A NEW CLASS OF STOCHASTIC VOLATILITY MODELS FOR PRICING OPTIONS BASED ON OBSERVABLES AS VOLATILITY PROXIES

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One basic assumption of the celebrated Black-Scholes-Merton PDE model for pricing derivatives is that the volatility is a constant. However, the implied volatility plot based on real data is not constant, but curved exhibiting patterns of volatility skews or smiles. Since the volatility is not observable, various stochastic volatility models have been proposed to overcome the problem of non-constant volatility. Although these methods are fairly successful in modeling volatilities, they still rely on the implied volatility approach for model implementation. To avoid such circular reasoning, we propose a new class of stochastic volatility models based on directly observable volatility proxies and derive the corresponding option pricing formulas. In addition, we propose a new GARCH (1,1) model, and show that this discrete-time stochastic volatility process converges weakly to Heston's continuous-time stochastic volatility model. Some Monte Carlo simulations and real data analysis are also conducted to demonstrate the performance of our methods.

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CHAPTER 1

INTRODUCTION

Since the derivation of an arbitrage-free and risk-neutral closed-form solution to European option pricing (Black and Scholes (1973)) [5], derivative trading activities have soared in the global financial markets. After that, option pricing has developed into one of the major research areas in modern finance theory. Black-Scholes (BS for short thereafter) formula has even boosted the development of the entire derivative market, according to Jarrow (1999) [23]. One major contribution of Black, Scholes and Merton's work was the application of risk-neutral option pricing that is independent of investors' risk preferences. The key advantage of risk-neutral measure is that portfolio discounted with risk-free rate is a martingale. This facilitates the pricing of any derivative security through conditioning the payoff at expiration. Because of the computational convenience of Black-Scholes-Merton formula based on the strong assumption of constant volatility of stock return, it wins great popularity in the finance field. However, the constant volatility assumption used in the Black-Scholes method seems unrealistic. Even at the time of their paper, Black and Scholes realized that the constant volatility assumption was a strong idealization. In an empirical paper Black and Scholes (1972) [4], the authors tested their option price formulas and concluded: "we found that using past data to estimate the variance caused the model to overprice options on high-variance stocks and under-price options on low-variance stocks." In fact, the overwhelming evidence from financial time-series data demonstrates that volatility exhibits unpredictable variation. Additionally, option prices exhibit a significant departure from the prices obtained by Black-Scholes (constant volatility) formula. Moreover, implied volatility of the stock prices inverted from Black-Sholes formula suggests that stochastic volatility models are more appropriate than constant volatility model. The three most popular stochastic volatility models are Hull and White (1987) model [22] , Stein and Stein model (1991) [39], and Heston(1993) model [20]. All these three models assume that the stock price follows a geometric Brownian motion, while volatility processes differ. The volatility process of Hull and White (1987) model follows a geometric Brownian motion. The volatility process of Stein and Stein (1991) model follows an arithmetic Ornstein-Uhlenbeck process. While, volatility process of Heston (1993) model follows a CIR - Cox, Ingersoll and Ross (1985) process [8]. The empirical results of Bakshi, Cao and Chen (1997) [2] show that stochastic volatility plays a significant role in option pricing. Other stochastic volatility option pricing models include Johnson and Shanno (1987) [24], Scott (1987) [38], Wiggins (1987) [40], Melino and Turnbull (1991) [29], Knoch (1992) [27], Duan (1995) [10], Nandi (1996) [30], Bates (1996) [3], Ritchken and Trevor (1999) [33], Heston and Nandi (2000) [21], Elliot, Siu and Chan (2006) [11], Christoffersen, Heston and Jacobs (2006) [7], and Mercuri (2008) [28].

There are two types of stochastic volatility models: continuous-time stochastic volatility models and discrete-time stochastic volatility-GARCH models. Pointed out by Heston and Nandi (2000), continuous-time stochastic volatility models work well in option pricing, but they are hard to implement, especially in volatility extraction. The continuous-time stochastic volatility models assume that the volatility is observable if filtering method is applied. However, there are difficulties in filtering a continuous volatility variable from discrete observations of the underlying asset prices in a continuous-time model, stated by Heston and Nandi (2000). There are two typical proxies for discrete-time volatility observation: historical volatility and implied volatility. Fleming (1998) [15] shows that implied volatility inverted from Black-Scholes formula has a stronger predictive power than historical volatility. However, implied volatility approach involves massive calculation of volatilities. For each strike price, one volatility for each date will be computed. And it is computationally time-consuming to get the implied volatilities from a long time-series of option prices. Continuous-time stochastic volatility models for option pricing requires a significant amount of work. What is worse, pointed out by Schroeder (2006) [37], it is a kind of circular reasoning when historical volatility is used for BS option pricing, and then inverted and returned as implied volatility.

Compared with continuous-time stochastic volatility option pricing models, discrete-time GARCH models have an apparent advantage that they are much easier to implement.

Before Heston and Nandi GARCH option pricing model (2000), there are no closed-form option pricing formulas for existing GARCH models. Those GARCH models, such as Engle and Mustafa (1992) [12], Amin and Ng (1993) [1], Satchell and Timmermann (1995) [35], and Duan (1995) are solved by simulation, which requires intensive computation. Among the GARCH models, Heston and Nandi (2000), Elliot, Siu and Chan (2006), Christoffersen, Heston and Jacobs (2006), and Mercuri (2008) developed closed form option pricing formula. Heston and Nandi (2000) was the first one provided a closed-form solution for its non-linear GARCH option pricing model. The lag one version of the Heston and Nandi (2000) GARCH (1,1) model on the S&P 500 index options data demonstrates significant pricing improvements over the Black-Scholes model even if the Black-Scholes option price is computed with implied volatility updated every period, while the option price from Heston and Nandi GARCH is computed with non-updated volatility filtered from the historical asset prices.

We develop an option pricing model with the underlying asset's continuously compounded returns following a GARCH (1,1) process. Like Heston and Nandi (2000) model, our new GARCH (1,1) model subsumes Heston's model as a diffusion limit with specified parameter values. Different from other GARCH option pricing models, which apply filtering method first and then estimate the parameters to extract the observable volatility, we present a new volatility proxy so that the volatility is directly observable without applying filtering method. In this way, we can estimate the parameters of the two equations separately, while other existing GARCH models estimate the parameters by combining the two equations together to estimate the parameters. Due to the new volatility proxy, we can price options without circular reasoning and with less computation for estimating the parameters of the continuous-time stochastic volatility models than the computation performed in parameter estimation using implied volatility. In real data application and analysis of Chapter 6, BS model and the Heston and Nandi GARCH (1,1) model will serve as our baseline models. We first show the prices of European call options by BS model and Heston Nandi model by only using stock price information. Then we show the prices of European options call obtained from a special case of Hull and White model, generalized Stein and Stein model (generalized by Schöbel and Zhu (1999) [36]), Heston model, two new models proposed by us (one is a continuous model, and the other is a discrete model), and other special cases of the continuous models together with the new volatility proxy. We show that the continuous models and their special cases with our new volatility proxy have better performance than BS model as continuous option pricing models in terms of root mean prediction error (RMPE) in real data application and analysis. And our new GARCH (1.1) model outperforms Heston and Nandi GARCH (1,1) model as discrete option pricing model in terms of RMPE both in simulation and real data analysis.

We organize the remaining material as follows: Chapter 2 will show the option pricing solution to the Black-Scholes model, Hull and White (1987) model, Heston (1993) model, Stein and Stein (1991) model, and the weakness of those models in option pricing. Chapter 3 will present a new method of volatility measure to overcome the unobservable volatility weakness of the models shown in Chapter 2. We also present a new continuous stochastic volatility model. The zero drift and uncorrelated special cases of Heston model, generalized Stein and Stein model, and new continuous model are also presented with the new method of measuring volatility. In Chapter 4, we will introduce Heston and Nandi GARCH (1,1) model and present a new GARCH (1,1) model, which has a close relationship with Heston and Nandi GARCH (1,1) model and Heston model. For Chapter 5, a simulation comparison among the new GARCH (1,1) model, Heston and Nandi GARCH (1,1) model, and Heston model with our new method is conducted, as well as more implied volatility comparison results. For Chapter 6, real data application and analysis is performed.

CHAPTER 2

RELATED EXISTING OPTION PRICING MODELS

Before we reveal the new method to measure volatility, let's first review some continuous stochastic volatility models and its corresponding solutions for option pricing. We first introduce the famous BS model, the first option pricing model with closed-form solution. After that, we review three of the most popular continuous stochastic volatility models: a special case of Hull and White model, generalized Stein and Stein model, and Heston model. These models are aiming to overcome the constant volatility problem of Black-Scholes model. Some disadvantages of these models and how the proposed new method will improve and simplify the estimation procedures will be illustrated as well. We also assume that the financial market is complete and arbitrage-free.

2.1. Black-Scholes Model

The most distinctive feature of BS model is the assumption of constant volatility, which can be estimated from the stock pricess process. Other quantities required as input of this model are observable from the market. Under physical measure $\mathbb{P}$, the stock price S_t is described by the following geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Under the risk-neutral measure $\mathbb{Q}$:

$$dS_t = r S_t dt + \sigma S_t dW_t^*$$

where

$$dW_t^* = \frac{\mu - r}{\sigma} dt + dW_t$$

t- Curent time;

S_t - Current stock price;

K - Option strike price;

r - Annual continuously compounded constant risk-free rate;

σ - Constant standard deviation of stock returns.

The Black-Scholes pricing formula for the price of a European call C_t is given by

$$C_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2).$$

where Φ is the cumulative distribution function of a standard normal variable. T is the time to maturity. And d_1, d_2 are given as follows:

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}} \cdot [\log(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)], \\ d_2 &= d_1 - \sigma\sqrt{T-t} \\ &= \frac{1}{\sigma\sqrt{T-t}} \cdot [\log(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})(T-t)], \end{aligned}$$

The price of a European put option P_t can be calculated through the following put-call parity:

$$C_t + K e^{-r(T-t)} = P_t + S_t$$

In practice, lots of companies pay dividend. If certain percentages q of stock price is paid as dividend, we need to replace r with $r-q$ for option price calculation. If certain dollar amount is paid, the present value of the dividend should be subtracted from the current stock price, that is to replace S_t with S_t- PV(div).

The biggest problem of this model is constant volatility. Scott (1987) and the relevant references show that volatility changes over time. Moreover, if we plot BS formula implied volatilities against strike prices, we will see a volatility skew or smile. This is against the expectation that a constant volatility should result a horizontal line if we plot implied volatility against strike price. And further empirical studies show that stochastic volatilities will better predict the option prices. Thereafter, people have developed numerous continuous stochastic volatility models, such as Scott (1987), Hull and White (1987), Wiggins (1987), Stein and Stein (1991), Heston (1993), Hagan, Kumar, Lesniewski, and Woodward (2002) [19]. Let's review those three most popular models mentioned at the beginning of the chapter and its corresponding option pricing solutions next.

2.2. A Special Case of Hull and White Model

The stochastic volatility model presented by Hull and White (1987) assumes that both the underlying stock price and its instantaneous variance follow Geometric Brownian Motion. Under physical measure $\mathbb{P}$, the model is given by:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_{1,t}$$

$$dV_t = aV_t dt + \sigma V_t dW_{2,t}$$

where $dW_{1,t}$ and $dW_{2,t}$ are Wiener processes correlated with coefficient ρ. μ may depend on S_t, V_t and t. a and σ may depend on V_t and t, but do not depend on S_t. Here, we consider a special case also presented by Hull and White (1987), where μ and σ are constants, and $a = \kappa(\theta - \sqrt{V_t})$. Then the special case of Hull and White Model is given by:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_{1,t}$$

$$dV_t = \kappa(\theta - \sqrt{V_t}) V_t dt + \sigma V_t dW_{2,t}$$

where $dW_{1,t}$ and $dW_{2,t}$ are independent Wiener processes. The reason we choose this special case is because it has one of the most important features of a volatility model - mean reversion, as pointed out by Engle and Patton (2001) [13].

Under the risk-neutral measure $\mathbb{Q}$, the special case of Hull and White model is given by:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW^*_{1,t}$$

$$dV_t = [\kappa(\theta - \sqrt{V_t}) V_t - \lambda_t] dt + \sigma V_t dW^*_{2,t}$$

Where λ_t stands for the market price of volatility risk. Hull and White (1987) postulates $\lambda_t = 0$ by assuming zero systematic risk for volatility. And we can get the the two Wiener processes under risk-neutral measure:

$$dW_{1,t}^{*} = dW_{1,t} + \frac{\mu - r}{\sqrt{V_t}}dt$$

$$dW_{2,t}^{*} = dW_{2,t} + \lambda_t dt$$

And the Wiener processes $dW_{1,t}^{*}$ and $dW_{2,t}^{*}$ are also independent under risk-neutral measure.

This model doesn't have an closed-form solution. Hull and White (1987) develops an approximate solution through Taylor expansion around $\kappa = 0$ as follows:

$$\begin{aligned}(1) \qquad f(S_t, \sigma_{BS}^2) =& C(\sigma_{BS}^2) \\ &+ \frac{1}{2}\frac{S\sqrt{T-t}\,\Phi'(d_1)(d_1d_2 - 1)}{4\sigma_{BS}^3} \times [\frac{2\sigma_{BS}^4(e^a - a - 1)}{a^2} - \sigma_{BS}^4] \\ &+ \frac{1}{6}\frac{S\sqrt{T-t}\,\Phi'(d_1)(d_1d_2 - 3)(d_1d_2 - 1) - (d_1^2 + d_2^2)}{8\sigma_{BS}^5} \\ &\times \sigma_{BS}^6[\frac{e^{3a} - (9 + 18a)e^a + (8 + 24a + 18a^2 + 6a^3)}{3a^3}]\end{aligned}$$

where $a = \sigma^2(T-t)$. And σ_{BS} is the constant volatility in BS model. $C(\sigma_{BS}^2)$ is the European call option price calculated with BS formula. Φ, d_1 and d_2 are defined in the same way as in BS model of Section 2.1.

For the second equation of the model under risk-neutral measure, if we apply Ito's lemma to $\log(V_t)$, we will get

$$d\log(V_t) = (\kappa\theta - \kappa\sqrt{V_t} - \frac{1}{2}\sigma^2)dt + \sigma dW_{2,t}^{*}$$

Integrate both sides, we will get for any $s \geq t$

$$V_s = V_t\,\exp[\kappa\theta(s-t) - \kappa\int_t^s \sqrt{V_u}\,du - \frac{1}{2}\sigma^2(s-t) + \sigma(W_{2,s}^{*} - W_{2,t}^{*})]$$

The above expression will guarantee the positivity of the volatility process given a positive initial volatility V_t. However, the major disadvantage of this model is that it has no closed-form solution. Hull and White (1987) provides the approximated option pricing formula (1) by arguing that the drift term of the volatility process is at least close to zero in practice.

Stein and Stein (1991) states that it is not sure whether the approximation is accurate enough when the drift term is significantly different from zero. Based on these disadvantages, Stein and Stein (1991) develops a stochastic volatility model with a closed-form solution, which is given in next section.

2.3. Generalized Stein and Stein Model

The stochastic volatility model developed by Stein and Stein assumes an Ornstein-Uhlenbeck model for its volatility process. Under physical measure $\mathbb{P}$, the model is given by:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_{1,t}$$

$$d\sqrt{V_t} = \kappa(\theta - \sqrt{V_t})dt + \sigma dW_{2,t}$$

where $dW_{1,t}$ and $dW_{2,t}$ are two independent Wiener processes under physical measure. Under the risk-neutral measure $\mathbb{Q}$, the model is given by:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW^*_{1,t}$$

$$d\sqrt{V_t} = (\kappa\theta - \kappa\sqrt{V_t} - \lambda_t)dt + \sigma dW^*_{2,t}$$

where $dW^*_{1,t}$ and $dW^*_{2,t}$ are two independent Wiener processes under risk-neutral measure, and λ_t is the market price of volatility risk. Like Hull and White (1987), Stein and Stein (1991) also assumes $\lambda_t = 0$. Stein and Stein (1991) states that market price of volatility risk is related to investor's risk preference. But in a risk-neutral world, investors are not concerned with risk preference. This will lead market price of volatility risk to be zero. Stein and Stein (1991) also develops a closed-form option pricing formula. Stein and Stein (1991) claims that volatility is strongly mean-reverting proven by empirical evidence and their model can deal with non-zero mean reversion parameter θ, and does not require any assumption about σ being close to zero based on its closed-form solution.

Empirical studies show that stock price and its volatility are usually correlated. Schöbel and Zhu (1999) [36] generalizes Stein and Stein model by assuming the two Wiener processes $dW_{1,t}$ and $dW_{2,t}$ are correlated with coefficient ρ (including zero). Here we call

Stein and Stein model with this generalization as Generalized Stein and Stein model thereafter. Schöbel and Zhu (1999) provides a different closed-form solution from Stein and Stein (1991) by inverting characteristic function through Fourier transform. Let $x(t)$ be the log stock price, that is $x(t) = \log S(t)$. Under risk-neutral probability measure $\mathbb{Q}$, the option pricing formula of the generalized Stein and Stein model from Schöbel and Zhu (1999) is given by

$$
\begin{aligned}
(2) \qquad C(S,t,T) &= \mathbb{E}^Q[e^{-r(T-t)}(S(T)-K)\cdot 1_{\{S(T)>K\}}] \\
&= \mathbb{E}^Q[e^{-r(T-t)}S(T)\cdot 1_{\{x(T)>\log K\}}] - e^{-r(T-t)}K\mathbb{E}^Q[1_{\{x(T)>\log K\}}] \\
&= S(t)\mathbb{E}_1^Q[1_{\{}x(T>\log K\}] - e^{-r(T-t)}K\mathbb{E}_2^Q[1_{\{}x(T>\log K\}] \\
&= S(t)P_1(S(T)>K) - e^{-r(T-t)}KP_2(S(T)>K).
\end{aligned}
$$

where P_1 and P_2 are given by the following Fourier inversion formula:

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re(f_j(\phi)\frac{\exp\{-i\phi\log K\}}{i\phi})d\phi,\ j=1,2.$$

In the above formula, $Re[]$ stands for the real part of a complex number. i is the imaginary unit. f_1, f_2 are characteristic functions of P_1, P_2 respectively and are given below:

$$
\begin{aligned}
f_1(\phi) = &\exp\{i\phi(r(T-t)+x(t)) + x(t) - \frac{1}{2}\rho(1+i\phi)[\sigma^{-1}v^2(t)+\sigma(T-t)]\} \\
&\times \exp\{\frac{1}{2}D(t,T;s_1,s_3)v^2(t) + B(t,T;s_1,s_2,s_3)v(t) + C(t,T;s_1,s_2,s_3)\}
\end{aligned}
$$

$$
\begin{aligned}
f_2(\phi) = &\exp\{i\phi(r(T-t)+x(t)) - \frac{1}{2}i\phi\rho[\sigma^{-1}v^2(t)+\sigma(T-t)]\} \\
&\times \exp\{\frac{1}{2}D(t,T;\hat{s_1},\hat{s_3})v^2(t) + B(t,T;\hat{s_1},\hat{s_2},\hat{s_3})v(t) + C(t,T;\hat{s_1},\hat{s_2},\hat{s_3})\}
\end{aligned}
$$

with

$$
\begin{aligned}
s_1 &= -\frac{1}{2}(1+i\phi)^2(1-\rho^2) + \frac{1}{2}(1+i\phi)(1-2\kappa\rho\sigma^{-1}),\\
s_2 &= (1+i\phi)\kappa\theta\rho\sigma^{-1},\\
s_3 &= \frac{1}{2}(1+i\phi)\rho\sigma^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{s_1} &= \frac{1}{2}\phi^2(1-\rho^2) + \frac{1}{2}i\phi(1-2\kappa\rho\sigma^{-1}),\\
\hat{s_2} &= i\phi\kappa\theta\rho\sigma^{-1},\\
\hat{s_3} &= \frac{1}{2}i\phi\rho\sigma^{-1},
\end{aligned}
$$

$$
\begin{aligned}
D(t,T) &= \frac{1}{\sigma^2}(\kappa - \gamma_1\frac{sinh\{\gamma_1(T-t)\} + \gamma_2 cosh\{\gamma_1(T-t)\}}{cosh\{\gamma_1(T-t)\} + \gamma_2 sinh\{\gamma_1(T-t)\}})\\
B(t,T) &= \frac{1}{\sigma^2\gamma_1}(\frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3) + \gamma_3(sinh\{\gamma_1(T-t)\} + \gamma_2 cosh\{\gamma_1(T-t)\})}{cosh\{\gamma_1(T-t)\} + \gamma_2 sinh\{\gamma_1(T-t)\}} - \kappa\theta\gamma_1)\\
C(t,T) &= -\frac{1}{2}\log(cosh\{\gamma_1(T-t)\} + \gamma_2 sinh\{\gamma_1(T-t)\}) + \frac{1}{2}\kappa(T-t)+\\
&\quad + \frac{(\kappa^2\theta^2{\gamma_1}^2 - {\gamma_3}^2)}{2\sigma^2\gamma_1^3}(\frac{sinh\{\gamma_1(T-t)\}}{cosh\{\gamma_1(T-t)\} + \gamma_2 sinh\{\gamma_1(T-t)\}} - \gamma_1(T-t))+\\
&\quad + \frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3)\gamma_3}{\sigma^2\gamma_1^3}(\frac{cosh\{\gamma_1(T-t)\} - 1}{cosh\{\gamma_1(T-t)\} + \gamma_2 sinh\{\gamma_1(T-t)\}})
\end{aligned}
$$

$$
\gamma_1 = \sqrt{2\sigma^2 s_1 + \kappa^2},\ \gamma_2 = \frac{1}{\gamma_1}(\kappa - 2\sigma^2 s_3),\ \gamma_3 = \kappa^2\theta - s_2\sigma^2.
$$

where v_t is the initial standard deviation. With the time dependent functions $D(t,T)$, $B(t,T)$ and $C(t,T)$, we can obtain the closed-form solutions for $f_j(\phi)$, and hence the closed-form option pricing formula.

2.4. Heston Model

Even though the generalized Stein and Stein model has a closed-form solution and requires less restrictive assumption than Hull and White model, its second equation doesn't guarantee the positivity of volatility. In order to overcome the positivity issue of volatility, Heston (1993) presents a stochastic volatility model with the instantaneous variance following a CIR process, which guarantees positivity of the variance if Feller's condition is satisfied. Under physical measure $\mathbb{P}$, Heston model is given by:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_{1,t}$$

$$dV_t = \kappa(\theta - V_t)dt + \sqrt{V_t}\sigma dW_{2,t}$$

where $dW_{1,t}$ and $dW_{2,t}$ are correlated with coefficient ρ under physical measure. And Feller's condition $2\kappa\theta > \sigma^2$ is assumed to be satisfied. According to Crisostomo (2014) [9], if we set $\kappa^* = \kappa + \lambda$, $\theta^* = \frac{\kappa\theta}{\kappa+\lambda}$, the model under risk-neutral measure $\mathbb{Q}$ can be given by:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW^*_{1,t}$$

$$dV_t = \kappa^*(\theta^* - V_t)dt + \sqrt{V_t}\sigma dW^*_{2,t}$$

where λV_t (which is the same as λ_t in previous section) stands for the market price of volatility risk, and λ represents unit price of volatility risk. $dW^*_{1,t}$ and $dW^*_{2,t}$ are Wiener processes correlated with coefficient ρ under risk-neutral measure. And they are given by:

$$dW^*_{1,t} = dW_{1,t} + \frac{\mu - r}{\sqrt{V_t}} dt$$

$$dW^*_{2,t} = \frac{\lambda\sqrt{V_t}}{\sigma} dt + dW_{2,t}$$

The closed form solution of option pricing under Heston model is given as follows:

$$C_t = S_t P_1 - Ke^{-r(T-t)} P_2 \tag{3}$$

where for $j = 1, 2$

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-i\phi \log K} f_j(\phi; t, x_t, v_t)}{i\phi}]d\phi$$

$$f_j(\phi; x_t, V_t) = \exp(C_j(\tau, \phi) + D_j(\tau, \phi)V_t + i\phi x_t)$$

and

$$\tau = T - t$$

$$x_t = \log(S_t)$$

$$C_j(\tau, \phi) = ri\phi\tau + \frac{a}{\sigma^2}[(b_j - \rho\sigma i\phi + d_j)\tau - 2\log(\frac{1 - g_j e^{d_j\tau}}{1 - g_i})]$$

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}})$$

such that

$$g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j}$$

$$d_j = \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)}$$

$$u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = \kappa\theta, b_1 = \kappa + \lambda - \rho\sigma, b_2 = \kappa + \lambda$$

Empirical evidence shows that no matter what kind of performance standard is applied, taking stochastic volatility into consideration is the foremost task in outperforming the BS formula, according to Bakshi, Cao and Chen (1997). Even though stochastic volatility improves option pricing on BS formula in general, they suffer from a common problem that volatility is not directly observable. Generally, historical volatility and implied volatility are two discrete-time volatility proxies for option pricing. According to Fleming (1998), implied volatility has stronger predictive power. The stronger predictive power makes implied volatility applied more often in stochastic volatility option pricing models. However, no matter how strong the predictive power of implied volatility is, the application of implied volatility is a kind of circular reasoning. In order to solve this circular reasoning problem, we will propose a new method to serve as volatility proxy in next chapter, which makes volatility directly observable. And Monte Carlo simulation of the new method for these three continuous models will be also conducted to show their performance in next chapter.

CHAPTER 3

A NEW METHOD TO MEASURE VOLATILITY FOR CONTINUOUS STOCHASTIC VOLATILITY MODELS

To overcome the unobservable volatility issue for the continuous stochastic models, we propose the following new method to measure volatility:

$$V_t = m * (TV_t)^p$$

where V_t is the instantaneous volatility. TV_t stands for total trading volume over the time interval $(t-dt, t)$. m is a positive constant, and p is a real-valued constant. The transformed trading volume to measure volatility works for the following reasons: (1) Trading volume is the second most important information right after stock price. It reflects the overall activity of investors towards the stock. And trading volume also reflects investor's future expectation, and very often showing some inside traders' reaction to inside information. (2) According to Chen, Firth and Rui (2001) [6], trading volume is correlated with all measures of volatility. (3) Stock price is made at the largest number of shares transacted at the moment. (4) Trading volume causes volatility, not vice versa, according to Paital and Sharma (2016) [32]. All these reasons serve as strong evidence for us to propose this new method to measure volatility. The identification of m and p will be explained in Chapter 6.

With the new method to measure volatility, we can estimate the parameters without involving circular reasoning. Moreover, we can estimate the parameters in the two diffusion equations of the stochastic volatility models separately with the new volatility proxy. The simulation performance will be shown for the rest of the chapter.

Before we start data simulation and parameter estimation, we set the market price of volatility risk $\lambda_t = 0$ in all the stochastic volatility models under risk-neutral measure by adopting the argument of Stein and Stein (1991), which means that investors are not concerned with risk preference in a risk-neutral world.

3.1. New Method for a Special Case of Hull and White Model

Since we set market price of volatility risk λ_t to be zero, the two equations of the new method for a special case of Hull and White (1987) model under risk-neutral measure is given by:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_{1,t}^*$$

$$dV_t = \kappa(\theta - \sqrt{V_t})V_t dt + \sigma V_t dW_{2,t}^*$$

And the new method is applied to measure volatility, that is to assume $V_t = m * (TV_t)^p$. The Wiener processes $dW_{1,t}^*$ and $dW_{2,t}^*$ are independent under risk-neutral measure.

3.1.1. Parameter Estimation of the New Method for Hull and White Model

There are four parameters κ, θ, σ and ρ to estimate in this model. We will apply MLE method with Monte Carlo Simulation to estimate the first three parameters, and method of moment to estimate ρ. Before estimation, we need to discretize the above processes. We discretize S_t and V_t as $\{S_0, S_\Delta, S_{2\Delta}..., S_{n\Delta}\}$ and $\{V_0, V_\Delta, V_{2\Delta}..., V_{n\Delta}\ \}$, where $n = \frac{T-t}{\Delta}$. Starting from here, we will omit the Δ in the subscripts for simplicity. Then the stock prirce process and volatility process are denoted as $\{S_i\}_0^n$and $\{V_i\}_0^n$. And we discretize the two diffusion equations of the model as below:

$$\log(S_i) - \log(S_{i-1}) = (r - \frac{1}{2}V_{i-1})\Delta + \sqrt{V_{i-1}\Delta} z_{1,i}^*$$

$$\log(V_i) - \log(V_{i-1}) = (\kappa\theta - \kappa\sqrt{V_{i-1}} - \frac{1}{2}\sigma^2)\Delta + \sigma\sqrt{\Delta} z_{2,i}^*$$

where $z_{1,i}^*$ and $z_{2,i}^*$ are independent standard normal variables. Then we have $\log(V_i) - \log(V_{i-1}) - (\kappa\theta - \kappa\sqrt{V_{i-1}} - \frac{1}{2}\sigma^2)\Delta = \sigma\sqrt{\Delta} z_{2,i}^*$. Let $U = \sigma\sqrt{\Delta} z_{2,i}^*$. Since $U = \sigma\sqrt{\Delta} z_{2,i}^* \sim N(0, \sigma^2\Delta)$, we get the probability density function of U below:

$$f_U(u) = \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{u^2}{2\sigma^2\Delta}}$$

Notice $\frac{dU}{dV_i} = \frac{1}{V_i}$. If we substitute U with $\log(V_i) - \log(V_{i-1}) - (\kappa\theta - \kappa\sqrt{V_{i-1}} - \frac{1}{2}\sigma^2)\Delta$ and apply chain rule, we will get the conditional density of V_i given V_{i-1} as follows:

$$f(V_i | V_{i-1}) = \frac{1}{\sqrt{2\pi V_i^2 \sigma^2 \Delta}} \exp(-\frac{X_i^2}{2\sigma^2\Delta})$$

where $X_i = \log(V_i) - \log(V_{i-1}) - (\kappa\theta - \kappa\sqrt{V_{i-1}} - \frac{1}{2}\sigma^2)\Delta$. Then, we get the likelihood function $L(\kappa, \theta, \sigma)$ below

$$L(\kappa, \theta, \sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi V_i^2 \sigma^2 \Delta}} \exp(-\frac{X_i^2}{2\sigma^2\Delta})$$

Thus, the log-likelihood function $l(\kappa, \theta, \sigma) = \log(L)$ is given by

$$l(\kappa, \theta, \sigma) = \sum_{i=1}^{n} \{-0.5[\log(2\pi V_i^2 \sigma^2 \Delta) + \frac{X_i^2}{2\sigma^2\Delta}]\}$$

The maximum likelihood estimation (MLE) of the parameters can be performed through "maxLik" package of R programming software.

3.1.2. Simulation Results

For the data generation, we set the risk-free rate $r = 0.03$, initial variance from transformed trading volume $V_0 = (32000000)^{-1/5}$, 1000 replications are simulated for different sample sizes. Time to maturity is 0.5 years. Initial stock price S_0 =\$100. After obtaining values of maximum likelihood (ML) estimators, we get estimated European call option prices for various strike prices in different sample sizes after plugging the values of ML estimators into the Taylor expansion formula (formula (1)). Estimated parameter values are compared with the true parameter values. And the estimated option prices are compared with the true option prices computed from formula (1) with true parameter values. The parameter estimation results and option price estimation results are shown in Table 3.1 and Table 3.2 respectively.

Table 3.1 and Table 3.2 show that as sample size increases, the estimated parameter values are getting closer and closer to the true parameter values. But estimated option prices have different behaviors. They are not sensitive to sample size. For any sample size shown in Table 3.2, the difference between the estimated option price and the true option price is within one cent. Form here, we can conclude that the new method for the special case of Hull and White model is reliable.

Table 3.1. Parameter Estimation of New Method for the Special Case of Hull and White Model (1000 replications)

$True\,Value$	$\kappa = 5$	$\theta = 0.2236$	$\sigma = 1$
$n = 50$	6.0262	0.2132	0.9730
$Bias$	1.0262	-0.0104	-0.0270
$RMSE$	2.3941	0.0517	0.1078
100	5.4234	0.2176	0.9834
$Bias$	0.4234	-0.006	-0.0166
$RMSE$	1.3010	0.0322	0.0746
200	5.2065	0.2214	0.9940
$Bias$	0.2065	-0.0022	-0.0060
$RMSE$	0.8600	0.0254	0.0510
300	5.1545	0.2209	0.9952
$Bias$	0.1545	-0.0027	-0.0048
$RMSE$	0.7065	0.0192	0.0414
400	5.0801	0.2226	0.9952
$Bias$	0.0801	-0.001	-0.0048
$RMSE$	0.5717	0.0158	0.0351
600	5.0607	0.2228	0.9987
$Bias$	0.0607	-0.0008	-0.0013
$RMSE$	0.4751	0.0127	0.0285
800	5.0533	0.2224	0.9974
$Bias$	0.0533	-0.0012	-0.0026
$RMSE$	0.3926	0.0127	0.0254
1000	5.0502	0.2226	0.9975
$Bias$	0.0502	-0.0010	-0.0025
$RMSE$	0.3734	0.0095	0.0223

$True\,Value$	$\kappa = 5$	$\theta = 0.2236$	$\sigma = 1$
2000	5.0198	0.2235	1.0001
$Bias$	0.0198	-0.0001	0.0001
$RMSE$	0.2601	0.0063	0.0158
5000	5.0103	0.2233	0.9996
$Bias$	0.0103	-0.0003	-0.0004
$RMSE$	0.1553	0.0032	0.0095
10000	5.0056	0.2235	0.9999
$Bias$	0.0056	-0.0001	-0.0001
$RMSE$	0.1140	0.0032	0.0063

TABLE 3.2. Option Price Estimation of New Method for the Special Case of Hull and White Model

Strike Price	$X = 80$	90	100	110	120
True Call Price	21.3542	12.391	5.6559	2.0097	0.6051
$n = 50$	21.3526	12.3927	5.6591	2.0125	0.6052
100	21.3532	12.3921	5.6578	2.0114	0.6052
200	21.3539	12.3914	5.6566	2.0103	0.6052
300	21.3539	12.3913	5.6564	2.0102	0.6051
400	21.3539	12.3913	5.6564	2.0102	0.6051
600	21.3542	12.3911	5.656	2.0098	0.6051
800	21.3541	12.3912	5.6562	2.0099	0.6051
1000	21.3541	12.3912	5.6562	2.0099	0.6051
2000	21.3542	12.3910	5.6559	2.0097	0.6051
5000	21.3542	12.3911	5.6559	2.0097	0.6051
10000	21.3542	12.3910	5.6559	2.0097	0.6051

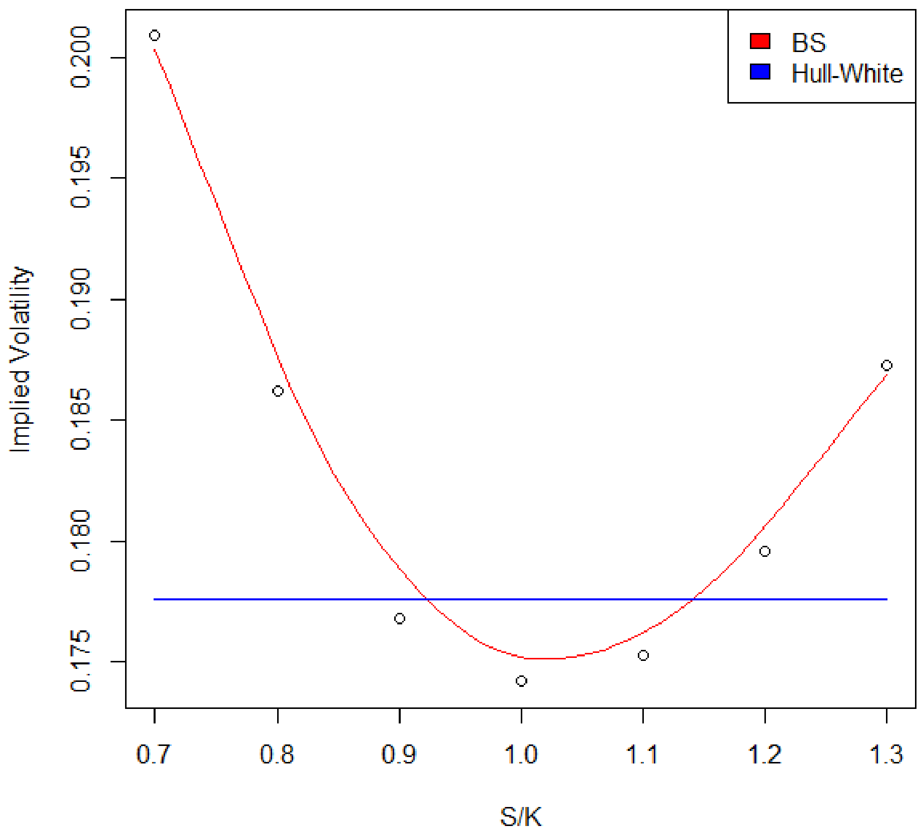

FIGURE 3.1. Hull-White VS BS Volatility Skew.

3.1.3. Implied Volatility Comparison between the Special Case of Hull and White Model and BS Model

The BS model implied volatility is obtained through the function "bscallimpvol" in R package "derivmkts". The Hull and White model implied volatility is obtained through the function "BFfzero" in R package "NLRoot". In this function, the Bisection method is applied. We apply the values of MLE from sample size n = 10000 to get the Hull and White model implied volatility.

From Figure 3.1, we can see that implied volatility inverted from the special case of

Hull and White model forms a horizontal line, while the BS model implied volatility forms a "volatility smile". This indicates that the special case of Hull and White model with our new method works better to reduce the volatility skew effect than BS model with constant volatility.

3.2. The New Method for Generalized Stein and Stein Model

In this section, we will apply the new method of measuring volatility for Generalized Stein and Stein model. Before we perform Monte Carlo simulation, we discretize the model with Euler's method first.

3.2.1. Discretization of Generalized Stein and Stein Model and Parameter Estimation

There are four parameters κ, θ, σ and ρ to estimate in this model. We will apply MLE method with Monte Carlo simulation to estimate the first three parameters, and method of moment to estimate ρ. Before estimation, we discretize the two processes S_t and $\sqrt{V_t}$ in the same way as previous section to $\{S_i\}_0^n$and $\{\sqrt{V_i}\}_0^n$, where $n = \frac{T-t}{\Delta}$. And we discretize the two equations under risk-neutral measure with Euler's method below:

$$\log(S_i) - \log(S_{i-1}) = (r - \frac{1}{2}V_{i-1})\Delta + \sqrt{V_{i-1}\Delta}z_{1,i}^*$$

$$\sqrt{V_i} - \sqrt{V_{i-1}} = \kappa(\theta - \sqrt{V_{i-1}})\Delta + \sigma\sqrt{\Delta}z_{2,i}^*$$

where $z_{1,i}^*$ and $z_{2,i}^*$ are standard normal variables correlated with coefficient ρ. Then we have $\sqrt{V_i} - \sqrt{V_{i-1}} - \kappa(\theta - \sqrt{V_{i-1}})\Delta = \sigma\sqrt{\Delta}z_{2,i}^*$. Let $U = \sigma\sqrt{\Delta}z_{2,i}^*$. Since $U = \sigma\sqrt{\Delta}z_{2,i}^* \sim N(0, \sigma^2\Delta)$, we get the probability density function of U as below:

$$f_U(u) = \frac{1}{\sqrt{2\pi\sigma^2\Delta}}e^{-\frac{u^2}{2\sigma^2\Delta}}$$

Notice $\frac{dU}{d\sqrt{V_i}} = 1$. If we substitute U with $\sqrt{V_i} - \sqrt{V_{i-1}} - \kappa(\theta - \sqrt{V_{i-1}})\Delta$, we will get the conditional density of $\sqrt{V_i}$ given $\sqrt{V_{i-1}}$ as follows:

$$f(\sqrt{V_i}\Big|\sqrt{V_{i-1}}) = \frac{1}{\sqrt{2\pi\sigma^2\Delta}}\exp(-\frac{X_i^2}{2\sigma^2\Delta})$$

where $X_i = \sqrt{V_i} - \sqrt{V_{i-1}} - \kappa(\theta - \sqrt{V_{i-1}})\Delta$. Then, we get the likelihood function $L(\kappa, \theta, \sigma)$ as below

$$L(\kappa, \theta, \sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2\Delta}} \exp(-\frac{X_i^2}{2\sigma^2\Delta})$$

Thus, the log-likelihood function $l(\kappa, \theta, \sigma) = \log(L)$ is given by

$$l(\kappa, \theta, \sigma) = \sum_{i=1}^{n} \{-0.5[\log(2\pi\sigma^2\Delta) + \frac{X_i^2}{2\sigma^2\Delta}]\}$$

The MLE of the parameters can be conducted through "maxLik" package of R programming software.

For the estimation of ρ, we adopt method of moment. Recall that $dW_{1,t}^*$ and $dW_{2,t}^*$ are correlated with coefficient ρ, that is

$$dW_{1,t}^* dW_{2,t}^* = \rho dt$$

where

$$dW_{1,t}^* = \frac{d\log(S_t) - (r - \frac{1}{2}V_t)dt}{\sqrt{V_t}}$$

$$dW_{2,t}^* = \frac{d\log(V_t) - (\kappa\theta - \kappa\sqrt{V_t} - \frac{1}{2}\sigma^2)dt}{\sigma}$$

The discrete form of the above two equations is given as

$$\sqrt{\Delta} z_{1,i}^* = \frac{\log(S_i) - \log(S_{i-1}) - (r - \frac{1}{2}V_{i-1})\Delta}{\sqrt{V_{i-1}}}$$

$$\sqrt{\Delta} z_{2,i}^* = \frac{\sqrt{V_i} - \sqrt{V_{i-1}} - \hat{\kappa}(\hat{\theta} - \sqrt{V_{i-1}})\Delta}{\hat{\sigma}}$$

where $\hat{\kappa}, \hat{\theta}, \hat{\sigma}$ are the corresponding ML estimators. As dt is discretized as Δ, we get

$$\rho = z_{1,i}^* z_{2,i}^*$$

Summate both sides of the above equation, we get

$$n\rho = \sum_{1}^{n} z_{1,i}^* z_{2,i}^*$$

Then the method of moment estimation of ρ is given as:

$$\hat{\rho} = \frac{1}{n}\sum_{1}^{n} z^*_{1,i} z^*_{2,i}$$

where

$$z^*_{1,i} = \frac{\log(S_i) - \log(S_{i-1}) - (r - \frac{1}{2}V_{i-1})\Delta}{\sqrt{\Delta}\sqrt{V_{i-1}}}$$

$$z^*_{2,i} = \frac{\sqrt{V_i} - \sqrt{V_{i-1}} - \hat{\kappa}(\hat{\theta} - \sqrt{V_{i-1}})\Delta}{\sqrt{\Delta}\hat{\sigma}}$$

3.2.2. Simulation Results

For data generation, we set the risk-free rate $r = 0.04$, initial standard deviation from transformed trading volume $\sqrt{V_0} = \sqrt{(10000000)^{-1/5}}$, and 1000 replications are simulated. Time to maturity is 0.5 years. Current stock price is assumed to be $S_0 = \$100$. During data generation process for V_t, negative values may be produced. We apply reflection method here. That is when a negative value of V_t is generated, it will be replaced by its absolute value. After obtaining ML estimators, we obtain estimated option prices in Table 3.4 after plugging the values of MLE into the closed-form option pricing formula given by (2).

Table 3.3 and Table 3.4 show that as sample size increases, the estimated parameter values are getting closer and closer to the true parameter values. And the estimated option prices show similar pattern of behaviors. As sample size increases, the estimated option prices are getting closer and closer to the true option prices. From here, we can conclude that the new method for the generalized Stein and Stein model is reliable.

Table 3.3. Parameter Estimation of New Method for Generalized Stein and Stein Model (1000 replications)

$True\,Value$	$\kappa = 10$	$\theta = 0.2$	$\sigma = 0.2$	$\rho = 0.7$
$n = 50$	32.8276	0.2389	0.1933	0.6747
$Bias$	22.8276	0.0389	-0.0067	-0.0253
$RMSE$	31.9191	0.5358	0.0201	0.1290
100	20.4512	0.2029	0.1965	0.6849
$Bias$	10.4512	0.0029	-0.0035	-0.0151
$RMSE$	16.0789	0.1265	0.0162	0.0867
200	15.2022	0.2005	0.1987	0.6956
$Bias$	5.2022	0.0005	-0.0013	-0.0044
$RMSE$	8.7047	0.0221	0.0096	0.0602
300	13.3608	0.1995	0.1988	0.6974
$Bias$	3.3608	-0.0005	-0.0012	-0.0026
$RMSE$	6.2384	0.019	0.0096	0.0507
400	12.6399	0.1999	0.1989	0.6964
$Bias$	2.6399	-0.0001	-0.0011	-0.0036
$RMSE$	5.0274	0.0158	0.0064	0.0444
600	11.5544	0.1994	0.1997	0.7001
$Bias$	1.5544	-0.0006	-0.0003	0.0001
$RMSE$	3.649	0.0127	0.0063	0.0348
800	11.3119	0.1995	0.1995	0.6976
$Bias$	1.3119	-0.0005	-0.0005	-0.0024
$RMSE$	3.0994	0.0127	0.0063	0.0317
1000	10.9414	0.1998	0.1996	0.6973
$Bias$	0.9414	-0.0002	-0.0004	-0.0027
$RMSE$	2.6519	0.0095	0.0032	0.0286

$True\,Value$	$\kappa = 10$	$\theta = 0.2$	$\sigma = 0.2$	$\rho = 0.7$
2000	10.4441	0.1998	0.1999	0.7010
$Bias$	0.4441	-0.0002	-0.0001	0.001
$RMSE$	1.7216	0.0063	0.0032	0.019
5000	10.1813	0.1998	0.1999	0.6995
$Bias$	0.1813	-0.0002	-0.0001	-0.0005
$RMSE$	1.0498	0.0032	0.0032	0.0127
10000	10.0814	0.1999	0.2000	0.7000
$Bias$	0.0814	-0.0001	0	0
$RMSE$	0.7476	0.0032	0	0.0095

Table 3.4. Option Price Estimation of New Method for Generalized Stein and Stein Model

$Strike\ Price$	$X = 80$	90	100	110	120
$Call\ True\ Value$	21.5868	12.9118	6.6469	3.0525	1.3045
$n = 50$	22.0128	13.7919	7.6606	3.8054	1.7215
100	21.7531	13.1064	6.7328	2.9935	1.1891
200	21.7236	13.0363	6.6807	2.9918	1.215
300	21.7116	13.0062	6.6603	2.995	1.2299
400	21.7099	13.0062	6.6712	3.0141	1.2481
600	21.5935	12.9215	6.6293	3.0092	1.2593
800	21.5931	12.9214	6.6321	3.0146	1.2646
1000	21.5922	12.9217	6.6399	3.0278	1.277
2000	21.5887	12.9146	6.6412	3.039	1.2905
5000	21.5873	12.9115	6.6419	3.0443	1.2968
10000	21.5869	12.9114	6.6445	3.0488	1.3011

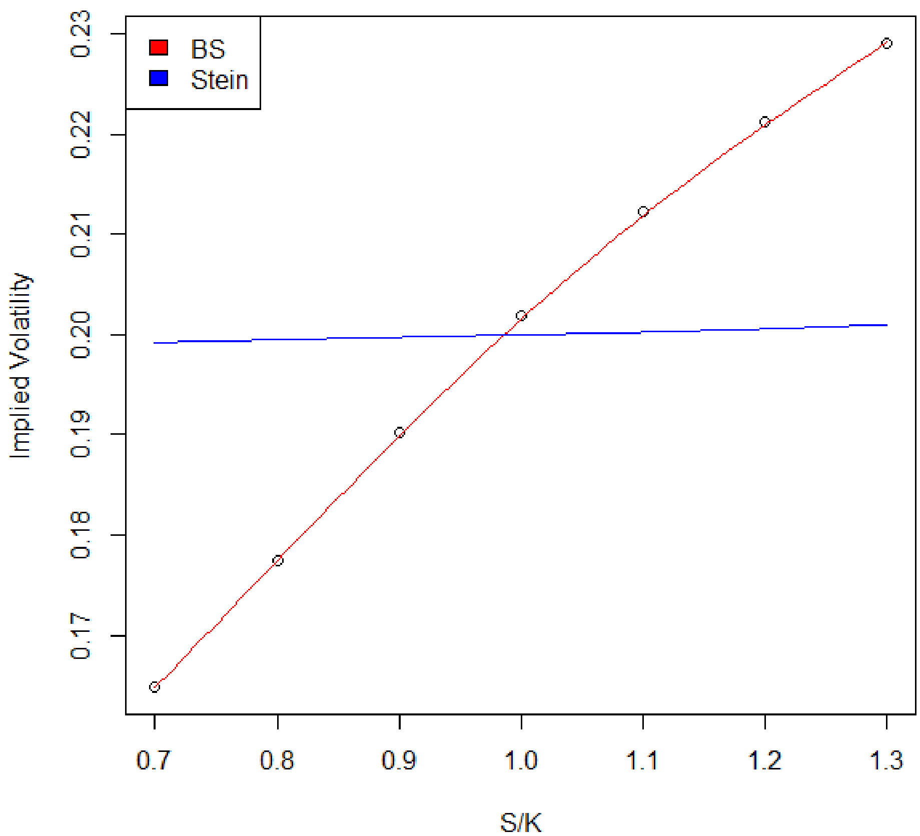

FIGURE 3.2. Stein and Stein VS BS Volatility Skew.

3.2.3. Implied Volatility Comparison between Generalized Stein and Stein Model and BS Model

The BS model implied volatility is obtained through the function "bscallimpvol" in R package "derivmkts". The generalized Stein and Stein model implied volatility is obtained through the function "BFfzero" in R package "NLRoot". In this function, the Bisection method is applied. We apply the values of MLE from sample size n = 10000 to get the generalized Stein and Stein model implied volatility.

From Figure 3.2, we can see that generalized Stein and Stein model implied volatility almost forms a horizontal line, while BS implied volatility forms a "forward skew". This tells us that generalized Stein and Stein model with our new method works better to reduce the volatility skew effect than BS model with constant volatility.

3.3. The New Method for Heston Model

Similar to what we have done to previous two stochastic volatility models, we apply the new method of measuring volatility to Heston model.

3.3.1. Heston Model Discretization and Parameter Estimation

There are four parameters κ, θ, σ and ρ to estimate in this model. We will apply MLE method with Monte Carlo Simulation to estimate the first three parameters, and method of moment to estimate ρ. Before estimation, we discretize the two processes S_t and $\sqrt{V_t}$ in the same way as previous section to $\{S_i\}_0^n$and $\{\sqrt{V_i}\}_0^n$, where $n = \frac{T-t}{\Delta}$. And we discretize the two equations under risk-neutral measure with Euler's method as below:

$$\log(S_i) - \log(S_{i-1}) = (r - \frac{1}{2}V_{i-1})\Delta + \sqrt{V_{i-1}\Delta}z_{1,i}^*$$

$$V_i - V_{i-1} = \kappa(\theta - \sqrt{V_{i-1}})\Delta + \sigma\sqrt{V_{i-1}\Delta}z_{2,i}^*$$

where $z_{1,i}^*$ and $z_{2,i}^*$ are standard normal variables correlated with coefficient ρ. Then we have $V_i - V_{i-1} - \kappa(\theta - \sqrt{V_{i-1}})\Delta = \sigma\sqrt{V_{i-1}\Delta}z_{2,i}^*$. Let $U = \sigma\sqrt{V_{i-1}\Delta}z_{2,i}^*$. Since $U = \sigma\sqrt{V_{i-1}\Delta}z_{2,i}^* \sim N(0, \sigma^2 V_{i-1}\Delta)$, we get the probability density function of U as below:

$$f_U(u) = \frac{1}{\sqrt{2\pi\sigma^2 V_{i-1}\Delta}} e^{-\frac{u^2}{2\sigma^2 V_{i-1}\Delta}}$$

Notice $\frac{dU}{dV_i} = 1$. If we substitute u with $V_i - V_{i-1} - \kappa(\theta - \sqrt{V_{i-1}})\Delta$, we will get the conditional density of V_i given V_{i-1} as follows:

$$f(V_i\,|V_{i-1}) = \frac{1}{\sqrt{2\pi\sigma^2 V_{i-1}\Delta}} \exp(-\frac{X_i^2}{2\sigma^2 V_{i-1}\Delta})$$

where $X_i = V_i - V_{i-1} - \kappa(\theta - \sqrt{V_{i-1}})\Delta$. Then, we get the likelihood function $L(\kappa, \theta, \sigma)$ as below

$$L(\kappa, \theta, \sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2 V_{i-1}\Delta}} \exp(-\frac{X_i^2}{2\sigma^2 V_{i-1}\Delta})$$

Thus, the log-likelihood function $l(\kappa, \theta, \sigma) = \log(L)$ is given by

$$l(\kappa, \theta, \sigma) = \sum_{i=1}^{n} \{-0.5[\log(2\pi\sigma^2 V_{i-1}\Delta) + \frac{X_i^2}{2\sigma^2 V_{i-1}\Delta}]\}$$

The MLE of the parameters can be performed through "maxLik" package of R programming software.

For the estimation of ρ, we adopt method of moment and follow the same procedures as in previous section. The method of moment estimation of ρ is given by:

$$\hat{\rho} = \frac{1}{n} \sum_{1}^{n} z_{1,i}^* z_{2,i}^*$$

where

$$z_{1,i}^* = \frac{log(S_i) - log(S_{i-1}) - (r - \frac{1}{2}V_{i-1})\Delta}{\sqrt{\Delta}\sqrt{V_{i-1}}}$$

$$z_{2,i}^* = \frac{V_i - V_{i-1} - \hat{\kappa}(\hat{\theta} - \sqrt{V_{i-1}})\Delta}{\sqrt{V_{i-1}\Delta}\hat{\sigma}}$$

And $\hat{\kappa}, \hat{\theta}, \hat{\sigma}$ are the corresponding ML estimators.

3.3.2. Simulation Results

For data generation, we set the risk-free rate $r = 0.02$, initial variance $V_0 = (3200000)^{-1/5}$, and 1000 replications are simulated. Time to maturity is 0.5 years. Initial stock price is assumed to be $S_0 = \$50$. During data generation process for V_t, reflection method is applied for negative values as in previous section. After obtaining ML estimators, we compute estimated option prices in Table 3.6 after plugging the values of MLE into the closed-form option pricing formula.

Table 3.5 and Table 3.6 show that as sample size increases, the estimated parameter values are getting closer and closer to the true parameter values. And the estimated option prices follow similar pattern As sample size increases, the estimated option prices are getting closer and closer to the true option prices. Form here, we can conclude that the new method for Heston model is reliable.

Table 3.5. Parameter Estimation of New Method for Heston Model (1000 replications)

$True\,Value$	$K = 12$	$\theta = 0.05$	$\sigma = 0.2$	$\rho = -0.6$
$n = 50$	35.4290	0.0588	0.1925	−0.5808
$Bias$	23.4290	0.0088	−0.0075	0.0192
$RMSE$	32.5598	0.1205	0.0204	0.1342
100	22.8212	0.0500	0.1960	−0.5888
$Bias$	10.8212	0	−0.004	0.0112
$RMSE$	16.6319	0.0063	0.0133	0.0955
200	16.7886	0.0501	0.1985	−0.5972
$Bias$	4.7886	0.0001	−0.0015	0.0028
$RMSE$	8.4015	0.0032	0.0096	0.0633
300	15.3433	0.0498	0.1991	−0.5972
$Bias$	3.3433	−0.0002	−0.0009	0.0028
$RMSE$	6.3818	0.0032	0.0095	0.0538
400	14.3311	0.0502	0.1992	−0.5958
$Bias$	2.3311	0.0002	−0.0008	0.0042
$RMSE$	5.0932	0.0032	0.0064	0.0445
600	13.6701	0.0499	0.1996	−0.6008
$Bias$	1.6701	−0.0001	−0.0004	−0.0008
$RMSE$	3.9731	0.0032	0.0063	0.038
800	13.3529	0.0500	0.1995	−0.5976
$Bias$	1.3529	0	−0.0005	0.0024
$RMSE$	3.3554	0.0032	0.0063	0.0317
1000	13.1405	0.0500	0.1995	−0.5978
$Bias$	1.1405	0	−0.0005	0.0022
$RMSE$	2.8473	0.0032	0.0032	0.0285

$True\,Value$	$K = 12$	$\theta = 0.05$	$\sigma = 0.2$	$\rho = -0.6$
2000	12.5101	0.0500	0.2001	-0.6004
$Bias$	0.5101	0	0.0001	-0.0004
$RMSE$	1.8855	0	0.0032	0.019
5000	12.2203	0.0500	0.2000	-0.5995
$Bias$	0.2203	0	0	0.0005
$RMSE$	1.1782	0	0.0032	0.0127
10000	12.1306	0.0500	0.2000	-0.5999
$Bias$	0.1306	0	0	0.0001
$RMSE$	0.8138	0	0	0.0095

Table 3.6. Option Price Estimation of New Method for Heston Model

Strike	$K = 40$	45	50	55	60
True Call Price	10.5980	6.4636	3.3676	1.4741	0.5406
$n = 50$	10.6575	6.6245	3.6198	1.7348	0.7349
100	10.5823	6.4482	3.371	1.4968	0.5668
200	10.5899	6.457	3.3725	1.4897	0.5571
300	10.5896	6.4532	3.3639	1.479	0.5486
400	10.5945	6.4626	3.3742	1.4865	0.5523
600	10.5936	6.4583	3.3659	1.4769	0.545
800	10.5948	6.4606	3.3684	1.4787	0.5459
1000	10.5953	6.4611	3.3683	1.4781	0.5451
2000	10.5968	6.4626	3.3679	1.4758	0.5425
5000	10.5974	6.4631	3.3678	1.4749	0.5415
10000	10.5977	6.4634	3.3677	1.4746	0.5411

3.3.3. Implied Volatility Comparison between Heston Model and BS Model

The BS model implied volatility is obtained through the function "bscallimpvol" in R package "derivmkts". The Heston model implied volatility is obtained through the function "BFfzero" in R package "NLRoot". In this function, the Bisection method is applied. We apply the values of MLE from sample size n = 10000 to get the Heston implied volatility.

From Figure 3.3, we can see that Heston implied volatility almost forms a horizontal line, while the BS model implied volatility forms a "reverse skew". This indicates that Heston model with our new method works better to reduce the volatility skew effect than BS model with constant volatility.

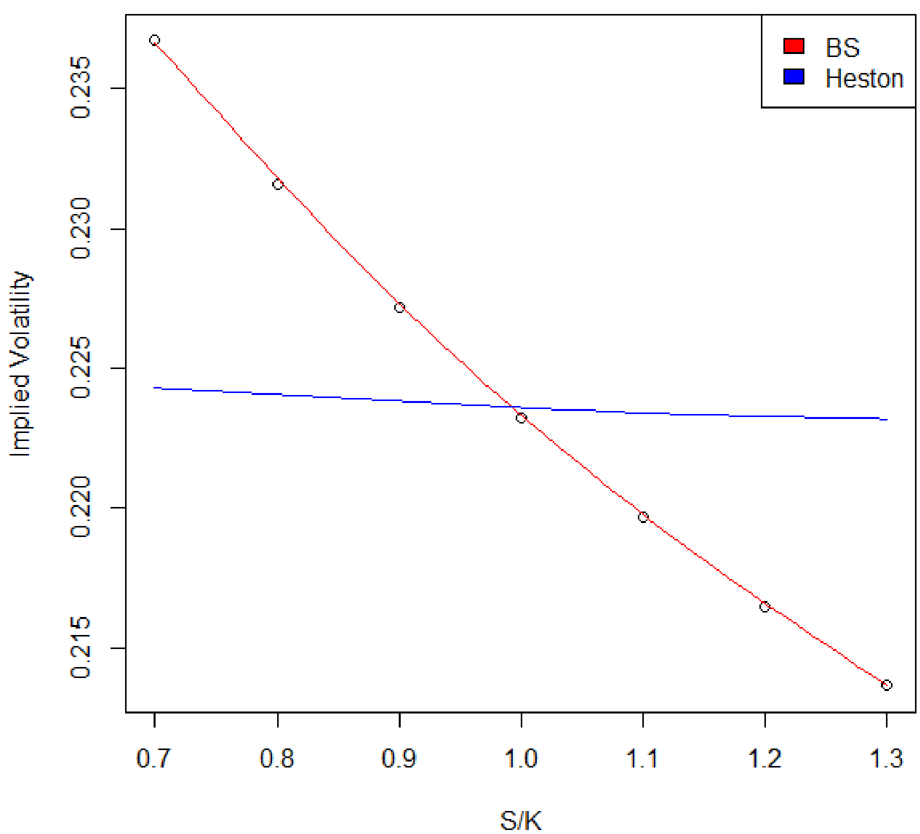

FIGURE 3.3. Heston VS BS Volatility Skew

Inspired by the Hull and White option pricing formula, which takes the drift term to be zero, we consider a special case of generalized Stein and Stein model by setting the drift term and correlation coefficient to be zero in the following section.

3.4. The New Method for a Special Case of Generalized Stein and Stein Model

Consider a special case of generalized Stein and Stein model under risk-neutral measure $\mathbb{Q}$

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW^*_{1,t}$$

$$d\sqrt{V_t} = \sigma dW^*_{2,t}$$

where $dW^*_{1,t}$ and $dW^*_{2,t}$ are independent Wiener processes under risk-neutral measure. Under risk-neutral probability measure $\mathbb{Q}$, the option pricing formula of the special case of generalized Stein and Stein model is given by

$$\begin{aligned}
C(S,t,T) &= \mathbb{E}^Q[e^{-r(T-t)}(S(T)-K)\cdot 1_{\{S(T)>K\}}] \\
&= \mathbb{E}^Q[e^{-r(T-t)}S(T)\cdot 1_{\{x(T)>\log K\}}] - e^{-r(T-t)}K\mathbb{E}^Q[1_{\{x(T)>\log K\}}] \\
&= S(t)\mathbb{E}^Q_1[1_{\{}x(T>\log K\}] - e^{-r(T-t)}K\mathbb{E}^Q_2[1_{\{}x(T>\log K\}] \\
&= S(t)P_1(S(T)>K) - e^{-r(T-t)}KP_2(S(T)>K).
\end{aligned} \tag{4}$$

Where P_1 and P_2 are given by the following Fourier inversion formula:

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re(f_j(\phi)\frac{\exp\{-i\phi\log K\}}{i\phi})d\phi,\ j=1,2.$$

In the above formula, f_1, f_2 are characteristic functions of P_1, P_2 respectively and are given as below:

$$\begin{aligned}
f_1(\phi) = \exp\{i\phi(r(T-t)+x(t))\} \\
\times \exp\{\frac{1}{2}D(t,T;s)v^2(t) + C(t,T;s)\}
\end{aligned}$$

$$f_2(\phi) = \exp\{i\phi(r(T-t) + x(t))\}$$
$$\times \exp\{\frac{1}{2}D(t,T;\hat{s})v^2(t) + C(t,T;\hat{s})\}$$

with

$$s = -\frac{1}{2}i\phi(1+i\phi),$$
$$\hat{s} = \frac{1}{2}\phi^2 + \frac{1}{2}i\phi,$$

$$D(t,T) = -\frac{\gamma}{\sigma^2}\tanh\{\gamma(T-t)\}$$
$$C(t,T) = -\frac{1}{2}\log(\cosh\{\gamma(T-t)\})$$

$$\gamma = \sqrt{2\sigma^2 s},$$

where v_t is the initial standard deviation. With functions $C(t,T)$ and $D(t,T)$, we obtain the closed-form solutions for $f_j(\phi)$, and hence the closed-form option pricing formula.

3.4.1. Model Discretization and Parameter Estimation

Since this is a special case of generalized Stein and Stein model, let $\kappa, \theta, \rho = 0$ in the generalized model, we will get the discretization as

$$\log(S_i) - \log(S_{i-1}) = (r - \frac{1}{2}V_{i-1})\Delta + \sqrt{V_{i-1}\Delta}z^*_{1,i}$$
$$\sqrt{V_i} - \sqrt{V_{i-1}} = \sigma\sqrt{\Delta}z^*_{2,i}$$

where $z^*_{1,i}$ and $z^*_{2,i}$ are independent standard normal variables. And the log-likelihood function $l(\sigma)$ is given by

$$l(\sigma) = \sum_{i=1}^{n}\{-0.5[\log(2\pi\sigma^2\Delta) + \frac{(\sqrt{V_i} - \sqrt{V_{i-1}})^2}{2\sigma^2\Delta}]\}$$

The MLE of the parameters can be performed through "maxLik" package of R programming software.

TABLE 3.7. Parameter Estimation of New Method for a Special Case of Generalized Stein and Stein Model (1000 replications)

$True\,Value$	$\sigma = 0.2$	$True\,Value$	$\sigma = 0.2$
$n = 50$	0.197	800	0.198
$Bias$	-0.003	$Bias$	-0.002
$RMSE$	0.0223	$RMSE$	0.0066
100	0.1966	1000	0.1981
$Bias$	-0.0034	$Bias$	-0.0019
$RMSE$	0.0162	$RMSE$	0.0037
200	0.1977	2000	0.1988
$Bias$	-0.0023	$Bias$	-0.0012
$RMSE$	0.0098	$RMSE$	0.0034
300	0.1976	5000	0.199
$Bias$	-0.0024	$Bias$	-0.001
$RMSE$	0.0098	$RMSE$	0.0033
400	0.1976	10000	0.1993
$Bias$	-0.0024	$Bias$	-0.0007
$RMSE$	0.0068	$RMSE$	0.0007
600	0.1983		
$Bias$	-0.0017		
$RMSE$	0.0065		

Table 3.8. Option Price Estimation of New Method for a Special Case of Generalized Stein and Stein Model

Strike Price	$K = 80$	90	100	110	120
Call True Value	21.7493	12.878	5.9186	2.2626	0.8722
$n = 50$	21.7435	12.8664	5.9041	2.2480	0.8607
100	21.7427	12.8649	5.9022	2.2461	0.8591
200	21.7448	12.8691	5.9074	2.2514	0.8634
300	21.7446	12.8687	5.9070	2.2509	0.8630
400	21.7446	12.8687	5.9070	2.2509	0.8630
600	21.7460	12.8714	5.9104	2.2543	0.8657
800	21.7454	12.8702	5.9089	2.2529	0.8645
1000	21.7456	12.8706	5.9094	2.2534	0.8649
2000	21.7470	12.8733	5.9128	2.2568	0.8676
5000	21.7474	12.8741	5.9138	2.2578	0.8684
10000	21.7480	12.8752	5.9152	2.2592	0.8695

3.4.2. Simulation Results

Set risk-free rate $r = 0.04$, initial standard deviation $\sqrt{V_0} = \sqrt{(90000000)^{-\frac{1}{5}}}$, and 1000 replications are simulated. Time to maturity is 0.5 years. During data generation process for $\sqrt{V_t}$, reflection method is applied. That is when a negative value of $\sqrt{V_t}$ is generated, it will be replaced by its absolute value. After obtaining MLE, we get estimated option prices in Table 3.8.

Table 3.7 and Table 3.8 show that as sample size increases, the estimated parameter values are getting closer and closer to the true parameter values. And the estimated option prices follow similar pattern. As sample size increases, the estimated option prices are getting closer and closer to the true option prices. Form here, we can conclude that the new method for the special case of generalized Stein and Stein model is reliable.

3.5. The New Method for a Special Case of Heston Model

Consider a special case of Heston model under risk-neutral measure $\mathbb{Q}$

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW^*_{1,t}$$

$$dV_t = \sigma \sqrt{V_t} dW^*_{2,t}$$

where $dW^*_{1,t}$ and $dW^*_{2,t}$ are independent Wiener processes under risk-neutral measure. Under risk-neutral probability measure $\mathbb{Q}$, the option pricing formula for the special case of Heston model is given by

$$C_t = S_t P_1 - Ke^{-r(T-t)} P_2 \tag{5}$$

where for $j = 1, 2$

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re[\frac{e^{-i\phi \log K} f_j(\phi; t, x_t, v_t)}{i\phi}] d\phi$$

$$f_j(\phi; x_t, V_t) = \exp(C_j(\tau, \phi) + D_j(\tau, \phi) V_t + i\phi x_t)$$

and

$$\tau = T - t, x_t = \log(S_t)$$

$$C_j(\tau, \phi) = ri\phi\tau$$

$$D_j(\tau, \phi) = \frac{d_j}{\sigma^2} (\frac{1 - e^{d_j \tau}}{1 + e^{d_j \tau}})$$

such that

$$d_j = \sqrt{-\sigma^2 (2u_j i\phi - \phi^2)}$$

$$u_1 = \frac{1}{2}, u_2 = -\frac{1}{2},$$

Table 3.9. Parameter Estimation of New Method for a Special Case of Heston Model (1000 replications)

$True\,Value$	$\sigma = 0.15$	$True\,Value$	$\sigma = 0.15$
$n = 50$	0.1223	800	0.1496
$Bias$	−0.0277	$Bias$	−0.0004
$RMSE$	0.0928	$RMSE$	0.0032
100	0.1337	1000	0.1494
$Bias$	−0.0163	$Bias$	−0.0006
$RMSE$	0.0684	$RMSE$	0.0032
200	0.1454	2000	0.1481
$Bias$	−0.0046	$Bias$	−0.0019
$RMSE$	0.0382	$RMSE$	0.0159
300	0.1479	5000	0.1463
$Bias$	−0.0021	$Bias$	−0.0037
$RMSE$	0.0254	$RMSE$	0.0102
400	0.1478	10000	0.1454
$Bias$	−0.0022	$Bias$	−0.0046
$RMSE$	0.0254	$RMSE$	0.0056
600	0.15		
$Bias$	0		
$RMSE$	0.0032		

3.5.1. Model Discretization and Parameter Estimation

Since this is a special case of Heston model, let $\kappa, \theta, \rho = 0$ in the general model, we will get the discretization as

$$\log(S_i) - \log(S_{i-1}) = (r - \frac{1}{2}V_{i-1})\Delta + \sqrt{V_{i-1}\Delta}z^*_{1,i}$$

$$V_i - V_{i-1} = \sigma\sqrt{V_{i-1}\Delta}z^*_{2,i}$$

where $z^*_{1,i}$ and $z^*_{2,i}$ are independent standard normal variables. And the log-likelihood function $l(\sigma)$ is given by

$$l(\sigma) = \sum_{i=1}^{n}\{-0.5[\log(2\pi\sigma^2 V_{i-1}\Delta) + \frac{(V_i - V_{i-1})^2}{2\sigma^2 V_{i-1}\Delta}]\}$$

The MLE of the parameters can be performed through "maxLik" package of R programming software.

TABLE 3.10. Option Price Estimation of New Method for a Special Case of Heston Model

Strike Price	$K = 80$	90	100	110	120
Call True Value	23.2409	16.0511	10.4895	6.5295	3.9046
$n = 50$	23.2408	16.0589	10.5022	6.5411	3.9107
100	23.2408	16.0559	10.4973	6.5366	3.9084
200	23.2409	16.0525	10.4918	6.5316	3.9057
300	23.2409	16.0518	10.4905	6.5305	3.9051
400	23.2409	16.0518	10.4906	6.5305	3.9052
600	23.2409	16.0511	10.4895	6.5295	3.9046
800	23.2409	16.0513	10.4897	6.5297	3.9047
1000	23.2409	16.0513	10.4898	6.5298	3.9048
2000	23.2409	16.0517	10.4904	6.5304	3.9051
5000	23.2409	16.0523	10.4913	6.5312	3.9055
10000	23.2409	16.0525	10.4918	6.5316	3.9057

3.6. Simulation Results

During the simulation process, we set the risk-free rate $r = 0.04$, initial variance $V_0 = (3200000)^{-1/7}$, and 1000 replications are simulated. Time to maturity is 0.5 years. Current stock price is assume to be $S_0 = \$100$. During data generation process for V_t, reflection method is applied. That is when a negative value of V_t is generated, it will be replaced by its absolute value. After obtaining ML estimators, we obtain estimated option prices in Table 3.10 after plugging the values of MLE into the closed-form option pricing formula with $\kappa, \theta, \rho = 0$.

Table 3.9 and Table 3.10 show that as sample size increases, the estimated parameter values are getting closer and closer to the true parameter values first until the sample size reaches 600. Then the estimated parameter values are increasingly deviating from the true value as the sample size increases. However, no matter how far away the estimated parameter value deviates from the true value, all the estimated option prices for any strike price have a difference less than one cent from the true option prices. We can claim that the model is reliable.

3.7. The New Method for a New Continuous Stochastic Volatility Model

Under physical measure $\mathbb{P}$, the new model is given by:

$$dS_t = \mu S_t dt + Y_t^{\frac{1}{4}} S_t dW_{1,t}$$

$$dY_t = \kappa(\theta - \sqrt{Y_t})\sqrt{Y_t}dt + Y_t^{\frac{3}{4}}\sigma dW_{2,t}$$

where $dW_{1,t}$ and $dW_{2,t}$ are correlated with coefficient ρ under physical measure. And $Y_t = V_t^2$, the square of the instantaneous variance. If we follow the argument of Stein and Stein (1991) about market price of volatility risk and set $\lambda_t = 0$, we get the model under the risk-neutral measure $\mathbb{Q}$:

$$dS_t = rS_t dt + Y_t^{\frac{1}{4}} S_t dW_{1,t}^*$$

$$dY_t = \kappa^*(\theta^* - \sqrt{Y_t})\sqrt{Y_t}dt + Y_t^{\frac{3}{4}}\sigma dW_{2,t}^*$$

where $\kappa^* = \kappa$, $\theta^* = \theta$. The two Wiener processes $dW_{1,t}^*$ and $dW_{2,t}^*$ are correlated with coefficient ρ under risk-neutral measure. And they are given by:

$$dW_{1,t}^* = dW_{1,t} + \frac{\mu - r}{\sqrt{V_t}}dt$$

$$dW_{2,t}^* = dW_{2,t}$$

Before we show the closed-form option pricing formula for this new continuous model, we introduce the following lemma.

LEMMA (Gil-Pelaez (1951) Inversion Theorem).

The cumulative distribution function (CDF) F_X of a random variable X and its characteristic function ϕ_X has the following relationship

$$F_X(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-iwx}\phi_X(w)}{iw}]dw \tag{6}$$

PROOF. See Appendix A.

THEOREM. The closed form solution of option pricing under the new continuous model is given as follows:

$$C_t = S_t P_1 - K e^{-r(T-t)} P_2 \tag{7}$$

where for $j = 1, 2$

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-i\phi \log K} f_j(\phi; t, x_t, v_t)}{i\phi}]d\phi$$

$$f_j(\phi; x_t, Y_t) = \exp(A_j(\tau, \phi) + B_j(\tau, \phi)\sqrt{Y_t} + i\phi x_t)$$

and

$$\tau = T - t$$

$$x_t = \log(S_t)$$

$$A_j(\tau, \phi) = ri\phi\tau + (\frac{2a}{\sigma^2} - \frac{1}{2})[(\frac{1}{2}b_j - \frac{1}{2}\rho\sigma i\phi + d_j)\tau - 2\log(\frac{1 - g_j e^{d_j\tau}}{1 - g_j})]$$

$$B_j(\tau, \phi) = \frac{2b_j - 2\rho\sigma i\phi + 4d_j}{\sigma^2}(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}})$$

such that

$$g_j = \frac{b_j - \rho\sigma i\phi + 2d_j}{b_j - \rho\sigma i\phi - 2d_j}$$

$$d_j = \frac{1}{2}\sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)}$$

$$u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = \kappa\theta, b_1 = \kappa - \rho\sigma, b_2 = \kappa$$

PROOF. We will adopt self-financing argument similar to Galiotos (2008) [17] to derive a PDE for our new continuous model .

First we construct a portfolio consisting of a risk-free asset with value B_t, Δ units of underlying stock, and ϕ units of an option U_t. And U_t is priced under our new continuous model framework. The portfolio has value P_t given as

$$P_t = \Omega B_t + \Delta S_t + \phi U_t \tag{7a}$$

and B_t satisfies

$$dB_t = rB_t dt \tag{7b}$$

Assume the portfolio is self-financing. Now we are trying to hedge the volatility risk for another option with price C_t by setting

$$P_t = C_t$$

The change in portfolio value is

$$dP_t = \Omega dB_t + \Delta dS_t + \phi dU_t \tag{7c}$$

Apply Ito's Lemma to C_t and U_t under physical measure , we get

$$\begin{aligned} dC_t = &\frac{\partial C_t}{\partial t}dt + \frac{\partial C_t}{\partial S_t}[\mu S_t dt + Y_t^{\frac{1}{4}} S_t dW_{1,t}] + \frac{\partial C_t}{\partial Y_t}[\kappa(\theta - \sqrt{Y_t})\sqrt{Y_t}dt + Y_t^{\frac{3}{4}}\sigma dW_{2,t}] \\ &+ (\frac{1}{2}\frac{\partial^2 C_t}{\partial {S_t}^2})Y_t^{\frac{1}{2}} S_t^2 dt + (\frac{1}{2}\frac{\partial^2 C_t}{\partial {Y_t}^2})\sigma^2 Y_t^{\frac{3}{2}} dt + \frac{\partial^2 C_t}{\partial S_t \partial Y_t}\rho\sigma S_t Y_t dt \end{aligned} \tag{8a}$$

$$\begin{aligned} dU_t = &\frac{\partial U_t}{\partial t}dt + \frac{\partial U_t}{\partial S_t}[\mu S_t dt + Y_t^{\frac{1}{4}} S_t dW_{1,t}] + \frac{\partial U_t}{\partial Y_t}[\kappa(\theta - \sqrt{Y_t})\sqrt{Y_t}dt + Y_t^{\frac{3}{4}}\sigma dW_{2,t}] \\ &+ (\frac{1}{2}\frac{\partial^2 U_t}{\partial {S_t}^2})Y_t^{\frac{1}{2}} S_t^2 dt + (\frac{1}{2}\frac{\partial^2 U_t}{\partial {Y_t}^2})\sigma^2 Y_t^{\frac{3}{2}} dt + \frac{\partial^2 U_t}{\partial S_t \partial Y_t}\rho\sigma S_t Y_t dt \end{aligned} \tag{8b}$$

Substitute (7b), (8b) and stock price equation dS_t under physical measure into (7c), we get

(8c)

$$dP_t = \phi[\frac{\partial U_t}{\partial t} + \frac{\partial U_t}{\partial S_t}\mu S_t + \frac{\partial U_t}{\partial Y_t}[\kappa(\theta - \sqrt{Y_t})\sqrt{Y_t} + \frac{1}{2}{Y_t}^{\frac{1}{2}} S_t^2 \frac{\partial^2 U_t}{\partial {S_t}^2} + \frac{1}{2}\sigma^2 {Y_t}^{\frac{3}{2}} \frac{\partial^2 U_t}{\partial {Y_t}^2} + \rho\sigma S_t Y_t \frac{\partial^2 U_t}{\partial S_t \partial Y_t}]dt$$

$$(\Omega r B_t + \Delta\mu S_t)dt + (\Delta {Y_t}^{\frac{1}{4}} S_t + \phi \frac{\partial U_t}{\partial S_t} {Y_t}^{\frac{1}{4}} S_t) dW_{1,t} + \phi \frac{\partial U_t}{\partial Y_t} {Y_t}^{\frac{3}{4}} \sigma dW_{2,t}$$

Set (8a) and (8c) equal, we should also set the coefficients of $dt, dW_{1,t}, dW_{2,t}$ equal between those two equations. Then we have

$$\phi = (\frac{\partial C_t}{\partial Y_t})/(\frac{\partial U_t}{\partial Y_t}), \Delta = -\phi\frac{\partial U_t}{\partial S_t} + \frac{\partial C_t}{\partial S_t} \tag{9a}$$

Since $P_t = C_t$, $\Omega B_t = P_t - \Delta S_t - \phi U_t = C_t - \Delta S_t - \phi U_t$, this will give the drift term of dP_t as

$$\begin{aligned} \text{drift}(dP_t) =& \phi[\frac{\partial U_t}{\partial t} + \frac{\partial U_t}{\partial S_t}\mu S_t + \frac{\partial U_t}{\partial Y_t}[\kappa(\theta - \sqrt{Y_t})\sqrt{Y_t} + \frac{1}{2}{Y_t}^{\frac{1}{2}} S_t^2 \frac{\partial^2 U_t}{\partial {S_t}^2} \\ & + \sigma^2 {Y_t}^{\frac{3}{2}} \frac{1}{2}\frac{\partial^2 U_t}{\partial {Y_t}^2} + \rho\sigma S_t Y_t \frac{\partial^2 U_t}{\partial S_t \partial Y_t}] + r(C_t - \Delta S_t - \phi U_t) + \Delta\mu S_t \end{aligned} \tag{9b}$$

And the drift of dC_t is given by

$$\begin{aligned} \text{drift}(dC_t) =& \frac{\partial C_t}{\partial t} + \mu S_t \frac{\partial C_t}{\partial S_t} + \frac{\partial C_t}{\partial Y_t}[\kappa(\theta - \sqrt{Y_t})\sqrt{Y_t} + \frac{1}{2}\sqrt{Y_t} S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} \\ & + \frac{1}{2}\sigma^2 {Y_t}^{\frac{3}{2}} \frac{\partial^2 C_t}{\partial Y_t^2} + \rho\sigma S_t Y_t \frac{\partial^2 C_t}{\partial S_t \partial Y_t} \end{aligned} \tag{9c}$$

Set the two drift terms (9b) and (9c) equal, and substitute (9a) into them, after some computation we will get

$$(\frac{\partial C_t}{\partial Y_t})^{-1}[-rC_t + \frac{\partial C_t}{\partial t} + rS_t\frac{\partial C_t}{\partial S_t} + \frac{\partial C_t}{\partial Y_t}[\kappa(\theta - \sqrt{Y_t})]\sqrt{Y_t} + \frac{1}{2}\sqrt{Y_t} S_t^2 \frac{\partial^2 C_t}{\partial S_t^2}$$

$$+ \frac{1}{2}\sigma^2 {Y_t}^{\frac{3}{2}} \frac{\partial^2 C_t}{\partial Y_t^2} + \rho\sigma S_t Y_t \frac{\partial^2 C_t}{\partial S_t \partial Y_t}]$$

$$= (\frac{\partial U_t}{\partial Y_t})^{-1}[-rU_t + \frac{\partial U_t}{\partial t} + rS_t\frac{\partial U_t}{\partial S_t} + \frac{\partial U_t}{\partial Y_t}[\kappa(\theta - \sqrt{Y_t})]\sqrt{Y_t} + \frac{1}{2}\sqrt{Y_t}S_t^2\frac{\partial^2 U_t}{\partial S_t^2}$$

$$+ \frac{1}{2}\sigma^2 Y_t^{\frac{3}{2}}\frac{\partial^2 U_t}{\partial Y_t^2} + \rho\sigma S_t Y_t\frac{\partial^2 U_t}{\partial S_t \partial Y_t}] \tag{9d}$$

And the left-hand side of the above equation is a function of C_t, but the right-hand side is a function of U_t. That means the above equation can be rewritten as a function of S_t, Y_t, t, name it as $\lambda(S_t, Y_t, t)$, which is defined as market price of volatility risk. Rewrite is as λ_t for simplicity. Then from the left-hand side of (9d), we have

$$(\frac{\partial C_t}{\partial Y_t})^{-1}[-rC_t + \frac{\partial C_t}{\partial t} + rS_t\frac{\partial C_t}{\partial S_t} + \frac{\partial C_t}{\partial Y_t}[\kappa(\theta - \sqrt{Y_t})]\sqrt{Y_t} + \frac{1}{2}\sqrt{Y_t}S_t^2\frac{\partial^2 C_t}{\partial S_t^2}$$

$$+ \frac{1}{2}\sigma^2 Y_t^{\frac{3}{2}}\frac{\partial^2 C_t}{\partial Y_t^2} + \rho\sigma S_t Y_t\frac{\partial^2 C_t}{\partial S_t \partial Y_t}] = \lambda_t$$

This will give us

$$[-rC_t + \frac{\partial C_t}{\partial t} + rS_t\frac{\partial C_t}{\partial S_t} + \frac{\partial C_t}{\partial Y_t}[\kappa(\theta - \sqrt{Y_t})\sqrt{Y_t} - \lambda_t] + \frac{1}{2}\sqrt{Y_t}S_t^2\frac{\partial^2 C_t}{\partial S_t^2}$$

$$+ \frac{1}{2}\sigma^2 Y_t^{\frac{3}{2}}\frac{\partial^2 C_t}{\partial Y_t^2} + \rho\sigma S_t Y_t\frac{\partial^2 C_t}{\partial S_t \partial Y_t}] = 0$$

In the above equation, set λ_t to be a special value 0, and reparametrize as $\kappa^* = \kappa, \theta^* = \theta$, we get the following PDE.

$$-rC_t + \frac{\partial C_t}{\partial t} + rS_t\frac{\partial C_t}{\partial S_t} + (\kappa^*\theta^* - \kappa^*\sqrt{Y_t})\sqrt{Y_t}\frac{\partial C_t}{\partial Y_t} + \frac{1}{2}\sqrt{Y_t}S_t^2\frac{\partial^2 C_t}{\partial S_t^2} + \frac{1}{2}\sigma^2 Y_t^{\frac{3}{2}}\frac{\partial^2 C_t}{\partial Y_t^2} + \rho\sigma S_t Y_t\frac{\partial^2 C_t}{\partial S_t \partial Y_t} = 0$$

For the remaining part of the proof, we will follow the framework of Rouah (2013) [34], but work with our new continuous model. Let $x_t = \log(S_t)$, then we have:

$$\frac{\partial C_t}{\partial S_t} = \frac{1}{S_t}\frac{\partial C_t}{\partial x_t}, \frac{\partial^2 C_t}{\partial S_t \partial V_t} = \frac{1}{S_t}\frac{\partial^2 C_t}{\partial x_t \partial V_t}$$

And apply product rule, we have:

$$\frac{\partial^2 C_t}{\partial S_t^2} = -\frac{1}{S_t^2}\frac{\partial C_t}{\partial x_t} + \frac{1}{S_t}\frac{\partial^2 C_t}{\partial S_t \partial x_t} = -\frac{1}{S_t^2}\frac{\partial C_t}{\partial x_t} + \frac{1}{S_t^2}\frac{\partial^2 C_t}{\partial x_t^2}$$

And rewrite the new model PDE in terms of x_t instead of S_t, then we get the new PDE for our continuous model:

$$\begin{aligned} &-rC_t + \frac{\partial C_t}{\partial t} + (r - \frac{1}{2}\sqrt{Y_t})\frac{\partial C_t}{\partial x_t} + (\kappa\theta - \kappa\sqrt{Y_t})\sqrt{Y_t}\frac{\partial C_t}{\partial Y_t} \\ &+ \frac{1}{2}\sqrt{Y_t}\frac{\partial^2 C_t}{\partial {x_t}^2} + \frac{1}{2}\sigma^2 {Y_t}^{\frac{3}{2}}\frac{\partial^2 C_t}{\partial Y_t^2} + \rho\sigma Y_t \frac{\partial^2 C_t}{\partial x_t \partial Y_t} = 0 \end{aligned} \tag{9e}$$

where κ^*, θ^* are replaced with κ, θ in the above equation. Notice that (9e) holds for any strike price $0 \leq K$, any stock price $0 \leq S_t$ any value of risk-free interest rate $0 \leq r$. Set $K = 0, S_t = 1$ in the option pricing formula will give us $C_t = P_1$. So P_1 satisfies (9e). Similarly, set $K = 1, S_t = 0, r = 0$ in the option pricing formula will give us $C_t = -P_2$. This shows $-P_2$ satisfies (9e), so does P_2. And we have

$$C_t = S_t P_1 - Ke^{-r(T-t)}P_2 = e^{x_t}P_1 - Ke^{-r(T-t)}P_2$$

Then we have the derivative of C_t with respect to t:

$$\frac{\partial C_t}{\partial t} = e^{x_t}\frac{\partial P_1}{\partial t} - Ke^{-r(T-t)}(rP_2 + \frac{\partial P_2}{\partial t}) \tag{10}$$

With respect to x_t:

$$\frac{\partial C_t}{\partial x_t} = e^{x_t}(P_1 + \frac{\partial P_1}{\partial x_t}) - Ke^{-r(T-t)}\frac{\partial P_2}{\partial x_t} \tag{11}$$

With respect to x_t^2:

$$\begin{aligned} \frac{\partial^2 C_t}{\partial x_t^2} &= e^{x_t}P_1 + 2e^{x_t}\frac{\partial P_1}{\partial x_t} + e^{x_t}\frac{\partial^2 P_1}{\partial x_t^2} - Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial x_t^2} \\ &= e^{x_t}(P_1 + 2\frac{\partial P_1}{\partial x_t} + \frac{\partial^2 P_1}{\partial x_t^2}) - Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial x_t^2} \end{aligned} \tag{12}$$

With respect to Y_t and Y_t^2:

$$\frac{\partial C_t}{\partial Y_t} = e^{x_t}\frac{\partial P_1}{\partial Y_t} - Ke^{-r(T-t)}\frac{\partial P_2}{\partial Y_t}, \frac{\partial^2 C_t}{\partial Y_t^2} = e^{x_t}\frac{\partial^2 P_1}{\partial Y_t^2} - Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial Y_t^2} \tag{13}$$

With respect to Y_t and x_t:

$$\frac{\partial^2 C_t}{\partial x_t \partial Y_t} = e^{x_t}(\frac{\partial P_1}{\partial Y_t} + \frac{\partial^2 P_1}{\partial x_t \partial Y_t}) - Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial x_t \partial Y_t} \tag{14}$$

In Equations (10) through (14), regroup terms and cancel e^{x_t} (since $e^{x_t} = S_t = 1$ in this set up), and substitute the terms into (9e), we get:

$$\begin{aligned} &-rP_1 + \frac{\partial P_1}{\partial t} + \frac{1}{2}\sqrt{Y_t}(P_1 + 2\frac{\partial P_1}{\partial x_t} + \frac{\partial^2 P_1}{\partial x_t^2}) + (r - \frac{1}{2}\sqrt{Y_t})(P_1 + \frac{\partial P_1}{\partial x_t}) + \rho\sigma Y_t(\frac{\partial P_1}{\partial Y_t} + \frac{\partial^2 P_1}{\partial x_t \partial Y_t}) \\ &+ \frac{1}{2}\sigma^2 Y_t^{\frac{3}{2}}\frac{\partial^2 P_1}{\partial Y_t^2} + (\kappa\theta - \kappa\sqrt{Y_t})\sqrt{Y_t}\frac{\partial P_1}{\partial Y_t} = 0 \end{aligned} \tag{15}$$

Simplify (15), we obtain:

$$\begin{aligned} &\frac{\partial P_1}{\partial t} + (r + \frac{1}{2}\sqrt{Y_t})\frac{\partial P_1}{\partial x_t} + \frac{1}{2}\sqrt{Y_t}\frac{\partial^2 P_1}{\partial x_t^2} + \rho\sigma Y_t\frac{\partial^2 P_1}{\partial x_t \partial Y_t} + \\ &[\kappa\theta\sqrt{Y_t} - (\kappa - \rho\sigma)Y_t]\frac{\partial P_1}{\partial Y_t} + \frac{1}{2}\sigma^2 Y_t^{\frac{3}{2}}\frac{\partial^2 P_1}{\partial Y_t^2} = 0 \end{aligned} \tag{16}$$

Similarly, in Equations (10) through (14), regroup terms and cancel $-Ke^{-r(T-t)}$ (since $K = 1, r = 0, -Ke^{-r(T-t)} = -1$ in this set up), and substitute the terms into (9e), we get:

$$\begin{aligned} &\frac{\partial P_2}{\partial t} + (r - \frac{1}{2}\sqrt{Y_t})\frac{\partial P_2}{\partial x_t} + \frac{1}{2}\sqrt{Y_t}\frac{\partial^2 P_2}{\partial x_t^2} + \rho\sigma Y_t\frac{\partial^2 P_2}{\partial x_t \partial Y_t} + \\ &[\kappa\theta\sqrt{Y_t} - \kappa Y_t]\frac{\partial P_2}{\partial Y_t} + \frac{1}{2}\sigma^2 Y_t^{\frac{3}{2}}\frac{\partial^2 P_2}{\partial Y_t^2} = 0 \end{aligned} \tag{17}$$

For convenience, combine Equation (16) and Equation (17) into one expression

$$\begin{aligned} &\frac{\partial P_j}{\partial t} + (r + u_j\sqrt{Y_t})\frac{\partial P_j}{\partial x_t} + \frac{1}{2}\sqrt{Y_t}\frac{\partial^2 P_j}{\partial x_t^2} + \rho\sigma Y_t\frac{\partial^2 P_j}{\partial x_t \partial Y_t} + \\ &[a\sqrt{Y_t} - b_j Y_t]\frac{\partial P_j}{\partial Y_t} + \frac{1}{2}\sigma^2 Y_t^{\frac{3}{2}}\frac{\partial^2 P_j}{\partial Y_t^2} = 0 \end{aligned} \tag{18}$$

for $j = 1, 2$ and where $u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = \kappa\theta, b_1 = \kappa - \rho\sigma, b_2 = \kappa$.

Let $f_j(\phi; x_t, Y_t)$ be the corresponding characteristic functions of P_j for $j = 1, 2$. When the characteristic functions $f_j(\phi; x_t, Y_t)$ are known, each in-the-money probability P_j can be recovered from the characteristic function via the Gil-Pelaez (1951) [18] inversion theorem, as

$$
\begin{aligned}
P_j =& Pr(\log(S_T) > \log(K)) \\
&= 1 - Pr(\log(S_T) < \log(K)) \\
&= 1 - (\frac{1}{2} - \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-i\phi\log(K)}f_j(\phi; x_t, Y_t)}{i\phi}]d\phi) \\
&= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-i\phi\log(K)}f_j(\phi; x_t, Y_t)}{i\phi}]d\phi
\end{aligned} \tag{19}
$$

At maturity, the probabilities are subject to the terminal condition

$$
P_j = \mathbb{1}_{x_T > \log K} \tag{20}
$$

where $\mathbb{1}$ is the indicator function. Equation (20) shows when $S_T > K$ at expiry, the probability of the call being in-the-money is unity. We postulate that the characteristic functions for the logarithm of the terminal stock price, $x_T = \log S_T$, are of the log linear form

$$
f_j(\phi; x_t, Y_t) = \exp(A_j(\tau, \phi) + B_j(\tau, \phi)\sqrt{Y_t} + i\phi x_t) \tag{21}
$$

where A_j and B_j are coefficients and $\tau = T - t$ is the time to maturity for the European call option.

The characteristic functions f_i will follow the PDE in Equation (18). This is a result of the Feynman-Kac theorem, which stipulates that, if a function $f(\mathbf{x}_t, t)$ of SDEs $\mathbf{x}_t = (x_t, Y_t) = (\log(S_t), Y_t)$ satisfies the PDE $\partial f/\partial t - rf + \mathcal{A}f = 0$, Where $\mathcal{A}$ is the generator defined as follows,

$$
\mathcal{A} = (r - \frac{1}{2}\sqrt{Y_t})\frac{\partial}{\partial x_t} + \frac{1}{2}\sqrt{Y_t}\frac{\partial^2}{\partial x_t^2} + \rho\sigma Y_t\frac{\partial^2}{\partial x_t \partial Y_t} +
$$

$$
[\kappa\theta\sqrt{Y_t} - \kappa Y_t]\frac{\partial}{\partial Y_t} + \frac{1}{2}\sigma^2 Y_t^{\frac{3}{2}}\frac{\partial^2}{\partial Y_t^2}
$$

Then the solution to $f(\mathbf{x}_t, t)$ is the conditional expectation

$$f(\mathbf{x}_t, t) = E[f(\mathbf{x}_T, T)|\mathcal{F}_t]$$

where $\mathcal{F}_t$ is the natural filtration generated by $W_{1,t}$ and $W_{2,t}$ jointly up to time t. Notice that the values of $\log(S_t)$ and Y_t can be completely determined by the information available up to time t. Using $f(\mathbf{x}_t, t) = \exp[i\phi \log(S_t)]$ produces the solution

$$f(\mathbf{x}_t, t) = E[e^{i\phi \log(S_T)}|x_t, Y_t]$$

which is the characteristic function for $x_T = \log(S_T)$. Hence, the PDE for the characteristic function is, from Equation (18)

$$-\frac{\partial f_j}{\partial \tau} + (r + u_j\sqrt{Y_t})\frac{\partial f_j}{\partial x_t} + \frac{1}{2}\sqrt{Y_t}\frac{\partial^2 f_j}{\partial x_t^2} + \rho\sigma Y_t \frac{\partial^2 f_j}{\partial x_t \partial Y_t} +$$

$$[a\sqrt{Y_t} - b_j Y_t]\frac{\partial f_j}{\partial Y_t} + \frac{1}{2}\sigma^2 {Y_t}^{\frac{3}{2}}\frac{\partial^2 f_j}{\partial Y_t^2} = 0 \tag{22}$$

Note the change from t to τ, which explains the negative sign in front of the first term in the PDE (22). The following derivatives are required to evaluate (22)

$$\frac{\partial f_j}{\partial \tau} = (\frac{\partial A_j}{\partial \tau} + \frac{\partial B_j}{\partial \tau}\sqrt{Y_t})f_j, \frac{\partial f_j}{\partial x_t} = i\phi f_j, \frac{\partial f_j}{\partial Y_t} = \frac{B_j}{2\sqrt{Y_t}}f_j$$

$$\frac{\partial^2 f_j}{\partial x^2} = -\phi^2 f_j, \frac{\partial^2 f_j}{\partial Y_t^2} = (\frac{B_j^2}{4Y_t} - \frac{B_j}{4\sqrt{Y_t^3}})f_j, \frac{\partial^2 f_j}{\partial Y_t \partial x_t} = \frac{i\phi B_j}{2\sqrt{Y_t}}f_j$$

Substitute these derivatives into (22) and drop the f_j terms to obtain

$$-(\frac{\partial A_j}{\partial \tau} + \sqrt{Y_t}\frac{\partial B_j}{\partial \tau}) + (r + u_j\sqrt{Y_t})i\phi + \frac{1}{2}\sqrt{Y_t}(-\phi^2) + \rho\sigma Y_t(\frac{i\phi B_j}{2\sqrt{Y_t}}) +$$

$$[a\sqrt{Y_t} - b_j Y_t]\frac{B_j}{2\sqrt{Y_t}} + \frac{1}{2}\sigma^2 {Y_t}^{\frac{3}{2}}(\frac{B_j^2}{4Y_t} - \frac{B_j}{4\sqrt{Y_t^3}}) = 0,$$

or equivalently

$$\sqrt{Y_t}(-\frac{\partial B_j}{\partial \tau} + u_j i\phi - \frac{1}{2}\phi^2 + \frac{\rho\sigma i\phi B_j}{2} - \frac{b_j B_j}{2} + \frac{\sigma^2 B_j^2}{8}) +$$

$$(-\frac{\partial A_j}{\partial \tau} + ri\phi + \frac{aB_j}{2} - \frac{\sigma^2 B_j}{8}) = 0$$

This gives two differential equations

$$\frac{\partial B_j}{\partial \tau} = u_j i\phi - \frac{1}{2}\phi^2 + \frac{\rho\sigma i\phi B_j}{2} - \frac{b_j B_j}{2} + \frac{\sigma^2 B_j^2}{8} \tag{23a}$$

$$\frac{\partial A_j}{\partial \tau} = ri\phi + \frac{aB_j}{2} - \frac{\sigma^2 B_j}{8} \tag{23b}$$

The equation in (23a) is a Riccati equation for B_j, while the equation in (23b) is an ordinary derivative for A_j that can be solved through direct integration once B_j is obtained. Solving these equations requires two initial conditions. Recall from (21) that the characteristic function is

$$f_j(\phi; x_t, v_t) = E[e^{i\phi x_T}] = \exp(A_j(\tau, \phi) + B_j(\tau, \phi)\sqrt{Y_t} + i\phi x_t). \tag{24}$$

At maturity $\tau = 0$, the value of $x_T = \log(S_T)$ is known, so the expectation in (24) will simply reduce to $\exp(i\phi x_T)$. This implies that the initial conditions at maturity are $B_j(0, \phi) = 0$ and $A_j(0, \phi) = 0$. At last ,when we compute the characteristic function, we use x_t as the log spot price of the underlying asset, and Y_t as its initial variance squared.

Solving the Riccati Equation

We will explain how the expressions in equations (23a) and (23b) can be solved to yield the call price. First, we introduce the Riccati equation and explain how its solution is obtained.

The Riccati Equation in a General Setting

Consider the generic Riccati equation for $y(t)$ with coefficients $P(t), Q(t)$, and $R(t)$ defined as follows:

$$\frac{dy(t)}{dt} = P(t) + Q(t)y(t) + R(t)y(t)^2. \tag{25}$$

The equation can be solved by considering the following second-order ordinary differential equation (ODE) for $w(t)$

(26a) $$w'' - [\frac{R'}{R} + Q]w' + PRw = 0$$

where $w(t)$ is given by the following relationship

(26b) $$y(t) = -\frac{w'(t)}{w(t)}\frac{1}{R(t)}$$

Notice that (26a) can be written as $w'' + bw' + cw = 0$, where $b = -(\frac{R'}{R} + Q)$, $c = PR$. The solution to Equation (25) is then given by (26b).

The ODE in (26a) can be solved through the characteristic equation $r^2 + br + c = 0$, which has two solutions r_1 and r_2 given by

$$r_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}, r_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

The solution to the second-order ODE in (26a) is

$$w(t) = Me^{r_1 t} + Ne^{r_2 t}$$

where M and N are constants. The solution to the Riccati equation is therefore

$$y(t) = -\frac{Mr_1 e^{r_1 t} + Nr_2 e^{r_2 t}}{Me^{r_1 t} + Ne^{r_2 t}}\frac{1}{R(t)}.$$

Solution to the Riccati Equation

For Equation (23a), this Riccati equation can be rewritten as

$$\frac{\partial B_j}{\partial \tau} = P_j - Q_j B_j + RB_j^2$$

where

$$P_j = u_j i\phi - \frac{1}{2}\phi^2, Q_j = \frac{b_j}{2} - \frac{\rho\sigma i\phi}{2}, R = \frac{1}{8}\sigma^2$$

The corresponding second-order ODE is

$$w'' + Q_j w' + P_j Rw = 0$$

and w here is given by the relationship $B_j = -\frac{1}{R}\frac{w'}{w}$. The characteristic equation is $r^2 + Q_j r + P_j R = 0$, which has roots

$$r_{1,j} = \frac{-Q_j + \sqrt{Q_j^2 - 4P_jR}}{2} = \frac{-Q_j + d_j}{2}$$

$$r_{2,j} = \frac{-Q_j - \sqrt{Q_j^2 - 4P_jR}}{2} = \frac{-Q_j - d_j}{2}$$

where

$$\begin{aligned} d_j &= r_{1,j} - r_{2,j} \\ &= \sqrt{Q_j^2 - 4P_jR} \\ &= \frac{1}{2}\sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)}. \end{aligned}$$

The solution to the Riccati equation is therefore

$$(27) \qquad B_j = -\frac{1}{R}\frac{w'}{w} = -\frac{1}{R}\frac{M_j r_{1,j}e^{r_{1,j}\tau} + N_j r_{2,j}e^{r_{2,j}\tau}}{M_j e^{r_{1,j}\tau} + N_j e^{r_{2,j}\tau}} = -\frac{1}{R}\frac{K_j r_{1,j}e^{r_{1,j}\tau} + r_{2,j}e^{r_{2,j}\tau}}{K_j e^{r_{1,j}\tau} + e^{r_{2,j}\tau}}$$

where $K_j = \frac{M_j}{N_j}$, and M_j , N_j are constants. The initial condition $B_j(0, \phi) = 0$ implies that, when $\tau = 0$ is substituted in (27), the numerator becomes $K_j r_{1,j} + r_{2,j} = 0$, from which $K_j = -\frac{r_{2,j}}{r_{1,j}}$. The solution for B_j becomes

$$B_j = -\frac{r_{2,j}}{R}\left(\frac{-e^{r_{1,j}\tau} + e^{r_{2,j}\tau}}{-g_j e^{r_{1,j}\tau} + e^{r_{2,j}\tau}}\right) = -\frac{r_{2,j}}{R}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right) = \frac{Q_j + d_j}{2R}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right)$$

where

$$g_j = -K_j = \frac{r_{2,j}}{r_{1,j}} = \frac{b_j - \rho\sigma i\phi + 2d_j}{b_j - \rho\sigma i\phi - 2d_j}$$

The solution for B_j can, therefore, be written as

$$B_j(\tau, \phi) = \frac{2b_j - 2\rho\sigma i\phi + 4d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right)$$

The solution for A_j is found by integrating (23b)

$$(28) \qquad A_j = \int_0^\tau ri\phi dy + \left(\frac{a}{2} - \frac{\sigma^2}{8}\right)\left(\frac{Q_j + d_j}{2R}\right)\int_0^\tau \left(\frac{1 - e^{d_j y}}{1 - g_j e^{d_j y}}\right)dy + K_0$$

where y is a dummy variable, and K_0 is a constant. The first integral is $ri\phi\tau$, and the second integral can be found by substitution. Let $x = \exp(d_j y)$, then $dx = d_j \exp(d_j y) dy$ and $dy = dx/(x d_j)$. Equation(28) becomes

$$A_j = ri\phi\tau + \frac{1}{d_j}(\frac{a}{2} - \frac{\sigma^2}{8})(\frac{Q_j + d_j}{2R}) \int_1^{\exp(d_j\tau)} (\frac{1-x}{1-g_j x})\frac{1}{x} dx + K_0. \tag{29}$$

The integral in (29) can be evaluated by partial fraction decomposition

$$\begin{aligned} \int_1^{\exp(d_j\tau)} \frac{1-x}{x(1-g_j x)} dx &= \int_1^{\exp(d_j\tau)} [\frac{1}{x} - \frac{1-g_j}{1-g_j x}] dx \\ &= [\log x + \frac{1-g_j}{g_j} \log(1-g_j x)]_{x=1}^{x=\exp(d_j\tau)} \\ &= [d_j\tau + \frac{1-g_j}{g_j} \log(\frac{1-g_j e^{d_j\tau}}{1-g_j})]. \end{aligned}$$

Substituting the integral back into (29), and substituting for d_j, Q_j, and g_j, produces the solution for A_j

$$A_j(\tau,\phi) = ri\phi\tau + (\frac{2a}{\sigma^2} - \frac{1}{2})[(\frac{1}{2}b_j - \frac{1}{2}\rho\sigma i\phi + d_j)\tau - 2\log(\frac{1-g_j e^{d_j\tau}}{1-g_j})] + K_0$$

where $a = \kappa\theta$. Note that we have the initial condition $A_j(0,\phi) = 0$, which leads to $K_0 = 0$ from the above equation. This will give the complete solution to A_j as follows:

$$A_j(\tau,\phi) = ri\phi\tau + (\frac{2a}{\sigma^2} - \frac{1}{2})[(\frac{1}{2}b_j - \frac{1}{2}\rho\sigma i\phi + d_j)\tau - 2\log(\frac{1-g_j e^{d_j\tau}}{1-g_j})]$$

A_j together with B_j completely gives the characteristic functions, and thus completes the proof of the theorem.

3.7.1. Relationship with Heston Model

Consider a special case of the second diffusion equation of Heston model

$$dV_t = -a_1 V_t dt + a_2 \sqrt{V_t} dW_{2,t}$$

That is to set the long-term mean to be zero for Heston model. Let $Y_t^* = V_t^2$, and apply Ito's Lemma

$$\begin{aligned} dY_t^* &= 2V_t dV_t + (dV_t)^2 \\ &= (a_2^2 V_t - 2a_1 V_t^2)dt + 2a_2 V_t^{\frac{3}{2}} dW_{2,t} \\ &= (a_2^2 - 2a_1\sqrt{Y_t^*})\sqrt{Y_t^*}dt + 2a_2(Y_t^*)^{\frac{3}{4}} dW_{2,t} \end{aligned}$$

And recall the second equation of the new continuous model,

$$dY_t = \kappa(\theta - \sqrt{Y_t})\sqrt{Y_t}dt + Y_t^{\frac{3}{4}}\sigma dW_{2,t}$$

If we set the coefficients of two equations dY_t^* and dY_t equal, we have

$$\begin{aligned} \kappa\theta &= a_2^2 \\ \theta &= 2a_1 \\ \sigma &= 2a_2 \end{aligned}$$

Put the above three qualities together and get

$$\kappa\theta = 4\sigma^2$$

We can see that if the coefficients of the new continuous model satisfy $\kappa\theta = 4\sigma^2$, the new continuous model will be reduced to a special case of Heston model given the same initial value for $\sqrt{Y_t}$ and V_t. However, if $\kappa\theta \neq 4\sigma^2$, the new continuous model cannot be reduced to the special case of Heston model.

Let's check the Feller condition for the new continuous model. Rewrite the second equation in the form of CIR process

$$dY_t = \kappa(\theta\sqrt{Y_t} - Y_t)dt + (\sigma Y_t^{\frac{1}{4}})\sqrt{Y_t}dW_{2,t}$$

Check Feller's condition to the above equation, we get

$$2\kappa\theta\sqrt{Y_t} > (\sigma Y_t^{\frac{1}{4}})^2$$

After simplification, we get

$$2\kappa\theta > \sigma^2$$

Then we can claim that if $2\kappa\theta > \sigma^2$, the volatility process of the new continuous model will be positive with probability 1.

3.7.2. Model Discretization and Parameter Estimation

Since we set market price of volatility risk to be zero, there are only four parameters κ, θ, σ and ρ to estimate in this model. We will apply MLE method with Monte Carlo Simulation to estimate the first three parameters, and method of moment to estimate ρ. Before estimation, we discretize the two processes S_t and Y_t in the same way as previous section to $\{S_i\}_0^n$and $\{Y_i\}_0^n$, where $n = \frac{T-t}{\Delta}$. And we discretize the two equations under risk-neutral measure with Euler's method as follows:

$$\log(S_i) - \log(S_{i-1}) = (r - \frac{1}{2}\sqrt{V_{i-1}})\Delta + (Y_{i-1})^{\frac{1}{4}}\Delta z_{1,i}^*$$

$$Y_i - Y_{i-1} = \kappa(\theta - \sqrt{Y_{i-1}})\sqrt{Y_{i-1}}\Delta + \sigma(Y_{i-1})^{\frac{3}{4}}\sqrt{\Delta}z_{2,i}^*$$

where $z_{1,i}^*$ and $z_{2,i}^*$ are standard normal variables correlated with coefficient ρ. Then we have $Y_i - Y_{i-1} - \kappa(\theta - \sqrt{Y_{i-1}})\sqrt{Y_{i-1}}\Delta = \sigma(Y_{i-1})^{\frac{3}{4}}\sqrt{\Delta}z_{2,i}^*$. Let $U = \sigma(Y_{i-1})^{\frac{3}{4}}\sqrt{\Delta}z_{2,i}^*$. Since $U = \sigma(Y_{i-1})^{\frac{3}{4}}\sqrt{\Delta}z_{2,i}^* \sim N(0, \sigma^2(Y_{i-1})^{\frac{3}{2}}\Delta)$, we get the probability density function of U below:

$$f_U(u) = \frac{1}{\sqrt{2\pi\sigma^2(Y_{i-1})^{\frac{3}{2}}\Delta}}\exp(-\frac{u^2}{2\sigma^2(Y_{i-1})^{\frac{3}{2}}\Delta})$$

Notice $\frac{dU}{dY_i} = 1$. If we substitute u with $Y_i - Y_{i-1} - \kappa(\theta - \sqrt{Y_{i-1}})\sqrt{Y_{i-1}}\Delta$, we will get the conditional density of Y_i given Y_{i-1} as follows:

$$f(Y_i\,|Y_{i-1}) = \frac{1}{\sqrt{2\pi\sigma^2(Y_{i-1})^{\frac{3}{2}}\Delta}}\exp(-\frac{X_i^2}{2\sigma^2(Y_{i-1})^{\frac{3}{2}}\Delta})$$

where $X_i = Y_i - Y_{i-1} - \kappa(\theta - \sqrt{Y_{i-1}})\sqrt{Y_{i-1}}\Delta$. Then, we get the likelihood function $L(\kappa, \theta, \sigma)$ below

$$L(\kappa, \theta, \sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2 (Y_{i-1})^{\frac{3}{2}}\Delta}} \exp(-\frac{X_i^2}{2\sigma^2 (Y_{i-1})^{\frac{3}{2}}\Delta})$$

Thus, the log-likelihood function $l(\kappa, \theta, \sigma) = \log(L)$ is given by

$$l(\kappa, \theta, \sigma) = \sum_{i=1}^{n} \{-0.5[\log(2\pi\sigma^2 (Y_{i-1})^{\frac{3}{2}}\Delta) + \frac{X_i^2}{2\sigma^2 (Y_{i-1})^{\frac{3}{2}}\Delta}]\}$$

The MLE of the parameters can be performed through "maxLik" package of R programming software.

For the estimation of ρ, we adopt method of moment and follow the same procedures as in Section 3.3. The method of moment estimation of ρ is given by:

$$\hat{\rho} = \frac{1}{n} \sum_{1}^{n} z_{1,i}^{*} z_{2,i}^{*}$$

where

$$z_{1,i}^{*} = \frac{\log(S_i) - \log(S_{i-1}) - (r - \frac{1}{2}\sqrt{Y_{i-1}})\Delta}{\sqrt{\Delta}(Y_{i-1})^{\frac{1}{4}}}$$

$$z_{2,i}^{*} = \frac{Y_i - Y_{i-1} - \hat{\kappa}(\hat{\theta} - \sqrt{Y_{i-1}})\sqrt{Y_{i-1}}\Delta}{\sqrt{\Delta}\hat{\sigma}(Y_{i-1})^{\frac{3}{4}}}$$

And $\hat{\kappa}, \hat{\theta}, \hat{\sigma}$ are the corresponding ML estimators.

Table 3.11. Parameter Estimation of New Method for the New Continuous Model (1000 replications)

$True\,Value$	$\kappa = 15$	$\theta = 0.05$	$\sigma = 0.2$	$\rho = 0.5$
$n = 50$	58.1536	0.0535	0.1933	0.4826
$Bias$	43.1536	0.0035	−0.0067	−0.0174
$RMSE$	60.3384	0.0444	0.0201	0.1371
100	35.8779	0.0507	0.1965	0.4881
$Bias$	20.8779	0.0007	−0.0035	−0.0119
$RMSE$	31.5806	0.0158	0.0162	0.0956
200	25.6668	0.0501	0.1987	0.4973
$Bias$	10.6668	0.0001	−0.0013	−0.0027
$RMSE$	16.4402	0.0063	0.0096	0.0665
300	22.2795	0.0498	0.1988	0.4985
$Bias$	7.2795	−0.0002	−0.0012	−0.0015
$RMSE$	12.0308	0.0032	0.0096	0.0569
400	20.9875	0.0498	0.1989	0.4969
$Bias$	5.9875	−0.0002	−0.0011	−0.0031
$RMSE$	10.0843	0.0032	0.0064	0.0475
600	18.9074	0.0498	0.1997	0.5008
$Bias$	3.9074	−0.0002	−0.0003	0.0008
$RMSE$	7.6072	0.0032	0.0063	0.038
800	18.1973	0.0498	0.1995	0.4977
$Bias$	3.1973	−0.0002	−0.0005	−0.0023
$RMSE$	6.5894	0.0032	0.0063	0.0317
1000	17.3305	0.0499	0.1997	0.4974
$Bias$	2.3305	−0.0001	−0.0003	−0.0026
$RMSE$	5.5792	0.0001	0.0032	0.0286

$True\,Value$	$\kappa = 15$	$\theta = 0.05$	$\sigma = 0.2$	$\rho = 0.5$
2000	16.4451	0.0499	0.1999	0.5011
$Bias$	1.4451	-0.0001	-0.0001	0.0011
$RMSE$	4.2414	0.0001	0.0032	0.0222
5000	15.5497	0.0499	0.2000	0.4994
$Bias$	0.5497	-0.0001	0	-0.0006
$RMSE$	2.4223	0.0001	0.0032	0.0127
10000	15.3011	0.0500	0.2000	0.5000
$Bias$	0.3011	0	0	0
$RMSE$	1.8618	0	0	0.0095

3.7.3. Simulation Results

For data generation, we set the risk-free rate $r = 0.03$, initial variance squared $Y_0 = ((6400000)^{-1/6})^2$, 1000 replications are simulated. Time to maturity is 0.5 years. Initial stock price is assumed to be $S_0 = \$100$. During data generation process for Y_t, reflection method is applied for negative values as in previous section. After obtaining ML estimators, we compute estimated option prices in Table 3.12 after plugging the values of MLE into the closed-form option pricing formula.

Table 3.11 and Table 3.12 show that as sample size increases, the estimated parameter values are getting closer and closer to the true parameter values. And the estimated option prices follow similar pattern. As sample size increases, the estimated option prices are getting closer and closer to the true option prices. Form here, we can conclude that the new method for the new continuous model is reliable.

TABLE 3.12. Option Price Estimation of New Method for the New Continuous Model

Strike Price	$K = 80$	90	100	110	120
True Value	21.3339	13.0494	6.921	3.1994	1.3138
$n = 50$	21.4249	13.2413	7.1447	3.3688	1.4063
100	21.3688	13.1137	6.9788	3.2245	1.3128
200	21.3511	13.0776	6.9394	3.1978	1.3007
300	21.343	13.0608	6.921	3.1853	1.295
400	21.3414	13.0585	6.9198	3.1859	1.2967
600	21.3381	13.0535	6.9173	3.1877	1.3004
800	21.3373	13.0521	6.9165	3.188	1.3013
1000	21.3372	13.0533	6.9202	3.193	1.3059
2000	21.3353	13.0505	6.9188	3.1941	1.308
5000	21.3337	13.0479	6.9174	3.1947	1.3097
10000	21.3345	13.0504	6.9216	3.1992	1.3132

3.7.4. New Continuous Model and BS Implied Volatility Skew

The BS model implied volatility is obtained through the function “bscallimpvol” in R package “derivmkts”. The new continuous model implied volatility is obtained through the function “BFfzero” in R package “NLRoot”. In this function, the Bisection method is applied. We apply the values of MLE from sample size n = 10000 to get the new continuous model implied volatility.

From the graph, we can see that new continuous model implied volatility forms a horizontal line, while the BS implied volatility forms a “forward skew”. This indicates that the new continuous model with the new method to measure volatility works better to reduce the volatility skew effect than BS model with constant volatility.

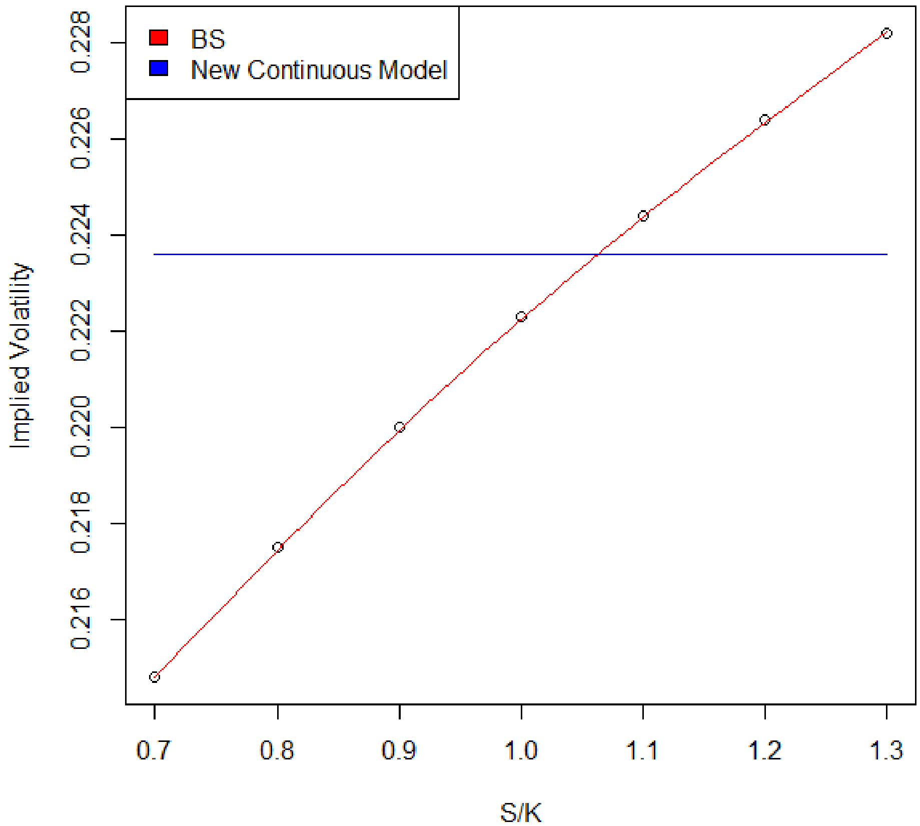

FIGURE 3.4. New Continuous VS BS Volatility Skew

3.8. The New Method for a Special Case of the New Continuous Model

Consider a special case of the new continuous model when $\kappa, \theta = 0$ and the two processes $dW^*_{1,t}$ and $dW^*_{2,t}$ are independent ($\rho = 0$). Under the risk-neutral measure $\mathbb{Q}$, the model is given by:

$$dS_t = rS_t dt + Y_t^{\frac{1}{4}} S_t dW^*_{1,t}$$

$$dY_t = Y_t^{\frac{3}{4}} \sigma dW^*_{2,t}$$

From the previous section, we can get the closed-from solution of option pricing for the special case as follow:

$$C_t = S_t P_1 - Ke^{-r(T-t)} P_2$$

where for $j = 1, 2$

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re[\frac{e^{-i\phi \log K} f_j(\phi; t, x_t, v_t)}{i\phi}]d\phi$$

(21) $$f_j(\phi; x_t, Y_t) = \exp(A_j(\tau, \phi) + B_j(\tau, \phi)\sqrt{Y_t} + i\phi x_t)$$

and

$$\tau = T - t$$

$$x_t = \log(S_t)$$

$$A_j(\tau, \phi) = ri\phi\tau - \frac{1}{2}d_j\tau + \log(\frac{1 + e^{d_j\tau}}{2})$$

$$B_j(\tau, \phi) = \frac{4d_j}{\sigma^2}(\frac{1 - e^{d_j\tau}}{1 + e^{d_j\tau}})$$

such that

$$d_j = \frac{\sigma}{2}\sqrt{\phi^2 - 2u_j i\phi}$$

$$u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}$$

3.8.1. Model Discretization and Parameter Estimation

Since this is a special case of the new continuous model, let $\kappa, \theta, \rho = 0$ in the general model, we will get the discretization as

$$\log(S_i) - \log(S_{i-1}) = (r - \frac{1}{2}\sqrt{V_{i-1}})\Delta + (Y_{i-1})^{\frac{1}{4}}\Delta z^*_{1,i}$$

$$Y_i - Y_{i-1} = \sigma(Y_{i-1})^{\frac{3}{4}}\sqrt{\Delta}z^*_{2,i}$$

where $z^*_{1,i}$ and $z^*_{2,i}$ are independent standard normal variables. Thus, the log-likelihood function $l(\kappa, \theta, \sigma) = \log(L)$ is given by

$$l(\kappa, \theta, \sigma) = \sum_{i=1}^{n}\{-0.5[\log(2\pi\sigma^2(Y_{i-1})^{\frac{3}{2}}\Delta) + \frac{(Y_i - Y_{i-1})^2}{2\sigma^2(Y_{i-1})^{\frac{3}{2}}\Delta}]\}$$

The MLE of the parameters can be performed through "maxLik" package of R programming software.

Table 3.13. Parameter Estimation of New Method for the Special Case of New Continuous Model (1000 replications)

$True\,Value$	$\sigma = 0.15$	$True\,Value$	$\sigma = 0.15$
$n = 50$	0.1784	800	0.1794
$Bias$	0.0284	$Bias$	0.0294
$RMSE$	0.038	$RMSE$	0.0296
100	0.1789	1000	0.1789
$Bias$	0.0289	$Bias$	0.0289
$RMSE$	0.0315	$RMSE$	0.0291
200	0.1799	2000	0.1761
$Bias$	0.0299	$Bias$	0.0261
$RMSE$	0.0314	$RMSE$	0.0269
300	0.1798	5000	0.1668
$Bias$	0.0298	$Bias$	0.0168
$RMSE$	0.0305	$RMSE$	0.0193
400	0.1796	10000	0.1609
$Bias$	0.0296	$Bias$	0.0109
$RMSE$	0.0303	$RMSE$	0.0126
600	0.1800		
$Bias$	0.0300		
$RMSE$	0.0307		

Table 3.14. Option Price Estimation of New Method for the Special Case of New Continuous Model

Strike Price	$X = 80$	90	100	110	120
Call True Value	21.7159	13.8356	7.9145	4.0834	1.9252
$n = 50$	21.7162	13.8365	7.9157	4.0846	1.9259
100	21.7161	13.8362	7.9153	4.0842	1.9257
200	21.7159	13.8357	7.9145	4.0835	1.9252
300	21.7159	13.8358	7.9146	4.0836	1.9253
400	21.716	13.8359	7.9148	4.0837	1.9254
600	21.7159	13.8356	7.9145	4.0834	1.9252
800	21.716	13.836	7.9149	4.0839	1.9254
1000	21.7161	13.8362	7.9153	4.0842	1.9257
2000	21.7166	13.8378	7.9175	4.0862	1.9269
5000	21.7181	13.8426	7.9246	4.0927	1.931
10000	21.7191	13.8455	7.9289	4.0966	1.9334

3.9. Simulation Results

During the simulation process, we set the risk-free rate $r = 0.03$, initial variance squared $Y_0 = ((10000000)^{-1/6})^2$, and 1000 replications are simulated. Time to maturity is 0.5 years. Current stock price is assume to be $S_0 = \$100$. During data generation process for Y_t, reflection method is applied. That is when a negative value of Y_t is generated, it will be replaced by its absolute value. After obtaining ML estimators, we obtain estimated option prices in Table 3.14 after plugging the values of MLE into the exact closed-form option pricing formula with $\kappa, \theta, \rho = 0$.

Table 3.13 shows that as sample size increases, the estimated parameter values are getting closer and closer to the true parameter values. The estimated option prices do not follow the same pattern. No matter what the sample size is, the estimated option prices have a difference within two cents of the true option prices for any strike price, according to

Table 3.14. Form here, we can conclude that the new method for the special case of the new continuous model is reliable.

CHAPTER 4

TWO DISCRETE VOLATILITY MODELS

Due to the implementation difficulties of the continuous stochastic volatility models, Heston and Nandi (2000) presents an easier to implement discrete GARCH model with a closed-form solution. Here we only focus on the one-lag GARCH model- GARCH (1,1). After introducing the Heston and Nandi GARCH (1,1) model, we will present a New GARCH (1,1) model and relevant theoretical results. At the end of the chapter, we will reveal the relationship among Heston and Nandi GARCH (1,1) model, our New GARCH (1,1) model, and (continuous) Heston model.

4.1. Heston and Nandi GARCH (1,1) Model

In this section, we will introduce the Heston and Nandi GARCH (1,1) model under both physical measure and risk-neutral measure as well as their continuous time limit processes. After that, the closed-form option pricing formula of this model will be given.

4.1.1. Model Specification and Continuous Time Limit

The spot asset price, S(t) (having incorporated the accumulated interest or dividends) is governed by the following process over step size Δ under physical measure $\mathbb{P}$:

$$\log(S(t)) = \log(S(t-\Delta) + r + \lambda h(t) + \sqrt{h(t)}z(t)$$

$$h(t) = \omega + \beta h(t-\Delta) + \alpha(Z(t-\Delta) - \gamma\sqrt{h(t-\Delta)})^2$$

where r is the constant risk-free rate and z(t) is a standard normal variable over one stepsize period. $h(t)$ is the conditional variance of the log return $R(t)$ ($R(t) = \log(S(t)) - \log(S(t-\Delta))$ between time $t-\Delta$ and t. It is determined by the information set at time $t-\Delta$. The parameters are constrained as $0 < \omega, \alpha, \beta < 1, 0 < \beta + \alpha\gamma^2 < 1$. Heston and Nandi (2000) shows that if particular values of $\omega, \alpha, \beta, \gamma$ are assigned, the Heston and Nandi GARCH (1,1) model converges weakly to the following continuous process under physical meansure $\mathbb{P}$:

$$d\log(S(t)) = (r + \lambda v(t))dt + \sqrt{v(t)}dz(t)$$
$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dz(t)$$

where $z(t)$ is a Wiener process. And the risk-neutral form of the model is given by:

$$\log(S(t)) = \log(S(t-\Delta)) + r - \frac{1}{2}h(t) + \sqrt{h(t)}z^*(t)$$
$$h(t) = \omega + \beta h(t-\Delta) + \alpha(Z^*(t-\Delta) - \gamma^*\sqrt{h(t-\Delta)})^2$$

where

$$z^*(t) = z(t) + (\lambda + \frac{1}{2})\sqrt{h(t)}$$
$$\gamma^* = \gamma + \lambda + \frac{1}{2}$$

Heston and Nandi (2000) shows that if $\omega, \alpha, \beta, \gamma$ are assigned with the same values as assigned in physical measure $\mathbb{P}$, the Heston and Nandi GARCH (1,1) model converges weakly to the following continuous process under risk-neutral measure $\mathbb{Q}$:

$$d\log(S(t)) = (r - \frac{1}{2}v(t))dt + \sqrt{v(t)}dz^*(t)$$
$$dv(t) = [\kappa(\theta - v(t)) + \sigma(\lambda + \frac{1}{2})v(t)]dt + \sigma\sqrt{v(t)}dz^*(t)$$

where $z^*(t) = (\lambda + \frac{1}{2})\sqrt{h(t)} + z(t)$, is a Wiener process under risk-neutral measure $\mathbb{Q}$.

4.1.2. Closed-form Formula

Let$C_t = C(S_t, V_t, t)$ be the price of European call option. The closed form solution of option pricing for Heston and Nandi GARCH (1,1) Model is given by Heston & Nandi (2000) as below:

$$C_t = e^{-r(T-t)}E^*[max(S(T) - K, 0)]$$
$$= \frac{1}{2}S_t + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty Re[\frac{K^{-i\phi}f^*(i\phi+1)}{i\phi}]d\phi - Ke^{-r(T-t)}(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{K^{-i\phi}f^*(i\phi)}{i\phi}]d\phi)$$

In this formula, Re[ ] stands for the real part of a complex number. and $f^*(i\phi)$ is the conditional characteristic function of log(S(T)) under the risk-neutral measure $\mathbb{Q}$. i is the imaginary unit $\sqrt{-1}$. The put option price could be calculated through put-call parity once the call price is obtained. Heston and Nandi (2000) shows that $f(\phi)$ - the conditional generating function of the terminal stock price S(T) given information set at time t takes the following log-linear form for the GARCH (1,1) model under physical measure $\mathbb{P}$. This is also the conditional moment generating function of the log(S(T)) under the physical measure $\mathbb{P}$:

$$f(\phi) = E_t[S(T)^\phi] = S(t)^\phi \exp(A(t; T, \phi) + B(t; T, \phi)h(t + \Delta))$$

Heston & Nandi (2000) shows the recursion formulas for the coefficients

$$A(t; T, \phi) = A(t + \Delta; T, \phi) + \phi r + B(t + \Delta; T, \phi)\omega - \frac{1}{2}\log(1 - 2\alpha B(t + \Delta; T, \phi)$$

$$B(t; T, \phi) = \phi(\lambda + \gamma) - \frac{1}{2}\gamma^2 + \beta B(t + \Delta; T, \phi) + \frac{(\phi - \gamma)^2}{2[1 - 2\alpha B(t + \Delta; T, \phi)]}$$

The two coefficients are computed recursively by working backward from the terminal conditions:

$$A(T; T, \phi) = 0$$

$$B(T; T, \phi) = 0$$

After obtaining $f(\phi)$, replace γ with γ^* (which is $\gamma + \lambda + \frac{1}{2}$ in $f(\phi)$, we will get the expression of $f^*(\phi)$

4.1.3. Simulation Results

Since Heston and Nandi GARCH (1,1) model is well known and widely used, we needn't show simulation results here.

4.2. A New GARCH (1,1) Model

We present a New GARCH (1,1) model in this section first. Then we will show more theoretical results and a closed-form option pricing formula later in this section.

4.2.1. Model Description and Continuous Time Limit

Assumption 1. The spot asset price, S(t) (having incorporated the accumulated interest or dividends) is governed by the following process over step size Δ under physical measure $\mathbb{P}$:

$$\log(S(t)) = \log(S(t-\Delta) + r + \lambda h(t) + \sqrt{h(t)}z(t) \tag{30}$$

$$h(t) = a - b^2 - ch(t-\Delta) + (bz(t-\Delta) - \sqrt{h(t-\Delta)})^2 \tag{31}$$

where r is the constant risk-free rate and z(t) is a standard normal variable over one step size period. $h(t)$ is the conditional variance of the log return $R(t)$, which is defined as $R(t)$ $:= \log(S(t)) - \log(S(t-\Delta))$, between time $t-\Delta$ and t. It is determined by the information set at time $t-\Delta$.

If we take expectation over (30), we will get mean log-return $E(R(t))$ equal to $r+\lambda h(t)$. We can see that the conditional variance $h(t)$ shows up in $E(R(t))$ as a return premium (Heston &Nandi, 2000). And $\lambda h(t)$ is the risk premium over the time period between $t-\Delta$ and t, according to Nelson (1990) [31].

When b= 0 and c=1, we get $h(t) = a$ from (31). Under this special condition, the model reduces to the BS model observed at discrete time intervals. From here, we need to constrain $a > 0$. The volatility process given by (31) stays stationary with finite mean and variance if $0 < c < 1$. Since z(t) is symmetric, we can always constrain b > 0. And also, we need to assume $a > b^2$. This is because if $(bz(t-\Delta) - \sqrt{h(t-\Delta)})^2 = 0$ in (31) (this equality could be achieved for some t), we have $a - b^2 = h(t) + ch(t-\Delta) > 0$. So, we have the joint constraints $a > 0, b > 0, 0 < c < 1, a > b^2$ for the model given in Assumption 1.

Equations (30) and (31) describe a discrete stochastic process over one stepsize period Δ, but they have a continuous-time limit. And we have the following convergence theorem.

Theorem 4.1. Under risk-neutral measure $\mathbb{P}$, as the stepsize Δ goes to zero, the variance process $v(t) := \frac{h(t)}{\Delta}$ and log stock price process will converge weakly to the continuous process given by Heston (1993) below if particular values of a, b, c are assigned:

$$d\log(S(t)) = (r + \lambda v(t))dt + \sqrt{v(t)}dz(t) \tag{32}$$

$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dz(t) \tag{33}$$

where z(t) is a Wiener process.

Proof. The details of proof are attached in Appendix(B).

Assumption 2. The risk-neutral measure $\mathbb{Q}$ satisfies Duan's (1995) locally risk-neutral valuation relationship(LRVNR), i.e., (1) measure $\mathbb{Q}$ is mutually absolutely continuous with respect to measure $\mathbb{P}$; (2) $\log(\frac{S(t)}{S(t-\Delta)}) \mid \mathcal{F}_{t-\Delta}$ is normally distributed under measure $\mathbb{Q}$, where $\mathcal{F}_t$, representing all market information up to time t, is a sequence of increasing σ-algebras of the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ associated with the model ; (3) $E^Q[\frac{S(t)}{S(t-\Delta)} \mid \mathcal{F}_{t-\Delta}] = e^{\Delta r}$; (4) $Var^Q[\log(\frac{S(t)}{S(t-\Delta)}) \mid \mathcal{F}_{t-\Delta}] = Var^P[\log(\frac{S(t)}{S(t-\Delta)}) \mid \mathcal{F}_{t-\Delta}]$ almost surely with respect to measure $\mathbb{P}$.

According to Heston& Nandi(2000), this assumption is equivalent to their assumption "The value of a call option with one period to expiration obeys the Black-Scholes-Rubinstein formula." Under this assumption, the spot price lognormally distributed over one stepsize period Δ and the volatility keeps unchanged over this interval. Hence, the BS formula can be applied. Following this assumption, we get the risk-neutral process for our model in the following proposition.

Theorem 4.2. Under risk-neutral measure $\mathbb{Q}$, the risk-neutral process of (30) and (31) is given by:

$$\log(S(t)) = \log(S(t-\Delta)) + r - \frac{1}{2}h(t) + \sqrt{h(t)}z^*(t) \tag{34}$$

$$h(t) = a - b^2 - ch(t-\Delta) + (bz^*(t-\Delta) - [b(\lambda + \frac{1}{2}) + 1]\sqrt{h(t-\Delta)})^2 \tag{35}$$

With $a > 0, b > 0, 0 < c < 1, a > b^2$. And

$$z^*(t) = z(t) + (\lambda + \frac{1}{2})\sqrt{h(t)})$$

Proof. We may follow Duan's (1995) idea, but work with different equations. Since $\log(\frac{S(t)}{S(t-\Delta)}) \mid \mathcal{F}_{t-\Delta}$ is normally distributed under measure $\mathbb{Q}$, it can be written as

$$\log \frac{S(t)}{S(t-\Delta)} = m_t + \epsilon_t$$

where m_t is the conditional mean and $\epsilon_t = \sqrt{h(t)}z^*(t)$ is a normal random variable under measure $\mathbb{Q}$. The conditinal mean of ϵ_t is zero and its conditional variance waits to be determined. First, we show that $m_t = r - \frac{1}{2}h(t)$. Since $log(\frac{S(t)}{S(t-\Delta)}) \mid \mathcal{F}_{t-\Delta}$ is normally distributed, $\frac{S(t)}{S(t-\Delta)} \mid \mathcal{F}_{t-\Delta}$ is log-normally distributed. And we have

$$\begin{aligned} E^Q(\frac{S_t}{S_{t-\Delta}}|\mathcal{F}_{t-\Delta}) &= E^Q(e^{m_t+\epsilon_t}|\mathcal{F}_{t-\Delta}) \\ &= e^{m_t+\frac{h(t)}{2}} \end{aligned}$$

where $h(t) = \text{Var}^P(\log(\frac{S(t)}{S(t-\Delta)})|\mathcal{F}_{t-\Delta}) = \text{Var}^Q(\log(\frac{S(t)}{S(t-\Delta)})|\mathcal{F}_{t-\Delta})$ by the LRNVR assumption of measure $\mathbb{Q}$. Since $E^Q(\frac{S(t)}{S(t-\Delta)}|\mathcal{F}_{t-\Delta}) = e^r$ by the LRNVR, we have $m_t = r - \frac{1}{2}h(t)$. By the above result and the first equation of our model under physical measure $\mathbb{P}$, we have

$$\log \frac{S(t)}{S(t-\Delta)} = r + \lambda h(t) + \sqrt{h_t}z_t = r - \frac{1}{2}h(t) + \epsilon_t$$

This implies that $\epsilon_t = (\lambda + \frac{1}{2})h(t) + \sqrt{h(t)}z(t)$. And recall that $\epsilon_t = \sqrt{h(t)}z^*(t)$, then

$$\sqrt{h(t)}z^*(t) = (\lambda + \frac{1}{2})h(t) + \sqrt{h(t)}z(t)$$

Factor out the term $\sqrt{h(t)}$ on the right side in the above equation

$$\sqrt{h(t)}z^*(t) = \sqrt{h(t)}[(\lambda + \frac{1}{2})\sqrt{h(t)} + z(t)]$$

Since $\sqrt{h(t)}$ is a non-negative random variable, from the above equation, we must have

$$z^*(t) = (\lambda + \frac{1}{2})\sqrt{h(t)} + z(t)$$

Substituting $z^*(t)$ into the second equation under measure $\mathbb{Q}$, we get the desired result.

After getting the risk-neutral form of the new GARCH (1,1) model, we can get the corresponding continuous time limit process, which is given by the following theorem.

Theorem 4.3. Under risk-neutral measure $\mathbb{Q}$, as the stepsize Δ goes to zero, the variance process $v(t) := \frac{h(t)}{\Delta}$ and log stock price process will converge weakly to the continuous process given by the following if particular values of a, b, c are assigned:

$$d\log(S(t)) = (r - \frac{1}{2}v(t))dt + \sqrt{v(t)}dz^*(t) \tag{36}$$

$$dv(t) = [\kappa(\theta - v(t)) + \sigma(\lambda + \frac{1}{2})v(t)]dt + \sigma\sqrt{v(t)}dz^*(t) \tag{37}$$

where $z^*(t) = (\lambda + \frac{1}{2})\sqrt{h(t)} + z(t)$, is a Wiener process under risk-neutral measure $\mathbb{Q}$.

Proof. The details of proof are attached in Appendix(C).

4.2.2. Closed-from Option Pricing Formula

Let $f(\phi)$ denote the conditional generating function of the terminal stock price S(T) given information set at time t under physical measure

$$f(\phi) = E_t[S(T)^{\phi}] \tag{38}$$

If we rewrite $S(T) = e^{\log(S(T))}$, we can see that f (ϕ) is also the conditional moment generating function of $\log(S(T))$. And we will use $f^*(\phi)$ to stand for the conditional generating function of the terminal stock price S(T) (hence the moment generating function of $\log(S(T))$) under risk-neutral measure $\mathbb{Q}$.

Proposition 1. $f(\phi)$, the conditional generating function of the terminal stock price, has the following solution of log-linear form

$$f(t;T,\phi) = S(t)^{\phi}\exp(A(t;T,\phi) + B(t;T,\phi)h(t+\Delta))$$

Where

$A(t;T,\phi) = A(t+\Delta;T,\phi) + \phi r + B(t+\Delta;T,\phi)(a-b^2) - \frac{1}{2}\log[1-2B(t+\Delta;T,\phi)b^2]$.

$B(t;T,\phi) = \phi\lambda + B(t+\Delta;T,\phi)(1-c) + \frac{1}{2-4b^2B(t+\Delta;T,\phi)}[\phi - 2bB(t+\Delta;T,\phi)]^2$

And the $A(t;T,\phi)$ and $B(t;T,\phi)$ can be calculated recursively from the terminal conditions

$$A(T;T,\phi) = 0$$

$$B(T;T,\phi) = 0$$

Proof. We follow Heston & Nandi's (2000) framework, but perform with our model. Let $x(t) = \log(S(t))$ and let $f(t;T,\phi)$ be the conditional generating function of S(T), or equivalently the conditional moment generating function of x(T), i.e.,

$$f(t;T,\phi) = E_t[\exp(\phi x(T))].$$

We shall guess that the moment generating function takes the log-linear form

$$f(t;T,\phi) = \exp(\phi x(t) + A(t;T,\phi) + B(t;T,\phi)h(t+\Delta)$$

and solve for the coefficients $A()$ and $B()$.

Since $x(T)$ is known at time T, equation (30)and (31) require the terminal condition

$$A(T;T,\phi) = B(T;T,\phi) = 0.$$

Applying the law of iterated expectations to $f(t;T,\phi)$,we get,

$f(t,T,\phi) = E_t[f(t+\Delta;T,\phi)] = E_t[\exp(\phi x(t+\Delta)+A(t+\Delta;T,\phi)+B(t+\Delta;T,\phi)h(t+2\Delta)$

Substituting related expressions of $x(t)$ in equations (30) and (31) shows

$f(t;T,\phi) = E_t[\exp(\phi(x(t) + r + \gamma h(t+\Delta) + \sqrt{h(t+\Delta)}z(t+\Delta)) + A(t+\Delta;T,\phi) + B(t+\Delta;T,\phi) \times (a - b^2 - ch(t+\Delta) + (bz(t+\Delta) - \sqrt{h(t+\Delta)})^2)$

Rearranging terms through completing squares and some algebra shows

(39)

$$\begin{aligned} f(t;T,\phi) =& E_t\{\exp(\phi(x(t)+r) + A(t+\Delta;T,\phi) + \phi\lambda h(t+\Delta) + B(t+\Delta;T,\phi)(a-b^2) \\ & + b^2 B(t+\Delta;T,\phi)[z(t+\Delta) + \frac{1}{2b^2}(\frac{\phi}{B(t+\Delta;T,\phi)} - 2b)\sqrt{h(t+\Delta)}]^2 \end{aligned}$$

$$+ B(t+\Delta;T,\phi)[1-c-\frac{1}{4b^2}(\frac{\phi}{B(t+\Delta;T,\phi)}-2b)^2 h(t+\Delta)]$$

For a standard normal variable z, we have the following result:

$$E[\exp(d_1(z+d_2)^2)] = \exp(-\frac{1}{2}\log(1-2d_1)+\frac{d_1 {d_2}^2}{1-2d_1}).$$

Substituting this result in (39) and subsequently equating terms in both sides of (39) shows

$A(t;T,\phi) = A(t+\Delta;T,\phi) + \phi r + B(t+\Delta;T,\phi)(a-b^2) - \frac{1}{2}log[1-2b^2 B(t+\Delta;T,\phi)]$.

$B(t;T,\phi) = \phi\lambda + B(t+\Delta;T,\phi)(1-c) + \frac{1}{2-4b^2 B(t+\Delta;T,\phi)}[\phi - 2bB(t+\Delta;T,\phi)]^2$

Some remark about Proposition 1. If we compare the two forms of the model under physical measure and risk-neutral measure, we can see that besides the difference between $z(t)$ and $z^*(t)$, another difference lies in the coefficient of $\sqrt{h(t-\Delta)}$ term. For physical measure form, the coefficient is one, but for risk-neutral form, the coefficient is $b(\lambda+\frac{1}{2})+1$. If we keep this different term and perform the reasoning in Proposition 1, we will get the expression of $f^*(t;T,\phi)$ as follows:

$$f^*(t;T,\phi) = S(t)^{\phi} exp(A^*(t;T,\phi) + B^*(t;T,\phi)h(t+\Delta))$$

Where

$$A^*(t;T,\phi) = A^*(t+\Delta;T,\phi) + \phi r + B^*(t+\Delta;T,\phi)(a-b^2) - \frac{1}{2}\log[1-2B^*(t+\Delta;T,\phi)b^2].$$

$$B^*(t;T,\phi) = \phi\lambda + B^*(t+\Delta;T,\phi)(d^2-c) + \frac{1}{2-4B^*(t+\Delta;T,\phi)b^2}[\phi - 2bdB^*(t+\Delta;T,\phi)]^2$$

$$d = b(\lambda+\frac{1}{2})+1$$

And the $A^*(t;T,\phi)$ and $B^*(t;T,\phi)$ can be calculated recursively from the terminal conditions

$$A^*(T;T,\phi) = 0$$

$$B^*(T;T,\phi) = 0$$

Proposition 2 (Heston and Nandi (2000)). If the characteristic function of the log spot price is $f(i\phi)$, then under physical measure we have

$$\begin{aligned} E_t[Max(S(T)-K,0)] =& f(1)(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re[\frac{K^{-i\phi}f(i\phi+1)}{i\phi f(1)}]d\phi) \\ & - K(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re[\frac{K^{-i\varphi}f(i\phi)}{i\phi}]d\phi), \end{aligned}$$

Proof. The details of proof are attached in Appendix(D).

Heston and Nandi (2000) presents this new inversion formula, different from that of Heston (1993) and other authors. On the right-hand side of the expression above, we only need to calculate one integral instead of two. An option value is the discounted expected value of the payoff, Max$(S(t)-K,0)$ calculated under risk-neutral measure $\mathbb{Q}$, i.e., applying the characteristic function $f^*(i\phi)$. By Heston and Nandi (2000), a European option value is given by the following corollary.

Corollary. At time t a European call option with strike price K that expires at time T is worth

$$\begin{aligned} C =& e^{-r(T-t)}E_t^*[Max(S(T)-K,0)] = \frac{1}{2}S(t) \\ & + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty Re[\frac{K^{-i\phi}f^*(i\phi+1)}{i\phi}]d\phi) \\ & - Ke^{-r(T-t)}(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re[\frac{K^{-i\phi}f^*(i\phi)}{i\phi}]d\phi). \end{aligned}$$

where $E_t^*[\,]$ denotes the expectation under the risk-neutral measure $\mathbb{Q}$. This finishes the option pricing formula.

Proof. Note that if we perform the same procedures as in Proposition 2 under risk neutral measure, we will get

$$
\begin{aligned}
C &= e^{-r(T-t)} E_t^*[Max(S(T)-K,0)] \\
&= e^{-r(T-t)} f^*(1)(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{K^{-i\phi} f^*(i\phi)}{i\phi f^*(1)}]d\phi) \\
&\quad - Ke^{-r(T-t)}(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{K^{-i\phi} f^*(i\phi)}{i\phi}]d\phi) \\
&= \frac{1}{2} e^{-r(T-t)} f^*(1) + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty Re[\frac{K^{-i\phi} f^*(i\phi)}{i\phi}]d\phi \\
&\quad - Ke^{-r(T-t)}(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{K^{-i\phi} f^*(i\phi)}{i\phi}]d\phi) \qquad (40)
\end{aligned}
$$

And under the risk-neutral distribution, follow the argument of Heston and Nandi (2000) (in the proof of Proposition 3), we have

$$e^{-r(T-t)} f^*(1) = e^{-r(T-t)} E^*[\exp(x(T))|S(t), h(t)] = e^{-r(T-t)} E^*[S(T)|S(t), h(t)] = S(t)$$

this together with (40) demonstrates the corollary.

4.2.3. Relation Between New GARCH (1,1) Model and Heston and Nandi GARCH (1,1) Model

Our model has a close relation with Heston and Nandi GARCH (1,1). The first relation between them is that both converge to the same continuous-time limit process (Heston model) under both physical measure and risk-neutral measure. Before revealing the second relation, let

$$a = \omega + \alpha, b = \alpha\gamma, c = 1 - \beta - \alpha\gamma^2$$

we will get

$$h_t = \omega + \beta h_{t-\Delta} + \alpha(z_{t-\Delta} - \gamma\sqrt{h_{t-\Delta}})^2 + \alpha(1 - z_{t-\Delta}^2)(1 - \alpha\gamma^2)$$

We can see that the difference between the above equation and the second diffusion equation of Heston and Nandi GARCH (1,1) model under physical measure is the last term $\alpha(1 - z_{t-\Delta}^2)(1 - \alpha\gamma^2)$. And the expectation of this term is zero. This means equation (3) is equal to the second equation of Heston and Nandi GARCH (1,1) model plus a second order term, which is asymptotically zero. This is the second relation between these two discrete models. Moreover, the second order term serves as a second-order correction to Heston and Nandi GARCH (1,1) model. Due to the second order correction, we can expect that the new GARCH (1,1) model should have better performance in option price estimation when the sample size is relatively small if the data set is generated from the new GARCH (1,1) model. When the sample size is large enough, we could expect that the estimated option prices obtained from those three models (including the continuous Heston model) should agree, since they converge to the same continuous-time process. The simulation results comparison among the new GARCH (1,1) model, Heston and Nandi GARCH (1,1) model, and (continuous) Heston model will be presented in next chapter, due to their special relationship.

CHAPTER 5

MORE SIMULATION RESULTS COMPARISON

In this chapter, we will present more simulation results and graphs based on some previous simulation results. First, we will show simulation comparison among new GARCH (1,1) model, Heston and Nandi GARCH (1,1) model, and Heston model. Secondly, we will show the graph of comparison between BS model and other stochastic volatility models. Before we reveal these two comparison results, we show method of parameter estimation for new GARCH (1,1) model with transformed trading volume serving as volatilities. And the parameter estimation of Heston and Nandi GARCH (1,1) model will be performed with an existing R programming software package.

5.1. New GARCH (1,1) Parameter Estimation Method

Since the volatilities are directly observable with our new method, we are able to estimate the parameters in each equation separately. And the second equation of the model doesn't depend on stock price, we can estimate the parameters in this equation first with MLE method.

5.1.1. Parameter Estimation for the Second Equation

Recall that the second equation of our new GARCH (1,1) model:

$$h(t) = a - b^2 - ch(t-\Delta) + (bz(t-\Delta) - \sqrt{h(t-\Delta)})^2$$

Let $U = |X| = \left|bz(t-\Delta) - \sqrt{h(t-\Delta)}\right| = \sqrt{h(t) - a + b^2 + ch(t-\Delta)}$. Since $bz(t-\Delta) - \sqrt{h(t-\Delta)} \sim N(-\sqrt{h(t-\Delta)}, b^2)$, $U = |X|$ follows folded normal distribution. Then the pdf of U is given by

$$f_U(u) = \frac{1}{\sqrt{2\pi b^2}}[\exp(-\frac{(u-\sqrt{h_{t-\Delta}})^2}{2b^2}) + \exp(-\frac{(u+\sqrt{h_{t-\Delta}})^2}{2b^2})]$$

Let $y_t = h(t) - a + b^2 + ch(t-\Delta)$. Substitute u in the above with $\sqrt{y_t}$ and apply chain rule, we get the conditional density of $h(t)$

$$f(h_t\,|h_{t-\Delta}) = \frac{1}{2b\sqrt{2\pi y_t}}[\exp(-\frac{(\sqrt{y_t}-\sqrt{h_{t-\Delta}})^2}{2b^2}) + \exp(-\frac{(\sqrt{y_t}+\sqrt{h_{t-\Delta}})^2}{2b^2})]$$
$$= \frac{1}{2b\sqrt{2\pi y_t}}\exp(-\frac{(\sqrt{y_t}-\sqrt{h_{t-\Delta}})^2}{2b^2})[1+\exp(-\frac{2\sqrt{y_t h_{t-\Delta}}}{b^2})]$$

Then we get the log-likelihood

$$l(\theta) = \sum[-\frac{1}{2}\log(b^2 y_t)] + \sum[-\frac{(\sqrt{y_t}-\sqrt{h_{t-\Delta}})^2}{2b^2}] + \sum\log[1+\exp(-\frac{2\sqrt{y_t h_{t-\Delta}}}{b^2})],$$

with constraints $a > 0, b > 0, 0 < c < 1, a > b^2$.

During parameter estimation for the second equation, $h(t)-a+b^2+ch(t-\Delta)$ becomes negative very likely during iteration (recall $\left|bz(t-\Delta)-\sqrt{h(t-\Delta)}\right| = \sqrt{h(t)-a+b^2+ch(t-\Delta)} \geq 0$). In order to solve the estimation issue, we multiply a large enough positive constant w to both sides of the second equation as below

$$wh(t) = wa - (\sqrt{w}b)^2 - cwh(t-\Delta) + (\sqrt{w}bz(t-\Delta) - d\sqrt{wh(t-\Delta)})^2$$

Let $\overline{h}(t) = wh(t), \overline{a} = wa, \overline{b} = \sqrt{w}b, \overline{c} = c$, then the above equation becomes

$$\overline{h}(t) = \overline{a} - (\overline{b})^2 - \overline{c}\,\overline{h}(t-\Delta) + (\overline{b}z(t-\Delta) - \sqrt{\overline{h}(t-\Delta)})^2$$

The criterion for choosing w is $wE(h(t)) \geq 0.1$, where $E(h(t))$-the expectation of $h(t)$ can be approximated with the sample mean of $h(t)$.

5.1.2. Parameter Estimation for the First Equation

Let $\hat{a}, \hat{b}, \hat{c}$ be the maximum likelihood estimator for a, b, c obtained previously. In order to estimate λ more accurately in the first equation, let's create a new time series for volatility with $\hat{a}, \hat{b}, \hat{c}$ recursively and call it $h^*(t)$ as follows:

$$h^*(t) = \hat{a} - \hat{b}^2 - \hat{c}h^*(t-\Delta) + (\hat{b}z(t-\Delta) - \sqrt{h^*(t-\Delta)})^2$$

$$z(t) = \frac{R(t) - r - \lambda * h^*(t)}{\sqrt{h^*(t)}}$$

Where

$$R(t) = \log(\frac{S(t)}{S(t-\Delta)})$$

And the inital value of $h^*(t)$ is set to be equal to the initial value of h(t).

Obviously, the time series $\{h^*(t)\}$ contains λ in every term. Now the first equation of our model becomes

$$R(t) = r + \lambda h^*(t) + \sqrt{h^*(t)}z(t)$$

where R(t) is the log-return, defined the same as above. Then $R(t) - r - \lambda h^*(t) = \sqrt{h^*(t)}z(t)$. Let U $= \sqrt{h^*(t)}z(t)$. Since, $\sqrt{h^*(t)}z(t) \sim N(0, h^*(t))$, we have the probability density of U:

$$f_U(u) = \frac{1}{\sqrt{2\pi h^*(t)}} e^{-\frac{u^2}{2h^*(t)}}$$

Then substitute u above with $R(t) - r - \lambda h^*(t)$ given $h^*(t-\Delta)$, we have the conditional probability density of $h^*(t)$:

$$f(h^*(t)\,|h^*(t-\Delta)) = \frac{1}{\sqrt{2\pi h^*(t)}} e^{-\frac{(R(t)-r-\lambda h^*(t))^2}{2h^*(t)}}$$

Let n $= \frac{T}{\Delta}$, and we get a sample time series of $\{h^*(t)\}$ as $\{h^*(\Delta), h^*(2\Delta), h^*(3\Delta), ..., h^*(i\Delta), ..., h^*(n\Delta)\}$ We get the likelihood function $L(\lambda)$:

$$L(\lambda) = \prod_{i=1}^{n} f(h^*(t)\,|h^*(t-\Delta)) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi h^*(i\Delta)}} e^{-\frac{(R(t)-r-\lambda h^*(t))^2}{2h^*(i\Delta)}}$$

Then the log-likelihood function $l(\lambda)$ is given as below

$$l(\lambda) = \sum_{i=1}^{n} \frac{1}{\sqrt{2\pi h^*(i\Delta)}} - \sum_{i=1}^{n} \frac{(R(t)-r-\lambda h^*(t))^2}{2h^*(i\Delta)}$$

The MLE of λ can be obtained numerically through the "maxLik" package in R programming language.

5.2. Heston and Nandi GARCH (1,1) Model Parameter Estimation

The parameters are estimated through MLE method. The likelihood function is given by

$$L = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi h(t)}} e^{-\frac{(R(t)-r-\lambda h(t))^2}{2h(t)}}$$

where h(t) is given by the second equation of Heston and Nandi GARCH (1,1) model recursively. That is

$$h_t = \omega + \beta h_{t-\Delta} + \alpha(Z_{t-\Delta} - \gamma\sqrt{h_{t-\Delta}})^2$$

Thus, the log-likelihood function is

$$l = \log(L) = \sum_{t=1}^{T} -0.5\{\log(2\pi h(t)) + \frac{(R(t)-r-\lambda h(t))^2}{h(t)}\}$$

The MLE of the parameters in Heston and Nandi GARCH (1,1) model will be performed numerically through "hngarhFit" function in "fOptions" package in R programming software.

Next, we will show simulation comparison among new GARCH (1,1) model, Heston and Nandi GARCH (1,1) model, and (continuous) Heston model due to the weak convergence relationship among them. First, We will generate data from our new GARCH (1,1) model with 1000 paths for different sample sizes, and then fit the data set with these three models. Then, we will compare the option price estimation results of these three models.

5.3. Simulation Results Comparison

For data generation from new GARCH (1,1) model, we set step-size $\Delta = 1$(day), and set the true values of the parameters as $\lambda = -2, a = 1.1E-04, b = 4E-04, c = 0.598$, initial stock price $S_0 = 100$, daily risk free rate $r = \frac{0.03}{252}$ (assume there are 252 trading days in a year), initial variance from transformed trading volume $h_0 = \frac{1}{252}(15000000)^{-\frac{1}{5}}$, and burn first 100 pieces of data generated. With this set up, we do Monte Carlo simulation with 1000 paths. And in table (5.1), we show the simulation results. For the MLE of the parameters in the second equation, we choose $w = 4000$ (multiply 4000 to both size of the equation to avoid iteration error).

From Table 5.1, we can see that as sample size increases, the estimation of parameters are becoming more and more close to the true values. From Table 5.4 and Table 5.5, we can see that the option price estimation for the new GARCH (1,1) model is very accurate. And the all the pricing errors are less than one cent. Hence, the new GARCH (1,1) model is reliable.

Table 5.4 shows that the new GARCH (1,1) model has the most accurate option price estimation among these three models. The new GARCH (1,1) slightly outperforms the Heston model with our new method. And both the new GARCH (1,1) model and Heston model with the new method obviously outperform Heston and Nandi GARCH (1,1) model. This is mainly because with new method of measuring volatilities, these two models incorporate more information than Heston and Nandi GARCH (1,1) during the parameter estimation process. The reason for our new GARCH (1,1) model outperforms Heston model is because our new GARCH (1,1) model doesn't have the discretization error as Heston model has.

Besides option pricing advantage of the new GARCH (1,1) model over Heston and Nandi GARCH (1,1) model, another advantage of the new GARCH (1,1) model over Heston and Nandi GARCH (1,1) model lies in parameter estimation. The new GARCH (1,1) model produces less negative values during the iterations of parameter estimation process. And it also converges much quicker than Heston and Nandi GARCH (1,1) model. In practice, the new GARCH (1,1) model converges very fast with a very broad range of initial values of parameters. This is mainly because with the new method of volatility proxy, we can estimate the parameters in these two diffusion equations of the new GARCH (1,1) model separately. However, parameters in Heston and Nandi GARCH (1,1) model cannot be estimated separately equation by equation without observable volatilities. Very rarely Heston and Nandi GARCH (1,1) model converges at the first time without updating the parameter values during iteration process of parameter estimation.

TABLE 5.1. Parameter Estimation of New Method for New GARCH (1,1) Model (1000 replications)

$True\,Value$	$\lambda = -2$	$a = 1.1E-04$	$b = 4E-04$	$c = 0.598$
$n = 50$	-2.0967	$1.19E-04$	$3.90E-04$	0.6466
$Bias$	-0.0967	$8.90E-06$	$-9.60E-06$	0.0486
$RMSE$	10.4992	$2.63E-05$	$4.18E-05$	0.1414
100	-1.8911	$1.13E-04$	$3.95E-04$	0.6164
$Bias$	0.1089	$3.40E-06$	$-5.50E-06$	0.0184
$RMSE$	7.3215	$1.70E-05$	$2.89E-05$	0.0935
200	-1.8384	$1.12E-04$	$3.98E-04$	0.6096
$Bias$	0.1616	$2.10E-06$	$-1.90E-06$	0.0116
$RMSE$	5.3119	$1.25E-05$	$1.99E-05$	0.0674
300	-1.7794	$1.11E-04$	$3.98E-04$	0.6052
$Bias$	0.2206	$1.30E-06$	$-2.40E-06$	0.0072
$RMSE$	4.3948	$1.05E-05$	$1.64E-05$	0.0574
400	-1.7748	$1.11E-04$	$3.98E-04$	0.6037
$Bias$	0.2252	$1.00E-06$	$-2.00E-06$	0.0057
$RMSE$	3.6688	$9.17E-06$	$1.46E-05$	0.0509
600	-1.6372	$1.11E-04$	$3.99E-04$	0.6029
$Bias$	0.3628	$9.00E-07$	$-1.10E-06$	0.0049
$RMSE$	3.1296	$7.19E-06$	$1.14E-05$	0.0383
800	-1.5646	$1.11E-04$	$4.00E-04$	0.6013
$Bias$	0.4354	$6.00E-07$	$-5.00E-07$	0.0033
$RMSE$	2.7792	$5.98E-06$	$9.56E-06$	0.0318
1000	-1.6187	$1.11E-04$	$4.00E-04$	0.6008
$Bias$	0.3813	$5.00E-07$	$-3.00E-07$	0.0028
$RMSE$	2.5521	$5.18E-06$	$8.76E-06$	0.0286

$True\ Value$	$\lambda = -2$	$a = 1.1E-04$	$b = 4E-04,$	$c = 0.598$
2000	-1.6346	$1.10E-04$	$4.00E-04$	0.599
$Bias$	0.3654	$2.00E-07$	$-2.00E-07$	0.0010
$RMSE$	1.7525	$3.71E-06$	$6.70E-06$	0.0190
5000	-1.5792	$1.10E-04$	$4.00E-04$	0.598
$Bias$	0.4208	0	0	0
$RMSE$	1.1458	$2.36E-06$	$4.22E-06$	0.0126
10000	-1.601	$1.10E-04$	$4.00E-04$	0.5986
$Bias$	0.3990	$1.00E-07$	$-2.00E-07$	$6.00E-04$
$RMSE$	0.8546	$1.70E-06$	$2.80E-06$	0.0095

TABLE 5.2. Parameter Estimation for HN GARCH (1,1) Model (1000 replications)

	λ	ω	α	β	γ
$n = 50$	-2.5865	$9.06E-05$	$1.38E-05$	0.3097	46.0814
SE	11.4633	$7.33E-05$	$2.65E-05$	0.3826	20.4663
100	-2.3486	$1.06E-04$	$9.02E-06$	0.2782	55.4369
SE	7.7033	$6.97E-05$	$1.72E-05$	0.3573	21.7375
200	-2.2825	$1.13E-04$	$5.52E-06$	0.2965	56.1677
SE	5.4423	$6.54E-05$	$9.81E-06$	0.3415	19.5587
300	-2.2389	$1.13E-04$	$4.50E-06$	0.3017	69.8806
SE	4.4809	$6.07E-05$	$7.75E-06$	0.3226	22.1106
400	-2.205	$1.14E-04$	$3.79E-06$	0.2999	77.0359
SE	3.7378	$5.83E-05$	$6.24E-06$	0.3131	25.4247
600	-2.0613	$1.12E-04$	$2.96E-06$	0.3157	94.7049
SE	3.1623	$5.54E-05$	$4.83E-06$	0.3004	25.8801
800	-2.007	$1.17E-04$	$2.55E-06$	0.2849	120.4533
SE	2.7670	$5.59E-05$	$4.12E-06$	0.2941	29.1941
1000	-2.056	$1.20E-04$	$2.33E-06$	0.2686	120.4503
SE	2.5361	$5.38E-05$	$3.72E-06$	0.2814	28.2075
2000	-2.0755	$1.25E-04$	$1.64E-06$	0.229	180.7783
SE	1.7140	$4.76E-05$	$2.53E-06$	0.2467	35.1456

Table 5.3. Parameter Estimation for Heston Model (1000 replications)

	κ	θ	σ	ρ
$n = 50$	160.13	0.0464	0.1965	0.0078
SE	32.6466	0.0006	0.0207	0.1444
100	155.52	0.0464	0.1992	-0.0014
SE	24.6446	0.0005	0.0145	0.1016
200	153.12	0.0463	0.2006	0.0005
SE	19.2569	0.0003	0.0106	0.0695
300	151.85	0.0464	0.201	0.0006
SE	14.1167	0.0003	0.0085	0.0585
400	152.03	0.0463	0.201	0.0022
SE	12.4054	0.0002	0.0072	0.0511
600	151.91	0.0464	0.2015	-0.0001
SE	10.3452	0.0002	0.0059	0.0415
800	151.36	0.0464	0.2014	0.0006
SE	8.2	0.0002	0.005	0.0357
1000	151.54	0.0464	0.2015	-0.0002
SE	9.4976	0.0002	0.0048	0.0320
2000	150.54	0.0464	0.2015	0
SE	9.0323	0.0001	0.0037	0.0216
5000	150.6	0.0464	0.2016	-0.0002
SE	6.8264	0.0001	0.0025	0.0147
10000	150.76	0.0464	0.2016	0.0005
SE	4.8615	0.0001	0.0017	0.0101

Table 5.4. New GARCH (1,1) VS HN GARCH (1,1) VS Heston

	$S = 100$	$K = 80$	90	100	110	120
	True Price	22.2719	13.7742	7.2959	3.2774	1.2572
$n = 50$	*New GARCH*	22.2711	13.7733	7.2962	3.2792	1.2594
	HN	22.1824	13.5015	6.8652	2.8629	0.9841
	Heston	22.2645	13.767	7.3029	3.2995	1.2818
100	*New GARCH*	22.2718	13.7745	7.2971	3.2792	1.2589
	HN	22.2042	13.5755	6.9893	2.9852	1.0644
	Heston	22.2646	13.767	7.3026	3.2991	1.2815
200	*New GARCH*	22.2717	13.7739	7.2958	3.2777	1.2577
	HN	22.2274	13.651	7.114	3.1095	1.1488
	Heston	22.263	13.7627	7.2962	3.293	1.2774
300	*New GARCH*	22.2718	13.7743	7.2962	3.278	1.2578
	HN	22.2312	13.662	7.1306	3.1249	1.1586
	Heston	22.2645	13.7669	7.3024	3.299	1.2814
400	*New GARCH*	22.2717	13.7739	7.2956	3.2773	1.2573
	HN	22.23	13.6594	7.1275	3.1227	1.1575
	Heston	22.263	13.7626	7.2962	3.293	1.2774
600	*New GARCH*	22.2722	13.7751	7.2973	3.2789	1.2584
	HN	22.2362	13.6776	7.1557	3.1494	1.1752
	Heston	22.2645	13.7669	7.3024	3.299	1.2814

	$S = 100$	$K = 80$	90	100	110	120
800	$New\,GARCH$	22.2722	13.7752	7.2974	3.279	1.2584
	HN	22.2412	13.6917	7.1765	3.1686	1.1874
	$Heston$	22.2645	13.7668	7.3024	3.2989	1.2814
1000	$New\,GARCH$	22.2721	13.7749	7.297	3.2785	1.2581
	HN	22.2396	13.6881	7.1724	3.1659	1.1863
	$Heston$	22.2645	13.7669	7.3024	3.2989	1.2814
2000	$New\,GARCH$	22.2723	13.7754	7.2977	3.2792	1.2585
	HN	22.242	13.695	7.1826	3.1752	1.1923
	$Heston$	22.2645	13.7668	7.3024	3.2989	1.2814
5000	$New\,GARCH$	22.2723	13.7753	7.2975	3.2789	1.2583
	HN	22.2381	13.6814	7.1586	3.1495	1.1734
	$Heston$	22.2645	13.7668	7.3024	3.2989	1.2814
10000	$New\,GARCH$	22.2722	13.7751	7.2972	3.2786	1.2581
	HN	22.2579	13.7376	7.2438	3.2305	1.2274
	$Heston$	22.2645	13.7668	7.3024	3.2989	1.2814

5.4. BS VS Stochastic Volatility

In this section, we will compare BS model implied volatility plot with plots of implied volatility inverted from those three stochastic volatility models with data generated from different models. Firstly, we make this comparison by generating data from Generalized Stein and Stein model with our new method. Secondly, we make the comparison by generating data from Heston model with our new method. The following two subsections will show more about the implied volatility comparison.

5.4.1. BS VS Stochastic Volatility with Generalized Stein and Stein Model Generated Data

For data generation from Stein and Stein model with our new method, we set $\kappa = 10, \theta = 0.2, \sigma = 0.2, \rho = 0.7$, the risk free rate $r = 0.04$, initial standard deviation from transformed trading volume $\sqrt{V_0} = \sqrt{(10000000)^{-1/5}}$, and 1000 replications are simulated. Time to maturity is 0.5 years. Current stock price is assumed to be $S_0 = \$100$. During data generation process for V_t, negative values may be produced. We apply reflection method here. That is when a negative value of V_t is generated, it will be replaced by its absolute value.

With the generated data set of sample size 10000, we fit the data set with generalized Stein and Stein model, Heston model, and new continuous model to get MLE of the parameters. Then with the MLE of the parameters, we can get the implied volatilities for those three stochastic volatility models based on the true option price calculated with the closed-form formula of Generalized Stein and Stein model. Figure 5.1 shows the comparison of the implied volatility vs S/K (ratio between stock price and strike price) for BS model and other three stochastic volatility models.

From Figure 5.1, we can see that all the stochastic volatility models with our new method have less steep implied volatility plot. Recall that the closer the implied volatility plot to a horizontal line, the better the model is from the perspective of reducing implied volatility skew effect. With this standard, we can see that all these three stochastic volatility models with the new method outperform BS model.

5.4.2. BS VS Stochastic Volatility with Heston Generated Data

For data generation from Heston model, we set $\kappa = 10, \theta = 0.05, \sigma = 0.2, \rho = -0.6$, the risk free rate $r = 0.02$, initial variance from transformed trading volume $V_0 = (3200000)^{-1/5}$, and 1000 replications are simulated. Time to maturity is 0.5 years. Initial stock price is assumed to be $S_0 = \$50$. During data generation process for V_t, reflection method is applied for negative values as in previous section. That is when a negative value of V_t is generated, it will be replaced by its absolute value.

With the generated data set of sample size 10000, we fit the data set with generalized Stein and Stein model, Heston model, and new continuous model to get MLE of the parameters. Then with the MLE of the parameters, we get the implied volatilities for those three stochastic volatility models based on the true option price calculated with the closed-form

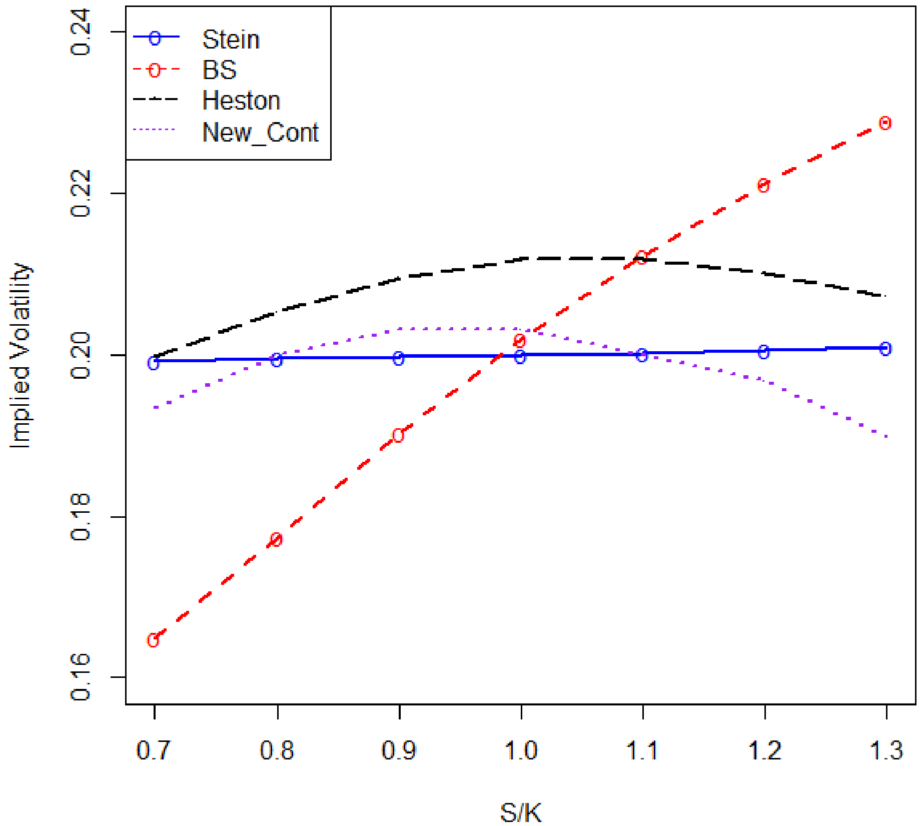

Figure 5.1. BS VS Stein VS Heston VS New Continuous

formula of Heston model. Figure 5.1 shows the comparison of the implied volatility vs S/K (ratio between stock price and strike price) for BS model and other three stochastic volatility models.

From Figure 5.2, we can see that all the stochastic volatility models with our new method have less steep implied volatility plot. Recall that the closer the implied volatility plot to a horizontal line, the better the model is from the perspective of reducing implied volatility skew effect. With this standard, we can see that all these three stochastic volatility models with the new method outperform BS model.

From the above two comparisons shown in Figure 5.1 and Figure 5.2, we can claim that those three stochastic volatility models outperform BS model in terms of implied volatility skew with our new method of measuring volatility.

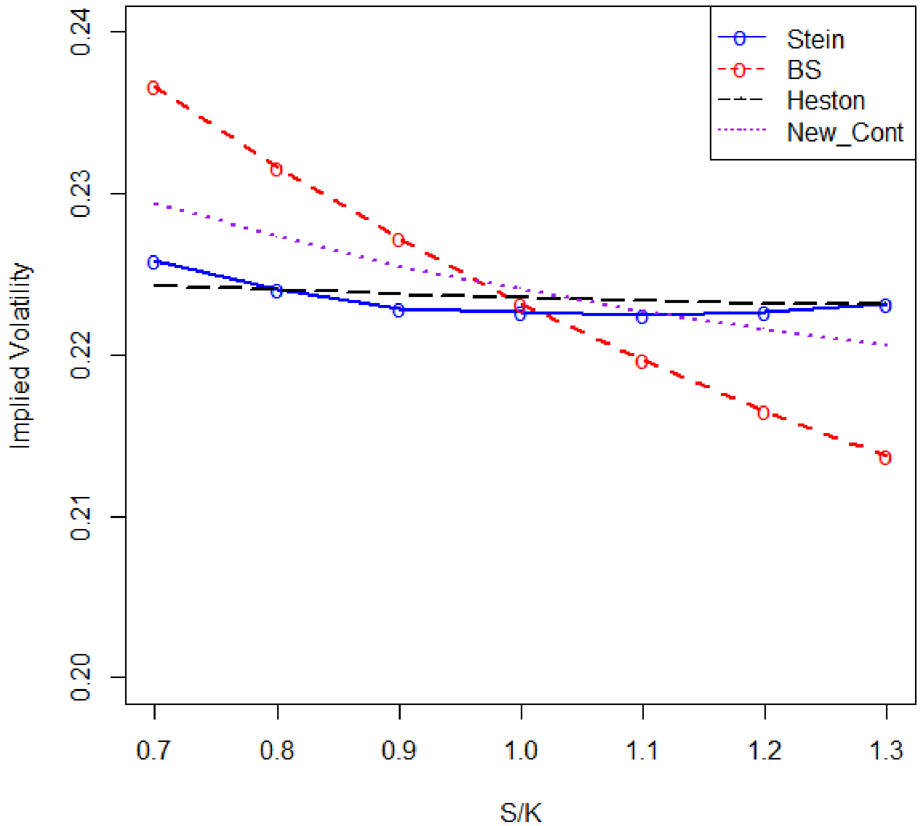

Figure 5.2. BS VS Heston VS Stein VS New Continuous

CHAPTER 6

REAL DATA APPLICATION AND ANALYSIS

In this chapter, we apply the volatility proxy: annualized variance $V_t = m * (TV_t)^p$, where $m = 1$ and p is identified as follows:

$$p = \begin{cases} -\frac{1}{6} & \text{if } 10^8 \leq \text{ATV} \\ -\frac{1}{5} & \text{if } 10^7 \leq \text{ATV} < 10^8 \\ -\frac{1}{4} & \text{if } 10^6 \leq \text{ATV} < 10^7 \end{cases}$$

where ATV is defined as the average trading volume of the stock in the chosen period of time. Here we use daily trading volume data. So ATV refers to average daily trading volume of the chosen trading volume sample data set. The above identification is based on the real market data of 2006. For other years, there may be slight adjustment. For 2006 data, we do not consider stocks with trading volume less than 10^6 since trading volume less than 10^6 is not liquid enough. For Daily variance $h_t = m * (TV_t)^p$, $m = 252$, and p is identified as above. For the rest of the chapter, we will see how this new method of volatility proxy is applied in read data. And the in sample performance and out of sample performance are measured by root mean prediction error (RMPE).

We choose stock price and trading volume information of Microsoft, Costco, Exxon Mobil, and JP Morgan to analyze. We first estimate parameters with MLE during the time period from January 1st to March 31st, 2006. For in sample performance, we choose the time period from March 31 to May 20. For out of sample performance, we choose the time period from April 7th to May 20th. And all the option prices shown in this chapter are about European call options. The corresponding European put option prices can be computed from put-call parity.

6.1. Microsoft Data Analysis and Option Price Forecasting

The average trading volume for Microsoft from January 1st to March 31st is between 10^7 and 10^8, so $p = -\frac{1}{5}$. For Heston and Nandi GARCH (1,1) model and the New GARCH

(1,1) model, we take stepsize as one day. This indicates m = 252. For all the continuous models, we take m = 1. We take the LIBOR $r = 0.051$ as annual risk-free rate. The rest three companies will adopt the same stepsize, m value, and risk-free rate as above mentioned. The ex-dividend date is May 15 with dividend \$0.09, which should be subtracted from stock price during option price computation.

6.1.1. MSFT Parameter Estimation

With the stock price information and transformed trading volume serving as volatility proxy, we perform MLE method to get parameter estimates in each model. The parameter estimation results are shown in Table 6.1.

6.1.2. MSFT In Sample Performance Comparison

After getting ML estimators for each model, we show forecasted option prices of different strike prices in Table 6.2, as well as observed market prices. Among those models, BS and Heston and Nandi GARCH (1,1) model serve as baselines. Forecasted option prices from all models with our new method have smaller RMPE than option prices calculated from two baseline models. The new method for the special case of generalized Stein and Stein model has the smallest RMPE and thus the best performance.

6.1.3. MSFT Out of Sample Performance Comparison

For out of sample performance, we show forecasted option prices of various strike prices in Table 6.3, as well as observed market prices. Forecasted option prices calculated from all non-baseline models with our new method have smaller RMPE compared with option prices calculated from two baseline models. Similar to the in sample performance, the new method for the special case of Stein and Stein model has the smallest RMPE and thus the best performance.

6.2. Costco Data Analysis and Option Price Forecasting

The average trading volume for Costco from January 1st to March 31st is between 10^6 and 10^7, so $p = -\frac{1}{4}$. The ex-dividend date is May 8 with dividend \$0.08, which should be subtracted from stock price during option price computation.

Table 6.1. MSFT Parameter Estimation

BS	$\sigma = 0.1566$				
SE	0.0142				
Hull-White	$\kappa = 109.1$	$\theta = 0.1663$	$\sigma = 0.0566$		
SE	4.201	0.0001	0.0051		
$Stein - Stein$	$\kappa = 188.5$	$\theta = 0.1670$	$\sigma = 0.0666$	$\rho = -0.1535$	
SE	5.949	0.0007	0.006		
$Heston$	$\kappa = 229.4$	$\theta = 0.0279$	$\sigma = 0.1405$	$\rho = -0.1503$	
SE	5.974	0.0002	0.0128		
$New\ Continuous$	$\kappa = 259.5$	$\theta = 0.0280$	$\sigma = 0.251$	$\rho = -0.1540$	
SE	4.195	0.0003	0.0227		
$Special\ Stein$	$\sigma = 0.0749$				
SE	0.0068				
$Special\ Heston$	$\sigma = 0.1496$				
SE	0.0135				
$Special\ New\ Continuous$	$\sigma = 0.2999$				
SE	0.02715				
$Heston - Nandi$	$\lambda = 0.2258$	$\omega = 9.35E - 05$	$\alpha = 9.149E - 30$	$\beta = 0.03834$	$\gamma = 4.975$
SE	5.9316	0	0	0	0
$New\ Garch(1,1)$	$\lambda = 0.1951$	$a = 5.636E - 05$	$b = 2.563E - 04$	$c = 0.5084$	
SE	16.7772	0	0	0.11	

6.2.1. COST Parameter Estimation

With the stock price information and transformed trading volume serving as volatility proxy, we perform MLE method to get parameter estimates in each model. The parameter estimation results are shown in Table 6.4.

TABLE 6.2. MSFT In Sample Performance

$S = 27.21$	$K = 22.5$	25	27.5	30	32.5	$RMPE$
Observed Market price(call)	4.85	2.45	0.6	0.1	0.05	
BS	4.7744	2.3318	0.5334	0.035	0.0006	0.0785
$Hull - White(New)$	4.7746	2.3438	0.5711	0.0462	0.0012	0.0679
$Stein - Stein(New)$	4.7759	2.3456	0.5731	0.0472	0.0023	0.0666
$Heston(New)$	4.7746	2.3452	0.5745	0.0471	0.0012	0.0671
New Continuous(New)	4.7746	2.3454	0.5748	0.0471	0.0012	0.0670
Special Stein − Stein(New)	4.7767	2.347	0.5708	0.0491	0.0033	0.0657
Special Heston(New)	4.7748	2.3449	0.5682	0.0475	0.0015	0.0676
Special New Continuous(New)	4.7747	2.3419	0.559	0.0447	0.0014	0.0699
$Heston - Nandi$	4.7744	2.3318	0.5332	0.0349	0.0006	0.0785
$New\ GARCH(1,1)$	4.7747	2.3476	0.5743	0.0449	0.001	0.0667

TABLE 6.3. MSFT Out of Sample Performance

$S = 27.25$	$K = 22.5$	25	27.5	30	32.5	$RMPE$
Observed Market price(call)	4.85	2.45	0.55	0.1	0.05	
BS	4.7917	2.3334	0.4923	0.0238	0.0002	0.07564
$Hull - White(New)$	4.7918	2.3474	0.5446	0.0371	0.0007	0.06377
$Stein - Stein(New)$	4.7944	2.3445	0.5284	0.0333	0.0027	0.06538
$Heston(New)$	4.7918	2.3437	0.531	0.0331	0.0005	0.06629
New Continuous(New)	4.7918	2.3441	0.5317	0.0332	0.0005	0.06611
Stein − Stein Special (New)	4.7946	2.3502	0.5431	0.0393	0.0032	0.06157
Heston Special (New)	4.7919	2.3483	0.5423	0.0381	0.0009	0.06328
New Continuous Special(New)	4.7919	2.3463	0.5353	0.0362	0.0008	0.06455
$Heston - Nandi$	4.7917	2.3334	0.4921	0.0237	0.0002	0.07569
$New\ GARCH(1,1)$	4.7918	2.346	0.5312	0.0313	0.0004	0.06593

Table 6.4. COST Parameter Estimation

BS	$\sigma = 0.1534$				
SE	0.0139				
$Hull - White$	$\kappa = 100.5$	$\theta = 0.1612$	$\sigma = 0.0840$		
SE	4.227	0.0001	0.0076		
$Stein - Stein$	$\kappa = 189.4$	$\theta = 0.1588$	$\sigma = 0.0944$	$\rho = -0.1477$	
SE	8.437	0.001	0.0086		
$Heston$	$\kappa = 126.2$	$\theta = 0.0253$	$\sigma = 0.1813$	$\rho = -0.1468$	
SE	5.933	0.0005	0.0164		
$New\ Continuous$	$\kappa = 475$	$\theta = 0.0253$	$\sigma = 0.4058$	$\rho = -0.1390$	
SE	4.198	0.0003	0.0368		
$Special\ Stein$	$\sigma = 0.1053$				
SE	0.0095				
$Special\ Heston$	$\sigma = 0.2651$				
SE	0.024				
$Special\ New\ Continuous$	$\sigma = 0.4221$				
SE	0.0382				
$Heston - Nandi$	$\lambda = 12.15$	$\omega = 1.229E - 07$	$\alpha = 7.875E - 18$	$\beta = 0.9987$	$\gamma = 5.086$
SE	2.966	0	0	0	0
$New\ Garch(1,1)$	$\lambda = 11.15$	$a = 4.945E - 05$	$b = 3.628E - 04$	$c = 0.4930$	
SE	9.686	0	0	0.1094	

6.2.2. COST In Sample Performance Comparison

After getting ML estimators for each model, we show forecasted option prices of different strike prices in Table 6.5, as well as the observed market prices . The forecasted option prices from all models with our new method have smaller RMPE than option prices calculated from two baseline models. The special case of Stein and Stein model has the smallest RMPE and thus the best performance.

Table 6.5. COST In Sample Performance

$S = 54.16$	50	52.5	55	57.5	60	$RMPE$
$Observed\ Market\ price(call)$	4.6	2.55	1	0.3	0.1	
BS	4.5057	2.4061	0.9693	0.2781	0.0555	0.0812
$Hull-White(New)$	4.5265	2.458	1.0332	0.3207	0.0721	0.0569
$Stein-Stein(New)$	4.5223	2.4434	1.0088	0.3101	0.0656	0.0613
$Heston(New)$	4.5213	2.4439	1.0143	0.3068	0.0662	0.0614
$New\ Continuous(New)$	4.52	2.4414	1.012	0.3057	0.0658	0.0625
$Special\ Stein-Stein\ (New)$	4.5398	2.4683	1.0344	0.3375	0.0819	0.0514
$Special\ Heston(New)$	4.5334	2.4487	1.0146	0.3192	0.0805	0.0560
$Special\ New\ Continuous\ (New)$	4.5191	2.4218	0.9837	0.2951	0.068	0.0697
$Heston-Nandi$	4.508	2.4144	0.9768	0.2829	0.0573	0.0768
$New\ GARCH(1,1)$	4.5308	2.458	1.0213	0.3049	0.0629	0.0550

6.2.3. COST Out of Sample Performance Comparison

For out of sample performance, we show forecasted option prices for various strike prices in Table 6.6, as well as observed market prices. And we can see that only Heston model, generalized Stein and Stein model, new continuous model, and new GARCH (1,1) model with our new method outperform the two baseline models. The remaining models with our new method, including special cases of Hull and White, Heston, Stein and Stein, and new continuous model, underperform the two baseline models. The main reason for this under performance is because the drift terms of those non-special models are not close enough to zero. And the new GARCH (1,1) model has the best performance in out of sample option price forecasting.

6.3. Exxon Mobil Data Analysis and Option Price Forecasting

The average trading volume for Exxon Mobil from January 1st to March 31st is between 10^7 and 10^8, so $p = -\frac{1}{5}$. The ex-dividend date is May 10 with dividend \$0.32, which should be subtracted from stock price during option price computation.

Table 6.6. COST Out of Sample Performance

$S = 55.7$	50	52.5	55	57.5	60	$RMPE$
$Observed\ Market\ price(call)$	6	3.65	1.8	0.6	0.15	
BS	5.9276	3.5775	1.6768	0.5571	0.1242	0.0751
$Hull-White(New)$	5.9262	3.5704	1.6621	0.5433	0.1177	0.0837
$Stein-Stein(New)$	5.9253	3.6049	1.7081	0.5885	0.1486	0.0569
$Heston(New)$	5.9319	3.5969	1.7137	0.5903	0.1397	0.055
$New\ Continuous(New)$	5.9317	3.5963	1.7134	0.5906	0.1401	0.0552
$Special\ Stein-Stein\ (New)$	5.9198	3.5947	1.6585	0.5474	0.1428	0.0804
$Special\ Heston(New)$	5.9312	3.5745	1.6483	0.5332	0.1235	0.0879
$Special\ New\ Continuous\ (New)$	5.9268	3.5593	1.6227	0.5083	0.109	0.105
$Heston-Nandi$	5.9282	3.5808	1.6835	0.5634	0.1272	0.0712
$New\ GARCH(1,1)$	5.936	3.6086	1.7257	0.5919	0.1358	0.0482

6.3.1. XOM Parameter Estimation

With the stock price information and transformed trading volume serving as volatility proxy, we perform MLE method to get parameter estimates in each model. The parameter estimation results are shown in Table 6.7.

6.3.2. XOM In Sample Performance Comparison

After getting ML estimators for each model, we show forecasted option prices of different strike prices in Table 6.8, as well as observed market prices. From this table, all the stochastic volatility models with the new method outperform the two baseline models. And the new GARCH (1,1) has the smallest RMPE and thus the best performance among all models.

6.3.3. XOM Out of Sample Performance Comparison

The out of sample performance of Exxon Mobil is shown in Table 6.9. We can see that all the stochastic volatility models with the new method outperform the two baseline

Table 6.7. XOM In Sample Performance

BS	$\sigma = 0.1655$				
SE	0.015				
$Hull - White$	$\kappa = 111.4$	$\theta = 0.1869$	$\sigma = 0.0344$		
SE	4.197	0	0.0031		
$Stein - Stein$	$\kappa = 162.7$	$\theta = 0.1872$	$\sigma = 0.0477$	$\rho = 0.1363$	
SE	5.9318	0.0006	0.0043		
$Heston$	$\kappa = 162.4$	$\theta = 0.0351$	$\sigma = 0.0953$	$\rho = 0.1362$	
SE	5.932	0.0002	0.0086		
$New\ Continuous$	$\kappa = 323.7$	$\theta = 0.0351$	$\sigma = 0.1904$	$\rho = 0.1335$	
SE	4.1944	0.0002	0.0172		
$Stein\ Special\ Case$	$\sigma = 0.0583$				
SE	0.0053				
$Heston\ Special\ Case$	$\sigma = 0.1165$				
SE	0.0106				
$New\ Cont\ Special$	$\sigma = 0.2328$				
SE	0.0211				
$Heston - Nandi$	$\lambda = 4.334$	$\omega = 1.291E - 115$	$\alpha = 1.828E - 05$	$\beta = 0.8305$	$\gamma = -13.0$
SE	5.932	0	0.00000507	0	0.0050127
$New\ Garch(1,1)$	$\lambda = 3.272$	$a = 8.9982E - 05$	$b = 1.8923E - 04$	$c = 6.4679E - 01$	
SE	8.389	0	0	0.1177	

models. The new method for the special case of generalized Stein and Stein model has the smallest RMPE and thus the best performance.

6.4. JP Morgan Data Analysis and Option Price Forecasting

The average trading volume for JP Morgan from January 1st to March 31st is between 10^7 and 10^8, so $p = -\frac{1}{5}$. The ex-dividend date is April 4 with dividend \$0.34, which should be subtracted from stock price during option price computation.

Table 6.8. XOM In Sample Performance

$S = 60.87$	55	57.5	60	62.5	65	$RMPE$
$Observed\ Market\ price(call)$	6.25	4	2.15	0.9	0.3	
BS	5.9919	3.7634	1.9875	0.8481	0.2857	0.1743
$Hull-White(New)$	6.0424	3.8851	2.1693	1.0259	0.4051	0.1293
$Stein-Stein(New)$	6.0427	3.8878	2.1717	1.0291	0.409	0.1301
$Heston(New)$	6.0434	3.8877	2.1735	1.0305	0.4087	0.1302
$New\ Continuous(New)$	6.0425	3.8857	2.1707	1.0278	0.4068	0.1298
$Special\ Stein-Stein\ (New)$	6.045	3.8893	2.1704	1.0276	0.4097	0.1289
$Special\ Heston(New)$	6.0438	3.8845	2.166	1.0229	0.4048	0.1282
$Special\ New\ Continuous\ (New)$	6.0405	3.8772	2.1555	1.0125	0.3976	0.1274
$Heston-Nandi$	5.9953	3.7524	1.9606	0.8359	0.2984	0.1823
$New\ GARCH(1,1)$	6.0471	3.8924	2.1758	1.0284	0.4043	0.1271

Table 6.9. XOM Out of Sample Performance

$S = 61.34$	55	57.5	60	62.5	65	$RMPE$
$Observed\ Market\ price(call)$	6.65	4.35	2.4	1	0.3	
BS	6.3729	4.0571	2.1459	0.8929	0.2821	0.2186
$Hull-White(New)$	6.41	4.1671	2.3341	1.09	0.4159	0.1529
$Stein-Stein(New)$	6.403	4.1551	2.3107	1.0658	0.4024	0.1561
$Heston(New)$	6.4047	4.1535	2.3127	1.0682	0.4008	0.1557
$New\ Continuous(New)$	6.4041	4.1519	2.3102	1.0657	0.399	0.1561
$Special\ Stein-Stein\ (New)$	6.4106	4.1718	2.3348	1.0902	0.4207	0.1523
$Special\ Heston(New)$	6.4112	4.1671	2.3318	1.0875	0.4155	0.1524
$Special\ New\ Continuous\ (New)$	6.4093	4.1622	2.3239	1.0793	0.4097	0.1532
$Heston-Nandi$	6.3778	4.0523	2.1199	0.8762	0.2936	0.2265
$New\ GARCH(1,1)$	6.4076	4.1582	2.3157	1.0666	0.3964	0.1526

6.4.1. JPM Parameter Estimation

With the stock price information and transformed trading volume serving as volatility proxy, we perform MLE method to get parameter estimates in each model. The parameter estimation results are shown in Table 6.10.

TABLE 6.10. JPM Parameter Estimation

BS	$\sigma = 0.1436$				
SE	0.013				
$Hull - White$	$\kappa = 224.7$	$\theta = 0.1988$	$\sigma = 0.0496$		
SE	4.196	0	0.0045		
$Stein - Stein$	$\kappa = 157.8$	$\theta = 0.2005$	$\sigma = 0.0797$	$\rho = -0.0014$	
SE	8.905	0.0012	0.0078		
$Heston$	$\kappa = 210.6$	$\theta = 0.0391$	$\sigma = 0.1525$	$\rho = 0.0077$	
SE	5.932	0.0003	0.0138		
$New\ Continuous$	$\kappa = 420.3$	$\theta = 0.0392$	$\sigma = 0.3037$	$\rho = 0.0065$	
SE	4.195	0.0003	0.0275		
$Stein\ Special\ Case$	$\sigma = 0.1007$				
SE	0.0091				
$Heston\ Special\ Case$	$\sigma = 0.2007$				
SE	0.0182				
$Special\ New\ Continuous$	$\sigma = 0.4009$				
SE	0.0363				
$Heston - Nandi$	$\lambda = 4.625$	$\omega = 3.197E - 08$	$\alpha = 1.104E - 54$	$\beta = 9.996E - 01$	$\gamma = 6.48$
SE	4.194	0	0	0	0
$New\ Garch(1,1)$	$\lambda = 2.543$	$a = 1.301E - 04$	$b = 3.033E - 04$	$c = 0.8386$	
SE	11.863	0.00001933	0.00002747	0.1249	

6.4.2. JPM In Sample Performance Comparison

After getting ML estimators for each model, we show forecasted option prices of different strike prices in Table 6.11, as well as observed market prices. From this table, not all the stochastic volatility models with the new method outperform the baseline models. For those models, which underperforms the baseline models, have slightly bigger RMPE than the baselines. But we still expect better performance of these models in out of sample performance. And the new method for the special case of the new continuous model has the smallest RMPE and thus the best performance.

TABLE 6.11. JPM In Sample Performance

$S = 41.64$	35	37.5	40	42.5	45	$RMPE$
$Observed\ Market\ price(call)$	6.65	4.2	2.05	0.6	0.1	
BS	6.5403	4.0767	1.8609	0.4979	0.0670	0.1221
$Hull-White(New)$	6.5485	4.1584	2.13	0.8151	0.2233	0.1264
$Stein-Stein(New)$	6.5492	4.1619	2.139	0.825	0.2296	0.1319
$Heston(New)$	6.5481	4.1563	2.125	0.8094	0.2199	0.1234
$New\ Continuous(New)$	6.548	4.1555	2.123	0.8072	0.2186	0.1222
$Special\ Stein-Stein\ (New)$	6.5512	4.1641	2.1349	0.8188	0.2305	0.1290
$Special\ Heston(New)$	6.5502	4.1607	2.1258	0.809	0.224	0.1235
$Special\ New\ Continuous\ (New)$	6.549	4.1538	2.1081	0.7887	0.2123	0.1131
$Heston-Nandi$	6.5402	4.074	1.8458	0.4791	0.0603	0.1310
$New\ GARCH(1,1)$	6.5489	4.1593	2.128	0.8082	0.2166	0.1224

6.4.3. JPM Out of Sample Performance Comparison

The out of sample performance of JP Morgan is shown in Table 6.12. We can see that all the stochastic volatility models with the new method outperform the two baseline models. The new method for the special case of Heston model and the new GARCH (1,1) model have the smallest RMPE and thus the best performance.

Table 6.12. JPM Out of Sample Performance

$S = 41.71$	35	37.5	40	42.5	45	$RMPE$
$Observed\ Market\ price(call)$	6.95	4.55	2.35	0.75	0.15	
BS	6.9149	4.4364	2.1158	0.5735	0.0709	0.1459
$Hull-White(New)$	6.9175	4.4813	2.3278	0.8746	0.2225	0.0736
$Stein-Stein(New)$	6.9185	4.484	2.3387	0.8875	0.2315	0.0788
$Heston(New)$	6.9175	4.481	2.3267	0.8733	0.2217	0.0731
$New\ Continuous(New)$	6.9175	4.4805	2.3251	0.8712	0.2204	0.0723
$Special\ Stein-Stein\ (New)$	6.9198	4.4845	2.333	0.8765	0.2291	0.0745
$Special\ Heston(New)$	6.9184	4.4836	2.3258	0.8694	0.2228	0.0715
$Special\ New\ Continuous\ (New)$	6.9179	4.4799	2.3133	0.8529	0.2131	0.0661
$Heston-Nandi$	6.9148	4.4352	2.1045	0.5551	0.0641	0.155
$New\ GARCH(1,1)$	6.9179	4.4834	2.3302	0.8727	0.2183	0.0715

6.5. Conclusion

From the real data application and analysis of those four companies, we can conclude the following: (1) All the stochastic volatility models with the new method have smaller average RMPE than BS model and Heston-Nandi GARCH (1,1) model. This further proves that besides being correlated with all measures of volatility, the trading volume itself can serve as a measure of volatility under some transformation. And the new method of measuring volatility is robust. (2) No matter for in sample data analysis or out of sample data analysis, the new GARCH (1,1) has the smallest average RMPE for both categories and thus the best performance. There are two major reasons for the new GARCH (1,1) to have the best performance. The first one is that the new method of measuring volatility carries more information into the option pricing process and thus provides more accurate option prices. The second reason is because the new GARCH (1,1) model is discrete with exact closed form solution and thus has no discretization error. (3) In real data analysis, the drift term of the continuous stochastic volatility models is close to zero most of the times. Even though this is not always true, the performance of the new method for the special case of Hull and White model, Stein-Stein model, Heston model, and the new continuous model shows this. This is why the new method for those special cases outperform BS model and Heston-Nandi GARCH (1,1) model.

Both the simulation and real data application and analysis show that the new method of measuring volatility works well. And the forecasted option prices outperform the two baseline models - BS model and Heston-Nandi GARCH (1,1) model in terms of RMPE. Further work for improvement could lie in more refined method of identifying m and p to get more accurate option price forecast.

APPENDIX A

GIL-PELAEZ INVERSION THEOREM

Proof of Gil-Pelaez (1951) Inversion Theorem (The proof is quoted from Crisostomo (2014) [9]).

Proof. The proof follows the reasoning in Kendall, Stuart and Ird (1994) [26] and Wu (2007) [41]. First we start with the integral

$$I = \int_0^\infty \frac{e^{iwx}\phi_X(-w) - e^{-iwx}\phi_X(w)}{iw} dw$$

Replacing each characteristic function by its integral form, the expression above becomes

$$\begin{aligned}
I &= \int_0^\infty \frac{e^{iwx}\int_{-\infty}^{\infty} e^{-iwz}dF(z) - e^{-iwx}\int_{-\infty}^{\infty} e^{iwz}dF(z)}{iw} dw \\
&= \int_0^\infty \int_{-\infty}^{\infty} \frac{e^{iwx}e^{-iwz} - e^{-iwx}e^{iwz}}{iw} dF(z)dw \\
&= \int_0^\infty \int_{-\infty}^{\infty} \frac{e^{iw(x-z)} - e^{-iw(x-z)}}{iw} dF(z)dw
\end{aligned}$$

Next, considering Euler's equality $\sin(\theta) = (e^{i\theta} - e^{-i\theta})/2i$, and using $\theta = w(x-z)$, it can be seen that $2\sin w(x-z) = (e^{iw(x-z)} - e^{-iw(x-z)})/i$. Notice that, for any real number δ, $\lim_{n\to\infty}\int_0^n \sin(\delta t)/t dt = (\pi/2)sgn(\delta)$. Therefore, applying Fubini's theorem and the above fact (replace δ with $x-z$, t with w) , the integral I simplifies to

$$\begin{aligned}
I &= \int_0^\infty \int_{-\infty}^{\infty} \frac{2\sin w(x-z)}{w} dF(z)dw \\
&= \int_{-\infty}^{\infty} \int_0^\infty \frac{2\sin w(x-z)}{w} dwdF(z) \\
&= \int_{-\infty}^{\infty} \pi sgn(x-z)dF(z) \\
&= \pi[F(x) + 0 - (1 - F(x))] \\
&= \pi[2F(x) - 1]
\end{aligned}$$

Consequently, solving for $F(x)$ and then substituting I by its original definition yields

$$
\begin{aligned}
F(x) &= \frac{1}{2} + \frac{1}{2\pi} \\
&= \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{e^{iwx}\phi_X(-w) - e^{-iwx}\phi_X(w)}{iw} dw
\end{aligned}
$$

Finally, since the desity of X is a real-valued function, using the properties of Fourier transforms, $\phi_X(w)$ has conjugate symmetry and $[\phi_X(w) + \phi_X(-w)]/2 = Re[\phi_X(w)]$. Therefore, the CDF of X can also be expressed as

$$
\begin{aligned}
F(x) &= \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{e^{iwx}\phi_X(-w) - e^{-iwx}\phi_X(w)}{2iw} dw \\
&= \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} Re[\frac{e^{-iwx}\phi_X(w)}{iw}] dw
\end{aligned}
$$

Here, we finish the proof of Gil-Pelaez Inversion Theorem.

APPENDIX B

CONVERGENCE TO CONTINUOUS TIME LIMIT UNDER PHYSICAL MEASURE

Convergence to Continuous Time Limit Under Physical Measure $\mathbb{P}$.

Proof. Notice that the first equation of our model (equation (30)) under physical measure is the Euler discretization of the continuous-time limit (equation (32)). According to Yan (2002) [42], Euler discretization is weakly convergent. This proves the first equation convergence.

Rewrite the second equation of our model under physical measure as follows

$$h_t = a - b^2 - ch_{t-\Delta} + (bz_{t-\Delta} - \sqrt{h_{t-\Delta}})^2 \tag{B1}$$

Notice that h_t is the conditional variance of the stock return between the time interval $t-\Delta$ and t. For notational clarity, rewrite it as ${}_\Delta h_{t-\Delta}$ with starting point ${}_\Delta h_0$.

Divide Δ to both sides of the above equation and let ${}_\Delta v_t = \frac{{}_\Delta h_t}{\Delta}$ for $t \geq 0$ with starting point ${}_\Delta v_0 = \frac{{}_\Delta h_0}{\Delta}$. Then ${}_\Delta v_{t-\Delta}$ is the variance per unit of time calculated from the time interval $(t-\Delta, t)$.

From (B1), we have

$$ {}_\Delta v_t = \frac{a-b^2}{\Delta} - c({}_\Delta v_{t-\Delta}) + (\frac{b}{\sqrt{\Delta}} z_{t-\Delta} - \sqrt{{}_\Delta v_{t-\Delta}})^2 \tag{B2}$$

Discretize the process ${}_\Delta v_t$ as follows:

$${}_\Delta v_t = {}_\Delta v_{j\Delta}$$

for $j\Delta \leq t < (j+1)\Delta$, where j is a natural number. Let $a = \kappa\theta\Delta^2$, $b = \frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4}$, $c = \kappa\Delta - \frac{\kappa^2\Delta^2}{4}$, and plug them into (B2) to get

$$\begin{aligned} {}_\Delta v_{j\Delta} = {} & \kappa(\theta - {}_\Delta v_{(j-1)\Delta})\Delta + (\frac{\kappa^2\Delta^2}{4} + 1){}_\Delta v_{(j-1)\Delta} + (\frac{\kappa\sigma\Delta^{\frac{3}{2}}}{2} - \sigma\sqrt{\Delta}) z_{t-\Delta}\sqrt{{}_\Delta v_{(j-1)\Delta}} \\ & + (\frac{\kappa^2\sigma^2\Delta^3}{16} + \frac{\sigma^2\Delta}{4} - \frac{\kappa\sigma^2\Delta^2}{4})(z^2_{t-\Delta} - 1) \end{aligned} \tag{B3}$$

with starting point ${}_\Delta v_0$. Let's choose a starting point v_0 for the continuous volatility process, which satisfies the following equality

$${}_\Delta v_0 = \frac{a-b^2}{\Delta} - c(v_0) + (\frac{b}{\sqrt{\Delta}} z_{t-\Delta} - \sqrt{v_0})^2 \tag{B4}$$

We assume the starting point v_0 follows a non-central chi-square distribution with mean μ_0 and variance σ_0^2. Also assume the non-central chi-square distribution that v_0 follows is generated by a normal variable that is independent of the normal variable that generates ${}_\Delta v_0$. we can see that the left-hand side of (B4) is a linear combination of shifted and scaled random variables of non-central chi-square distributions. Thus, we can conclude that ${}_\Delta v_0$, on the right-hand side of (B4), also follows a non-central chi-square distribution.

Do the substitution $a = \kappa\theta\Delta^2$, $b = \frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4}$, $c = \kappa\Delta - \frac{\kappa^2\Delta^2}{4}$ in (B4), we can see that ${}_\Delta v_0$ and v_0 also satisfy (B3). That is

(B5)
$$ {}_\Delta v_0 = \kappa(\theta - v_0)\Delta + (\frac{\kappa^2\Delta^2}{4} + 1)v_0 + (\frac{\kappa\sigma\Delta^{\frac{3}{2}}}{2} - \sigma\sqrt{\Delta})z_{t-\Delta}\sqrt{v_0} $$

$$ + (\frac{\kappa^2\sigma^2\Delta^3}{16} + \frac{\sigma^2\Delta}{4} - \frac{\kappa\sigma^2\Delta^2}{4})(z_{t-\Delta}^2 - 1) $$

Take expectation for both sides of (B5), we get

$$ E[{}_\Delta v_0] = \kappa(\theta - \mu_0)\Delta + (\frac{\kappa^2\Delta^2}{4} + 1)\mu_0 $$

where $\mu_0 = E[v_0]$. Let $\Delta \to 0^+$, we have

(B6)
$$ \lim_{\Delta\to 0^+} E[{}_\Delta v_0] = \mu_0 $$

Take variance for both sides of (B3) and take the limit, we will get

(B7)
$$ \lim_{\Delta\to 0^+} Var[{}_\Delta v_0] = \lim_{\Delta\to 0^+} [(\frac{\kappa^2\Delta^2}{4} + 1)v_0] $$

We get the above equality is because when $\Delta \to 0^+$, the variance terms and covariance terms containing Δ will go to zero. Take out the coefficient on the right side of (B7), we get

(B8)
$$ \lim_{\Delta\to 0^+} Var[{}_\Delta v_0] = \lim_{\Delta\to 0^+} (\frac{\kappa^2\Delta^2}{4} + 1)^2 Var[v_0] $$

Since $Var[v_0] = \sigma_0^2$, and $\lim_{\Delta\to 0^+}(\frac{\kappa^2\Delta^2}{4} + 1)^2 = 1$, we get

(B9)
$$ \lim_{\Delta\to 0^+} Var[{}_\Delta v_0] = Var[v_0] = \sigma_0^2 $$

From (B5), we get

(B10) $$\lim_{\Delta\to 0^+} {}_\Delta v_0 = v_0$$

Let F_Δ and F be the cumulative distribution functions of ${}_\Delta v_0$ and v_0 respectively. Since the cumulative functions of non-central chi-square distributions are continuous on positive real line, together with (B6), (B9), (B10), we can get

(B11) $$\lim_{\Delta\to 0^+} F_\Delta(y) = \lim_{\Delta\to 0^+} P({}_\Delta v_0 \le y) = F(y)$$

From (B3), we have

$$\frac{1}{\Delta}E[{}_\Delta v_t - {}_\Delta v_{t-\Delta}|{}_\Delta v_{t-\Delta} = y] = \kappa(\theta - {}_\Delta v_{t-\Delta}) + \frac{\kappa^2\Delta}{4}{}_\Delta v_{t-\Delta}$$

$$\frac{1}{\Delta}Var[{}_\Delta v_t - {}_\Delta v_{t-\Delta}|{}_\Delta v_{t-\Delta} = y] = \sigma^2{}_\Delta v_{t-\Delta} + (\frac{\sigma^4}{8} - \sigma^2\kappa{}_\Delta v_{t-\Delta} + \frac{\sigma^2\kappa^2}{4}{}_\Delta v_{t-\Delta}\Delta)\Delta$$
$$- \frac{\kappa\sigma^4\Delta^4}{16} + \frac{5\kappa^2\sigma^4\Delta^5}{64} - \frac{\kappa^3\sigma^4\Delta^6}{32} + \frac{\kappa^4\sigma^4\Delta^7}{256}$$

So, if we let $\Delta \to 0^+$, the conditional expectation and variance above become

(B12) $$\lim_{\Delta\to 0^+} \frac{1}{\Delta}E[{}_\Delta v_t - {}_\Delta v_{t-\Delta}|{}_\Delta v_{t-\Delta} = y] = \kappa(\theta - {}_\Delta v_{t-\Delta})$$

(B13) $$\lim_{\Delta\to 0^+} \frac{1}{\Delta}Var[{}_\Delta v_t - {}_\Delta v_{t-\Delta}|{}_\Delta v_{t-\Delta} = y] = \sigma^2{}_\Delta v_{t-\Delta}$$

For $\delta = 1$, from (B3) we have

$$|{}_\Delta v_t - {}_\Delta v_{t-\Delta}|^{2+\delta} = |(\sqrt{\Delta})^3 g(\Delta)^3|$$

where

$$g(\Delta) = \kappa(\theta - {}_\Delta v_{(j-1)\Delta})\sqrt{\Delta} + \frac{\kappa^2\Delta^{\frac{3}{2}}}{4}{}_\Delta v_{(j-1)\Delta} + (\frac{\kappa\sigma\Delta}{2} - \sigma)z_{t-\Delta}\sqrt{({}_\Delta v_{(j-1)\Delta})}$$
$$+ (\frac{\kappa^2\sigma^2\Delta^3}{16} + \frac{\sigma^2\Delta}{4} - \frac{\kappa\sigma^2\Delta^2}{4})(z_{t-\Delta}^2 - 1)$$

Then

$$\lim_{\Delta\to 0^+}\frac{1}{\Delta}E[|_\Delta v_t - {}_\Delta v_{t-\Delta}|^{2+\delta}|_\Delta v_{t-\Delta}=y] = \lim_{\Delta\to 0^+}\frac{1}{\Delta}E[|(\sqrt{\Delta})^3 g(\Delta)^3|]$$

$$= \lim_{\Delta\to 0^+}\sqrt{\Delta}E[|g(\Delta)^3|]$$

Since $g(\Delta)$ only involves one random variable $z_{t-\Delta}$, whose any moment is finite, and other quantities are finite, we can conclude that $E[|g(\Delta)^3|$ is finite. Then together with the above equality, we can get

$$\text{(B14)} \qquad \lim_{\Delta\to 0^+}\frac{1}{\Delta}E[|_\Delta v_t - {}_\Delta v_{t-\Delta}|^{2+\delta}|_\Delta v_{t-\Delta}=y] = 0$$

Based on (B11), (B12), (B13), and (B14), by Theorem 2.1 in Foster and Nelson (1994) [16], this proves that (31) converges to (33) weakly. This completes Theorem 4.1.

APPENDIX C

CONVERGENCE TO CONTINUOUS TIME LIMIT UNDER RISK-NEUTRAL MEASURE

Convergence to Continuous-time Limit Under Risk-neutral Measure $\mathbb{Q}$

Proof. Since (34) is the Euler discretization of (36), by Yan (2002), we can conclude that (34) converges to (36) weakly.

Rewrite the second equation of our model under risk-neutral measure as follows

$$\text{(C1)} \qquad h(t) = a - b^2 - ch(t-\Delta) + (bz^*(t-\Delta) - [b(\lambda + \frac{1}{2}) + 1]\sqrt{h(t-\Delta)})^2$$

As in **Appendix B**, , rewrite $h(t)$ as ${}_\Delta h_{t-\Delta}$ with starting point ${}_\Delta h_0$.

Divide Δ to both sides of the above equation and let ${}_\Delta v_t = \frac{{}_\Delta h_t}{\Delta}$ for $t \geq 0$ with starting point ${}_\Delta v_0 = \frac{{}_\Delta h_0}{\Delta}$. From (C1), we have

$$\text{(C2)} \qquad v_t = \frac{a - b^2}{\Delta} - cv_{t-\Delta} + (\frac{b}{\sqrt{\Delta}} z^*_{t-\Delta} - [b(\lambda + \frac{1}{2}) + 1]\sqrt{v_{t-\Delta}})^2$$

Discretize the process ${}_\Delta v_t$ as follows:

$${}_\Delta v_t = {}_\Delta v_{j\Delta}$$

for $j\Delta \leq t < (j+1)\Delta$, where j is a natural number. Let $a = \kappa\theta\Delta^2$, $b = \frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4}$, $c = \kappa\Delta - \frac{\kappa^2\Delta^2}{4}$, and plug them into (C2) to get

$$\text{(C3)} \quad {}_\Delta v_{j\Delta} = \frac{1}{\Delta}[\kappa\theta\Delta^2 - (\frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4})^2] - (\kappa\Delta - \frac{\kappa^2\Delta^2}{4})_\Delta v_{(j-1)\Delta} + \{\frac{1}{\sqrt{\Delta}}(\frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4})z^*_{t-\Delta}$$

$$- [(\frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4})(\lambda + \frac{1}{2}) + 1]\sqrt{{}_\Delta v_{(j-1)\Delta}}\}^2$$

with starting point ${}_\Delta v_0$. Let's choose a starting point v_0 for the continuous volatility process, which satisfies the following equality

$$\text{(C4)} \qquad {}_\Delta v_0 = \frac{a - b^2}{\Delta} - cv_0 + (\frac{b}{\sqrt{\Delta}} z^*_{t-\Delta} - [b(\lambda + \frac{1}{2}) + 1]\sqrt{v_0})^2$$

We assume the starting point v_0 follows a non-central chi-square distribution with mean μ_0 and variance σ_0^2. Also assume the non-central chi-square distribution that v_0 follows is generated by a normal variable that is independent of the normal variable that generates ${}_\Delta v_0$. we can see that the left-hand side of (C4) is a linear combination of shifted and scaled random variables of non-central chi-square distributions. Thus, we can conclude that ${}_\Delta v_0$, on the right-hand side of (C4), also follows a non-central chi-square distribution.

Do the substitution $a = \kappa\theta\Delta^2$, $b = \frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4}$, $c = \kappa\Delta - \frac{\kappa^2\Delta^2}{4}$ in (C4) , we can see that ${}_\Delta v_0$ and v_0 also satisfy (C3). That is

$$\text{(C5)} \qquad {}_\Delta v_0 = \frac{1}{\Delta}[\kappa\theta\Delta^2 - (\frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4})^2] - (\kappa\Delta - \frac{\kappa^2\Delta^2}{4})v_0 + \{\frac{1}{\sqrt{\Delta}}(\frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4})z^*_{t-\Delta}$$
$$- [(\frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4})(\lambda + \frac{1}{2}) + 1]\sqrt{v_0}\}^2$$

Take expectation for both sides of (C5) and let $\Delta \to 0^+$, we get

$$\text{(C6)} \qquad \lim_{\Delta\to 0^+} E[{}_\Delta v_0] = E[v_0] = \mu_0$$

Take variance for both sides of (C3) and let $\Delta \to 0^+$, we get

$$\text{(C7)} \qquad \lim_{\Delta\to 0^+} Var[{}_\Delta v_0] = \lim_{\Delta\to 0^+} Var\{[[(\frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4})(\lambda + \frac{1}{2}) + 1]\sqrt{v_0}]^2\} = Var[v_0] = \sigma_0^2$$

We get the above equality is because when $\Delta \to 0^+$, the variance terms and covariance terms containing Δ will go to zero.

From (C5), we get

$$\text{(C8)} \qquad \lim_{\Delta\to 0^+} {}_\Delta v_0 = v_0$$

Let F_Δ and F be the cumulative distribution functions of ${}_\Delta v_0$ and v_0 respectively. Since the cumulative functions of non-central chi-square distributions are continuous on positive real line, together with (C6), (C7), (C8), we can get

$$\text{(C9)} \qquad \lim_{\Delta\to 0^+} F_\Delta(y) = \lim_{\Delta\to 0^+} P({}_\Delta v_0 \le y) = F(y)$$

From (C3), we get

$$\text{(C10)} \qquad \lim_{\Delta\to 0^+} \frac{1}{\Delta} E[{}_\Delta v_t - {}_\Delta v_{t-\Delta} | {}_\Delta v_{t-\Delta} = y] = \kappa(\theta - {}_\Delta v_{t-\Delta}) + \sigma(\lambda + \frac{1}{2}){}_\Delta v_{t-\Delta}$$

$$\text{(C11)} \qquad \lim_{\Delta\to 0^+} \frac{1}{\Delta} Var[{}_\Delta v_t - {}_\Delta v_{t-\Delta} | {}_\Delta v_{t-\Delta} = y] = \sigma^2 {}_\Delta v_{t-\Delta}$$

For $\delta = 1$, from (C3) we have

$$|_{\Delta}v_t - {}_{\Delta}v_{t-\Delta}|^{2+\delta} = |(\sqrt{\Delta})^3 g(\Delta)^3|$$

where

$$\begin{aligned}
g(\Delta) =&[\kappa\theta\sqrt{\Delta} - (\frac{\sigma\Delta^{\frac{1}{4}}}{2} - \frac{\kappa\sigma\Delta^{\frac{5}{4}}}{4})^2] + [(\frac{\sigma\Delta^{\frac{1}{4}}}{2} - \frac{\kappa\sigma\Delta^{\frac{5}{4}}}{4})z^*_{t-\Delta}]^2 \\
&- 2[(\frac{\sigma\Delta^{\frac{1}{4}}}{2} - \frac{\kappa\sigma\Delta^{\frac{5}{4}}}{4})z^*_{t-\Delta}][(\frac{\sigma\Delta}{2} - \frac{\kappa\sigma\Delta^2}{4})(\lambda + \frac{1}{2}) + 1]\sqrt{{}_{\Delta}v_{(j-1)\Delta}} \\
&+ 2(\frac{\sigma\Delta^{\frac{1}{2}}}{2} - \frac{\kappa\sigma\Delta^{\frac{3}{2}}}{4})(\lambda + \frac{1}{2})\sqrt{{}_{\Delta}v_{(j-1)\Delta}} \\
&+ (\frac{\sigma\Delta^{\frac{3}{4}}}{2} - \frac{\kappa\sigma\Delta^{\frac{7}{4}}}{4})(\lambda + \frac{1}{2})\sqrt{{}_{\Delta}v_{(j-1)\Delta}}
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{\Delta\to 0^+} \frac{1}{\Delta} E[|_{\Delta}v_t - {}_{\Delta}v_{t-\Delta}|^{2+\delta} | {}_{\Delta}v_{t-\Delta} = y] &= \lim_{\Delta\to 0^+} \frac{1}{\Delta} E[|(\sqrt{\Delta})^3 g(\Delta)^3|] \\
&= \lim_{\Delta\to 0^+} \sqrt{\Delta} E[|g(\Delta)^3|]
\end{aligned}$$

Since $g(\Delta)$ only involves one random variable $z_{t-\Delta}$, whose any moment is finite, and other quantities are finite, we can conclude that $E[|g(\Delta)^3|$ is finite. Then together with the above equality, we can get

(C12) $$\lim_{\Delta\to 0^+} \frac{1}{\Delta} E[|_{\Delta}v_t - {}_{\Delta}v_{t-\Delta}|^{2+\delta} | {}_{\Delta}v_{t-\Delta} = y] = 0$$

Based on (C9), (C10), (C11), and (C12), by Theorem 2.1 in Foster and Nelson (1994) [16], this proves that (35) converges to (37) weakly. This completes Theorem 4.3.

APPENDIX D

PROOF OF PROPOSITION 2

Proof of Proposition 2 (Heston and Nandi (2000)).

Proof. Let $f(\phi)$ denote the conditional moment generating function of the conditional probability density, $p(x(T)|S(t),h(t))$, where $x(T)$ is the logarithm of the terminal asset price ($x(T) = \log S(T)$). Let $g(x(T)) = \frac{1}{f(1)}\exp(x(T))p(x(T)|S(t),h(t))$. It is easy to see that it is a valid probability density because it is non-negative and $f(1) = E_t[\exp(x(T))]$ from equation (38). Notice that

$$\begin{aligned} g(x(T)) =& \frac{1}{f(1)}\exp(x(T))p(x(T)|S(t),h(t)) \\ =& \frac{1}{E_t[\exp(x(T))]}\exp(x(T))p(x(T)|S(t),h(t)) \\ =& \frac{1}{\int_{-\infty}^{\infty}\exp(x(T))p(x(T)|S(t),h(t))dx(T)}\exp(x(T))p(x(T)|S(t),h(t)) \end{aligned}$$

Notice that, the above expression shows that all the computations about $g(x(T))$ are , in fact, conditional on $S(t)$ and $h(t)$. And this conditioning is realized through $p(x(T)|S(t),h(t))$. And the moment generating function for $g(x(T))$ is

$$\begin{aligned} \int_{-\infty}^{\infty}\exp(\phi x(T))g(x(T))dx(T) &= \frac{1}{f(1)}\int_{-\infty}^{\infty}\exp((\phi+1)x(T))p(x(T)|S(t),h(t))dx(T) \\ &= \frac{f(\phi+1)}{f(1)} \end{aligned}$$

Since the terminal asset price is $\exp(x(T))$, the conditional expectation of a call option payoff separates into two terms with probability integrals.

$$\begin{aligned} E[Max(e^{x(T)}-K,0)|S(t),h(t)] =& \int_{\log(K)}^{\infty}\exp(x(T))p(x(T)|S(t),h(t))dx(T) \\ & - K\int_{\log(K)}^{\infty}p(x(T)|S(t),h(t))dx(T) \\ \text{(D1)} \qquad =& f(1)\int_{\log(K)}^{\infty}g(x(T))dx(T) - K\int_{\log(K)}^{\infty}p(x(T)|S(t),h(t))dx(T) \end{aligned}$$

Since $\frac{f(\phi+1)}{f(1)}$ is the moment generating function corresponding to $g(x(T))$, due to the relationship between characteristic function and moment generating function, $\frac{f(i\phi+1)}{f(1)}$ is the characteristic function corresponding to $g(x(T))$. Feller (1971) [14] and Kendall and Stuart (1977) [25] show how to recover the "probabilities" from the characteristic function below

$$\int_{\log(K)}^{\infty} g(x(T))dx(T) = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} Re[\frac{e^{-i\phi\log(K)}f(i\phi+1)}{i\phi f(1)}]d\phi \tag{D2}$$

And $f(\phi)$ is the moment generating function corresponding to $p(x(T)|S(t),h(t))$, then $f(i\phi)$ is the characteristic function corresponding to $p(x(T)|S(t),h(t))$. We can recover the "probabilities" from the characteristic function below

$$\int_{\log(K)}^{\infty} p(x(T)|S(t),h(t))dx(T) = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} Re[\frac{e^{-i\phi\log(K)}f(i\phi)}{i\phi}]d\phi \tag{D3}$$

Substituting equation (D2) and equation (D3) into (D1) proves the proposition.

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