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**PHOTON SPECTRUM FROM BREMSSTRAHLUNG**

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**UNCLASSIFIED****PHOTON SPECTRUM FROM BREMSSTRAHLUNG****Abstract**

The number of photons of given energy radiated in Bremsstrahlung collision of electrons with nuclei is calculated relativistically. Expansions are obtained for energies small compared to  $mc^2$ . The photon spectrum is averaged over a Maxwell distribution for the electrons. Numerical results are included.

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### I. Introduction

An electron gas in the presence of nuclei can be expected to produce photons by means of the Bremsstrahlung process. The photons thus produced absorb energy from the electrons by various methods. One of these methods is the inverse Compton effect, which has been treated in LA-301a and Yale(LA)2. The present work concerns itself with the spectrum of the Bremsstrahlung photons.

In Section II the photon spectrum is calculated for specific incident electron energies. A series expansion is given and is used in conjunction with the Bethe-Heitler formula<sup>1</sup>. The spectrum is averaged over a Maxwell distribution of electron energies in Section III.

Numerical calculations are described in Section IV, and tables of numerical results are included. In particular, the photon spectrum is given in Table II for electrons of specified energy and in Table VI averaged over a Maxwell distribution of electron energies.

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<sup>1</sup> c.f. W. Heitler, Quantum Theory of Radiation, Oxford University Press, 2nd edition, p. 165. Because of the relative complexity of the reduction of the traces and integrations over angles leading to Eq. (16) p. 165, the work was repeated independently and a check was obtained.

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### Symbols and Notation

- $E_0$  Total energy of the incident electron<sup>2</sup>.
- $k$  Energy of the emitted photon.
- $E$  Total energy of the scattered electron.
- $p_0, p$  Momenta of the incident and scattered electrons, respectively.
- $\bar{\sigma} = \frac{Z^2 (e^2/mc^2)^2 (e^2/hc)}$ .
- $Z$  Nuclear charge.
- $d\sigma$  Cross section for the emission of a photon in the energy interval  $dk$ .
- $p_0, p$  Non-relativistic momenta =  $[2(E_0 - 1)]^{1/2}$ ,  $[2(E - 1)]^{1/2}$ , respectively.
- $s = p / p_0$ .
- $\epsilon_0$  Kinetic energy of the incident electron =  $E_0 - 1$ .
- $T$  Electron temperature expressed in units of  $mc^2$ .
- $s(T, \epsilon_0) = p^2 \exp(-\epsilon_0/T) (dp/dE)$  = electron energy distribution function giving relative number of electrons per unit energy range; this quantity yields the number of electrons per  $cm^3$  with kinetic energy between  $\epsilon_0$  and  $\epsilon_0 + d\epsilon_0$  after inclusion of factors available in Eq. (6) and is expressed in terms of  $\epsilon_0$  in Eq. (6.1).

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<sup>2</sup> All energies are expressed in units of  $mc^2$ . In addition, the constants  $\hbar$  and  $c$  will be set equal to unity except where their specific use clarifies the expression; e.g., in  $(e^2/mc^2)$ ,  $(e^2/hc)$ , etc.

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- $n_e$  Number of electrons per  $\text{cm}^3$ .
- $N(T)$  Normalization factor for electron energy distribution.
- $K_n(x)$  Modified Bessel function of the second kind as defined by Whittaker and Watson.
- $n_t$  Number of target nuclei per  $\text{cm}^3$ .
- $dN_\gamma$  Number of photons emitted per  $\text{cm}^3$  in the energy interval  $dk$  averaged over the electron energy distribution.
- $\mathcal{N}(T,k)$  Number of photons emitted per  $\text{cm}^3$  in the energy interval  $dk$ , apart from constants, as defined in Eq. (9.1).
- $R(T,k)$  Ratio of  $\mathcal{N}(T,k)$  to its non-relativistic limit.
- NR The subscript NR denotes the non-relativistic limit of the quantity with which it is used.

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## II. Photon Spectrum for Electrons of Given Energy.

The cross section for the emission of a photon of energy  $k^2$  in a collision of an electron of total initial energy  $E_0$  with a nucleus at rest is given by the Bethe-Heitler formula<sup>1</sup> as

$$d\sigma = \bar{\sigma} (dk/k) (p/p_0) (X + LY), \quad (1)$$

where

$$\begin{aligned} X &= (4/3) - [2 E E_0 (p^2 + p_0^2)/p^2 p_0^2] + (\ell_0 E/p_0^3) + (\ell E_0/p^3) - (\ell \ell/p p_0), \\ Y &= (8 E E_0/3 p p_0) + [k^2 (E_0^2 E^2 + p_0^2 \ell^2)/p_0^3 p^3] + (k/E p^4 p_0^4) [\ell_0 p^5 (E E_0 + \\ &+ p_0^2) - \ell p_0^5 (E E_0 + p^2) + 2 k E_0 E p_0 p], \end{aligned}$$

$$L = \ln \left( \frac{E E_0 - 1 + p p_0}{E E_0 - 1 - p p_0} \right), \quad \ell_0 = \ln \left( \frac{E_0 + p_0}{E_0 - p_0} \right), \quad \ell = \ln \left( \frac{E + p}{E - p} \right), \quad (1.1)$$

$$\bar{\sigma} = Z^2 (e^2/mc^2)^2 (e^2/4\pi c).$$

The calculations in this report are subject to the same limitations as those underlying the original derivation of the Bethe-Heitler formula. Since it is highly improbable that refinements which have become of interest to theoretical physicists since the Bethe-Heitler paper can be of importance in the

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practical application they are omitted. It may do no harm however to mention some of them.

The distortion of the electron wave function is taken into account only in the first Born approximation. This omission cannot be of importance for the small values of the nuclear charge which enter the present problem. The effect of higher order corrections has been recently worked on by Maximon and Bethe<sup>3</sup>. Their results clearly indicate that the effects under discussion are important only for heavier elements. A change in their results has been recently recorded by Bethe, Maximon and Low<sup>4</sup>. This change has been noted in a related (unclassified) connection at this laboratory without publication before the present report was completed. It does not modify any of the present results.

Radiative corrections also affect the results in principle but they are very small in the present case.

The expression in Eq. (1.1) is the result of an integration of the differential cross section over the angles of the photon and of the final electron momentum. Conservation of energy requires that

<sup>3</sup> L. Maximon and H. A. Bethe, Phys. Rev. 87, 156 (1952).

<sup>4</sup> Bethe, Maximon and Low, Phys. Rev. 91, 417 (1953).

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$$E_0^2 = E + k, \quad (1.2)$$

where  $E$  is the total energy of the scattered electron. The momenta  $p_0$  and  $p$  are related to the total energies by the relations

$$E_0^2 = p_0^2 + 1, \quad E = p^2 + 1. \quad (1.3)$$

For  $p=0$ ,  $E_0 - 1 = k$  and the quantity  $X+LY$  reduces to

$$(X+LY)_{p=0} = (16/3k) \left\{ 1 + (1/8) k + (3/8) k^2 - (3/8)(1-k) \left[ 1 - (\sinh^{-1} \sqrt{k(2+k)}) / \sqrt{k(2+k)} \right] \right\}. \quad (2)$$

The quantity  $X+LY$  in Eq. (1.1) may be expanded in powers of the non-relativistic momenta  $P_0$ ,  $P$  and yields

$$\begin{aligned} X+LY &\approx \frac{1}{PP_0} \left[ \frac{16}{3} + 2(P^2 + P_0^2) + \frac{39P^4 - 38P^2P_0^2 + 39P_0^4}{120} \right. \\ &+ \left. \frac{309P^6 - 589P^4P_0^2 - 589P^2P_0^4 + 309P_0^6}{6720} \right] \ln \left( \frac{P_0 + P}{P_0 - P} \right) - \frac{49(P^2 + P_0^2)}{60} \\ &- \frac{629P^4 - 518P^2P_0^2 + 629P_0^4}{3360}, \end{aligned} \quad (3)$$

where  $P_0^2 = 2(E_0 - 1)$  and  $P^2 = 2(E - 1)$ .

When the factor  $(p/p_0)$  appearing in Eq. (1) is included, the expansion becomes

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$$(p/p_0)(x+LY) \cong \frac{1}{p_0^2} \left\{ \left[ \frac{16}{3} + \frac{4p_0^2 + 8p^2}{3} + \frac{3p_0^4 - 8p_0^2 p + 8p^4}{15} + \right. \right.$$

$$\left. \frac{32p^6 - 32p^4 p_0^2 + 10p^2 p_0^4 + 11p_0^6}{420} \right] \ln \left( \frac{p_0 + p}{p_0 - p} \right) + \frac{p}{p_0} \left[ -4 + \frac{70p^2 + 19p^4}{60} - \right.$$

$$\left. - \frac{181p^4 - 728p^2 p_0^2 + 1287p_0^4}{3360} \right]. \quad (4)$$

The non-relativistic limiting form is obtained directly from Eqs.(1) and (3) as the lowest order term in  $p_0^2$ , giving

$$\frac{d\sigma}{dk} = \bar{\sigma} (dk/k) (16/3 p_0^2) \ln \left[ (p_0 + p) / (p_0 - p) \right]. \quad (5)$$

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### III. Photon Spectrum for a Maxwell Distribution of Electron Energies.

The incident electrons are assumed to have a Maxwell-Boltzmann distribution in energy given by

$$dN_e = n_e N(T) s(T, \varepsilon_0) d\varepsilon_0, \quad (6)$$

where

$$s(T, \varepsilon_0) = (1 + \varepsilon_0)^{-1} (2\varepsilon_0 + \varepsilon_0^2)^{1/2} \exp(-\varepsilon_0/T) \quad (6.1)$$

and T is the temperature of the equilibrium distribution of electrons in units of  $mc^2$ ,  $\varepsilon_0$  is the kinetic energy of the electrons given by  $\varepsilon_0 = E_0 - 1$ ,  $n_e$  is the number of electrons per  $\text{cm}^3$  and  $N(T)$  is the normalization factor adjusted so that  $\int dN_e = n_e$ . This leads to the equation

$$\frac{1}{N(T)} = \int_0^\infty (1 + \varepsilon_0)^{-1} (2\varepsilon_0 + \varepsilon_0^2)^{1/2} \exp(-\varepsilon_0/T) d\varepsilon_0. \quad (7)$$

If one expresses the integrand in Eq. (7) in terms of  $E_0 = \varepsilon_0 + 1$  and integrates by parts, one obtains

$$\frac{1}{N(T)} = \frac{\exp(1/T)}{ST} \int_1^\infty (E_0^2 - 1)^{3/2} \exp(-E_0/T) dE_0. \quad (7.1)$$

The normalization factor may thus be expressed <sup>3</sup> in terms of

<sup>3</sup> Use has been made of the integral representation of the Bessel function  $K_n(z)$  given in Eq. (4), p. 172 of Watson's Bessel Functions. The notation, however, is that of Whittaker and Watson, Modern Analysis.

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the Bessel function of imaginary argument of the second kind as

$$H(T) = [e^{-1/T} / \pi K_2(1/T)] \approx (2/\pi)^{1/2} (T)^{-3/2} [1 + (15/8)T + (105/128)T^2 - (315/1024)T^3 + \dots]^{-1}. \quad (7.2)$$

The series in Eq. (7.2) results from the use of the asymptotic expansion of  $K_2(1/T)$  for small  $T$ .

If  $n_t$  is the number of target nuclei per  $\text{cm}^3$ , the number of photons emitted per  $\text{cm}^3$  per sec in the energy interval between  $k$  and  $k+dk$  is

$$\begin{aligned} dN_\gamma &= n_t \int_k^\infty v_0 (d\sigma) (dN_e) \\ &= n_t \left\{ \int_k^\infty \frac{d\sigma}{dk} \frac{dN_e}{d\varepsilon_e} v_0 d\varepsilon_e \right\} dk, \end{aligned} \quad (8)$$

where  $v_0$  is the velocity of the incident electron given by  $p_0/E_0$ . The integral is carried out for constant photon energy  $k$  over the electron energy  $\varepsilon_e$  between the limits  $k$  and  $\infty$ , corresponding to the range within which conservation of energy allows photon emission to take place. Substitution of Eqs. (1), (6) and (6.1) into Eq. (8) yields

$$dN_\gamma \approx n_t \eta_2(T, k) dk, \quad (9)$$

where

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$$\eta(T, k) = N(T) k^{-1} \int_k^{\infty} (2\epsilon_0 + \epsilon_0^2)^{1/2} \exp(-\epsilon_0/T) p(X+LY)d\epsilon_0, \quad (9.1)$$

with  $p^2 = (\epsilon_0 - k)(2 + \epsilon_0 - k)$  and  $X, L$  and  $Y$  as in Eq. (1.1).

The non-relativistic photon spectrum is obtained by combining Eqs. (5), (6) and (8), giving

$$(dN_p)_{NR} = n_e n_t \bar{\eta}_{NR}(T, k) dk \quad (10)$$

where

$$\eta_{NR}(T, k) = (2/\pi)^{1/2} (T)^{-3/2} (16/3k) \int_k^{\infty} \exp(-\epsilon_0/T) \ln[(P_0 + P)/(P_0 - P)] d\epsilon_0. \quad (10.1)$$

The integral in Eq. (10.1) can be put in simpler form if one changes the independent variable from  $\epsilon_0$  to

$$x = (2\epsilon_0 - k)/k$$

and integrates by parts. The result of this change is

$$\int_k^{\infty} \exp(-\epsilon_0/T) \ln[(P_0 + P)/(P_0 - P)] d\epsilon_0 = T \exp(-k/2T) \int_1^{\infty} \exp(-kx/2T) (x^2 - 1)^{-1/2} dx. \quad (11)$$

The integral on the right side of Eq. (11) is one of the representations of the Bessel function  $K_0(k/2T)$  given by Watson<sup>5</sup>

<sup>5</sup> Watson, Bessel Functions, p. 172, Eq. (4).

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Eq. (10.1) may they be written as

$$\eta_{NR}(T, k) = (2/\pi)^{1/2} T^{-1/2} (16/3k) \exp(-k/2T) K_0(k/2T). \quad (12)$$

The ratio of the photon spectrum calculated relativistically to that calculated non-relativistically is of interest in indicating the importance of relativistic corrections for various temperatures. It may be written as

$$R(T, k) = \eta(T, k) / \eta_{NR}(T, k). \quad (13)$$

This ratio may be written as a series expansion in powers of  $T$  and  $k$ . If one starts with Eq. (4) for  $(p/p_0)(X+LY)$  the integration indicated in Eq. (9.1) may be written as

$$\begin{aligned} k\eta(T, k)/N(T) &= \int_k^\infty d\varepsilon_0 \left(1 + \frac{\varepsilon_0}{2}\right) \exp(-\varepsilon_0/T) \left[ (p/p_0)(X+LY) \right] d\varepsilon_0 \\ &\approx \int_k^\infty d\varepsilon_0 \left(1 + \frac{\varepsilon_0}{2}\right) \exp(-\varepsilon_0/T) \left\{ \left[ \frac{16}{3} + \frac{24\varepsilon_0 - 16k}{5} + \frac{20\varepsilon_0^2 - 40\varepsilon_0 k + 32k^2}{15} \right. \right. \\ &\quad \left. \left. - \frac{70\varepsilon_0^3 - 140\varepsilon_0^2 k - 16\varepsilon_0 k^2 + 64k^3}{105} \right] \ln \left( \frac{\sqrt{\varepsilon_0} + \sqrt{\varepsilon_0 - k}}{\sqrt{\varepsilon_0} - \sqrt{\varepsilon_0 - k}} \right) + \left[ -8 + \frac{98\varepsilon_0 - 19k}{15} \right. \right. \\ &\quad \left. \left. - \frac{740\varepsilon_0^2 + 366\varepsilon_0 k + 181k^2}{420} \right] \frac{\sqrt{\varepsilon_0 - k}}{\sqrt{\varepsilon_0}} \right\} \approx \end{aligned}$$

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$$\begin{aligned} & \approx \int_k^{\infty} d\epsilon_0 \exp(-\epsilon_0/T) \left\{ \left[ \frac{16}{3} + \frac{32\epsilon_0 - 16k}{3} + \frac{80\epsilon_0^2 - 80\epsilon_0 k + 32k^2}{15} + \frac{128\epsilon_0^3 k - 64k^3}{105} \right] \right. \\ & \times \left. \ln \left( \frac{\epsilon_0 + \sqrt{\epsilon_0 - k}}{\sqrt{\epsilon_0} - \sqrt{\epsilon_0 - k}} \right) + \left[ -8 + \frac{38\epsilon_0 - 19k}{15} + \frac{632\epsilon_0^2 - 632\epsilon_0 k - 181k^2}{420} \right] \frac{\sqrt{\epsilon_0 - k}}{\sqrt{\epsilon_0}} \right\}. \end{aligned} \quad (14)$$

One may integrate the terms containing the logarithm by parts, and all the integrals in Eq. (14) may then be written in the form

$$I_n = \int_k^{\infty} \frac{\epsilon_0^n}{\sqrt{\epsilon_0} \sqrt{\epsilon_0 - k}} \exp(-\epsilon_0/T) d\epsilon_0. \quad (15)$$

The substitution  $\epsilon_0 = k(1+x)/2$  allows one to express the integral in Eq. (15), for  $n = 0$ , in terms of the Bessel function of imaginary argument of the second kind as before<sup>3</sup>, giving

$$\int_k^{\infty} \frac{\exp(-\epsilon_0/T)}{\sqrt{\epsilon_0} (\epsilon_0 - k)} d\epsilon_0 = \exp(-k/2T) \int_1^{\infty} \frac{\exp(-kx/2T)}{\sqrt{x^2 - 1}} dx = \exp(-k/2T) K_0(k/2T). \quad (15.1)$$

The integral in Eq. (15) may be evaluated for other values of  $n$  by taking derivatives of Eq. (15.1) with respect to  $(-1/T)$ . One can then express the derivatives of the Bessel functions in terms of  $K_0$  and  $K_1$  by means of the well known recurrence relations. Using the first non-vanishing term to give  $N_{NR}(T, k)$ , one obtains

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for  $R(T,k)$  to third power in either  $T$  or  $k$ ,

$$R(T,k) = \left\{ [1+2T+2T^2] + k^2 [(83/160)+(493/1120)T] - [k_1(k/2T)/k_0(k/2T)] \right. \\ \left. \times k [(1/4)+(79/40)T+(237/280)T^2+(331/4480)k^2] \right\} / [1+(15/8)T+(105/128)T^2 - \\ - (315/1024)T^3], \quad (16)$$

where the denominator comes from the normalization factor in Eq. (7).

For  $k=0$ , the ratio may be obtained for any  $T$  by substitution of the appropriate limit for  $(p/p_0)(X+LY)$  from Eq. (1) into Eq. (9.1). Since  $L \rightarrow \infty$  as  $k \rightarrow 0$ , the leading term in the ratio comes only from the LY terms and may be written as

$$\lim_{k \rightarrow 0} [k R(T,k)/N(T)] = \lim_{k \rightarrow 0} \int_0^{\infty} d\varepsilon_0 (1+\varepsilon_0)^2 \exp(-\varepsilon_0/T) (16/3) \ln [(2\varepsilon_0 + \varepsilon_0^2)/mk].$$

The leading term is diverging and contains the factor  $\ln(1/k)$ . For this term the lower limit of the integral may be set equal to zero. One then obtains for the ratio  $R(T,0)$  only the ratio of the coefficients of the  $\ln(1/k)$  terms in the expansions of  $N(T,k)$  and  $N_{NR}(T,k)$  near  $k=0$  giving

$$R(T,0) = (1+2T+2T^2) (\pi T/2)^{1/2} \exp(-1/T) / I_0(1/T), \quad (16.1)$$

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in agreement with setting  $k=0$  in Eq. (16).

For  $T=0$ ,  $R(T,k)$  becomes

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$$\mathcal{R}(0,k) \leq 1 + (1/4)k + (83/160)k^2 + (331/4480)k^3. \quad (16.2)$$

The exact value of  $\mathcal{R}(0,k)$  for any  $k$  may be obtained by taking the limit as  $T \rightarrow 0$  in Eq. (9.1). In this limit, contributions to the integral come only from  $\varepsilon_0 = k$  ( $p=0$ ). Using Eq. (2), one then obtains

$$\mathcal{R}(0,k) = (1+k)^{1/2} \left\{ 1 + (1/8)k + (3/8)k^2 - (3/8)(1-k)[1 - \frac{\sinh^{-1}\sqrt{k(2+k)}}{\sqrt{k(2+k)}}] \right\}. \quad (16.3)$$

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IV. Numerical Calculations.

The quantity  $X+LY$  in Eq. (1.1) was calculated for values of  $k$  between  $k = .05$  and  $k = .5$  in intervals of .05, and numerical results are given in Table I. For each  $k$  the values of  $E_0 - 1$  used were  $E_0 - 1 = k$ ,  $k + .005$ ,  $.05$ ,  $.065$ ,  $.10$ ,  $.15$ ,  $.20$ ,  $.25$ ,  $.30$ ,  $.60$ ,  $.90$ . For  $E_0 - 1 = k$ , the final electron momentum is zero and Eq. (2) was used. For  $(E_0 - 1) < k$ ,  $X+LY$  does not exist as a physical quantity. The exact expression for  $X+LY$  in Eq. (1.1) and the expansion in Eq. (3) were used in those ranges of the parameters where each was suitable for numerical work. The approximation in Eq. (3) was such that, for values of  $E_0 - 1$  up to .3, Eqs. (1.1) and (3) differed by less than 0.5%.

The tabulated values given in Table I were plotted against  $E_0 - 1 = \xi_0$  for each value of  $k$ . In order to carry out the integral over the energy distribution indicated in Eq. (9.1), the values of  $X+LY$  given in Table II were read from the graphs for each value of  $k$  and for values of  $\xi_0$  between  $\xi_0 = .045$  and  $\xi_0 = .915$  in steps of .050. These values of  $\xi_0$  were chosen in order to permit using the statistical factor,  $S(T, \xi_0)$ , given in Eq. (6.1), which was calculated for the inverse Compton effect in Yale (IA). The values of  $S(T, \xi_0)$  used are given in Table III for values of  $T$  between .0250 and .1500 in intervals of .0125. The integration in Eq. (9.1) was carried out numerically, using Simpson's rule up to  $\xi_0 = .885$ .

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Corrections for the tail of the energy distribution were made by assuming a linear variation for  $p(X+LY)$  beyond  $\xi_0 = .885$ . The correction was most important for large  $T$  and ranged from 2% to 10% when  $k$  went from .03 to .30. The estimated accuracy of the final result for  $\mathcal{N}(T,k)$  obtained from numerical integration is 0.5% for  $T > .05$ . For  $T = .0500, .0375, .0250$ , the relatively large intervals used for  $\xi_0$  resulted in an estimated uncertainty of as much as 5%. Using the values of  $\mathcal{N}_{\text{RR}}(T,k)$  in Table IV, which were calculated by means of Eq. (12), one obtains the ratio  $R(T,k)$  in Eq.(15) with the same limits of accuracy discussed for  $\mathcal{N}(T,k)$ .

Calculations were also performed using the expansion of the ratio  $R(T,k)$  given in Eq. (16). From the convergence of the expansion, these results are estimated to be uncertain to less than 0.2% over the entire range of parameters used above. Because of this, values of  $\mathcal{N}(T,k)$  were obtained from these values of the ratio  $R(T,k)$  together with the values of  $\mathcal{N}_{\text{RR}}(T,k)$  in Table IV.

Tables V and VI contain the values of  $R(T,k)$  and  $\mathcal{N}(T,k)$ , respectively, obtained in this way. In addition, values of the ratio were calculated for  $k = .006$  and in the limits  $T=0$  from Eqs. (16.2), (16.3) and  $k=0$  from Eq. (16.1). The value at  $k = .006$  was included because of the relatively rapid variation of  $R(T,k)$  between  $k = .03$  and  $k = 0$ .

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If an electron energy distribution other than Maxwellian is desired, the values of  $X + LY$  in Table II should be useful.

Fig. 1 contains plots of the values of  $\bar{R}(T,k)$  in Table V vs. T for various values of k shown on the graph. Fig. 2 contains plots of the values of  $\bar{\eta}(T,k)$  in Table VI vs. T for similar values of k.

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Table I

X+LY calculated from Eqs. (1.1) and (3).

$E_0 - L$	k	.03	.06	.09	.12	.15	.18	.21	.24	.27	.30
.050		177.9									
.055		160.1									
.060		128.6									
.065			89.06								
.070			109.6	84.91							
.075					59.52						
.080					57.91						
.085						63.40	64.66	56.08			
.090						44.79					
.095						45.86					
.100						65.72	58.48	45.03	39.89	35.99	
.105						56.19	45.26	39.06	34.73	31.43	28.76
.110										26.00	
.115										25.77	
.120											22.91
.125											22.75
.130											22.58
.135											20.58
.140											20.42
.145											19.87
.150											18.65
.155											18.56
.160											18.55
.165											18.75
.170											
.175											
.180											
.185											
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.470											
.475											
.480											
.485											
.490											
.495											
.500											

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Table II

X+LY obtained from graphs of the values in Table I.

$\frac{S_{\text{MB}} - 1}{S_{\text{MB}}}$	k	.05	.06	.09	.12	.15	.18	.21	.24	.27	.30
SECUR.	.045	156.8									
NO.	.075	100.5	77.4								
235	.105	60.9	65.0	54.6							
022	.135	69.8	65.4	47.6	42.1						
	.165	62.3	49.9	42.9	38.5	34.4					
	.195	56.9	45.9	39.5	35.3	31.7	29.1				
	.225	52.8	48.8	56.9	53.0	29.9	27.5	25.3			
	.255	49.8	40.4	55.0	31.3	28.4	26.1	24.1	22.4		
	.285	47.3	38.5	55.5	29.6	27.2	25.0	23.2	21.6	20.2	
	.315	45.3	37.0	32.1	28.9	26.5	24.2	22.4	20.9	19.6	18.4
	.345	43.6	35.7	31.1	28.0	25.5	23.5	21.9	20.4	19.1	18.0
	.375	42.1	34.7	30.2	27.2	24.8	22.9	21.4	19.9	18.7	17.7
	.405	40.8	33.7	29.4	26.6	24.5	22.4	20.9	19.6	18.4	17.4
	.435	39.8	32.9	28.6	26.1	23.9	22.1	20.7	19.5	18.1	17.1
	.465	38.9	32.3	28.4	25.6	23.5	21.8	20.5	19.0	17.9	16.9
	.495	38.2	31.7	28.0	25.5	23.2	21.5	20.1	18.8	17.8	16.8
	.525	37.5	31.2	27.6	25.0	22.9	21.3	19.9	18.7	17.6	16.7
	.555	37.0	30.8	27.3	24.7	22.7	21.1	19.8	18.6	17.5	16.6
	.585	36.5	30.5	27.0	24.5	22.5	21.0	19.7	18.5	17.5	16.5
	.615	36.1	30.1	26.8	24.5	22.4	20.9	19.6	18.4	17.4	16.5
	.645	35.7	29.9	26.6	24.1	22.5	20.8	19.5	18.4	17.4	16.5
	.675	35.3	29.7	26.4	24.0	22.2	20.7	19.4	18.3	17.4	16.5
	.705	34.9	29.4	26.2	23.9	22.1	20.6	19.4	18.3	17.4	16.5
	.735	34.7	29.2	26.0	23.8	22.0	20.6	19.3	18.3	17.4	16.5
	.765	34.4	29.1	25.9	23.7	22.0	20.6	19.3	18.3	17.4	16.6
	.795	34.2	28.9	25.8	23.7	22.0	20.5	19.3	18.3	17.4	16.6
	.825	34.0	28.8	25.8	23.6	21.9	20.5	19.4	18.3	17.4	16.6
	.855	33.9	28.6	25.8	23.6	21.9	20.5	19.4	18.3	17.5	16.7
	.885	33.8	28.7	25.7	23.6	21.9	20.5	19.4	18.3	17.5	16.7
	.915	33.7	28.7	25.7	23.6	21.9	20.6	19.4	18.3	17.6	16.8

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Table III

S(T<sub>0</sub>S<sub>0</sub>) for numerical averaging over electron energy distribution.

T S <sub>0</sub>	.1500	.1375	.1250	.1125	.1000	.0875	.0750	.0625	.0500	.0375	.0250
.045	.2349	.2286	.2212	.2126	.2022	.1896	.1740	.1544	.1289	.09549	.05242
.075	.2572	.2456	.2328	.2177	.2003	.1800	.1560	.1277	.09462	.05739	.02112
.105	.2580	.2421	.2245	.2043	.1818	.1565	.1281	.09685	.06364	.03159	.00779
.135	.2478	.2283	.2070	.1836	.1580	.1303	.1007	.07026	.04096	.01665	.00276
.165	.2307	.2097	.1860	.1606	.1388	.1056	.07715	.04969	.02568	.00855	.00095
.193	.2151	.1893	.1645	.1381	.1113	.08480	.05806	.03452	.01582	.00451	.00032
.225	.1934	.1687	.1453	.1173	.09136	.06624	.04315	.02368	.00963	.00215	.00010
.255	.1739	.1489	.1239	.09869	.07451	.05162	.03176	.01609	.00581	.00106	.00004
.285	.1551	.1305	.1051	.08234	.05998	.03992	.02320	.01085	.00347	.00052	.000012
.315	.1375	.1136	.09035	.06828	.04812	.03068	.01684	.00727	.00207	.00025	.000004
.345	.1213	.09836	.07657	.05635	.03841	.02346	.01216	.00485	.00122	.00018	.0000018
.375	.1065	.08486	.06461	.04630	.03052	.01785	.00874	.00322	.00071	.00006	.0000004
.405	.0932	.07294	.05431	.03788	.02415	.01355	.00696	.00212	.00042	.00005	.00000018
.435	.0813	.06306	.04662	.03091	.01907	.01023	.00447	.00140	.00025	.00001	.0000004
.465	.0707	.05333	.03802	.02814	.01499	.00772	.00318	.00093	.00014	.00001	.00000001
.495	.0613	.04536	.03167	.02040	.01176	.00580	.00226	.00060	.00008	.00000	.00000000
.525	.0530	.03865	.02634	.01651	.00922	.00435	.00160	.00040	.00005	.00000	.00000000
.555	.0468	.03278	.02187	.01533	.00780	.00386	.00113	.00026	.00005	.00000	.00000000
.585	.0395	.02768	.01811	.01076	.00561	.00244	.00080	.00017	.00002	.00000	.00000000
.615	.0339	.02358	.01497	.00866	.00458	.00182	.00056	.00011	.00001	.00000	.00000000
.645	.0292	.01977	.01235	.00696	.00339	.00135	.00040	.00007	.00001	.00000	.00000000
.675	.0260	.01665	.01017	.00559	.00263	.00101	.00026	.00004	.00000	.00000	.00000000
.705	.0214	.01389	.00858	.00447	.00205	.00075	.00020	.00003	.00000	.00000	.00000000
.735	.0185	.01181	.00689	.00357	.00157	.00051	.00014	.00002	.00000	.00000	.00000000
.765	.0157	.00975	.00570	.00285	.00128	.00041	.00010	.00001	.00000	.00000	.00000000
.795	.0134	.00829	.00456	.00227	.00094	.00030	.00007	.00001	.00000	.00000	.00000000
.825	.0114	.00697	.00382	.00181	.00072	.00022	.00005	.00001	.00000	.00000	.00000000
.855	.0097	.00580	.00315	.00145	.00055	.00017	.00005	.00000	.00000	.00000	.00000000
.885	.0082	.00482	.00256	.00134	.00042	.00018	.00002	.00000	.00000	.00000	.00000000

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Table IV

$\eta_{HR}$  ( $T, k$ ) calculated from Eq. (12).

$\frac{T}{k}$	.1500	.1375	.1250	.1125	.1000	.0875
.05	804.8	803.0	799.7	795.5	784.5	767.9
.06	262.8	256.8	249.5	240.0	228.0	212.9
.09	124.1	118.9	112.7	105.5	96.59	86.14
.12	68.40	64.22	59.40	54.06	47.65	40.61
.15	41.07	37.85	34.24	30.26	25.88	21.11
.18	26.05	23.55	20.85	17.91	14.79	11.55
.21	17.16	15.22	13.17	11.02	8.794	6.571
.24	11.65	10.15	8.565	6.969	5.578	3.849
.27	8.053	6.882	5.692	4.508	3.584	2.305
.30	5.673	4.755	3.846	2.967	2.140	1.404

$\frac{T}{k}$	.0750	.0625	.0500	.0375	.0250	.0125
.05	743.5	704.9	645.1	547.2	382.8	151.7
.06	193.5	168.5	135.3	95.04	45.03	4.041
.09	75.66	58.91	41.85	25.42	7.213	0.2025
.12	52.90	41.22	15.21	6.948	1.429	0.01200
.15	16.04	10.89	6.052	2.258	0.3105	0.0007826
.18	8.282	5.180	2.550	0.7777	0.07150	0.00006437
.21	4.448	2.564	1.118	0.2788	0.01716	0.000003905
.24	2.657	1.507	0.5051	0.1050	0.004242	0.0000008905
.27	1.383	0.6815	0.2853	0.05882	0.001074	0.0000002115
.30	0.7985	0.3615	0.1097	0.01487	0.0002767	0.0000000273

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Table V

 $R(T, k)$  calculated from Eq. (16).

$k \backslash T$	.1500	.1375	.1250	.1125	.1000	.0875
0	1.0355	1.0316	1.0277	1.0241	1.0206	1.0172
.005	1.0680	1.0613	1.0647	1.0480	1.0417	1.0355
.030	1.0890	1.0811	1.0731	1.0652	1.0577	1.0502
.060	1.1084	1.0995	1.0906	1.0820	1.0733	1.0653
.090	1.1265	1.1171	1.1077	1.0982	1.0890	1.0801
.120	1.1446	1.1343	1.1241	1.1140	1.1042	1.0946
.150	1.1630	1.1524	1.1417	1.1309	1.1203	1.1103
.180	1.1820	1.1708	1.1595	1.1483	1.1374	1.1268
.210	1.2016	1.1898	1.1783	1.1667	1.1555	1.1442
.240	1.2219	1.2097	1.1975	1.1853	1.1737	1.1622
.270	1.2430	1.2305	1.2178	1.2050	1.1927	1.1810
.300	1.2649	1.2523	1.2396	1.2262	1.2135	1.2012
$k \backslash T$	.0750	.0625	.0600	.0375	.0250	.0125
0	1.0141	1.0111	1.0084	1.0059	1.0037	1.0017
.005	1.0297	1.0244	1.0193	1.0145	1.0092	1.0057
.030	1.0438	1.0364	1.0302	1.0242	1.0185	1.0131
.060	1.0574	1.0499	1.0425	1.0355	1.0289	1.0227
.090	1.0717	1.0632	1.0550	1.0474	1.0401	1.0331
.120	1.0854	1.0765	1.0681	1.0600	1.0521	1.0444
.150	1.1009	1.0913	1.0822	1.0734	1.0650	1.0570
.180	1.1167	1.1067	1.0970	1.0878	1.0789	1.0702
.210	1.1334	1.1229	1.1128	1.1031	1.0937	1.0844
.240	1.1511	1.1401	1.1296	1.1193	1.1094	1.0998
.270	1.1693	1.1582	1.1475	1.1366	1.1262	1.1161
.300	1.1890	1.1774	1.1660	1.1549	1.1441	1.1334

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Table VI

$\eta(T,k)$  obtained from the values in Tables IV and V.

$k \backslash T$	.1500	.1375	.1250	.1125	.1000	.0875
.03	875.8	868.1	858.2	845.0	829.8	806.4
.06	291.5	282.4	271.9	259.7	244.7	226.8
.09	159.8	132.8	124.8	115.6	105.2	95.04
.12	78.29	72.84	66.87	60.22	52.84	44.67
.15	47.76	43.60	39.09	34.22	28.99	23.44
.18	30.79	27.57	24.15	20.57	16.82	13.01
.21	20.62	18.11	15.52	12.86	10.16	7.519
.24	14.21	12.25	10.26	8.260	6.312	4.472
.27	10.01	8.468	6.938	5.432	4.012	2.720
.30	7.176	5.958	4.770	3.638	2.597	1.686

$k \backslash T$	.0750	.0625	.0500	.0375	.0250	.0125
.03	775.4	730.6	664.6	560.4	389.9	123.3
.06	204.6	176.7	141.1	96.34	44.27	4.133
.09	78.94	62.64	44.15	24.53	7.502	0.2090
.12	55.71	26.07	16.25	7.365	1.503	0.01255
.15	17.66	11.88	6.549	2.424	0.3305	0.0008272
.18	9.249	5.733	2.797	0.8460	0.07714	0.00005797
.21	5.039	2.879	1.244	0.3075	0.01877	0.000004259
.24	2.828	1.490	0.5706	0.1153	0.004706	0.0000003198
.27	1.625	0.7890	0.2677	0.04424	0.001210	0.00000002469
.30	0.9492	0.4256	0.1297	0.01729	0.0003166	0.00000001945

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Fig. 1 Values of  $\Omega(T,k)$  from Table V plotted against electron temperature T for photon energy  $k = 0, .005, .03, .06, .09, .12, .15, .18, .21, .24, .27, .30.$

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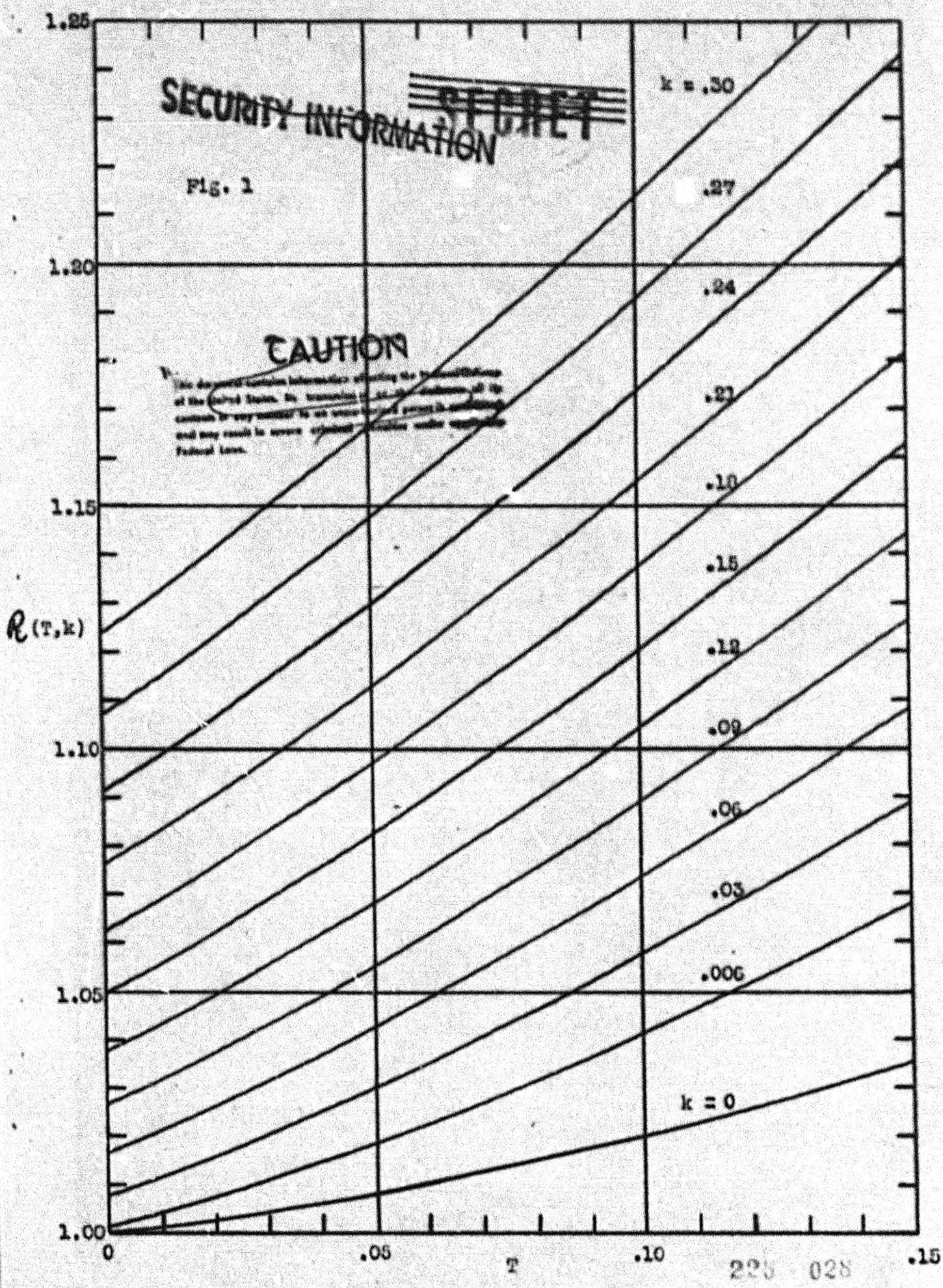


Fig. 2 Values of  $\eta(T,k)$  from Table VI plotted against electron temperature  $T$  for photon energy  $k = .03, .06, .09, .12, .15, .18, .21, .24, .27, .30.$

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