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**ELEMENTARY THEORY OF NEUTRON SOURCES
IN REACTORS**

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ELEMENTARY THEORY
OF
NEUTRON SOURCES IN REACTORS

A. V. H. Masket

Little is lost in principle in the following discussion if we employ elementary diffusion rather than transport theory, consider a bare reactor rather than a reflected reactor, and restrict ourselves to one group theory - all neutrons are assumed to be created and absorbed at thermal energy. In addition, only the steady state of the reactor - the equilibrium condition after a sufficiently long time following any change in the reactor such as the introduction of a neutron source - will be considered. Under these general conditions, the flux in the reactor is determined as the solution of a boundary value problem in differential equations given by the following:

$$(A) \quad \begin{cases} \Delta \varphi(r^0) + B^2 \varphi(r^0) + S(r^0) = 0, \\ \varphi(r_s^0) = 0, \quad B^2 > 0, \quad S \geq 0, \\ \varphi(r^0) \text{ is non-negative definite within } S. \end{cases}$$

The symbols in (A) have these definitions:

$\varphi(r^0)$ is the thermal neutron flux, a scalar function of the position vector, r^0 , inside the reactor;

$$\Delta \varphi(r^0) = \nabla^2 \varphi(r^0) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \varphi(r^0)$$

is proportional to the net rate of loss of neutrons from the

infinitesimal volume element at the position $r^0(x, y, z)$ by diffusion;

$B^2 \varphi(r^0)$ is proportional to the net rate of gain of neutrons in the infinitesimal volume element at $r^0(x, y, z)$ inside the reactor as a result of neutron creation by fissionable material and losses due to absorption;

B^2 is a positive definite constant associated with the material content of a homogeneous reactor called the "material buckling" or "material laplacian" of the reactor to be distinguished from a constant known as the "geometrical buckling" or "geometrical laplacian" associated with the shape of the reactor;

$S(r^0)$ is proportional to the rate of neutron production in the infinitesimal volume element at $r^0(x, y, z)$ inside the reactor by non-fissionable sources and is a non-negative scalar function of position vector $r^0(x, y, z)$ everywhere inside the reactor.

$\varphi(r_s)$ represents the magnitude of the flux on the (extrapolated) boundary or bounding surface, S , of the reactor which is designated by the position vector, r_s .

The neutron balance equation in (A),

$$\Delta \varphi + B^2 \varphi + S = 0,$$

is an inhomogeneous equation which is compatible with the boundary condition

$$\varphi(r_2) = 0,$$

that is to say, the solution of the balance equation exists and is finite and vanishes on $\mathcal{S}$, provided certain conditions are placed on B^2 and S . We must take S to be non-negative definite if it is to represent a source density and not a sink density anywhere in the reactor. As for B^2 it must be a positive definite constant if the reactor is to show any neutron multiplication, that is, if the probability for absorption producing fission is large enough to overcome non-fissionable absorption of any kind. The central problem is therefore to determine just what range of values B^2 must have in the practical case of operating a reactor. To do this we find the general representation of the solution for the problem (A) in what follows.

Consider first the problem for the critical reactor:

$$(B) \quad \begin{cases} \Delta \psi(r) + B^2 \psi(r) = 0, \\ \psi(r_2) = 0, \psi(r) \not\equiv 0, B^2 > 0, \\ \psi(r) \text{ is non-negative definite within } \mathcal{S}, \end{cases}$$

defined over the same reactor as (A). We know that this problem has a countably infinite number of positive discrete values of B^2 which we designate as $\{B_k^2\}$ (and which we can arrange in

ascending order of magnitude: $0 < B_1^2 < B_2^2 < \dots < B_k^2 < \dots < \infty$,
corresponding to which there is the sequence of linearly in-
dependent functions $\{\psi_k\}$ such that for each (B_k, ψ_k)
the problem is solved:

$$(B^1) \quad \begin{cases} \Delta \psi_k(r) + B_k^2 \psi_k(r) = 0, \\ \psi_k(r_j) = 0, \psi_k(r_0) \neq 0, B_k^2 > 0. \end{cases}$$

However, of all the solutions of the sequence $\{\psi_k\}$, only ψ_1 ,
corresponding to the smallest of the sequence $\{B_k^2\}$, namely B_1^2 ,
according to our arrangement, does not change sign anywhere inside
the (extrapolated) boundary $\mathcal{S}$ of the region defined by the
reactor. (The proof that such a positive definite solution in
the interior exists for arbitrary shapes is a mathematical prob-
lem which we shall assume and may be found in mathematical
treatises.) Notice further, that, because problem (B) is com-
pletely homogeneous, each solution ψ_k may be multiplied by
an arbitrary constant, positive or negative, and still remain a
perfectly valid mathematical solution. Physically, however, the
only acceptable solution can be a non-negative solution for the
flux. Consequently, we take as the only physically possible
solution for (B) function:

$$(0) \quad c \cdot \Psi(B_1^2, r) = c \cdot \frac{\psi_1(r)}{\left| \int_V \psi_1^2(r) dV \right|} = c \cdot \Psi_1$$

where Ψ_1 is in normalized form and c is an arbitrary non-negative constant.

The constant, B_1^2 , is called the geometrical buckling or geometrical laplacian mentioned above. It is the only value of B^2 , other than zero, which can be used in the homogeneous equation of (B) so that the solution can fit the conditions of vanishing on the boundary and being non-negative in the interior. We shall call $\Psi(B_1^2, r^0)$, the fundamental solution. Similarly, we normalize all other solutions of the sequence by writing:

$$\Psi_k = \Psi(B_k^2, r^0) = \Psi_k(r^0) \cdot \left| \int_V \Psi_k^2(r^0) dV \right|^{-1/2}.$$

Hence, the sequence $\{\Psi_k\}$ constitutes an orthonormal set of functions in terms of which any piecewise continuous function may be represented in the Fourier manner:

$$(D) \quad f(r^0) = \sum_{n=1}^{\infty} \Psi(B_n^2, r^0) \int_V f(r^0) \Psi(B_n^2, r^0) dV.$$

Consider next the original problem (A). In accordance with (D), let us assume:

$$\varphi = \sum_{n=1}^{\infty} b_n \Psi_n, \quad S = \sum_{n=1}^{\infty} \Psi_n \int_V \Psi_n S dV.$$

We find from (A) that

$$\sum_{n=1}^{\infty} \left[b_n (-B_n^2 + B^2) + \int_V \Psi_n S dV \right] \Psi_n = 0,$$

which requires

$$b_n = \frac{1}{B_n^2 - B^2} \int_V \Psi_n S dV, \quad n=1,2,3,\dots$$

Consequently

$$(B) \quad \varphi(r) = \sum_{n=1}^{\infty} \frac{\Psi(B_n^2, r)}{B_n^2 - B^2} \int_V \Psi_n(B_n^2, r) S(r) dV$$

is a representation of the desired solution for (A). The mathematical properties of the solution (B) will determine what values of B^2 are physically acceptable for the operation of a reactor in a steady state.

Case 1: $0 < B^2 < B_1^2 < B_2^2 < \dots \infty$, The Sub-critical Reactor.

In this case, we have taken the material buckling B^2 , as determined by the material of the reactor, to be less than the geometrical buckling B_1^2 , as determined by the shape of the reactor, so that the material buckling is less than any of the mathematically permissible constants B_k^2 of the sequence $\{B_k^2\}$. This means that each term of the infinite series

$$(F) \quad \sum_{n=1}^{\infty} \Psi(B_n^2, \rho^0) \int_V \Psi(B_n^2, \rho^0) S(\rho^0) dV$$

is divided by a positive constant, $B_k^2 - B^2$, and these new constants form an infinite sequence $\{B_k^2 - B^2\}$ with unbounded limit just as does the sequence $\{B_k^2\}$. Now the infinite series in (F) is just the development of the source function $S(\rho^0)$ in the Fourier manner:

$$(F1) \quad S = \sum_{n=1}^{\infty} \Psi_n \int_V \Psi_n S dV$$

which is either uniformly or conditionally convergent to a non-negative function by hypothesis. In the series (E), each term of the series in (F1) has been divided by an ever increasing constant so that the convergence must be improved in (E) over that in (F1) to a non-negative definite function, in fact, uniformly convergent. We have, therefore, the important result:

When a finite, homogeneous, bare reactor contains a distributed non-negative finite source of neutrons, with a material buckling which is positive and less than the geometrical buckling, the final steady state of the flux in the reactor is finite everywhere in the reactor.

We obtain two further conclusions from this analysis. First, suppose $(B_1^2 - B^2) \longrightarrow 0$, then the first term of (B) which is necessarily positive, becomes indefinitely large, in fact, as large as one pleases as $B_1^2 - B^2$ is made as small as one pleases. The question arises whether the steady state flux also increases indefinitely. We prove the answer is affirmative, for consider the integral

$$\begin{aligned} & \int_V \Psi_1 \phi dV \\ (G) = & \int_V \Psi_1 \left[\sum_{n=1}^{\infty} \frac{\Psi_n}{B_n^2 - B^2} \int_V \Psi_n S dV \right] dV \\ = & \frac{1}{B_1^2 - B^2} \int_V \Psi_1 S dV \end{aligned}$$

which follows from the orthonormality relation

$$(H) \quad \int_V \Psi(B_m^2, r_0) \Psi(B_n^2, r_0) dV = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases}$$

among the functions of the sequence $\{\Psi_k\}$. The result expressed by (6) is the analytical formulation of the weighted steady state flux integrated over the reactor. The weighting function is the fixed fundamental solution $\Psi(B_1^2, r^0)$, non-negative definite. Hence, the result of the integration is that, since

$$\frac{1}{B_1^2 - B^2} \int_V \Psi_1 S dV$$

can become as large as one pleases by making B^2 sufficiently close to B_1^2 , the magnitude of φ must increase as large as one pleases and, being a necessarily continuous function, it will increase everywhere except on $\mathcal{S}$ as long as S is non-negative definite. Thus, any large allowable flux level in a sub-critical reactor can be attained by use of a source if the material buckling is properly chosen.

Next, suppose $B^2 = 0$. Then, the solution for the problem

$$(A^1) \quad \begin{cases} \Delta \varphi(r^0) + S(r^0) = 0 \\ \varphi(r_s^0) = 0 \end{cases}$$

is

$$(E^1) \quad \varphi(r^0) = \sum_{n=1}^{\infty} \frac{\Psi(B_n^2, r^0)}{B_n^2} \int_V \Psi(B_n^2, r^0) S(r^0) dV$$

and gives the flux distribution in the equivalent of a pure diffuser of thermal neutrons which has a perfectly absorbing boundary. As this has no illustration in practice, it is only of academic interest.

A source of neutrons other than fissionable material is seen to be of immediate practical utility in approaching a critical state for finite (relatively small), enriched reactors because it enables the operator to determine relatively quickly and conveniently how close the reactor may be to criticality without the danger associated with exceeding it. We mention, in passing, the well-known, but never explicitly stated, fact that a sub-critical pile cannot be made critical by a non-fissionable source of neutrons no matter how dense that source may be.

The above analysis indicates that any practical flux level may be achieved in a sub-critical reactor by variation of B^2 , B_1^2 , or B .

The material buckling, B^2 , is most easily varied (although not homogeneously) by means of control rods of high thermal neutron cross-section for "thermal" reactors. Truly homogeneous fluid reactors may conceivably have the material buckling varied by change in concentration of the fissionable isotope. Geometrical buckling, B_1^2 , is controlled readily in a fluid or slurry pile but this is not feasible for fixed reactors of thermal type.

However, in reactors called fast or intermediate, which operate

essentially on neutrons of epithermal energies for which there are no effective absorbers other than fissionable materials, control is mandatory by variation of B_1^2 . Essentially this is accomplished by bringing sub-critical portions of the reactor closer or farther apart. Finally, the variation of S itself is the least practical but its location in the reactor is quite effective in governing the effectiveness of neutron supply rate.

We may summarize the discussion in a concise manner. Let us suppose that on the whole a reactor is characterized by generating k neutrons net following the introduction of one neutron counting all possible phenomena. The consequence will be that eventually there will have been generated

$$k + k^2 + \dots + k^n + \dots = K$$

neutrons. This sum is convergent if $k < 1$, and

$$(I) \quad K = (1 - k)^{-1}k, \quad k < 1,$$

is called the multiplication of the reactor. Hence, when a reactor is sub-critical, $k < 1$, the introduction of a source producing S neutrons/second will eventually result in the reactor producing the steady state

$$(J) \quad A = K \cdot S \text{ neutrons/second} = \text{Activity},$$

which will have any desired value by proper choice of k . This gives the important result again that a non-fissionable source of neutrons permits the achievement of any flux level without having to reach criticality, $k = 1$. The practical procedure in assembling a reactor to criticality is to plot the reciprocal activity versus mass of fissionable isotope. When the reciprocal activity, $1/A$, is sufficiently close to zero, the operator can estimate the critical mass by extrapolation. This will guide the final assembly.

Case 2: $B^2 = B_1^2$ or $k = 1$; The Critical Reactor

It is evident from (E) or (G) that $\varphi \rightarrow \infty$ as $B^2 \rightarrow B_1^2$ because Ψ_1 , and S are non-negative definite. We conclude that a critical reactor with a source ultimately results in catastrophe - infinite flux! The practical consequence is that when a reactor is brought to criticality, the source must be removed so that the reactor may be operated self-sustaining at the level attained.

We have seen that when the reactor is critical, then flux distribution is given by

$$(c) \quad \varphi(r^0) = c \cdot \Psi(B_1^2, r^0) \quad , \text{ Reactor Critical}$$

where c is an arbitrary non-negative constant, whereas when the reactor is sub-critical,

$$(2) \quad \varphi(r) = \sum_{n=1}^{\infty} \frac{\Phi_n}{B_n^2 - B^2} \int \Phi_n S dV, \text{ Reactor Sub-Critical.}$$

The distinction between these two cases primarily concerns the following:

- (a) safety,
- (b) flux distribution at activity or flux level desired,
- (c) control.

Other factors being equal, it is obvious that it is preferable to operate the reactor sub-critical for safety reasons - the sub-critical reactor cannot of itself increase in flux level to dangerous values. The flux distribution may become an important consideration in the operation of a reactor at lower activity levels because the source may distort the flux distribution considerably from that expressed by the fundamental solution (C).

In such cases, a reactor may have to be operated critical at very low flux levels in order to determine reactor characteristics.

At very high flux levels, a source, in the practical case, will introduce a local distribution distortion which is negligible, particularly if the source is small dimensionally. With regard to control at very high flux levels, when k is very nearly unity there is not much distinction to be made. However, at moderate levels the control problem when $k < 1$ is not as severe because

it may be feasible to overcome statistical fluctuations in reactor activity and still remain below criticality, if not entirely, at least a major fraction of the time.

As a final remark, we note that the analytic result $\varphi \rightarrow \infty$ as $B^2 \rightarrow B_1^2$ is consistent with the expression in (J):

$$\Lambda = (1 - k)^{-1} \cdot k \cdot S \rightarrow \infty \quad \text{as } k \rightarrow 1.$$

Also, if $S = 0$ and $k = 1$, then Λ is indeterminate which is another way of expressing the arbitrary nature of the constant $\underline{c}$ in (U). We cannot consider the possibility of an indeterminacy arising from

$$\int_V \Psi_1 S dV = 0, \quad B_1^2 = B^2,$$

in the first term of (E) because the function S would have to assume negative values in a finite region in order to be orthogonal to an everywhere non-negative function: $\Psi_1 \geq 0$, and an S with negative values is physically excluded.

Case 3: $B^2 > B_1^2$ or $k > 1$; The Super-Critical Reactor

For purposes of mathematical completeness, we distinguish between

the cases where $B_1^2 < B^2 \neq B_k^2$, $k = 2, 3, 4, \dots$ and $B_1^2 < B^2 = B_k^2$, $k = 2, 3,$

a) $B_1^2 < B^2 \neq B_k^2$, $k = 2, 3, 4, \dots$

While a mathematical solution exists for (A) in this case, it is physically meaningless because as we will show directly, the solution, while finite everywhere must take on negative values over a finite region not including the boundary $\mathcal{S}$. To prove this consider again the relation given above in (G):

$$(G) \quad \int_V \Psi_1 \varphi dV = \frac{1}{B_1^2 - B^2} \int_V \Psi_1 S dV.$$

When $B^2 > B_1^2$, the right-hand side of (G) is negative definite because Ψ_1 and S are non-negative definite. In the left-hand side, the factor φ must be negative over a finite region in order to have a negative integral when weighted by a non-negative function Ψ_1 . As this is physically impossible, it follows that a steady state balance equation such as (A) does not exist for a super-critical reactor. In other words, a super-critical reactor can be described properly only as a time-dependent system.

$$(b) \quad B_1^2 < B^2 \leq B_k^2, \quad k = 2 \text{ or } 3 \text{ or } 4 \text{ ---}$$

To handle this case, we must turn again to the general solution

$$(B) \quad \varphi(r^0) = \sum_{n=1}^{\infty} \frac{\Psi(B_n^2, r^0)}{B_n^2 - B^2} \int_V \Psi(B_n^2, r^0) S(r^0) dV.$$

Once again we find the solution is unbounded if S is finite, piecewise continuous provided, however, that

$$\int_V \Psi(B_k^1, r) \cdot S(r) dV \neq 0,$$

for, otherwise, the k^{th} term would become an indefinite form, $0/0$, whose limit could only be given for actual cases. Nevertheless, should the right hand side of (E) still have a finite result for the possibility that

$$\int_V \Psi(B_k^1, r) S(r) dV = 0, \quad B_k^1 = B^1,$$

the result would still be physically unacceptable by the same argument advanced above to show Φ would be required to be negative in a finite region. Therefore, from every viewpoint we find $B^2 > B_1^2$ leads to no possibility of a steady state.

The generalization of these results to multi-group theory and to the consideration of the transient state in a reactor with non-fissionable sources is beyond the original objective of this discussion and may be found elsewhere.

APPENDIX I

A simple illustration in concrete form of the result that the solution for (A) is positive definite in the interior and vanishes on the boundary is given here. Assume a slab reactor so that its thickness is very small compared with the length and breadth. Let the opposite large faces normal to the thickness

be located at $x = 0$ and $x = l$. We have here virtually one dimensional geometry and (A) can be written:

$$(1a) \quad \frac{d^2 \varphi(x)}{dx^2} + B^2 \varphi(x) + S = 0, \quad B > 0,$$

$$S = \text{constant},$$

$$(1b) \quad \varphi(0) = \varphi(l) = 0.$$

The solution of (1a) subject to conditions (1b) is readily verified to be

$$(2) \quad \varphi(x) = \frac{S}{B^2} \left\{ \frac{\cos B(x - l/2)}{\cos Bl/2} - 1 \right\}.$$

When the slab is critical, the sequence of permissible solutions is $\{\Psi(B_k^2, x)\} = \{(2/l)^{1/2} \sin \frac{k\pi x}{l}\}$ and the sequence of permissible values is

$$\{B_k^2\} = \left\{ \left(\frac{k\pi}{l} \right)^2 \right\}.$$

Hence, the fundamental solution is

$$(3a) \quad \Psi_1 = \left(2/l \right)^{1/2} \sin \frac{\pi x}{l}$$

and the geometrical buckling is

$$(3b) \quad B_1^2 = \pi^2/l^2.$$

For the subcritical slab $B < \pi/l$. Let us assume

$$B = \epsilon \cdot \pi/l, \quad 0 < \epsilon < 1$$

Substituting this into (2), we are led to the expression

$$(4) \quad \varphi(x) = \frac{S}{B^2} \left\{ \frac{\cos \frac{\pi(2x-l)}{2l}}{\cos \frac{\pi}{2}} - 1 \right\}$$

where it is seen on inspection that, for $0 \leq x \leq l$, $\varphi(x)$ vanishes for conditions (1b), satisfies (1a), and is positive definite for $0 < x < l$.

The formal equivalence of (2) and the series solution given by (B) is the relation

$$(5) \quad \frac{\cos B(x-l/2)}{\cos Bl/2} = 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x/l}{(2n-1) \left(\frac{(2n-1)^2 \pi^2}{l^2 B^2} - 1 \right)}$$

APPENDIX II

Another derivation of the relation (G)

$$\int_V \varphi(r) \Psi(B^2, r) dV = \frac{1}{B_1^2 - B^2} \int_V S(r) \Psi(B^2, r) dV$$

is given here by means of Green's Theorem in 3 dimensions. We start with the equations (A):

$$\begin{aligned} \Delta \varphi + B^2 \varphi + S &= 0, \\ \varphi(r_0) &= 0, \end{aligned}$$

and the fundamental solution from (B'):

$$\begin{aligned} \Delta \Psi_1 + B_1^2 \Psi_1 &= 0, \\ \Psi_1(B_1^2, r_0) &= 0. \end{aligned}$$

We form

$$\begin{aligned} \Psi_1 \Delta \varphi + B^2 \Psi_1 \varphi + \Psi_1 S &= 0, \\ \varphi \Delta \Psi_1 + B_1^2 \varphi \Psi_1 &= 0, \end{aligned}$$

and subtract and integrate over the volume of the reactor, so that

$$\int_V (\varphi \Delta \Psi_1 - \Psi_1 \Delta \varphi) dV + (B_1^2 - B^2) \int_V \varphi \Psi_1 dV = \int_V \Psi_1 S dV$$

By Green's Theorem:

$$\int_V (\varphi \Delta \Psi_1 - \Psi_1 \Delta \varphi) dV = \int_S n \cdot (\varphi \nabla \Psi_1 - \Psi_1 \nabla \varphi) dS = 0$$

because φ and Ψ_1 both vanish on the boundary S . This gives again (G):

$$\int_V \varphi \Psi_1 dV = \frac{1}{B_1^2 - B^2} \int_V \Psi_1 S dV$$

so that no use was made of the actual solution for φ given by (E).

A concluding remark in this connection is to point out that relation (G) cannot be used to establish the non-negative character of φ when $0 < B^2 < B_1^2$ because from (G), when the right hand side is positive, it does not necessarily follow that φ must be non-negative although it does necessarily follow that φ must be negative over a finite region when the right hand side is negative. Because (G) was not sufficient to establish the non-negative character of φ for non-negative S , we were forced to consider the representation in (E) in order to establish this obvious physical fact in a rigorous mathematical way.

APPENDIX III

For the sake of completeness we record the representation of the flux distribution in a bare reactor when the source is supposed to be contracted to a point but of such "infinite" density as to be a finite supply rate of neutrons. The simplest thing to do is to assume a supply rate of 1 neutron per second. How can such a source density be represented mathematically? From the point of view of the physicist such a function is extremely convenient, although not essential to the mathematician. The point-source has been introduced into physical literature by several physicists. According to P.A.M. Dirac, it is called the delta-function, and is defined to have the properties:

- (a) $\delta(r, r') = 0, \quad r \neq r';$
- (b) $\int_V \delta(r, r') dV = \begin{cases} 1, & \text{if } V \text{ includes } r', \\ 0, & \text{if } V \text{ does not include } r', \end{cases}$
- (c) $\int_V \delta(r, r') S(r) dV = \begin{cases} S(r'), & \text{if } V \text{ includes } r', \\ 0, & \text{if } V \text{ does not include } r'. \end{cases}$

A. Sommerfeld has called the point-source the "sackenfunction" and when applied to vibration theory as a force-function, it has been called a point-load by A. G. Webster.

We are led to consider the following problem:

$$(1) \quad \begin{cases} \Delta \Gamma(r, r') + B^2 \Gamma(r, r') + \delta(r, r') = 0, \\ \Gamma(r, r') = 0, \quad 0 < B^2 < B_1^2, \end{cases}$$

where $\Gamma(r, r')$ designates the flux distribution for the δ -function source at r' .

To obtain a solution, assume a series solution

$$(2) \quad \Gamma(r, r') = \sum_{n=1}^{\infty} \gamma_n \cdot \Psi(B_n^2, r),$$

in terms of the complete orthonormal sequence $\{\Psi(B_k^2, r)\}$.

This leads to the equation:

$$(3) \quad \sum_{n=1}^{\infty} (-\gamma_n B_n^2 \Psi_n + \gamma_n B^2 \Psi_n) + \delta(r, r') = 0.$$

Multiply (3) by Ψ_k and integrate over the reactor volume V :

$$\sum_{n=1}^{\infty} \gamma_n (B_n^2 - B^2) \int_V \Psi_k \Psi_n dV = \int_V \Psi_k \delta(r, r') dV.$$

By the orthonormality relation:

$$(4) \quad \int_V \Psi_k \Psi_n dV = \begin{cases} 1, & k = n, \\ 0, & k \neq n, \end{cases}$$

Hence, (4) together with (3) and

$$(5) \quad \int_V \Psi(B_k^2, r) \cdot \delta(r, r') dV = \Psi(B_k^2, r')$$

leads to $\gamma_k = \frac{\Psi(B_k^2, r')}{B_k^2 - B^2}$ and

$$(6) \quad \Gamma(r, r') = \sum_{n=1}^{\infty} \frac{\Psi(B_n^2, r) \cdot \Psi(B_n^2, r')}{B_n^2 - B^2}.$$

For the problem $B^2 = 0$,

$$\Delta G(r, r') + \delta(r, r') = 0,$$

$$G(r, r') = 0,$$

G would be the distribution for pure diffusion, and in a manner similar to the above, or directly from (6), we get

$$(7) \quad G(r, r') = \sum_{n=1}^{\infty} \frac{\Psi(B_n^2, r) \cdot \Psi(B_n^2, r')}{B_n^2}$$

The representations (6) and (7) are known as the Green's Functions or Influence Functions. The latter designation is expressive of the flux arising in the steady state at point r due to a unit source at r' . Hence, if the source in the reactor is continuously distributed with source density $S(r)$ throughout the reactor, the solutions (E) and (E') represent precisely the integrated effect of the distributed source density function $S(r)$. For this reason the solutions (E) and (E') are said to be represented sourcewise. The Green's Function is a very powerful method of handling boundary value problems of the sort discussed and can be extended to time dependent problems very effectively. Notice that both $G(r, r')$ and $\Gamma(r, r')$ are completely symmetric in r, r' .

The formal representation of $\delta(r, r')$ as a series is given by

$$(8) \quad \delta(r, r') = \sum_{n=1}^{\infty} \Psi(B_n^2, r) \cdot \Psi(B_n^2, r')$$

as is easy to verify by the property (a).

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