UNCLASSIFIED**MASTER****HYDROMAGNETIC INSTABILITIES CAUSED BY
RUNAWAY ELECTRONS**

by

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The possibility of the excitation of hydromagnetic instabilities caused by a beam of runaway electrons is investigated. It is found that if the wave length and frequency of the wave are such that the Doppler shift of the frequency due to the runaways' motion causes the electrons to see their cyclotron frequency, and if the electric field rotates in the same direction as the electrons do about the confining magnetic field, then an instability occurs.

It has been observed in the operation of the B-1 machine at Matterhorn that when the accelerating voltage is crowbarred during the rise of the current, the current time plot is similar to that shown in Fig. 1. Upon crowbar, the current immediately drops to a value which is from 5 to 10 per cent of the maximum current. It continues at this level for a considerable length of time (several milliseconds) and then goes to zero by passing through a series of steps. It seems likely that when $t > t_2$, the current is due to runaway electrons which are produced during the application of the voltage. The steps must be due to some kind of instability in the stream of runaway electrons. Tom Coor has suggested that this instability might be due to the catastrophic generation of hydromagnetic waves by the runaways.

The problem has been investigated both from the linearized Boltzmann equation, and from the linearized hydrodynamic equations in the limit where the stress tensor is negligible and the various classes of particles in the equilibrium can be considered to be monoenergetic streams moving along the equilibrium magnetic field. The two methods lead to essentially the same results when the spread in velocity of the runaway electrons is small compared with the phase velocity of the wave under consideration. The latter analysis is far simpler analytically and more easily amenable to physical interpretation. Consequently the following discussion will be confined to it.

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The equations of motion in E. M. U. are given in equations 1 through 5.

$$m_i \left[\frac{\partial \underline{v}_i}{\partial t} + \underline{v}_i \cdot \frac{\partial \underline{v}_i}{\partial z} \right] = \frac{e_i}{c} (\underline{E} + \underline{v}_i \times \underline{B}_0) + \frac{e_i \underline{v}_i}{c} \times \underline{B} \quad (1)$$

$$\nabla \cdot \underline{E} = 4\pi c \sum_i e_i (n_i - n_{i0}) \quad (2)$$

$$\nabla \cdot \underline{B} = 0 \quad (3)$$

$$\nabla \times \underline{B} = \frac{1}{c} \frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c} \sum_i n_{i0} e_i \underline{v}_i \quad (4)$$

$$\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} \quad (5)$$

Equation (1) is the equation of motion of the i th beam; it may be either a beam of ions or electrons. The quantities entering in this equation have the following meanings: m_i is the mass of the particles in the i th beam; $\underline{v}_i$ is the velocity of the particles in the i th beam due to a wave or instability; $\underline{v}_i$ is the zero order velocity of the particles in the i th beam and is taken to be in the same direction as the zero order magnetic field $\underline{B}_0 = B_0 \underline{e}_z$; e_i is the charge on the particles in the i th beam; $\underline{E}$ and $\underline{B}$ are the (first order) electric and magnetic fields due to the wave and/or instability; Equations 2 through 5 are Maxwell's equations; m_i and n_{i0} are respectively the first and zeroth order number densities of the i th beam. These equations are correct to first order in the perturbations. They are capable of taking into account thermal motions along the lines but they do not include any zero order motions perpendicular to the lines. The normal modes of interest are circularly polarized transverse waves traveling along the zero order magnetic field. That is, $\underline{v}$, $\underline{E}$ and $\underline{B}$ are perpendicular to $\underline{B}_0$ and of the form

$$\underline{v}_i = \underline{v}_i e^{i(\omega t - kz)}$$

$$\underline{E} = \underline{E} e^{i(\omega t - kz)}$$

$$\underline{B} = \underline{B} e^{i(\omega t - kz)}$$

$$\underline{B}_0 \parallel \underline{z}, \underline{v}_i \perp \underline{B}_0, \underline{E} \perp \underline{B}_0, \underline{B} \perp \underline{B}_0$$

When these equations are substituted in equations (1) through (5) and the algebra is carried out, one finds the following results.

$$c^2 k^2 = \omega^2 \cdot \sum_i \frac{\omega_{pi}^2 [\omega - k V_i]}{[\omega - k V_i \pm \omega_{ci}]^2} \quad (6)$$

$$\omega_{pi} = \frac{4\pi e^2 n_{oi}}{m_i} \quad , \quad \omega_{ci} = \frac{e B_0}{m_i c}$$

$$v_i = \frac{\frac{B_0 e_i}{m_i c} \left[\frac{\omega}{k} - v_i \right]}{[\omega - k V_i \pm \omega_{ci}]} \quad (7)$$

The upper sign $\pm$ is to be taken for positive ions and the lower sign for electrons when the wave is right circularly polarized with respect to B_0 , and vice-versa when it is left circularly polarized.

Observe from eq. (7), which gives the perturbed velocity in terms of the perturbed magnetic field, that there is a singularity when the denominator of the right hand side vanishes. Now kV_i is just the Doppler shift in the wave frequency seen by the particles. Thus the singularity occurs when the particles in the frame of reference which moves along B_0 with them see the wave as having their gyration frequency, the polarization of the wave being such that its associated electric field is gyrating in phase with the particles. This, physically, is just the situation where one would expect a maximum coupling between wave and particles. Hence, if for a given k there is a frequency ω which satisfies eq. (6), and which almost satisfies $\omega - k v \pm \omega_{ci} = 0$, one anticipates the possibility of an instability.

The simplest way to analyze the dispersion relation is to make a plot similar to that shown in Figure 2. Fig. 2 represents the dispersion relation when there is one beam of fast electrons passing through a cold plasma. The quantity ϵ is the ratio of the number density of electrons in the beam to the number density of electrons in the main plasma. ϵ is assumed to be small compared to 1.

The curves in Fig. 2 are obtained by fixing k and then plotting the left and right hand sides of the dispersion relation separately vs. ω . The left hand side is ω^2 and yields simply the parabola shown in the figure. The other curve is the right hand side of the dispersion relation.² It has singularities at $-\omega_{ce}$, ω_{ci} and $kV - \omega_{ce}$. It crosses the curve ω^2 at five places. Each of these is a possible ω for the given k . Since eq. (8) is equivalent to seeking the roots of a polynomial, if one reduces it to a common denominator, these are all the roots of the equation.

² A dispersion relation of the form of 6 has been obtained by J.R. Pierce, Traveling Wave Tubes, D. VanNostrand Co. Inc., N. Y. Chapter XIII.

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Consider now what happens when either the velocity of the beam or k is increased so that $kV - \omega_{ce}$ moves towards ω_{ci} . A situation like that shown in Fig. 3 will ultimately occur. The diagram is very similar to Fig. 2. However, the region of the curve representing the right hand side of the dispersion relation and lying between $kV - \omega_{ce}$ and ω_{ci} has been compressed and its maximum has become lower. There will be a value of V for which it no longer intersects the parabola. When this happens, one loses two real roots and gains two complex conjugate roots. One of these represents a growing wave, the other a damped wave. The damped wave is just the reverse motion of the growing wave. If one has an initial small disturbance it will be composed of both the growing and damped waves. However, after a long time only the growing wave will be noticeable.

When the density of the fast beam of electrons is very low, then the value of $\omega - kV$ for which the instability occurs is very nearly equal to $-\omega_{ce}$. This means that the electrons see their cyclotron frequency since $\omega - kV$ is the Doppler shifted frequency for the beam. Because of this, the perturbation of the beam is very large. That is, v_1 for the beam is very large. If one computes the ratio of the energy going into the beam as compared to that going into the main body of the plasma one finds that it is in the ratio of the ion mass to the electron mass. That is the beam gets by far the larger part of the instability energy. Since this energy must be supplied from the energy of motion of the beam along B_0 , the process is essentially one of scattering the beam by the wave.

When the thermal motion of the ions and the velocity spread of the fast electrons is taken into account the instabilities are modified by Landau damping. The resulting instabilities will grow more slowly or if the velocity spreads are large enough they will not grow at all.

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Figure 1.

$$\omega^2 = c^2 k^2 + \frac{\omega_{pi}^2 \omega}{\omega - \omega_{ci}} + \frac{\omega_{pe}^2 \omega}{\omega + \omega_{ce}} + \frac{\epsilon \omega_{pe}^2 [\omega - kV]}{\omega - kV + \omega_{ce}}$$

Fig. 2

Fig. 3

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