Magnetobound positronium and protonium

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The formation of magnetobound positronium and protonium is investigated via classical trajectory simulations of binary point charge interactions in an external magnetic field. A magnetobound state is a predicted pair-particle system that is temporarily bound due to the presence of an external magnetic field. The magnetic field constrains the motion of charged particles in the direction perpendicular to it, while allowing them to move freely in the parallel dimension. At large separations, each particle undergoes helical motion with an adiabatically invariant magnetic moment. As the charges approach each other, the electric interaction breaks the adiabatic constant of the motion, and the particles may temporarily behave as a highly correlated pair. The results of computer simulations of the fully three-dimensional trajectories of classical and non-relativistic point charges with the same mass, equal charge magnitude, and opposite sign are reported. The simulations show the formation of magnetobound positronium and protonium. The results yield formation cross sections, which are compared to analytical expressions. Additionally, the results reveal that magnetobound states drift across magnetic field lines. Observations on the drift distance, lifetime, and drift speed of simulated magnetobound states are reported. © 2014 AIP Publishing LLC.
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I. INTRODUCTION

A magnetobound state is a pair particle-antiparticle system of positive energy that is temporarily bound due to the presence of an external magnetic field. Here, the potential energy of the binary charge system is defined as zero when the particles are at infinite separation. These systems are studied via classical trajectory simulations. The three-dimensional numerical trajectory solutions show the formation of magnetobound states.

Quasibound states are a similar phenomenon to the magnetobound states predicted here but concern particles of disparate masses, while magnetobound states concern particles of equal masses. Quasibound states of antihydrogen (or hydrogen) are positive energy systems of a lepton and a nucleus that are temporarily bound due to the presence of a magnetic field.¹ Quasibound states may form when positrons and antiprotons interact in antihydrogen production experiments at the European Organization for Nuclear Research (CERN).

Magnetized positronium (Ps) atoms (negative energy states), magnetized protonium (Pn) atoms, and guiding center drift atoms are two-particle systems that are similar to the magnetobound states described here. A guiding center drift atom is a loosely bound state of a magnetized lepton and a nucleus.² A magnetized Ps atom³ is a loosely bound electron and positron pair in which, similarly to guiding center drift atoms,⁴ the lepton oscillates axially along a magnetic field line about the other particle (due to that particle's electric potential well) with a frequency that is much larger than the lepton's frequency of azimuthal drift rotation about the other particle. Magnetized Ps atoms have been reported when low

energy positrons are impacted onto a gas covered target or cold surfaces immersed in a magnetic field.^{3,5} In experiments being conducted at CERN that aim at the production and study of cold antihydrogen, the formation of Pn atoms has been identified inside strong magnetic fields.⁶ In addition, the interaction of positrons or antihydrogen atoms with the trap electrodes may lead to Ps atom production.⁷ Furthermore, the formation of excited states of Ps atoms inside strong magnetic fields has been identified as a possible avenue for the production of antihydrogen.^{8–10}

Section II describes the equations of motion and methodology used to carry out numerical studies regarding the formation of magnetobound states. Section III shows the derivation of an analytical expression for the formation cross section of magnetobound states. Section IV presents the results of the numerical studies and includes comparisons with analytical expressions. Also, a brief report is provided regarding some patterns in the behavior of magnetobound states. Section V discusses the possibility of using the formation of magnetobound states as an intermediate step in antihydrogen synthesis in the laboratory. Finally, some concluding remarks are provided in Sec. VI.

II. CLASSICAL TRAJECTORY SIMULATIONS

Consider a charged particle in an external magnetic field $\mathbf{B}$ and in the electric field produced by a point charge Q . Classically, its trajectory is governed by the Lorentz force

$$\ddot{\mathbf{r}}_u = \frac{\kappa_u q_u Q_u}{m_u} \frac{\hat{\mathbf{R}}}{R_u^2} + \frac{k_{Lu} q_u}{m_u} \dot{\mathbf{r}}_u \times \mathbf{B}_u. \quad (1)$$

Unnormalized coordinates, physical constants, and the magnetic field are denoted with the subscript u . m is mass, $q = Ze$ is charge, e is the charge of a positron, Z is a positive or

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negative integer, κ is the Coulomb force constant [e.g., $\kappa_u = (4\pi\epsilon_0)^{-1}$ in SI units, where ϵ_0 is the permittivity of free space], k_L is the Lorentz force constant (e.g., $k_{Lu} = 1$ in SI units), $\mathbf{R}$ is a vector identifying one particle's position relative to the other particle, $\hat{\mathbf{R}}$ is the corresponding unit vector, $\mathbf{r}$ is a vector identifying a particle's position from the origin of a coordinate system, and the dots denote derivatives with respect to the unnormalized time t_u . A dimensionless form of Eq. (1)

$$\mathbf{r}'' = \frac{q_u Q_u}{q_n^2} \frac{\hat{\mathbf{R}}}{R^2} + \frac{q_u}{q_n} \mathbf{r}' \times \mathbf{B}, \quad (2)$$

can be obtained, where the dimensionless system of coordinates and parameters are defined such that

$$\chi = \frac{\chi_u}{\chi_n}. \quad (3)$$

Here, the normalization factors are signified by the subscript n , and the primes denote derivatives with respect to the normalized time t . The normalization factors are

$$\begin{aligned} B_n &= \frac{\kappa_u^2 m_u^2 e_u^3}{k_{Lu} \hbar_u^3}, \\ r_n &= \frac{\hbar_u^2}{\kappa_u m_u e_u^2}, \\ t_n &= \frac{r_n}{\alpha c_u}, \end{aligned} \quad (4)$$

where $\hbar$ is the reduced Planck constant, α is the fine structure constant, c is the speed of light, and $\alpha c_u = \kappa_u e_u^2 / \hbar_u$. The normalization factor for charge, q_n , equals e_u , the unnormalized charge of the positron. Additionally, the normalization factor for mass, m_n , is the unnormalized mass of the positron or proton, depending on whether electrons and positrons or protons and antiprotons are being considered. Other normalization factors can be constructed from the normalization factors already specified. For instance, the normalization factors for speed, acceleration, force, and energy, respectively, are as follows: $v_n = r_n / t_n$, $a_n = r_n / t_n^2$, $F_n = m_n a_n$, and $K_n = F_n r_n$. Here, the SI system of units is employed. All normalization factors are written in terms of four unnormalized quantities: m_u , e_u , κ_u , and $\hbar_u$. Thus, the corresponding normalized quantities are unity, $m = e = \kappa = \hbar = 1$ and $k_L = 1$. The corresponding normalization factors are $m_n = m_u$, $q_n = e_u$, $\kappa_n = \kappa_u$, $\hbar_n = \hbar_u$, and $k_{Ln} = 1$.

Consider a system of two particles where $m = m_+ = m_-$, $Z_\pm = \pm 1$, $\mathbf{B} = B\hat{\mathbf{z}}$, and $\hat{\mathbf{z}}$ is a unit vector parallel to a uniform magnetic field. The $+$ and $-$ signs in the subscript $\pm$ correspond to the positive and negative charge, respectively. Let $\mathbf{R}(t) \equiv \mathbf{R}_+(t)$, $\hat{\mathbf{R}}_\pm(t) = \pm \hat{\mathbf{R}}(t)$, and $R_\pm^2(t) = R^2(t)$. Then, Eq. (2) becomes

$$\mathbf{r}''_\pm = \mp \left(\frac{\hat{\mathbf{R}}}{R^2} - \mathbf{r}'_\pm \times \mathbf{B} \right), \quad (5)$$

where $\mathbf{R}(t) \equiv \mathbf{r}_+(t) - \mathbf{r}_-(t)$, and $\mathbf{r}_\pm$ is the normalized vector from the origin to the positive or negative particle (i.e., $\mathbf{r}_+$ or

$\mathbf{r}_-$). Thus, in Cartesian coordinates, the equations of motion in each of the three orthogonal directions are

$$x''_\pm(t) = \mp \left(\frac{X(t)}{R^3(t)} - y'_\pm(t) B \right), \quad (6)$$

$$y''_\pm(t) = \mp \left(\frac{Y(t)}{R^3(t)} + x'_\pm(t) B \right), \quad (7)$$

and

$$z''_\pm(t) = \mp \frac{Z(t)}{R^3(t)}. \quad (8)$$

Here, $R(t) = \sqrt{X^2(t) + Y^2(t) + Z^2(t)}$, $X(t) = x_+(t) - x_-(t)$, $Y(t) = y_+(t) - y_-(t)$, and $Z(t) = z_+(t) - z_-(t)$.

The second order coupled differential equations of motion, Eqs. (6)–(8), are solved numerically. The solution is cast in parametric form and used to render a three-dimensional trajectory for each of the particles. The equations of motion have only one parameter, B , which is directly related to the magnetic field strength. Therefore, in order to produce a unique trajectory solution, six initial conditions and one parameter need to be specified. The results are presented as they pertain to the interaction of electron-like masses. Nonetheless, the results apply equally well to the interaction of a proton and an antiproton since Eqs. (6)–(8) apply to any system of two particles of equal mass, equal charge magnitude, and opposite charge sign.

Figure 1 depicts the initial positions of two particles with respect to a Cartesian coordinate system. Let quantities with a subscript 0 refer to values assigned at $t = 0$. Numerous binary trajectory simulations were carried out. Typical initial conditions are

$$\begin{aligned} x_{0\pm} &= \pm \frac{b}{2}, \\ y_{0\pm} &= \pm r_c, \\ z_{0\pm} &= \pm \frac{z_0}{2}, \\ x'_{0\pm} &= v_{\perp 0}, \\ y'_{0\pm} &= 0, \\ z'_{0\pm} &= \mp v_{z0}. \end{aligned} \quad (9)$$

Here, $v_{\perp 0}$ is the initial speed associated with motion perpendicular to the local magnetic field, b is the guiding center

FIG. 1. Schematic of initial positions and velocities of a particle and an anti-particle. The magnetic field direction is out of the page, and it is parallel to the z axis. The circle segments indicate the cyclotron orbit of the corresponding particle for a particular value of magnetic field strength. The dots indicate the initial positions of the particles in the xy -plane, and the $(+)$ and $(-)$ denote the charge sign of the corresponding particle. The axial position of the particles is $z_0/2$ for the positive particle and $-z_0/2$ for the negative particle.

impact parameter, r_c is the cyclotron radius, and z_0 is the initial axial separation between the particles. Particles would undergo cyclotron motion near $t=0$, provided that $v_{\perp 0} \neq 0$ and z_0 is sufficiently large. Cyclotron motion is the gyration that charged particles experience in a magnetic field. The axis of the cyclotron motion of a particle is referred to as the guiding center of that particle.

The initial phase of cyclotron motion is implicitly defined by the choice of $x_{0\pm}$ and $y_{0\pm}$. The initial phase of the particles was uniformly sampled in two of the three computational studies reported here.

The initial conditions are chosen such that the guiding center of each particle is on the xz -plane. The guiding center motion of a particle is the motion that is obtained when averaging over the cyclotron period. Guiding center theory hinges on the assumption that the magnetic moment associated with each particle's cyclotron motion is an adiabatic constant. The guiding center impact parameter b is the perpendicular (to the direction of the magnetic field $\hat{z}$) distance between the initial guiding centers of the charges. Hereafter, the term "impact parameter" is used in place of "guiding center impact parameter" for brevity. The initial cyclotron radius is $r_c = v_{\perp 0}/\omega_c$, where $\omega_c = k_L q B/m$ and $q = |q_{\pm}|$.

For large enough values of the initial axial separation, the results of the simulation studies are independent of z_0 . The initial conditions, Eq. (9), chosen are consistent with that expectation. The simulation begins when the particles are far enough from each other to ensure that they undergo guiding center motion. For very large separations, the electric potential between the particles is very small due to its inverse dependence on R . The simulated motion begins when $K_0 = (1 + \epsilon)K_{\infty}$, where $\epsilon \ll 1$ is a chosen constant, $K_0 = mv_0^2$, $K_{\infty} = mv_{\infty}^2$, K denotes the kinetic energy of the two-particle system, the subscript ∞ refers to quantities when the particles are an infinite distance apart, and the subscript zero denotes initial time, $t=0$ (i.e., at the beginning of the simulation). Energy conservation requires that $U_0 + K_0 = K_{\infty}$, where U denotes the potential energy of the guiding centers. Solving the energy conservation equation with $U_0 = -\kappa q^2/R_0$ gives

$$z_0 = \sqrt{\frac{\kappa^2 q^4}{\epsilon^2 m^2 v_{\infty}^4} - b^2}. \quad (10)$$

In deriving Eq. (10), the distance between the particles is approximated using the separation between their guiding centers. This is no more consequential than choosing a slightly different ϵ . For the simulations, $\epsilon = 1/100$ was chosen. Additionally, it was required that $b \geq r_c$. $\epsilon \ll 1$ ensures that $|U_0| \ll K_0$. This, along with the range of values of b chosen, ensures that $z_0 \gg b$. For all trajectory solutions, it was verified that $[v_{\perp}(\tau/5) - v_{\perp}(t=0)]/v_{\perp}(t=0) \lesssim 1/10\,000$. Here, τ is the simulation duration. $\tau = 2\tau_0$ is chosen, where $\tau_0 = z_0/v_{z0}$ is twice the time that the particles would require to reach zero axial separation if they did not interact with one another and v_{z0} is the axial speed of each particle at the beginning of the simulation.

FIG. 2. Example of a magnetobound state. The lines represent the trajectories of a particle and an antiparticle in a magnetic field, with the positive particle approaching from the positive z -direction and the negative particle approaching from the negative z -direction. Here, coordinates are normalized according to Eq. (3).

Figure 2 shows a sample solution that resulted in a magnetobound state. The equations of motion are solved numerically for $0 \leq t \leq \tau$. The numerical differential equation solver utilized to solve the equations of motion progressively reduces the step size until it is able to track the solution well over the entire region.¹¹ A magnetobound state is defined to have been formed when both particles reverse the sign of their axial velocity component two or more times. An unmagnetized particle will experience such sign changes at most once. For a magnetobound state, that number can be much higher. This is illustrated in Fig. 3, which shows the axial position versus time for a sample solution.

Three numerical studies were carried out to investigate the formation of magnetobound states. Each study consists of seven series of simulations. Each series of simulations consists of numerous trajectory solutions to the equations of

FIG. 3. Plot of the normalized axial positions with respect to normalized time for two particles that form a magnetobound state. The normalization convention is the same as that of Fig. 2. Each line represents the trajectory of one of the particles, with the positive particle approaching from the positive z -direction and the negative particle approaching from the negative z -direction. Initially, the guiding centers of the particles lie on the xz -plane. Each time the trajectories cross the horizontal axis, the particles are at approximately zero axial separation.

motion for two leptons of opposite charge, equal charge magnitude, and the same mass in an external magnetic field. Each series of simulations differs from the other series only by the strength of the magnetic field. Each simulation within a series is identical to the other simulations in its series except for the initial impact parameter. b is increased in multiples of r_c from twice the cyclotron radius to b_{max} , which is chosen to be a length beyond which magnetobound states are not expected.

The initial conditions of the simulations impart equal kinetic energy to each particle. The first two studies also impart equal speeds in each of the three orthogonal dimensions for each particle. In contrast, the third study assigns most of the initial kinetic energy of each particle to that particle's axial velocity component.

The parameters and initial conditions for the first computational study are as follows: $1 \text{ T} \leq B_u \leq 7 \text{ T}$; $2r_c \leq b \leq b_{max}$, in increments of r_c ; $v_{z0} = v_0/\sqrt{3}$; $v_{\perp 0} = v_0\sqrt{2/3}$; $v_0 = v_\infty \sqrt{(1+\epsilon)}$; and $v_\infty = \sqrt{k_B T/m}$, where k_B is the Boltzmann constant and $T_u = 5 \text{ K}$. The second study considers $T_u = 15 \text{ K}$. The third study considers particles with $T_u = 15 \text{ K}$ and near-zero speed perpendicular to the magnetic field. In all other aspects, the third study is the same as the second study.

III. ANALYTICAL FORMATION CROSS SECTION

A. Formation cross section, mean free path, and formation timescale

Consider a particle and its antiparticle interacting in a strong magnetic field. Here, the particles are treated as non-relativistic classical point charges with $r_c \ll b$, and the magnetic field is assumed to be uniform and extend over all space. Let r_{mbs} be a formation impact parameter such that $b \leq r_{mbs}$ leads to a magnetobound state being formed (expressions for r_{mbs} will be derived in the following subsection). A formation cross section of the form

$$\sigma_{mbs} = \pi r_{mbs}^2 \quad (11)$$

is constructed.

Consider the interaction of a test particle and a mono-energetic collection of its corresponding anti-particles (field particles) uniformly distributed over space interacting in a strong magnetic field. The field particles are assumed to be uncorrelated and do not interact among themselves or drift relative to each other. Given a formation cross section of the form of Eq. (11), the probability that a magnetobound state will form during a sufficiently small time period Δt is

$$P_{mbs} = n\sigma_{mbs}\Delta s. \quad (12)$$

Here, Δs is the distance travelled by the test particle's guiding center relative to the guiding centers of the field particles during a time Δt , and n is the field particle density. The mean free path for the formation of a magnetobound state is arrived at by setting the probability in Eq. (12) equal to one,

$$l = (n\sigma_{mbs})^{-1} \quad (13)$$

(i.e., $\Delta s = l$ when $P_{mbs} = 1$). Once l is known, the formation timescale can be obtained as

$$t_{mbs} = l/g_{\parallel}, \quad (14)$$

where $g_{\parallel}$ is the relative speed between the guiding centers of the particles (which is assumed to be parallel to the magnetic field due to the strength of the magnetic field). Equation (14) is used in Sec. IV B for a comparison of timescales.

B. Formation impact parameter

The adiabatic invariance of the magnetic moments of charged particles in a constant magnetic field can be broken as the charges approach each other. Guiding center motion ceases, and the particles behave as a correlated system. Otherwise, the particles reflect or pass by one another axially, resuming guiding center motion at sufficient separation.

The formation impact parameter of magnetobound states, r_{mbs} , is hypothesized to be approximately equal to the distance between the particles at zero axial separation at which the magnitude of the electric force on each charge equals the maximum magnitude of the magnetic force on either particle. This would guarantee the breaking of the invariance of the magnetic moment. With the assumption $r_{mbs} \gg r_c$, the condition is expressed as $k_L q v_{\perp,b} B \approx \kappa q^2 / r_{mbs}^2$ or

$$r_{mbs} \approx \sqrt{\frac{\kappa q}{k_L B v_{\perp,b}}}. \quad (15)$$

Here, $v_{\perp,b}$ is the perpendicular speed of either particle at the point of zero axial separation (i.e., when the position vector from one charge to the other is perpendicular to the magnetic field). If different values of r_{mbs} are obtained depending on which particle is considered, the value that yields the largest r_{mbs} should be chosen. Particles of equal charge magnitude are being considered. Additionally, the guiding center impact parameter of the particles is assumed to remain unaffected during the portion of the trajectories prior to the breaking of the magnetic moment invariance. If the breaking of the magnetic moment invariance does not take place at approximately zero axial separation (i.e., $R = b$ in the guiding center approximation), then it is assumed to not occur at larger impact parameters since the particles experience the largest electric field at approximately zero axial separation.

The kinetic and potential energies of the particles will be considered using a coordinate system whose origin coincides with the center of mass of the system when the particles are an infinite distance apart, before they interact. (However, the present work does not utilize the center of mass equations or the reduction of a two particle system to a single reduced mass and a center of force.) The total energy of the system equals the instantaneous kinetic energy of the particles at infinity, $K_\infty = (1/2)m_1 v_{1,\infty}^2 + (1/2)m_2 v_{2,\infty}^2 = (1/2)\mu g_\infty^2$, where $\mu = m_1 m_2 / (m_1 + m_2)$. Here, g denotes the relative speed between the particles at infinity, m_1 and m_2 denote the masses of particle 1 and particle 2, respectively. If the particles have different masses, the subscript 2 denotes the particle with the largest mass. The particles are first considered to have disparate masses to illustrate the connection between the theory being presented and prior theory.

When the invariance of the magnetic moment of a particle is broken, the perpendicular speed of one or both

particles can change drastically. However, energy conservation limits the speeds of the particles such that

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{\kappa q^2}{R} = K_\infty, \quad (16)$$

where v_1 and v_2 are the speeds of particle 1 and particle 2, respectively. Thus, the maximum speed possible is the limit in which the energy $K_\infty + \kappa q^2/R$ is assigned to the lightest of the two masses at $R=b$, where the electric force is at its maximum. In this limit, the magnetic force is at its maximum if, additionally, $v_{1,b} = v_{\perp 1,b}$ (i.e., the total speed of particle 1 at a separation b from particle 2, $v_{1,b}$, equals its perpendicular speed at that same separation, $v_{\perp 1,b}$). Thus, with $m_2 \gg m_1$, $\mu \approx m_1$, the speed to utilize for the hypothesized condition for the formation impact parameter of magnetobound states is

$$v_{\perp,b}^2 \simeq g_\infty^2 + \frac{2\kappa q^2}{m_1 b}. \quad (17)$$

Using Eq. (17) in Eq. (15), one obtains the following expression for the formation impact parameter of magnetobound states, r_{mbs1} ,

$$\frac{\kappa q^2}{r_{mbs1}^2} = k_L q B \sqrt{g_\infty^2 + \frac{2\kappa q^2}{m_1 r_{mbs1}}}. \quad (18)$$

Hereafter, r_{mbsN} symbols with N being an even number refer to equal masses, while an odd N subscript refers to cases of disparate masses. Thus, the N subscript is used to differentiate different expressions for the magnetobound state formation impact parameter, which differ due to the different limiting cases being considered.

Equation (18) has two interesting limiting cases, that for which $g_\infty^2 \ll 2\kappa q^2/(m_1 r_{mbs})$ and the reverse. That is, these cases represent the extremes of how the initial kinetic energy of the particles in the center of mass frame at infinity, $(1/2)\mu g_\infty^2$, compares to the magnitude of the potential energy of the particles at the distance of closest approach, $\kappa q^2/r_{mbs}$. The first of these choices leads to

$$r_{mbs3} = \left(\frac{\kappa m_1}{2k_L^2 B^2} \right)^{1/3} \approx \left(\frac{\kappa m_1}{k_L^2 B^2} \right)^{1/3}. \quad (19)$$

The scale length given by Eq. (19) has been obtained in Ref. 2 in association with guiding center drift atoms. Such atoms satisfy the condition that the cyclotron frequency is much greater than the axial bounce frequency, which in turn is much greater than the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency.² This condition leads to the scale length given by Eq. (19),² which has been used with a different multiplicative factor in experimental reports pertaining magnetized Ps atoms.³ A timescale analysis leads to Eq. (19) in Ref. 1, which regards quasibound states of antihydrogen. It is interesting that Eq. (19) can be derived from different physical considerations: the comparison of maximum electric and magnetic forces experienced by the particles, or an analysis of timescales. Another interesting aspect of Eq. (19) is that it is independent of speed.

Considering disparate masses and $g_\infty^2 \gg 2\kappa q^2/(m_1 r_{mbs})$ in Eq. (18) leads to

$$r_{mbs5} = \sqrt{\frac{\kappa q}{k_L g_\infty B}}. \quad (20)$$

In Ref. 12, Eq. (20) was obtained for describing the formation of quasibound states of antihydrogen by comparison of maximum forces, but neglecting the change in potential energy.

For equal masses, $m_1 = m_2 = m$, $\mu = m/2$, and Eq. (16) becomes $m(v_1^2 + v_2^2)/2 - \kappa q^2/R = (1/4)mg_\infty^2$. The maximum speed for any particle in the binary system considered here must be equal-to-or-smaller-than the speed obtained when the kinetic energy of one of the particles is zero (e.g., $v_2 = 0$). Similarly to Eqs. (17) and (18), one obtains $v_{\perp,b}^2 \simeq g_\infty^2/2 + \frac{2\kappa q^2}{mb}$ and

$$\frac{\kappa q^2}{r_{mbs}^2} = k_L q B \sqrt{\frac{g_\infty^2}{2} + \frac{2\kappa q^2}{mr_{mbs}}}. \quad (21)$$

Equations (18) and (21) are quartic functions. They can be cast in the form $\mathcal{A}r_{mbs}^4 + \mathcal{B}r_{mbs}^3 + \mathcal{E} = 0$. Here, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{E}$ are real coefficients which guarantees real solutions. The exact solutions can be found, of which the positive real solution must be chosen. For Eq. (21), $\mathcal{A} = 1$, $\mathcal{B} = 4\kappa q^2/(mg_\infty^2)$, $\mathcal{E} = -2\kappa^2 q^2/(k_L^2 B^2 g_\infty^2)$. Thus,

$$r_{mbs} = \frac{1}{4} \left(-\mathcal{B} - \sqrt{\mathcal{B}^2 + \delta} + \sqrt{\frac{2\mathcal{B}^3}{\sqrt{\mathcal{B}^2 + \delta}} + 2\mathcal{B}^2 - \delta} \right), \quad (22)$$

where $\delta = 2^{13/3} 3^{-1/3} \mathcal{E} \gamma^{-1} + 2^{5/3} 3^{-2/3} \gamma$ and $\gamma = (9\mathcal{B}^2 \mathcal{E} + \sqrt{81\mathcal{B}^4 \mathcal{E}^2 - 768\mathcal{E}^3})^{1/3}$. Equation (22) will be evaluated and compared to the simulation results that will be discussed in Sec. IV.

Equation (21) also has two limiting cases. Considering the limit in which $g_\infty^2/2 \ll 2\kappa q^2/(mr_{mbs})$ leads to

$$r_{mbs2} = \left(\frac{\kappa m}{2k_L^2 B^2} \right)^{1/3}. \quad (23)$$

The limit $g_\infty^2/2 \gg 2\kappa q^2/(mr_{mbs})$ in Eq. (21) leads to

$$r_{mbs4} = 2^{1/4} \sqrt{\frac{\kappa q}{k_L g_\infty B}}, \quad (24)$$

which, neglecting a factor of $\sqrt[4]{2} \approx 1.19$, is the same as Eq. (20).

The theory presented here, when applied to disparate masses and certain limiting cases, has produced Eqs. (19) and (20), which have been previously found to apply to guiding center drift atoms and/or quasibound states. When the same theory has been applied to the case of a charged particle and its antiparticle, Eqs. (22), (23), and (24) have resulted. Equations (23) and (24) are more easily used than Eq. (22). These equations apply to magnetobound states, which have not been previously predicted or empirically

FIG. 4. Plot of the results of the numerical study regarding the formation impact parameter of magnetobound states considering $T_u = 5$ K. The vertical axis represents the normalized impact parameter and the horizontal axis represents the magnetic field strength in SI units. The normalization convention is the same as that of Fig. 2. The squares are the initial impact parameter below which at least 84% of trajectories resulted in magnetobound states. The dots are the impact parameter above which no magnetobound trajectories were observed. The dotted line is the prediction from Eq. (22). The solid line is the prediction from Eq. (23). The dotted-dashed line is the prediction from Eq. (24). No fitting parameter was used.

shown to our knowledge. In the following section, Eqs. (22), (23), and (24) are compared to numerical simulations.

IV. NUMERICAL RESULTS

A. Formation impact parameter

The first numerical study consisted of 360 classical trajectory simulations of binary point charge interactions in an external magnetic field. The results of this numerical study regarding the formation impact parameter are summarized in Fig. 4, where the horizontal axis represents magnetic field strength, and the vertical axis represents the initial impact parameter. The dotted line is the prediction from Eq. (22). The solid line is the prediction from Eq. (23). The dotted-dashed line is the prediction from Eq. (24). The solid dots and squares represent the numerical results for the formation impact parameter of magnetobound states. The solid squares mark the initial impact parameter below which at least 84% of trajectories resulted in magnetobound states, while the solid dots mark the impact parameter above which no magnetobound trajectories were observed. (This is similar to the antihydrogen quasibound states studied in Ref. 1, where most but not all trajectories below a certain threshold-impact parameter were found to result in quasibound states.) The initial phase of the particles was only varied for the trajectories that lie in the region between the squares and the dots. There, the initial phase of one particle was varied between

TABLE I. Formation impact parameter percent errors from the numerical study considering $T_u = 5$ K. The data chosen for the comparisons are that which is represented by the solid squares in Fig. 4.

Formation Cross Section	Avg. (%)	Max. (%)	ι	Avg. (%)	Max. (%)
Eq. (22)	74.4	76.4	3.82	5.9	8.4
Eq. (23)	73.9	76.0	3.83	6.5	9.2
Eq. (24)	31.6	39.6	1.49	5.1	10.1

zero and 2π in increments of $\pi/4$. The phase seemed to affect the formation of magnetobound states insofar as it may change the phase of each particle as they approach zero axial separation. Thus, the initial phase can change the particles' distance of closest approach by up to $4r_c$. The average and maximum error percentages between the data represented by the seven solid squares in Fig. 4 and Eqs. (22), (23), and (24) are summarized in Table I. The percent error is calculated as $\text{Error} = \frac{|\text{Numerical} - \text{Prediction}|}{\text{Numerical}} \times 100\%$, where Numerical represents the values observed in the numerical study and Prediction represents the values given by the corresponding analytical expression. Using a multiplicative fitting factor ι (i.e., $f_1 = \iota f_0$), all analytical predictions for the magnetobound state formation cross section agreed to within 10% with the numerical results. In the figures and in the equations, the fitting parameter is $\iota = 1$ while the error tables use a variety of values. A simple multiplicative factor is used with the analytical predictions because the electric-to-magnetic-force ratio at which the electric force between the charges breaks the adiabatic invariance of the magnetic moment of either particle (due to the external magnetic field) is not precisely known, and the breaking of this invariance forms the basis for the analytical predictions presented here.

A second numerical study was performed using a different value for the initial speeds of the particles, but otherwise the same as the first. The initial speeds of the particles were chosen to correspond to $T_u = 15$ K. All other parameters remained unchanged. The results of this study are presented in Fig. 5, which is similar to Fig. 4 in its labeling conventions. In a similar fashion to Table I, the average and maximum error percentages between Eqs. (22), (23), and (24) and the data represented by the seven solid squares in Fig. 5 are summarized in Table II.

Thus far, the computational studies assumed that the initial kinetic energy of the particles was equally divided between the orthogonal components of the motion when averaged over the initial cyclotron period. A final study was conducted in which the perpendicular component of the initial speed of each particle corresponded to 1/11th of the total initial speed, and the rest was allocated to the axial component of the motion. In all other aspects, the third study was identical to the second numerical study, with one exception: the initial phase of the particles was not varied. The

FIG. 5. Same as Fig. 4, except $T_u = 15$ K.

TABLE II. Formation impact parameter percent errors from the numerical study considering $T_u = 15$ K. The data chosen for the comparisons are that which is represented by the solid squares in Fig. 5.

Formation cross section	Avg. (%)	Max. (%)	ι	Avg. (%)	Max. (%)
Eq. (22)	69.0	71.2	3.07	8.4	11.9
Eq. (23)	68.4	70.9	2.98	9.7	13.4
Eq. (24)	37.2	41.7	1.57	5.1	8.5

increments of the impact parameter were very small for the third study. The initial impact parameter was increased in multiples of the initial cyclotron radius, which is very small due to the low initial perpendicular speeds of the particles.

The results of the third study, which pertains to particles with a small ratio of perpendicular to parallel initial speeds, ($v_{\perp 0}/v_0 = 1/11$), are presented in Fig. 6. This figure retains the labeling conventions of Figs. 4 and 5. Table III lists the average and maximum error percentages between the data represented by the seven solid squares in Fig. 6 and Eqs. (22)–(24). Here, the data points below which 84% are only shown because they were very close to the data points above which no magnetobound states were observed. An observation that is not shown in Fig. 6 or Table III (that is, for the study pertaining $T_u = 15$ K and $v_{z0} \gg v_{\perp 0}$) is that magnetobound states did not form if the initial impact parameter was below $7r_c$. Each of these cases resulted in a π -radian reflection of the particles. A similar observation was made for 12% of the cases with impact parameters below $7r_c$ in the study pertaining $T_u = 15$ K. For the study pertaining $T_u = 5$ K, magnetobound states were consistently observed for small impact parameters as small as $b = 2r_c$ (the lower limit of the range).

B. Lifetime, drift, and drift speed

All magnetobound states observed in the simulations drifted across the magnetic field. The drift, drift speed, and lifetime of a subset of the magnetobound trajectories of the

FIG. 6. Plot of the results of the numerical study pertaining to particles initially with $v_{z0} \gg v_{\perp 0}$. The vertical axis represents the normalized impact parameter and the horizontal axis represents the magnetic field strength in SI units. The normalization convention is the same as that of Fig. 2. The squares are the initial impact parameter below which at least 84% of trajectories resulted in magnetobound states. The dotted line is the prediction from Eq. (22). The solid line is the prediction from Eq. (23). The dotted-dashed line is the prediction from Eq. (24). No fitting parameter was used.

TABLE III. Formation impact parameter percent errors from the numerical study considering $T_u = 15$ K and $v_{z0} \gg v_{\perp 0}$. The data chosen for the comparisons are that which is represented by the solid squares in Fig. 6.

Formation cross section	Avg. (%)	Max. (%)	ι	Avg. (%)	Max. (%)
Eq. (22)	65.5	69.1	2.87	7.2	11.3
Eq. (23)	65.7	68.7	2.79	8.2	12.8
Eq. (24)	32.0	34.4	1.46	2.7	4.2

$T_u = 5$ K study were analyzed. Limitations of the numerical root solver employed required the lifetimes of magnetobound states to also be checked by eye. Consequently, the number of magnetobound trajectories analyzed for these features is small, and the sample is not regarded as statistically significant. The simulated data to be discussed in this section are shown in Figs. 7–9 and were obtained from analyzing 98 trajectory simulations that resulted in magnetobound states. Of these, 10 trajectories were analyzed for the data points associated with $B_u = 1$ T, 16 trajectories for $B_u = 2$ T, 11 for

FIG. 7. Plot of the ratio τ_{mbs}/τ_c versus the strength of the magnetic field as observed in the $T_u = 5$ K computational study. Here, τ_{mbs} is the average lifetime of the magnetobound states and it is defined as the time between the first and last time that the charges of a magnetobound state pass each other axially. τ_c is the initial cyclotron period, which is itself a function of the magnetic field strength.

FIG. 8. Plot of the average drift speed (perpendicular to the magnetic field) of magnetobound states as a fraction of the total initial speed of each particle versus magnetic field strength in SI units as observed in the $T_u = 5$ K computational study. The dots represent the average drift speed of the magnetobound states that formed at that particular value of the magnetic field strength.

FIG. 9. Plot of the average drift perpendicular to the magnetic field of a magnetobound state as a function of the magnetic field strength as observed in the $T_u = 5$ K computational study. The dots represent the average drift of the magnetobound states that formed at that particular value of the magnetic field strength.

$B_u = 3$ T, 13 for $B_u = 4$ T, 12 for $B_u = 5$ T, 12 for $B_u = 6$ T, and 24 trajectories were analyzed for the $B_u = 7$ T data points. For each value of magnetic field strength, the values of b were chosen to approximately uniformly cover the range between $b = 2r_c$ and the points marked by the solid squares in Fig. 4.

Figure 7 shows the ratio of τ_{mbs}/τ_c , where τ_c is the initial cyclotron period. The lifetime of a magnetobound state, τ_{mbs} , was defined as the time between the first and last time that the particles of a magnetobound state reverse their axial motion. The average τ_{mbs}/τ_c was calculated to be approximately 3.3×10^3 . The average τ_{mbs} was calculated to be between 346 (for $B_u = 1$ T) and 6000 (for $B_u = 7$ T) times larger than τ_c . The average τ_{mbs} was calculated to be about three orders of magnitude smaller than t_{mbs} , the magnetobound state formation time as given by Eq. (14) considering $n = 10^{12} \text{ m}^{-3}$.

The drift of a magnetobound state across magnetic field lines in a direction perpendicular to the plane of the initial guiding centers can be seen in Fig. 2. As a system, magnetobound states are neutral. Every magnetobound state observed in the simulations displayed said drift. The direction of the magnetobound state drift seemed to be determined by the initial conditions. That is, if the direction of the magnetic field defines the positive z axis and the plane of the initial guiding centers defines the xz -plane, then whether the particles drifted in the positive or negative y -direction depended only upon whether the positive charge approached the yz -plane from the positive or negative z -direction (and the negative charge thus approached the mid-plane from the complementary direction). These statements regarding the direction of said drift are an observation that held true for every magnetobound state seen in the simulations, and they were checked by reversing the initial conditions of about twenty trajectory simulations that led to magnetobound states.

Figure 8 shows the average drift speed observed in the simulations for each value of B_u , while Fig. 9 shows the total drift. The speed of the drift of magnetobound states across magnetic field lines is a large fraction of the initial speed of the particles. This drift may be detrimental to the radial confinement of the charges that form a magnetobound state.

V. ANTIHYDROGEN SYNTHESIS

Magnetobound states may be useful as an intermediate step in the production of neutral antimatter. Two scenarios will be very briefly speculated in this section.

In antihydrogen experiments being conducted at CERN, a nested Penning trap is used to confine antiprotons and positrons. In a nested Penning trap, radial confinement is provided by a magnetic field, and an electric field produced by biased cylindrical electrodes provides the axial confinement of the particles. A nested Penning trap uses five or more electrodes to produce nested electric wells of opposite sign. Therefore, charges of opposite sign can be stored in different sections of the trap. In order to achieve overlap between charges of opposite sign, the energy of one of the species is increased such that the particles gain enough kinetic energy to overcome the potential barrier separating them from the other species. Antihydrogen may be produced primarily via three-body recombination. However, the antihydrogen that is produced is energetic due to the process that is used to achieve overlap. This makes these antihydrogen atoms difficult to confine given the limited magnetic field well depth available to confine neutral atoms via their magnetic dipoles.

Consider a nested Penning trap, with antiprotons in the inner well and positrons in the outer wells. Allow the two species to remain at low temperatures and in their respective wells. Introduce an electron beam traveling axially through the trap. As the electron beam encounters the positrons of the well closest to the origin of the beam, magnetobound Ps may be produced. Magnetobound Ps may overcome the electric potential separating the positrons from the antiprotons. This is because magnetobound states are neutral systems. The magnetobound Ps would then travel axially, due the momentum provided by the beam's particle (and radially due to the drift discussed in the previous section), and interact with the species in the inner well. Thus, cold antihydrogen may be produced via three-body recombination between magnetobound Ps and the antiprotons of the inner well.

The radial drift observed in the simulations may be detrimental to the scenario discussed thus far. However, it may be useful for antihydrogen synthesis in a modified scheme. Consider a hollow antiproton plasma confined in a Penning trap, as well as an electron plasma confined in the volume within the inner radius of the antiproton plasma. If positrons are sent to stream through the electron plasma, magnetobound Ps may form. As the magnetobound Ps drifts axially and radially, it may undergo three-body recombination within the antiproton plasma and result in the production of cold antihydrogen atoms.

These schemes for achieving antihydrogen synthesis via magnetobound states utilize the ability of magnetobound states to drift across magnetic fields and possibly electric fields. In antihydrogen experiments such as those conducted at CERN, this may represent a practical way to overcome the difficulties related to matching the kinetic energy of the species in the outer wells to the values necessary to overcome the electric potential of the inner well without reaching levels that prevent the production of cold antihydrogen. Thus, further studies of these aspects of magnetobound states are

desirable. Additionally, further studies regarding the formation of magnetobound and quasibound states at larger energies (~ 50 K) may be useful, as these states may have already been formed in experiments working near those energy levels. It is of interest to note that, under certain conditions, the scenarios discussed may also work if the particles are exchanged for their oppositely signed counterparts. In such a case, magnetobound Pn would be produced instead of magnetobound Ps.

VI. CONCLUDING REMARKS

The interaction of a charged particle and its antiparticle in a strong magnetic field has been investigated. As a charged particle and its antiparticle approach one another, the electric field does work on the particles. This results in an increased kinetic energy and changes in each particle's pitch angle. The adiabatic invariance of the magnetic moment may be broken as the electric force between the charges approaches or overcomes the magnetic force due to the external field. The particles may then behave as a temporarily bound system.

Three numerical studies of the interactions of a particle and an antiparticle of the same mass under a strong magnetic field have been conducted. The formation of temporarily bound states of positive energy, called magnetobound states, was revealed. Several analytical expressions for the formation cross section of magnetobound states were developed and compared to numerical results. In one of its limiting cases and considering disparate masses, the theory presented here gave rise to a scale length that has been associated with guiding center drift atoms in other work.² In another limiting case and also considering disparate masses, the formation impact parameter that has been associated with quasibound states of antihydrogen was found.¹

Magnetobound states may have important implications. For instance, the formation of magnetobound states increases the time that two particles remain close together, which may allow participation in three body recombination of said particles. In three body recombination, the two particles that are near each other would experience a collision with a third particle, and one of the particles may take sufficient energy away to result in the formation of an atom. A magnetobound state is a neutral system. As would be consistent with that expectation, the study revealed that magnetobound states drift across magnetic field lines. Observations on the drift distance, lifetime, and drift speed of the magnetobound states formed in one of the computational studies were reported. The formation of magnetobound states may thus reduce the confinement times of plasmas where pair particles interact. However, magnetobound states may be useful as an intermediate step in the production of antihydrogen. Two possible scenarios were briefly discussed that may open opportunities for the production of neutral antimatter at lower temperatures than currently achieved.

Prior to the formation of a magnetobound state, and after its disassociation, the center of mass of the two particle system experiences zero net displacement in the guiding center approximation. However, during the lifetime of a magnetobound state, the simulations showed that the center of mass acquires a drift velocity that is perpendicular to the magnetic field.

In summary, the formation of temporarily bound systems of positive energy formed by a charged particle and its antiparticle have been studied via full trajectory simulation of point charges in a strong magnetic field. Several analytical expressions for the formation impact parameter of magnetobound states were proposed and found to be in good agreement with the numerical results over the range of conditions considered. In addition, the observation of cross-field drift distance, lifetime, and drift speed of the magnetobound states formed in the simulations was reported. Finally, possible avenues for the use of magnetobound states as an intermediate step in antihydrogen synthesis were described.

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