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METHOD OF DETERMINING SOLID ANGLES
(Applicable to Critical Mass Studies)

Raymond Murray
George W. Schmidt

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INTRODUCTION

There is no rigorous theory available to our present knowledge, that allows the accurate calculation of critical conditions (mass, dimensions, or concentration of fissionable materials) of separated, untamped containers of uranium solutions. A conservative approximation developed by R. P. Feynman utilizes the concept that the "interaction" of assemblies, i.e., the mutual effect of supplying neutrons one to the other, is dependent principally on the solid angle "seen" from points on the surface of one container to the surface of adjacent bodies. The basis of this viewpoint is discussed in another report.*

The frequency with which problems occur in the Y-12 plant in which critical conditions must be determined for arrays of adjacent containers indicates the need for rapid dependable methods for computing solid angles, as well as a compilation of formulas used in such computations. No claim to originality for the compilation is made. Three cases are discussed: 1. point to rectangle; 2. point to disc; 3. cylinder to cylinder.

DEFINITIONS

The solid angle subtended by a surface S at a point P is given by

$$\Omega = \int_S \frac{d\vec{S} \cdot \vec{r}}{r^3} = \int_S \frac{dS \cos \theta}{r^2} \quad (A)$$

Where $d\vec{S}$ represents the direction of the normal vector to the surface element dS and θ is the angle between the radius vector and this normal.

* Interaction of Solutions of Uranium Salts, Raymond Murray and George Schmidt

An equivalent definition (B) states that the solid angle is the area on a sphere of unit radius formed by projecting the surface S on the sphere. Derivations based on the two definitions take different forms but give identical answers. The fractional solid angle, $\Omega_f = \frac{\Omega}{4\pi}$, is the ratio of the solid angle seen by a point to the maximum solid angle. This parameter is the one actually used in the solution of interaction problems.

1. Point to Rectangle

Derivation A

$$d\Omega = \frac{d\vec{S} \cdot \vec{r}}{r^3} = \frac{dS \cos \theta}{r^2} = \frac{dx dy}{r^2} \frac{h}{r}$$

$$\Omega = \int_0^a \int_0^b \frac{h dx dy}{(x^2 + y^2 + h^2)^{3/2}} = h \int_0^b dy \left[\frac{x}{(y^2 + h^2)(x^2 + y^2 + h^2)^{1/2}} \right]_0^a = ah \int_0^b \frac{dy}{(y^2 + h^2)(a^2 + y^2 + h^2)^{1/2}}$$

Putting $y^2 + h^2 = u$, $dy = \frac{du}{2\sqrt{u - h^2}}$

$$\begin{aligned}
 \Omega &= \frac{ah}{2} \int_{y=0}^{y=b} \frac{du}{u\sqrt{u-a^2}\sqrt{u-h^2}} = \frac{ah}{2} \int_{h^2}^{b^2+h^2} \frac{du}{u\sqrt{u^2-(a^2-h^2)u-a^2h^2}} \\
 &= \frac{ah}{2} \frac{1}{ah} \left[\sin^{-1} \frac{(a^2-h^2)u-2a^2h^2}{u(a^2+h^2)} \right]_{h^2}^{b^2+h^2} = \frac{\pi}{4} + \frac{1}{2} \sin^{-1} \left[\frac{(a^2-h^2)(b^2+h^2)-2a^2h^2}{(b^2+h^2)(a^2+h^2)} \right] \\
 \Omega &= \frac{1}{2} \cos^{-1} \left[\frac{h^4+a^2h^2+b^2h^2-a^2b^2}{h^4+a^2h^2+b^2h^2+a^2b^2} \right]
 \end{aligned}$$

Derivation B

Using the same diagram, divide the rectangle into two parts by means of the diagonal, with the boundaries:

Part 1 - $\varphi = 0, \tan^{-1} \frac{b}{a}$

$$\theta = \sin^{-1} \frac{a}{\sqrt{a^2+h^2\cos^2\varphi}}$$

Part 2 - $\varphi = \tan^{-1} \frac{b}{a}, \frac{\pi}{2}$

$$\theta = \sin^{-1} \frac{b}{\sqrt{b^2+h^2\sin^2\varphi}}$$

$$\Omega = \int_0^{\tan^{-1} \frac{b}{a}} \int_0^{\sin^{-1} \frac{a}{\sqrt{a^2+h^2\cos^2\varphi}}} \sin\theta \, d\theta \, d\varphi + \int_{\tan^{-1} \frac{b}{a}}^{\frac{\pi}{2}} \int_0^{\sin^{-1} \frac{b}{\sqrt{b^2+h^2\sin^2\varphi}}} \sin\theta \, d\theta \, d\varphi$$

since the surface element on the unit sphere is $\sin\theta \, d\theta \, d\varphi$.

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$$\Omega = \int_0^{\tan^{-1} \frac{h}{a}} \left[\frac{h \cos \varphi d\varphi}{\sqrt{a^2 + h^2 \cos^2 \varphi}} - d\varphi \right] + \int_{\tan^{-1} \frac{h}{b}}^{\frac{\pi}{2}} \left[-\frac{h \sin \varphi d\varphi}{\sqrt{b^2 + h^2 \sin^2 \varphi}} - d\varphi \right]$$

The integrals are of the type $\int \frac{du}{\sqrt{k^2 - u^2}} = \sin^{-1} \frac{u}{k}$ where $u = h \sin \varphi$ or $h \cos \varphi$, $k^2 = a^2 + h^2$ or $b^2 + h^2$

$$\begin{aligned} \Omega &= \left[\sin^{-1} \frac{h \sin \varphi}{\sqrt{a^2 + h^2}} - \varphi \right]_0^{\tan^{-1} \frac{h}{a}} - \left[\sin^{-1} \frac{h \cos \varphi}{\sqrt{b^2 + h^2}} + \varphi \right]_{\tan^{-1} \frac{h}{b}}^{\frac{\pi}{2}} \\ &= \sin^{-1} \frac{hb}{\sqrt{(a^2 + b^2)(a^2 + h^2)}} + \sin^{-1} \frac{ha}{\sqrt{(a^2 + b^2)(b^2 + h^2)}} - \frac{\pi}{2} \end{aligned}$$

This can be put in the form of Ω as given by definition A.

In order to calculate the solid angle due to any rectangle perpendicular to the line h , the area can be divided into sections, each having a corner on the line and the solid angles added (or subtracted).

It will be noted that Ω_f depends only on the ratios a/h and b/h , allowing use of a convenient graph (Figure 1). For problems dealing with arrays of containers, the solid angle "seen" by the central container is merely the sum of the solid angles subtended by the various containers at the center one.

2. Point to Disc

The solid angle subtended at P by a disc whose normal at the center passes through P is obtained as follows, using the fact that Ω is the same as that of a spherical shell with the disc boundaries:

$$d\Omega = \frac{dS \cdot \cos \theta}{r^2} = \frac{(2\pi r \sin \theta) (r d\theta)}{r^3} = 2\pi \sin \theta d\theta$$

$$\Omega = 2\pi \int_{\theta=0}^{\theta=\theta_{\max}} \sin \theta d\theta = -2\pi \cos \theta \Big|_0^{\theta_{\max}} = 2\pi(1 - \cos \theta_{\max})$$

In terms of the dimensions of the disc and separation, the fractional solid angle is

$$\Omega_f = \frac{1}{2}(1 - \cos \theta_{\max}) = \frac{\sqrt{h^2 + R^2} - h}{2\sqrt{h^2 + R^2}}$$

An approximation to this formula which can be used for calculating with more difficult geometries comes from the assumption that

$$\Omega = \frac{\overline{\cos \theta} S}{r^2}$$

where the bars indicate averages over the disc area S.

$$\cos \theta = \frac{\int_{y=0}^{y=R} \frac{h}{r} 2\pi y \, dy}{R^2} = \frac{2h}{R^2} (\sqrt{h^2 + R^2} - h)$$

and

$$\overline{r^2} = \frac{\int_{y=0}^{y=R} r^2 2\pi y \, dy}{\pi R^2} = \frac{R^2}{2} + h^2$$

Upon solving we get

$$\Omega_f \approx \frac{\sqrt{h^2 + R^2} - h}{\frac{R^2}{h} + 2h}$$

If one expands the denominator of the true fractional solid angle in powers of R/h , the result is

$$\Omega_f = \frac{\sqrt{h^2 + R^2} - h}{\frac{R^2}{h} + 2h - \frac{h}{4} \frac{R^4}{h^4} + \dots} \quad (\text{where } R < h)$$

differing from the approximate value by terms of higher order in R/h . From a separation only slightly more than one radius, the error is less than 6%.

3. Cylinder to Cylinder

There are a number of ways in which the solid angle subtended at one cylinder by an adjacent cylinder can be determined.

a. To find the exact expression for the average solid angle subtended at all points on the surface of cylinder A by the surface of cylinder B, one needs to evaluate the expression

$$\Omega = \frac{\int_A \left[\int_B \frac{d\vec{S}_B \cdot \vec{r}_{AB}}{r_{AB}^2} \right] dS_A}{\int_A dS_A}$$

This integration is difficult without approximations.

b. One may assume that the planes defined by the intersecting tangent planes to the cylinders are "equivalent" in area to area of the cylinder effective in so far as the solid angle is concerned and that the separation of the planes so defined is "equivalent" to the effective separation of the cylinders. The inherent errors in this method are three:

(1) The surfaces cannot be made equivalent to rectangular planes for the cylinder ends present a curved boundary to the observer, which becomes more pronounced as the separation decreases.

(2) The solid angle goes to zero with decreased separation while it should go to a finite large value. This sets a lower limit on the validity of the approximation.

(3) The choice of the point on one plane from which to measure the solid angle subtended by the other is ambiguous. The center of one plane is not a representative point.

It will be found however that despite these errors, some of which tend to compensate each other, a value of Ω accurate enough for critical mass calculations is obtained. Tests on the validity of the method were made by calculations of the solid angle by means of the following approximations:

- (1) Observations taken from every 10° around the middle of the cylinder to the visible surface of the other cylinder.
- (2) Determination of end effects along the nearest vertical line on the circumference of the cylinder.
- (3) Determination of the variation in solid angle with observations off center.
- (4) Calculation of an approximate value of Ω for a point at the center of one cylinder to the diametral plane of the other.

It is concluded that the equivalent plane method is useful for our purpose for all dimensions such that the separation is equal to the radius of the cylinders and/or the heights are no larger than four diameters.

Convenient formulas for the half width and separation of equivalent planes in terms of radii and separations can be derived.

From the geometry of the figure we have

$$\sin \alpha = \frac{R}{\frac{S}{2} + R}, \quad \cos \alpha = \frac{a}{R}, \quad \tan \alpha = \frac{a}{h/2}$$

and
$$a = R\sqrt{1 - \sin^2 \alpha} = \frac{R\sqrt{S^2 + 4SR}}{S + 2R}$$

$$h = \frac{a}{\tan \alpha} = \frac{R \cos^2 \alpha}{\sin \alpha} = \frac{R(1 - \sin^2 \alpha)}{\sin \alpha} = \frac{S^2 + 4SR}{S + 2R}$$

For the purpose of calculating the total solid angle of an array the ratios A/h and B/h appearing in the graph, Figure 1, may be expressed as

$$\frac{A}{h} = \frac{2a}{h} = \frac{2R}{\sqrt{S^2 + 4SR}}$$

and if $k = S/R$, i.e., the separation of the cylinders in units of R , then

$$\frac{A}{h} = \frac{2}{\sqrt{k(k+4)}}$$

Also, if H is the cylinder height, we have

$$\frac{B}{h} = \frac{H}{\frac{S^2 + 4SR}{S + 2R}} = \frac{H}{R} \frac{k+2}{k(k+4)}$$

EXAMPLES

To illustrate the use of graph, Figure 1, and the method of adding solid angles due to both symmetrical and non-uniform arrays of solution containers, two examples are provided.

Example 1:

Find the approximate solid angle subtended at the central drum of a rectangular array of twenty-five by the remainder of the array with the drums separated by one diameter. The height of the drum is 50 cm. and the radius 25 cm.

We consider one quarter of the system, and classify drums according to types. Also note $H/R = 2$. We use Figure 1, with a calculation table. A total of 8 drums are shielded, i.e., the central drum does not "see" them.

Type	k	A/h	B/h	Ω_f	Number of Type	Total
A	2	.577	.567	.028	4	.112
B	$4\sqrt{2} - 2$	.378	.404	.013	4	.052
C	$4\sqrt{5} - 2$	.229	.235	.004	8	.032

TOTAL .196

Thus the total fractional solid angle is approximately 0.20.

Example 2:

Find the value of the largest solid angle subtended at any solution container by the other ones in a room containing seven vessels arranged as shown in the sketch.

Plan

Elevation

12X44

It is apparent that reactor C-1 "sees" the greatest solid angle; hence, in view of the large separations, calculations are made with measurements from its center to diametral planes of the other vessels. We tabulate the necessary data and use Figure 1, as needed.

Container	Distance Center of C-1 to Diametral Plane, H (cm)	Width of Plane A (cm)	Height of Plane B (cm)	A/H	B/H	Ω_f
B	300	100	250	.333	.833	.021
A-1	250	30	60	.120	.240	.001
A-2	200	30	60	.150	.300	.003
A-3	175	30	60	.171	.343	.004
C-2	200	60	160	.300	.750	.017
C-3	200	60	100	.300	.500	.012
						.058

Thus the total fractional solid angle is .058.

Raymond Murray
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George H. Schmidt
George H. Schmidt

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Figure I
Fractional Solid Angle - Point to Plane

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