(x) \in \overline{Orb(x^{\perp})}$.

Let a, b be such that for all $k \in \mathbb{Z}$ we have $Skel_{p_{t_0}}(x)[kp_{t_0}, (k+1)p_{t_0} = a * b$. Since for each t, $Skel_{p_t}(x^{\perp})[kp_t, (k+1)p_t)$ also has exactly one hole, so does $Skel_{p_t}(y)[kp_t, (k+1)p_t)$. So there exist c, d such that

$$\pi(a * b) = c * d, \text{ where } * \text{ is 0 or 1.}$$

Case 1. $\pi(a0b) = c0d$, and $\pi(a1b) = c1d$.

Now let $t_1 \geq t_0$ be the least integer such that the filling from $Skel_{p_t}(x)[0, p_{t+1})$ to $Skel_{p_{t+1}}(x)[0, p_{t+1})$ is not of the first symmetry. Let a', b' be such that for all $k \in \mathbb{Z}$

$$Skel_{p_{t_1}}(x)[kp_{t_1}, (k+1)p_{t_1} = a'a * bb'.$$

Let $\theta \in Sym(2^{p_{t_1}})$ be such that for all $k \in \mathbb{Z}$,

$$\begin{aligned}\theta(x[kp_{t_1}, (k+1)p_{t_1})) &= \hat{\pi}(x)[kp_{t_1}, (k+1)p_{t_1})) \\ &= \pi(x[kp_{t_1}, kp_{t_1} + p_{t_0}))^\wedge \ldots ^\wedge \pi(x[(k+1)p_{t_1} - p_{t_0}, (k+1)p_{t_1}))\end{aligned}$$

Let

$$c' = \pi(a'[0, p_{t_0}))^\wedge \pi(a'[p_{t_0}, 2p_{t_0}))^\wedge \ldots ^\wedge \pi(a'[|a'| - p_{t_0}, |a'|)),$$

$$d' = \pi(b'[0, p_{t_0}))^\wedge \pi(b'[p_{t_0}, 2p_{t_0}))^\wedge \ldots ^\wedge \pi(b'[|b'| - p_{t_0}, |b'|)).$$

Then, we have $\theta(a'a0bb') = c'c0dd'$, and $\theta(a'a1bb') = c'c1dd'$.

Let $e_1 = bb'a'a$, $e_2 = dd'c'c$. Since $\hat{\pi}(x) \in \overline{Orb(x^{\perp})}$, e_2 and $e_1(p_{t_1}-2)e_1(p_{t_1}-3)\ldots e_1(0)$ are the unique block between two consecutive p_{t_1} holes of $Skel_{p_{t_1}}(x^{\perp})$, we have $e_2 = e_1(p_{t_1} - 2)e_1(p_{t_1} - 3)\ldots e_1(0)$. Let $m \in [0, p_{t_1+1})$ be the unique unfilled position in $Skel_{p_{t_1+1}}(x)$. Let $z(0), z(1), \ldots, z(m-1), z(m+1), \ldots, \frac{z(p_{t_1+1})}{p_{t_1}} - 1)$ be such that

$$Skel_{p_{t_1+1}}[0, p_{t_1+1}) = a'az(0)e_1z(1)e_1 \ldots e_1z(m-1)e_1 * e_1z(m+1)\ldots e_1z(\frac{p_{t_1+1})}{p_{t_1}} - 1))bb'.$$

Note that the block between two consecutive p_{t_1+1} holes of $Skel_{p_{t_1+1}}(x)$ is

$$w_0 := e_1z(m+1)\ldots e_1z(\frac{p_{t_1+1})}{p_{t_1}} - 1))e_1z(0)e_1z(1)e_1 \ldots e_1z(m-1)e_1,$$

the block between two consecutive p_{t_1+1} holes of $Skel_{p_{t_1+1}}(x^\perp)$ is

$$w_1 := e_2z(m-1)e_2z(m-2)\ldots e_2z(0)e_2z(\frac{p_{t_1+1})}{p_{t_1}}-1))e_2z(\frac{p_{t_1+1})}{p_{t_1}}-2)e_2\ldots e_2z(m+1)e_2,$$

and the block between two consecutive p_{t_1+1} holes of $Skel_{p_{t_1+1}}(th\hat{e}ta(x))$ is

$$w_2 := e_2z(m+1)\ldots e_2z(\frac{p_{t_1+1})}{p_{t_1}}-1))e_2z(0)e_2z(1)e_2\ldots e_2z(m-1)e_2.$$

Since the filling at this step is not of the first symmetry, we have $W_1 \neq W_2$, that is $\hat{\theta}(x) \notin \overline{Orb(x^\perp)}$, a contradiction!

Case 2. $\pi(a0b) = c1d$, and $\pi(a1b) = c0d$.

The proof is similar with that in case 1 and we omit it here. □

Combininge the lemma 5.8 and lemma 5.9, we get the following characterization of the flip invariant property on single hole Toeplitz subshifts.

Theorem 5.10. *A Toeplitz subshift over $\{0, 1\}$ with a single hole structure is flip invariant if and only if it has a nice symmetric filling property.*

Remark 5.11. (1) First, we know that there are uncountably many Toeplitz subshifts which are flip invariant since the single hole Toeplitz subshifts described in section 4.2 all have the nice symmetric filling property. Meanwhile, there are many Toeplitz subshifts which are not flip invariant. For example, consider a Toeplitz subshift defined recursively as follows:

At step 0, let α_0 be the completely unfilled sequence marked as $\alpha_0(k) = *$ for all $k \in \mathbb{Z}$.

At step $2n+1$, fill the leftmost unfilled position and the rightmost unfilled position of $\alpha_{2n}[k3^{2n+1}, (k+1)3^{2n+1})$ by 1 for all $k \in \mathbb{Z}$, and denote the sequence we obtained by α_{2n+1}.

At step $2n+2$, fill the leftmost unfilled position and the rightmost unfilled position of $\alpha_{2n+1}[k3^{2n+2}, (k+1)3^{2n+2})$ by 0 for all $k \in \mathbb{Z}$, and denote the sequence we obtained by α_{2n+2}.

The diagram below shows our first three constructions:

$st.1$: $...1 * 11 * 11 * 11 * 11 * 11 * 11 * 11 * 11 * 11 * 11 * 11 * 11 * 11 * 11 * 11 * 1...$

$st.2$: $...1011 * 11011011 * 11011011 * 11011011 * 11011011 * 11011011 * 1101...$

$st.3$: $...10111111011011 * 110110111110110111111011011 * 1101101111101...$

The limit of the sequences we construct above is a Toeplitz sequence having single hole with respect to $(3^n)_{n\in\mathbb{N}}$ and such a filling is not a nice symmetric filling. Therefore, the Toeplitz subshift containing this sequence is not flip invariant.

(2) Since whether a Toeplitz subshift X is flip invariant or not is independent from the choice of its period structures, it follows that having nice symmetric filling property is also independent from the choice of period structures.

(3) Theorem 5.10 generalizes a result of Simon Thomas in [33], who showed that the Teoplitz subshifts introduced in section 4.2 is flip invariant.

REFERENCES

[1] Scott Adams and Alexander S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, Journal of the American Mathematical Society 13 (2000), no. 4, 909–943.

[2] Sergey Bezuglyi and Konstantin Medynets, *Full groups, flip conjugacy, and orbit equivalence of Cantor minimal systems*, Colloquium Mathematicum 110 (2008), 409–429.

[3] John D. Clemens, *Isomorphism of subshifts is a universal countable Borel equivalence relation*, Israel Journal of Mathematics 170 (2009), no. 1, 113–123.

[4] Randall Dougherty, Steve Jackson, and Alexander S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Transactions of the American Mathematical Society 341 (1994), no. 1, 193–225.

[5] T. Downarowicz, J. Kwiatkowski, and Y. Lacroix, *A criterion for Toeplitz flows to be topologically isomorphism and applications*, Colloquium Mathematicum 68 (1995), no. 2, 219–228.

[6] Tomasz Downarowicz, *Survey of odometers and Toeplitz flows*, vol. 385, Providence, RI: American Mathematical Society, 2005, pp. 7–38.

[7] Jacob Feldman and Calvin C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Transactions of the American Mathematical Society 234 (1977), no. 2, 289–324.

[8] Harvey M. Friedman and Lee J. Stanley, *A Borel reducibility theory for classes of countable structures*, The Journal of Symbolic Logic 54 (1989), no. 3, 894–914.

[9] Su Gao, *The action of $SL(2,\mathbb{Z})$ on the subsets of $\mathbb{Z}^2$*, Proceedings of the American Mathematical Society 129 (2001), no. 5, 1507–1512.

[10] ———, *Invariant Descriptive Set Theory*, vol. 293, Chapman and Hall/CRC, 2008.

[11] Su Gao and Aaron Hill, *Topological isomorphism for rank-1 systems*, Journal d'Analyse Mathématique 128 (2016), no. 1, 1–49.

[12] Su Gao, Steve Jackson, and Brandon Seward, *Group Colorings and Bernoulli Subflows*, vol. 241, American Mathematical Society, 2016.

[13] Thierry Giordano, Ian F Putnam, and Christian F Skau, *Full groups of Cantor minimal systems*, Israel Journal of Mathematics 111 (1999), no. 1, 285–320.

[14] Leo A. Harrington, Alexander S. Kechris, and Alain Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, Journal of the American Mathematical Society 3 (1990), no. 4, 903–928.

[15] Greg Hjorth and Alexander S. Kechris, *Rigidity Theorems for Actions of Product Groups and Countable Borel Equivalence Relations*, vol. 833, American Mathematical Society, 2005.

[16] Steve Jackson, Alexander S. Kechris, and Alain Louveau, *Countable Borel equivalence relations*, Journal of Mathematical Logic 2 (2002), no. 1, 1–80.

[17] K. Juschenko and N. Monod, *Cantor systems, piecewise translations and simple amenable groups*, Annals of Mathematics 178 (2012), 775–787.

[18] Burak Kaya, *Cantor minimal systems from a descriptive perspective*, Ph.D. Thesis, Rutgers The State University of New Jersey C New Brunswick (2016).

[19] ———, *The complexity of the topological conjugacy problem for Toeplitz subshifts*, Israel Journal of Mathematics 220 (2017), no. 2, 873–897.

[20] Alexander S. Kechris, *Classical Descriptive Set Theory*, vol. 156, Springer Science & Business Media, 2012.

[21] Alexander S. Kechris, *The Theory of Countable Borel Equivalence Relations*, Manuscript, 156pp, available at http://www.math.caltech.edu/ kechris/papers/lectures on CBER03.pdf (2019).

[22] Douglas Lind and Brian Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, Cambridge, 1995.

[23] Ashok Maitra, *Selectors for Borel sets with large sections*, Proceedings of the American Mathematical Society 89 (1983), no. 4, 705–708.

[24] Hiroki Matui, *Some remarks on topological full groups of Cantor minimal systems II*, Ergodic Theory and Dynamical Systems 33 (2013), no. 5, 1542–1549.

[25] Julien Melleray, *Dynamical simplices and Borel complexity of orbit equivalence*, Israel Journal of Mathematics 236 (2020), no. 1, 317–344.

[26] Vibeke Quorning, *Superrigidity, profinite actions and Borel reducibility*, Master's Thesis, University of Copenhagen (2015).

[27] Marcin Sabok and Todor Tsankov, *On the complexity of topological conjugacy of Toeplitz subshifts*, Israel Journal of Mathematics 220 (2017), no. 2, 583–603.

[28] Scott Schneider and Brandon Seward, *Locally nilpotent groups and hyperfinite equivalence relations*, arXiv preprint arXiv:1308.5853 (2013).

[29] Atle Selberg, *An elementary proof of Dirichlet's theorem about primes in an arithmetic progression*, Annals of Mathematics 50 (1949), no. 2, 297–304.

[30] Sashi Mohan Srivastava, *A Course on Borel Sets*, vol. 180, Springer Science & Business Media, 2008.

[31] Simon Thomas, *Superrigidity and countable Borel equivalence relations*, Annals of Pure and Applied Logic 120 (2003), no. 1-3, 237–262.

[32] ———, *Topological full groups of minimal subshifts and just-infinite groups*, Proceedings of the 12th Asian Logic Conference (2013), 298–313.

[33] ———, *Topological full groups of minimal subshifts and the classification problem for finitely generated complete groups*, Groups, Geometry, and Dynamics 13 (2018), no. 1, 327–347.

[34] Susan Williams, *Toeplitz minimal flows which are not uniquely ergodic*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 67 (1984), no. 1, 95–107.