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TO: R.A. Charpie

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SUBJECT: Criticality Studies of Interacting Cylindrical Systems

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INTRODUCTION

In the steadily expanding field of Atomic Energy, material handling and storage of fissionable material has often caused concern over whether a critical condition might result. The probability of such an event occurring is dependent upon the geometry of the system, total mass, distribution and concentration of fissionable material and poison present, and, except in very simple systems, is quite difficult to predict. Consequently, present methods of fissionable material handling and storage are always very conservative on the safe side. Such a procedure is quite inefficient and will often lead to uneconomical methods of handling. For example, reactor fuel elements must be stored before insertion and after removal from the reactor. It is highly desirable to provide the minimum storage volume for such elements and still maintain the entire system subcritical.

Theoretical and experimental investigations have been carried out on single elements of fissionable material so that it is usually possible

to foretell when a given element will become critical. The more realistic problem, however, would involve a study of a system containing more than one element; the elements stored near enough to each other so that neutron leakage from one results in an appreciable neutron gain in the others. Criticality considerations in such interacting systems of fissionable elements are difficult to determine experimentally or to calculate rigorously, however some preliminary work is being started. As a part of this program, this report presents the results of a study of an interacting system in which the conditions for criticality are given in terms of the concentration of fissionable material and the geometry of the system. In the limited time available, only a single type of geometry could be considered, i.e., an infinite system in which the fissionable material was contained in cylinders spaced in an equilateral triangular pattern. The cylinder diameter and distance between cylinders thus define the specific geometry of the system. The cylinders were chosen to contain U^{235} (as UO_2F_2) dissolved in light water, and the free space between cylinders contained light water only. Since it was important to determine only the conditions which prevail when the multiplication factor equals unity (i.e., the system becomes critical), the results are expressed as a function of geometry and U^{235} concentration only.

The general method of calculation consisted of breaking the lattice into unit cylindrical cells. The core of each cell was the cylinder of fissionable material and the surrounding material then composed a light water reflector. Each cell was then treated as a subcritical homogeneous reactor with a reflector by the two group method. Coupling between cells was accomplished by choosing an appropriate boundary condition at the outer cell boundary; namely, that the gradient of the flux equaled zero.

THEORY

A. Description of the System

An infinite system (in all directions) of cylinders of radius R_0 was considered. The cylinders were spaced in an equilateral triangular pattern with a distance, b , between centers and contained a concentration, N , of U^{235} present as UO_2F_2 . The free volume between cylinders, as well as the solvent in the cylinders, was chosen as light water.

B. Simplifying Assumptions Made in the Calculations

The system described above will become critical when the variables R_0 , b , and N have specific values which may be denoted as 'critical values'. For all other values the system will be either sub or supercritical and of no interest here.

It is desired to have a treatment which is relatively simple to calculate. A more rigorous treatment is planned by Dr. R.A. Charpie

in the near future and the results of the present treatment are to be compared with those of Fr. Charpie. If the results compare favorably, then the rigorous treatment will be abandoned, for calculational purposes, in favor of the simplified treatment. Thus, in the following treatment many simplifying assumptions are made to calculate the relationship between 'critical values'. The assumptions are presented below and some of them are rationalized. The final validity of the assumptions will naturally rest on the comparison of the results obtained with those obtained from a more rigorous treatment.

Absorption Only at Thermal Energies

It is assumed that all fissions are caused by thermal neutrons and that all absorption of neutrons occur at thermal energies. Thus the fast fission factor, ϵ , and the resonance escape probability, p , both of which are ordinarily considered in criticality studies, are taken to be unity in the present treatment. The number of fast fissions is usually only a few per cent of the total fissions in a thermal system (ϵ for natural uranium ~ 1.03 (2)) and the larger contribution to fast fission is usually due to U-238 instead of U-235. Hence in a pure U-235 system little error is introduced by neglecting this factor, especially in low concentration systems. The resonance escape probability is very important in U-238-U-235 systems but becomes less important in a pure U-235 case such as is considered here. The assumption that $p = 1$ tends to increase the probability of a given system becoming critical in direct contrast to the decrease in probability introduced by assuming $\epsilon = 1$.

Equivalent Cell Representation

The cross section of the lattice is divided into a number of identical hexagonal unit cells and it is assumed that these can be replaced by cells with circular cross-sections of the same area. The unit hexagonal cells are shown in Figure 1 and the equivalent cell of circular cross-section is indicated. Equivalent unit cell approximations are common practice in the representation of heterogeneous reactor systems and the use of these approximations has yielded quite satisfactory results. Since the system considered here is essentially an infinite heterogeneous reactor, this approximation seems warranted.

Simple Diffusion Theory

Simple diffusion theory is assumed to be applicable. This assumption is only strictly valid in regions away from boundaries and neutron sources where the angular distribution of neutron velocity vectors is isotropic. A rigorous treatment would involve the use of transport theory of which simple diffusion theory is an asymptotic form. In view of the extreme complexity which would be introduced by the introduction of the transport equations, since the case considered is a hypothetical one to begin with, and since this treatment is only the first step in the solution of the problem, it is certainly reasonable to use simple diffusion theory here.

$$R_1 = \left(\frac{\text{AREA OF HEXAGON}}{\pi} \right)^{1/2} = \left(\frac{\sqrt{3}}{2\pi} \right)^{1/2} b$$

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EQUIVALENT UNIT CELL

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Two Group Method

The two group method is used to determine the 'critical values' for the unit cells which are considered as water-reflected subcritical reactors. In this method all neutrons are taken as being in one of two energy groups, namely, thermal and fast. That is, all neutrons of thermal energy constitute one group while all those of higher energy constitute the other group. A better approximation would be made if the energy range were split into many segments instead of just two and, of course, in the limit of an infinite number of groups, an exact solution would be obtained. The two group method is a first approximation and cannot be expected to give exact answers.

In the two group method all neutrons are assumed to originate from a monoenergetic source with a high energy E_0 (~ 2 Mev). The neutrons retain their source energy until they have undergone the average number of scattering collisions to reduce their energies to the thermal value, E_{th} . The fast neutrons are then instantaneously degraded to the thermal energy group.

Since the fast neutrons have, in reality, energies ranging from E_0 to E_{th} , average values over this energy range are used for the cross sections, diffusion lengths, diffusion coefficients, etc. (See Appendix, Section B).

C. Derivation of Critical Equation for the System

The following derivation follows closely the two group treatment of a homogeneous reactor with a reflector given by Glasstone and Edlund (2). Using similar nomenclature the subscript 1 will refer to fast neutrons, the subscript 2 to thermal neutrons, the subscript c will refer to the cylinder of fissionable material, and the subscript r to the reflector. For convenient reference in following the derivation, a table of nomenclature is presented here.

Nomenclature

- A, A', A'' - constants of solution of neutron diffusion equations (dimensionless).
- A_0 - atomic weight (gm./gm. atom).
- B^2 - buckling (cm.^{-2}).
- b - center to center separation of cylinders (cm.).
- C, C', C'' - constants of solution of neutron diffusion equations (dimensionless).
- D - diffusion coefficient (cm.).
- E_0 - fast neutron energy (~ 2 Mev).
- E_{th} - slow neutron energy (0.025 ev.).

- F - constant of solution of neutron diffusion equations (dimensionless).
- G - constant of solution of neutron diffusion equations (dimensionless).
- I_p - modified Bessel function of order p.
- J_p - Bessel function of order p.
- k - multiplication factor (dimensionless).
- K_p - modified Bessel function of the second kind, of order p.
- L - diffusion length (cm.).
- N - concentration (number of atoms/cu.cm.).
- p - resonance escape probability.
- P - constant of solution of wave equation.
- Q - constant of solution of wave equation.
- r - space variable (cm.).
- R_0 - radius of core (cm.).
- R_1 - outer radius of reflector (cm.).
- S - coupling coefficient which gives the relationship between fast and thermal fluxes.
- V - volume (cu.cm.).
- X - solution of the wave equation for the cylinder.
- Y - solution of the cylinder wave equation.
- Y_p - Bessel function of the second kind of order p.
- Z - solution of the reflector flux (neutrons/(cm.²)(sec.)).

- ϵ - fast fission factor.
- Δ - value of the determinant.
- ∇^2 - Laplacian operator.
- μ^2 - solution of B^2 from critical equation (cm.⁻²).
- $\bar{\mu}_0$ - average cosine of the scattering angle $\approx \frac{2}{3A}$.
- $-\mu^2$ - solution of B^2 from critical equation (cm.⁻²).
- κ^2 - Σ/D (cm.⁻²).
- τ - Fermi Age (cm.²).
- $\bar{\nu}$ - number of fast neutrons emitted per fission.
- λ_{tr} - transport mean free path (cm.).
- Σ - macroscopic absorption cross section (cm.⁻¹).
- Σ_1 - slowing down macroscopic cross section (cm.⁻¹).
- Σ_f - macroscopic fission cross section (cm.⁻¹).

- Σ_s - scattering macroscopic cross section (cm.^{-1}).
 Σ_{tr} - transport macroscopic cross section (cm.^{-1}).
 ϕ - neutron flux (neutrons/ $\text{cm.}^2/\text{sec.}$).
 ξ - mean average logarithmic energy decrement per collision.

Subscripts:

- H_2O - refers to water.
 u - refers to uranium

Primes on X, Y, or Z refer to differentiation of the function with respect to the space variable.

Derivation

Consider the cylinder first. The number of fast neutrons produced by thermal fissions per cm.^3 per sec. is $k\Sigma_{2c}\phi_{2c}$. The fast neutrons lost per cm.^3 per sec. by leakage is $D_{1c}\nabla^2\phi_{1c}$ and those lost by being degraded from the fast into the thermal group is $-\Sigma_{1c}\phi_{1c}$. $\Sigma_{1c}\phi_{1c}$ also represents the number of thermal neutrons produced per cm.^3 per sec. The steady state diffusion equation (neutron balance) for the fast neutrons in the cylinder is thus

$$D_{1c}\nabla^2\phi_{1c} - \Sigma_{1c}\phi_{1c} + k\Sigma_{2c}\phi_{2c} = 0 \quad (1)$$

-(Leakage) - (Absorption) + (Production)

Similarly the steady state diffusion equation for thermal neutrons in the cylinder is

$$D_{2c}\nabla^2\phi_{2c} - \Sigma_{2c}\phi_{2c} + \Sigma_{1c}\phi_{1c} = 0 \quad (2)$$

In the reflector no fissions take place and hence there is no production term for fast neutrons. Thus the two diffusion equations are:

$$D_{1r}\nabla^2\phi_{1r} - \Sigma_{1r}\phi_{1r} = 0 \quad (3)$$

and

$$D_{2r}\nabla^2\phi_{2r} - \Sigma_{2r}\phi_{2r} + \Sigma_{1r}\phi_{1r} = 0 \quad (4)$$

Since the system is infinite and since the boundary of a unit cell is a point of symmetry between any two neighboring cylinders, the gradients of the fast- and slow-neutron fluxes are zero at the reflector boundary ($r = R_r$). Taking this boundary condition into consideration, the permissible solutions for the reflector fluxes are (Appendix, Section A):

$$\phi_{1r} = F Z_1 \quad (5)$$

$$\phi_{2r} = G Z_2 + S_3 F Z_1 \quad (6)$$

$$\text{where: } Z_n = I_0(\kappa_{nr} r) + \frac{I_1(\kappa_{nr} R_1)}{K_1(\kappa_{nr} R_1)} K_0(\kappa_{nr} r)$$

$$\kappa_{nr}^2 = \Sigma_{nr} / D_{nr}$$

$$S_3 = (\Sigma_{1r} / D_{2r}) (\kappa_{2r}^2 - \kappa_{1r}^2)^{-1}$$

The solutions of the two cylinder diffusion equations (1) and (2) are (See Appendix, Section A):

$$\phi_{1c} = AX + CY \quad (7)$$

$$\text{and } \phi_{2c} = AS_1 X + CS_2 Y \quad (8)$$

$$\text{where: } X = J_0(\mu r)$$

$$Y = I_0(\gamma r)$$

$$\mu^2 = \frac{1}{2} \left[-\left(\frac{1}{L_c^2} + \frac{1}{L_{2c}^2}\right) + \sqrt{\left(\frac{1}{L_c^2} + \frac{1}{L_{2c}^2}\right)^2 + \frac{4(k-1)}{L_c L_{2c}}} \right]$$

$$\gamma^2 = \frac{1}{2} \left[-\left(\frac{1}{L_c^2} + \frac{1}{L_{2c}^2}\right) - \sqrt{\left(\frac{1}{L_c^2} + \frac{1}{L_{2c}^2}\right)^2 + \frac{4(k-1)}{L_c L_{2c}}} \right]$$

$$\text{where: } S_1 = \frac{\Sigma_{1c}}{D_{2c}\mu^2 + \Sigma_{2c}} \quad S_2 = \frac{\Sigma_{1c}}{\Sigma_{2c} - D_{2c}\gamma^2}$$

k = multiplication factor for the bare cylinder

$$\bar{\nu} \Sigma_{fc} = \frac{\Sigma_{fc}}{\Sigma_{2c}}$$

The four flux equations, (5), (6), (7), and (8), contain four unknowns, i.e., A, C, F, and G, but there are four continuity conditions which must be met, namely, that the fast- and slow-neutron fluxes and current densities are continuous at the cylinder-reflector interface. In equation form these boundary conditions are:

$$\phi_{1c} = \phi_{1r}$$

$$\phi_{2c} = \phi_{2r}$$

$$D_{1c} \frac{d\phi_{1c}}{dr} = D_{1r} \frac{d\phi_{1r}}{dr}$$

$$\text{and } D_{2c} \frac{d\phi_{2c}}{dr} = D_{2r} \frac{d\phi_{2r}}{dr}$$

all evaluated at $r = R_0$. These conditions lead directly to the following equations:

$$\begin{aligned} XA + YC - Z_1 F &= 0 & (9) \\ (S_1 X)A + (S_2 Y)C - (S_3 Z_1)F - Z_2 G &= 0 & (10) \\ (D_{1c} X^1)A + (D_{1c} Y^1)C - (D_{1r} Z_1^1)F &= 0 & (11) \\ (S_1 D_{2c} X^1)A + (S_2 D_{2c} Y^1)C - (S_3 D_{2r} Z_1^1)F - (D_{2r} Z_2^1)G &= 0 & (12) \end{aligned}$$

In order to satisfy the conditions that lead to a critical system, it is seen that the above set of homogeneous equations in A, C, F, and G must be consistent; however, these equations will only have a consistent and non-trivial set of solutions for the four constants if the determinant of the coefficients is zero (Cramer's rule). Hence,

$$\begin{vmatrix} X & Y & -Z_1 & 0 \\ S_1 X & S_2 Y & -S_3 Z_1 & -Z_2 \\ D_{1c} X^1 & D_{1c} Y^1 & -D_{1r} Z_1^1 & 0 \\ S_1 D_{2c} X^1 & S_2 D_{2c} Y^1 & -S_3 D_{2r} Z_1^1 & -D_{2r} Z_2^1 \end{vmatrix} = 0 \quad (13)$$

Denoting the value of the determinant as Δ , it is found that:

$$\begin{aligned} \frac{\Delta}{XYZ_1 Z_2} &= (D_{1c} \frac{X^1}{X} - D_{1r} \frac{Z_1^1}{Z_1}) (S_2 D_{2c} \frac{Y^1}{Y} - S_3 D_{2r} \frac{Z_1^1}{Z_1} - (S_2 - S_3) D_{2r} \frac{Z_2^1}{Z_2}) - \\ &\quad (D_{1c} \frac{Y^1}{Y} - D_{1r} \frac{Z_1^1}{Z_1}) (S_1 D_{2c} \frac{X^1}{X} - S_3 D_{2r} \frac{Z_1^1}{Z_1} - (S_1 - S_3) D_{2r} \frac{Z_2^1}{Z_2}) \quad (14) \end{aligned}$$

This equation is the critical equation for the system and gives the critical conditions when $\Delta = 0$.

Values which satisfy equation (14) can easily be found by a trial and error method of solution. The following is a suggested procedure:

1) For a given concentration of U-235 in the cylinder, the values of $S_1, S_2, S_3, X_1, X_2, \mu^2$ and μ^2 are computed for a given cell radius R_1 . (Since R_1 is proportional to b , the cylinder separations, this is equivalent to computing these values for a given N and b).

2) Several values of the cylinder radius, R_0 , are now used to compute X, Y, Z , and their derivatives.

3) Values from (1) and (2) are substituted in equation (14) and the value of R_0 determined which makes $\Delta = 0$.

Repeating the above procedure for various N and R_1 's, a compilation of values of N, R_1 , and R_0 can be made and plotted to enable a quick determination of the conditions when a system of this type becomes critical. One method of graphical presentation of the results that would be useful would be to plot the cylinder radius versus concentration at the critical conditions with various separation distances, b , as a parameter.

CONCLUSIONS

It is thought by the authors that the expression derived fulfills the requirements that it be a relatively simple solution to calculate. Numerical solutions, which are desired for comparison, are to be obtained by machine calculations in the near future at ORNL under the direction of Dr. Chappie.

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APPENDIX

A. Solution of Diffusion Equations

The solutions below follow closely those of Glasstone & Edlund (2) and are repeated here only for convenience of the reader.

Since the cylinders are infinitely long and have symmetry about the axis, the diffusion equations are one dimensional. The cylinder diffusion equations (1) and (2) are:

$$\nabla^2 \phi_{1c} - \kappa_{1c}^2 \phi_{1c} + k \frac{\Sigma_{2c}}{D_{1c}} \phi_{2c} = 0 \quad (1)$$

$$\text{and } \nabla^2 \phi_{2c} - \kappa_{2c}^2 \phi_{2c} + \frac{\Sigma_{1c}}{D_{2c}} \phi_{1c} = 0 \quad (2)$$

where the diffusion coefficients D_{1c} and D_{2c} have been divided out, $\frac{\Sigma_{1c}}{D_{1c}}$ replaced by κ_{1c}^2 , and $\frac{\Sigma_{2c}}{D_{2c}}$ replaced by κ_{2c}^2 .

To solve these equations, it is assumed that solutions of the type

$$\nabla^2 \phi_{1c} + B^2 \phi_{1c} = 0 \quad (1A)$$

$$\text{and } \nabla^2 \phi_{2c} + B^2 \phi_{2c} = 0 \quad (2A)$$

exist and the conditions necessary for these solutions to apply are determined. As a first step it can be shown that the constant B^2 is the same in the two equations. Solving for ϕ_{2c} from (1) and substituting in (2), it is found that

$$\phi_{2c} = -(\nabla^2 \phi_{1c} - \kappa_{1c}^2 \phi_{1c}) \frac{D_{1c}}{\kappa_{2c}^2}$$

Therefore:

$$-\frac{D_{1c}}{\kappa_{2c}^2} \nabla^2 (\nabla^2 \phi_{1c}) + \frac{\Sigma_{1c}}{\kappa_{2c}^2} \nabla^2 \phi_{1c} + \frac{D_{1c}}{\kappa_{2c}^2} \nabla^2 \phi_{1c} - \frac{\Sigma_{1c}}{\kappa_{2c}^2} (1 - k) \phi_{1c} = 0 \quad (1B)$$

Solving for ϕ_{1c} from (2) and substituting in (1)

$$\phi_{1c} = (-\nabla^2 \phi_{2c} + \kappa_{2c}^2 \phi_{2c}) \frac{D_{2c}}{\Sigma_{1c}}$$

therefore,

$$\frac{\epsilon_{2c}}{\epsilon_{1c}} \nabla^2 \phi_{2c} - \frac{D_{2c}}{\epsilon_{1c}} \nabla^2 (\nabla^2 \phi_{2c}) - \frac{\epsilon_{2c}}{D_{1c}} \phi_{2c} + \frac{D_{2c}}{D_{1c}} \nabla^2 \phi_{2c} + k \frac{\epsilon_{2c}}{D_{1c}} \phi_{2c} = 0$$

Multiplying through by $(\frac{\epsilon_{1c}}{\epsilon_{2c}})(\frac{D_{1c}}{D_{2c}})(\frac{1}{k})$ the result is

$$-\frac{D_{1c}}{k\epsilon_{2c}} \nabla^2 (\nabla^2 \phi_{2c}) + \frac{\epsilon_{1c}}{k\epsilon_{2c}} \nabla^2 \phi_{2c} + \frac{D_{1c}}{kD_{2c}} \nabla^2 \phi_{2c} - \frac{\epsilon_{1c}}{kD_{2c}} (1-k) \phi_{2c} = 0 \quad (2B)$$

It is seen that equation (1B) is identical to equation (2B) with ϕ_{1c} replaced by ϕ_{2c} . Hence if wave equations are written for ϕ_{1c} and ϕ_{2c} (equation (1A) and (2A)), the constant B^2 must be the same in both equations.

It becomes necessary now to find the conditions under which the solutions of the type (1A) and (2A) apply. By substituting $-B^2 \phi_{1c}$ for $\nabla^2 \phi_{1c}$ in (1) and $-B^2 \phi_{2c}$ for $\nabla^2 \phi_{2c}$ in (2) it is seen that

$$-D_{1c} B^2 \phi_{1c} - \epsilon_{1c} \phi_{1c} + k \epsilon_{2c} \phi_{2c} = 0$$

$$\text{and } -D_{2c} B^2 \phi_{2c} - \epsilon_{2c} \phi_{2c} + \epsilon_{1c} \phi_{1c} = 0$$

In order that these two equations have a non-trivial solution it is necessary that the determinant of the coefficients vanish. That is,

$$(D_{1c} B^2 + \epsilon_{1c})(D_{2c} B^2 + \epsilon_{2c}) - k \epsilon_{1c} \epsilon_{2c} = 0$$

or dividing by $\epsilon_{1c} \epsilon_{2c}$

$$(\frac{D_{1c}}{\epsilon_{1c}} B^2 + 1)(\frac{D_{2c}}{\epsilon_{2c}} B^2 + 1) = k \quad (3A)$$

Replacing D_{1c}/ϵ_{1c} by τ_2 and D_{2c}/ϵ_{2c} by L_{2c}^2 , equation (3A) becomes

$$(1 + \tau_2 B^2)(1 + L_{2c}^2 B^2) = k$$

$$\text{or } B^4 + (\frac{1}{\tau_2} + \frac{1}{L_{2c}^2}) B^2 - \frac{(k-1)}{\tau_2 L_{2c}^2} = 0$$

By solving the equation as a quadratic in B^2 , it is seen that two solutions for B^2 exist. Representing one solution by μ^2 and the other by $-\nu^2$ so that both μ^2 and ν^2 are positive quantities, the quadratic formula gives

$$\mu^2 = \frac{1}{2} \left[-\left(\frac{1}{r_c} + \frac{1}{L_{2c}^2}\right) + \sqrt{\left(\frac{1}{r_c} + \frac{1}{L_{2c}^2}\right)^2 + \frac{4(k-1)}{r_c L_{2c}^2}} \right]$$

and

$$-\nu^2 = \frac{1}{2} \left[-\left(\frac{1}{r_c} + \frac{1}{L_{2c}^2}\right) + \sqrt{\left(\frac{1}{r_c} + \frac{1}{L_{2c}^2}\right)^2 + \frac{4(k-1)}{r_c L_{2c}^2}} \right]$$

Considering now equations (1A) and (2A), let the solutions for the flux corresponding to the two values μ^2 and $-\nu^2$ for B^2 be X and Y respectively. The equations can be written then as

$$\nabla^2 X + \mu^2 X = 0 \quad (4A)$$

and $\nabla^2 Y - \nu^2 Y = 0 \quad (5A)$

In cylindrical coordinates these become

$$\frac{d^2 X}{dr^2} + \frac{1}{r} \frac{dX}{dr} + \mu^2 X = 0$$

and $\frac{d^2 Y}{dr^2} + \frac{1}{r} \frac{dY}{dr} - \nu^2 Y = 0$

Both equations are Bessel equations and have solutions of the form

$$X = A' J_0(\mu r) + A'' Y_0(\mu r)$$

$$Y = C' I_0(\nu r) + C'' K_0(\nu r)$$

At $r = 0$, Y_0 and K_0 both approach infinity; therefore $A'' = C'' = 0$ and

$$X = A J_0(\mu r) \quad (6A)$$

and $Y = A' I_0(\nu r) \quad (7A)$

The most general solutions for the core are linear combinations of X and Y , namely,

$$\phi_{1c} = AX + CY \quad (8A)$$

$$\phi_{2c} = PX + QY \quad (9A)$$

These solutions appear to have four arbitrary constants, but only two are independent. Permissible solutions for ϕ_{1c} and ϕ_{2c} are

$$\phi_{1c} = AX$$

$$\phi_{2c} = PX$$

even though the general solutions are given by (8A) and (9A). Since $\nabla^2 X + \mu^2 X = 0$, $\nabla^2 \phi_{2c} = -\mu^2 PX$ and by substituting in equation (2) the result is

$$-D_{2c}\mu^2 PX - E_{2c}PX + E_{1c}AX = 0$$

Letting $P/A = S_1$, the coupling coefficient, it is seen that $S_1 = \frac{E_{1c}}{D_{2c}\mu^2 + E_{2c}}$ (10A)

Similarly substitution of CY and QY results in $S_2 = \frac{Q}{C} = \frac{E_{1c}}{E_{2c} - D_{2c}\mu^2}$ (11A)

Therefore the fuel cylinder equations can be written as

$$\phi_{1c} = AX + CY \quad (12A)$$

$$\phi_{2c} = S_1 AX + S_2 CY \quad (13A)$$

The reflector equations can be written as

$$\nabla^2 \phi_{1r} - \kappa_{1r}^2 \phi_{1r} = 0 \quad (3)$$

$$\text{and } \nabla^2 \phi_{2r} - \kappa_{2r}^2 \phi_{2r} + \frac{E_{1r}}{D_{2r}} \phi_{1r} = 0 \quad (4)$$

where D_{1r} and D_{2r} have been divided through, $\frac{E_{1r}}{D_{1r}}$ replaced by κ_{1r}^2 and $\frac{E_{2r}}{D_{2r}}$ replaced by κ_{2r}^2 .

Equation (3) is homogeneous and can be solved explicitly in terms of Bessel functions. The solution of such an equation is

$$\phi_{1r} = F I_0(\kappa_{1r} r) + F' K_0(\kappa_{1r} r)$$

At the outer cell boundary, the gradient of the flux must equal zero since this boundary is a point of symmetry and the system is infinite. Thus,

$$\left. \frac{d\phi_{1r}}{dr} \right|_{r=R_1} = 0 = F \kappa_{1r} I_1(\kappa_{1r} R_1) - F' \kappa_{1r} K_1(\kappa_{1r} R_1)$$

from which

$$F' = \frac{F I_1(\kappa_{1r} R_1)}{\kappa_1(\kappa_{1r} R_1)}$$

Therefore,

$$\phi_{1r} = F \left[I_0(\kappa_{1r} r) + \frac{I_1(\kappa_{1r} R_1)}{\kappa_1(\kappa_{1r} R_1)} K_0(\kappa_{1r} r) \right]$$

$$\text{Letting } Z_n = \left[I_0(\kappa_{nr} r) + \frac{I_1(\kappa_{nr} R_1)}{\kappa_1(\kappa_{nr} R_1)} K_0(\kappa_{nr} r) \right] \quad (14A)$$

$$\phi_{1r} \text{ can be rewritten as } \phi_{1r} = F Z_1 \quad (15A)$$

The homogeneous part of equation (4) has a similar solution, i.e., the solution of

$$\nabla^2 \phi_{2r} - \kappa_{2r}^2 \phi_{2r} = 0$$

$$\text{is } \phi_{2r} = G Z_{2r} \quad (16A)$$

To obtain the particular solution of equation (4) a solution of the type $S_3 \phi_{1r}$ is tried and S_3 determined. Substituting $S_3 \phi_{1r}$ in equation (4) the result is

$$S_3 \nabla^2 \phi_{1r} - S_3 \kappa_{2r}^2 \phi_{1r} + \frac{E_{1r}}{D_{2r}} \phi_{1r} = 0$$

but since $\nabla^2 \phi_{1r} = \kappa_{1r}^2 \phi_{1r}$ from equation (3)

$$S_3 \kappa_{1r}^2 \phi_{1r} - S_3 \kappa_{2r}^2 \phi_{1r} + \frac{E_{1r}}{D_{2r}} \phi_{1r} = 0$$

$$\text{and } S_3 = \frac{E_{1r}}{D_{2r}} \left(\frac{1}{\kappa_{2r}^2 - \kappa_{1r}^2} \right) \quad (17A)$$

$$\text{Therefore } \phi_{1r} = F Z_1 \quad (15A)$$

$$\phi_{2r} = G Z_2 + S_3 F Z_1 \quad (18A)$$

B. Constants for Fast Neutron Group

The group constants for the fast neutron group are defined by the following formulae as given by Glasstone & Edlund (2):

$$\bar{\Sigma}_s = \frac{\int_{E_{th}}^{E_0} \Sigma_s(E) \phi(E) dE}{\int_{E_{th}}^{E_0} \phi(E) dE}$$

$$\Sigma_1 = \frac{\bar{\Sigma}_s}{\int_{E_{th}}^{E_0} \ln E_0/E_{th}}$$

$$D_1 = \frac{\int_{E_{th}}^{E_0} \frac{1}{3} \lambda_{tr}(E) \frac{dE}{E}}{\ln(E_0/E_{th})}$$

where $\lambda_{tr} = \frac{1}{\Sigma_{tr}} \approx \frac{1}{\bar{\Sigma}_s (1 - \bar{\mu}_0)}$,

$\bar{\mu}_0 = \frac{2}{3A_0}$, and A_0 is the atomic mass of the nuclide doing the scattering.

For a multinuclide system

$$\bar{\Sigma}_{tr} = \bar{\Sigma}_{s1} (1 - \bar{\mu}_{01}) + \bar{\Sigma}_{s2} (1 - \bar{\mu}_{02}) + \dots + \bar{\Sigma}_{sn} (1 - \bar{\mu}_{0n})$$

The Fermi age for the reflector is given as

$$\tau_{lr} = D_{lr} / \Sigma_{lr}$$

since the reflector is a pure substance.

Since τ for uranium is so much greater than τ for water, the Fermi age for uranium can be taken as infinity. Then (1)

$$\tau_c = \frac{V_{H_2O} + V_u}{V_{H_2O}} \tau_{lr}$$

where V_{H_2O} = volume of water in the cylinder

and V_u = volume of uranium in the cylinder.

C. Literature Citations

(1) Benedict, Manson, "Notes for Nuclear Reactor Engineering", Subject N-22, Massachusetts Institute of Technology, Cambridge, Mass., Spring, 1954.

(2) Glasstone, S. and Edlund, M.C., "The Elements of Nuclear Reactor Theory", Toronto, New York, and London, D. Van Nostrand Company, 1952.