OPTIMIZATION THEORY IN ADMINISTRATIVE ANALYSIS

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OPTIMIZATION THEORY IN ADMINISTRATIVE ANALYSIS

DISSERTATION

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CHAPTER I

INTRODUCTION

The intensive conceptual development of mathematical models and practical solution techniques, as applied to administrative problems, began with the development of operations research during World War II. Over the past two decades the use of mathematical models and the implementation of the appropriate mathematical techniques for finding optimal solutions have become quite extensive as aids to solving administrative problems in both private industry and the government. This increased utilization of mathematical analysis has brought about deeper inquiries into the feasibility of applying mathematical techniques to administrative problems. At the same time this utilization has brought an increased awareness that the suitability and validity of mathematical analysis--both conceptually and computationally--depend upon the validity of the assumptions made in formulating the problem and the data used in obtaining the solution.

The necessary conceptual development can only be obtained by careful investigation of problem characteristics, classes of problems, and available methodologies. Given the proper development of the problem, availability of valid data, and implementation of a suitable computational technique, optimal solutions can be obtained to the problem under investigation--provided those solutions exist. However, the valid conceptual development of a problem requires a well defined familiarity with the classes of problems that must be analyzed and solved. This familiarity comes only through careful investigation of various categories of problems and their respective characteristics. Once the general class of problems has been identified, there must be an awareness of the solution methods that are suitable for the class of problems under investigation. This awareness can be achieved only by a match between the generalized problem area and the optimization techniques suited for that class of problems.

At the present time, the literature is neither explicit nor extensive in the generalizing process. Although a fairly extensive literature exists, the required generalizations have not been made. Extensive as it is, the literature relates to specific applications and does not contain a general classification of problem type and solution technique. For example, linear programming has been applied to several classes of administrative problems. These applications include transportation problems, product-mix problems, budgeting problems, and various resource allocation problems. In addition research has been done on the relevance of nonlinear programming and simulation as conceptual, as well as practical, methods for analyzing administrative problems. This research, coupled with conceptual departures from linear programming, has made feasible the use of nonlinear optimization as a conceptual and methodological tool for solving administrative problems. However, the full potential of nonlinear optimization cannot be realized until the general classes of problems to which optimization theory can be applied are identified. This requirement can only be met by analyzing known applications and generalizing existing theory for the purpose of extending the areas of application.

Statement of the Problem

The problem to be investigated in this study is best described in terms of two basic needs that must be met before theoretical development can be satisfactorily applied.

(1) The general characteristics of the problems that can be solved must be identified;

(2) The theoretical development must be examined for the purpose of deriving computational algorithms that enhance the application of the technique contained therein.

With its development as a tool of nonlinear analysis, modern optimization theory is a natural extension of classical optimization theory. This study is a part of the transition that must be made from abstract conceptualization of technique to its application.

The problem characteristics required for modern optimization theory, although reasonably well developed in mathematical theory, have not been identified clearly in terms of administrative analysis. Extensive application of modern optimization theory has been largely confined to the physical sciences and selected areas of engineering--in particular, systems design and development of industrial processes. This restricted applicability has been of such a nature that the solution techniques and the problems to which these techniques are appropriate have not been analyzed and structured with sufficient formality to permit their extension to problems of an administrative nature. The extension of the application of modern optimization theory to the area of administration will be achieved by (1) identifying general classes of administrative problems amenable to the techniques of modern optimization theory, (2) developing general computational algorithms suitable for administrative application, (3) relating applicable techniques to the general classes of problems, and (4) demonstrating the use of the algorithms by presenting detailed solutions to selected administrative problems which exhibit characteristics that identify them as belonging to a certain problem classification.

Classical optimization theory is well developed with respect to technique and application. This provides a frame of reference from which the techniques and applications of modern optimization theory can be identified and explained. By presenting new theoretical developments and identifying classes of problems, this study will serve to extend the relating of existing theory with new applications by helping to provide a basis for matching problem classifications to solution techniques.

Definitions and Symbols

Definitions

Definition 1.--The optimum solution to a given problem is the solution which best satisfies a defined criterion of effectiveness.

Definition 2.--Optimization theory is the quantitative study of optima, with modifications to the point of deriving methods for finding optimal values.

Definition 3.--Classical optimization theory is defined as that set of concepts and techniques readily suited to linear analysis and the use of the differential calculus and the Lagrange multiplier.

Definition 4.--Modern optimization theory is defined as that set of concepts and techniques readily suited to both linear and nonlinear analysis. Stemming from the Kuhn-Tucker conditions, this set includes iterative techniques and the multivariable calculus.

Symbols

$\forall$: for every, for all

$\in$: belongs to the set

$\rightarrow$: implies

$\exists$: there exists

$\prod$: the product of the following terms

Thesis

The thesis of this study is that modern optimization theory is a natural extension of classical optimization theory. As such, modern optimization theory will be applied to administrative problems only after interpretive studies are made that provide (1) an explanation of the general theoretical development of the techniques of modern optimization theory, (2) computational algorithms for implementing the techniques of modern optimization theory, (3) detailed demonstrations of the computational aspects of each technique and its corresponding algorithm, and (4) an identification of the types of problems to which these techniques are applicable.

Validation of this thesis will be accomplished by tracing the development of optimization theory from its roots in classical optimization theory to its frontiers in modern optimization theory. This presentation will provide a base from which administrative problems amenable to modern optimization theory can be readily identified and associated with a suitable solution technique. The applications of modern optimization theory to problems of an administrative nature will be further enhanced by elaborating on existing computational techniques and by developing and demonstrating additional computational algorithms applicable to administrative problems.

Classical optimization theory, in the context of this study, is defined more as an approach to a problem rather than a specific content of methods and models. This approach includes the use of linear models and those models suitable to the maximum-minimum concepts of the differential calculus. These problems (e.g., input-output analysis, cost-profit analysis, inventory, production activity analysis, etc.), framed with the restrictions imposed by linear assumptions and the maximum-minimum criterion of the calculus, represent the initial applications of optimization techniques to administrative problems. These applications, however, have been restricted in terms of conceptual development and implementation of solution techniques.

Modern optimization theory, on the other hand, refers to the conceptual development of models and solution techniques that are not subject to the restrictions imposed by the classical approach. In this respect, modern optimization theory represents an area of development that is concerned with both linear and nonlinear, univariable and multivariable analysis. It is an area of analysis that attempts to provide the analyst with a more flexible, highly practical means of analyzing and solving existing problems.

The incorporation of modern optimization theory as a tool of administrative analysis permits the addressing of the nonlinearities of the real world. These techniques are of such sophistication, however, that the elaboration of this study is necessary if a proper foundation is to be developed which permits future application. Since an understanding of technique is essential to proper implementation, this study is developed in such a way that it provides for the necessary conceptual understanding. This is evidenced by the detailed discussion and presentation of the techniques and applications of classical and modern optimization theory.

A corollary of this study is that classical and modern optimization theory can provide management with more realistically developed problems and better solutions to those problems. This will be demonstrated by identifying the general characteristics of those problems amenable to classical and modern optimization theory, matching solution techniques to problems, and demonstrating the computational aspects of each technique in a manner not currently available in the literature.

Background and Significance of Study

Background

In Mathematical Methods of Operations Research, Thomas L. Saaty presents a brief discourse on optimization, the subject of this study.[1] In this discourse Saaty indicates that the theory of finding optimal solutions has progressed rapidly since its intrusion into applied administrative areas at the close of World War II; but at no point is found a

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[1] Thomas L. Saaty, Mathematical Methods of Operations Research (New York, 1959), pp. 96-162.

definition or explanation of what is meant by optimization and the theory of optima.

Wilde and Beightler, in discussing "best" and "optimum," note that

> optimum has become a technical term connoting quantitative measurement and mathematical analysis, whereas "best" remains a less precise word more suitable for everyday affairs.[2]

In this regard, the meaning of optimization, hence optimization theory, is found to stem from the root word "optimize."

> The technical verb _optimize_, a stronger word than "improve," means to achieve the optimum, and _optimization_ refers to the act of optimizing. Thus, optimization theory encompasses the quantitative study of optima and methods for finding them.[3]

Optimization theory can be described as the study of classes of problems for which two or more possible solutions exist. The selection of the optimal value, optimization, refers to the process of choosing from among a set of possible alternatives. The optimum value, the end result of optimization, is optimal in the sense that of all feasible alternatives, it yields the best value as determined by the criterion of effectiveness.

Optimization differs from problem solving in that it is a basic decision process which requires the following: (1) definition of the problem, (2) identification of feasible alternatives, and (3) selection of the solution that is optimal in terms of the criterion of effectiveness.

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[2]Douglas J. Wilde and Charles S. Beightler, _Foundations of Optimization_ (Englewood Cliffs, 1967), p. 1.

[3]_Ibid._

If the function is constrained, the optimal solution is optimal in that it is the best of the feasible solutions. If the function is unconstrained, the optimal solution is the value at which the function achieves its true optimum. Whereas problem solving is associated with unconstrained functions, optimization is associated with constrained functions.

Significance

The significance of this investigation is that it will provide an interpretive study of classical and modern optimization theory and will document the applicability of modern optimization theory to problems of an administrative nature. Although classical optimization theory is well documented with respect to administrative applications, available solution techniques, and examples, modern optimization theory is not. At present there has been a lack of administrative research on the techniques of modern optimization theory and their use as a tool of administrative analysis. The need for such research has been noted by others.

> It must be realized, however, that not everything is linear and that occasionally we come across constraints which are mathematically pathological types. . . . We need a great deal more basic research on optimization methods in the universities and industrial research laboratories.[4]

This study will help to satisfy this need.

Before existing problems can be solved, there must be an awareness of the characteristics which define classes of problems. These problems must be conceptually developed with regard to their particular characteristics to the point of applying a valid solution technique. Given

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[4]W. W. Garvin, H. W. Crandall, J. B. John, and R. A. Spellman, "Applications of Linear Programming in the Oil Industry," Mathematical Studies in Management Science, edited by Arthur F. Veinott, Jr. (New York, 1965), pp. 127-128.

this development, the optimizing process is achieved by matching the problem under investigation to similar ones with known characteristics and selecting the computational technique that best fits the problem description. If the data base from which the problem is formulated is sufficient, proper utilization of this matching process will result in solutions that are truly optimal.

Although identifying the types of administrative problems to which modern optimization theory is applicable and matching these problems with the computational techniques that are available is significant in itself, this study will be further enhanced by the development of computational algorithms that are based upon the mathematical theory of modern optimization theory. These algorithms will then be demonstrated by application to specific problems. In this way, this study will (1) provide an insight into the use that can be made of these new techniques as tools of administrative analysis, (2) provide algorithms for implementing these new techniques, and (3) provide a detailed demonstration of the computational aspects of each algorithm.

Scope of the Study

This study will be limited to the general characteristics, components, and sufficiency conditions for implementation and use of both classical and modern optimization theory. As an interpretive study, it is not intended to be an exhaustive presentation of the many techniques and modifications to both classical and modern optimization theory. Rather, the intent of this study is centered on the general conceptual and methodological developments of these two areas of optimization, with particular attention being given to administrative applications.

Since administrative applications of modern optimization theory are not readily available, detailed problem examination will be limited to selected examples. This lack of current administrative application is attributed to the newness of the computational techniques of modern optimization theory and the lack of an adequate data base from which problems can be formulated. Hence, some abstract problems are utilized to demonstrate the type of problem formulation being investigated and the solution technique employed. The next step to proper administrative application is the identifying of problems that can be solved by the solution techniques that are discussed. By relating these problems to those of classical optimization theory, it is possible to infer areas of similar administrative applicability. From this development, further inquiry can be initiated that leads to the investigation and development of the data that is necessary for practical application.

Methodology

The methodology to be employed in the course of this study includes a comprehensive review of developed theory and published applications. The purpose of this review of the literature is to (1) establish the general characteristics of the problems to which both classical and modern optimization theory can be applied, (2) present the characteristics of the general solution techniques that are available, and (3) describe the administrative applications suited to the use of these techniques.

In this way the classes of administrative problems amenable to both areas of optimization theory will be identified and correlated with appropriate solution techniques.

The review of the literature relevant to optimization theory and its application as a problem solving tool will include a comprehensive study of the techniques of optimization theory. Since one of the major features of this study is the extraction of existing algorithms and the development of algorithms by which new techniques can be applied, this review of the literature will provide a basis from which this can be accomplished. The detail and manner of presentation of extracted material and developed algorithms will be such that administrative application will be enhanced. These algorithms will then be demonstrated in a depth that is not available in current literature.

As noted previously, administrative applications of classical optimization theory are well documented in current literature. This documentation is evidenced by the many conceptual and empirical studies which utilize the computational techniques of classical optimization theory. However, administrative applications of modern optimization theory have not received the degree of documentation that has been given to classical optimization theory. This lack of documentation has been attributed to the newness of the computational techniques of modern optimization theory and the lack of empirical data. Hence, until empirical studies have provided a sufficient data base, it will be necessary to identify the administrative applications of modern optimization theory from the basis of conceptual development.

CHAPTER II

CLASSICAL OPTIMIZATION THEORY

Introduction

The use of mathematical models for decision analysis serves a twofold purpose:

(1) The essential variables of the problem are brought together in one model accounting for the constraints and the function to be optimized.

(2) The problem [via the mathematical formulation] is now given a familiar structure which can be analyzed for solutions, their existence, uniqueness, and construction.[1]

These models, properly derived, can serve to better the decision making process by enabling the analyst to examine carefully the problem formulation and, based upon this formulation, select the most appropriate solution technique.

From the viewpoint of historical development, the first class of problems generally considered by the analyst are those characterized by non-iterative, strictly defined solution techniques. Throughout this discussion, this class of optimization problems will be considered the context of classical optimization theory.

Definition 2.1.--Classical optimization theory is defined as an association of solution technique and problem formulation for which alternative solutions do not exist and for which the search for the

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[1] Thomas L. Saaty, Mathematical Models of Operations Research (New York, 1959), p. 99.

optimum solution is a pseudo search. In this sense, classical optimization theory describes a straight-forward application of a formula or a non-iterative algorithm that yields, in a direct manner, a unique solution to a given problem.

Examples of classical optimization problems include breakeven analysis, rate of return, average length of a queue, determining maximum and/or minimum values of a given nonlinear function, and some types of product-mix problems. Solutions to these problems are classified as pseudo search because they are defined by equality conditions and are of such a nature that unique solutions result.

Classical optimization techniques include the use of linear systems, matrix algebra, max-min calculus, Lagrange multipliers, and queueing theory. Each provides a distinct means of formulating and solving classes of problems, and each can be categorized on this basis. However, before a problem can be solved by a given technique, the problem itself must be recognized and formulated. Given this formulation, the necessary technique for solving can be applied and solutions obtained (provided those solutions exist).

As a means of providing a proper base from which to discuss the applications of decision analysis, the search techniques of classical optimization theory will be categorically presented. Following this presentation the applications will be discussed.

Techniques of Classical Optimization Theory

Algebraic Equations

Initially the analysis of business systems began with algebraic expressions representing the particular problem under investigation. Characterized by a single-variable expression of the form

$$f(x) = a_o x^n + a_1 x^{n-1} + \ldots + a_n,$$

this attempt at functional representation forced all products into a homogeneous category. The result, for $a_o \neq 0$, was the n^{th} degree polynomials defined by f(x). When equated to zero, f(x) can be solved for the n roots (solutions) by means of direct formulas taken from numerical analysis and the theory of equations.[2]

Classical algebraic analysis of administrative problems assumes an algebraic expression that is at most third degree (cubic), but generally second degree (quadratic) or first degree (linear). For the second degree equation, by far the most common, the normal solution technique is the quadratic formula. With the exception of non-univariate functions (i.e., functions with more than one independent variable), application of the quadratic formula yields numerical solutions. For multivariable quadratic functions, the solutions are dependent.

Linear Systems

The need for systems of equations becomes apparent when the problem under investigation is such that the products are not homogeneous and

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[2]These methods include the following: Newton's method for integral roots, application of Cardan's formulas for solving reduced cubic equations, Horner's approximation method, and the method of "regula falsi."

cannot be considered as belonging to the same set (for example, a plant operation that produces four distinct products: chairs, tables, desks, and beds). Such problems arise when the composite elements of the problem are distinct and cannot be considered in terms of a single variable. In such a case, it is necessary to describe the problem by a system of n-variable simultaneous equations. In describing such problems, the following definition is of value.

<u>Definition 2.2</u>.--The multivariable linear system problem is defined as one in which the objective is to find the set of x_j $(j = 1, 2,\ldots,n)$ satisfying the m linear equations

$$\sum_{i=1}^{m} a_{ij}x_j = b_i, \qquad j = 1, 2,\ldots,m.$$

The system is said to consist of m equation with n unknowns (i.e., m x n).

In this formulation it is assumed that total consumption is required of all resources and a linear relationship exists among the variables. This approach allows more flexibility because contributions (as well as expenses) of individual elements can be considered.

Techniques for solving such algebraic systems include the substitution method, the elimination method, the application of Cramer's rule, or the use of matrices. With the exception of matrix theory, a distinct applied tool in its own right due to its wide applicability, each of these represents a standard technique for solving linear systems. In the discussion to follow, these basic solution techniques will be briefly examined. Because of the elementary nature of some of these techniques, the discussion, in some parts, will be limited to a direct statement of technique. In particular, this approach will be utilized on the explanation of the substitution method and the elimination method.

Solution by substitution.--The substitution method, although applicable for any linear system, is practical only when m and n are at most three. When m = n exceeds three, the substitution method becomes unwieldy and better techniques can be applied. An algorithmic approach to the substitution method can be described as follows:

(1) Given an m x n linear system, select one of the variables for the purpose of substituting it into the remaining equations.

(2) Solve one of the m equations for the selected variable.

(3) Substitute the result of (2) into the remaining m - 1 equations. For m > 2 the process will have to be repeated until a unique solution is obtained, a solution which can then be used to determine the other values, or a dependent solution is obtained.

Solution by elimination.--Solving a given linear system by elimination involves selecting a variable to be eliminated, multiplying as needed to equate coefficients, and then subtracting or adding in such a way that the chosen variable is eliminated from the system. The process is repeated until the variables are numerically determined, are defined in terms of one or more of the variables, or the system is identified as being inconsistent (no solution).

Cramer's rule.--Cramer's rule defines a mathematical technique for solving simultaneous linear equations.[3] The problem to be solved is one

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[3]Nelson Bush Conkwright, Introduction to the Theory of Equations (New York, 1957), p. 132.

which is described by n simultaneous equations in n unknowns x_1, $x_2, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n. \end{aligned}$$

Let $\underline{D}$ denote the determinant defined by the coefficients of the linear systems; i.e.,

$$\underline{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Let $\underline{D}_j$ $(j = 1, 2, \ldots, n)$ denote the determinant obtained by replacing the elements of the j^{th} column of $\underline{D}$ by the column of constants b_1, $b_2, \ldots, b_n$; for example, for $j = 1$,

$$\underline{D}_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Then, utilization of the following rule provides a means for solving the simultaneous system for each of the n variables.

Rule 2.1 (Cramer's Rule).--If $\underline{D} \neq 0$, the simultaneous equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= bn \end{aligned}$$

are satisfied by a unique (one, and only one) set of values of the n variables. The values of the n variables are given by the following set of formulas:

$$x_1 = \frac{D_1}{D}; \quad x_2 = \frac{D_2}{D}; \quad \ldots; \quad x_n = \frac{D_n}{D} .$$

The use of Cramer's rule provides a means for solving an n x n system without resorting to the tedious techniques of substitution or elimination. Its application requires the evaluation of (n + 1) determinants and the evaluation of n ratios and is applied as follows:

1. Determine the value of the determinant defined by the coefficients of the set of simultaneous equations; i.e., evaluate

$$\underline{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} .$$

2. Determine the value of the determinant $\underline{D}_j$, defined as the determinant obtained by replacing the j^{th} column of $\underline{D}$ by the column of constants b_1, $b_2, \ldots, b_n$; i.e., evaluate

$$\underline{D}_j = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}, \qquad j = 1, 2, \ldots, n.$$

3. For $D \neq 0$, apply the solution formula for determining the value of the j^{th} variable

$$x_j = \frac{D_j}{D}, \qquad j = 1, 2, \ldots, n.$$

Repeated application of the third step results in n unique values, one for each of the n variables. Thus, given the value of D, the solution set is obtained by evaluating n additional determinants and calculating the necessary quotients. As noted in the statement of Cramer's rule, the calculated values are the unique (one, and only one) solutions to the given system of linear equations. That is, the solution set is the unique set of points at which all of the n equations intersect each other.

An obvious drawback of Cramer's rule is the infeasibility of application when the linear system is large. This drawback is also common to both solution by substitution and solution by elimination. For large systems, the use of matrix theory provides a more suitable method of obtaining the values of the solution set.

Matrix Theory

The matrix approach to simultaneous equations provides a means for describing a given system in an abbreviated form. Defining a matrix as an array consisting of m rows and n columns (with m not necessarily equal to n), a simultaneous set of equations given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned}$$

can be written in the form

$$\underline{AX} = \underline{B},$$

where

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}; \quad \underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}; \text{ and } \underline{B} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}.$$

Proper implementation of the notational aspects of matrix theory provides an efficient means of describing multivariable linear problems. Although it is not restricted to describing linear systems, the use of the matrix notation in classical optimization theory centers on the simultaneous linear system. In illustration, consider the following problem: A mining company has demand for 1,000 tons of iron ore, 2,000 tons of crushed rock, and 500 tons of dirt. Three products, x_1, x_2, and x_3, can be made by blending the ore, rock and dirt. Product x_1 requires 5 tons of iron ore, 10 tons of crushed rock, and 10 tons of dirt. Product x_2 requires 5 tons of iron ore, 8 tons of crushed rock, and 5 tons of dirt. Product x_3 requires 15 tons of iron ore, 4 tons of crushed rock, and 2 tons of dirt. Demand is such that it must be met. Assuming a linear relationship between x_1, x_2, and x_3, it is possible to describe this problem by the following set of simultaneous equations:

$$5x_1 + 5x_2 + 15x_3 = 1000,$$
$$10x_1 + 8x_2 + 4x_3 = 2000,$$
$$10x_1 + 5x_2 + 2x_3 = 500.$$

Applying matrix notation, the problem takes on the form

$$\begin{bmatrix} 5 & 5 & 15 \\ 10 & 8 & 4 \\ 10 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \\ 500 \end{bmatrix}.$$

In addition to the abbreviated form, the use of matrices provides a method by which systems with the proper characteristics can be solved more feasibly and more efficiently than by any other technique. Solutions to the given system, expressed by the solution vector

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \equiv \underline{X}^T = [x_1, x_2, \ldots, x_n]^T,$$

can be obtained by multiplying the abbreviated system on the left by the inverse of the matrix of coefficients; i.e., denoting the inverse of $\underline{A}$ by $\underline{A}^{-1}$,

$$\underline{A}^{-1}(\underline{AX}) = A^{-1}B$$

yields

$$\underline{X} = \underline{A}^{-1}\underline{B}.$$

It is assumed that the inverse matrix, $\underline{A}^{-1}$, exists. If it does, $\underline{A}^{-1}$ is the unique matrix such that multiplication of the coefficient matrix $\underline{A}$ by A^{-1} reduces $\underline{A}$ to the identity matrix $\underline{I}$. The identity matrix is defined as the n x n matrix whose diagonal elements are equal to unity and elements off the diagonal are equal to zero; i.e.,

$$\underline{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Thus, solving a system of simultaneous linear equations using matrix theory requires four basic operations: (1) write the given linear system in matrix form; (2) calculate the inverse of the coefficient matrix $\underline{A}$; (3) multiply the column vector of constants, $\underline{B}$, on the left by the inverse of $\underline{A}$; (4) write the solution vector that results: $\underline{X} = \underline{A}^{-1}\underline{B}$. In this discussion it is tacitly assumed that the inverse exists.[4]

Before the matrix-inverse approach to solving simultaneous equations can be applied, several conditions must be satisfied. These conditions are given by reference to some of the theorems of matrix theory. For this reference, the following definitions of terminology are required.

Definition 2.3.--If $\underline{A}$ denotes the m x n matrix of coefficients of

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{matrix}$$

[4]The existence of an inverse yields a unique universe; i.e., for any nonsingular square matrix A, there is only one inverse for A.

the augmented matrix C is defined as the m x (n + 1) matrix obtained by attaching the column of constants, B, to the matrix A; i.e.,

$$\text{if } \underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix},$$

$$C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

Definition 2.4.--The matrix A is said to be nonsingular if there exists a matrix D such that DA = I = AD, where I is the identity matrix. If D exists, then $\underline{D} = \underline{A}^{-1}$. If D does not exist, the matrix A is said to be singular.

Definition 2.5.--The row rank of the matrix

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij}), \quad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n,$$

is defined as the number of nonzero rows of A when A is expressed in row equivalent simplified form; i.e., the matrix A has been reduced by a series of row operations to a matrix such that:

(a) the first r rows, for some $r \geq 0$, are nonzero and all remaining rows, if any, are zero;

(b) the first nonzero element in the i^{th} row $(i = 1, 2, \ldots, r)$ is equal to unity, the column in which it occurs being numbered c_i;

(c) $c_1 < c_2 < \ldots < c_r$;

(d) the only nonzero element in column c_i is the 1 in row i.

The matrix to which A has been reduced is said to be the row equivalent matrix of A.[5]

The term "row operation" refers to any one of three operations on a matrix:

(1) interchange of two rows;

(2) multiplication of any row by a nonzero constant; or,

(3) replacement of the j^{th} row by the sum of the j^{th} row and k times the i^{th} row, where $i \neq j$, and k is any nonzero constant.

The purpose of the row operation is to reduce a given matrix to the form described in Definition 2.5. For example, the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

can be reduced by a series of row operations to the identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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[5]Sam Perlis, Theory of Matrices (Reading, 1958), pp. 42-45.

Definition 2.5 then states that the original matrix has row rank, r, equal to 3; i.e., the row equivalent form of the original matrix contains three nonzero rows.

The concept of row operation, in conjunction with the preceding definitions, provides the necessary base from which the conditions required for obtaining solutions to given linear systems can be written. The following conditions, found in any standard text on the theory of matrices, are taken from Perlis.

> 1. A system of simultaneous linear equations has a solution if and only if the [row] rank of the augmented matrix equals the [row] rank of the coefficient matrix.
>
> 2. A homogeneous system of linear equations [a system such that all of the constant terms are zero] has a nontrivial solution [all $x_i \neq 0$] if and only if the number of unknowns exceeds the row rank of the coefficient matrix.[6]

As noted previously, a given linear system, if it has a solution, can be solved by multiplying the column vector of constants by the inverse of the coefficient matrix. The multiplication is achieved by multiplying on the left. Symbolically, if $\underline{A}\underline{X} = \underline{B}$, and $\underline{A}^{-1}$ exists,

$$\underline{A}^{-1}(\underline{A}\underline{X}) = \underline{A}^{-1}\underline{B},$$

$$(\underline{A}^{-1}\underline{A})\underline{X} = \underline{A}^{-1}\underline{B},$$

$$\underline{I}\underline{X} = \underline{A}^{-1}\underline{B},$$

$$\underline{X} = \underline{A}^{-1}\underline{B}.$$

This inverse can be uniquely determined from the coefficient matrix by reducing the coefficient matrix to the identity matrix. The series of row operations which reduces the coefficient matrix to the identity matrix,

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[6] Ibid., pp. 45-62.

if performed on a similar identity matrix, will transform it into the unique inverse of the coefficient matrix. The procedure is as follows:

(1) set up the partitioned matrix $[\underline{A}|\underline{I}]$, $\underline{A}$ any n x n coefficient matrix and $\underline{I}$ any identity matrix of equivalent size;

(2) reduce the matrix $\underline{A}$ to the identity matrix $\underline{I}$ by a series of row operations; at the same time, perform the same series of row operations on the augmented matrix $\underline{I}$;

(3) the resulting partition has the form: $[\underline{A}^{-1}\underline{A}|\underline{A}^{-1}\underline{I}] \equiv [\underline{I}\ |\underline{A}^{-1}]$, the matrix to which the identity matrix has been transformed being the inverse of the coefficient matrix $\underline{A}$.

As a mathematical tool, the theory of matrices provides a means whereby problems formulated as linear systems can be written in an abbreviated form. In addition, the use of matrices provides a more efficient method for solving a given system of simultaneous equations through the use of the inverse. Since the system being investigated defines a set of simultaneous linear equations, the resulting solution vector is unique; i.e., it defines the only set of solutions which satisfies all of the equations simultaneously.

As a tool of classical optimization theory the application of the theory of matrices "solves" the defining system in a direct manner. No attempt is made to determine whether or not alternative solutions exist since the use of the equality implies one and only one solution set. Examples of problems suitable for this method of analysis include budget allocations between departments of a firm, allocation of labor to productive endeavor when fixed amounts of production are required, and allocation of investment opportunities to yield a fixed dollar (or percentage) return.

Max-min Calculus

The use of the calculus as a tool for solving administrative problems can be traced to the mathematical school of economic thought. Although isolated application to administrative problems can be found prior to this time--notably the work of Cournot, Jevons, and Walras in profit analysis, marginal utility and marginal productivity of capital, and marginal utility and demand analysis, respectively--the accepted use of mathematics (especially the calculus) as a tool of decision analysis received its greatest impetus from the work of these pioneering applications. In its capacity as a classical optimization technique, the max-min calculus has been used in applications such as inventory analysis, cost-profit analysis, and the study of isoquants and their related families of curves.

The techniques of the calculus are divided into two major divisions: (1) the single variable case, where the dependent variable is defined as a function of a single independent variable and (2) the multivariable case, where the dependent variable is defined as a function of n independent variables. In the first case, the approach for determining maxima or minima is based upon the use of the derivative. In the second case, the approach for determining maxima or minima is based upon the use of the partial derivative. Because of the nature of the applications, the same division of topics will be used in this study.

Functions of a single variable.--The development of the calculus as a tool of analysis is based on the following concepts: limit, continuity, derivative, relative maximum (minimum), and absolute maximum (minimum).

These concepts are defined and explained as a basis for later discussion.

Definition 2.6.--Let f(x) be any function defined for the independent variable x. The function f(x) approaches a limit L as the variable x approaches the value c if and only if for every positive number, ε, there is another number $\delta > 0$, such that, whenever the absolute difference between x and the point being approached is less than δ, the absolute value of the difference between f(x) and L is less than ε; i.e.,

$$\lim_{x \to c} f(x) = L \leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \ni 0 < | x - c | < \delta \rightarrow | f(x) - L | < \varepsilon.$$

The meaning of this concept can be explained in the following manner: Consider the function f(x) defined over the relevant range of x. As the independent variable, x, approaches the value c (from either side) in such a way that the absolute difference between x and c is infinitely small, the value of the function, f(x), approaches a fixed value (i.e., as the value of the independent variable approaches a given value, the difference between the value of f(x) evaluated at the point in question, c, and the limiting value, L, becomes infinitely small). Thus, as x approaches (or is limited by) c, the function, f(x), approaches L as its limit.

Definition 2.7.--Let f(x) be any function defined for the independent variable x. The function f(x) is said to be continuous at x = c (where c is in the acceptable domain of f(x)) if and only if the following conditions are satisfied:

(1) f(x) is defined at x = c;

(2) as x approaches c, the limit of f(x) exists; and,

(3) the limit of f(x) as x approaches c equals f(x) evaluated at x = c.

The importance of continuity to the calculus centers on the derivative. In addition to guaranteeing a smooth curve, continuity must be established before the existence of the derivative can be assumed. Continuous functions are functions containing no gaps (i.e., the curve is not kinked or stepped) and functions for which the derivative can be defined.

Definition 2.8.--Let y = f(x) be any function defined over a given interval. The derivative of y = f(x), denoted by y' = f'(x), is defined by

$$y' = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$ provided the limit exists.

Upon examining this definition it is seen that the derivative defines the average rate of change in f(x) as this rate of change becomes infinitely small (approaches zero). Further insight into the meaning of the derivative can be obtained by considering the ratio

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

The numerator, $f(x+\Delta x) - f(x)$, is the difference between two values of the dependent variable; as such, it represents the amount of change produced in the dependent variable due to a change of magnitude Δx in the independent variable. Dividing the amount of change in the dependent variable by the amount of change in the independent variable, $\frac{f(x+\Delta x) - f(x)}{\Delta x}$, results in the average rate of change of f(x) over the interval $(x, x+\Delta x)$.

As a graphic illustration consider the function shown in Figure 2.1, a discussion of which follows. In this illustration the variable y is said to be a function of the variable x. Incremental changes in x will be denoted by Δx.

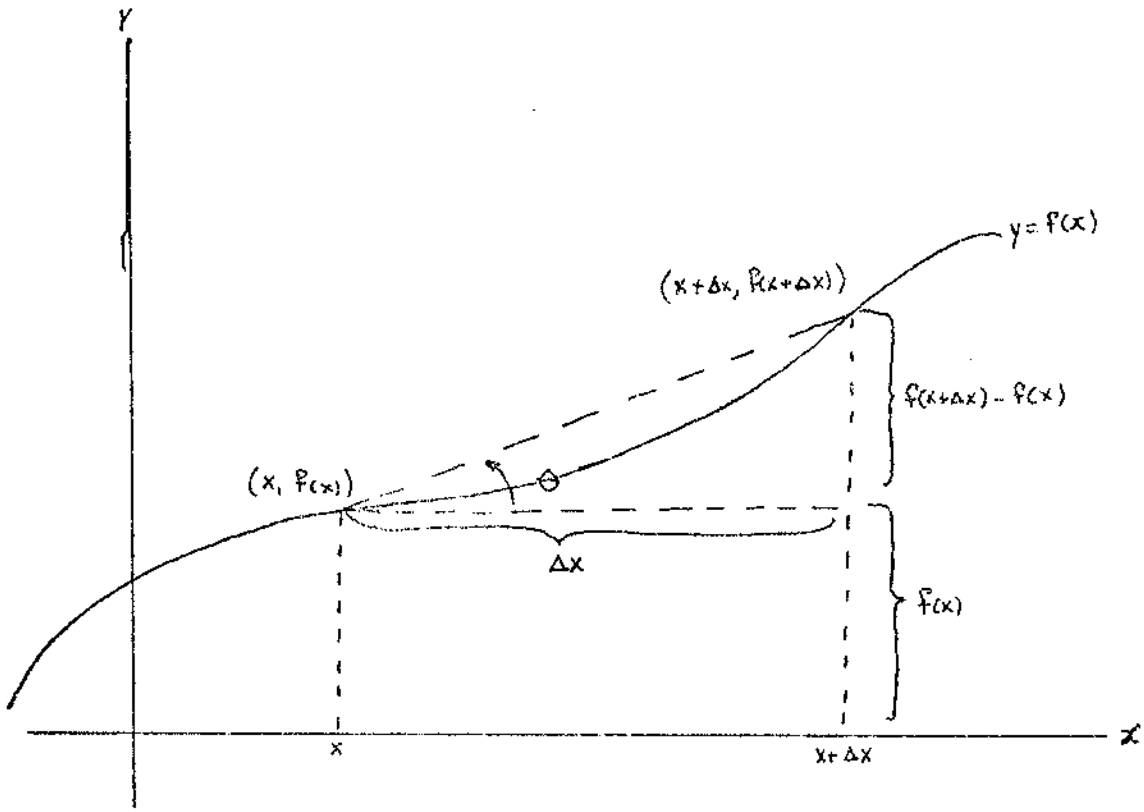

Fig. 2.1--The derivative

As the change in x becomes increasingly small ($\Delta x \to 0$), the value of Δx is said to approach zero as a limit. This limiting process, applied to the ratio, $\frac{f(x+\Delta x) - f(x)}{\Delta x}$, is called the instantaneous rate of change of f(x) with respect to x; thus f'(x) is the derivative of f(x) with respect to the changing (independent) variable x. It is the rate of rise (or fall) of the function at any point defined for that function; and, as such it defines the slope of the tangent to the curve at any given point.[7] This is easily demonstrated by constructing a right triangle as shown in Figure 2.1. Since the tangent of the angle is defined as the ratio between the line segment opposite the angle $(f(x+\Delta x))$ and the line segment adjacent to the angle (Δx), the tangent of the angle formed (Θ) is $\frac{f(x+\Delta x) - f(x)}{\Delta x}$. For Δx taken in successively small increments the curve is approximated by a series of straight lines, the slope of which is given by the tangent value. Thus, as $\Delta x \to 0$ the line tangent to $f(x)$ at $x+\Delta x$ has slope

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

which is the derivative of $f(x)$.

The derivative is one of the most important concepts of the differential calculus. In addition to its use as a tool for determining points of maxima and minima, the derivative (as a measure of the average rate of change) defines the marginal rate of change in the dependent variable due to a change in the independent variable. In this use the derivative can be used to describe such concepts as

(1) marginal profit--the average rate (or amount) of change in profit due to a change in the variable which defines profit (say, units produced);

(2) marginal utility--the average rate (or amount) of change in utility due to a change in some defined independent variable;

(3) marginal demand--the average rate (or amount) of change in demand due to a change in some independent variable (say, price);

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[7]Gordon Fuller and Robert M. Parker, Analytic Geometry and Calculus (Princeton, 1964), pp. 66-67.

(4) marginal consumption--the average rate (or amount) of change in consumption due to a change in some independent variable (say, income); and

(5) marginal output (productivity)--the average rate (or amount) of change in output due to a change in the input of productive factors. From this limited selection it is easily seen that the derivative functions as a very important tool for solving those problems amenable to its implementation.

The application of the differential calculus includes the determination of points of maximum and/or minimum value of given differentiable functions. In classical optimization theory this is accomplished by the direct application of a set of definitions and theorems, a set that completely defines the conditions by which the maximum and/or minimum values are identified. These conditions, as well as the necessary terminologies, are presented as definitions or theorems.

Definition 2.9.--The critical point (or points) of a given continuous function is defined as that point (or set of points) such that the derivative vanishes at zero; i.e., for a given continuous function f(x), the critical point c is defined as follows:

$$c = \{x \mid f'(x) = 0\}.$$

In an operational context the critical point of a given function represents the point at which the slope of the line tangent to the curve at that point has a value of zero. Thus, the critical point represents the value of the independent variable at which the line tangent to the curve at that point is parallel to the axis defined by the independent variable. Recalling that the derivative of a function (if it exists) represents the slope of the line tangent to the function at every point defined along the curve, the relationship between the derivative and the critical point can be easily seen: the critical point identifies the point at which the line parallel to the axis representing the independent variable is tangent to the curve.

Definition 2.10.--Let f(x) be a function defined over the open interval $a < x < b$. Let x_1 and x_2 be any two points of this interval. Then, for $x_1 < x_2$,

(1) f(x) is said to be increasing on the interval $a < x < b$ if $f(x_1) < f(x_2)$;

(2) f(x) is said to be decreasing on the interval $a < x < b$ if $f(x_1) > f(x_2)$.

The meaning of this definition, illustrated in Figure 2.2, is described as follows.

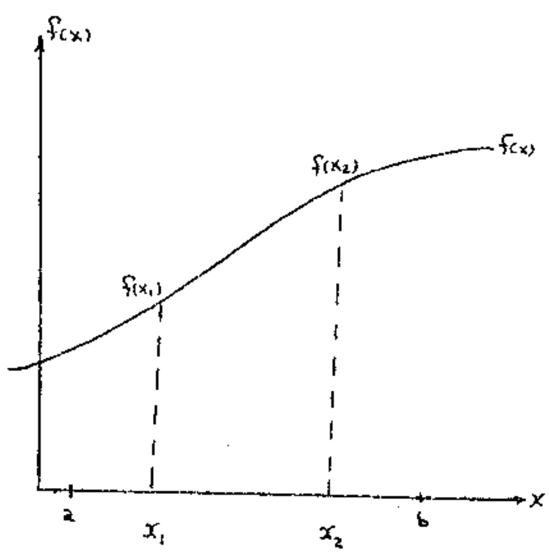

Fig. 2.2(a)--Increasing function

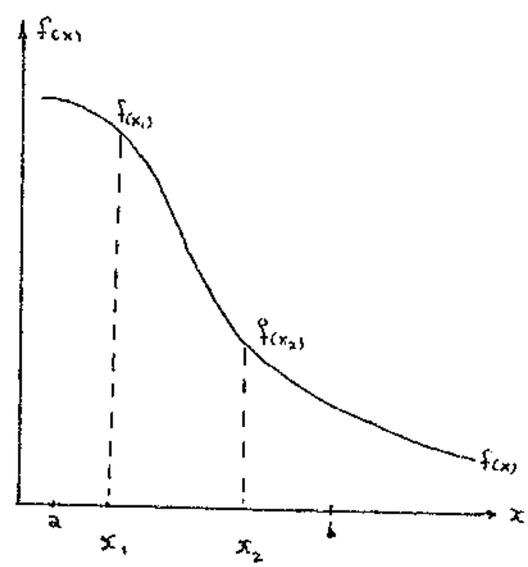

Fig. 2.2(b)--Decreasing function

(1) Consider Figure 2.2(a). Let x_1 and x_2 be any two points belonging to the open interval (a, b) such that $x_1 < x_2$. Evaluating f(x) at $x = x_1$ and $x = x_2$, the resulting values, $f(x_1)$ and $f(x_2)$, respectively, are such that $f(x_1) < f(x_2)$. As x increases in value (from x_1 to x_2) the value of the function, f(x), increases (from $f(x_1)$ to $f(x_2)$). Therefore, as x increases in value the function defined for x increases; f(x) is said to be increasing for $x_1 < x_2$.

(2) Consider Figure 2.2(b). Let x_1 and x_2 be any two points belonging to the open interval (a, b) such that $x_1 < x_2$. Evaluating f(x) at $x = x_1$ and $x = x_2$, the resulting values, $f(x_1)$ and $f(x_2)$, respectively, are such that $f(x_1) > f(x_2)$. As x increases in value (from x_1 to x_2) the value of the function, f(x), decreases (from $f(x_1)$ to $f(x_2)$). Therefore, as x increases in value the function defined for x decreases; f(x) is said to be decreasing for $x_1 < x_2$.

Assuming that the given function, f(x), is continuous over the open interval (a, b) the derivative can be readily applied to determine whether f(x) is increasing or decreasing over (a, b). When using the derivative to determine the nature of the function (increasing or decreasing), the following theorem is useful.

<u>Theorem</u> <u>2.1</u>.--Let f(x) be any continuous function defined over the interval (a, b). The function f(x) is increasing or decreasing according to the sign of the derivative of the function. If the derivative is positive the function is increasing; if the derivative is negative the function is decreasing.

Symbolically,

(1) $f'(x) > 0$ indicates $f(x)$ to be increasing;

(2) $f'(x) < 0$ indicates $f(x)$ to be decreasing.

An intuitive interpretation of this theorem can be obtained by referring to Figure 2.3. In the figure the given function, $y = f(x)$, is assumed to be continuous over the interval (a, b). As a means of simplifying the illustration $y = f(x)$ is assumed to be parabolic (unimodal).

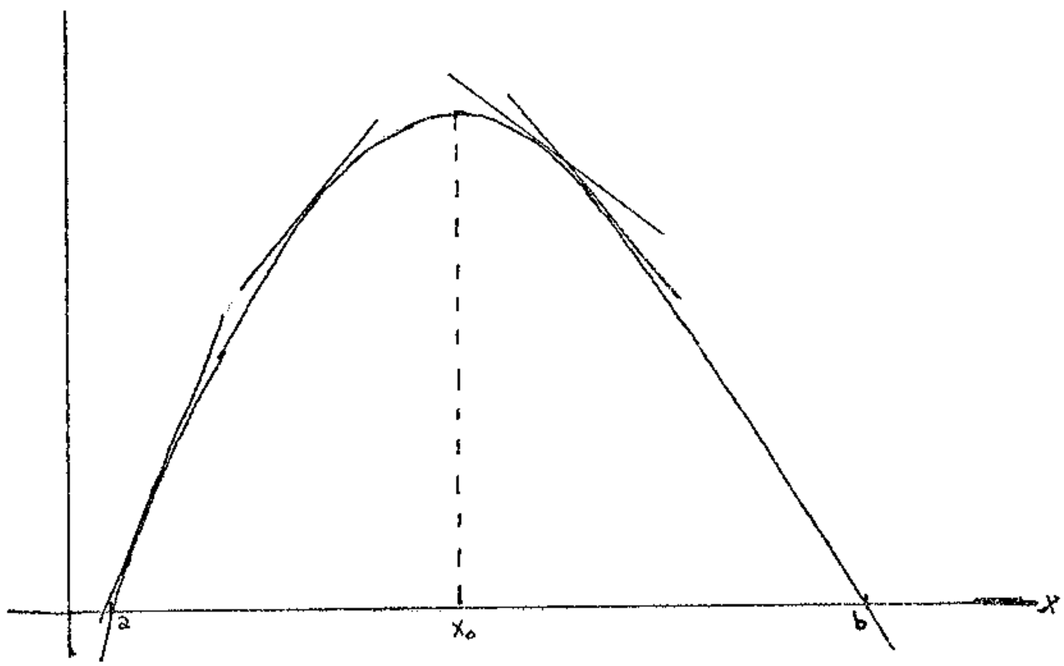

Fig. 2.3--Graphic representation of Theorem 2.1

For values of x such that $a \leq x < x_o$, $y = f(x)$ is an increasing function. At every point along the curve for the given interval, $f'(x) > 0$. This means that for all x lying within the interval $a \leq x < x_o$ the line tangent to $y = f(x)$ has positive slope. For values of x such that $x_o > x \geq b$, $y = f(x)$ is a decreasing function. At every point along the curve for the given interval, $f'(x) < 0$. This means that for all x lying within the interval $x_o > x \geq b$ the line tangent to $y = f(x)$ has negative slope.

In conjunction with the concept of an increasing and/or decreasing function is the concept of the inflection point. This concept relates to the concavity of the given function and is defined in the following manner.

Definition 2.11.--The inflection point of a given continuous function is defined as the point at which the graph of the function changes its direction of concavity.

In this context the inflection point of a given continuous function is simply that point at which the function changes its opening; i.e., if the curve initially opens upward the inflection point is that point where the curve begins to open downward (or vice versa). As a means of obtaining further insight into the concept of the inflection point,[8] consider the graph of the function shown in Figure 2.4. It is assumed

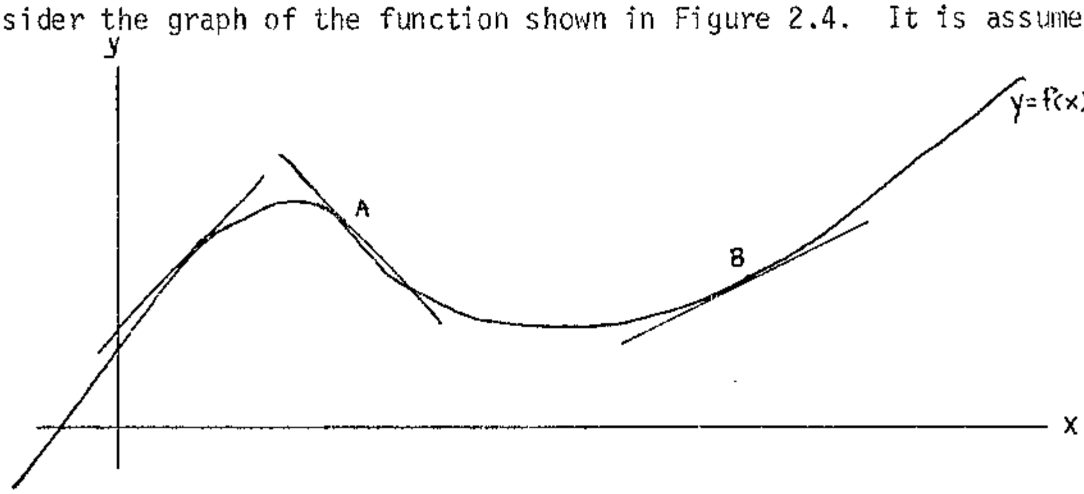

Fig. 2.4--Inflection point

[8] Ibid., pp. 103-104.

that the function is continuous, given by $y = f(x)$, and is such that the line tangent to $y = f(x)$ moves in a positive direction (left to right). As the tangent line rotates in a clockwise direction to point A, the direction of rotation is positive. At point A the tangent line reverses its direction of rotation; i.e., as the tangent line rotates along $y = f(x)$, point A is the point at which the curve changes direction. As the tangent line follows the curve between points A and B the rotation is counter-clockwise. At point B the tangent line again begins a clockwise rotation due to a change in the concavity of the curve.

It is possible, using the derivative, to determine the concavity of a given function. In this application the function is assumed to be differentiable through at least the second derivative, an assumption which, if satisfied, leads to the following operational definitions.

<u>Definition 2.12</u>.--Let $f(x)$ be any continuous function differentiable through at least the second derivative. Let $f'(x)$ decrease within the interval where x increases. Then the function $f(x)$ is defined as being concave downward.

<u>Definition 2.13</u>.--Let $f(x)$ be any continuous function differentiable through at least the second derivative. Let $f'(x)$ increase within the interval where x increases. Then the function $f(x)$ is defined as being concave upward.

Theorem 2.2.--If $y = f(x)$ has a second derivative on an interval, then the abscissa of any inflection point of the interval will be a root of $y'' = f''(x) = 0$.[9]

The meaning of Theorem 2.3 is inherently clear: to determine the critical values for which a continuous function may have points of inflection, solve $f''(x) = 0$ for all possible roots. These roots are the points at which the given function may have a change in the concavity of its curvature. Although the condition $f''(x) = 0$ does not guarantee points of inflection, it does represent a prerequisite for a given point to be a point of inflection, a result that can be used to assist in determining concavity. This result is summarized by the following:

(1) if $f''(x) < 0$, the graph of the function at the point x is concave downward;

(2) if $f''(x) > 0$, the graph of the function at the point x is concave upward;

(3) if $f''(x) = 0$, the function must be tested for $(x \pm \Delta x)$ so as to apply (1) and (2).[10]

The procedure employed in the method is explained as follows: To determine the inflection points of a given function, set the second derivative equal to zero and solve. This will yield critical points for the given function which may be inflection points. It is then necessary to test the roots of $f''(x) = 0$ to determine any change in direction of $f(x)$. A common method is to simply straddle the roots of $f''(x) = 0$:

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[9] Ibid., p. 104.

[10] Chris A. Theodore, Applied Mathematics: An Introduction (Homewood, 1965), p. 433.

suppose x_m is a root of $f''(x) = 0$; obtain $x_m \pm \Delta x$ and substitute into $f''(x)$. If $f''(x)$ changes sign within the defined interval, x_m is an inflection point with value $f(x_m)$.

As an example, consider the function $f(x) = x^3 - 6x^2 + 9x$, $0 < x < \infty$. Test $f(x)$ for inflection points.

Solution: The necessary first and second derivatives, $f'(x)$ and $f''(x)$, respectively, are given by

$$f'(x) = 3x^2 - 12x + 9,$$

$$f''(x) = 6x - 12.$$

For $f''(x) = 0$, $x = 2$, a single critical point for $f''(x)$. To test for any change in concavity, let $\Delta x = 1$; evaluate $f''(x \pm \Delta x)$:

(1) at $x = 2$, $f''(x + \Delta x) \equiv f'(3) = 6 > 0$.

(2) at $x = 2$, $f''(x - \Delta x) = f''(1) = -6 < 0$.

As x approaches the value 2 from the left the function f(x) is concave upward; as x approaches 2 from the right, f(x) is concave downward. Thus the point $(x, f(x)) \equiv (2, 2)$ is an inflection point; it is the point at which the function shifts from concave upward to concave downward. (See Figure 2.5.)

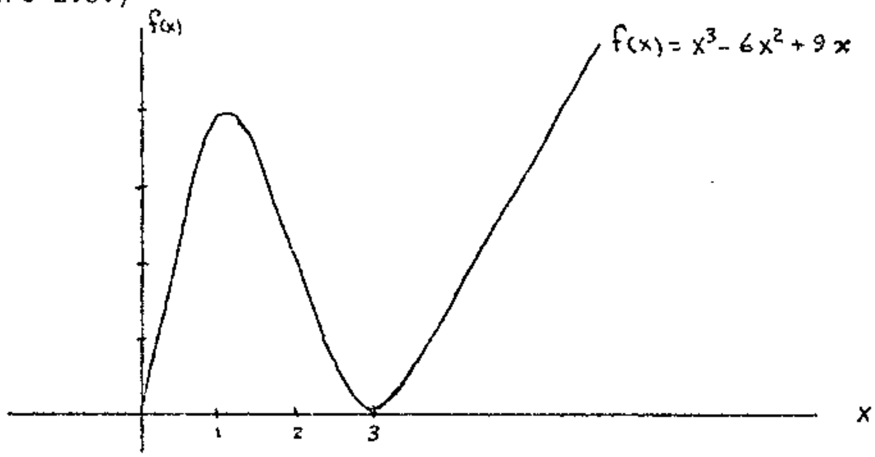

Fig. 2.5--Graph of $f(x) = x^3 - 6x^2 + 9x$

An intuitive look at the preceding example leads directly to the concepts of maxima and minima for a given function. At $x = 1$ the function has a value of $y = 4$; at $x = 3$ the function has a value of $y = 0$, and, for $x > 3$, y is an increasing function. Observationally, a maximum exists at the point (1, 4) and a minimum at (3, 0). However, this same observational approach reveals that for $x > 3$, $y = f(x)$ is increasing without limit. Thus, the point (1, 4) represents a relative maximum and (3, 0) an absolute minimum within the defined interval.

Conditions under which maxima and/or minima exist are rigidly defined in the mathematical theory of the calculus. These conditions, expressed as theorems and definitions, are presented in the following discussion. Since the intent here is interpretation, proofs are not included.

Theorem 2.3 (Necessary Condition for a Maximum).--Let $f(x)$ be any continuous differentiable function defined over the closed interval $a \leq x \leq b$. Let c be any point in the open interval (a, b) such that $a < c < b$. Let $f(a) < f(c)$, and $f(b) < f(c)$ for $a < c < b$. Then there exists at least one value X, $a < X < b$, such that

(1) $f(x) \leq f(X)$, $a \leq x \leq b$,

(2) $f'(X) = 0$.[11]

The conditions $f(a) < f(c)$ and $f(b) < f(c)$ guarantee that $f(x)$ will have at least one maximum point in the interval (a, b). This point(s) x defined for the closed interval $[a, b]$, is (are) such that, for at least one X belonging to (a, b), $f(x)$ is at most equal to $f(X)$, the maximum point of

[11] David V. Widder, Advanced Calculus (Englewood Cliffs, 1961), p. 119.

the curve. In addition, the derivative of $f(x)$ is such that it equals zero, guaranteeing a horizontal tangent line at all points for which $f'(x) = 0$.

Theorem 2.4 (Necessary condition for a minimum).--Let $f(x)$ be any continuous differentiable function defined over the closed interval $a \leq x \leq b$. Let c be any point in the open interval (a, b) such that $a < c < b$. Let $f(a) > f(c)$, and $f(b) > f(c)$ for $a < c < b$. Then there exists at least one number X, $a < X < b$, such that

(1) $f(x) \geq f(X)$, $a \leq x \leq b$,

(2) $f'(x) = 0$.

The conditions $f(a) > f(c)$ and $f(b) > f(c)$ guarantee that $f(x)$ will have at least one minimum point in the interval (a, b). This point(s) x defined for the closed interval [a, b] is (are) such that, for at least one x belonging to [a, b], $f(x)$ is at least equal to $f(X)$, the minimum point of the curve. In addition, the derivative of $f(x)$ is such that it equals zero, guaranteeing a horizontal tangent line at all points for which $f'(x) = 0$.

The primary contribution of these two theorems is the statement of conditions necessary for a given continuous differentiable function to have at least one maximum (minimum) value. The given assumptions are regarded as necessary prerequisites before the "then" clause of the "if-then" phrases and must be satisfied before conditions are suitable for a maximum (minimum); given this satisfaction, the theorems go on to state that a horizontal tangent exists ($f'(x) = 0$) and the function is such that for the point X (within the defined interval) is a point of maximum (minimum) value.

Having presented the conditions necessary for a maximum (minimum) point (or set of such points) on a given continuous curve, it is necessary to define what is meant by a maximum (minimum) point. This definition, however, is not as simple as it might seem as a given function may be such that it has more than one maximum (minimum) point. Such conditions are handled by defining two types of maximum (minimum) points, relative and absolute.

Definition 2.14.--Let $f(x)$ be a given function, defined for the closed interval $a \leq x \leq b$. Let $f(x_o)$ be the function $f(x)$ evaluated at $x = x_o$, where x_o is contained in $[a, b]$. The point x_o is a point of relative maximum if the value of $f(x)$ at $x = x_o$ is at least as great as $f(x)$ evaluated at any point in the neighborhood of x_o;

i.e., $x = x_o$ is a point of relative maxima if $\exists \delta > 0$ such that $f(x_o) \geq f(x)$ for all $x \varepsilon [a, b]$ such that $|x - x_o| < \delta$.

Definition 2.15.--Let $f(x)$ be a given function, defined for the closed interval $a \leq x \leq b$. Let $f(x_o)$ be the function $f(x)$ evaluated at $x = x_o$, where x_o is contained in $[a, b]$. The point x_o is a point of relative minimum if the value of $f(x)$ at $x = x_o$ is at most equal to $f(x)$ evaluated at any point in the neighborhood of x_o;

i.e., $x = x_o$ is a point of relative minima if $\exists \delta > 0$ such that $f(x_o) \leq f(x)$ for all $x \varepsilon [a, b]$ such that $|x - x_o| < \delta$.

Definition 2.16.--Let $f(x)$ be a given function, defined for the closed interval $a \leq x \leq b$. Let $f(x_1)$ be the function $f(x)$ evaluated at $x = x_1$, $a \leq x_1 \leq b$. The point $x = x_1$ is an absolute maximum if the value of $f(x_1)$ is at least as great as the value of $f(x)$ at any other point in the given interval.

i.e., $x = x_1$ is an absolute maximum if $f(x_1) \geq f(x)$ for all x belonging to $[a, b]$.

Definition 2.17.--Let $f(x)$ be a given function, defined for the closed interval $a \leq x \leq b$. Let $f(x_1)$ be the function $f(x)$ evaluated at $x = x_1$, $a \leq x_1 \leq b$. The point $x = x_1$ is an absolute minimum if the value of $f(x_1)$ is at most equal to the value of $f(x)$ at any other point in the given interval.

i.e., $x = x_1$ is an absolute minimum if $f(x_1) \leq f(x)$ for all x belonging to $[a, b]$.

Theorem 2.5 (Sufficient Conditions for Relative Maxima).--Let $f(x)$ belong to the $2n^{th}$ class of continuous functions defined over the closed interval $[a, b]$ such that $f^{2n}(x)$, the $2n^{th}$ derivative of $f(x)$, exists. Let X be any value contained in the open interval (a, b). Let the k^{th} derivative ($k = 1, 2, \ldots, 2n-1$) equal zero for some X belonging to the open interval $a < X < b$. Let the $2n^{th}$ derivative of $f(x)$ evaluated at X be less than zero. Then there exists some positive number ε such that for any small neighborhood around X less than ε the value of the function at any point exceeds the value of the function at X. Symbolically,

$$\text{let } f(x) \in C^{2n},\ a \leq x \leq b;$$

$$\text{let } f^{k}(X) = 0,\ a < X < b;$$

$$\text{let } f^{2n}(X) < 0;$$

then, there exists $\varepsilon > 0$ such that $f(x) > f(X)$ when $0 < |x - X| < \varepsilon$.[12]

Theorem 2.6 (Sufficient Conditions for Relative Minima).--Let f(x) belong to the $2n^{th}$ class of continuous functions defined over the closed interval [a, b] such that the $2n^{th}$ derivative of f(x) exists. Let X be any value contained in the open interval (a, b). Let the k^{th} derivative ($k = 1, 2, \ldots, 2n-1$) equal zero for some X belonging to the open interval $a < X < b$. Let the $2n^{th}$ derivative of f(x), evaluated at X, be greater than zero. Then there exists some positive number ε such that for any small neighborhood around X less than ε the value of the function at any other point is less than the value of the function at X. Symbolically,

$$\text{let } f(x) \in C^{2n},\ a \leq x \leq b;$$
$$\text{let } f^{k}(X) = 0,\ k = 1, 2, \ldots, 2n-1;\ a < X < b;$$
$$\text{let } f^{2n}(X) > 0;$$

then there exists some positive number ε such that $f(x) < f(X)$ when $0 < |x - X| < \varepsilon$.

The two theorems cited previously give the conditions that are sufficient for a given function to have a maximum (minimum) value at some point in the interval over which it is defined. Solving the k^{th} derivative for those values of X, $a < X < b$, for which $f^{k}(X) = 0$ yields critical points to be tested by the $2n^{th}$ derivative. The sign of the $2n^{th}$ derivative, when $f^{2n}(x)$ is evaluated at $x = X$, can then be used to determine whether or not the point X is a maximum or minimum point within the defined interval.

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[12] Ibid., pp. 119-120.

Because of the use made of these concepts in more advanced analysis, a conceptual development is in order. This development will serve to illustrate the distinction between points of relative and absolute maximum (minimum) value, a distinction that is analogous to points of local and global maximum (minimum) value. Consider first the necessary conditions for a point of maximum or minimum.

In a given problem the desired effect is usually known: the function is either being solved for maximum values or minimum values. For a function to have a maximum (or minimum) the graph of the function must be concave downward, falling away from the maximum point (concave upward, sloping upward away from the minimum point). This condition is met by the requirement that, for $a < c < b$, $f(a) < f(c)$ and $f(b) < f(c)$ for a maximum; for $a < c < b$, $f(a) > f(c)$ and $f(b) > f(c)$ for a minimum. This guarantees the existence of at least one point on the graph of $f(x)$ that represents a maximum (or minimum).

With the existence of the extremum (maximum or minimum) established, it remains to determine the type of extremum (maximum or minimum) which has been reached. In this respect the emphasis will be on whether or not a given extremum is a relative maximum (or minimum) or an absolute maximum (or minimum).

Consider first the concept of maximal value, given a maximum point for a function. This maximum point can either be an absolute maximum or a relative maximum, depending upon its relationship with the other acceptable points lying within the defined interval. If the point in question yields a value for the given function that exceeds (or equals) the value of the function evaluated at every other point within the interval, then the point in question is the point of relative maximum; i.e., it is a maximum within a given interval. If the interval is changed, the maximum point may change. If the point in question is a maximum for all points at which the function is defined, then the point in question is an absolute maximum. (See Figure 2.6.)

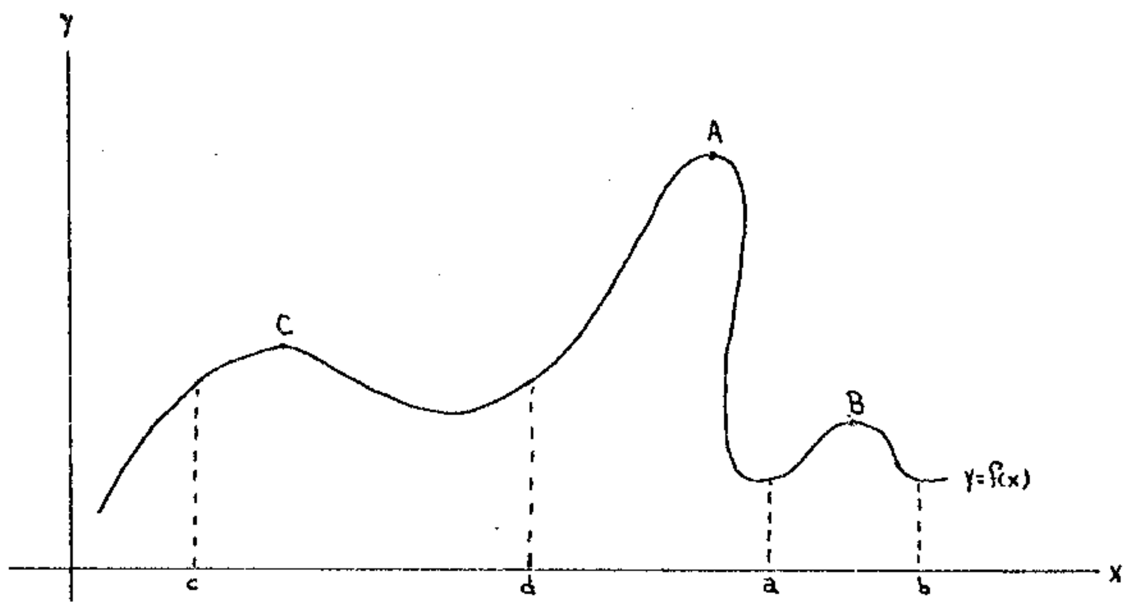

Fig. 2.6--Relative and absolute maxima

In the above figure point A represents an absolute maximum for the function y = f(x), defined for the set of real numbers. Points B and C represent points of relative maxima: point B for the interval [a, b], point C for the interval [c, d]. An interesting application of this concept is the production function where the defined closed intervals ([a, b] and [c, d]) represent levels of production activity; although the function yields maximal values within the given intervals, maximum production is not achieved unless the firm is capable of production within the closed interval [d, a].

The concept of minimal value can be developed in a similar manner. Suppose a given function is to be minimized, and some minimal point has been determined. As was the case in the discussion of maximum values, the minimal point can either be an absolute minimum or a relative minimum, depending upon its relationship with the other acceptable points lying within the defined interval. If the point in question yields a value for the given function that is less than (or equal to) the value of the function evaluated at every other point within the interval, then the point in question is the point of relative minimum; i.e., it is a minimum within a given interval. If the interval is changed, the minimum point may change. If the point in question is a minimum for all points at which the function is defined, then the point in question is an absolute minimum. (See Figure 2.7.)

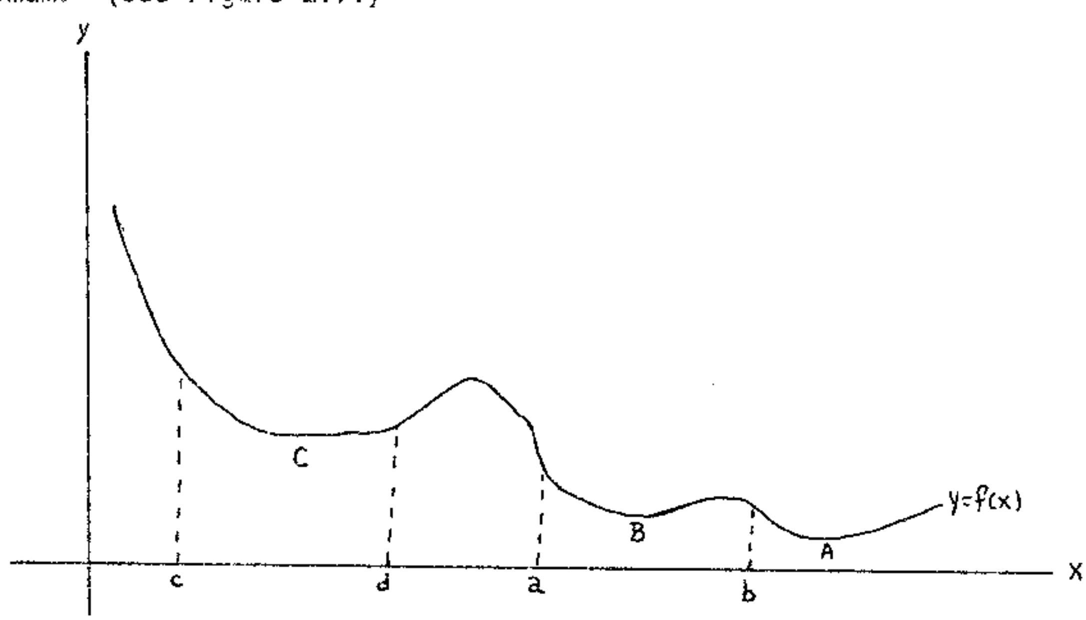

Fig. 2.7--Relative and absolute minima

In Figure 2.7 point A represents an absolute minimum for the function $y = f(x)$, defined for the set of real numbers. Points B and C represent points of relative minima: point B for the interval [a, b], point C for the interval [c, d]. An interesting application of this concept is a curvilinear cost function where the defined closed intervals ([a, b] and [c, d]) represent levels of activity yielding minimum cost figures; but, these cost values (B and C) do not yield the minimum cost for the defined function. The minimum cost value (absolute minimum) occurs at point A, a point which requires activity in excess of that achieved at activity level b.

The sufficiency conditions for relative extrema can be explained in the following manner: Over the defined closed interval the given function is continuous (hence differentiable) through the defined class of continuous functions, and the k^{th} derivative is equated to zero and solved for the critical points. These critical points lie in the open interval (a, b), defined by $a < X < b$. The $(2n)^{th}$ derivative of the given function is then evaluated at each critical point and the results interpreted as follows: (1) if the $(2n)^{th}$ derivative is less than zero, there exists some neighborhood of X of radius $\varepsilon > 0$ such that the function f(x) is a maximum; (2) if the $(2n)^{th}$ derivative is greater than zero, there exists some neighborhood of X of radius $\varepsilon > 0$ such that the function f(x) is a minimum.

The results of the discussion of single variable maxima and minima can be summarized in the following manner. If $y = f(x)$ is a function of one variable, it achieves a maximum at the point $x = X$ if $f(X) \geq f(x)$

for all values of x lying within an ϵ neighborhood about X. It is not necessary that f(X) exceed f(x) outside the ϵ neighborhood.[13] The function $y = f(x)$ achieves a minimum at the point $x = X$ if $f(X) \leq f(x)$ for all values of x lying within an ϵ neighborhood about X. It is not necessary that f(X) be less than f(x) outside the ϵ neighborhood.

Intuitively,

> [a] function that has a maximum (or minimum) is, by definition, neither increasing nor decreasing at its extreme point. But the first derivative is the function's rate of increase...therefore [it must] equal zero at an extreme point [indicating a zero rate of increase]. [In the case of a maximum point] a function first increases, becomes stationary, and then decreases. Thus the second derivative (the rate of change of the first derivative) is [less than zero] at a maximum.[14]

These concepts can be extended to a minimum point in the following manner. In the case of a minimum point a function first decreases, becomes stationary, and then increases. Thus, the second derivative (the rate of change of the first derivative) is greater than zero at a minimum.

A similar summary can be made on the necessary and sufficient conditions for a maximum (or minimum)

> [the function] f(x) attains a maximum (minimum) at [x = X] if and only if (1) dy/dx [or f'(x)] = 0 at [x = X], (2) the first (n-1) (where n is even) derivatives are all zero and the first nonzero derivative (the nth) is negative (positive) at [x = X].[15]

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[13] James M. Henderson and Richard E. Quandt, Microeconomic Theory: A Mathematical Approach (New York, 1958), p. 265.

[14] Ibid., pp. 265-266.

[15] Ibid., p. 267.

The existence of a maximum or minimum point for a continuous function on a defined closed interval is guaranteed. This guarantee is framed within the context of two of the basic theorems of the differential calculus.[16]

<u>Theorem 2.7</u>.--Let f(x) be a continuous function, defined on the closed interval $a \leq x \leq b$. Then the function f(x) has an absolute minimum and an absolute maximum.

<u>Theorem 2.8</u> (<u>Rolle's Theorem</u>).--Let f(x) be a continuous function with the following properties:

(1) f(x) is continuous on the closed interval $a \leq x \leq b$.

(2) $f(a) = f(b) = 0$.

(3) $f'(x)$ exists on the open interval $a < x < b$.

Then there exists at least one value c, $a < c < b$, such that $f'(c) = 0$.

The first of these two theorems guarantees that the set of values for which f(x) is defined have an upper bound; i.e., the set of values determined by f(x) for the interval $a \leq x \leq b$ has both a greatest value and a least value. The second, Rolle's theorem, guarantees the existence of a point c in the open interval (a, b) such that the value of the derivative at that point is zero ($f'(c) = 0$). Thus, for any function satisfying the conditions of Rolle's theorem there is at least one point such that the line tangent to the curve there is horizontal to the x-axis. This horizontal tangent is such that it lies between any two points satisfying $f(x) = 0$. (See Figure 2.8.)

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[16]Fuller and Parker, <u>op. cit.</u>, p. 130.

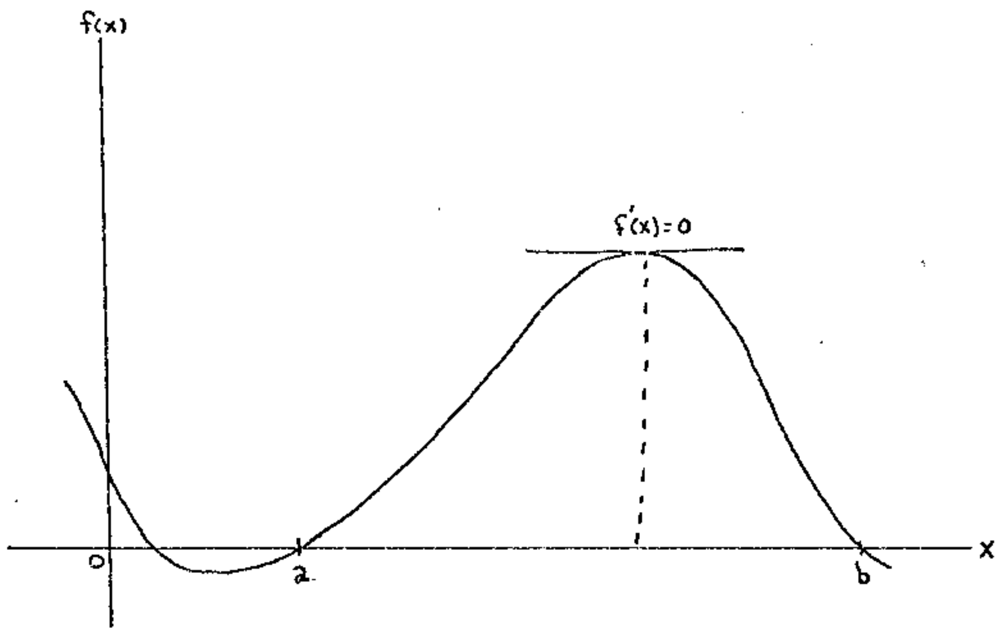

Fig. 2.8--Rolle's theorem

Having surveyed the theorems and definitions from which the application of the max-min calculus stems, it is now feasible to point out the methodology by which a given continuous, differentiable function is tested for max-min values. In administrative use the function is generally of such a nature that it is one which is known, for example, to be a minimizing function (cost) or a maximizing function (profit); however, for functions of degree greater than two (as might be found in inventory analysis with fluctuating order levels) there exists the possibility of relative minima or relative maxima. In such a case the function must be tested for minima and/or maxima.

There are two fundamental methodologies by which a given function is tested for points of maximum-minimum value, the first derivative test and the second derivative test. Of the two, the second derivative test is the more direct, even though it requires the existence of the second derivative. Both methodologies are presented here because there might exist, in practical application, situations in which the use of the second derivative test is either impractical or impossible.

Although both the first derivative test and the second derivative test are given by two separate theorems, both of these tests stem from the same source. This source provides the base on which these two tests are built and is presented as an introduction to the first and second derivative tests for extrema.

Theorem 2.9.--Let f(x) be any function defined on the closed interval $a \leq x \leq b$. Let f(x) be such that a relative extremum (maximum or minimum) exists at $x = c$, where $a < c < b$. Let f(x) be differentiable at $x = c$. Then, $f'(c) = 0$.

This theorem serves to verify that a point of extremum has a tangent line passing through this point parallel to the x-axis. Since the derivative of a function represents the slope of the line tangent to the function, (for $x = c$, $f'(x) = f'(c)$) its value at the extreme point must be zero, a condition which indicates that the curve is no longer increasing or decreasing.[17]

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[17] Ibid., p. 97.

Theorem 2.10 (First Derivative Test).--Let f(x) be a given function, differentiable in the interval $c - \delta < c < c + \delta$, $\delta > 0$.

(1) In this interval let $f'(c) = 0$. The function f(x) has a relative maximum at the point x = c if

(a) $f'(x) > 0$ when $x < c$, and

(b) $f'(x) < 0$ when $x > c$.

(2) In this interval let $f'(c) = 0$. The function f(x) has a relative minimum at the point x = c if

(a) $f'(x) < 0$ when $x < c$, and

(b) $f'(x) > 0$ when $x > c$.[18]

A graphic interpretation is shown in Figure 2.9. It is assumed that f(x) is continuous and differentiable.

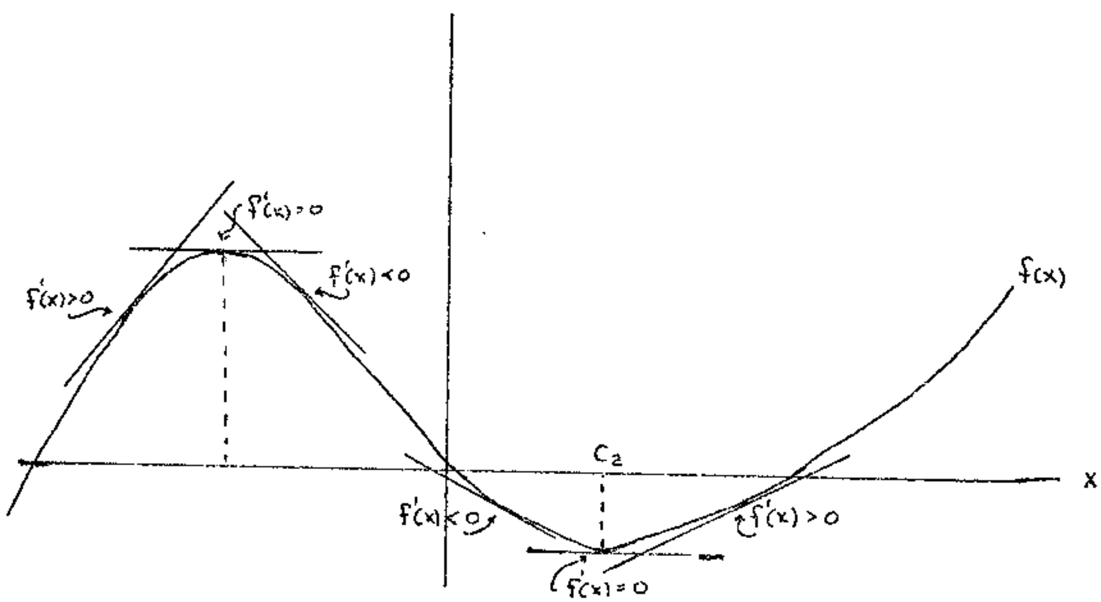

Fig. 2.9--First derivative test

[18] Ibid., p. 98.

From Theorem 2.10 it is known that if $f(x)$ has a relative extremum at $x = c$, the value of $f'(x)$ is equal to zero at that point. Solving $f'(x) = 0$ yields critical points that must be examined for max-min value. If a given critical point, $x = c$, is a maximum then for values of x less than c, $f'(x)$ will be positive; for values of $x = c$, $f'(x)$ will equal zero (Theorem 2.10); and, for values of x greater than c, $f'(x)$ will be negative. This case is shown by $x = c_1$. If a given critical point, $x = c$, is a minimum then for values of $x < c$, $f'(x)$ will be negative; for values of $x = c$, $f'(x)$ will equal zero; and, for values of x greater than c, $f'(x)$ will be positive. This case is shown by $x = c_2$. Thus, a given function has a relative maximum at the critical point $x = c$ if $f'(x)$ changes from a positive value to a negative value as x passes through c from left to right (i.e., $c - \delta < c < c + \delta$); a given function has a relative minimum at the critical point $x = c$ if $f'(x)$ changes from a negative value to a positive value as x passes through c from left to right. This result is summarized by the following algorithm.

Algorithm 2.1 (First Derivative Test).--Let $f(x)$ be a given continuous, differentiable function. Let $f'(x)$ exist at $x = c$.

Step 1. Calculate $f'(x)$; set $f'(x) = 0$ and solve for the roots of $f'(x) = 0$. [Denote these roots by the letter c.]

Step 2. For each root of $f'(x) = 0$, determine the sign of $f'(x)$ for values of x which are less than the root c and for values of x which are greater than the root c.

Step 3. Apply the following criterion:

(a) if $f'(x)$ changes sign from + to - as x increases through c (left to right) the point c is a relative maximum with value $f(c)$;

(b) if $f'(x)$ changes sign from - to + as x increases through c (left to right) the point c is a relative minimum with value $f(c)$;

(c) if $f'(x)$ does not change sign as x increases through c, the point c is neither a maximum nor minimum.

Theorem 2.11 (Second Derivative Test).--Let $y = f(x)$ be a given function, differentiable in the interval $a \leq x \leq b$. Let c be any critical point of $f(x)$; i.e., $f'(x) = 0$ at $x = c$.

(1) The function $f(x)$ has a relative maximum at $x = c$ if $f''(c) < 0$;

(2) The function $f(x)$ has a relative minimum at $x = c$ if $f''(c) > 0$.

The logical consequence of this theorem stems from both Theorems 2.9 and 2.10. It is known that at points of relative extremum $(x = c)$ the value of $f'(x)$ equals zero. When $f''(x) < 0$, $f'(x)$ is decreasing as x increases. Since $f'(c) = 0$, $f'(x)$ is positive for $x < c$ and negative for $x > c$; hence, $f''(c) < 0$ implies a maximum value at $x = c$. When $f''(x) > 0$, $f'(x)$ is increasing as x increases. Since $f'(c) = 0$, $f'(x)$ is negative for $x < c$ and positive for $x > c$; hence, $f''(c) > 0$ implies a minimum value at $x = c$. Thus the criterion for extremum: (1) $f''(c) < 0$ yields a maximum at $x = c$; (2) $f''(c) > 0$ yields a minimum at $x = c$. As was the case for the first derivative test, the use of the second derivative test can be expressed in algorithmic form.

Algorithm 2.2 (Second Derivative Test).--Let $f(x)$ be a continuous, differentiable function. Let $f'(x)$ and $f''(x)$ both exist at $x = c$.

Step 1. Calculate $f'(x)$; set $f'(x) = 0$ and solve for the roots of $f'(x) = 0$. [Denote these roots by the letter c.]

Step 2. Calculate $f''(x)$. Substitute each value of $x = c$ into $f''(x)$ and evaluate.

Step 3. Apply the following criterion:

(a) if $f''(x) < 0$ at $x = c$, the given function achieves a maximum at $x = c$ with value $f(c)$;

(b) if $f''(x) > 0$ at $x = c$, the given function achieves a minimum at $x = c$ with value $f(c)$;

(c) if $f''(x) = 0$ at $x = c$, the test fails and the first derivative test must be used.

Functions of two or more variables.--The extension of the differential calculus to functions of two or more variables is done in a manner analogous to that for functions of one variable. The concepts involved are, basically, the same (for example, relative maximum, relative minimum, etc.). For this reason the necessary tools for multivariable analysis are presented without additional context; the underlying conceptual developments have not been changed, just the magnitude of application.

Definition 2.18 (Function).--Suppose there is a collection of points $(x_1, x_2,\ldots,x_n)$ in n-dimensional space and to each of these points there is a uniquely determined number *y* yielding an n-tuple $(x_1, x_2,\ldots,x_n, y)$. The collection of n-tuples thus established is called a function of the independent variables $(x_1, x_2,\ldots,x_n)$. The collection of points $(x_1, x_2,\ldots,x_n)$ is called the domain of the function; the set of corresponding values of y is called the range of the function. Notationally, $y = f(x_1, x_2,\ldots,x_n)$. (As a matter of note, administrative application of algebraic functions and/or systems requires that the domain of the function be restricted to the set of x_i, $i = 1, 2, \ldots, n$, such that x_i is at least zero.)

Definition 2.19 (Limit).--The limit of the function $y = f(x_1, x_2, \ldots, x_n)$ as x_1 approaches a_1, x_2 approaches $a_2, \ldots, x_n$ approaches a_n, is L if for any $\varepsilon > 0$

$$|f(x_1, x_2, \ldots, x_n) - L| < \varepsilon$$

whenever $|x - a_1|$, $|x - a_2|, \ldots, |x_n - a_n|$ are sufficiently small but different from zero. Symbolically,

$$\lim f(x_1, \ldots, x_n) = L \text{ as } x_1 \to a_1,\ x_2 \to a_2, \ldots, x_n \to a_n.$$

Definition 2.20 (Continuity).--The function $y = f(x_1, x_2, \ldots, x_n)$ is continuous at $a_1, a_2, \ldots, a_n)$ if each of the following three conditions is satisfied:

(1) $f(x_1, x_2, \ldots, x_n)$ is defined at $(a_1, a_2, \ldots, a_n)$;

(2) $\lim f(x_1, x_2, \ldots, x_n)$ exists as $x_1 \to a_1$, $x_2 \to a_2, \ldots, x_n \to a_n$; and

(3) $\lim f(x_1, x_2, \ldots, x_n) = f(a_1, a_2, \ldots, a_n)$ as $x_1 \to a_1$, $x_2 \to a_2, \ldots,$ $x_n \to a_n$.

With these concepts in hand, the use of the derivative in multivariable functions can be defined. In this particular case the use of the derivative as a tool of optimization involves two applications: (1) the partial derivative and (2) the total differential.

Definition 2.21 (Partial Derivative).--Let $y = f(x_1, x_2, \ldots, x_n)$ be a function defined over a given interval. Then the partial derivative of y with respect to x_i (i = 1, 2,...,n) is defined as the function obtained by differentiation with respect to x_i alone, all other independent variables being held constant.

i.e., $$\frac{\partial y}{\partial x_i} = \lim_{\partial x_j \to 0} \frac{y(x_1, x_2, \ldots, x_{i-1}, x_i + \partial x_i, \ldots, x_n) - y(x_1, x_2, \ldots, x_n)}{\partial x_i} .$$

The partial derivative is thus a measure of the average rate of change in the given function with only one independent variable allowed to vary. Such techniques are useful in production functions relating multiproducts or cost functions exhibiting multivariable tendencies or interrelationships among the cost elements. Economic theory utilizes this concept as a means of interpreting marginal product, marginal cost, marginal utility, etc.

The geometric interpretation of the partial derivative is demonstrated graphically in Figure 2.10. For purposes of simplicity the given function is expressed in terms of two independent variables, x_1 and x_2. The analysis can be generalized to partial derivatives of functions with n independent variables.

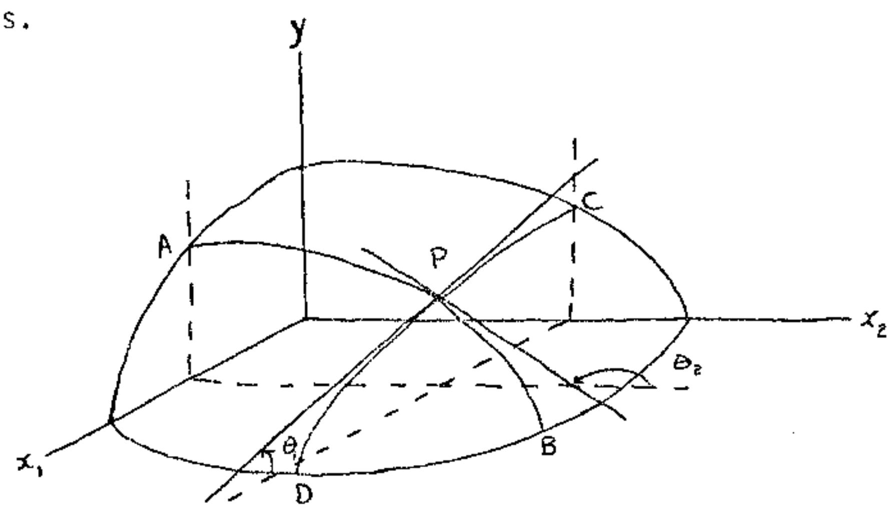

Fig. 2.10--Geometric interpretation of the partial derivative

Let $y = f(x_1, x_2)$ be represented by the graphic surface in Figure 2.10. Let P represent any set of coordinates (x'_1, x'_2, y) on the given surface. The plane $y = x'_1$ intersects the surface $y = f(x_1, x_2)$ along the curve APB; the plane $y = x'_2$ intersects the surface $y = f(x_1, x_2)$ along the curve CPD. Along the curve CPD the value of x_2 is constant, making y a function of one independent variable x_1. Thus, $\frac{y}{x_1} = f_1(x_1, x_2)$ represents the slope of CPD at point P. Similarly, $\frac{\partial y}{\partial x_2} = f_2(x_1, x_2)$ is the slope of APB at point P. Since the value of the slope at a given point for a given function is identical to the tangent of the angle made by the line tangent to the curve at that point,

$$\tan \Theta_1 = \frac{\partial y}{\partial x_1} = f_1(x_1, x_2);$$

$$\tan \Theta_2 = \frac{\partial y}{\partial x_2} = f_2(x_1, x_2).$$

This type of analysis has been applied to several areas of economic analysis.[19] In particular, suppose total profit is defined in terms of the average cost per unit and the level of sales. If y denotes total profit and x_1 and x_2 denote the average cost per unit and the level of sales, respectively, the profit function is written $y = f(x_1, x_2)$. A graphic representation of this relationship would result in a hill similar to that in Figure 2.10. The partial derivative $\frac{\partial y}{\partial x_1}$ can be used to determine the change in profit due to a change in the average cost per unit, with the level of sales held constant. Similarly, the partial derivative $\frac{\partial y}{\partial x_2}$

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[19] William J. Baumol, Economic Theory and Operations Analysis (Englewood Cliffs, 1961), pp. 51-58.

can be used to determine the change in profit due to a change in the level of sales, with the average cost per unit held constant.

With the understanding that the application of the differential calculus to functions of two or more variables is, in essence, an extension of the techniques for a function of one variable, the techniques for determining maximum or minimum values will be presented without discussion. Included in this presentation will be the concepts of the saddle-point, a relationship somewhat similar to the inflection point, and the Jacobian, which is primarily a computational technique for later use. For this presentation $f_i(x_1,\ldots,x_n)$, $(i = 1, 2,\ldots,n)$, will denote the derivative of $f(x_1,\ldots,x_n)$ with respect to the i^{th} variable. The expression $f_{ij}(x_1,\ldots,x_n)$, $(i, j = 1, 2,\ldots,n)$, will denote the second partial derivative of $f(x_1,\ldots,x_n)$,

$$f_{ij}(x_1, x_2,\ldots,x_n) = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f(x_1, x_2,\ldots,x_n),$$

$$i = 1, 2,\ldots,n; \; j = 1, 2,\ldots,n.$$

The extension of this concept to the n^{th} partial derivative is accomplished in a similar manner. It is assumed that $f(x_1, x_2,\ldots,x_n)$ is differentiable through the n^{th} partial derivative.

The conditions by which points of maxima-minima are identified will be presented with reference to two cases. The first will be for a function defined in terms of two independent variables. The second will be for a function defined in terms of three or more independent variables.

<u>Theorem</u> <u>2.12</u>.--Let $y = f(x_1, x_2)$, where the partial derivatives exist through at least the second partial derivatives of both x_1 and x_2. Then

$y = f(x_1, x_2)$ has a relative maximum at (X_1, X_2) if and only if the following conditions are satisfied:

(a) $f_1(x_1, x_2) = f_2(x_1, x_2) = 0$ at the critical points (X_1, X_2);

(b) $f^2{}_{12}(x_1, x_2) - f_{11}(x_1, x_2) \cdot f_{22}(x_1, x_2) < 0$ at (X_1, X_2); and,

(c) $f_{11}(x_1, x_2) < 0$ at (X_1, X_2).

Theorem 2.13.--Let $y = f(x_1, x_2)$, where the partial derivatives exist through at least the second partial derivatives of x_1 and x_2. Then $y = f(x_1, x_2)$ has a relative minimum at (X_1, X_2) if and only if the following conditions are satisfied:

(a) $f_1(x_1, x_2) = f_2(x_1, x_2) = 0$ at the critical points (X_1, X_2);

(b) $f^2{}_{12}(x_1, x_2) - f_{11}(x_1, x_2) \cdot f_{22}(x_1, x_2) < 0$ at (X_1, X_2); and,

(c) $f_{11}(x_1, x_2) > 0$ at (X_1, X_2).

Because of the nature of multivariable functions, it is possible for a given function to contain both positive and negative values in every neighborhood of a given set of critical points (X_1, X_2). If the point approached from one side tends to decrease the value of the defined function, it exhibits properties similar to those which identify a minimum. If the point is approached from the other side and tends to increase the value of the defined function, it exhibits properties similar to those which identify a maximum. Such a condition is described by the term saddle-point and is shown in Figure 2.11.

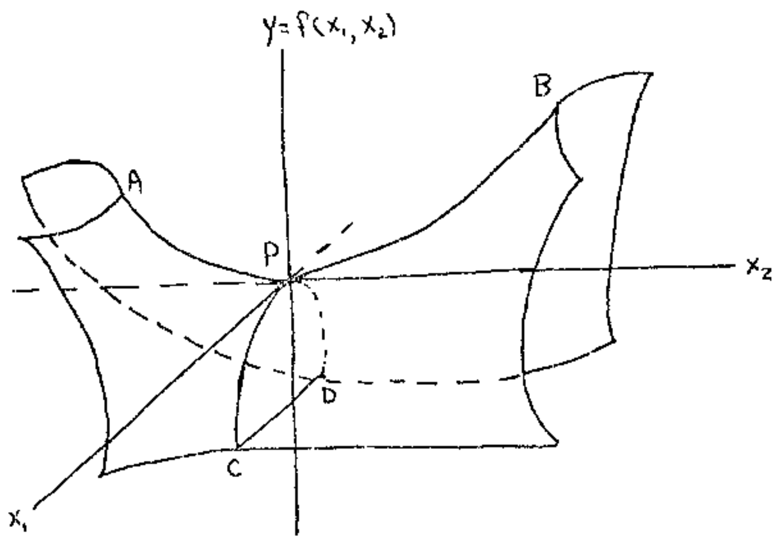

Fig. 2.11--Saddlepoint

A cursory examination of Figure 2.11 reveals that the saddle-point concept identifies a point that is a minimum point with respect to one variable and a maximum point with respect to the other variable. For example, any point along APB will indicate that a minimum occurs at P. However, any point along CPD will indicate that a maximum occurs at P. Thus, P is a saddle-point for the illustrated function since it is both a maximum and a minimum for neighborhoods around P. The conditions that must be satisfied if a given point is a saddle-point are contained in the following theorem.

<u>Theorem</u> <u>2.14</u>.--Let $f(x_1, x_2)$ be continuous through at least second partial derivatives. Then $f(x_1, x_2)$ has a saddle-point at (X_1, X_2) if the following conditions are satisfied:

(a) $f_1(x_1, x_2) = f_2(x_1, x_2) = 0$ at the critical points (X_1, X_2); and,

(b) $f^2_{12}(x_1, x_2) - f_{11}(x_1, x_2) \cdot f_{22}(x_1, x_2) > 0$ at (X_1, X_2).[20]

The importance of the saddle-point concept can be illustrated in the consideration of game theory. Game theory describes a decision process which is concerned with the determination of optimal strategies (courses of action) in competitive situations. In this form of decision analysis, the game situation is one similar to collective bargaining, business competition, and conflict situations. The objective is to select that strategy which minimizes maximum loss and maximizes minimum return. Such a strategy is called minimax and corresponds to the saddle-point, that point which represents a minimum on one side and a maximum on the other.

Much of the work done in classical optimization is restricted to the quadratic function. For example, a cost curve for a multiproduct firm (in this case three cost inputs) can be assumed quadratic. Such a function has the quadratic form

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}x_ix_j$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + a_{22}x^2_2 + a_{23}x_2x_3 + a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3^2.$$

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[20] If the test described here is such that $f^2_{12}(x_1, x_2) - f_{11}(x_1, x_2) \cdot f_{22}(x_1, x_2) = 0$ at (X_1, X_2), the given function, $f(x_1, x_2)$ must be investigated near $x_1 = X_1$ and $x_2 = X_2$.

In this context the a_{ij} values represent constants (cost coefficients in this example), the values of which are fixed. The analysis of this particular class of functions requires that the given function be of a given quadratic form; i.e., positive definite, positive semi-definite, negative definite, or negative semi-definite. Definitions of these terms follow.

<u>Definition 2.22</u>.--Let $f(x_1, x_2,\ldots,x_n)$ be a given function in n variables. Then $f(x_1, x_2,\ldots,x_n)$ is positive definite (negative definite) if, and only if, $f(x_1, x_2,\ldots,x_n) > 0$ (< 0) except when $x_i = 0$ for $i = 1, 2,\ldots,n$.

<u>Definition 2.23</u>.--Let $f(x_1, x_2,\ldots,x_n)$ be a given function in n variables. Then $f(x_1, x_2,\ldots,x_n)$ is positive semi-definite (negative semi-definite) if, and only if, $f(x_1, x_2,\ldots,x_n) \geq 0$ (≤ 0), with the equality holding for certain values of $x_i \neq 0$ for $i = 1, 2,\ldots,n$.

Fortunately, the function $f(x_1, x_2,\ldots,x_n)$ can be written in a matrix form. This form simplifies the problem of determining the definiteness of a given function. As an example, consider the two variable function

$$f(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2.$$

This function can be written

$$\begin{aligned} f(x_1, x_2) &= a_{11}x_1^2 + a_{12}x_1x_2 \\ &\quad + a_{21}x_2x_1 + a_{22}x_2^2 \\ &= [x_1, x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} . \end{aligned}$$

The function $f(x_1, x_2)$ is said to be positive definite if, and only if,

$$a_{11} > 0, \text{ and } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0.$$ [21]

The use of the double bars, | |, indicates the determinant of the enclosed values.

The results of this example can be easily extended to the three variable case. In this extension the function being examined has the form

$$\begin{aligned} f(x_1, x_2, x_3) &= \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}x_i x_j \\ &= a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + a_{22}x_2^2 \\ &\quad + a_{23}x_2x_3 + a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3^2 \\ &= a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 \\ &\quad + a_{21}x_2x_1 + a_{22}x_2^2 + a_{23}x_2x_3 \\ &\quad + a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3^2 \\ &= [x_1, x_2, x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} . \end{aligned}$$

The function $f(x_1, x_2, x_3)$ is said to be positive definite if, and only if,

$$a_{11} > 0; \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0; \text{ and, } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0.$$

[21] Widder, _op. cit._, p. 132.

Similar operations can be used to extend these results to functions defined in terms of more than three variables.

These results have defined positive definite forms as determined by the coefficients and determinants of the given function. When the other forms are being considered, it is better to resort to the analysis of the coefficient matrix.[22]

Functions with three or more variables are examined for points of maxima or minima by considering the determinant of second partial derivatives. It is assumed that the function is continuous, with existing second partial derivatives. Each of the second partial derivatives is evaluated at the critical points of the given function.

Theorem 2.15.--Let $f(x_1, x_2, x_3)$ be continuous through at least second derivatives. Let

$$f_{ij} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f(x_1, x_2, x_3) \text{ , } (i, j = 1, 2, 3),$$

be evaluated at the critical point (X_1, X_2, X_3).

Then $f(x_1, x_2, x_3)$ has a relative minimum at (X_1, X_2, X_3) if the following conditions are satisfied:

(a) $f_1(x_1, x_2, x_3) = f_2(x_1, x_2, x_3) = f_3(x_1, x_2, x_3) = 0$ at (X_1, X_2, X_3).

(b) $f_{11} > 0; \quad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0; \quad \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} > 0.$

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[22]Franklin A. Graybill, An Introduction to Linear Statistical Models, Vol. I (New York, 1961), pp. 3-4.

Application of Theorem 2.15 requires that $\frac{\partial}{\partial x_1}$ $f(x_1, x_2, x_3)$, $\frac{\partial}{\partial x_2} f(x_1, x_2, x_3)$, and $\frac{\partial}{\partial x_3} f(x_1, x_2, x_3)$, respectively, be equated to zero and solved for critical points (X_1, X_2, X_3). The second partial derivatives of $f(x_1, x_2, x_3)$ are then obtained and evaluated at the critical point(s) of $f(x_1, x_2, x_3)$. If (X_1, X_2, X_3) defines a point of relative minima, it is necessary that $f_{11}(x_1, x_2, x_3)$ exceed zero at (X_1, X_2, X_3). In addition, it is also necessary that the determinants $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$ and $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$ exceed zero in value. The numerical value of each of the elements of the respective determinants is obtained by evaluating the indicated second partial derivative at (X_1, X_2, X_3).

The conditions for determining relative maxima are similar to those of Theorem 2.15. The difference is found in the sign of f_{11} and the three by three determinant.

<u>Theorem 2.16</u>.--Let $f(x_1, x_2, x_3)$ be any continuous function through at least second derivatives. Let

$$f_{ij} = \frac{\partial}{\partial x_i} \quad \frac{\partial}{\partial x_j} \; f(x_1, x_2, x_3) \;, \; i = 1, 2, 3; \; j = 1, 2, 3,$$

be evaluated at the critical point (X_1, X_2, X_3). Then $f(x_1, x_2, x_3)$ has a relative maximum at (X_1, X_2, X_3) if the following conditions are satisfied:

(a) $f_1(x_1, x_2, x_3) = f_2(x_1, x_2, x_3) = f_3(x_1, x_2, x_3) = 0$ at (X_1, X_2, X_3).

(b) $f_{11} < 0$; $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0$; and $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} < 0.$

The Jacobian

The Jacobian is a specialized notation for a determinant whose elements are partial derivatives and is especially useful in multivariable analysis. Its use provides a criterion for determining the functional dependence of two or more functions as well as a means for analyzing problems which require a change of variable.[23] In this application the Jacobian provides a necessary and sufficient condition for two or more functions to be related via some identifical function.

Theorem 2.17.--Let $u = u(x_1, x_2)$ and $v = v(x_1, x_2)$ be any two functions in x_1 and x_2. A necessary and sufficient condition that $u = u(x_1, x_2)$ and $v = v(x_1, x_2)$ be connected by an identical relation $f(u, v) = 0$ is that the Jacobian vanish (= 0). It is assumed that $f(u, v)$ has no stationary value in the domain under consideration. The existence of $f(u, v) = 0$ indicates a functional dependence between u and v.

As noted previously, the Jacobian is a determinant whose elements are partial derivatives. As a determinant, the Jacobian functions as a computational tool. It is defined as follows.

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[23] Widder, op. cit., p. 48.

Definition 2.24.--Consider the system of simultaneous equations

$$f^1(x_1, x_2,\ldots,x_n) = y_1,$$
$$f^2(x_1, x_2,\ldots,x_n) = y_2,$$
$$\cdots\cdots\cdots\cdots$$
$$f^n(x_1, x_2,\ldots,x_n) = y_n.$$

The Jacobian, the determinant of the first partial derivatives of $f^k(x_1, x_2,\ldots,x_n)$, is defined by

$$J = \frac{\partial(y_1, y_2,\ldots,y_n)}{\partial(x_1, x_2,\ldots,x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} & \cdots & \frac{\partial y_1}{\partial x_1} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \frac{\partial y_n}{\partial x_3} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

Evaluation of the determinant defined by the Jacobian provides a means of examining a system of simultaneous equations for the existence of unique solutions. In addition, evaluation of the Jacobian provides a means of determining the existence of dependent solutions.[24]

Theorem 2.18.--Let $f^k(x_1, x_2,\ldots,x_n)$, $(k = 1, 2,\ldots,n)$, be a set of n continuous functions with continuous first partial derivatives. Let $(\overset{\circ}{x}_1, \overset{\circ}{x}_2,\ldots,\overset{\circ}{x}_n)$ be any point within the domain of definition for which the system of equations

[24] Henderson and Quandt, op. cit., p. 275.

$$f^1(x_1, x_2, \ldots, x_n) = y_1,$$
$$f^2(x_1, x_2, \ldots, x_n) = y_2,$$
$$\cdots\cdots\cdots$$
$$f^n(x_1, x_2, \ldots, x_n) = y_n,$$

is satisfied. A necessary and sufficient condition for the system defined by $f^k(x_1, x_2, \ldots, x_n) = y_k$, $(k = 1, 2, \ldots, n)$, to have a solution $x_k = \phi^k(y_1, y_2, \ldots, y_n)$, $(k = 1, 2, \ldots, n)$ is that the Jacobian,

$$J = \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \frac{\partial y_n}{\partial x_3} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix},$$

not equal zero in a neighborhood about $(\overset{\circ}{x}_1, \overset{\circ}{x}_2, \ldots, \overset{\circ}{x}_n)$.

Theorem 2.19.--Let $f^k(x_1, x_2, \ldots, x_n)$, $(k = 1, 2, \ldots, n)$, be a set of n continuous functions with continuous first partial derivatives. A necessary and sufficient condition for the existence of a function, $G(y_1, y_2, \ldots, y_n) = 0$, which expresses functional dependence among the system of equations

$$f^1(x_1, x_2, \ldots, x_n) = y_1,$$
$$f^2(x_1, x_2, \ldots, x_n) = y_2,$$
$$\cdots\cdots\cdots$$
$$f^n(x_1, x_2, \ldots, x_n) = y_n,$$

is for the Jacobian to vanish identically or to vanish at every point in a neighborhood about a point $(\overset{\circ}{x}_1, \overset{\circ}{x}_2, \ldots, \overset{\circ}{x}_n)$.

If the system of equations is linear, Theorem 2.18 is identical to the Cramer's Rule requirement that the determinant of coefficients be nonzero. Theorem 2.19 is identical to the requirement that a vanishing determinant of coefficients implies dependent solutions for the given system. For example, consider the functions

$$x_1^2 - 2x_2 - 2 = y_1$$

$$x_1^4 - 4x_1^2 x_2 + 4x_2^2 = y_2.$$

The Jacobian is given by

$$J = \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} 2x_1 & -2 \\ 4x_1^3 - 8x_1x_2 & -4x_1^2 + 8x_2 \end{vmatrix}.$$

Evaluating the determinant,

$$\begin{aligned} 2x_1(-4x_1^2 + 8x_2) - [(-2)(4x_1^3 - 8x_1x_2)] &= -8x_1^3 + 16x_1x_2 - [-8x_1^3 + 16x_1x_2] \\ &= -8x_1^3 + 16x_1x_2 + 8x_1^3 - 16x_1x_2 \\ &= 0. \end{aligned}$$

Since the Jacobian vanishes, the given system of equations will have a set of dependent solutions. For the given system, a solution set is

$$x_1^2 = \frac{6x_2^2 - y_2}{2 - y_1}, \quad x_2 \text{ independent.}$$

If $x_1^2 - 2x_2 - 2 = y_2$ is solved for x_2 and the solution is substituted into $x_1^4 - 4x_1^2x_2 + 4x_2^2 = y_2$, the resulting expression, $(y_1 + 2)^2 - y_2 = 0$, defines the functional dependence for the system.

The Total Differential

The use of the total differential can be shown by reference to a utility function defined by $U° = f(x_1, x_2)$, where $U°$ is a given (assigned) constant. By changing the values of $U°$, a series of positive quadrant curves (for $x_1, x_2 > 0$) can be obtained as shown in Figure 2.12. This set of curves is defined as an indifference map with each curve representing the locus of all commodity combinations from which the consumer derives the same level of personal satisfaction.

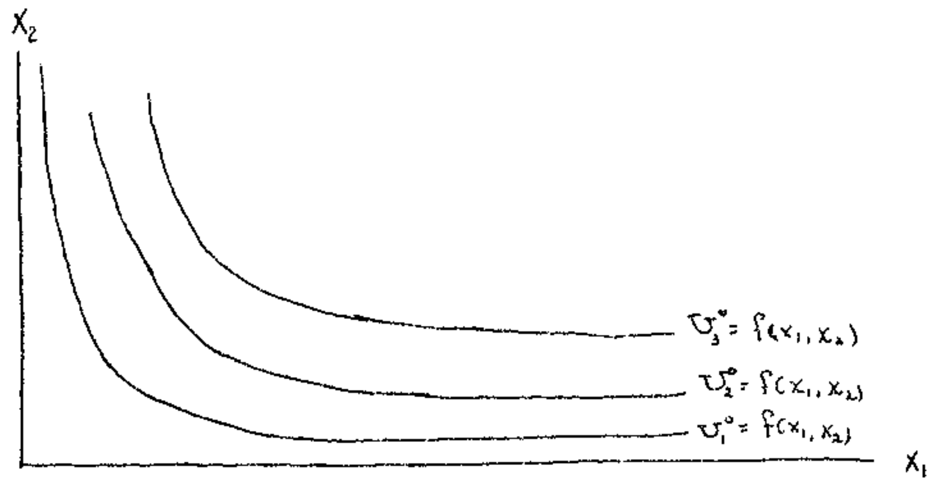

Fig. 2.12--Utility curves

Of particular interest in this type of analysis is the rate at which a given consumer substitutes x_1 for x_2, or x_2 for x_1.

The analysis of functions similar to those of Figure 2.12 utilizes the total differential.

Definition 2.25.--Let $y = f(x_1, x_2, \ldots, x_n)$ be a continuous function in n variables. Then the total differential of the function $y = f(x_1, x_2, \ldots, x_n)$ is defined by

$$dy = f_1(x_1,\ldots,x_n)dx_1 + f_2(x_1,\ldots,x_n)dx_2 +\ldots+ f_n(x_1,\ldots,x_n)dx_n.$$

The expression defined by the total differential is

> the general form of the equation of the tangent plane (or hyperplane) to the surface (or hypersurface) defined by $y = f(x_1, x_2,\ldots,x_n)$. It also provides an approximate value of the change in the function when all variables are permitted to vary, provided that the variation in the independent variables is small. The total derivative of the function $[y = f(x_1,\ldots,x_n)]$ with respect to x_i is
>
> $$\frac{dy}{dx_i} = f_1 \frac{dx_1}{dx_i} + f_2 \frac{dx_2}{dx_i} +\ldots+ f_i +\ldots+ f_n \frac{dx_n}{dx_i}$$
>
> or the rate of change of y with respect to x_i when all other variables are permitted to vary and where all x_j are specified functions of x_i.[25]

In the previous example, the rate at which the consumer will substitute x_1 for x_2 (or vice versa) is expressed by

$$\frac{dx_2}{dx_1} = - \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}.$$ [26]

This rate of substitution holds so long as the consumer chooses to maintain a given level of utility.

The Lagrange Multiplier

As a tool of classical optimization theory, the Lagrange multiplier technique represents an extension of the multivariable calculus and a step toward a more realistic representation of problem situations. This

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[25] Ibid., pp. 269-270.

[26] For $U = f(x_1, x_2)$, the total change in utility tends to approach zero as a limit. Thus, $dU = 0$. Applying the concept of the total differential $f_1dx_1 + f_2dx_2 = 0$. Solving for $\frac{dx_2}{dx_1}$ gives $\frac{dx_2}{dx_1} = -\frac{f_1}{f_2}$.

is accomplished by structuring the mathematical formulation in such a way that any competitive or restrictive conditions are written into the functional model as side restrictions. These side restrictions are written as equalities and are incorporated into the function to be optimized through the use of the Lagrange multiplier.

The mathematical formulation of a problem requiring the use of the Lagrange multiplier is characterized by a function to be optimized subject to a defined set of equality constraints. The use of the Lagrange multiplier provides a method for analyzing optimization problems that removes the necessity of considering, in a special way, variables which are regarded as independent.[27]

In the traditional unconstrained optimization problem, the "optimal" solution is achieved by partially differentiating the unconstrained function with respect to each variable, equating these partial derivatives to zero, and solving the resulting set of simultaneous equations. Such a system of equations has as many equations as variables. The existing solutions, consequently, are unique.

When the decision problem consists of a function to be optimized subject to a set of constraints expressed as equalities, the partial differentiation process of the unconstrained problem results in a set of simultaneous equations that has more equations than unknowns. The use of the Lagrange multiplier technique avoids this problem. Utilization of the Lagrange multiplier increases the number of variables in such a way that the system of partial derivative equations has as many unknowns as equations. This system then has a unique set of solutions.

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[27]Widder, op. cit., p. 135.

Definition 2.26.--Lagrange multipliers are defined as arbitrary parameters introduced into an optimizing process to increase the number of variables to equality with the number of partial derivative equations and constraint equations. The purpose of the Lagrange multiplier is to serve as an agent to transform a restricted (or constrained) problem into an unrestricted (or unconstrained) problem.

The role of the Lagrange multiplier is illustrated by the following non-numeric example. Let $f(x_1, x_2, \ldots, x_n)$ be an n-variable differentiable function representing cost for a particular firm. The problem is to minimize $f(x_1, x_2, \ldots, x_n)$ subject to m differentiable equality restrictions. The restrictions are imposed by labor, cost of materials, budget, etc., and are defined by $g_j(x_1, x_2, \ldots, x_n) = 0$, $(j = 1, 2, \ldots, m;\ m \leq n)$. Let $\lambda_j (j = 1, 2, \ldots, m)$ denote the Lagrange multiplier for the j^{th} restriction. Incorporation of the m Lagrange multipliers as part of the cost function $f(x_1, x_2, \ldots, x_n)$ results in the following Lagrange function:

$$F(x_1, x_2, \ldots, x_n; \lambda_1, \lambda_2, \ldots, \lambda_m) = f(x_1, x_2, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j g_j(x_1, x_2, \ldots, x_n).$$

The expanded form of the Lagrangian function is given by

$$\begin{aligned} F(x_1, x_2, \ldots, x_n; \lambda_1, \lambda_2, \ldots, \lambda_m) = f(x_1, x_2, \ldots, x_n) &+ \lambda_1 g_1(x_1, x_2, \ldots, x_n) \\ &+ \lambda_2 g_2(x_1, x_2, \ldots, x_n) + \ldots + \\ &\lambda_m g_m(x_1, x_2, \ldots, x_n). \end{aligned}$$

This function is minimized by direct application of the partial derivative.[28]

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[28] Ibid., p. 136.

The Lagrange multiplier technique is a tool of analysis that is suitable for optimizing a function subject to a set of constraint equalities. Under certain conditions it is applicable to functions that are to be optimized subject to a set of constraint inequalities. When applied to inequalities, the Lagrange multiplier technique treats the inequalities as though they were equalities. When the optimal solution is obtained, the sign of the Lagrange multiplier is used to determine whether or not the inequality restriction is actually limiting the optimum value of the given function.

The application of the Lagrange multiplier technique utilizes two of the basic tools of classical optimization theory. First, it uses the differential calculus to construct a set of simultaneous equations. Then, it uses Cramer's rule or the techniques of matrix theory to solve the constructed system. As a computational technique, the use of the Lagrange multiplier requires the following sequence of steps:

(1) set up the restricting j^{th} functional relationship such that its value is zero when multiplied by a variable, the Lagrange multiplier, λ_j;

(2) add the product of (1) to the original function that is to be optimized;

(3) optimize the resulting function described in (2) by equating the partial derivatives of all the variables, including λ_j, to zero;

(4) if the restrictions are inequalities, and the given function is to be maximized, interpret as follows:

(a) $\lambda_j > 0$ indicates that the j^{th} inequality is not restrictive to the maximum value, and the given function can be optimized without regard to the j^{th} restriction;

(b) $\lambda_j \leq 0$ indicates that the j^{th} inequality is restricting the optimum value and setting this inequality equal to zero will result in an optimal solution.

The meaning associated with the Lagrange multiplier depends upon the function being optimized. However, the use of the Lagrange multiplier removes the necessity of stipulating the independent variable in the solution set. That is, the use of the Lagrange multiplier transforms a problem that would have a set of dependent solutions into one that has a unique set of solutions.

As a means of illustrating the Lagrange multiplier technique, consider the following problem. It is assumed that both the objective function and the constraint equations are differentiable. Output (y) is defined to be a function of two input factors (x_1 and x_2). The input factors have unit costs c_1 and c_2, respectively. In addition, it is assumed that a fixed cost (c_o) exists for all other input factors, assumed fixed. The problem is to maximize the production function, $y = f(x_1, x_2)$, subject to the cost restriction $C = c_o + c_1x_1 + c_2x_2$, where C is fixed by budget restrictions. A graphic illustration of this problem is shown in Figure 2.13.

Applying the Lagrange multiplier technique, the objective of the problem will be to find values of x_1, x_2, and λ such that

$$y = f(x_1, x_2) + \lambda(c_o + c_1x_1 + c_2x_2 - c) = 0$$

achieves a maximum. This is accomplished by equating $\frac{\partial y}{\partial x_i}$ (i = 1, 2) and $\frac{\partial y}{\partial \lambda}$ to zero and solving the resulting system of simultaneous equations.

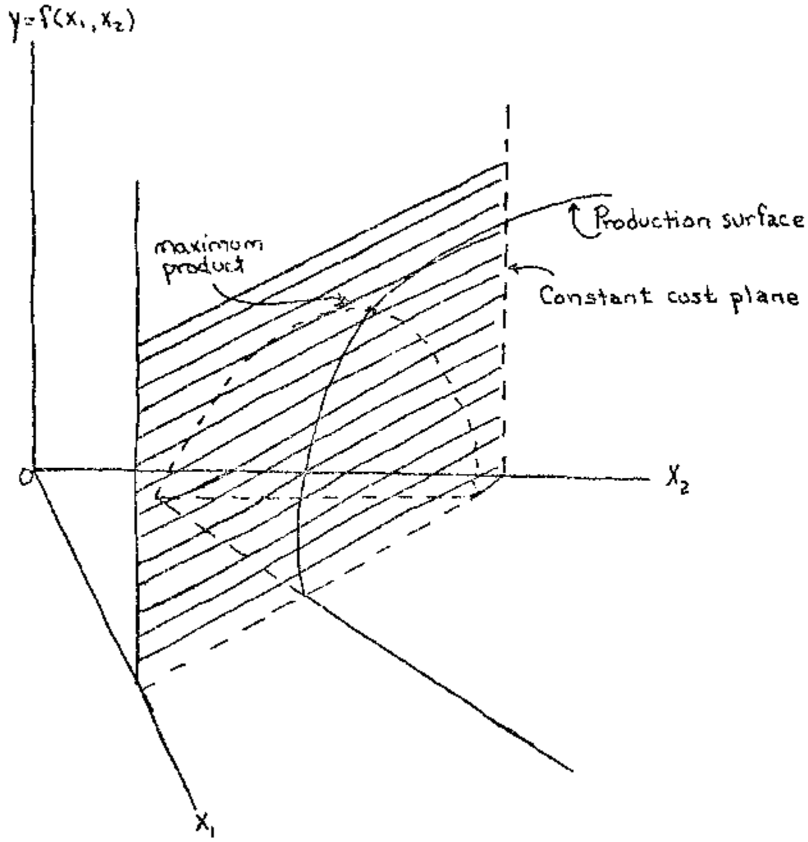

Fig. 2.13--Production function with cost restriction

This procedure is written symbolically as

$$\frac{\partial y}{\partial x_1} = f_1(x_1, x_2) + \lambda c_1 = 0$$

$$\frac{\partial y}{\partial x_2} = f_2(x_1, x_2) + \lambda c_2 = 0$$

$$\frac{\partial y}{\partial \lambda} = c_o + c_1 x_1 + c_2 x_2 - C = 0,$$

where c_o, c_1, c_2, and C are constants. A cursory investigation of this system of equations reveals the following relationship between $\frac{\partial y}{\partial x_i}$ and

c_i $(i = 1, 2)$:

$$\frac{\partial y/\partial x_1}{c_1} = \frac{\partial y/\partial x_2}{c_2} = -\lambda.$$

Since $\partial y/\partial x_i$ represents the marginal product of x_i, the optimum combination of inputs occurs at the point where the marginal product per dollar is the same for all input factors. Thus,

> if the marginal product per dollar is the same for two inputs, a dollar taken out of one factor and spent on the other will cause no change in total product. If the marginal products per dollar are not equal, a dollar taken out of the factor with the smaller ratio $\frac{\partial y/\partial x_i}{c_i}$ and budgeted to the factor with the larger ratio will cause an increase in total product.[29]

Queueing Theory: Models

Unlike the preceding tools/techniques of classical optimization theory, queueing theory is not a deterministically oriented approach to the analysis of decision problems. Rather, queueing theory is a time-oriented approach to solving a certain class of problems characterized by congested points of entry or service. These points of congestion (examples of which include bottlenecks in production operations, delays in the shipment of materials (logistics problems), and service facilities, etc.) are of such a nature that a waiting line results. The development of the waiting line is due to the fact that the elements arrive in such a manner that the capacity of the entry point is exceeded.

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[29] Donald J. Clough, Concepts in Management Science (Englewood Cliffs, 1963), pp. 150-153.

Queueing theory is a mathematical approach to the study of waiting lines (queues) and provides a formal structure through which various classes of time-service oriented problems are analyzed. Queue systems represent situations in which a customer (person, product, unit, etc.) arrives sequentially over time at a service facility (store, work station, bank cage, etc.). Such systems are completely described by four major characteristics: (1) arrival patterns, (2) service patterns, (3) the number of entry/service channels, and (4) priority ratings. The wide variety of queueing problems that are found in decision problems result from variations in these four basic characteristics.

Queueing theory is used to examine and to describe the random character of a given queue system. This is accomplished by an analysis of the system's variables. The variables most commonly considered for this analysis are the following: (1) the state of the system (e.g., the queue itself where the number of customers[30] waiting in line at any given time is a random variable), (2) customer waiting time,and (3) the idle time of the facility (idle time, generally measured as the percentage of facility utilization, occurs when the rate of service exceeds the rate of arrivals). These variables are defined and described by three numerically determined sets of data: (1) the average length of the waiting line and its average waiting time, (2) the average arrival rate and the associated average service rate, and (3) the probability that a given number of elements are in the waiting line. The data from which the waiting line

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[30]The term "customer" is used in a generic sense and refers to any entity (person, product, etc.) that is waiting for service at a given facility.

values are derived are taken from records or surveys. This data is then analyzed to determine the characteristic behavior of the problem under study.

Of particular interest and application are queueing models which describe the following generalized problem: (1) the system being investigated provides a specific service; (2) units (customers, products, etc.) arrive at random to receive the service made available; (3) there exists an expected or average rate of arrival of units for service, with a distribution of arrival rates around the mean arrival rate; and (4) there exists an expected or average rate of servicing units, with a distribution of service rates around the mean service rate. These models are of such a nature that they require four basic inputs: (1) the probability distribution of service times, (2) the probability distribution of arrival times, (3) the number of channels or points of service, and (4) the queue discipline (conditions of arrival-service).

The problem in queue formation arises because of two major factors: (1) at any given time, it is possible for more units to arrive for service than had been expected, or (2) the rate of service for a run of units will be less than the average service rate, thus causing a backlog (queue). Under such circumstances a queue can develop even though the system has sufficient capacity and is capable of providing more service than is normally demanded. At other times, less than the expected number of units can arrive or shorter than average service rates can occur, a situation which results in idle time for the service facility. An obvious indication of these comments is that the arrival rate and service rate interact with each other. This is particularly true when the output of one queue constitutes an input to another queue. It is highly feasible that arrival and service distributions will depend on each other. For example, if an assembly line is composed of sequential stations, the output of one station serves as an input to the next station. In such a case, the service rate of one station is a definite factor in determining the arrival rate for the next station.

The primary use made of queueing models has been the reduction of costs. Costs, in this sense, are not necessarily dollar costs, but include time, service, etc. Since overloaded or idle facilities tend to increase costs, the use of the queueing model centers on evaluating the following:

(1) the average number of elements in the queue, denoted L_q;

(2) the average number of elements in the system, denoted L (includes the number of elements in the queue and the number of elements being serviced);

(3) the average waiting time or delay before service begins, denoted W_q;

(4) the average time spent by an element in the system, denoted W (includes the delay before service begins and the time required to complete service);

(5) the probability that any delay will occur, denoted $Pr(N > M)$, where N represents the number of elements in the system, M the number of service facilities;

(6) the probability that the total delay, W, will exceed some value of t, denoted $Pr(W > t)$;

(7) the probability that all service facilities will be idle, denoted Pr_o;

(8) the expected per cent idle time of the total service facility, I, where $\bar{I} = \frac{M}{M} Pr_o + \frac{M-1}{M} Pr_1 + \ldots + \frac{M-n}{M} Pr_n + \frac{M-M}{M} Pr_M$, and Pr_n denotes the probability that n elements will be in the system, both waiting and receiving service from M facilities;

(9) the probability of turn-aways, where turn-aways result from a lack of suitable queue accommodations.

The selection of a suitable queueing model (Erlang, Poisson, etc.) can be made only by comparing the characteristics of the given problem with those of defined models. A selected sample of the basic queueing models is presented in the discussion that follows.

Deterministic Queues.--Deterministic queueing models are characterized by known arrival rates and a known (exact length) service time. As such, these models are described by known, nonrandom variables with strictly defined relationships.[31]

The deterministic queueing model, in general, is based upon two basic assumptions: (1) a constant rate of arrival (λ) and (2) a constant rate of service (μ). The numerical values associated with λ and μ are determined inversely by the regular time interval of arrival (denoted by a) and the length of the service interval (denoted by b), respectively; i.e., $\lambda = 1/a$ and $\mu = 1/b$.

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[31] Thomas L. Saaty, Elements of Queueing Theory (New York, 1961), pp. 27-30.

If the regular time of arrival exceeds the length of the service interval ($a > b$), the service rate (μ) will exceed the arrival rate (λ). When $\mu > \lambda$, a queue will not be formed since service on units currently at the service point will be completed before additional units arrive. If the regular time of arrival equals the length of the service interval ($a = b$), the arrival rate (λ) will equal the service rate (μ). When $\lambda = \mu$, a queue will result only if there were a queue at the beginning of the process. If the interval of service exceeds the interval of arrival ($a < b$), the arrival rate (λ) will exceed the service rate (μ). When $\lambda > \mu$, the arrival rate exceeds the service rate, forming a queue of arriving units.

Probabilistic Queues.--In general, the probabilistic queueing problem is such that the amount of time a customer spends in service is not known. However, it is possible that the relative frequencies with which a customer requires service of a given length can be obtained. With these relative frequencies, it is possible to obtain a ratio between the number of times a customer requires a certain length of service and the total number of times a customer required any service. The ratio can be used to define the probability that customer service time will last up to a given length of time.

The probabilistic queueing model is concerned with arrival and service rates of unknown time durations. Since the values corresponding to these time-duration intervals are free to assume any value within a given interval, the time-duration variables are defined as random (or chance) variables. For a probabilistic queueing model, these variables are such that they exhibit five characteristics: (1) the time interval is continuous (i.e., free to assume any value; (2) the probability of occurrence of any value, x, lying in an interval of length dx can be described by $f(x)dx$; (3) X denotes a chance variable, free to assume any of the possible values of service-time duration; (4) $Pr(X \leq x) = F(x) = \int_0^x f(y)dy$; and, (5) the only parameter (descriptive measure) involved is time.

The probabilistic queueing model is such that the "habits"[32] of a unit (customer) vary over time. The probability of time-duration of service is described by a probabilistic variable that is a function of time. The durations are described by a random (or chance) variable, denoted X_t, that depends on time. As such, the durations can be described by a probability distribution function defined at time t by $Pr(X_t \leq x) = F(x; t)$. For each value of t, the random variable at time t, X_t, has a defined probability distribution. For a range of values of t, the cumulative function $F(x; t)$ describes a family of random variables. A family of random variables that depends upon a parameter defines a stochastic process.[33] If there is no reason to distinguish between the service-time durations of different customers,

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[32]The term "habit" refers to the service-time duration exhibited by each arrival to the service station.

[33]A stochastic process is defined as a collection of random variables defined at some point in time, indicating that a stochastic process is any process such that the outcomes of the process are not known with certainty. Such a process is one in which the changes that occur are related by the laws of probability and are such that events occur in random fashion at random (or fixed) intervals.

the same stochastic process can be used to describe the times of all those arriving units which require service. If it is necessary to distinguish between service-time durations, then a stochastic process must be defined for each unit (customer) being served. By defining a stochastic process for each unit to be served, the identity of the entering unit and its time of entry into the service station is described. This individual assignment is denoted by X(k, t) and indicates that the distribution of the random variable X is defined by the time at which the k^{th} customer entered service.

As a means of illustrating the concept described in the preceding discussion, consider a production process which consists of a series of paint operations. The items to be painted include door panels, tables, and other various pieces of furniture. Each item entering the paint station requires a different amount of time to complete service. Assuming that arrivals occur at random and are defined by a stochastic process, the item which arrives to be painted is the "customer" for that service station. Since the arriving units require different lengths of time to complete service, the stochastic process which describes the entrance of a unit for painting will be defined in terms of its number (k) and the time it entered service (t).

The use of probabilistic models in describing waiting line phenomena is based on four basic models: the Poisson probability model, Erlang's model, the exponential probability model, and the gamma probability model. As a means of providing an adequate survey of suitable time-oriented models, each of these four models is discussed in this study.

Poisson distribution: The Poisson probability model is based on three postulates which stipulate the conditions under which a particular probability distribution is described by a Poisson process. These postulates, taken from Hogg and Craig,[34] follow.

Definition 2.27.--Let $g(x, h)$ denote the probability of x changes in an interval of length L. Let $T(h)$ denote any function such that the limit as h approaches zero of the quotient $\frac{T(h)}{h}$ equals zero. Let λ be the arrival (or service) rate. A Poisson process is one such that

(1) $g(1, h) = \lambda h + T(h)$ where $\lambda > 0$ and $h > 0$;

(2) $\sum_{x=2}^{\infty} g(x, h) = T(h)$; and,

(3) the number of changes in nonoverlapping intervals is stochastically independent.[35]

A Poisson process is one such that the probability of one change in a short interval h is independent of changes in other nonoverlapping intervals. As indicated by postulates (1) and (3), this probability is approximately proportional to the length of the interval. In addition, postulate (3)

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[34]Robert V. Hogg and Allen T. Craig, Introduction to Mathematical Statistics (New York, 1965), pp. 87-89.

[35]Stochastic independence is defined as follows: Let the random variable X_1 and X_2 have the joint probability density function $f(X_1, X_2)$ and the marginal density functions $f_1(X_1)$ and $f_2(X_2)$, respectively. The random variables X_1 and X_2 are said to be stochastically independent if, and only if, $f(X_1, X_2) = f_1(X_1) \cdot f_2(X_2)$. This definition of stochastic independence simply states that variables are stochastically independent if their joint probability density function can be written as a product of the variables' respective functions.

indicates that the probability of two or more changes in the same short interval h is essentially equal to zero.

A discrete distribution, the Poisson probability function is independent of previous events. In addition, the Poisson distribution, in defining the probability of a number of arrivals or length of service as a function of time, requires but two quantitative measures: (1) the elementary probability of an arrival per unit of time and (2) the finite period of time in which the study is made (total time). Given these two units of measure, the Poisson distribution for n arrivals at rate λ during the time period T is given by

$$f(n;\ \lambda,\ T) = \frac{(\lambda T)^n e^{-\lambda T}}{n!},\ 0 \leq n \leq \infty \text{ and } 0 \leq T \leq \infty.$$

As a matter of note, the function defined by $f(n;\ \lambda,\ T)$ is a discrete distribution and the only admissable values of n are integer values (i.e., n = 0, 1, 2,...). For time t and arrival rate λ, the probability that n items arrive at the queue by time t (denoted $Pr_n(t)$) is given by

$$Pr_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \ldots$$

$$= 0 \quad \text{elsewhere.}$$

Although the unit interval of time is free to assume any value, practical application dictates that this interval be set equal to unity (i.e., t = 1). The unit interval of time is defined to be equal to 1 because, regardless of the actual length of the time interval, the unit intervals will be of equal length. For example, although the time interval between observations may be 2 hours, a scale of 2 hours per unit of time would yield t = 1.

Definition 2.28.--Let $\lambda > 0$ be the mean of a Poisson process, expressed per unit of time ($t = 1$). The Poisson distribution function, defined in terms of λ, gives the probability of exactly x occurrences of some given form. This probability is obtained by evaluating

$$Pr(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$
$$= 0 \quad \text{elsewhere.}$$

The form shown in Definition 2.28 is the one normally associated with the Poisson distribution. It should be remembered that this form is applicable when $t = 1$. For $t \neq 1$,

$$Pr_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \ldots$$
$$= 0 \quad \text{elsewhere.}$$

When $t \neq 1$, the mean number of arrivals by time t (denoted by L) and the variance are given by

$$L = \sum_{n=0}^{\infty} nPr_n(t) = \lambda t;$$
$$\text{var} = \sum_{n=0}^{\infty} (n - L)^2 Pr_n(t) = \lambda t.$$

The widespread use of the Poisson distribution for describing random arrival processes is due to the fact that the three postulates of Definition 2.27 are "approximately fulfilled in all phenomena involving random arrivals."[36] In addition, there exists a relationship between service rate distributions and Poisson arrival distributions. This relationship is given by the following:

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[36]Arnold Kaufman, Methods and Models of Operations Research (Englewood Cliffs, 1963), p. 84.

(1) when the arrival rates are distributed according to a Poisson process, the interarrival rates are distributed according to a related exponential distribution; and,

(2) when the service rates are distributed according to an exponential distribution, the arrival rates for service are distributed according to a Poisson distribution.[37]

Erlang distribution: The Erlang queueing model first appeared as a means of explaining problems relating to telephone communications. Since the appearance of this model, its formulas have been used to calculate a number of different average dimensions. In addition, its formulas have been generalized to provide descriptions of stationary cases in transient or permanent systems.[38] Although there exists a set of distributions called Erlang distributions, distributions of the form

$$f(x) = (r\mu x)^{r-1} \frac{e^{-r\mu x}}{(r-1)!} r\mu,$$

the most common Erland model is that model for which $r = 1$. This reduces the function to $f(x) = \mu e^{-\mu x}$ which is the exponential distribution.

Problems suitable for application of the Erlang model follow strictly defined characteristics. Of these characteristics, the following are the most stringent:

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[37]The exponential distribution is defined by the probability density function

$$f(x) = \alpha e^{-\alpha x}, \quad x \geq 0$$
$$= 0 \text{ elsewhere.}$$

[38]Kaufman, op. cit., p. 340.

(1) An operation starts with no items (units) waiting in line;

(2) Inputs to the system follow a Poisson distribution with parameter λ;

(3) Holding time (service) follows an exponential distribution with parameter μ;

(4) Arrivals are serviced on a first-come, first-served basis;

(5) Arriving units can enter the system at only one location; and,

(6) The probability of n units being in the system at time $t + \Delta t$ is given by

(a) for $n \geq 1$,

$$Pr_n(t + \Delta t) = Pr_n(t)[1 - (\lambda+\mu)\Delta t] + Pr_{n-1}(t)\ \Delta t + Pr_{n+1}(t)\mu\Delta t.$$

(b) for n = 0,

$$Pr_o(t + \Delta t) = Pr_o(t)(1 - \lambda\Delta t) + Pr_1(t)\mu\Delta t.$$ [39]

Utilizing a traffic intensity factor (or utilization factor) defined by $p = \lambda/\mu$, the Erlang model is extremely useful for calculating the expected number of elements in the system, the variance of the distribution defining the queue, and the expected number in the queue. Given the arrival rate λ, the service rate μ, and the steady-state probability p_n, these values are given by direct application of the following set of unique value formulas:

(1) the expected number in the system, L,

$$L = \sum_{n=0}^{\infty} np_n = \frac{p}{1-p}\ ;$$

[39]Verbally, the probability of n items in the system at time $t + \Delta t$ equals the probability of n items in the system at time t multiplied by the probability of no arrivals and no departures, plus the probability of n-1 items in the system at time t multiplied by the probability of one arrival and no departures, plus the probability of n+1 items in the system at time t multiplied by the probability of a single departure and no arrivals.

(2) the variance of the distribution, V,

$$V = \frac{p}{1-p} + \frac{p^2}{(1-p)^2} \text{ ; and,}$$

(3) the expected number in the queue, L_q,

$$L_q = \sum_{n=0}^{\infty} (n-1)p_n = \frac{p^2}{1-p} .$$

Exponential distribution: The exponential distribution is well suited to the analysis of waiting lines. In this application it is particularly useful in determining the length of time spent waiting when the sequence of occurrences follow a Poisson distribution.

The exponential queueing model is described by the exponential probability distribution. This distribution is defined in terms of the random variable x and the constant $\propto$. It is assumed that admissable values of x are those values such that x is at least zero and $\propto$ is positive.

Definition 2.29.--Let $\propto > 0$. The exponential distribution, defined in terms of $\propto$, gives the probability of exactly x occurrences of some given form. This probability is obtained by evaluating

$$\begin{aligned} Pr(x) &= \propto e^{-\propto x} \quad \text{for } x \geq 0, \\ &= 0 \quad \text{elsewhere.} \end{aligned}$$

Unlike the Poisson distribution, which is a discrete function, the exponential distribution is a continuous function and is amenable to the calculus. The use of the calculus is especially suited for cumulative problems but is not necessary. For example, if it were necessary to evaluate the exponential function for all x less than or equal to 6, this could be accomplished in one of two ways:

(1) evaluate $Pr(x \leq 6) = \int_0^6 \alpha e^{-\alpha x} dx$, or

(2) evaluate $Pr(x \leq 6) = 1 - e^{-\alpha x}$, for $x = 6$.

With the availability of tables for exponential functions, the second approach is the most practical.

The mean and the variance of the exponential distribution are determined once α is known. This is because the mean and the variance of the exponential distribution are $\frac{1}{\alpha}$ and $\frac{1}{\alpha^2}$, respectively. The numerical value associated with α is the rate at which the queue elements arrive (or are serviced).

As a matter of note, the exponential distribution offers an advantage over the Poisson model. This advantage is evidenced by the fact that, whereas the exponential distribution is equally suited for both arrival rates and service rates, the Poisson distribution applies only to arrival rates.

> The crucial distinction is that, when there are no elements in the system, none can be serviced. The average service time...is the average time taken for servicing while servicing is going on. Idle time is not counted... the distinction between arriving and servicing is that arriving is always going on, and time between arrivals is counted in computing the average arrival rate. Servicing...takes place only when one or more elements are in the system. Thus, the average service rate...is the average number of elements serviced per unit time of continuous servicing.[40]

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[40] Samuel B. Richmond, Operations Research for Management Decisions (New York, 1968), p. 412.

Gamma distribution: The gamma distribution, also a continuous probability function, is suited for the same type of analysis as the exponential distribution. In particular, the gamma distribution can be used to find the total arrival time for any number of consecutive arrivals.

The gamma function, denoted $\Gamma(\cdot)$, is defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

This function is defined in terms of the constant α and the random variable x. Although defined as an integral function, it is not necessary to utilize the integral calculus. Rather, this integral is evaluated by applying the property that, for $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)!$

Examination of the literature reveals that the gamma function can be defined in a variety of ways. The most common of these definitions is shown below. Although these definitions appear different, this difference is only in the manner in which the definitions are written.

Definition 2.30(a).--Let $\alpha > 0$, $\beta > 0$ be defined parameters. Let x be a continuous random variable. Let f(x) be the defined gamma function. Then,

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty, \ \alpha = 1, 2, \ldots$$
$$= 0 \text{ elsewhere.}$$ [41]

Definition 2.30(b).--Let $\lambda > 0$, $r > 0$, be defined parameters. Let x be a continuous random variable. Let f(x) be the defined gamma function.

[41] Hogg and Craig, op. cit., p. 91.

Then,

$$f(x) = \frac{\lambda}{(r-1)!} (\lambda x)^{r-1} e^{-\lambda x}, \quad \text{for } x \geq 0,\ r = 1,2,\ldots$$
$$= 0 \quad \text{elsewhere.}^{42}$$

Definition 2.30(c).--Let $\lambda > 0$, $n > 0$ be defined parameters. Let x be a continuous random variable. Let f(x) be the defined gamma function. Then,

$$f(x) = \frac{\lambda(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!}, \quad \text{for } y \geq 0,\ n = 1, 2,\ldots$$
$$= 0 \text{ elsewhere.}^{43}$$

As noted previously, the gamma distribution function can be used to determine the probability of total arrival time for any number of consecutive arrivals. This application is made by considering the random variable x as representing the sum of n independent values. These n independent values are taken from the defined gamma function and define the sum of the total intervals for any n consecutive intervals. Thus,

$$\Pr\left[\begin{matrix}\text{total interval for any} \\ \text{n consecutive arrivals} \leq T\end{matrix}\right] = \int_0^T f(x)dx = 1 - \sum_{k=1}^{n-1} \frac{(\lambda T)^k e^{-\lambda T}}{k!}, \text{ where}$$

$$f(x) = \frac{\lambda(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!}, \quad x \geq 0,\ n > 0,$$
$$= 0 \quad \text{elsewhere.}$$

The mean and variance of the gamma distribution are defined in terms of the parameters which describe the function. For example, if the gamma

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[42] Emanuel Parzen, Modern Probability Theory and Its Applications (New York, 1960), p. 180.

[43] Harvey M. Wagner, Principles of Operations Research (Englewood Cliffs, 1969), p. 845.

function is defined by

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty, \ \alpha = 1, 2, \ldots$$
$$= 0 \quad \text{elsewhere,}$$

the mean and variance are $\alpha\beta$ and $\alpha\beta^2$, respectively. Similar expressions can be written for 2.30(b) and 2.30(c) by equating like terms.

Having briefly considered the four major distribution functions suited for the analysis of waiting lines, it is necessary to give consideration to other factors which would influence the selection of a particular formula. Chief among these other considerations is the type of queue that has developed, single-channel or multiple-channel. In the final analysis, it is this decision that dictates the formula to be employed as a means of solving a given problem.

Single-channel queues.--Single-channel queueing problems are characterized by only one service facility. Arriving units are compelled to enter service at only one station which may or may not be capable of meeting service demand. When the arrival rate exceeds the service rate, a queue results. These comments lead to the following definition.

Definition 2.31.--Single-channel queues are queues such that arriving units (customers) are able to receive service at only one service facility (station).

Other characteristics of the single-channel queue include random arrivals and service time that is independent of queue length. Units are admitted on a first-come, first-served basis, such as service at a service station, service at teller windows in banks, or units passing along an assembly line. (See Figure 2.14.) Additional insight into the single-channel queue can be obtained by considering the work of di Roccaferrera.

> When the arrivals and service time are both at random, this type of problem is also called a single-exponential channel in accordance with the Poisson distribution of values indicating the time of arrivals and the time of service. When the service is busy, the incoming element waits in line in order of arrival until the previous element leaves the channel at the end of its service. This rule is called strict queue discipline. . . . The elements in line do not always form a sort of line which proceeds in unchanged order toward the station. Sometimes an element in line may leave the queue and the line is shortened. This case is defined as a "queueing problem with impatient customers. . . ." The expected escape rate is treated as a function of the limited time of the availability of the element in line (customer). The element leaving the system is a lost customer or a lost potential customer.[44]

Fig. 2.14--Single-channel queue

[44]Giuseppe M. Ferrero di Roccaferrera, Operations Research Models for Business and Industry (Cincinnati, 1964), p. 324.

Multiple-channel queues.--Whereas the single-channel queue restricts the arriving unit to a single point of service, the multiple-channel queue permits the arriving unit to select any one of a number of serving facilities. In this respect, the multiple-channel queue is characterized by n service facilities, with arrivals unrestricted in queue selection.

Definition 2.32.--Multiple-channel queues are queues such that arriving units (customers) are permitted to obtain service at any one of a number of serving stations.

The primary consideration of the multiple-channel queue is directed at two points of interest: (1) the type of queue (finite or infinite) and (2) the arrangement of the serving stations (in parallel or in sequence). These points greatly affect the manner in which a queueing problem is analyzed and must be determined prior to actual analysis due to the influence made on the queue distribution. Because of their importance, each is defined and described in the discussion to follow.

Multiple-channel queues in parallel: Many queueing situations (e.g., restaurants, customers at shops) consist of a finite number of service channels arranged in parallel. For completely random arrivals and service times and no departures from the system except for service, the arriving unit will be serviced immediately if a service channel is idle. A queue is formed when the number of arrivals exceeds the number of channels or service points. Elements (or units) belonging to the queue are free to move from one queue to another until they are serviced.

It is to be noted that when an element cannot pass from one line to another, even if the system is apparently composed of two or more stations, the problem is not a multiple-channel problem. Instead, it is a single-channel problem composed of several dichotomous problems. In this particular case, it is assumed that the formation of each queue is independent of the others. Once an element selects a queue, it becomes part of the single-channel system.[45]

As an example of the multiple station, single-channel queue concept, consider the intersection of several routes where each route has a separate entry station. Each of these routes lead to a different location. (See Figure 2.15.)

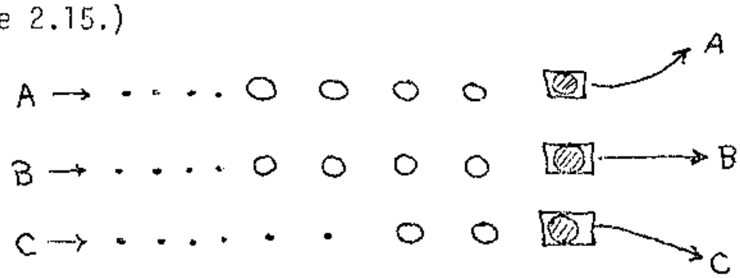

Fig. 2.15--Parallel stations, single-channel queue

Reference to Figure 2.15 reveals that an arrival in channel A is not free to receive service at stations B or C and then return to channel A. In this case, the waiting line is not multiple channel. It consists of a series of parallel stations, with single-channel queues leading to each of the stations. Elements entering the system are not free to change lines.

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[45] Ibid., p. 837.

Thus, the primary consideration to be made in defining (or describing) a problem involving multiple-channel queues in parallel is the manner in which elements in the queue gain admittance to serving stations. If the queue elements are unable to pass from one queue to another (e.g., cars entering a toll bridge station), the queue is not a multiple-channel queue but one that is single channel, multi-problem. If the queue is one such that the queue elements are free to change channels at will until served, then the queue is a multiple-channel queue (e.g., shoppers at a checkstand), as shown in Figure 2.16.

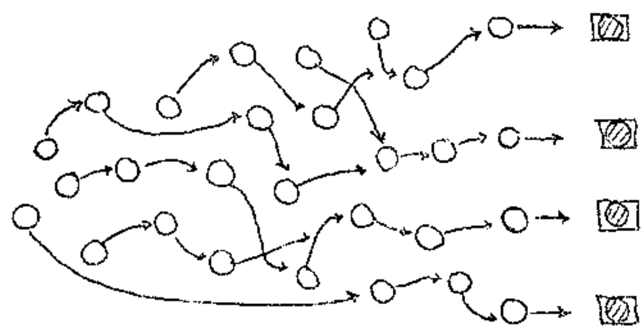

Fig. 2.16--Multiple-channel queue, four stations in parallel

From this discussion can be drawn several characteristics of multiple-channel queues in parallel: (1) the queue is such that free mobility among queue elements is allowed; (2) elements in the queue can be served equally well at any available station; and, (3) there exist at least two serving stations to which arriving elements may go. These points are formalized in the following definition.

Definition 2.33.--Queues are said to be multiple channel in parallel when entry into the system allows free distribution to two or more service stations; i.e., elements in the queue can be served equally well by more than one station.

Multiple-channel queues in sequence: As previously noted, multiple-channel queues can be, and usually are, categorized on the basis of the type of queue considered. Queues in parallel are those queues such that the service facilities (stations) are arranged in parallel, with free mobility of queue elements. In contrast, queues in sequence are those queues such that the output of one service facility serves as an input into another service facility; in addition, the service at each stage must be performed before another service can begin. The multiple-channel queue in sequence can then be formally defined in the following manner.

Definition 2.34.--Queues are said to be multiple channel in sequence when the output of one service facility is an input to another service facility, and the service that is performed on the unit must be completed before another service begins.

Sequential queues thus represent queueing problems in which the output of one stage serves as the input to the next stage, an example of which is an assembly line. Such queues are generally found to be of two types: (1) multiple-channel entry with a fixed sequence of service, or (2) multiple-channel entry with a multiple-channel service sequence.

In illustration, consider the queueing problem with multiple-channel entry and a fixed sequence of service. Such a system is characterized by a free selection of the entry station. Once the system is penetrated, subsequent queues are single channel. (See Figure 2.17.) This case is illustrated by a cafeteria serving line with multiple ports of entry. Until the serving line reaches the food station the persons forming the queues are free to move from line to line. Once the serving of the food begins, the person (queue element) is fixed with regard to the particular line.

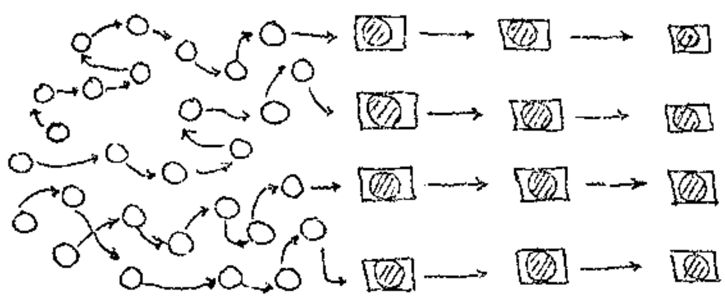

Fig. 2.17--Multiple-channel entry, fixed sequence

Consider next the queueing problem with multiple-channel entry and multiple-channel service. Entry into the system is such that each element is free to select any available station. Once the system is penetrated, the output of one station serves as an input to another station. Each input is free to change channels up to the point of actual service. (See Figure 2.18.) This case is illustrated by the job shop, where a particular job proceeds through a series of operations on different machines. Each machine represents an independent service center. The jobs arrive at random intervals from other machines in the system. Each machine may have a queue of jobs competing for the service of that facility.

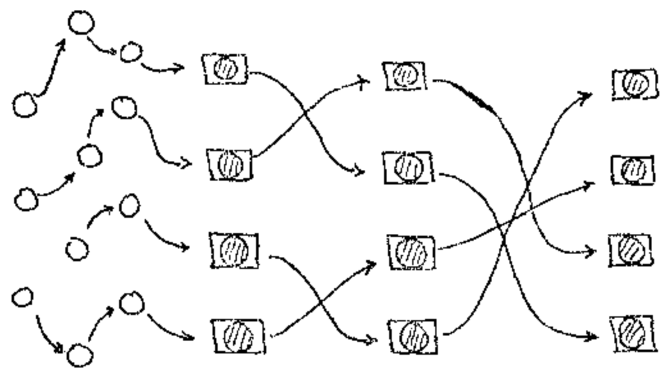

Fig. 2.18--Multiple-channel entry, multiple sequence

The preceding discussion has centered on both the classification of queues according to the type of model describing the system and the particular type of channel (single- or multiple-) that is employed. Although these two classifications are of primary importance, consideration should be given the following if the problem is to be properly described. The context of these other factors is taken from Saaty.[46]

1. Types of arrivals and service-time distributions. Arrivals into a queue occur by assumption according to a certain frequency distribution, as do the intervals between arrivals. These arrivals may be independently distributed for application purposes, or they may be dependent.

2. Initial-input variations. The initial number of units in a system when an operation begins may be given by a distribution because it is different for each complete run of the operation. The input to

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[46]Saaty, op. cit., pp. 9-12.

a queue may be from a limited (finite) or an unlimited (infinite) population which may also consist of several categories (populations) of customers, each of which may arrive by a different distribution, singly or in batches, and may queue in a prescribed order.

3. Unit behavior.

a. Balking. Unit behavior can vary. Arriving units may balk (i.e., not join the queue) because of the length of the existing queue, or simply because they have to wait; these units are lost to the queue.

b. Influence of incomplete information. For many problems a decision may be required as to which line of a multiple-queue operation to join when information about only a few is immediately available--a case of incomplete information.

c. Unit adaptation to queue conditions. On the basis of experience, passengers may learn to travel earlier or later to avoid intolerable queueing, and such measures, when adequately studied, may even relieve congestion. An item may join a large waiting line at closing hours for fear that a short line which it encounters may be closed suddenly. But there are situations in which an item which arrives before another must go into service before the following one. Also, there are cases in which each service facility has its own specialty--consequently, its own queue [e.g., stamp-sale counter and money-order or special registry counter at the Post Office].

d. Collusion, jockeying, and reneging. Several units may be in collusion whereby only one person waits in line while the rest are free to attend to other business. Some may even arrange to take turns waiting. Units may jockey from one line to another; or, a customer may lose patience and leave the queue.

4. Queue and channel variations.

a. Full or limited availability. Service channels may be available to any unit waiting in a system (full availability) or may be available only to some waiting units. Other units are blocked and must wait until a channel that can provide the required service becomes available.

b. Service procedures or discipline. While in line customers may be chosen for service by allocation to the channels in an ordered first-come, first-served manner or at random, they may be assigned priorities with errors committed when initially it is not clear which priority to assign, or the priority assignment may change in time. Priorities may be preempted if higher priorities arrive, or they may be allowed to finish the service. Finally, items may be chosen for service on a last-come, first-served basis.

c. Specialized service channels. Some of the service channels may specialize while others remain general. Parallel channels may all cooperate to serve the many needs of each customer. Customers may cycle by returning to the waiting line for additional service.

d. Queue interference. Two (or more) queues may interfere with each other. Such situations could present themselves on narrow, one car per time bridges, four-way stops, etc.

5. The output of a queue. The output of a queue may also be of importance, particularly when it forms an input to another queue in series with the first one. Arrival and service distributions may depend on each other.

Queueing Theory: Techniques

Although queueing theory contains both deterministic and probabilistic applications, these applications are such that they belong to the techniques of classical optimization theory. This classification of queueing theory is based upon the fact that solutions to given problems are obtained by direct application of a given formula. In addition, the solutions obtained are unique. For a given set of input data, alternative solutions do not exist.

As a means of presenting the techniques of queueing theory, the following format will be followed: (1) the notation for infinite source queues will be presented; (2) the deterministic queue is reviewed and applicable formulas presented; (3) the probabilistic queue is reviewed and applicable formulas presented. The section on probabilistic queues is further divided into infinite source and finite source analysis. Since the finite queue is a special case of the infinite queue, the notation required for the study of finite queues is presented as it is needed. Both infinite and finite source queues are further subdivided according to the number of service channels in the system.

The following set of symbols will be used for the study of infinite source queues. These symbols are presented here to facilitate the discussion to follow.

λ = mean arrival rate (the number of arrivals per unit time)

μ = mean service rate per channel

s = number of service channels

n = number of units in the system

p = traffic intensity, or utilization factor

$Pr_n(t)$ = the probability that, at time t, there will be exactly n units in the system, both waiting and in service

Pr_n = steady-state (time-dependent) probability that there will be n units in the system, both waiting and in service

$$\sum_{n=0}^{\infty} Pr_n(t) = \sum_{n=0}^{\infty} P_n = 1.$$

Pr(=0) = the probability of no waiting

Pr(>0) = the probability of any waiting

Pr(>T) = the probability of waiting greater than T

L = the average number of units in the system, both waiting and in service

L_q = the average number of units waiting in the queue

W = the average waiting time in the system

b = time length of service interval

a = time length of arrival interval

Deterministic queues with an infinite source.--In formulating a deterministic queueing problem, it is necessary to make some assumptions regarding the arrival and service rates. These assumptions are the following:

(1) arrival times and service times are fixed;

(2) the single channel arrival rate λ equals the reciprocal of the regular time interval of arrival "a"; i.e., $\lambda = 1/a$;

(3) the service rate μ equals the reciprocal of the regular time interval of service b; i.e., $\mu = 1/b$; and,

(4) both arrivals and services are regular in occurrence.

If the time interval of service is less than the time interval of arrival, the arrival rate will be less than the service rate; i.e.,

$$\text{if } b < a, \text{ then } \lambda < \mu.$$

When $\lambda < \mu$, no queue will be formed. If the time interval of service exceeds the time interval of arrival, the arrival rate will exceed the service rate; i.e.,

$$\text{if } b > a, \text{ then } \lambda > \mu.$$

When $\lambda > \mu$, a queue will form. If the time interval of service equals the time interval of arrival, and no queue existed at the start of the process, the arrival rate will equal the service rate; i.e.,

$$\text{if } b = a, \text{ then } \lambda = \mu.$$

When $\lambda = \mu$, and there was no queue at the start of the process, arrivals will be serviced immediately, with no idle time between arrivals and service.

In developing the basic formulas for the deterministic queueing model, two cases will be presented. In the first case, it will be assumed that a queue of size $n \geq 2$ is already in existence. In the second case, it is assumed that servicing is begun with one of the initial n queue members being admitted to service. Arriving units are serviced as they arrive, with the queue member admitted to service between arrivals. In both cases, it is assumed that the service rate exceeds the arrival rate ($\mu > \lambda$).

Case I: Suppose the operation begins with a line consisting of n units waiting to receive some service, $n \geq 2$. If $n = 1$, service on any given unit will be completed before a new unit arrives for service. All n of the queue elements will have been served by the end of a time interval of length nb. By this time $\left[\frac{nb}{a}\right]$ customers would have arrived and would have to wait. (Note: the brackets indicate the greatest integer k, such that $k \leq \frac{nb}{a}$.) The last customer to arrive will wait for the service facility to complete its service.

The probability that there exists a queue of length n at time t is expressed by

$$P_n(0) = \begin{cases} 1 & \text{for } n = j; \\ 0 & \text{for } n \neq j. \end{cases}$$

The value of j equals the length of the queue at the start of the process. The length of time required to serve all of the customers that have waited is given by

$$T = b(j + \left[\frac{jb-a}{a-b}\right] + 1) \equiv b\left[\frac{jb-b}{a-b}\right].$$

Case II: Suppose the operation begins with the admission of one of the n initial queue members to service. Additional queue members will be admitted according to the following criterion: if a customer arrives when the service facility is being used, the arriving customer will be admitted to service ahead of the initial queue members. Since the service rate is assumed to exceed the arrival rate, this amounts to providing service during excess service time in such a way that the initial queue will be eliminated. As a means of clarification, assume that the service facility is being used. During this period of service, it is possible that a new customer will arrive. Since the service rate, μ, exceeds the arrival rate, λ, the time interval of service is less than the time interval of arrival; i.e.,

$$\mu > \lambda \text{ implies } a > b.$$

Thus, the possibility that the system has a new arrival prior to the completion of the service in process depends upon the magnitude of a - b. The arriving customer will enter service once the facility becomes vacant, and all customers who may have arrived will also be serviced. When no arrivals are waiting, another of the original n customers is admitted into service. Additional arrivals are again served once the facility is empty, and so on until all n customers go into service. When the last of the initial customers goes into service, there will be no one waiting. If anyone arrives during this service time he waits and is then served, etc., until arriving customers do not have to wait.

Since $b < a$, there is a gain of a - b time units from each of the arriving customers. Theoretically, since the service time is b, b/(a-b) customers must arrive to generate enough slack time to service one of the n - 1 waiting customers who would be there after the operation begins. At this point the first customer in the queue has already gone into service. In order to vacate the line $\frac{b(n-1)}{(a-b)}$ customers must arrive.

The total number of customers who have waited in the line after the operation began is $\frac{a(n-1)}{(a-b)}$.

The total number of customers who have waited in the line after the operation began, including the initial customer in service, is $\frac{na-b}{(a-b)}$.

The total time until the service facility first becomes idle is

$$T = \left[\frac{na-b}{a-b}\right] b.$$

At time t the number of customers ahead of an arrival is

$$t\left[\frac{1}{a} - \frac{1}{b}\right] + (n - 1).$$

The waiting spent in the line by a customer arriving at the first multiple of a past t is

$$W(t) = \begin{cases} 0 & T \leq t \\ (t\,\frac{b-a}{ab} + n)b, & 0 < t \leq T \\ (k-1)b & t = 0 \end{cases}$$

where W(0) is the time in the line for the k^{th} member of the initial group of n customers.

The total time spent in the system by a customer arriving at the first multiple of a past is

$$W_r(t) = \begin{cases} b & T \leq t \\ (\frac{t}{a} + n - \frac{t}{b})\ b & 0 < t \leq T \\ kb & t = 0 \end{cases}$$

This includes the time spent in service.

Probabilistic queues.--In general, the queueing problem is of such a nature that the amount of time an arriving customer will spend in service is not known. In such situations, it is often possible to obtain the relative frequency with which customers require services of given lengths; i.e., it is possible to determine the ratio between the number of times a customer requires a certain length of service and the total number of times a customer required any service. In obtaining these relative frequencies, however, it is necessary to give consideration to whether or not the customers arrive from an infinite source or a finite source. Because of the need for such consideration, the techniques of probabilistic queue analysis will be subdivided into techniques based upon an infinite source and a finite source.

Infinite source: The term source is used as a means of defining the population from which customers emanate. A source is considered infinite if it is satisfactorily large or so large that it is uncountable.

The use of either of the four basic distributions of queueing theory yields the probability that a given number of occurrences will be observed. An occurrence in this sense can be interpreted as the arrival of a customer for service or the servicing of a customer. For example, the gamma distribution is used to determine the probability that the total arrival time for n consecutive arrivals is at most T. The Poisson distribution can be used to determine the probability that exactly x occurrences will be observed at a given time. If the probability associated with the Poisson distribution is concerned with the probability of the total number of arrivals, this can be obtained by summing the Poisson distribution over the range of arrivals; i.e., if the arrival distribution is Poisson, then the probability of at most k arrivals during time period t is given by

$$\sum_{n}^{k} \frac{(\lambda t)^n e^{-\lambda t}}{n!} .$$

In application, queueing theory centers on answering such questions as the probability that no elements will be in the queue, the average length of the waiting line (queue), or the average waiting time of a member of the queue. As such, the solution to a given problem, in many instances, requires the use of a specific formula. It is in this sense that queueing theory is a tool of classical optimization theory.

As a means of providing a tool, applied queueing theory utilizes a breakdown into single-channel and multiple-channel queues. This format will be utilized here.

Single-channel queues: Single-channel queues have been defined as queues served by only one service facility. Formulas suited for analyzing single-channel queues require two numerical values: λ, the arrival rate, and μ, the service rate. Since this study is only concerned with technique, the formulas for single-channel queue analysis are presented without proof or derivation.

The probability that the queue will be empty, P_o, is given by $P_o = 1 - \frac{\lambda}{\mu}$.

The facility utilization factor, p, is given by $p = \frac{\lambda}{\mu}$.

The probability that the queue contains one unit, P_1, (excluding the unit in service) is given by $P_1 = \frac{\lambda}{\mu} P_o$.

The probability that the queue contains n units, P_n, (excluding the unit in service) is given by $P_n = P_o(\frac{\lambda}{\mu})^n$.

The probability of having a queue of length n, $n \geq 0$, is given by $\rho_n = \rho^n(1 - \rho)$, where $\rho = \frac{\lambda}{\mu}$ and $\frac{\lambda}{\mu} < 1$.

The average (mean) number of units in the system, L, (both waiting and in service) is given by

$$L = \frac{\lambda/\mu}{1 - /}, \quad \text{for } \lambda/\mu < 1.$$

The expression for L can be written as $L = \frac{\lambda}{\mu - \lambda} \equiv \frac{\rho}{1 - \rho}$.

The average (mean) length of the waiting line, L_q, (excluding the unit being serviced) is given by

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}, \quad \text{where } \lambda/\mu < 1.$$

The mean time between arrivals is given by $1/\lambda$.

The average (mean)waiting time of a unit in the system, W, is given by

$$W = \frac{\lambda}{\mu(\mu - \lambda)} \equiv \frac{\rho}{\mu(1 - \rho)}, \quad \text{where } \lambda/\mu < 1.$$

As a means of demonstrating the single channel, infinite source queueing problem, consider a hypothetical firm engaged in both shipping and receiving activities. Since rapid and efficient service has been a prime factor in maintaining sales and good customer relations, management is interested in maintaining their current image. As such, suggestions which might improve this image are always given careful consideration.

It has been noted that a rough comparison between the cost of idle trucks and drivers and the costs of the idle service facility indicates that there are too many trucks waiting for too long a period of time. It has been suggested that loading and unloading of trucks be standardized and that uniform quantities be carried on each truck. In addition, it has been suggested that palletizing the loads and using lift trucks will substantially reduce the waiting time at the docks.

A study of the current loading and unloading procedure was made. This study provided the following information relative to the current loading and unloading procedure:

(1) The arrival distribution is a Poisson distribution.

(2) The service-time distribution is exponential.

(3) Infinite waiting lines are theoretically possible.

(4) The queue discipline is first-come, first-served.

(5) Trucks arrive at the plant every 40 minutes (on the average). This yields an average arrival rate of 1.5 trucks per hour.

(6) The loading and unloading dock and workers can unload or load a truck, on the average, every 30 minutes. This yields an average service rate of two trucks per hour.

This study further revealed that the current situation was such that the three criteria for a queue situation were present: (1) a distribution of inputs or arrivals at the service facility, (2) a number of servers whose service time could be described by a distribution function, and (3) a defined queue discipline.

With the results of this study in hand, management decided to evaluate the current loading and unloading procedure by obtaining answers to the following questions:

(1) What is the average number of trucks in the waiting line?

(2) What is the average number of trucks in the waiting line, including the one being serviced?

(3) What is the average waiting time of trucks in line?

(4) What is the average waiting time of trucks in line, including the one being serviced?

(5) What is the probability that the loading and unloading dock and workers will be idle?

The potential improvement offered by standardizing loading and unloading, using uniform quantities on trucks, and using lift trucks is to be evaluated on the basis of two additional questions and answers.

(6) What reductions are possible if loading and unloading is standardized and uniform quantities are placed on the trucks?

(7) What reductions are possible if lift trucks are used?

Solution: Let λ equal the average arrival rate. Let μ equal the average service rate. Let n equal the number of trucks being serviced. From the study, the numerical values of λ and μ were found to be 1.5 and 2.0, respectively.

(1) The average number of trucks in the waiting line, L_q, is found by applying $L_q = \frac{\lambda^2}{\mu}(\mu - \lambda)$. Substituting for λ and μ,

$$Lq = \frac{(1.5)^2}{2.0(2.0 - 1.5)}$$

$$= \frac{2.25}{2(.5)}$$

$$= 2.25.$$

There is an average of 2.25 trucks waiting in line.

(2) The average number of trucks in the waiting line, including the one being served, L, is found by applying $L = \frac{\lambda}{\mu - \lambda}$. Substituting for λ and μ,

$$L = \frac{1.5}{(2.0 - 1.5)}$$

$$= \frac{1.5}{(.5)}$$

$$= 3.0.$$

There is an average of 3.0 trucks waiting in line, including the one in service.

(3) The average waiting time of trucks in line, W_q, is found by applying $W_q = \frac{\lambda}{\mu(\mu - \lambda)}$. Substituting for λ and μ,

$$W_q = \frac{1.5}{2.0(2.0 - 1.5)}$$

$$= \frac{1.5}{2.0(0.5)}$$

$$= 1.5.$$

The average waiting time per truck is 1.5 hours.

(4) The average waiting time of trucks in line, including the truck in service, W, is found by applying $W = \frac{1}{\mu - \lambda}$. Substituting for λ and μ,

$$W = \frac{1}{2.0 - 1.5}$$

$$= \frac{1}{0.5}$$

$$= 2.$$

The average waiting time of trucks in line, including the one in service, is 2 hours.

(5) The probability that the loading and unloading dock and workers will be idle, P_n, is found by applying $P_n = (1 - \lambda/\mu)(\lambda/\mu)^n$. Substituting for λ and μ,

$$P_n = (1 - \frac{1.5}{2})(\frac{1.5}{2})^n$$
$$= (1 - .75)(0.75)^n$$
$$= (0.25)(0.75)^n.$$

For the facility to be idle, n must equal zero. Therefore,

$$P_n = P_o = (0.25)(0.75)^o$$
$$= 0.25.$$

The probability that the service facility will be idle is 0.25.

(6) The reduction achieved by standardizing loading and unloading and using uniform quantities on trucks is calculated as follows: Assuming that loading and unloading of trucks is standardized and that uniform quantities are carried on the trucks, it is possible that the service times would become constant instead of exponential. Assuming no other changes in the basic assumptions, the average number of trucks in the waiting line is given by $L_q = \frac{\lambda^2}{2\mu(\mu - \lambda)}$, and the average waiting time is given by $W_q = \frac{\lambda}{2\mu(\mu - \lambda)}$. Substituting for λ and μ,

$$L_q = \frac{(1.5)^2}{2(2)(2.0 - 1.5)}$$
$$= \frac{2.25}{4(.5)}$$
$$= \frac{2.25}{2}$$
$$= 1.125,$$

and

$$W_q = \frac{1.5}{2(2)(2.0 - 1.5)}$$

$$= \frac{1.5}{4(.5)}$$

$$= \frac{1.5}{2.0}$$

$$= 0.75.$$

The incorporation of constant service times reduces the number of trucks waiting from 2.25 trucks to 1.125 trucks. In addition it also reduces the waiting time per truck from 1.5 hours to 0.75 hours.

(7) The reduction achieved by using lift trucks is based upon the reduction made in the service time per truck. Suppose this reduction in service time is a constant 15 minutes per truck, or 4 services per hour. The use of the lift truck will reduce the length of the waiting line, L_q, and the waiting time per truck, W_q. Assuming these are the only measures of interest, these can be determined by applying

$L_q = \frac{\lambda^2}{2\mu(\mu - \lambda)}$ and $W_q = \frac{\lambda}{2\mu(\mu - \lambda)}$. In this case, $\lambda = 1.5$ and $\mu = 4$. Substituting for λ and μ,

$$L_q = \frac{(1.5)^2}{2(4)(4 - 1.5)}$$

$$= \frac{2.25}{8(2.5)}$$

$$= \frac{2.25}{20}$$

$$= 0.1125,$$

and

$$W_q = \frac{1.5}{2(4)(4 - 1.5)}$$

$$= \frac{1.5}{8(2.5)}$$

$$= \frac{1.5}{20}$$

$$= 0.075.$$

The use of lift trucks--under the given assumptions--reduces the number of trucks waiting in line from 2.25 to 0.1125. It also reduces the waiting time per truck from 1.5 hours to 0.075 hours, i.e., a time saving of 85.5 minutes.

Multiple-channel queues: When the service channels are composed of s parallel stations, the state of the system n (the number of elements present in the system at a certain point in time t) can assume one of these values:

(1) $n \leq s$: There is no queue because all elements are being served.

(2) $n > s$: A queue is formed of length $n - s$ (provided the formation of the waiting line is permitted by the nature of the problem).

When n of the s channels are occupied (for $n < s$), the rate of change from n elements present to $n - 1$ (because one element has completely received a service) is "$n\mu P_n dt$" (where P_n = the probability of having n elements in the system) since any one of the n occupied stations may finish its job in the next instant dt.

When $n > s$, a queue is formed. The system is defined as being in state E_n if there are n elements present in the system--waiting and/or in service. Hence, a queue is created when the system is in state E_n with $n > s$ and $n - s$ elements waiting in the queue.

As in the case for the single-channel queue, the formulas applicable to multiple-channel queues will be presented without proof or derivation. It is assumed that the system consists of s service facilities and that each of the s service facilities has a mean service rate μ. It is further assumed that units arrive at the service facilities at a mean arrival rate equal to λ.

The relationship between the mean service rate and the s service stations is given by $\mu_n = \mu s$, $n \geq s$.

The utilization factor, ρ_s, for multiple-channel system is equal to the ratio between the mean arrival rate λ and the maximum possible rate of service of all the s channels; i.e., $\rho_s = \frac{\lambda}{\mu s}$.

For $n < s$ and $s \geq 2$, the probability that there are n elements in the system, P_{sn}, is given by

$$P_{sn} = \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n, \quad n = 0, 1, 2, \ldots, s - 1.$$

The probability that there are no elements in the system, P_{so}, is given by

$$P_{so} = \frac{1}{\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \frac{\mu s}{\mu s - \lambda}}, \quad \mu s > \lambda,\ s > 1.$$

If $s = 1$, the probability that there are no elements in the system is the same as the single-channel case, $P_o = 1 - \frac{\lambda}{\mu}$. The probability that an arrival must wait for service is given by $P_n = \lambda/\mu$.

The probability that an element approaching the service facilities has to wait to receive service, $P_{sn'}$, is given by

$$P_{sn'} = \frac{\mu(\lambda/\mu)^s}{(s - 1)!(\mu s - \lambda)} P_{so},$$

where n' = any value between s up to and including n, and $\mu s > \lambda$.

The average (mean) length of the queue, L_{qs}, (excluding the elements under service) is given by

$$L_{qs} = \frac{\lambda\mu(\lambda/\mu)^s}{(s - 1)!(\mu s - \lambda)} P_{so},$$

where $\mu s > \lambda$.

The average number of elements in the system, L, is given by

$$L = \frac{\lambda\mu(\lambda/\mu)^s}{(s - 1)!(\mu s - \lambda)^2} \cdot P_{so} + \frac{\lambda}{\mu},$$

where $\mu s > \lambda$.

(1) For T = 1, $f(n; \lambda) = \frac{\lambda^n e^{-\lambda}}{n!}, \quad 0 \leq n < \infty;$

(2) The mean arrival rate is the expected number of arrivals occurring in a unit interval of time T. The mean time between arrivals is $1/\lambda$.

(3) The density function of T is given by the expression

$$f(T; \lambda) = \lambda e^{-\lambda T}.$$

The average (mean) waiting time of an element that has arrived in the system, W_s, is given by

$$W_s = \frac{\mu(\lambda/\mu)^s}{(s - 1)!(\mu s - \lambda)^2} \cdot P_{so},$$

where $\mu s > \lambda$.

The average (mean) time that an arrival spends in the system, W_{as}, is given by

$$W_{as} = \frac{\mu(\lambda/\mu)^s}{(s - 1)!(\mu s - \lambda)^2} \cdot P_{so} + \frac{1}{\mu} ,$$

where $\mu s > \lambda$.

As an example of the multiple channel queue with an infinite source, consider the problem of determining (1) the average number of arrivals per average serving time, (2) the average waiting time, (3) the expected number of arrivals, (4) the total daily service time, and (5) the total cost of waiting for a firm which utilizes a tool crib for storage and supply purposes. As a means of facilitating storage and supply, the firm has k of these cribs located throughout the plant. Each crib is maintained by two clerks who hand out the tools as the mechanics request them and take them back when the tools are returned to the crib.

Arrival and service distributions were obtained by noting the time at which a man arrived for service (request a tool) and left with his requested tool. This timing process facilitated the determining of both service and arrival distributions at each crib. The totality of these individual distributions provided a distribution which described the arrival distribution and the service distribution for all of the k cribs.

The queue nature of the problem became evident when it was noted that the varying nature of the service time and the random nature of arrivals resulted in the formation of a line at the counters. If the mechanic arrived at a counter at the very moment the mechanic receiving service completed service, there was no queue. Without a queue, there was no time loss due to idle mechanics. The lack of arrivals or mechanics receiving service produced idle time on the part of the clerks.

Analysis of the arrival distribution revealed that a mechanic arrived for service every 35 seconds. The average service time was found to be 50 seconds per mechanic.

Solution: (1) The average arrival rate, λ, is determined by the ratio between the number of arrivals per time unit and the average serving time per time unit. Substituting as indicated.

$$\lambda = (1/35) \div (1/50)$$
$$= 50/35$$
$$= 1.43.$$

The average number of arrivals per average serving time is 1.43 mechanics. It is assumed that the average service rate for the time interval, μ, is equal to unity.

(2) The average waiting time, T_w, is found by applying

$$T_w = \frac{P_0}{s\mu(s!)[1 - \lambda/\mu]^2} \left(\frac{\lambda}{\mu}\right)^s,$$

where

$$P_0 = \frac{1}{\sum_{n=0}^{s-1} (\lambda/\mu)^n/n! + \{(\lambda/\mu)^s/[s!(1 - \lambda/\mu s)\}} .$$

For $s = 2$, $\lambda = 1.43$, and $\mu = 1$,

$$P_0 = \frac{1}{\left(\frac{1.43}{1}\right)^0/0! + \left(\frac{1.43}{1}\right)^1/1 + \{\left(\frac{1.43}{1}\right)^2/2(1 - \frac{1.43}{2(1)})\}}$$

$$= \frac{1}{(1.00 + 1.43) + \left(\frac{2.0449}{.57}\right)}$$

$$= \frac{1}{(2.43) + (3.58)}$$

$$= \frac{1}{6.01}$$

$$= 0.166.$$

The probability of no mechanics waiting is approximately 0.166. For $s = 2$, $\lambda = 1.43$, $\mu = 1$, and $P_0 = 0.166$,

$$T_W = \frac{0.166}{2(1)2![1 - \frac{1.43}{(2)(1)}]^2} (\frac{1.43}{1})^2$$

$$= \frac{0.166}{4(0.285)^2} (1.43)^2$$

$$\vdots$$

$$= 1.04.$$

There are 1.04 units of average waiting time. With an average serving time of 50 seconds per mechanic, the average waiting time is 52 seconds ($1.04 \times 50 = 52$).

(3) The expected number of arrivals, $E(n)$, is determined by the ratio between total seconds worked per work day and the average arrival rate. Substituting as indicated for a 7.5 hour work day,

$$E(n) = \frac{(7.5)(3600)}{35}$$

$$= \frac{27000}{35}$$

$$= 770.$$

In a 7.5 hour work day, there are 770 expected arrivals for service.

(4) The total daily service time, TDST, is the ratio between total expected service time and the total seconds per hour ...

rate of 50 seconds per arrival, 770 expected arrivals, and 3600 seconds per hour,

$$\text{TDST} = \frac{(770)(50)}{3600}$$

$$= \frac{38,500}{3600}$$

$$= 10.7.$$

The total daily time spent in service is 10.7 hours.

(5) The total cost of waiting is the sum of the idle time cost for the two clerks and the idle time cost for the mechanics. The idle time cost for the two clerks is the total hourly cost paid during idleness. The idle time cost for mechanics is the product of the hourly wage per mechanic and the expected waiting time of the mechanics in a 7.5 hour work day.

Two clerks furnish 15 hours of service time per day. Of this 15 hours, only 10.7 hours is spent in service. This results in 4.3 hours of idle time for the two clerks. At c_1 dollars per hour, total daily cost for clerk idle time is $4.3c_1$.

With an expected waiting time of 52 seconds per arrival and 770 expected arrivals per day, the expected daily waiting time is 11.1 hours. This is calculated as follows: $\frac{770 \times 52}{3600}$. At c_2 dollars per hour, total daily cost for mechanic idle time is $11.1c_2$.

With total cost of waiting equal to the sum of total daily cost for clerk idle time and total daily cost for mechanic idle time, the total cost of waiting is given by the sum $(4.3c_1 + 11.1c_2)$. Given c_1 and c_2, this cost is easily determined.

Finite source: When consideration is given to the possible existence of a finite population of queue inputs, it becomes necessary to modify the techniques of the previous discussion. Finite queueing problems have been classified as queueing problems with limited inputs or finite-input-sources. Regardless of the classification, the study of the finite queueing problem involves the use of a finite number of arrivals. The importance of such a study is evidenced by the applications made of finite queue analysis in the textile industry, machine repair, and inventory analysis.

As an example of this problem, consider the machine interference problem which follows. A machine operator tends a set of N machines. If a machine fails and the operator is free, it is immediately repaired. If a machine fails and the operator is busy, the repairing of the disabled machine is delayed until the operator is again free. When a broken machine must wait to be repaired, a loss of production results. This loss is attributed to interference. If the operator is required to service a large number of machines, the loss of production due to interference is large. If the operator is required to service a smaller number of machines, the loss of production will decrease; but, the cost of operation will increase. The problem then becomes one of minimizing production cost. This is accomplished by balancing the cost of operator per machine against the loss of production resulting from interference.

A cursory analysis of this problem reveals that the "customer" is the machine to be repaired. The service facility is the operator. The arrival time is the rate at which machine breakdown occurs and causes an "arrival" at the repair facility. The service time is the rate at which the operator is able to repair broken machines. The input source is finite since the operator is responsible for a fixed number of machines. Arrival rates can be determined by calculating the mean breakdown rate. Service rates can be determined by calculating the mean service rate. If such data is available, these calculations can be made from historical data. If historical data is not available, the values associated with arrival and service rates can be estimated. Although the distributions which describe the arrival pattern and the service pattern can be estimated by curve fitting, it is generally assumed that arrivals follow a Poisson distribution and that service is either constant or distributed exponentially.[47]

Single channel: The single channel, finite source queueing problem is one in which there is one service facility and all arrivals to the service facility come from a finite source. Arrivals are assumed to be distributed according to a Poisson distribution with parameter λ. Service rates are assumed to be constant or exponentially distributed.

Constant service rate: Under the assumption of a constant rate of service, Cox indicates that "the rate of [service] increases as the dispersion of service-time decreases."[48] This indicates that the constant service rate concept is, in reality, a limit value. This inference is supported by Saaty.

> ...suppose that service is accomplished in k stages, each of which is exponentially distributed with mean $1/\mu$. The total service time will be a chi-squared

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[47]Saaty, op. cit., p. 323.

[48]D. R. Cox and Walter L. Smith, Queues (London, 1961), p. 102.

> distribution with 2k degrees of freedom and mean k/μ. If k and μ become infinite, so that k/μ remains constant, we have. . .the constant-service-time case.[49]

Exponential service rate: Under the assumption of an exponential service rate, units leave the queue according to the exponential distribution. The analysis of the finite queue with an exponential service rate requires the identification of two parameter values: (1) λ, the mean number of units leaving the queue source and (2) μ, the mean number of units returned to the queue source. For example, in the machine interference problem μ is equal to the mean number of repairs completed by the operator per unit of his repair time. With λ equal to the mean number of units leaving the queue source, $1/\lambda$ is equal to the mean time for units to remain in the queue source. With μ equal to the mean number of units returned to the queue, $1/\mu$ is equal to the mean service time.

In defining λ, the following assumptions are made:

(1) All elements in the queue source have the same mean time in the queue, $1/\lambda$;

(2) The system consisting of the queue source and the service facility is in a state of statistical equilibrium.[50]

(3) The periods that queue members remain in the queue source are independent random variables with an exponential distribution of the form

$$P(w_i \leq t) = 1 - e^{-\beta t},$$

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[49] Saaty, op. cit., p. 329.

[50] Statistical equilibrium is defined as a state in which the probabilities associated with arrival rate distribution and service rate distribution are independent of time. (See Ibid., p. 15.)

where β is a constant, w_i is the length of the i^{th} period spent in the queue source, and t is the unit of time.

(4) The service periods follow the discrete distribution

$$P(u = u_i) = \psi_i \quad (i = 1, 2, \ldots),$$

where u denotes the i^{th} service period, and ψ_i denotes the exponential distribution which describes the i^{th} service rate. As a matter of note, the mean service time, $1/\mu$, is equal to $\sum_i u_i \psi_i$.

Having briefly discussed the two types of service distributions, the formulas applicable to the analysis of single channel, finite queues will be presented. These formulas are presented without proof or derivation. As a means of providing a common notational base, the symbols used in describing the formulas for analyzing finite queueing problems are defined below.

λ = the mean number of units leaving the finite source

μ = the mean number of units returned to the finite source

x_m = the mean length of the service period

$\bar{x}_m$ = the mean number of services performed during a service period

$P_{m,r}(t)$ = the probability that if, at some instant, m units are in the finite source, there will be exactly r units remaining after a time t

P_0 = the probability that there will be no units in the service queue

P_n = the probability that there will be n units in the service queue

$\bar{v}$ = the mean number of units in the queue

$\bar{t}_f$ = the mean time on the waiting line n = the number of units in the service facility (either undergoing service or awaiting service or in need of service)

m = the number of units in the finite source

ρ = the traffic intensity (facility utilization factor)

As previously noted, this study is primarily concerned with applied technique. Therefore, the formulas applicable to finite source, single-channel queues are presented without proof or derivation.

The traffic intensity (facility utilization factor) is given by the ratio $\rho = \frac{\lambda}{\mu}$.

The probability that there will be no units waiting for service, P_0, is given by

$$P_0 = \frac{1}{1 + \sum_{n=1}^{m} \frac{m!\rho^m}{(m - n)!}},$$

where $\sum_{n=0}^{m} P_n = 1$.

The probability that there will be n units in the service queue, P_n, is given by

$$P_n = \frac{m!}{(m - n)!} \rho^m P_0.$$

The mean number of units in the queue, $\overline{v}$, is given by

$$\overline{v} = m - \frac{1 + \rho}{\rho} (1 - P_0).$$

The mean time spent in the queue, $\overline{t}_f$, is given by

$$\overline{t}_f = \frac{\overline{v}}{\lambda(m - n)} = \frac{1}{\mu}\left(\frac{m}{1 - P_0} - \frac{1 + \rho}{\rho}\right).$$

The probability that if, at some instant, m units are in the finite source, there will be exactly r units remaining after a time t is given by

$$P_{m,\ r} = \frac{m!}{r!(m - r)!} e^{-r\lambda t}(1 - e^{-\lambda t})^{m-r}.$$

The mean number of services performed during a service period is given by

$$\overline{X} = \mu x_m.$$

Multiple channels: The introduction of multiple service channels (facilities) is an extension of the single channel case. With a finite source, arrivals to the multiple facilities come from a fixed number of arrivals. For example, consider the problem involving the repair of m machines with s operators available. Arrivals to the repair facilities cannot exceed m in number since the source of arrivals is fixed at m. Machines requiring repair are the customers for the operator (service facility). Assuming that there are fewer service facilities than source elements (i.e., $s < m$) and that n elements are in need of service, $1 \leq n \leq s$ implies that $s - n$ service facilities are idle. With $s - n$ service facilities idle and n units receiving service, no queue exists. If $s \leq n \leq m$, there are s units receiving service; $n - s$ units are waiting to enter service. These waiting units form a queue of length $n - s$.

Although arrival rates can follow any probability distribution, Poisson arrivals are the most common. Service rates tend to be exponentially distributed. The use of these distributions provides tools for calculating probabilities similar to those of the preceding sections.

In working with multiple-channel queues, consideration is given to the state of the queueing system. The "state of the system" concept relates to the number of units of the original m units which remain in the queue source. If it is found that m - n units remain in the queue source, the system is said to be in state E_n. The possible states of the system are then E_0 (no units in service or waiting), E_1 (m - 1 units remaining in the queue source),...,E_m (no units remaining in the queue source). In this light the system is in state E_n if m - n units remain in the queue source and either (1) $n > s$ or (2) $n \leq s$. If $n < s$, s units are receiving service; and n - s units are waiting. If $n \leq s$, n units are receiving service. It is assumed that service periods, u, are independent random variables. The service periods are described by the exponential distribution, $1 - e^{-\mu u}$, where μ is the service rate.

Having briefly discussed the finite source, multiple-channel queueing problem, the formulas applicable to the study of these problems will now be presented. In keeping with the procedure of the previous presentations, these formulas are given without proof or derivation. The symbols used in writing these formulas are given below.

λ = the mean number of units leaving the finite source (arrival rate)

μ = the mean number of units leaving the service facility (service rate)

ρ = the traffic intensity (facility utilization) ratio

k_1 = the coefficient of unavailability for units in the finite source

k_2 = the coefficient of service facility idleness

$\overline{v}$ = the mean number of units in the queue

$\overline{n}$ = the mean number of units in all of the queues and in service

m = the number of units in the queue source

s = the number of service facilities

n = the number of units leaving the queue source

P_n = the probability that there will be n units waiting for service

P_o = the probability that there will be no units waiting for service

ζ = the mean number of vacant (idle) service facilities

T = time interval of delay due to waiting

a_n = the ratio between P_n and P_o (i.e., $a_n = P_n/P_o$, with $a_o = 1$)

$\overline{t}_f$ = the mean waiting time for service

L_s = the mean number of units receiving service

$\xi(\overline{t}_f)$ = the expected waiting time

P = the probability of waiting (i.e., of a delay)

These symbols are utilized in writing the following set of formulas. These formulas apply only to finite source, multiple-channel queueing problems.

The traffic intensity ratio (facility utilization factor), , is given by $\rho = \frac{\lambda}{\mu}$.

The value of a_n, defined as $a_n = P_n/P_o$, is given by

$$a_n = \begin{cases} \frac{m - n + 1}{n} \cdot \rho \cdot a_{n-1}, & 0 \leq 0 \leq s - 1 \\ \frac{m - n + 1}{s} \cdot \rho \cdot a_{n-1}, & s \leq n \leq m, \end{cases}$$

with $a_o = 1$.

The probability that there will be no units waiting for service, P_o, is given by

$$P_o = \frac{1}{1 + \sum_{n=1}^{m} a_n} .$$

The probability that there will be n units waiting for service, P_n, is given by

$$P_n = \begin{cases} \dfrac{m!}{(m-n)!n!} \rho^n P_o, & 0 \le n \le s \\ \dfrac{n!}{s!s^{n-s}} \cdot \dfrac{m!}{(m-n)!n!} \rho^n P_o, & s \le n \le m, \end{cases}$$

with $\sum_{n=0}^{m} P_n = 1$.

The coefficient of unavailability for units in the finite source, k_1, is given by

$$k_1 = \frac{\text{the mean number of units waiting for service}}{\text{total units in the queue source}} = \frac{\bar{v}}{m} .$$

The coefficient of service facility idleness, k_2, is given by

$$k_2 = \frac{\text{the mean number of idle service facilities}}{\text{total number of service facilities}} = \frac{\zeta}{s} .$$

The mean number of units waiting for service, $\bar{v}$, is given by

$$\bar{v} = \sum_{n=s+1}^{m} (n - s)P_n .$$

The mean number of vacant (idle) service facilities, ζ, is given by

$$\zeta = \sum_{n=0}^{s} (s - n)P_n .$$

The mean number of units in the queue and in service, $\bar{n}$, is given by

$$\bar{n} = s + \sum_{n=s+1}^{m} (n - s)P_n - \sum_{n=0}^{s} (s - n)P_n .$$

The probability of a delay, $P(>0)$, is given by

$$P(>0) = \sum_{n=s}^{m} P_n .$$

The mean waiting time for service, $\bar{t}_f$, is given by

$$\bar{t}_f = \frac{\bar{v}}{\lambda(m - \bar{n})} \equiv \frac{1}{\lambda(m - \bar{n})} \cdot \sum_{n=s+1}^{m} (n - s)P_n.$$

The mean number of units receiving service, L_s, is given by

$$L_s = s - \sum_{n=0}^{s-1} (s - n)P_n.$$

The expected waiting time, $\xi(\bar{t}_f)$, is given by

$$\xi(\bar{t}_f) = \frac{\bar{t}_f}{P(>0)}.$$

The economic function in queue analysis has been defined as the total cost of the customer's waiting time and the service facility's idle time. This is the mathematical expectation of expenses caused by delays to both customers and service facilities.[51]

In calculating the mathematical expectation of expense caused by delays to both customers and service facilities, use is made of two numerical values. The two values are (1) the cost of customer delay and (2) the cost of service facility idle time. These two values are obtained as follows:

cost of customer delay = (cost per unit of customer time)(mean time lost by customers);

cost of idle services = (cost per unit of service time)(mean time lost due to idle services).

The mean time lost by customers is the product of the time interval of delay, T, and the mean number of customers waiting in line, $\bar{v}$; i.e.,

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[51]Kaufman, op. cit., p. 118.

$$\text{mean time lost by customers} = \bar{v}T.$$

The mean time lost due to idle services is the product of the time interval of delay, T, and the mean number of idle services, ζ; i.e.,

$$\text{mean time lost due to idle services} = \zeta T.$$

If c_1 and c_2 denote the costs per unit of customer time and the cost per unit of service time, respectively,

$$\text{cost of customer delay} = c_1\bar{v}T;$$

$$\text{cost due to idle services} = c_2\zeta T.$$

Therefore, the total cost of delay for s stations in parallel, $\Upsilon(s)$, is given by

$$\begin{aligned}\Upsilon(s) &= c_1\bar{v}T + c_2\zeta T \\ &= T[c_1\bar{v} + c_2\zeta] \\ &= T[c_1 \sum_{n=s+1}^{m} (n - s)P_n + c_2 \sum_{n=0}^{s} (s - n)P_n].\end{aligned}$$

The total cost of delay per unit of time is given by

$$\frac{\Upsilon(s)}{T} = c_1 \sum_{n=s+1}^{m} (n - s)P_n + c_2 \sum_{n=0}^{s} (s - n)P_n.$$

This cost function, which is to be minimized, is defined in terms of the number of service facilities, s.

As an example of the finite source queue, consider a numerical example of the machine interference problem. A maintenance man is responsible for servicing four machines. Machines tend to need repair at a mean rate of two per eight hour day. It is assumed that arrivals follow a Poisson distribution. Machine repair requires 30 minutes per machine. It is assumed that service follows an exponential distribution, and the order of service is first-come, first-served. Determine (1) the mean arrival rate, (2) the mean service rate, (3) the probability of n units waiting or being serviced, (4) the expected length of the queue, and (5) the expected waiting time.

Solution: (1) The mean arrival rate, λ, is the ratio between the expected daily arrivals and the hours of service available. Substituting as indicated,

$$\lambda = \frac{2}{8} = 0.25 \text{ units per hour.}$$

(2) The mean service rate, μ, is the ratio between the time unit and the repair time. Since repair time is expressed in minutes, the time unit per hour is 60 minutes. Substituting these values as indicated,

$$\mu = \frac{60}{30} = 2 \text{ units served per hour.}$$

(3) The probability of n units waiting or being serviced, P_n, is given by

$$P_n = \frac{1}{1 + \sum_{n=1}^{m} \frac{1}{n!} (\mu/\lambda)^n}.$$

Substituting for m, μ, and λ,

$$P_n = \frac{1}{1 + \sum_{n=1}^{4} \frac{1}{n!} (8)^n}.$$

The probability of n units waiting or being serviced is calculated by substituting the appropriate value of n into the relation defining P_n.

(4) The expected length of the queue, $\bar{v}$, is given by

$$\bar{v} = m - \frac{1+\rho}{\rho}(1 - P_0),$$

where m = the number of possible inputs, $\rho = \lambda/\mu$, and

$$P_0 = \frac{1}{1 + \sum_{n=1}^{m} \frac{m!\rho^m}{(m-n)!}} .$$

For m = 4 and $\rho = .25/2 = 1/8$,

$$\bar{v} = 4 - \frac{1 + \frac{1}{8}}{\frac{1}{8}} \quad 1 - \frac{1}{1 + \sum_{n=1}^{4} \frac{4!}{(4-n)!} (\frac{1}{8})^n}$$

$$= 4 - 9 \quad 1 - \frac{1}{1 + \frac{4!}{3!}(\frac{1}{8}) + \frac{4!}{2!}(\frac{1}{8})^2 + \frac{4!}{1!}(\frac{1}{8})^3 + \frac{4!}{0!}(\frac{1}{8})^4}$$

$$= 4 - 9 \quad 1 - \frac{1}{1 + \frac{4}{8} + \frac{12}{64} + \frac{24}{512} + \frac{24}{4096}}$$

$$= 4 - 9(1 - \frac{1}{\frac{7128}{4096}})$$

$$= 4 - 9(1 - \frac{4096}{7128})$$

$$= 4 - 9(\frac{3032}{7128})$$

$$= 4 - 3.82$$

$$= 0.18 \text{ units.}$$

The expected length of the queue is 0.18 units.

(5) The expected waiting time, $\overline{t}_f$, is given by

$$\overline{t}_f = \frac{\overline{v}}{\lambda(m - n)} \quad \frac{1}{\mu}\left(\frac{m}{1 - P_o} - \frac{1 + \rho}{\rho}\right).$$

Substituting for $m = 4$, $\mu = 2$, $P_o = \frac{4096}{7128}$, and $\rho = 1/8$,

$$\overline{t}_f = \frac{1}{2}\left(\frac{4}{1 - \frac{4096}{7128}} - \frac{1 + \frac{1}{8}}{\frac{1}{8}}\right)$$

$$= \frac{1}{2}\left(\frac{4}{(3032/7128)} - 9\right),$$

$$= \frac{1}{2}\left[4\left(\frac{7128}{3032}\right) - 9\right],$$

$$= \frac{1}{2}\left[\frac{28512 - 27288}{3032}\right],$$

$$= \frac{1}{2}\left[\frac{1224}{3032}\right],$$

$$= \frac{612}{3032},$$

$$= 0.20 \text{ hours.}$$

The expected waiting time is 0.20 hours. This is equivalent to 12 minutes.

Applications of Classical Optimization Theory

Introduction

Mathematical approaches to decision-making activities utilize some form of model. The model itself is then used to represent reality. As such, the model is restricted only by the ability of the decision-maker and the manner in which reality can be described.

Classical optimization theory utilizes a class of models that can be defined as problem solving, or unique solution, models. The purpose of the classical optimization theory model is to determine the relationships between input and output factors, describe these relationships with some functional expression, and then solve the resulting expression with an appropriate solution technique. It is important to note that the objective of classical optimization theory is not determining an ideal level of activity. The nature of the expressions and the manner in which solution techniques are applied do not take into account the possibility of alternative solutions.

The tools and models of classical optimization theory are ideally suited for the problems to which they have been applied. As a direct, computational approach to decision-making, the use of classical optimization theory provides the solution to a given problem. Since all of the problems are formulated in terms of equalities, the solution achieved is the only solution. There are no alternatives, a result guaranteed by the formulation of the problem itself.

In the applications to follow, it is to be noted that all of the problems seek the answer, not an answer. The mathematical expressions are such that there is but one answer to a given problem. These expressions are then solved by a straightforward computational formula (for example, the quadratic formula).

Algebraic Equations

Although the algebraic equation has been used to describe functional relationships between sales and revenue, costs and labor, sales and profit etc., the classic use of the algebraic equation in classical optimization theory is breakeven analysis. The typical type of mathematical function utilized in breakeven analysis is a linear function. This linear function is generally used to represent such concepts as profit, total revenue, total costs, and to analyze economic trends via time series analysis. The linear assumptions are important for several reasons, the most prominent being the following:

(1) the rate of increase (or decrease) per unit change is constant;

(2) there exists a one-to-one relationship between the dependent variable and the independent variable (i.e., if $y = f(x)$ defines a linear function, for every value of x there exists a unique value of y, and vice versa);

(3) graphic representations are practical (in two-space);

(4) linear equations are easily understood.

In addition to these four basic assumptions, the use of linear functions as a means of defining revenue and cost guarantee that equilibrium (if it is achieved) exists at a single point.

Although linear analysis has been a major tool for classical breakeven analysis, the validity of the linear function must be carefully considered. With an increasing cost function (i.e., a cost function with a positive slope), cost increases at a constant rate. With an increasing revenue function (i.e., a revenue function with positive slope), revenue increases at a constant rate, a situation which prohibits diminishing return. The use of linear function results in a constant change in the dependent variable for every unit change in the independent variable. This constant change is equal to the value of the slope of the linear function. For example, suppose $y = f(x) = 350 + 5x$, where y equals total cost and x equals units produced. This function indicates that the phenomenon being described has a fixed cost of 350 dollars and a variable cost of 5 dollars per unit produced. If production increases, total cost will increase 5 dollars for each unit produced. If production decreases, then total cost will decrease 5 dollars per unit decrease in production until the fixed cost of 350 dollars is reached. At the 350 dollar mark, total cost is incurred for cost factors other than variable cost.

It is important to note that the use of economic cost is for convenience only. The input-output relationship defined by the algebraic equation depends upon the entity described. The coefficients represent some type of contribution (cost, revenue, profit, utility, etc.). The input defines some type of characteristic by which output is described (for example, labor costs-hours worked, total sales--units sold, job replacement--aptitude test score, etc.).

Breakeven analysis has been extensively applied as a tool for managerial and economic planning. In this application, it has been used to determine the relative profitability of products and the effect of various sales mixes on profits. In addition, it has been used as a means of determining the point at which sales revenue is sufficient to meet all costs for various productive activities.[52]

Although generally associated with sales-cost relationships, breakeven analysis can be traced to concepts in classical economics. In particular,

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[52] Theodore, op. cit., p. 210.

breakeven analysis can be traced to the classical concepts regarding the equilibrium level of employment and the theory of interest.[53]

As an example of classical breakeven analysis, consider the revenue and cost functions shown in Figure 2.19. It is assumed that both revenue and cost are linear functions defined in terms of units produced and sold.

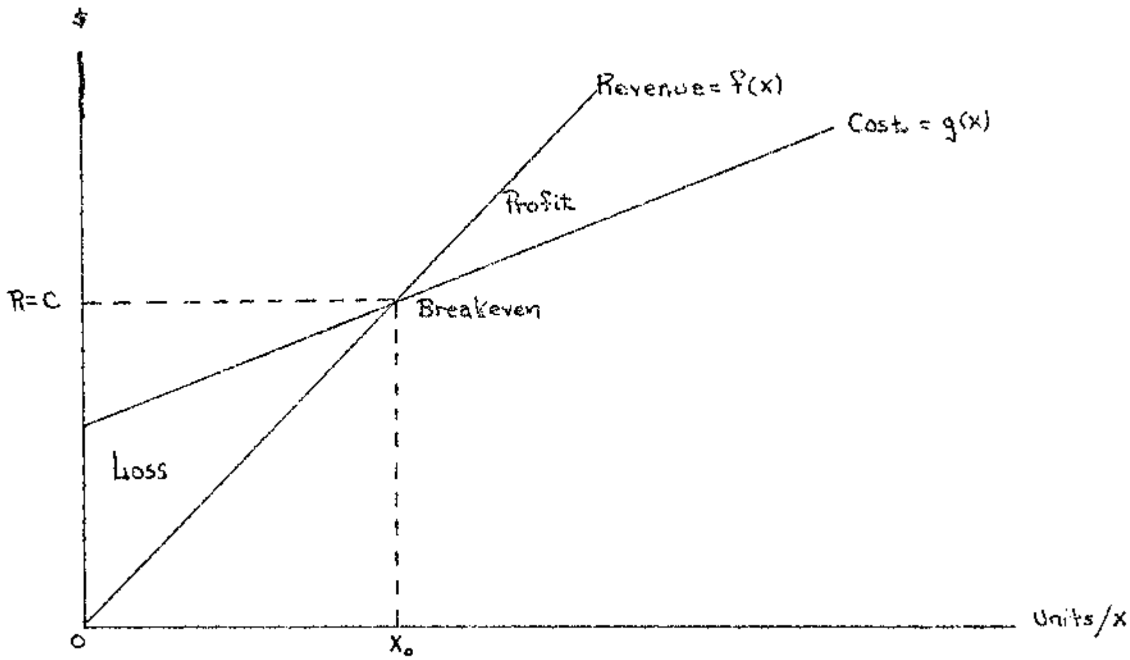

Fig. 2.19--Linear breakeven analysis

The breakeven point occurs at x_o. At x_o units, total revenue equals total costs. If the firm produces and sells $x_o + k$ units, the firm will have a profit equal to $f(x_o + k) - g(x_o + k)$.

If the assumption is made that revenue is nonlinear (say, quadratic) and cost is linear, the result is a graph similar to that shown in Figure 2.20. In this particular illustration, the firm has two breakeven points, x_1 and x_2. Maximum profit will occur somewhere between x_1 and x_2.

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[53] Wallace C. Peterson, Income, Employment, and Economic Growth (New York, 1967), pp. 91-100.

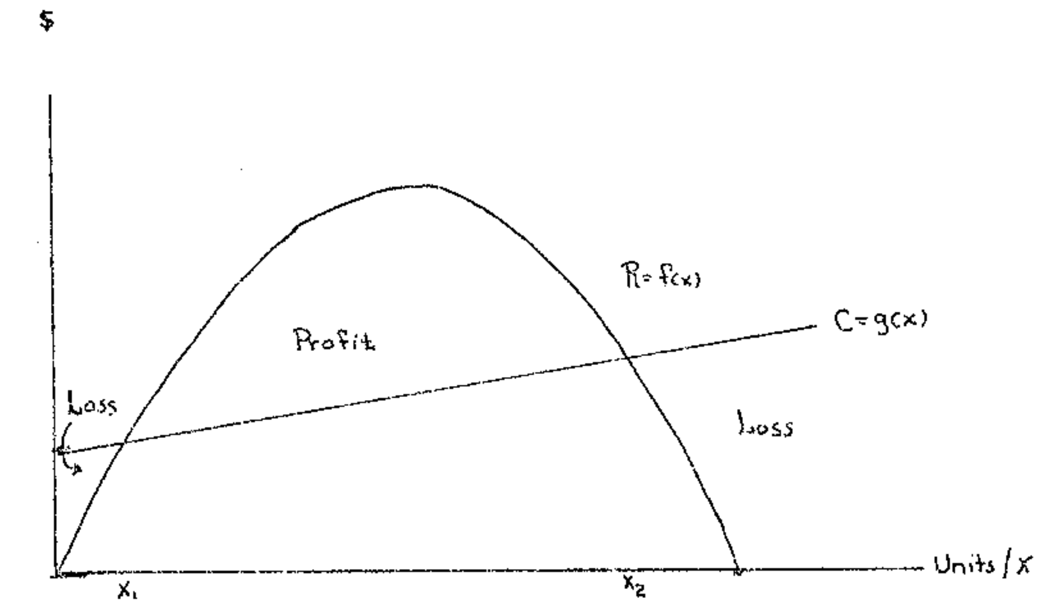

Fig. 2.20--Nonlinear revenue with linear cost

If the assumption is made that both revenue and cost are nonlinear, the resulting problem is similar to that of Figure 2.21. In this illustration it is assumed that both revenue and cost are quadratic functions.

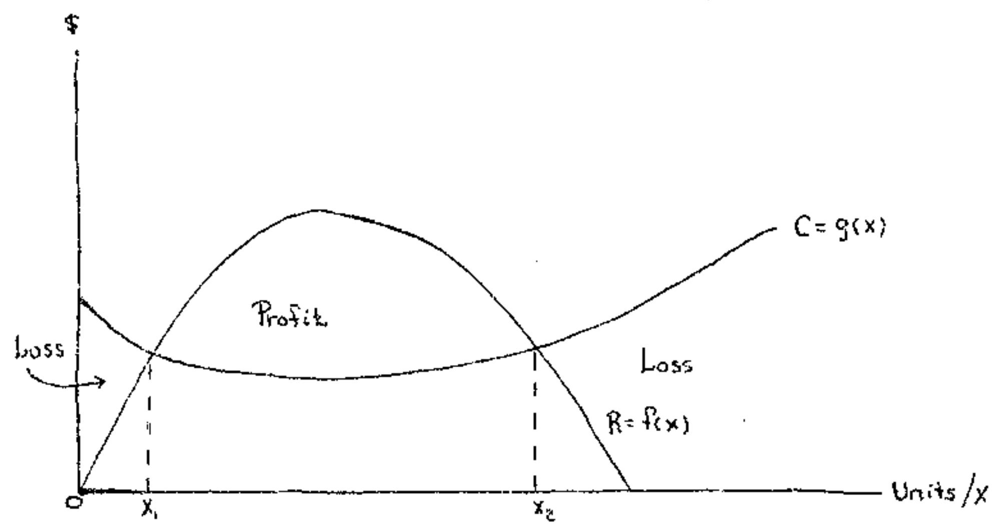

Fig. 2.21--Nonlinear breakeven analysis

Under these assumptions, there are two breakeven points, x_1 and x_2. Maximum profit will occur somewhere between x_1 and x_2. An immediate consequence of these graphic illustrations is the realization that the technique is the same regardless of the functions involved. Nonlinearity in the revenue or cost function or both simply introduces the possibility of more than one point of economic equilibrium, or breakeven. At each point of intersection for the graphed equations, the return equals the cost incurred.

Although graphic breakeven analysis is generally limited to two-space, the technique is a useful tool of classical optimization theory. However, in application, certain limiting factors must be given careful consideration. These factors include the following:

(1) Breakeven analysis requires valid data relative to the quantities being studied.

(2) Linear assumptions are realistic only over narrow ranges of output.

(3) All inputs are grouped in a homogeneous manner (i.e., multiple inputs are not considered). This presumes any quantity of a homogeneous product at a single price--dollars, time, or effort.

(4) Because of the possibility of "lumped inputs," breakeven analysis tends to present an overly-simplified picture of reality.

(5) Breakeven analysis, at best, is fixed with respect to time. Because of this, it is best used only for short-run problem-solving.

As a means of describing the applications that have been made of breakeven analysis, three primary areas of application have been selected for presentation. These three areas are: (1) economics, (2) production, and (3) cost-volume analysis. This selection was based on the criteria that breakeven analysis is primarily concerned with the determining of points of equilibrium. Equilibrium is achieved by balancing (equating) return to cost. The return may be dollars, satisfaction, efficiency, etc.

Economic applications.--Jevons, Walras, Cournat, and Marshall contributed much toward the utilization of mathematics as a tool of economic analysis. As a means of figuratively explaining the relationships which existed between supply and demand, cost and price, these early mathematically inclined economists stressed the applicability of the basic concepts of mathematical functions to solving these problems. Their work laid the foundation on which future applications would be built.

The primary use made of breakeven analysis as an economic tool centers on the areas of demand-supply analysis and market supply-price analysis. In these two areas the classical approach to the problem-solving activity is through the use of linear relationships. By defining strictly linear functions as being representative of the problem being investigated, the intersection of the defining equations yields the single solution to the problem. This is true so long as the defining equations are not parallel lines.

As a means of illustrating the use of breakeven analysis as a tool of demand-supply analysis, consider the problem of determining the equilibrium level of employment from the breakeven point of aggregate supply and aggregate demand. This relationship is graphically depicted in Figure 2.22. Both axes are defined in terms of dollars. The vertical axis measures the expenditures of the economy. The horizontal axis measures the revenue from the economy (the money paid for employment). Both aggregate supply and aggregate demand are expressed as linear functions of real income.

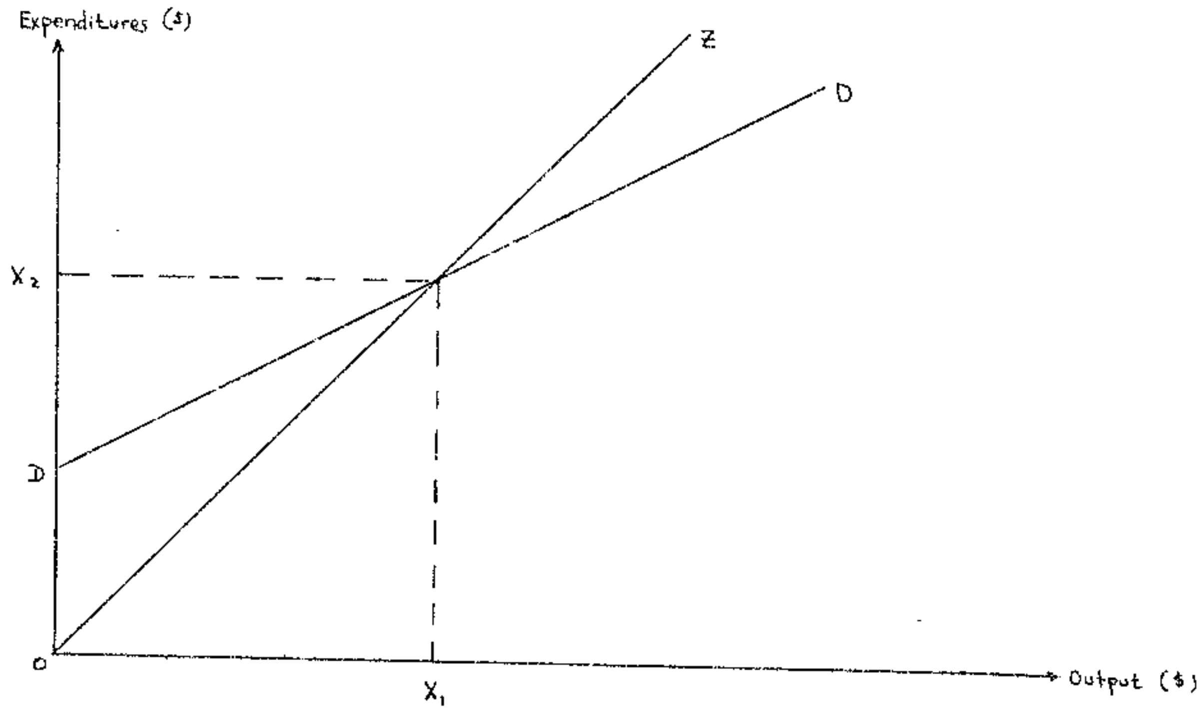

Fig. 2.22--Aggregate demand-aggregate supply

In Figure 2.22, line OZ defines the aggregate supply function. Line DD defines the aggregate demand function. Points along OZ represent the productive output of the economy. Points along DD represent the spending decisions of the economy. When DD exceeds OZ (the interval from 0 to x_1), the productive capacity of the economy will expand to meet an excess of demand. When DD is less than OZ (the interval from x_1 to ∞), the productive capacity will decrease since more is being produced than is being demanded.

Economic balance is achieved at coordinate (x_1, x_2), the breakeven point. Point (x_1, x_2) thus defines the level of income at which the expenses of the economy (the money paid for employment) are equal to the revenues from the economy (the expenditure by units in the economy).[54] This point is reached when aggregate supply (OZ) is equal to aggregate demand (DD). If OZ were defined as $g(x)$ and DD were defined as $f(x)$, the equilibrium level of employment is that level such that $f(x) = g(x)$. This is reached for $f(x_1) = g(x_2)$.

When applied to problems relating market supply and market demand, it is necessary to redefine the independent variable. In this case the common variable is the price that consumers are willing to pay for a given number of units of a given product. The breakeven point (or point of equilibrium) then defines the price at which supply equals demand. As in the previous case, the market supply and market demand functions are assumed to be linear.

Production applications.--The primary use of breakeven analysis in production activity is to give management an improved understanding of the relationships between sales income, costs, and profits at different volumes of production and sales. As a managerial tool it has been used to answer such questions as

(1) What will be the effect on profit if the company raises or lowers prices?

(2) What will be the effect on profit of increases or decreases in costs such as taxes, rent, salaries, supplies, and equipment?

(3) How much will profits increase with an increase in production and sales?

(4) Should the company go through with a proposed plant-expansion program?

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[54] Ibid., pp. 122-123.

(5) How much increased volume will be needed to cover the cost of a wage increase?

(6) Is the company's budget in line?

(7) Where is the "fine-line" between profit and loss for the company?[55]

Although not restricted to linear functions, the primary use of break-even analysis in production work has been based on assumptions of linearity. This is partly because linear approximations are reasonably valid for short periods of time and linear analysis is more easily understood and explained than nonlinear analysis.

Administrative applications.--Of particular importance to management is a valid cost-volume-profit analysis. This particular area is of special importance because it "provides attention-directing and problem-solving background for important planning decisions such as selecting distribution channels, pricing, special promotions, and personnel hiring."[56] Knowledge of cost behavior patterns can be used as a valuable aid in planning both short-run and long-run operations.[57]

In addition to the regular assumptions applicable to linear breakeven analysis (e.g., constant prices, costs, homogeneous products, etc.) there are some additional assumptions to be considered. Among these are the following:

(1) Expenses may be classified into variable and fixed categories. Total variable expenses vary directly with volume. Total fixed expenses do not change with volume.

(2) Efficiency and productivity will be unchanged.

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[55] Richard J. Hopeman, Production: Concepts, Analysis, and Control (Columbus, 1965), pp. 80-81.

[56] Charles T. Horngren, Accounting for Management Control: An Introduction (Englewood Cliffs, 1965), p. 162.

[57] Ibid., p. 172.

(3) Sales mix will be constant. The sales mix is the relative combination of quantities of a variety of company products that compose total sales.

(4) The difference in inventory level at the beginning and at the end of a period is insignificant.[58]

As in the previous examples, the approach to the problem via breakeven analysis is to first express a cost-sales relationship in terms of a common variable. In this case the common variable is the product that is produced and sold. It then becomes a problem in determining the volume (in terms of units) at which total sales equals total costs. This is the breakeven point.

Systems of Equations

Simultaneous linear equations represent a collection of one-to-one mappings subject to one of three possibilities: (1) unique solutions exist, in which case the system is said to be consistent, (2) no solution exists, in which case the system is said to be inconsistent, or (3) multiple solutions exist, in which case the solutions are dependent. These three possibilities indicate that a given system of simultaneous linear equations has exactly one set of solutions, no solution, or an infinite number of solutions.

Applications of simultaneous equations overlap those of simple algebraic equations. Their use permits multivariable descriptions of given problems and a better analysis of problems for which homogeneous groupings are not valid. Problems amenable to simultaneous equations include product-mix analysis, input-output analysis, cost allocation analysis, and financial

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[58] Ibid., p. 174.

investment allocations. Each of these applications is characterized by the same set of generalizations: (1) the relationships are linear, (2) there exist interrelationships between the variables of the system, and total consumption (or use) is guaranteed. In addition, there are no restrictions placed on the variables with respect to possible numerical values.

The input-output model developed by Wassily Leontief represents an application of simultaneous linear equations to a static economic structure. The purpose of the model is to determine industrial production, given changes in final demand. Solutions to input-output models are obtained by applying matrix algebra to the defining system. The model itself provides a method of recording systematically the input factors that are used by all industries in a given economic system.

> Assume that an economy is divided into n industries and that each industry produces only one type of output. Industries are usually interconnected in the sense that one must use some of the others' product in order to operate. An economy must usually produce some finished products for final demand as well as for the use of other industries. One of the basic problems of input-output analysis is determining the production of each of the industries if final demand changes, assuming the structure of the economy does not change.[59]

In input-output analysis, use is made of a tabular set of data recording n producers, n users, final demand, and total output. This data is tabulated in general form in Table 2.1.[60]

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[59] Jean E. Draper and Jane S. Klingman, Mathematical Analysis (New York, 1967), p. 491.

[60] Ibid., pp. 491-492.

TABLE 2.1

GENERAL INPUT-OUTPUT TABLE

	User	Final	Total
Producer	1 2 . . . n	demand	output
1	b_{11} b_{12}. . .b_{1n}	h_1	x_1
2	b_{21} b_{22}. . .b_{2n}	h_2	x_2
.	. . .	.	.
.	. . .	.	.
.	. . .	.	.
n	b_{n1} b_{n2}. . .b_{nn}	h_n	x_n

Table element b_{ij} (i_j = 1, 2,...,n) denotes the dollar amount of the products of industry i consumed by industry j. Table element h_i denotes the final demand for industry i. Total output for industry i, denoted by x_i, is obtained by summing the components of row i; i.e.,

$$x_i = h_i + \sum_{j=1}^{n} b_{ij}, \qquad i = 1, 2, \ldots, n.$$

Associated with the input-output table is a technological matrix. This technological matrix is given by $\underline{A} = (a_{ij})$, where $a_{ij} = \frac{b_{ij}}{x_j}$. The value associated with a_{ij} is the dollar value of the output of industry i that industry j must purchase to produce one dollar's worth of its own products. To meet the needs of all industries, the i^{th} industry ($i = 1, 2, \ldots, n$) must produce a level of output that is equal to $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$. The interindustry demand is given by the linear system

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n. \end{array}$$

Since interindustry demand and final demand for the i^{th} industry must equal the output for the i^{th} industry,

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + h_i, \qquad i = 1, 2, \ldots, n.$$

This form can be expanded into the linear system

$$\begin{array}{c} x_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + h_1, \\ x_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + h_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + h_n. \end{array}$$

Solving for h_i ($i = 1, 2, \ldots, n$),

$$\begin{array}{c} (1-a_{11})x_1 - a_{12}x_2 - \ldots - a_{1n}x_n = h_1, \\ -a_{21}x_1 + (1-a_{22}x_2) - \ldots - a_{2n}x_n = h_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -a_{n1}x_1 - a_{n2}x_2 - \ldots - (1-a_{nn})x_n = h_n. \end{array}$$

If the vector $\underline{X} = (x_1, x_2, \ldots, x_n)^T$ denotes the output for the industry and the matrix $\underline{A}$ denotes the technological matrix, the interindustry demand can be written as the product matrix $\underline{AX}$, where

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots a_{nn} \end{bmatrix} \quad \text{and } \underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

If $\underline{H} = (h_1, h_2, \ldots, h_n)^T$ denotes the final demand, the output system for the industry can be written as $\underline{X} = \underline{AX} + \underline{H}$.

The solution system for final demand,

$$\begin{aligned} (1-a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_n &= h_1, \\ -a_{21}x_1 - a_{22}x_2 - \cdots - a_{2n}x_n &= h_2, \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ -a_{n1}x_1 - a_{n2}x_2 - \cdots - (1-a_{nn})x_n &= h_n, \end{aligned}$$

can then be written in the matrix form

$$(\underline{I} - \underline{A})\underline{X} = \underline{H},$$

Where $\underline{I}$ is an appropriate n x n identity matrix. The solution to the system, given by the $\underline{X}$ vector, can be obtained by calculating the inverse of $(\underline{I} - \underline{A})^{-1}$, and multiplying on the left; i.e.,

$$(\underline{I} - \underline{A})^{-1}[(\underline{I} - \underline{A})] = (\underline{I} - \underline{A})^{-1}\underline{H}.$$

This operation expresses the solution vector $\underline{X}$ as

$$\underline{X} = (\underline{I} - \underline{A})^{-1}\underline{H}.$$

Although the input-output model is somewhat long, it typifies the use of simultaneous linear equations in administrative analysis. It demonstrates the linearity that is generally assumed, the interrelationships among the independent variables, and the total consumption requirement. The formulation of problems relative to cost allocations between products or departments, product-mix allocations, budget analysis, etc., proceed in a similar manner. With the exception that solutions must be nonnegative, there are no restrictions other than all equations be satisfied simultaneously.

The use of simultaneous linear equations is characterized by several assumptions regarding the formulation of the problem-expression. These characteristics reflect the assumptions made when describing any class of allocation-distribution problem and are as follows:

1. Linearity. As was the case in linear breakeven analysis, describing the problem in terms of linear equations requires that the relationships among the variables be proportional; i.e., the rate of substitution among the variables is constant.

2. Certainty. The use of linear equations makes the assumption that all of the relationships among the variables of the problem are known. (This is an inherent assumption of any deterministic model.)

3. Total Consumption (total use). The use of the equality indicates that the resource is to be totally consumed.

4. Additivity. The total amount specified by the system as a whole equals the summation of the various inputs (inflows) minus the summation of the various outputs (outflows). The result is a "material balance equation."[61]

5. Nonnegativity. Since business activity is not defined for negative input (or output), the linear systems must have nonnegative solutions. In Cartesian coordinate space this places all allowable solutions in the upper right quarter of the coordinate system.

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[61] George B. Dantzig, Linear Programming and Extensions (Princeton, 1963), p. 33.

6. The use of the simultaneous system permits consideration of multivariable phenomena. This provides a useful expression for any type of linear distribution/allocation type problem.

Max-Min Calculus

As a tool of classical optimization theory, the max-min calculus is associated with nonlinear analysis. In this application the function being investigated is a continuous, nonlinear cost or profit function and is not subject to a system of limiting constraints. Many of the basic descriptions of business activity (for example, revenue and cost curves, production functions, and learning curves) can be described by nonlinear functions. The most common nonlinear functions are quadratic functions, cubic functions, and exponential functions. As in the classical use of the algebraic equation, the typical nonlinear function utilizes a single independent variable. In this application, all inputs are described by a homogeneous term. For example, costs may be defined in terms of hours worked, regardless of the product; revenue may be defined in terms of total units sold, with no effort being made to distinguish the sale of different products.

Various applications of nonlinear, max-min analysis exist in administrative areas. For example, a production function is generally described by a parabolic curve opening downward. In addition, isoquant and isocost analysis is based on nonlinear assumptions. The isoquant curve is used to show the different combinations of resources a firm can use to produce equal amounts of a given product. The isocost curve is used to show the different combinations of resources which the firm can purchase, given the price per unit of resource and the cost outlay made by the firm. Of particular interest to the administrator is the analysis of a profit function to determine maximum profit or the analysis of a function to determine minimum cost.

The classical approach to solving problems of this type is through the use of the differential calculus. As an administrative tool, the differential calculus is employed to identify points of maximum or minimum value (or both) on given continuous and differentiable functions. Functions are generally assumed to be continuous and differentiable, as well as unimodal. These assumptions guarantee the existence of the necessary derivatives and the existence of a maximum or a minimum. (It should be noted that such max-min points may be local and not global.)

Consider the following inventory problem: A company has a contract to supply R units of product per month at a uniform daily rate. Monthly storage rates are given as c_1 dollars per unit of product held in storage. Setup costs are fixed at c_2 dollars. It is assumed that production is instantaneous and shortages do not occur. Determine the number of units to be made at each production run that will minimize the total average monthly cost.

Solution: Let x denote the number of units produced per production run. Average inventory is given by $x/2$. Average storage cost is $c_1x/2$. Each production run lasts x/R months, yielding an average setup cost (the cost to set up to produce x units) of c_2R/x. Assuming total cost is defined as average storage cost plus average setup cost, total cost is given by

$C(x) = c_1x/2 + c_2R/x.$

The number of units, x, to be produced so as to minimize C(x) is that quantity for which $C'(x) = 0$ and $C''(x) > 0$. Therefore, assuming differentiability, the total cost function can be minimized as follows:

$$C(x) = \left(\frac{C_1}{2}\right)x + (C_2R)x^{-1};$$

$$C'(x) = \frac{C_1}{2} - (C_2R)x^{-2};$$

$$C''(x) = (2c_2R)x^{-3}.$$

Setting $C'(x) = 0$ and solving for x,

$$\frac{c_1}{2} - (c_2R)x^{-2} = 0,$$

$$\frac{c_1}{2} = (c_2R)x^{-2},$$

$$\frac{c_1}{2} = \frac{c_2R}{x^2},$$

$$x^2 = \frac{2}{c_1}(c_2R),$$

$$X = \pm\sqrt{2c_2R/c_1}.$$

Although the solution to $C'(x) = 0$ is both positive and negative, only positive values of x are permissable.

To minimize total average monthly cost, select x such that $\frac{2c_2R}{x^3} > 0$.

Since c_1, c_2, and R are known in advance, the value of x is completely determined. This value of x then indicates the number of units to be made at each production run to minimize total average monthly cost. Because of the quadratic nature of the defined cost function, $x = \sqrt{2c_2R/_c}$, is the only point of minization. That is, it is the production quantity that produces a minimum value for the given cost function.

This example can be used to distinguish several characteristics peculiar to problems similar to this one.

(1) The function to be maximized or minimized is described in terms of one independent variable. This forces all product units to be grouped into a single identifying set.

(2) For purposes of differentiation the functional relationship is assumed to be continuous and differentiable through at least the second derivative.

(3) The function is assumed to be unimodal; i.e., the function has an absolute maximum or an absolute minimum. This assumption generally limits the functional relationship to a quadratic expression.

Although the classic nonlinear textbook problem is generally univariable, multivariable cases do appear (e.g., demand defined as a function of both supply and price). Cost (or profit) functions can be defined in terms of the various products contributing cost (or profit). The end result is a nonlinear, multivariable expression that is solved by applying the concepts of partial derivatives. This application usually results in a set of simultaneous equations which can be solved algebraically or by matrix algebra. Examples of such problems include multimarket equilibrium,[62] multiperiod production functions,[63] demand surface analysis,[64] and multivariable profit-earning functions.[65]

As a means of demonstrating multivariable max-min applications, two general areas will be explored. These two areas are pricing decisions and revenue-cost-productivity analysis. It should be remembered that these areas are specific cases of a general class of problems. Any problem possessing characteristics similar to these specific applications is amenable to the solution techniques used in solving the demonstrated problems.

(1) Pricing decisions. Common to both economic and marketing areas, pricing problems usually evolve in response to supply-and-demand relationships. Since every pricing decision involves a balancing of cost and demand considerations, it is necessary to find a price "in between that which drives most customers away and that which does not cover costs."[66] This is accomplished in classical analysis in the following manner: A function is derived (univariable or multivariable) that depicts the relationship between demand for a product and its selling price. Applying the differential calculus (derivative or partial derivatives), the price

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[62] Henderson and Quandt, op. cit., pp. 126-163.

[63] Ibid., pp. 241-243.

[64] If two related commodities exist for which the quantities demanded are x and y and the respective prices are p and q, then the demand functions can be defines by $x = f(p, q)$ and $y = g(p, q)$, assuming that the quantities demanded depend only on the prices of the two commodities. If a demand function of two independent variables is continuous and single-valued, it can be represented by a surface, defined as a demand surface. (Note: Draper and Klingman, op. cit., pp. 292-297.)

[65] Theichroew, op. cit., pp. 266-294.

[66] Baumol, op. cit., p. 306.

yielding maximum demand (or some satisfactory combinations of price-demand-supply) can be uniquely determined.

As an example of this approach to problem-solving, consider the following: The forecasted sales function for a commodity is given by $S_{10} = (1,000,000)\ (1 - x^{-2})$, where $x \geq \$1.5$ represents the advertising budget in millions of dollars and S_{10} is the annual sales in units at a price of $10 per unit. The annual production cost function, assumed linear, is given by: $C = 200,000 + 5S$, $S \leq 1,200,000$. Here S is the number of units produced per year.

Solution 1: Assuming that no other information is available, it is possible to determine the level of advertising necessary for the firm to achieve maximum profit. Since profit is given by the difference between total revenue and total cost, the profit function can be written

Profit = Revenue - (Production Cost + Advertising Cost),

$P(x, S) = 10S - [200,000 + 5S + 1,000,000x]$,

$P(x, S) = 5S - 200,000 - 1,000,000x$.

It is assumed that potential sales reach a maximum of 1,000,000 units.

The profit function, as shown, is a function in two variables. However, there is a defining function for S which can be used to express P(x, S) as a function of one variable. This function is $S = 1,000,000\ (1 - x^{-2})$. If this is substituted into P(x, S), the result is a function which defines profit in terms of the single independent variable, advertising cost (x).

$P(x) = -200,000 + 5,000,000\ (1 - x^{-2}) - 1,000,000\ x$, where x represents the level of advertising.

Since profit is to be maximized, the max-min criterion will be as follows: (1) set $P'(x) = 0$ and solve for the critical points; (2) evaluate $P''(x)$ for all x such that $P'(x) = 0$; (3) $P''(x) < 0$ indicates a maximizing solution at the x value for which $P''(x) < 0$. The required derivatives are

$$P'(x) = -1{,}000{,}000 + 10{,}000{,}000x^{-3},$$

and

$$P''(x) = -30{,}000{,}000x^{-4}.$$

Setting $P'(x) = 0$ and solving for x,

$$-1{,}000{,}000 + 10{,}000{,}000x^{-3} = 0,$$

$$10{,}000{,}000/x^3 = 1{,}000{,}000,$$

$$x^3 = 10,$$

$$x = \sqrt[3]{10},$$

$$x = 2.1544.$$

At $x = 2.1544$, $P''(x) = -30{,}000{,}000x^{-4} < 0$. This implies that maximum profit is realized with an advertising budget of $2,154,400.

By adopting an advertising budget of $2,154,400, the firm will realize a profit (at a selling price of $10 per unit) of $1,568,300. Note that this profit is achieved with a fixed selling price of $10 per unit and a maximal advertising budget of $2,154,400, based on an assumed 1,000,000 units. Actually, the number of units sold, S, is given by

$$S = 1{,}000{,}000\ (1 - x^{-2}).$$

At $x = 2.1544$, $S = 784{,}500$ units.

Solution 2: Assume that price is to be one of the variables to be considered, where $5 \leq p \leq 20$. The demand function, as a function of price, is given by:

$$S_p = S_{10}\left(\frac{20 - p}{10}\right).$$

In this expression, p denotes the price per unit, S_{10} denotes the quantity sold at a price of \$10, and S_p denotes the quantity sold at price p.

The profit function is again given by the difference between total revenue and total cost:

Profit = Total Revenue - (Production Cost + Advertising Cost)

$$P(p, S_p, x) = pS_p - (200{,}000 + 5S_p) - 1{,}000{,}000x.$$

$P(p, S_p, x)$ defines profit in terms of three independent variables. However, it is possible to express the profit function in terms of two independent variables. This is accomplished by utilizing the relationship between S_p and S_{10} and substituting into $P(p, S_p, x)$.

Since $S_{10} = 1{,}000{,}000\ (1 - x^{-2})$ and $S_p = S_{10}\left(\frac{20 - p}{10}\right)$, S_p can be written as

$$S_p = 1{,}000{,}000\ (1 - x^{-2})\ \left(\frac{20 - p}{10}\right).$$

Substituting this relation for S_p into $P(p, S_p, x)$ will express the profit function in terms of p and x:

$$P(x, p) = p\left[1{,}000{,}000\ (1 - x^{-2})\ \left(\frac{20 - p}{10}\right)\right] - 200{,}000 - 5{,}000{,}000\ (1 - x^{-2})$$

$$\left(\frac{20 - p}{10}\right) - 1{,}000{,}000x.$$

This expression reduces to

$$P(x, p) = \left[1{,}000{,}000\ (1 - x^{-2})\ \left(\frac{20 - p}{10}\right)\right] (p - 5) - 1{,}000{,}000x - 200{,}000.$$

In order to determine the values of x and p for which $P(x, p)$ is maximized, it is necessary to use the partial derivative concept. This requires setting $\partial P/\partial x$ and $\partial P/\partial p$ equal to zero and simultaneously solving the resulting system for the critical values of x and p. The criteria for a maximum is given by

(1) $\frac{\partial}{\partial x} P(x, p) = \frac{\partial}{\partial p} P(x, p) = 0;$

(2) $\frac{\partial^2}{\partial x \partial p} P(x, p) - \frac{\partial 2}{\partial x^2} P(x, p) \cdot \frac{\partial 2}{\partial p^2} P(x, p) < 0$ at values of x and p for which $\frac{\partial}{\partial x} P(x, p) = \frac{\partial}{\partial p} P(x, p) = 0$; and,

(3) $\frac{\partial^2}{\partial x^2} P(x, p) < 0$ at values of x and p for which $\frac{\partial}{\partial x} P(x, p) = \frac{\partial}{\partial p} P(x, p) = 0$.

Differentiating $P(x, p)$ with respect to x and p yields

$$\frac{\partial P}{\partial x} = 1{,}000{,}000\ (2/x^3)\ \left(\frac{20 - p}{10}\right)(p - 5) - 1{,}000{,}000, \text{ and}$$

$$\frac{\partial P}{\partial p} = 1{,}000{,}000\ (1 - x^{-2})\ \left(\frac{20 - p}{10}\right) + (p - 5)\ (1{,}000{,}000)\ (1 - x^{-2})\ \left(\frac{-1}{10}\right).$$

Setting the partial derivatives equal to zero gives the simultaneous system

$$1{,}000\ (2/x^{-3})\ \left(\frac{20 - p}{10}\right)(p - 5) - 1{,}000{,}000 = 0,$$

$$1{,}000{,}000\ (1 - x^{-2})\ \left(\frac{20 - p}{10}\right) + (p - 5)\ (1{,}000{,}000)\ (1 - x^{-2})\ \left(\frac{-1}{10}\right) = 0.$$

This sytem can be solved for values of x and p which simultaneously satisfy the given system.

$$\frac{2}{x^3}\left(\frac{20 - p}{10}\right)(p - 5) - 1 = 0,$$

$$(1 - x^{-2})\left(\frac{20 - p}{10}\right) - (p - 5)(1 - x^{-2})\left(\frac{1}{10}\right) = 0.$$

This system can be reduced to

$$2(2 - .10p)(p - 5) - x^3 = 0,$$

$$\left(\frac{20 - p}{10}\right) - (p - 5)\left(\frac{1}{10}\right) = 0.$$

Further reduction yields

$$-x^3 = .20p^2 + 5p - 20 = 0,$$

$$2 - .10p - .10p + .50 = 0.$$

Collecting like terms,

$$-x^3 - .20p^2 + 5p - 20 = 0,$$

$$-.20p + 2.50 = 0.$$

The values of x and p for which $\frac{\partial}{\partial x} P(x, p) = \frac{\partial}{\partial p} P(x, p) = 0$ are obtained by simultaneously solving the reduced system. Since $-.20p + 2.50 = 0$ is linear in p, this equation will be solved for p and the result substituted into the other equation. With this approach, $\frac{\partial}{\partial x} P(x, p) = \frac{\partial}{\partial p} P(x, p) = 0$ at $p = 12.50$ and $x = 2.24$.

If $P(x, p)$ is maximized when $p = 12.50$ and $x = 2.24$, $\frac{\partial^2}{\partial x^2} P(x, p)$ and $\frac{\partial^2}{\partial x \partial p} P(x, p) - \frac{\partial^2}{\partial x^2} P(x, p) \cdot \frac{\partial^2}{\partial p^2} P(x, p)$ will both be less than zero.

The necessary partial derivatives are

$$\frac{\partial^2}{\partial x^2} P(x, p) = -3{,}000{,}000\left(\frac{2}{x^4}\right)\left(\frac{20 - p}{10}\right)(p - 5);$$

$$\frac{\partial^2}{\partial x \partial p} P(x, p) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial p} P(x, p)\right] = \frac{-2{,}000{,}000(2 - .10p)}{x^3} - 200{,}000x^3(p - 5);$$

$$\frac{\partial^2}{\partial p^2} P(x, p) = -200{,}000 + \frac{200{,}000}{x^2} .$$

At $x = 2.24$ and $p = 12.50$, $\frac{\partial^2}{\partial x^2} P(x, p) = -1{,}345{,}265$, $\frac{\partial^2}{\partial x \partial p} P(x, p) = -16{,}933{,}929$, and $\frac{\partial^2}{\partial x^2} P(x, p) = -160{,}141$. Using these numbers,

$$\frac{\partial^2}{\partial x^2} P(x, p) = -1{,}345{,}265 < 0, \text{ and}$$

$$\frac{\partial^2}{\partial x \partial p} P(x, p) - \frac{\partial^2}{\partial x^2} P(x, p) \cdot \frac{\partial^2}{\partial p^2} P(x, p) = -142{,}804{,}706{,}294 < 0.$$

Thus, at $x = 2.24$ and $p = 12.50$, $P(x, p)$ achieves a maximum value.

Based upon the results of the preceding paragraph, the price per unit of product will be \$12.50. The advertising budget will be \$2.24 per unit. At a price of \$12.50 per unit, sales will total 600,700 units. The number of units sold is obtained by evaluating

$$S_p = 1{,}000{,}000(1 - x^{-2})(\frac{20 - p}{10})$$

at $p = 12.50$ and $x = 2.24$. At $p = 12.50$ and $x = 2.24$, the amount of profit will be \$2,064,000. The amount of profit to be realized is obtained by evaluating

$$P(x, p) = [1{,}000{,}000(1 - x^{-2})(\frac{20 - p}{10})](p - 5) - 1{,}000{,}000x - 200{,}000$$

at $p = 12.5$ and $x = 2.24$.

This example incorporates advertising expenditures into the problem and demonstrates the use of classical max-min calculus. In the first solution, price per unit was not an independent variable. The only independent variable was the advertising expense per unit of potential sales. In the second solution, both price per unit and advertising expense per unit were allowed to vary. The approach shown in the second solution resulted in a price per unit that fluctuated in response to fluctuations in advertising expense per unit.

(2) Revenue-Cost-Productivity Analysis. The use of the calculus as a tool of maximal-minimal analysis is well documented in economic theory, particularly microeconomic theory. This documentation is evidenced by the application of marginal analysis to the following functional expressions:

(a) the production function, $Q = f(x)$, which records how the required quantity of labor or some raw material, x, varies with the production level, Q, of some commodity;

(b) the cost function, $C = g(Q)$, which records the total cost, C, associated with some production level Q;

(c) the demand function, $P = F(Q)$, which shows how high a price, P, can be changed per unit in order to sell Q units (i.e., the demand function indicates the expected quantity at different price levels);

(d) the revenue function, $R = G(Q, P)$, which shows the total income (or revenue) accruing to the firm from the sale of Q units at price P; and,

(e) the utility function, $U(Q)$, which measures the pleasure that the individual derives from the possession of some quantity, Q, of some commodity.[67]

The most practical tool for marginal analysis is the derivative, previously defined as the average rate of change in the dependent variable per unit change in the independent variable as the change in the independent variable approaches zero as a limit. In this application, the derivative is interpreted as follows:

(a) marginal product, $\frac{dQ}{dx}$;

(b) marginal cost, $\frac{dc}{dQ}$;

(c) marginal price, $\frac{dP}{dQ}$;

(d) marginal revenue, $\frac{dR}{dQ}$; and,

(e) marginal utility, $\frac{dU}{dQ}$.

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[67] *Ibid.*, pp. 59-60.

Each of these interpretations measures the rate of change in the respective function for every unit change in the independent (input) variable.

To illustrate this application and use of the calculus, consider the production function $y = f(x)$. This function defines the relationship existing between a given firm's outputs and its inputs. Assuming the production function is defined in terms of a single independent variable, the relationship can be plotted in Cartesian coordinate space as shown in Figure 2.23. The various inputs are plotted on the horizontal axis and are represented by the variable x. The output associated with a given input is plotted on the vertical axis and is identified by the variable y.

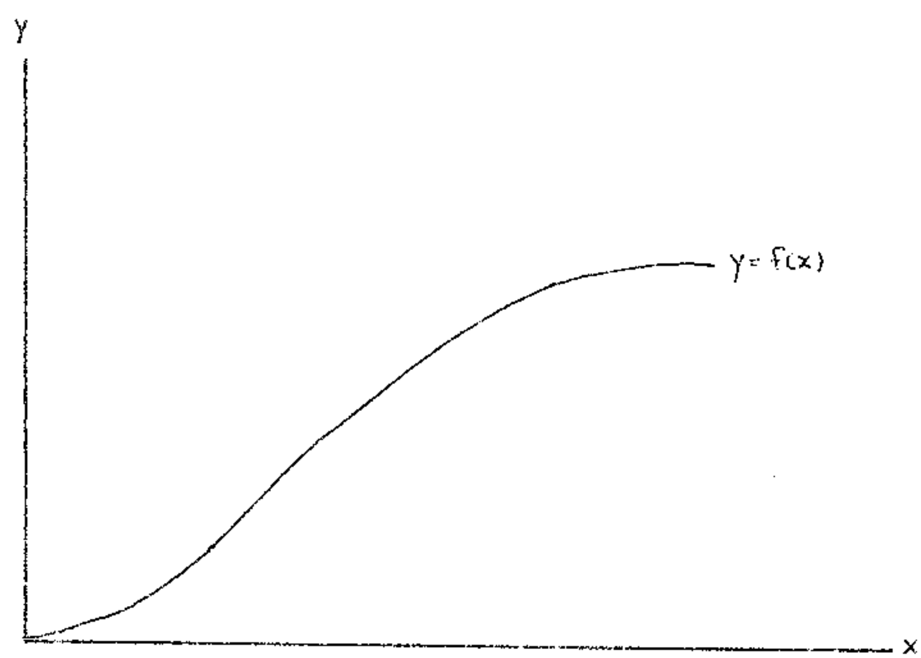

Fig. 2.23--The production function

In describing the production function several characteristics are to be noted:

(1) The production function is expressed free of any cost. The input factor is measured in factors of production (generally technological factors such as man-hours or units of raw material) while output is measured in units of commodity production.

(2) Different technological factors define different production functions. The defined relationship applies to one given productive process and is valid for a given state of technological development.

(3) All input factors not considered as independent variables are defined as fixed factors of production.

(4) The usual shape (one independent variable) of the production function is parabolic, opening downward. (This is not a necessity, as a production function can be described by any degree equation.) This description better describes the concept of diminishing returns, the point beyond which increases in input result in smaller changes in output.

(5) The point of maximum productivity is defined as the point where marginal productivity, dy/dx, equals zero. So long as $dy/ds > 0$ the firm can increase productivity by increasing input. When $dy/dx < 0$ the firm is decreasing productivity by increasing input.[68]

Although these characteristics are not all inclusive, they serve to bring out certain points of particular interest. Among these are the following:

(1) As required for functional analysis, the univariable case results in a "lumping together" of factors. This is necessary to establish a common input. Multivariable assumptions allow for better analysis, but these result in complex functions that are not easily (if ever) graphed.

(2) The point of maximum productivity is achieved at the level of input for which the line tangent to $y = f(x)$ has zero slope. This relationship is handled in the multivariable case by equating the n partial derivatives of $y = f(x_1, \ldots, x_n)$, $\partial y/x_i (i = 1, 2, \ldots, n)$ to zero. This defines the point

[68]Clough, op. cit., p. 146.

at which the surface tangent to the production surface equals zero.

(3) The analysis is static, i.e., valid only at that point in time.

From these features, additional characteristics applicable to the class of functions described here can be derived. If these characteristics are satisfied by a given problem, the tools of the max-min calculus can be effectively utilized.

(1) The required functions are defined in terms of n input variables. If $n = 1$, the function is univariable and, assuming continuity, amenable to the differential calculus. If $n \geq 2$ and the function is assumed to be continuous, the function is amenable to the use of partial derivatives.

(2) The required functions are generally assumed to be unimodal. This assumption guarantees the existence of only one maximum point or one minimum point.

(3) If the derivative exists, marginal analysis is applicable. In addition, the concept of diminishing returns is satisfied.

(4) A common characteristic is the parabolic nature of the functions. This is, in part, due to the fact that the functions are generally quadratic (i.e., second degree functions).

(5) With the exception of the cost function (which has a minimum), all of the applications are maximal in nature. The desire is to locate the point at which profit, demand, utility, etc., is maximum; for the cost function the desire is to locate the point at which cost is minimal in value.

(6) Although the curve is defined to be unimodal (or the surface to have one peak), the maximum utilization of input factors is defined for each of the cases in the following manner.

(a) maximum profit: that input-output combination such that marginal cost and marginal revenue are equal; i.e., $dR/dQ = dC/dQ$.

(b) maximum utility: that point at which the consumer's marginal utility per dollar's worth of goods is equal for all goods.

(c) maximum productivity with a given cost outlay: that point on the total product surface (or curve) such that the marginal product of a dollar's worth of one resource (input) equals the marginal product of a dollar's worth of every resource used.

These definitions relate the respective derivatives or partial derivatives of the defined functions in proportion to their respective unit costs. Of all of these, perhaps the most common (from the viewpoint of the calculus) is the assumption that marginal revenue equated to marginal cost yields maximum profit. This assumption forces the economic requirement that maximum profit occur at the point of minimum cost.

Consider the following numerical example. Price per unit for a particular product is assumed to be linearly related to weekly production in the following manner: $p = f(q) = 100 - 0.01\,q$, where p equals the price per unit, and q equals the amount of weekly production. Cost of production, C, is assumed to be linearly related to weekly production and is defined by the function $C = g(q) = 50q + 30{,}000$. Determine the price and quantity for which profit is maximized.

Solution 1: The profit function is defined by the relation Profit = Revenue - Cost. Revenue is defined as unit price times units sold and is functionally described by $pq = q \cdot f(q) = q(100 - 0.01q)$. Given the defined cost function, the profit function, P, is written in terms of the single variable q.

$$\begin{aligned} P(q) &= (100 - .01q)q - (50q + 30{,}000) \\ &= -.01q^2 + 100q - 50q - 30{,}000 \\ &= -.01q^2 + 50q - 30{,}000. \end{aligned}$$

To test for maximization, set $P'(q) = 0$ and solve for q. If $P''(q) < 0$ for values of q such that $P'(q) = 0$, then $P(q)$ is maximized at that value of q. The derivative of $P(q)$, $P'(q)$, is given by

$$P'(q) = -.02q + 50.$$

Setting $P'(q) = 0$ and solving, $q = 2500$.

$$P''(q) = -.02.$$

At $q = 2500$, $P''(q) < 0$, indicating that the profit function is maximized at $q = 2500$.

At $q = 2500$, $p = 75$ units of cost to the consumer, say dollars. The cost incurred is \$155,000. The profit realized is \$32,500.

Solution 2: If it is assumed that marginal revenue $(\frac{dR}{dq})$ is equal to marginal cost $(\frac{dC}{dq})$, it is necessary to obtain functions for $\frac{dR}{dq}$ and $\frac{dC}{dq}$.

$$\frac{dR}{dq} = \frac{d}{dq}\,(100q - .01q^2) = 100 - 0.02a;$$

$$\frac{dC}{dq} = \frac{d}{dq}\,(50q + 30{,}000) = 50.$$

Equating marginal revenue and marginal cost and solving for q,

$$100 - 0.02q = 50,$$

$$-0.02q = -50,$$

$$q = 2500.$$

At q = 2500, the price, cost, and profit are the same as in Solution 1.

Solution 3: Had the intent of the problem been to minimize cost, then quantity produced would have been zero (i.e., q = 0). At this value of q, there would have been no profit. Rather, there would have been a loss of $30,000.

Solution 3 demonstrates an important phenomena of cost-profit analysis. This phenomena relates to the fact that the point of maximum profit is not always equal to the point of minimum cost. A particular case is a cubic sales (or revenue) function with a quadratic cost function. Only when it is assumed that marginal revenue equals marginal cost is it guaranteed that the point of minimum cost will equal the point of maximum profit.

Analogous applications of these two classes of problems can be found in all phases of business activity. Pricing problems are found in accounting, finance, marketing, economics, and management. The "price" is given in terms of some other consideration (for example, goodwill, labor, interest, etc.). Revenue-cost-productivity problems can be translated into such contexts as cost analysis for accounting, advertising expenditure and/or consumer exposure for marketing, sales maximization for management, or maximization of investment returns. The major point is that all of these applications are specific examples of a general class of administrative problems. This general class of problems is identified as unconstrained, continuous functions which describe defined relationships between the variables of the problem. Such functions can be either univariable or multivariable.

The Lagrange Multiplier

The Lagrange multiplier is applicable to the same class of problems as the max-min calculus. In fact, the Lagrange multiplier technique is an extension of the max-min calculus. It allows the incorporation of a minimum number of constraint functions into the problem. These constraint functions are written as equalities.

The only difference between the use of the Lagrange multiplier and the traditional max-min calculus is the inclusion of the restrictive conditions. Examples of such restrictions are maximinization of consumer utility subject to a fixed budget constraint, maximization of profit subject to cost constraints, and minimization of cost subject to fixed labor requirements.

Since the only difference between the use of the Lagrange multiplier and the traditional max-min calculus is the inclusion of the constraint equations, the application area for the Lagrange multiplier is the same as that for the max-min calculus. Utilization of the Lagrange multiplier provides a means of incorporating constraints into the function to be maximized or minimized. Given this incorporation, the solution technique duplicates that of the max-min calculus.

Consider the problem of maximizing $f(x_1, x_2) = 5x_1^2 + 6x_2^2 - x_1x_2$ subject to the linear constraint $g(x_1, x_2) = x_1 + 2x_2 - 24 = 0$. The Lagrange multiplier technique requires the construction of the Lagrangian function

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2).$$

Only one λ is required since there is only one constraint. The function defined by $L(x_1, x_2, \lambda)$ is a multivariable function and is tested for a minimum value according to the traditional max-min calculus. However, since the original function only contains two variables, the criterion for minimization is

(1) $L_1\ (x_1, x_2, \lambda) = L_2\ (x_1, x_2\ \lambda) = L_3\ (x_1, x_2, \lambda) = 0$

(2) $L_{11} > 0$ and $\begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} > 0.$

L_i denotes the partial derivative of $L(x_1, x_2, \lambda)$ with respect to variable i, (i = 1, 2, 3). L_{ij} denotes the second partial derivative of $L(x_1, x_2, \lambda)$ with respect to variables i and j, (i = 1, 2, 3; j = 1, 2, 3).

The required partial derivatives are as follows:

$$\begin{aligned} L_1(x_1, x_2, \lambda) &= \frac{\partial}{\partial x_1}[5x_1{}^2 + 6x_2{}^2 - x_1x_2 - \lambda(x_1 + 2x_2 - 24)] \\ &= 10x_1 - x_2 - \lambda. \end{aligned}$$

$$\begin{aligned} L_2(x_1, x_2, \lambda) &= \frac{\partial}{\partial x_2}[5x_1{}^2 + 6x_2{}^2 - x_1x_2 - \lambda(x_1 + 2x_2 - 24)] \\ &= 12x_2 - x_1 - 2\lambda\ . \end{aligned}$$

$$\begin{aligned} L_3(x_1, x_2, \lambda) &= \frac{\partial}{\partial \lambda}[5x_1{}^2 + 6x_2{}^2 - x_1x_2 - \lambda(x_1 + 2x_2 - 24)] \\ &= -(x_1 + 2x_2 - 24). \end{aligned}$$

$$\begin{aligned} L_{11}(x_1, x_2, \lambda) &= \frac{\partial}{\partial x_1} L_1(x_1, x_2, \lambda) \\ &= \frac{\partial}{\partial x_1}(10x_1 - x_2 - \lambda) \\ &= 10. \end{aligned}$$

$$L_{12}(x_1, x_2, \lambda) \equiv L_{21}(x_1, x_2, \lambda) = \frac{\partial}{\partial x_1} L_2(x_1, x_2, \lambda)$$
$$= \frac{\partial}{\partial x_1} (12x_2 - x_1 - 2\lambda)$$
$$= -1.$$

$$L_{22}(x_1, x_2, \lambda) = \frac{\partial}{\partial x_2} L_2(x_1, x_2, \lambda)$$
$$= \frac{\partial}{\partial x_2} (12x_2 - x_1 - 2\lambda)$$
$$= 12.$$

The system of equations defining the critical values of x_1, x_2, and λ is given by

$$10x_1 - x_2 - \lambda = 0,$$
$$-x_1 + 12x_2 - 2\lambda = 0,$$
$$x_1 + 2x_2 = 24.$$

The solution to this system is the unique solution-set $x_1 = 6$, $x_2 = 9$, and $\lambda = 51$.

The numerical value associated with each $L_{ij}(x_1, x_2, \lambda)$, $(i,j = 1, 2, 3)$ is obtained by substituting for x_1, x_2, and λ. This operation yields the following numerical values: $L_{11} = 10$; $L_{22} = 12$; $L_{33} = 0$; $L_{12} = L_{21} = -1$; $L_{13} = L_{31} = -1$; $L_{23} = L_{32} = -2$; $L_{33} = 0$. Applying the criterion for minimization,

$$L_{11} = 10 > 0;$$
$$\begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 10 & -1 \\ -1 & 12 \end{vmatrix} = 10(12) - (-1)(-1) = 120 - 1 = 119 > 0.$$

Since $L_{11}(x_1, x_2, \lambda)$ and $(L_{11}L_{22} - L_{12}L_{22})$ are both positive at the critical points, $f(x_1, x_2) = 5x_1^2 + 6x_2^2 - x_1x_2$ is minimized at $x_1 = 6$, $x_2 = 9$. The minimum value is $f(6, 9) = 612$.

An interesting consequence of this example is the role played by the Lagrange multiplier. The criterion used to determine whether or not the critical points identify points of maximization or points of minimization does not incorporate the Lagrange multiplier other than in the evaluation of the various partial derivatives. It is not part of the determinant system defined in Theorem 2.15. The role played by the Lagrange multiplier is that of defining the critical points which satisfy the defined function and the limiting constraints.

Administrative problems solvable by the method of the Lagrange multiplier exhibit some common characteristics, among which are the following:

(1) The functional relationship describing the problem can be univariable or multivariable, depending upon the problem.

(2) Functional relationships can be linear or nonlinear; but, the functions are assumed to be differentiable.

(3) Constraint functions are generally equalities, indicating that the resource is such that it is totally consumed. If an inequality condition exists, the Lagrange multiplier technique requires a minimal number of such restrictions. (This is true for equality constraints, also.)

(4) Optimization via the use of the Lagrange multiplier method results in an optimum value which lies on the boundary of the solution space. This is due to the fact that all constraints are described by equalities (inequalities are converted to equalities by introducing appropriate slack or artificial variables).

(5) The final solution is obtained from a system of equations, usually linear, that is solved algebraically or with matrix algebra. This final solution is generally based upon the size of the resulting system.

Queueing Theory

The application of queueing theory requires less mathematical expertise than any other tool of classical optimization theory. This is due to the general nature of queueing applications. These applications generally require the direct application of a specific formula as a means of answering a specific question (for example, the average length of a queue, the average time spent by a queue element in the system, or the probability that all service facilities will be idle). In the application, it is generally assumed that the queueing problem is described by a suitable probability distribution such as the Poisson or Erlang model.

When analyzing queueing problems, an important point to remember is that arrival and service distributions may not be described by the "established" probability distributions. If the arrival and/or service distributions are not distributed according to a Poisson, Erlang, exponential, or gamma distribution, it is necessary that the defining probability distributions be determined. This can be accomplised by employing the chi-square test for goodness of fit.[69] In this way, the most appropriate probability distribution can be determined for queueing phenomena that does not follow a definite distribution.

Consider the following problem: A firm is confronted with the problem of determining the number of loading and unloading docks that will be needed in shipping and receiving department of the firm. As a means of of answering the questions necessary for solving the problem, it is assumed

[69] Charles T. Clark and Lawrence L. Schkade, Statistical Methods for Business Decisions (Cincinnati, 1969), pp. 426-430.

that the firm wants to know how many trucks will be waiting for loading and unloading, as well as the probability that the loading and unloading dock and workers will be idle. Given the necessary assumptions regarding the arrival distribution, the service-time distribution, finite or infinite queues, queue discipline, etc., these questions can be answered by the application of a specific formula.

It is assumed that the arrival pattern follows a Poisson distribution with a mean arrival rate equal to λ. The service rate follows an exponential distribution with a mean service rate equal to μ. Infinite queues are theoretically possible. The queue discipline is first-come, first served. The number of trucks being serviced is equal to n. Under these assumptions,

1. the average number of trucks in the queue, denoted L_q is given by $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$;

2. the average number of trucks in the waiting line, including the one being serviced, L, is given by

$$L = \frac{\lambda}{\mu-\lambda};$$

3. the average waiting time of trucks in the line, W_q, is given by

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)};$$

4. the average waiting time of trucks in line, including the truck being serviced, L, is given by

$$L = \frac{1}{\mu-\lambda} \text{ ; and,}$$

5. the probability that the loading and unloading dock and workers will be idle, Pn, is given by

$$Pn = (1-\lambda/\mu)\ (\lambda/\mu)^n,\ n \geq 1.$$

For $n = 0$, $P_o = 1 - \lambda/\mu$.

Although this example is an elementary application, it serves to point out the nature of classical queue analysis. Once the values of λ and μ are known, the answers to the given questions are easily determined by direct use of a given formula. This is true except for the irregular arrival and service distributions sometimes encountered in queueing phenomena. When these irregular distributions are encountered, the formulas shown do not apply. Rather, it is necessary to consider each problem on its own merit.[70]

The irregular queueing problem can be demonstrated in the following manner. A single channel queue receives arrivals at random with rate α. The service time probability distribution is given by

$$\frac{(k/b_1)\ (kx/b_1)^{k-1}\ e^{-kx/b_1}}{(k-1)!},$$

where k equals the number of stages of service, b_1 equals the mean-service time, and x is the random variable. If the service distribution were exponentially distributed, the state of the system could be defined by the number of units in the system. However, in order to work with this distribution, the state of the system must be defined in terms of the current state of service. The state of the system at any point in time must be derived, and applicable formulas developed (if possible) before the analysis of the system can be accomplished.

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[70] Cox and Smith, op. cit., p. 110.

Let $p_{ni}(t)$ denote the probability that at time t there are n units in the system and the customer in service is in the i^{th} stage of service ($n = 1, 2, \ldots; i = 1, 2, \ldots, k$). Let $p_o(t)$ denote the probability that there are no units in the system at time t. If arrivals occur in groups of k each with an arrival rate of α, and if the service-time is exponential with parameter σ, the equilibrium probability p_r of there being r units in the system at an arbitrary point in time is given by

$$\alpha p_o = \sigma p_1,$$

$$(\alpha+\sigma)p_r = \sigma p_{r+1} + \alpha p_{r-k} \quad (r = 1, 2, \ldots, k),$$

$$p_{r-k} = 0 \text{ for } r < k.$$

In this form, $r = nk - i + 1$.[71] Further modifications can be utilized to determine computational schemes for calculating the long-run proportion of time for which the service facility is occupied and the waiting time of a unit in the system.

The point of this example is to demonstrate the fact that not all queueing situations can be neatly categorized into one of the four basic distributions. When the probability distribution associated with a given queueing process is exponential or Poisson, the analysis of the system is greatly simplified. This is due to the existence of specific formulas applicable to the given problem. Saaty discusses irregular distributions, with particular attention being given to non-Poisson queues[72], Markov chains[73], general-input and arbitrary-service-time-distributions[74], and general independent-input exponential-service times.[75]

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[71] Ibid., pp. 111-114.

[72] Saaty, op. cit., pp. 153-170.

[73] Ibid., pp. 171-190. [74] Ibid., pp. 193-196. [75] Ibid., pp. 198-216.

The literature is well documented with queueing applications: persons passing through checkout counters, machine breakdown and operation, restaurant service, customers entering and leaving supermarkets, bakeries, etc., to mention a few. Specific applications are described below.

Machine breakdown and repair.--A company finds it necessary to minimize the number of machines not available for use. This is accomplished by careful assignment of mechanics to repair machines so that production losses from the lack of adequate equipment is minimized. The machines which are not available for use constitute a queue waiting for repairs; the mechanics constitute the available service facility. There is a point such that the cost of having mechanics available for machine repair exceeds the potential loss incurred for lost production. By building a distribution table (varying the input data and collecting the output) it is possible to determine the number of mechanics which would allow the best use of the company's cost resources.[76] The end result is a minimization of the total cost of the service and a maximization of utilization of mechanic's time.

Flows in production.--Items in a production line are in a queue. They may arrive at varying rates, or they may arrive at a constant rate. Holding time is assumed to be constant. Items in the queue may go through a number of parallel channels or through a single channel. Service facilities may be carried on in parallel or in multi-stage systems. Single channel queues may be serviced in sequence. Although not restricted to inventory and application problems, business use of queueing theory in this area has been

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[76]Saaty, op. cit., p. 365.

extensive, the data obtained via the queueing application being used as input data for some other model.[77]

Servicing problem.--Servicing problems are characterized by a given rate of arrival and a given departure rate following receipt of the service. Servicing problems are generally concerned with achieving an economical balance between the annoyance of waiting units and the number of available service facilities. As an example, consider the following:

> Customers arrive at and leave a restaurant at known rates. Both the arrival rate and the service rate are measured. If the collection of these data is performed in such a way that it is possible to detect the behavior of both events, such measurements can be converted into a probability distribution. From this distribution the probability of arrivals can be computed. The ratios of the average arrival rate over the average service rate, and of the average number of persons in line waiting to be accommodated at the tables per unit of time over the average waiting time required to deliver the service per unit of time are two essential elements for determining the theoretical behavior of the waiting line. [Obviously] . . . the waiting time spent by customers decreases as the number of facilities (tables, waiters, and equipment in the kitchen) increases. . . to increase the facilities [it is necessary] to consider a means of balancing the cost of losing patrons, if the line and the waiting service. For any number of facilities added, there is a corresponding reduction in the average time spent in waiting. When the cost per unit of waiting time as well as the cost of operating each facility known, the total cost of each facility can be computed. The optimum solution is secured when the optimal number of facilities is determined and simultaneously the minimum cost is achieved.[78]

Other applications.--Other applications of queueing theory include the analysis of buffer stocks (components waiting for machining), the

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[77] di Roccaferrera, op. cit., p. 345.

[78] Ibid., p. 812.

analysis of warehouse stocks (finished goods waiting for customers), cars in line at service stations, cars in line at toll booths, the analysis of the optimum number of bank tellers during given periods of the day, and the determination of the most economical combination among type and number of machines to maintain a given level of productivity. Inherent in all of these is a central characterizing theme: there exists something which requires some type of service from at least one of a limited number of available facilities and for which there is a cost associated with any delay caused by waiting for the desired facility. Given this characteristic, the problem is amenable to queueing theory and its techniques of analysis.[79] (Note: Certain types of inventory control problems can be treated as queueing problems: associated with the condition "out-of-stock" is a cost for waiting. Given this cost it is possible to determine the amount of stock (service units) necessary to minimize waiting costs and inventory costs.)

In addition to these applications, queueing theory has been used to determine the optimum number of inspectors required to inspect a product while in process,[80] to determine the amount of oil individual pumping stations should handle[81], and to determine whether or not the service of a new product should be added to the line.[82] In the latter application, the problem is multidimensional and involves priority considerations.

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[79]David W. Miller and Martin K. Starr, Executive Decisions and Operations Research (Englewood Cliffs, 1965), p. 397.

[80]L. F. Sespaniak, "An Application of Queueing Theory for Determining Manpower Requirements for an In-Line Assemble Inspection Department," Journal of Industrial Engineering, vol. 4 (July-August, 1959), 265-267.

[81]Saaty, op. cit., p. 351.

[82]N. K. Jaiswal, Priority Queues (New York, 1968), pp. 70-82

CHAPTER III

MODERN OPTIMIZATION THEORY: BASIC TECHNIQUES OF OPTIMAL SEARCH

Introduction

As an administrative tool, mathematical models which seek to optimize the use of limited resources when these resources are subjected to competing demands are described by the term mathematical programming. The problems to which this particular class of models is applied are generally expressed as functional relationships in which the independent variables (or unknowns) represent the resource quantity (or quantities) that defines the units being demanded. Typical problems suited to the techniques of mathematical programming include the following:

1. A certain machine shop has a variety of lathes. In a certain time period, a set of jobs is assigned to the lathe department for processing. Each job may be routed alternatively to more than one type of lathe. There is insufficient lathe capacity to assign each job to the lowest cost process. The problem is to determine the optimum assignment of jobs so that total processing costs are minimized.

2. A saw mill produces a variety of end products. The input to the mill is a variety of types and sizes of logs. The problem is to decide how the logs are to be allocated to the production of end products. Each end product will have a given cost, depending on the log type and process, and a given market price. The objective is to maximize the price of the output over a given time period.

3. A company produces a set of products with seasonal variation. The problem is to decide on a production plan that will designate month by month how much of each product is to be produced, amounts of overtime, inventory level, and employment levels, such that the total cost of the plan is minimized.

4. An oil company wishes to determine an optimum plan for its refinery operations, given estimates of external conditions, such as market demands for end products, crude oil availability, etc., and given internal company directive, such as refinery unit construction, minimum contracted crude, etc.[1]

In the preceding chapter, optimization of such problems was achieved under the assumption that resources were unlimited and all resources were totally consumed. In actual practice, however, the firm usually finds itself operating in an environment in which resources are limited in terms of quantity and availability. For example a production company may be able to sell every product it produces. However, because of limited manpower, daily production is restricted. Any attempt to optimize profit must take into consideration the limited availability of manpower. Problems formulated under restrictive conditions will take on one of the following forms:

(1) Find the values of $x_j (x_j \geq 0)$ which maximize the function $f(x_1, x_2, \ldots, x_n)$ subject to the constraint equations $g_i(x_1, x_2, \ldots, x_n) \leq b_i$ $(i = 1, 2, \ldots, m)$. The b_i are constants, and the system involves an m x n system, i.e., m constraint equations with n variables. Examples of such problems are (a) maximizing total profit (the function to be optimized) subject to the constraint equations relating budget allocations, production allocations, and (b) maximizing output subject to inventory restrictions, labor hours available, and component-mix.

(2) Find the values of $x_j (x_j \geq 0)$ which minimize the function $f(x_1, x_2, \ldots, x_n)$ subject to the constraint equations $g_i(x_1, x_2, \ldots, x_n) \geq b_i$,

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[1]J. William Gavett, Production and Operations Management (New York, 1968), pp. 58-59.

($i = 1, 2, \ldots, m$). The b_i are constants, and the constraint equations are $m \times n$ in size. Examples of such problems are (a) minimizing a given cost function subject to restraint equations relating man-hours available, minimal output required, and minimum time available on machines at fixed costs, and (b) minimizing transportation costs when the constraint equations are defined by minimal accepted lot shipments at both warehouse and factory.

(3) Find the values of $x_j (x_j \geq 0)$ which optimize (maximize or minimize) a given objective function $f(x_1, x_2, \ldots, x_n)$ subject to the mixed constraint equations $g_i(x_1, x_2, \ldots, x_k) \leq b_i$ and $g_\ell(x_{k+1}, x_{k+2}, \ldots, m) \geq a_\ell$, ($i = 1, \ldots, k$; $\ell = k+1, \ldots, m$), with the b_i and a_ℓ constant. Such systems are characterized by k constraint equations with at most $b_i (i = 1, 2, \ldots, k)$ units available and $m - x_k$ constraint equations with at least a_ℓ ($\ell = k+1, \ldots, m$) units available. The function to be optimized can be one of maximization (profit) or one of minimization (costs).

These mathematical formulations can be described in terms of either linear programming, quadratic programming, geometric programming, or dynamic problem. In terms of basic optimal search, the classification is dictated by the objective function. For example, a linear objective function with linear constraints is a linear programming problem while a quadratic objective function with linear constraints is a quadratic programming problem. Give this problem classification, a suitable selection technique can be selected for determining the optimal solution to the problem under study. Solution techniques for the types of problems cited are presented in this study.

Basic Techniques of Optimal Search

Kuhn-Tucker Conditions

The use of the Lagrange multiplier as a tool of classical optimization theory requires equality constraints on the function to be optimized. In addition, the function under investigation is assumed to be continuous. However, in practical application the constraints are not necessarily defined by equalities. The function being optimized is optimized subject to a system of constraints that is defined by inequalities or a mixture of inequalities and equalities.

For optimization problems subject to one inequality constraint, the use of the Lagrange multiplier is satisfactory. When the constraining system contains more than one inequality, however, proof of optimization becomes more difficult. As a means of describing conditions under which multiple inequality constraints lead to optimum solutions, mathematicians Kuhn and Tucker initiated a study into the possibility of extending the Lagrange multiplier technique. The result of their work is summarized as the Kuhn-Tucker conditions. This set of conditions defines optimality for a function which is restricted by a set of inequalities. These conditions provide the base on which the concepts and techniques of mathematical programming are built.

Kuhn-Tucker conditions for maximization.--A point $(x_1, x_2,\ldots,x_n)$ maximizes a function $f(x_1, x_2,\ldots,x_n)$ subject to $g_j(x_1, x_2,\ldots,x_n) \leq 0$, $j = 1, 2,\ldots,m$ if there exists a set of λ_j, $j = 1, 2,\ldots,m$, $\lambda_j \geq 0$, such that

$$h_i = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n_i;$$

$$\lambda_j g_j(x_1, x_2, \ldots, x_n) = 0; \text{ and,}$$

$$g_j(x_1, x_2, \ldots, x_n) \leq 0.$$[2]

Since maximization involves consideration of both local maxima and global maxima, Kuhn and Tucker defined sufficiency conditions under which the point in question could be tested for local or global maxima. These conditions are summarized by the following:

Local maximum: Let $f(x_1, x_2, \ldots, x_n)$ be a function of n variables subject to the constraint $g(x_1, x_2, \ldots, x_n) \leq 0$. A point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is a local maximum of $f(x_1, x_2, \ldots, x_n)$ subject to $g(x_1, x_2, \ldots, x_n)$ only if there exists $\lambda \geq 0$ such that λ and $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ satisfy

$$h_i = \frac{\partial f}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i} = 0; \quad i = 1, 2, \ldots, n;$$

$$\lambda g(x_1, x_2, \ldots, x_n) = 0; \text{ and,}$$

$$g(x_1, x_2, \ldots, x_n) \leq 0.$$[3]

These conditions are also sufficient if $f(x_1, x_2, \ldots, x_n)$ is concave and the constraint is convex.

Global maximum: Let $f(x_1, x_2, \ldots, x_n)$ be a function of n variables subject to the j constraints $g_j(x_1, x_2, \ldots, x_n) \leq 0$, $j = 1, 2, \ldots, m$. A point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is a global maximum if there exists a set of

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[2]Daniel Teichroew, An Introduction to Management Science: Deterministic Models (New York, 1964), p. 523.

[3]Jean E. Draper and Jane S. Klingman, Mathematical Analysis: Business and Economic Applications (New York, 1967), pp. 514-515.

nonnegative $\lambda_1, \lambda_2, \ldots, \lambda_m$, such that

$$h_i = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0;$$

$$\lambda_j g_j = 0;$$

$$g_j \leq 0.$$ [4]

These conditions are sufficient for a global maximum if both $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$, $j = 1, 2, \ldots, m$, are concave and differentiable. If the x_i must satisfy $x_i \geq 0$, the necessary conditions for a global maximum are

$$h_i = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} \leq 0;$$

$$h_i x_i = 0;$$

$$g_j \lambda_j = 0;$$

$$g_j \leq 0; \text{ and,}$$

$$x_i \geq 0.$$

At this point several points require clarification. The first of these is the distinction between convex and concave functions.

Definition 3.1.--A function is said to be strictly concave if a line segment drawn between any two points on its graph falls entirely below the graph. (See Figure 3.1a.)

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[4] Teichroew, op. cit., p. 563.

Definition 3.2.--A function is said to be strictly convex if a line segment drawn between any two points on its graph lies entirely above the graph. (See Figure 3.1b.)

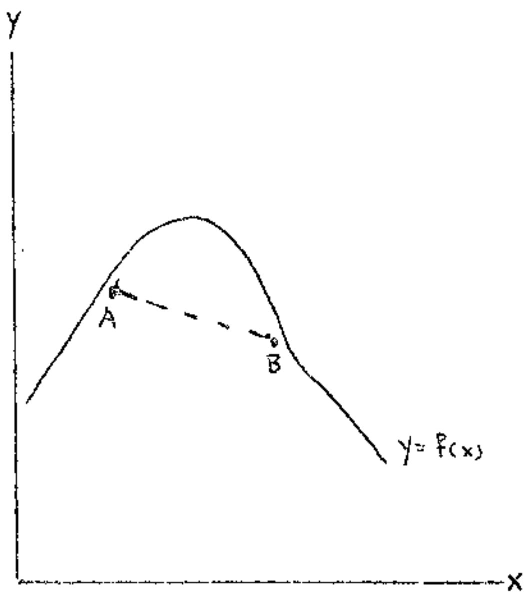

Fig. 3.1a--Concave function

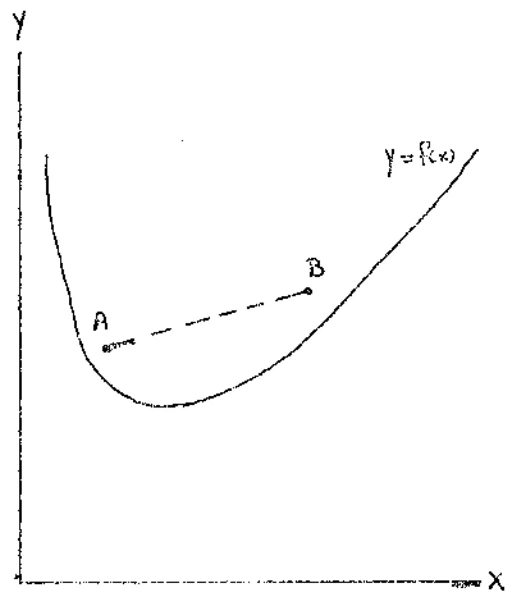

Fig. 3.1b--Convex function

Strictly concave (convex) simply means that all points contained in the defined region do not lie on the boundary (i.e., strict inequality prevails). These functions are such that concavity or convexity (for functions whose second derivative exists) can be determined by investigating the value of the second derivative. For example, let $y = f(x)$ be examined at the point $x = a$. If the second derivative y, d^2y/dx^2, identifies the slope (at $x = a$) of the curve defined by dy/dx, the following holds:

(1) if $d^2y/dx^2 < 0$ at $x = a$, dy/dx is decreasing at $x = a$, and $y = f(x)$ is said to be concave;

(2) if $d^2y/dx^2 > 0$ at $x = a$, dy/dx is increasing at $x = a$, and $y = f(x)$ is said to be convex.

The function $y = f(x)$ is then concave within a given interval if and only if $y'' = f''(x) \leq 0 \ \forall \ x$ lying in the interval; if $y'' = f''(x) \geq 0 \ \forall \ x$ lying in the interval, the function is convex. Note that concavity is associated with maximization, convexity with minimization. A global maximum occurs if a given function is concave throughout a defined region and has a single stationary point. Similarly, a global minimum occurs if a given function is convex throughout a defined region and has a single stationary point.

The second point relates to the interpretation of the Kuhn-Tucker conditions. In developing these conditions, Kuhn and Tucker brought out the following facts:

> (1) For a wide class of [mathematical] programming problems (including all linear problems and all diminishing-returns nonlinear problems) a Lagrangian expression can be formed in exactly the [same] way as. . .for the calculus case, and this Lagrangian expression will have the same useful property--whatever values of the variables maximize (minimize) the value of the original objective function subject to its equality or inequality constraints will maximize (minimize) the value of the Lagrangian expression.
>
> (2) Suppose [one is] dealing with a maximization problem and it turns out that the optimal values of the variables $x_1, x_2, \ldots, x_n$ are. . .numbers which [are] designated by $x_1^*, x_2^*, \ldots, x_n^*$ . . . . Suppose. . .these numbers [are substituted] for the x's in the Lagrangian expression. If the Lagrange multipliers are then treated as variables [the Lagrange multipliers being denoted by λ] and the Lagrangian expression minimized with respect to the λ's, the minimizing values of the λ's are the same as the constant Lagrangian multipliers required for the solution of the original maximization problem. The original problem is said to be solved when and only when the values of the x's which maximize the Lagrangian expression and the values of

the λ's which minimize the Lagrangian expression have been found. [Such a point is called a saddle point.]

(3) If the function to be optimized [the objective function] and the constraining expressions are all linear, the Lagrangian multipliers are the optimal values of the associated dual function. Given any primal and its dual, the Lagrange expressions for both are identical.

(4) This duality relationship leads to the saddle-point property. Consider a primal problem seeking to maximize profit. The dual of the primal is one which seeks to minimize cost. Letting P, C, and L denote the profit (primal), cost (dual), and Lagrangian expressions, respectively, the following analysis results: The values of the primal variables, $x_1, x_2, \ldots, x_n$, which maximize P must also be those which maximize L. Similarly, the values of the dual variables, the Lagrange multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$, which minimize C must also minimize L. The result is that of minimax (saddle point): if one finds a combination of x's and λ's which constitute solutions to the primal and dual problems, respectively, the Lagrangian expression will--for these values--have the minimum value possible for any λ and the maximum value for any x.[5]

With these points in mind, the meaning (and full importance) of the Kuhn-Tucker conditions follows: these conditions serve as existence theorems for optimization under inequality conditions. For the class of programming problems for which they are valid, the Kuhn-Tucker theorems state that a solution exists for a given problem if and only if the corresponding Lagrange expression is simultaneously satisfied.

For example, consider a function $f(x_1, x_2)$ that represents the relationship existing between commodities x_1 and x_2 and their respective contribution to profit. This profit function is to be maximized subject to the constraint equations $g_1(x_1, x_2) \leq 0$ and $g_2(x_1, x_2) \leq 0$. The

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[5]William J. Baumol, Economic Theory and Operations Analysis (Englewood Cliffs, 1961), pp. 51-58.

Kuhn-Tucker conditions then state that for a point (x_1^*, x_2^*) to be a maximum, the following must hold:

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = 0$$

$$\lambda_1 g_1(x_1, x_2) = 0$$

$$\lambda_2 g_2(x_1, x_2) = 0$$

$$g_1(x_1, x_2) \leq 0$$

$$g_2(x_1, x_2) \leq 0.$$

If all $\lambda \geq 0$, the point in question is a local maximum;[5] if $f(x_1, x_2)$, $g_1(x_1, x_2)$, and $g_2(x_1, x_2)$ are all concave (and differentiable), the point in question is a global maximum.[7] If the set of admissable values of the x_i is restricted to non-negative values, as would be the case for profit maximization, the necessary conditions are given by

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_1}{\partial x_1} \leq 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} \leq 0$$

$$x_1 \frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = 0$$

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[6] Teichroew, _op. cit._, p. 307.

[7] _Ibid._, p. 563.

$$\lambda_1 g_1(x_1, x_2) = 0$$

$$\lambda_2 g_2(x_1, x_2) = 0$$

$$g_1(x_1, x_2) \leq 0$$

$$g_2(x_1, x_2) \leq 0$$

$$x_1 \geq 0$$

$$x_2 \geq 0.$$

The results of the Kuhn-Tucker investigations, although given for maximization of a concave function subject to concave constraints, are extended to minimization of convex functions subject to convex constraints by noting that a maximum point of $f(x_1, x_2,\ldots,x_n)$ is a minimum point of $-f(x_1, x_2,\ldots,x_n)$. This extension then takes the following form.

Kuhn-Tucker conditions for minimization.--A point $(x_1, x_2,\ldots,x_n)$ minimizes a function $f(x_1, x_2,\ldots,x_n)$ subject to $g_j(x_1, x_2,\ldots,x_n) \geq 0$, $j = 1, 2,\ldots,m$ if there exists a set of λ_j, $j = 1, 2,\ldots,m$, $\lambda_j \geq 0$, such that

$$h_i = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_1} = 0, \quad i = 1, 2,\ldots,n;$$

$$\lambda_j g_j(x_1, x_2,\ldots,x_n) = 0; \text{ and}$$

$$g_j(x_1, x_2,\ldots,x_n) \geq 0.$$

Sufficiency conditions for local and global minima are similarly extended.

Linear Programming

Linear programming has been defined as

> . . .a technique for specifying how to use limited resources or capacities of a business to obtain a particular objective, such as least cost, highest margin, or least time, when those resources have alternate uses. It is a technique that systematizes for certain conditions the process of selecting the most desirable course of action from a number of available courses of action, thereby giving management information for making a more effective decision about the resources under its control.[8]

Since its introduction as a tool of the operations researcher (or the administrator), linear programming has received extensive attention from both the mathematician and the practitioner. Although the intent of this study centers on the application made of linear programming in administrative analysis, the definitions, theorems, and explanations necessary for a full understanding of the technique are presented in the discussion that follows.

The general linear programming problem can be described in the following manner: Assume that a given amount of resources (manhours, machine-time, quantity of resource, etc.) are available for use in a productive activity. Given the output per unit of resource consumed and the return per unit of output consumed, determine that combination of inputs which optimizes some defined function (e.g., maximizes profit or minimizes cost). It is further assumed that the function to be optimized and its constraining functions are linear expressions of the various quantities produced by the productive activity.

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[8]Richard A. Johnson, Fremont E. Kast, and James E. Rosenzweig, The Theory and Management of Systems (New York, 1967), p. 291, citing Robert O. Ferguson and Lauren F. Sargent, Linear Programming, p. 3.

Thus, the linear programming problem can be of such a nature that the objective is to maximize the return of the productive process or to minimize the effort required to complete the productive process. The constraining functions can be equalities or inequalities or combinations of both. This type of problem is defined in the following manner.

Definition 3.3.--Let $f(x_1, x_2,\ldots,x_n)$ denote a defined linear objective function of the form $f(x_1, x_2,\ldots,x_n) = c_1x_1 + c_2x_2+\ldots+c_nx_n$ where the c_i $(i = 1, 2,\ldots,n)$ represent the associated return (if $f(x_1, x_2,\ldots,x_n)$ is to be maximized) per unit output or the associated cost (if $f(x_1, x_2,\ldots,x_n)$ is to be minimized) per unit input. Let

$\sum_{i,j=1}^{n} a_{ij}x_i$ $(\leq, =, \geq, <, >)$ b_i denote the conditions under which

$f(x_1, x_2,\ldots,x_n)$ is to be optimized (maximized or minimized as needed). In the system defined by the constraint functions, only one of the symbols $(\leq, =, \geq, <, >)$ appears, and all of the constraint functions are linear.

The linear programming problem is then defined as a problem of the following type: Optimize (maximize or minimize)

$$f(x_1, x_2,\ldots,x_n) = c_1x_1 + c_2x_2+\ldots+c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2+\ldots+a_{1n}x_n \ (\leq, =, \geq, <, >)b_1$$

$$a_{21}x_1 + a_{22}x_2+\ldots+a_{2n}x_n \ (\leq, =, \geq, <, >)b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2+\ldots+a_{mn}x_n \ (\leq, =, \geq, <, >)b_m.$$

The function to be optimized is said to be the objective function, and the conditions under which the objective function is to be optimized are called the constraints. A feasible solution is a set of values defined by the n component vector $\underline{x} = (x_1, x_2, \ldots, x_n)$ satisfying all of the constraints of the problem. The solution of the problem refers to any one of three possibilities: (1) an optimal solution has been determined, along with the corresponding value of the objective function; (2) it has been proved that no feasible solution exists; or, (3) it has been proved that, although feasible solutions exist, there is no optimal solution.[9]

Special cases.--Although the term linear programming has been used to define a general class of allocation problems, there are some special cases of the general case that deserve particular mention. These special cases are such that "the coefficients in the constraints have special forms,"[10] and there exist specialized computational techniques for handling these special problems. The most common are presented in this discussion.

(A) Transportation problem. The transportation problem describes a particular sub-class of the linear programming problem that is concerned with shipping a given product (or set of products) from m sources to n destinations. These problems are of special importance "both because

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[9] William R. Smythe, Jr. and Lynwood A. Johnson, Introduction to Linear Programming, with Applications (Englewood Cliffs, 1966), p. 68.

[10] Teichroew, op. cit., p. 503.

they occur often in practice, and because they can be solved by algorithms which are more efficient for this class of problems."[11]

The classical description of the transportation problem involves the determination of an optimal schedule for product shipments that

(1) originate at sources (supply houses, warehouses, etc.) where the total amount of product available for shipment is fixed;

(2) are sent directly to the final point of disposition (retail stores, demand source, etc.) where the total amount of product required is known, although it may vary from destination to destination;

(3) exhaust the source of supply and fill the demand quantity; this forces the condition that total supply equals total demand.

The cost incurred for the operation is such that

(4) it is linear; i.e., the cost of each shipment is proportional to the amount shipped, and the total cost of the operation is given by summing the individual costs involved.[12]

Thus, the transporation problem is concerned with the shipping of a homogeneous product (or set of homogeneous products) from m sources (or origins) to n destinations (or demand points). Each origin furnishes a fixed amount of the product, and each destination requires a fixed amount of the product. This relationship requires that total supply equal total demand. The amount stored at the source is completely exhausted in such a way that total demand is completely satisfied.

The mathematical definition of the transporation problem follows from this discussion.

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[11] Douglas J. Wilde and Charles S. Beightler, Foundations of Optimization (Englewood Cliffs, 1967), pp. 187-188.

[12] George B. Dantzig, Linear Programming and Extensions (Princeton, 1963), p. 299.

Definition 3.4.--Let a_i denote the total quantity of product available for shipment from source i (i = 1, 2,...,m). Let b_j denote the total quantity of product demanded at destination j (j = 1, 2,...,n). Let c_{ij} denote cost of shipping (or transporting) the product from source i to destination (or demand point) j. Let x_{ij} denote the quantity of product that is shipped from source i to destination j. Then, the transporation problem is defined as: minimize the total cost of shipping x_{ij} units of product from source i to destination j in such a way that total demand equals total supply, i.e.,

$$\min \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2,\ldots,m;$$

$$\sum_{i=1}^{n} x_{ij} = b_j, \quad j = 1, 2,\ldots,n;$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j; \text{ and}$$

$$x_{ij} \geq 0 \text{ for all i and j.}$$

The notation given in the definition can be expanded in the manner shown below.

$$\min (c_{11}x_{11} + c_{12}x_{12} + c_{1m}x_{1m})$$
$$+ (c_{21}x_{21} + c_{22}x_{22}+\ldots+c_{2m}x_{2m})$$
$$+\ldots+(c_{m1}x_{m1}+\ldots+c_{mn}x_{mn})$$

subject to

$$x_{11} + x_{12} + x_{13} + \ldots + x_{1m} = a_1$$

$$x_{21} + x_{22} + x_{23} + \ldots + x_{2m} = a_2$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$x_{m1} + x_{m2} + x_{m3} + \ldots + x_{mn} = a_m$$

$$x_{11} + x_{21} + x_{31} + \ldots + x_{n1} = b_1$$

$$x_{12} + x_{22} + x_{32} + \ldots + x_{n2} = b_2$$

$$x_{1n} + x_{2n} + x_{3n} + \ldots + x_{mn} = b_n$$

$$a_1 + a_2 + \ldots + a_m = b_1 + b_2 + \ldots + b_n$$

and $x_{ij} \geq 0$ for all i and j.

In practical application, total supply and total demand are not necessarily equal. Such a situation can be handled by the use of a dummy (non-existent) supply or dummy demand point that supplies or demands the quantity necessary for equality. Another approach to the same situation (unequal supply and demand totals) is explained by di Roccaferrera.

> When the total items required [demanded] is more than production capacity [supply], this difference can be distributed among all the receiving points (by reducing their demand), thus satisfying the objective function [minimizing total transportation cost]. In the opposite case, when total production [supply] exceeds requirements [demand], an inventory is created.[13]

(B) The assignment problem. The assignment problem describes a sub-class of linear programming problems that is concerned with the allocation of n jobs to n facilities. In this particular class of

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[13]Giuseppe M. Ferrero di Roccaferrera, Operations Research Models for Business and Industry (Cincinnati, 1964), p. 345.

problem, it is assumed that (1) the value of assigning the i^{th} person to the j^{th} job can in some way be determined (subjectively or objectively) and (2) the individuals (or machines) vary in their suitability for a given job.[14] The objective is to minimize the cost of completing all of the jobs and to utilize all available resources, i.e., each of the individuals (or machines) will be assigned to exactly one job.

Although structured in the same manner as the transportation problem, the assignment problem differs in that it assigns exactly n non-zero and unit-valued variables in any feasible solution. In addition, existing algorithms are such that degenerate possibilities are avoided.[15] The mathematical definition of the assignment problem follows.

Definition 3.5.--Let c_{ij} denote the cost of job i being done by individual (or machine) j. Let x_{ij} denote the assignment of job i to individual (or machine) j. Then the assignment problem is defined as: minimize the cost of assigning job i to individual (or machine) j in such a way that each job is assigned to exactly one individual (or machine), i.e.,

$$\text{minimize } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

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[14]Dantzig, op. cit., p. 316.

[15]The case for degeneracy is defined as follows: degeneracy occurs whenever one or more of the basic variables are zero. The term "basic variable" refers to the dependent variable (or variables) of the given problem. (Note Dantzig, op. cit., p. 81.)

$$\sum_{j=1}^{n} x_{ij} = 1$$

and

$$\sum_{i=1}^{n} x_{ij} = 1$$

where

$$x_{ij} = \begin{cases} 1 & \text{if job } i \text{ is performed by } j \\ 0 & \text{otherwise} \end{cases}$$

$$i, j = 1, 2, \ldots, n.$$

(C) Network problems. Also known as the shortest-route problem, the class of problems known as network problems can be described as one in which the objective is to find the shortest path between a series of connected points. These points are connected by arcs of predetermined, fixed length, with the arcs ending at a desired terminal point.

> Consider a network comprised of a set of nodes [events or states of a system], certain pairs of which are connected by directed (i.e., direction-oriented) arcs. . . one node is distinguished as the terminal. Then the problem is to find the shortest path to the terminal from at least one other designated point, and sometimes from every other point. The amount c_{ij} represents the length associated with traversing the arc that starts at node i and ends at node j. . . . the c_{ij} actually may be measured in units other than distance. To illustrate, c_{ij} may denote the cost of going from node i to node j. In that case, the problem is to find the least-cost path. Or c_{ij} may represent the time to travel between the nodes. Then the objective is to find a minimum-duration path.[16]

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[16]Harvey M. Wagner, Operations Research (Englewood Cliffs, 1969), p. 178.

Although generally associated with the minimizing objective, the network problem can be of such a nature that the intent is the maximization of the flow of material, information, etc. This approach is the converse of the shortest-path, minimum-duration problem in that maximum flow is generally associated with the shortest-path minimum-duration problem.[17] Regardless of the particular approach involved, the network problem can be defined as follows.

Definition 3.5.--Let x_{ij} denote the branch from node i to node j.

$$\text{Let } x_{ij} = \begin{cases} 1 \text{ if the branch from } i \text{ to } j \text{ is in the solution,} \\ 0 \text{ if the branch from } i \text{ to } j \text{ is not in the solution.} \end{cases}$$

Let c_{ij} denote the associated cost measure. The network problem for minimizing the cost of shipping through the network is given by the following:

$$\text{minimize} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, n;$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, n;$$

and

$$x_{vr} = 1.$$ [18]

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[17] Gavett, op. cit., p. 84.

[18] Robert W. Llewellyn, Linear Programming (New York, 1966), pp. 336-339.

(D) The dual problem.

> For every linear programming problem there is another intimately related linear programming problem. . . . For every maximization (or minimization) problem in linear programming, there is a unique similar problem of minimization (or maximization) involving the same data which describe the original problem.[19]

Thus, for every linear programming problem there exists a corresponding problem: If the original problem (the primal) is one of maximization, its dual is one of minimization; if the original problem is one of minimization, its dual is one of maximization. The significance of this relationship can be seen when the primal problem presents considerable difficulty in obtaining a solution: it is possible to bypass many difficulties by formulating and solving the dual problem.[20] The selection of the primal problem, when given a set of data, is a matter of individual judgment. (For example a problem defined by a set of cost and profit data could be approached as follows: minimize cost or maximize profit. In either case, the data describing the problem remain the same.)

According to di Roccaferrera,

> . . .the reckoning of the dual instead of the primal may offer the advantage of shortening the time of computation, or may allow a check on the primal problem. When very large problems are analyzed to determine the sets of original data, the computation time represents an important factor to be considered carefully. . . . The interpretation of the solution [to the dual] can be referred back to the primal provided there exists a strict connection (generally equality) with the solution.[21]

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[19]di Roccaferrera, _op. cit._, p. 683.

[20]_Ibid._, p. 683.

[21]_Ibid._, pp. 683-684.

From this statement comes one of the most important contributions of the dual problem: the dual can result in faster, more efficient solutions. In addition when the dual problem has been solved, the primal problem has been solved.[22]

As a means of illustrating the relationship which exists between the primal problem and its dual, consider the following:

$$\text{maximize } f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{mn}x_n \leq b_n$$

where all $x \geq 0$.

Applying standard notation, the problem statement is condensed into the form

$$\text{maximize } \sum_{j=1}^{n} c_jx_j$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_j, \quad i = 1, 2, \ldots, m,$$

and $x_j \geq 0$ for $j = 1, 2, \ldots, n$.

The dual is written

$$\text{minimize } g(u_1, u_2, \ldots, u_m) = b_1u_1 + b_2u_2 + \ldots + b_mu_m$$

subject to

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[22] Wagner, op. cit., pp. 134-139.

$$a_{11}u_1 + a_{21}u_2 + \ldots + a_{m1}u_m \geq c_1$$

$$a_{12}u_1 + a_{22}u_2 + \ldots + a_{m2}u_m \geq c_2$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$a_{1n}u_1 + a_{2n}u_2 + \ldots + a_{mn}u_m \geq c_m$$

where all $u \geq 0$.

Condensing the form,

$$\text{minimize} \sum_{i=1}^{m} b_i u_i$$

subject to $$\sum_{i=1}^{m} a_{ij}u_i \geq c_j, \quad j = 1, 2, \ldots, n,$$

and $u_i \geq 0$ for $i = 1, 2, \ldots, n$.

With reference to these two forms,[23] it is to be noted that

(1) the column coefficients of the primal are the row coefficients of the dual;

(2) the row coefficients of the primal are the column coefficients of the dual;

(3) the coefficients of the primal objective function become the restraint constants of the dual (with no change in order); and,

(4) the restraint constants of the primal become the coefficients of the dual objective function (with no change in order).

If matrix notation is applied, the computational technique of moving from primal to dual is evident.

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[23]An important point to note is that in the illustration the inequality signs are not mixed; i.e., the primal (a maximization problem) is completely described by "at least" conditions, leading to a dual completely described by "at most" conditions. Thus, the inequality signs of the dual are the reverse of the primal.

PRIMAL	DUAL
max $f(x) = \underline{CS}$	min $g(U) = \underline{B}^T\underline{U}$
subject to	subject to
$\underline{AX} \leq \underline{B}$	$\underline{A}^T\underline{U} \geq \underline{C}^T$
$\underline{X} \geq 0$	$\underline{U} \geq 0$

In this form the superscript T denotes transpose; A is an m x n, nonzero coefficient matrix; B is m x 1 matrix and C is 1 x n matrix. The significance of these results rest on the duality theorem of linear programming.

Theorem 3.1.--If one of the two problems

Maximize $\underline{CX}$ subject to the constraints $\underline{AX} \leq \underline{B}$, $\underline{X} \geq 0$

Minimize $\underline{B}^T\underline{U}$ subject to the constraints $\underline{A}^T\underline{U} \geq \underline{C}^T$, $\underline{U} \geq 0$

has a solution, then so does the other. In addition, if solutions exist, then the objective function values are equal.[24]

The importance of this theorem lies in the fact that it guarantees equivalence of primal-dual solutions. In fact,

(a) if both the primal and dual problems possess feasible solutions, then the primal problem has an optimal solution x_j^*, $j = 1, 2, \ldots, n$, the dual problem has an optimal solution u_i^*, $i = 1, 2, \ldots, m$, and

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i u_i^*;$$

(b) if either the primal or dual problem possesses a feasible solution with a finite optimal objective function value, then the other problem possesses a feasible solution with the same optimal objective function value.[25]

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[24]Thomas L. Saaty, Mathematical Methods of Operations Research (New York, 1959), p. 118.

[25]Wagner, op. cit., p. 135.

As a means of illustrating the dual formulation of the standard linear programming problem, consider the following example.[26] A firm operates two separate production lines, each of which produces the same three products. For simplicity, denote the production lines by P and Q, the products by A, B, and C. Because of technological restrictions the relative daily output is fixed at the quantities shown below.

	Line P	Line Q
Product A	300	100
Product B	100	100
Product C	200	600

It is assumed that the firm is not operating at full capacity even though it has orders for 2,400 units of product A, 1,600 units of product B, and 4,800 units of product C. These orders are to be completed within the coming month. Based upon past experience management has assumed that line P incurs a daily variable cost of $600, line Q a daily variable cost of $400. As a means of determining the optimum number of days to operate each of the production lines, it is assumed that production costs for the month should be minimized.

Solution: Let x_1 denote product line P and x_2 product line Q in terms of operating days. Then the primal problem can be written

$$\text{minimize } f(x_1, x_2) = 600x_1 + 400x_2$$

subject to

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[26]The context of this example was taken from Donald J. Clough, Concepts in Management Science (Englewood Cliffs, 1963), pp. 326-328.

$$300x_1 + 100x_2 \geq 2{,}400$$

$$100x_1 + 100x_2 \geq 1{,}600$$

$$200x_1 + 600x_2 \geq 4{,}800$$

$$x_1 \geq 0$$

$$x_2 \geq 0.$$

The dual is given by

$$\text{maximize } g(u_1, u_2, u_3) = 2{,}400u_1 + 1{,}600u_2 + 4{,}800u_3$$

subject to

$$300u_1 + 100u_2 + 200u_3 \leq 600$$

$$100u_1 + 100u_2 + 600u_3 \leq 400$$

$$u_1 \geq 0$$

$$u_2 \geq 0$$

$$u_3 \geq 0$$

where the u_i $(i = 1, 2, 3)$ denote the shadow prices of products A, B, and C.

In this example the primal function has as its objective the minimizing of the monthly production cost; the dual function has as its objective the maximizing of the shadow prices associated with products A, B, and C. Shadow prices represent the true accounting value of the associated commodity. Whereas the primal constraints indicate that the firm must operate a sufficient number of days to meet minimum monthly orders, the dual constraints indicate that the accounting value of the daily output of a production line cannot exceed the daily cost of operating the individual lines. (As a matter of note, a u_i value of zero, i = 1, 2, 3, indicates that the associated product is produced in surplus as a by-product.)

Examination of these special cases reveals that each one exhibits the basic characteristics of the linear programming problem; i.e., (1) there is some objective to be attained; (2) there are a large number of variables to be handled simultaneously; (3) there are many interactions between the variables; (4) there exist objectives that conflict with the principal objective of the problem; and (5) all existing relationships are linear. Each of these special cases can be solved by the same techniques that are applicable to the general linear programming problem. Of these available techniques, the foremost is the simplex method of George Dantzig.

Although the simplex method is the most widely used technique for solving linear programming problems, it is not the only solution technique that is available. Modifications to the simplex procedure have produced more efficient techniques, notably the revised simplex and the MINIT (minimum iteration) method, for solving linear programming problems. In addition special computational techniques have been developed for solving transportation problems (the modified distribution method and the Vogel approximation method) and assignment and network problems (the Hungarian method).

This study will be limited to an analysis of the simplex method and the simplex algorithm. These two techniques are the principal techniques for solving linear programming problems and are the base from which the other techniques stem.

The simplex method.--The simplex method represents the basic technique for solving linear programming problems. This method is distinguished from the simplex algorithm on the basis of its form and content. Whereas the simplex algorithm defines an iterative technique that consists of a series of pivot operations which systematically introduce selected variables into the solution until the optimal combination of solution points is achieved, the simplex method defines the procedure for solving a system of simultaneous linear equations for non-negative values which optimize a given function. This is accomplished by (1) expressing all inequalities as equalities, (2) solving the resulting system of simultaneous linear equations for non-negative values, and (3) testing the system for optimality. The approach taken by the simplex method thus operates under the same solution criterion imposed upon the linear algebraic system. For the given set of simultaneous linear functions, now defined as equalities, three solution possibilities exist: (1) the given system is inconsistent and has no solution; (2) the given system is consistent and has a set of unique solutions; (3) the given system has an infinite number of solutions only one of which minimizes the objective function.

An algorithmic approach to the simplex method requires that a systematic procedure be developed for obtaining all possible solution combinations for the variables of the problem. If the system of equations to which the algorithm is applied is inconsistent (no solution), the work is minimal. If the system is consistent (unique solutions), the work may be tedious but it terminates in the optimal solution. If the system is dependent (an infinite number of solutions), the work is not only tedious but infeasible without some electronic assistance from a computer.

The following algorithm represents a technique for applying the simplex method to linear programming problems. This algorithm is derived from the general theory of the simplex method and has proved to be effective when the number of independent variables is not excessive.

Algorithm 3.1 (Simplex Method).--Step 1. Locate all corners of the solution space by treating the constraint functions as equalities and solving for all points of intersection.

Step 2. Check all solution points for the linear system to determine whether or not the constraints are violated. This is accomplished by substituting the solution points, the points of intersection, into the constraint functions.

Step 3. Eliminate all solution points which violate the constraint set.

Step 4. Substitute the admissable corner points into the objective function to test for optimality.

Step 5. The optimal solution is the solution points which corresponds to the optimal value of the objective function.

Inspection of this set of rules reveals that use of the simplex method identifies all possible solutions to the constraint system. These solutions are then checked to guarantee that the solutions to be evaluated for optimality satisfy all of the restricting conditions. With non-admissable solutions eliminated, the objective function is evaluated at the remaining solution points. The optimal solution is then selected by observation.

As a means of illustrating the simplex method, consider the following two-dimensional problem. A firm is engaged in the manufacture of two products, A and B. Product A contributes \$2 profit per unit while product B contributes \$3 profit per unit. Maximum time available for manufacture is 60 hours per week. Product A requires 5 hours per unit, and product B requires 6 hours per unit. Product B requires twice as much inspection time as does product A, with only 16 total hours available for inspection. Current market demand for A prohibits more than 6 units per week. Determine the product mix for A and B that will maximize total profit.

Solution: The functional representation of the problem is given by

$$\max f(A, B) = 2A + 3B$$

subject to

$$A \geq 0$$

$$B \geq 0$$

$$A \leq 10$$

$$B \leq 6$$

$$5A + 6B \leq 60$$

$$A + 2B \leq 16.$$

Applying the simplex method,

Step 1. Locate all corner solutions by treating the constraint functions as though they were equalities and solving for points of intersection.

$$e_1: \quad A + 0B = 0 \sim A = 0.$$
$$e_2: \quad 0A + B = 0 \sim B = 0.$$
$$e_3: \quad A + 0B = 10 \sim A = 10.$$
$$e_4: \quad 0A + B = 6 \sim B = 6.$$
$$e_5: \quad 5A + 6B = 60.$$
$$e_6: \quad A + 2B = 16.$$

Corner combinations (A, B) are given by (0, 0); (0, 6); (0, 10); (0, 8); (10, 0); (12, 0); (16, 0); $(10, \frac{5}{3})$; (10, 3); (10, 6); $(\frac{24}{5}, 6)$; (4, 6); and (6, 5). These corner combinations are obtained by (1) letting A = 0, solving for B; (2) letting B = 0, solving for A; (3) letting A = 10, solving for B; (4) letting B = 6, solving for A; (5) solving e_5 and e_6 for intersection points. As a matter of note, e_1 and e_2 define the axes of the Cartesian system, any point lying on the axes being a possible solution; e_3 defines the line passing through the point A = 10, B = 0, parallel to the B axis; e_4 defines the line passing through A = 0, B = 6, parallel to the A axis. Were it not for the fact that only corner solutions need be considered, it would be necessary to consider all possible points lying on the axes. The same would apply to the space bounded by the constraint functions, defined as the feasible region. (See Figure 3.2.)

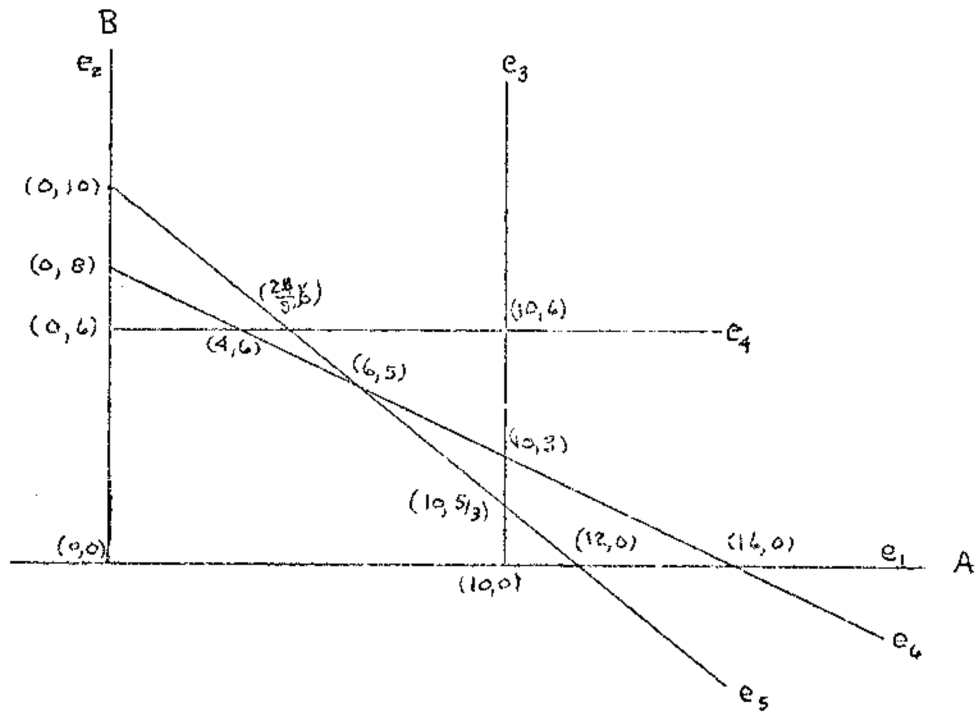

Fig. 3.2--Graphical representation of the simplex method

Step 2. Check all solution points to determine whether or not they satisfy all of the constraint functions simultaneously.

Solution Point	Satisfies Constraints?	Admissable
(0, 0)	Yes	Yes
(0, 6)	Yes	Yes
(0, 8)	No--violates e_4	No
(0, 10)	No--violates e_4 and e_6	No
(10, 0)	Yes	Yes
(12, 0)	No--violates e_3	No
(16, 0)	No--violates e_3	No
(10, $\frac{5}{3}$)	Yes	Yes

(10, 3)	No--violates e_5	No
(10, 6)	No--violates e_5 and e_6	No
$(\frac{24}{5}, 6)$	No--violates e_6	No
(4, 6)	Yes	Yes
(6, 5)	Yes	Yes

Step 3. Eliminate all corner points which violate the constraint set. From the tabulation it is seen that the only admissable corner combinations are (0, 0); (0, 6); (10, 0); $(10, \frac{5}{3})$; (4, 6); and (6, 5).

Step 4. Substitute the admissable corner points into the objective function to test for optimality.

Solution Point	Value of objective function = 2A + 3B
(0, 0)	0
(0, 6)	18
(10, 0)	20
$(10, \frac{5}{3})$	25
(4, 6)	26
(6, 5)	27

Step 5. Select the optimal combination by inspection of the results of Step 4. Since the objective function is to be optimized, A = 6, B = 5 is the desired combination; i.e., the firm should produce 6 units of product A and 5 units of product B in order to achieve maximum profit of $27.

Although limited to the two-dimensional case, this examples does illustrate the mechanics of the simplex method. These mechanics require the evaluation of all corner points (points of intersection) and then the evaluating of these points for feasibility. Although all possible solution points are considered at the outset, only feasible (admissable) points are tested for optimality.

The simplex algorithm.--The simplex algorithm is an iterative procedure whereby a problem formulated as a linear programming problem is systematically investigated for the optimal solution. It has been shown that if a feasible optimal solution exists it is located at a corner of the convex set described by the linear system. The simplex algorithm, through its iterative search, selects this optimal solution from among the set of feasible solutions to the problem. This selection is made by systematically considering the corner points of the solution space of feasible solution points. Only those points contained in the feasible set are evaluated. Points outside the feasible set are not considered. The algorithm terminates at the optimal solution.

The simplex algorithm initiates its iterative process by assigning value only to an appropriately selected set of variables that have been introduced into the problem. This approach is based upon the initial assumption that the primary variables of the problem are all zero in value. This assumption is analogous to starting the iterations at the point of origin of the defining coordinate system. From this initial assignment, the algorithm selects the variable which is to enter the solution according to the rule of steepest ascent, a rule which asserts that the variable to enter the solution is the one contributing the most to the desired optimal value. The variable to be replaced by the entering variable is the one creating the bottleneck to the optimal solution. This procedure is repeated until no further improvements can be made. At this point the algorithm terminates at the optimal solution or indicates that the given problem has no solution.

Although the rules governing the simplex algorithm apply to both maximization and minimization problems, the method by which the entering solution variable is selected is different. In the maximization problem the entering variable is identified as the variable which contributes the most to the value of the objective function. In the minimization problem the entering variable is identified as the variable which contributes the least "cost" to the objective function. This process has been identified as the rule of steepest ascent and is defined as follows.

Definition 3.7.--Let x_j denote the j^{th} variable. Let c_j be the cost (or profit) associated with the variable x_j. Let z_j be the total cost (or profit) associated with the variables in the current solution; i.e., z_j is the cost (or profit) obtained by multiplying the cost (or profit) associated with the variable currently in the solution by the corresponding element in the j^{th} column of the structured linear programming problem and then column summing the individual products obtained for the elements in each of the j columns. The rule of steepest ascent then identifies the variable to enter the solution as that variable satisfying

(1) $\max (c_j - z_j) > 0$ for maximization,

(2) $\min (c_j - z_j) < 0$ for minimization.

The simplex algorithm can be applied to any linear programming problem which requires maximization or minimization of a defined objective function. However, the distinction between the use of the simplex algorithm for maximization or minimization is not presented in the same manner nor in the same detail as will be done in this study. Since the intent of this chapter is the examination and explanation of the basic techniques of optimal search, algorithms for both maximization and minimization will be presented.

<u>Algorithm 3.2</u> (<u>Simplex algorithm for maximization</u>).--The problem to be solved has the form

$$\max f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$$

and $x_i \geq 0$ for $i = 1, 2, \ldots, m$.

This system is converted to a system of linear equations by adding a slack variable, s_i, to each linear inequality. The slack variable is assumed to yield a contribution to the objective function of value 0 and is so included in the objective function. Addition and inclusion of the appropriate slack variable yields the following form:

$$\max f(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0s_1 + 0s_2 + \ldots + 0s_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 + 0 \sum_{k=2}^{n} s_k = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + 0s_1 + s_2 + 0 \sum_{k=3}^{n} s_k = b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + 0 \sum_{k=1}^{n-1} s_k + s_n = b_n.$$

Inspection of this linear system reveals the following points: (1) for maximization, the contribution of the slack variable, representing unused capacity, is zero; (2) the slack variables are augmented to the original, basic variables of the problem in such a way that the original matrix of coefficients, $\underline{A} = (a_{ij})$, is augmented by an identity matrix of the appropriate size; (3) the addition of the k slack variables transforms the original m x n system of linear inequalities into an m x (n + k) system of linear equalities. In matrix notation, this transformation is written as follows:

$$\max f(\underline{x}) = \underline{Cx}$$

subject to

$$\underline{Ax} \leq b,$$

where

$$\underline{C} = [c_1, c_2, \ldots, c_n];$$

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix};$$

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \text{; and,}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} .$$

Introduction of the slack variables results in the following:

$$\max f(\underline{x}, \underline{s}) = \underline{Cx} + 0 \sum_{k=1}^{n} s_k$$

subject to

$$(\underline{A}, \underline{I})(\underline{x}, \underline{s}) \leq b$$

where

$(\underline{A}, \underline{I})$ = the original coefficient matrix augmented by the n-dimensional identity matrix;

$(\underline{x}, \underline{s}) = (x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_n)^T$

$b = (b_1, b_2, \ldots, b_n)^T.$

This linear system is converted for algorithmic application by constructing a table consisting of $m + 4$ rows and $(n + k + 4)$ columns. This arrangement provides row and column space for the following: (1) the objective function coefficients, c_j, (2) the variables contained in the transformed formulation, (3) the z_j values, (4) the $(c_j - z_j)$ computations, (5) the contribution of the variables in the solution, (6) the solution mix--the basis, (7) the constants representing the value of the corresponding solution variable, and (8) the locator for the variable which is to leave the solution. (See Table 3.1.) At each iteration of the algorithm a new table is constructed. The process is repeated until the optimal solution is reached. At the optimal solution the value of the objective function is taken from the intersection of the z_j row and the column of constants.

Given this tabular formulation, the application of the simplex algorithm requires but five steps. However, there are some undefined terms that require introduction since they are part of the algorithm. These terms are as follows and apply to both maximization and minimization:

(1) pivot column: the column headed by the variable which is to enter the solution and is identified by the rule of steepest ascent;

(2) pivot row: the row corresponding to the variable currently in the solution but destined to be replaced by the variable identified by the pivot column;

(3) pivot element: the element located at the intersection of the pivot row and the pivot column;

(4) row element: the individual elements in each of the m rows of the body matrix;

(5) replacement pivot row: the row elements obtained by dividing each element in the original pivot row by the pivot element.

TABLE 3.1

INITIAL BASIC SOLUTION

Contribution	Objective Coefficients: c_j	c_1	c_2	...	c_n	0	0	0	...	0	Constants	Variable Leaving Solution Locator
	Solution Set	x_1	x_2	...	x_n	s_1	s_2	s_3	...	s_n	$b_i \equiv P_o$	
0	s_1	a_{11}	a_{12}	...	a_{1n}	1	0	0	...	0		
0	s_2	a_{21}	a_{22}	...	a_{2n}	0	1	0	...	0		
⋮	⋮	. .	.	. . .	. .	. .	. . .	. .	. . .	. .		
0	s_n	a_{m1}	a_{m2}		a_{mn}	0	...	0	...	1		
	z_j											
	$c_j - z_j$											

In the steps to follow, it is assumed that the linear programming problem has been converted to equalities and arranged in the form of Table 3.1. In addition it is assumed that the necessary c_j, z_j, and $c_j - z_j$ calculations have been made. The form represented in Table 3.1 is the one associated with the initial solution to the given maximization problem. All variables of the original problem are equal to zero. The only nonzero variables are the slack variables.

Algorithm 3.2 (Simplex Algorithm for Maximization).--Step 1. Locate the pivot column. The pivot column is identified as the column corresponding to $\max_j (c_j - z_j)$ for $c_j > z_j$.

Step 2. Locate the pivot row. The pivot row is identified as the row corresponding to the minimum positive quotient obtained by dividing the row constant, b_i, by the corresponding element in the pivot column; i.e.,

$$\min_i \left[\frac{P_o \text{ element}}{\text{Pivot column element}} \right] > 0.$$

The remaining steps refer to the construction of a new table of values showing the coefficients that result from a change in the solution set. This change is recorded in a table similar to that of Table 3.1. At each iteration it is necessary to construct a table that summarizes the operations performed on the current solution matrix-table. The constructed table is then checked to determine if an optimal solution has been achieved. If it has, the process stops; if it has not, the iterations continue.

Step 3. Replace the pivot row by dividing the pivot element into each element of the pivot row (including the pivot element itself).

Step 4. Replace all non-pivot row elements by applying the following:

$$\begin{pmatrix}\text{Elements of Row}\\ \text{to be}\\ \text{Replaced}\end{pmatrix} - \begin{pmatrix}\text{Element at the}\\ \text{Intersection of Pivot}\\ \text{Column and the Row}\\ \text{Being Replaced}\end{pmatrix} \cdot \begin{pmatrix}\text{Corresponding Elements}\\ \text{of the Replacement}\\ \text{Pivot Row}\end{pmatrix}.$$

Step 5. Recalculate the z_j values. Recalculate $c_j - z_j$ for each of the j columns. The maximal value has been achieved with the new solution mix if and only if $(c_j - z_j) \leq 0$ for all $c_j - z_j$ values. If $(c_j - z_j) \not\leq 0$ for all $c_j - z_j$, repeat steps 1 through 5. If $(c_j - z_j) > 0$ but $a_{ij} < 0$ for all a_{ij} in column j, as dictated by $c_j - z_j > 0$, the objective function is unbounded and no unique maximum exists.

As a means of illustrating this algorithm, consider the illustration of the simplex method. In that example the problem was one of combining two products in such a way that profit was maximized. With $x_1 \equiv A$ and $x_2 \equiv B$, the functional expression is written

$$\max f(x_1, x_2) = 2x_1 + 3x_2$$

subject to

$$\begin{aligned} x_1 &\leq 10 \\ x_2 &\leq 6 \\ 5x_1 + 6x_2 &\leq 60 \\ x_1 + 2x_2 &\leq 16 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

It is necessary to find that combination of nonnegative values of x_1 and x_2 that maximize the given objective function.

Since the objective function is restricted by four constraint functions, it is necessary to introduce four slack variables. It is important to note that the form of the constraints is uniform; i.e., each of the constraint functions is defined by a "less than or equal to" inequality. This is the necessary form (or at least "less than") if only a slack variable is to be introduced. Introduction of the necessary four slack variables s_1, s_2, s_3, and s_4 redefines the problem as follows:

$$\max f(x_1, x_2, s_1, s_2, s_3, s_4) = 2x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

subject to

$$x_1 + 0x_2 + s_1 + 0s_2 + 0s_3 + 0s_4 = 10$$

$$0x_1 + x_2 + 0s_1 + s_2 + 0s_3 + 0s_4 = 6$$

$$5x_1 + 6x_2 + 0s_1 + 0s_2 + s_3 + 0s_4 = 60$$

$$x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3 + s_4 = 16.$$

The initial-solution is shown in Table 3.2.

TABLE 3.2

INITIAL SOLUTION

	c_j	2	3	0	0	0	0		
Contribution	Solution Set	x_1	x_2	s_1	s_2	s_3	s_4	P_0	P_0/a_{ij}
0	s_1	1	0	1	0	0	0	10	∞
0	s_2	0	1	0	1	0	0	6	6
0	s_3	5	6	0	0	1	0	60	10
0	s_4	1	2	0	0	0	1	16	8
	z_j	0	0	0	0	0	0	0	
	$c_j - z_j$	2	3	0	0	0	0		

The values shown in the contribution column were taken directly from the objective coefficients shown in the c_j row corresponding to the variables in the solution set. The z_j values were obtained by first multiplying each element in each of the variable columns by the corresponding row contribution and then totaling the products for each column. The respective column total is the z_j value for the j^{th} column. For example,

$$z_1 = 0(1) + 0(0) + 0(5) + 0(1) = 0;$$

$$z_2 = 0(0) + 0(1) + 0(6) + 0(2) = 0;$$

etc.

Application of the $(c_j - z_j)$ rule for maximization reveals the variable of column 2 with a $c_j - z_j$ value of +3 should enter the solution. This indicates that variable x_2 will contribute more to the objective function than the variables currently in the solution. At the next iteration, variable x_2 will enter the solution set. It will contribute a per unit value of 3 as shown in the c_j row. This contribution value will be placed in the contribution column, replacing the contribution of the variable leaving the solution. The simplex algorithm will be used to construct the next solution table.

Step 1. Locate the pivot column. The pivot column is identified as the column corresponding to max $(c_j - z_j) > 0$. Sinc the $c_j - z_j$ value of +3 is the maximum of the $(c_j - z_j)$ values, column 2 is identified as the pivot column.

Step 2. Locate the pivot row. The pivot row is identified as the row corresponding to the minimum positive quotient $\frac{\text{constraint constant}}{\text{pivot column element}}$.

These quotients are shown in the column labeled P_o/a_{ij}. Inspection of this column reveals that the minimum positive quotient is given by $P_o/a_{22} = 6/1 = 6$. This value corresponds to row 2. Therefore, the pivot row is row 2. Variable x_2 will enter the solution set in place of slack variable s_2.

Step 3. Replace the pivot row by dividing the pivot element into each element of the pivot row. The pivot element is the element located at the intersection of the pivot row and pivot column. For this iteration, the pivot element is 1. Replaced pivot row = (0, 1, 0, 1, 0, 0, 6).

Step 4. Replace all non-pivot row elements by applying the following:

$$\begin{pmatrix}\text{Elements of Row}\\ \text{to be}\\ \text{Replaced}\end{pmatrix} - \begin{pmatrix}\text{Element at the}\\ \text{Intersection of Pivot}\\ \text{Column and the Row}\\ \text{Being Replaced}\end{pmatrix} \cdot \begin{pmatrix}\text{Corresponding Elements}\\ \text{of the Replacement}\\ \text{Pivot Row}\end{pmatrix}.$$

Row 1. Intersection Element: 0

$1 - (0)(0) = 1$

$0 - (0)(1) = 0$

$1 - (0)(0) = 1$

$0 - (0)(1) = 0$

$0 - (0)(0) = 0$

$0 - (0)(0) = 0$

$10 - (0)(6) = 10$

Row 3. Intersection Element: 6

$5 - (6)(0) = 5$

$6 - (6)(1) = 0$

$0 - (6)(0) = 0$

$0 - (6)(1) = -6$

1 - (6)(0) = 1

0 - (6)(0) = 0

60 - (6 6) = 24

Row 4. In section Element: 2

1 - (2)(0) = 1

2 - (2)(1) = 0

0 - (2)(0) = 0

0 - (2)(1) = -2

0 - (2)(0) = 0

1 - (2)(0) = 1

16 - (2)(6) = 4

The results of these calculations is summarized in Table 3.3., Iteration I. The fifth step of the simplex algorithm is applied to this summary table to determine whether or not the solution set is optimal.

TABLE 3.3

RESULTS OF FIRST ITERATION

Contribution	Solution Set	2	3	0	0	0	0		
	c_j								
Contribution	Solution Set	x_1	x_2	s_1	s_2	s_3	s_4	P_o	P_o/a_{ij}
0	s_1	1	0	1	0	0	0	10	10
3	x_2	0	1	0	1	0	0	6	∞
0	s_3	5	0	0	-6	1	0	24	24/5
0	s_4	1	0	0	-2	0	1	4	4
	z_j	0	3	0	3	0	0	18	
	$c_j - z_j$	2	0	0	-3	0	0		

Step 5. Recalculate the z_j values. Recalculate $c_j - z_j$. The solution set is maximal if and only if $(c_j - z_j) \leq 0$ for all values of the $c_j - z_j$. Inspection of Table 3.3 reveals that $c_1 - z_1 = 2 > 0$. All other $c_j - z_j$ values are at most zero. Since there exists a positive $c_j - z_j$ value, the solution is not maximal. It is necessary to repeat the algorithm using the values contained in Table 3.3.

Reapplication of the simplex algorithm indicates that variable x_1 will enter the solution set and replace slack variable s_4. Application of the simplex algorithm yields the results shown in Table 3.4.

TABLE 3.4

RESULTS OF SECOND ITERATION

	c_j	2	3	0	0	0	0		
Contribution	Solution Set	x_1	x_2	s_1	s_2	s_3	s_4	P_o	P_o/a_{ij}
0	s_1	0	0	1	2	0	-1	6	3
3	x_2	0	1	0	1	0	0	6	6
0	s_3	0	0	0	4	1	-5	4	1
2	x_1	1	0	0	-2	0	1	4	
	z_j	2	3	0	-1	0	2	26	
	$c_j - z_j$	0	0	0	1	0	-2		

Inspection of Table 3.4 reveals that the indicated solution is not the maximal combination. This is evidenced by the +1 $c_j - z_j$ value for the

s_2 column. Application of the simplex algorithm to this table of values indicates that slack variable s_3 will be replaced by slack variable s_2 in the solution set. The results are summarized in Table 3.5.

TABLE 3.5

RESULTS OF THIRD ITERATION

	c_j	2	3	0	0	0	0	
Contribution	Solution Set	x_1	x_2	s_1	s_2	s_3	s_4	P_o
0	s_1	0	0	1	0	$-\frac{1}{2}$	$\frac{3}{2}$	4
3	x_2	0	1	0	0	$-\frac{1}{4}$	$\frac{5}{4}$	5
0	s_2	0	0	0	1	$\frac{1}{4}$	$-\frac{5}{4}$	1
2	x_1	1	0	0	0	$\frac{1}{2}$	$-\frac{3}{2}$	6
	z_j	2	3	0	0	$\frac{1}{4}$	$\frac{3}{4}$	27
	$c_j - z_j$	0	0	0	0	$-\frac{1}{4}$	$-\frac{3}{4}$	

Inspection of Table 3.5 reveals that $(c_j - z_j) \leq 0$ for all values of $c_j - z_j$. Thus, the solution set is the one for which the given objective function is maximized. This solution set yields $s_1 = 4$; $x_2 = 5$; $s_2 = 1$; and $x_1 = 6$. In order to achieve maximum profit, it is necessary to produce 6 units of x_1 and 5 units of x_2. This combination will result in a maximum profit of \$27. The s_1 value of 4 indicates that there will be 4 units of unsatisfied demand for x_1 as given in the original formulation of the problem. The s_2 value of 1 indicates that the market for x_2 will have one unsatisfied unit. Production and inspection time will be totally consumed.

A graphic representation of the problem solved by the simplex algorithm is shown in Figure 3.3. Although this graph is identical to that shown with the simplex method, it is reproduced here along with the appropriate change of variables, so that the simplex algorithm can be visually demonstrated.

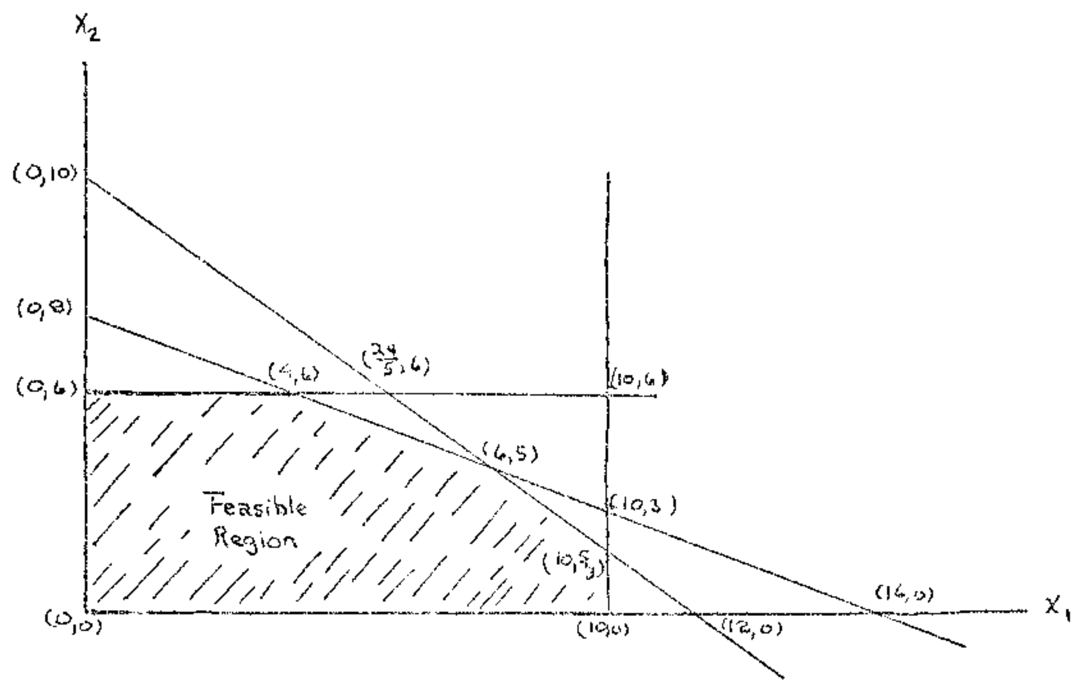

Fig. 3.3--Graphic solution to constrained maximum

By assigning values to slack variables only in the initial solution, the simplex algorithm started at the origin. At this point the algorithm indicated that a maximal solution had not been achieved. Application of the rule of steepest ascent indicated that variable x_2 should be placed into the solution set in lieu of slack variable s_2. The effect of this was to "move" from the origin to the point $x_1 = 0$, $x_2 = 6$. This solution was found to be non-maximal; and according to the rule of steepest ascent, variable x_1 should replace slack variable s_4. The effect of this was to "move" from the coordinate (0, 6) to the coordinate (4, 6). At this point, $x_1 = 4$, $x_2 = 6$, the objective function was not maximized. Application of the rule of steepest ascent indicated that slack variable s_2 should reenter the solution, replacing slack variable s_3. This "moved" the solution set from coordinate (4, 6) to coordinate (6, 5). Coordinate (6, 5), indicating that the required units of x_1 and x_3 were 6 and 5, respectively, was then shown to be the maximal combination.

An important point of this demonstration is the fact that at no time did the algorithm consider solution combinations outside the region bounded by all of the constraints. Whereas the simplex method requires consideration of all possible corners (points of intersection), the simplex algorithm considers only those corner points which simultaneously satisfy all of the constraint functions. The algorithm simply moves from one corner to another until the optimal solution is achieved. However, these moves are neither random nor haphazard. Each move from one corner point to another is based upon the criterion of the rule of steepest ascent. This guarantees that the new solution point will be an improvement relative to the current solution point. The "moving process" continues until the algorithm terminates in the optimal solution or the problem is found to have no optimal solution.

Simplex algorithm for minimization: The problem to be solved has the form

$$\min f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_1$$

and $x_i \geq 0$ for $i = 1, 2, \ldots, m$.

This system is converted to a system of linear equations by subtracting a slack variable, s_i, from each linear inequality. The slack variable is assumed to yield a unit contribution of zero to the value of the objective function. However, for every slack variable that is placed into the system it is necessary to also include an artificial variable, d_i, so that the initial solution is defined for non-negative solutions. In this way an initial basic feasible solution is guaranteed. In order to eliminate the artificial variable from the solution, each artificial variable is assigned a high cost coefficient of value M and is included in the objective function. The result is a linear system of the following form:

$$\min f(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m, A_1, A_2, \ldots, A_m)$$
$$= c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0s_1 + 0s_2 + \ldots + 0s_m$$
$$+ M_1d_1 + M_2d_2 + \ldots + M_md_m$$

subject to

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - s_1 + d_1 = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n - s_2 + d_2 = b_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n - s_m + d_m = b_m.
\end{array}
$$

As in the case of the maximization problem, the initial solution is constructed from the non-negative, non-basic variables of the constructed system of linear equalities. For the minimization problem defined here, this initial solution will be composed of artificial variables only. (See Table 3.6.)

Each z_j value for Table 3.6 is calculated in the same manner as for the maximization problem; i.e., $z_j = \sum_{j=1}^{m} M_i a_{ij}$, $i = 1, 2, \ldots, n$.

The $c_j - z_j$ row is then obtained by subtracting each z_j value from its corresponding c_j value, exactly as in the case for maximization. Application of the rule of steepest ascent will serve to indicate the variable which is to enter the solution. This application of the rule of steepest ascent, however, is made with some modification:

(1) select that variable corresponding to max $[-(c_j - z_j) > 0]$; or,

(2) select that variable corresponding to min $[(c_j - z_j) < 0]$.

In either case the result is the introduction into the solution the variable contributing the "most" to the minimization process.

With the new solution variable identified, the simplex algorithm is applied to determine the body matrix coefficients and the value of the solution variables. This is accomplished by proceeding in the same manner as dictated by Steps 2, 3, and 4 of the simplex algorithm for maximization.

TABLE 3.6

INITIAL BASIC SOLUTION

	c_j	c_1	c_2	c_3	...	c_n	0	0	0	...	0	M	M	...	M	Constraints
Cost	Solution Set	x_1	x_2	x_3	...	x_n	s_1	s_2	s_3	...	s_m	d_1	d_2	...	d_m	$b_i \equiv P_o$
M	d_1	a_{11}	a_{12}	a_{13}	...	a_{1n}	-1	0	0	...	0	M	0	...	0	b_1
M	d_2	a_{21}	a_{22}	a_{23}	...	a_{2n}	0	-1	0	...	0	0	M	...	0	b_2
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
M	d_m	a_{m1}	a_{m2}	a_{m3}	...	a_{mn}	0	0	0	...	-1	0	0	...	M	b_m
	z_j															
	$c_j - z_j$															

The difference in the two algorithms, minimization and maximization, is found in the solution set itself. Whereas the simplex algorithm for maximization systematically replaces slack variables with basic variables, the simplex algorithm for minimization replaces artificial variables with basic variables or slack variables. Slack variables remaining in the solution set indicate unused or excessive capacity or resource.

As previously noted, the simplex algorithm for minimization utilizes three steps of the algorithm for maximization. The difference between the algorithms is the manner by which the pivot column is located and the manner by which an optimal solution is indicated.

In the algorithm to follow it is assumed that the problem has been expressed as a system of linear equations. The necessary slack and artificial variables have been introduced and a table of the form shown by Table 3.6 constructed. Although this initial construct represents the initial solution to the problem, the algorithm proceeds from solution to solution in the same manner as the algorithm for maximization.

Algorithm 3.3 (Simplex Algorithm for Minimization).--Step 1. Locate the pivot column. The pivot column is identified as the column corresponding to $\min_j (c_j - z_j) < 0$. This is equivalent to $\max_j [-(c_j - z_j) > 0]$.

Step 2. Locate the pivot row. The pivot row is identified as the row corresponding to the minimum positive quotient obtained by dividing each row constant, $b_i \equiv P_0$, by the corresponding element in the pivot column; i.e.,

$$\min_i \left[\frac{P_0 \text{ element}}{\text{Pivot column element}}\right] > 0.$$

Step 3. Replace the pivot row by dividing the pivot element into each element of the pivot row (including the pivot element itself).

Step 4. Replace all non-pivot row elements by applying the following:

$$\begin{pmatrix}\text{Elements of Row}\\ \text{to be}\\ \text{Replaced}\end{pmatrix} - \begin{pmatrix}\text{Element at the}\\ \text{Intersection of Pivot}\\ \text{Column and the Row}\\ \text{Being Replaced}\end{pmatrix} \cdot \begin{pmatrix}\text{Corresponding Elements}\\ \text{of the Replacement}\\ \text{Pivot Row}\end{pmatrix}.$$

Step 5. Recalculate the z_j values. Recalculate $c_j - z_j$ for each of the j columns. The minimal value has been achieved with the new solution mix if and only if $c_j - z_j \geq 0$ for all $c_j - z_j$. If $c_j - z_j \ngeq 0$ for all $c_j - z_j$, repeat steps 1 through 5. If $c_j - z_j < 0$ but $a_{ij} < 0$ for all a_{ij} in column j, the objective function is unbounded and no unique minimum exists.

As a means of illustrating this algorithm, consider the following simplified example. A firm operates two production lines. Each line produces three products, A, B, and C. Relative daily outputs are fixed as follows:

	Line 1	Line 2
Product A	300	100
Product B	100	100
Product C	200	600

Minimal orders for A, B, and C are 2,400 units, 1,600 units, and 4,800 units, respectively. Production costs for both lines are assumed to be variable: $600 for line 1 and $400 for line 2. Determine the number of days that the two lines are to run in order to satisfy order requirements and minimize total production costs.

Solution: Let x_1 denote the number of days line 1 is to operate. Let x_2 denote the number of days line 2 is to operate. The problem can then be described by the following linear system.

$$\min f(x_1, x_2) = 600x_1 + 400x_2$$

subject to

$$300x_1 + 100x_2 \geq 2,400;$$
$$100x_1 + 100x_2 \geq 1,600;$$
$$200x_1 + 600x_2 \geq 4,800;$$
$$x_1 \geq 0; \text{ and,}$$
$$x_2 \geq 0.$$

The introduction of the necessary slack and artificial variables (s_i and d_i, respectively, i = 1, 2, 3) results in

$$\min f(x_1, x_2, s_1, s_2, s_3, d_1, d_2, d_3)$$
$$= 600x_1 + 400x_2 + 0s_1 + 0s_2 + 0s_3 + Md_1 + Md_2 + Md_3$$

subject to

$$300x_1 + 100x_2 - s_1 + d_1 = 2,400;$$
$$100x_1 + 100x_2 - s_2 + d_2 = 1,600;$$
$$200x_1 + 600x_2 - s_3 + d_3 = 4,800.$$

The initial solution is shown in Table 3.7.

Since M represents a very large number, the $c_j - z_j$ row contains values of $c_j - z_j$ that are negative. This indicates that the existing solution is not optimal. Application of the $c_j - z_j$ rule indicates that variable x_2 can enter the solution and reduce the cost from its present level of 8800M.

TABLE 3.7

INITIAL SOLUTION

Cost	Solution Set c_j	600 x_1	400 x_2	0 s_1	0 s_2	0 s_3	M d_1	M d_2	M d_3	P_o	P_o/a_{ij}
M	d_1	300	100	-1	0	0	1	0	0	2400	24
M	d_2	100	100	0	-1	0	0	1	0	1600	16
M	d_3	200	600	0	0	-1	0	0	1	4800	8
	z_j	600M	800M	-M	-M	-M	M	M	M	8800M	
	$c_j - z_j$	600-600M	400-800M	M	M	M	0	0	0		

Step 1. Locate the pivot column. The pivot column is identified as the column corresponding to $\min_j (c_j - z_j) < 0$. Since the 400 - 800M value of column 2 is less than that of 600 - 600M for large M, column 2 is identified as the pivot column.

Step 2. Locate the pivot row. The pivot row is identified as the row corresponding to the minimum positive quotient obtained by dividing each row constant, $b_i \equiv P_o$, by the corresponding element in the pivot column. These quotients are shown in the column headed P_o/a_{ij}. Inspection of this column indicates that variable x_2 will replace d_3 in the solution set. The minimum P_o/a_{ij} value is the 8 corresponding to d_3.

Step 3. Replace the pivot row by dividing the pivot element into each element of the pivot row. The pivot element is the element located at the intersection of the pivot row and the pivot column. For this iteration the pivot element is 600.

Replaced pivot row = $(\frac{1}{3}, 1, 0, 0, \frac{-1}{600}, 0, 0, \frac{1}{600}, 8)$.

Step 4. Replace all non-pivot row elements by applying the following:

$$\begin{pmatrix}\text{Elements of Row}\\ \text{to be}\\ \text{Replaced}\end{pmatrix} - \begin{pmatrix}\text{Element at the}\\ \text{Intersection of Pivot}\\ \text{Column and the Row}\\ \text{Being Replaced}\end{pmatrix} \cdot \begin{pmatrix}\text{Corresponding Elements}\\ \text{of the Replacement}\\ \text{Pivot Row}\end{pmatrix}.$$

Row 1. Intersection Element: 100

$$300 - (100)(\tfrac{1}{3}) = \tfrac{800}{3}$$

$$100 - (100)(1) = 0$$

$$-1 - (100)(0) = -1$$

$$0 - (100)(0) = 0$$

$$0 - (100)(\tfrac{-1}{600}) = \tfrac{1}{6}$$

$$1 - (100)(0) = 1$$

$$0 - (100)(0) = 0$$

$$0 - (100)(\tfrac{1}{600}) = -\tfrac{1}{6}$$

$$2400 - (100)(8) = 1600$$

Row 2. Intersection Element: 100

$$100 - (100)(\tfrac{1}{3}) = \tfrac{200}{3}$$

$$100 - (100)(1) = 0$$

$$0 - (100)(0) = 0$$

$$-1 - (100)(0) = -1$$

$$0 - (100)(\tfrac{-1}{600}) = \tfrac{1}{6}$$

$$0 - (100)(0) = 0$$

$$1 - (100)(0) = 1$$

$$0 - (100)(\frac{1}{600}) = \frac{1}{6}$$

$$1600 - (100)(8) = 800$$

These results are summarized in Table 3.8.

TABLE 3.8

RESULTS OF FIRST ITERATION

Cost	c_j / Solution Set	600 x_1	400 x_2	0 s_1	0 s_2	0 s_3	M d_1	M d_2	M d_3	P_o
M	d_1	$\frac{800}{3}$	0	-1	0	$\frac{1}{6}$	1	0	$\frac{-1}{6}$	1600
M	d_2	$\frac{200}{3}$	0	0	-1	$\frac{1}{6}$	0	1	$\frac{-1}{6}$	800
400	x_2	$\frac{1}{3}$	1	0	0	$-\frac{1}{600}$	0	0	$\frac{1}{600}$	8
	z_j	$\frac{1}{3}(400+1000M)$	400	-M	-M	$\frac{1}{3}M-\frac{2}{3}$	M	M	$\frac{2}{3}-\frac{M}{3}$	3200 +
	$c_j - z_j$	$\frac{1400-1000M}{3}$	0	M	M	$\frac{2}{3}-\frac{M}{3}$	0	0	$\frac{4M}{3}-\frac{2}{3}$	2400M

Step 5. Recalculate the z_j values. Recalculate $c_j - z_j$ for all j. The minimal value has been achieved with the new solution mix if and only if $c_j - z_j \geq 0$ for all $c_j - z_j$. Inspection of the $c_j - z_j$ row reveals that $c_1 - z_1 < 0$ for large M and $c_5 - z_5 < 0$ for large M. Therefore, the current solution is not optimal. The cost figure of 3200 + 2400M can be reduced still further.

Reapplication of the simplex algorithm yields the values shown in Table 3.9. Application of Step 5 indicates that the minimum cost has not been achieved with the new solution set: column 3, identified as s_1, has a $c_j - z_j$ value of $\frac{1}{4}(7-M)$; column 5, identified as s_3, has a $c_j - z_j$ value of $\frac{1}{8}(3-M)$. For large M, both of these are negative. The current cost of 6000 + 400M can be reduced.

TABLE 3.9

RESULTS OF SECOND ITERATION

Cost	Solution Set	x_1	x_2	s_1	s_2	s_3	d_1	d_2	d_3	P_o
	c_j	600	400	0	0	0	M	M	M	
600	x_1	1	0	$-\frac{3}{800}$	0	$\frac{1}{1600}$	$\frac{3}{800}$	0	$-\frac{1}{1600}$	6
M	d_2	0	0	$\frac{1}{4}$	-1	$\frac{1}{8}$	$-\frac{1}{4}$	1	$-\frac{1}{8}$	400
400	x_2	0	1	$\frac{1}{800}$	0	$\frac{-9}{4800}$	$-\frac{1}{800}$	0	$\frac{9}{4800}$	6
	z_j	600	400	$\frac{M-7}{4}$	-M	$\frac{M-3}{8}$	$\frac{7-M}{4}$	M	$\frac{3-M}{8}$	6000 + 400M
	$c_j - z_j$	0	0	$\frac{7-M}{4}$	M	$\frac{3-M}{8}$	$\frac{5M-7}{4}$	0	$\frac{9M-3}{8}$	

According to the $c_j - z_j$ rule the variable to enter the solution is s_1. This slack variable will replace artificial variable d_2. The computations of the simplex algorithm are summarized in Table 3.10.

TABLE 3.10

RESULTS OF THIRD ITERATION

Cost	c_j / Solution Set	600 x_1	400 x_2	0 s_1	0 s_2	0 s_3	M d_1	M d_2	M d_3	P_o
600	x_1	1	0	0	$-\frac{12}{800}$	$\frac{4}{1600}$	0	$\frac{12}{800}$	$-\frac{4}{1600}$	12
0	s_1	0	0	1	-4	$\frac{1}{2}$	-1	4	$-\frac{1}{2}$	1600
400	x_2	0	1	0	$\frac{4}{800}$	$-\frac{12}{4800}$	0	$-\frac{4}{800}$	$\frac{12}{4800}$	4
	z_j	600	400	0	-7	.5	0	7	-.5	8800
	$c_j - z_j$	0	0	0	7	-.5	M	M-7	M+.5	

Since all $c_j - z_j$ in the $c_j - z_j$ row are not positive or zero, a better solution exists. Reapplication of the simplex algorithm results in slack variable s_3 replacing slack variable s_1. The new solution set is given by x_1, x_2, and s_3 and is shown in Table 3.11.

Since $c_j - z_j \geq 0$ for all $c_j - z_j$ values, the solution set is optimal. Minimum cost is achieved by operating Line 1 for 4 days, Line 2 for 12 days. This operating schedule produces a minimum cost of $7,200. This operacing schedule will produce an excess of 3,200 units for Product C during the operating period. This indicates that the production facility is producing C as a by-product of the total production process. The quantities produced for A and B are the minimal; i.e., the production facility is capable of meeting minimal orders for A and B. It is producing 8000 units of C, with a minimum requirement of 4,800 units.

TABLE 3.11

FINAL SOLUTION

Cost	c_j / Solution Set	600 x_1	400 x_2	0 s_1	0 s_2	0 s_3	M d_1	M d_2	M d_3	P_o
600	x_1	1	0	$-\frac{4}{800}$	$\frac{4}{800}$	0	$\frac{4}{800}$	$\frac{-4}{800}$	0	4
0	s_3	0	0	2	-8	1	-2	8	-1	3200
40	x_2	0	1	$\frac{24}{4800}$	$-\frac{72}{4800}$	0	$\frac{-24}{4800}$	$\frac{72}{4800}$	0	12
	z_j	600	400	-1	-3	0	1	3	0	7200
	$c_j - z_j$	0	0	1	3	0	M-1	M-3	M	

The techniques shown were applied to situations in which the problem description followed the strict definition of the given linear programming problem. Maximization was achieved subject to "less than or equal to" constraints, and minimization was achieved subject to "greater than or equal to" constraints. Such strictness is not necessary since the algorithm for either maximization or minimization utilizes the same basic technique. Initial solutions are written in terms of the slack or artificial variables. All that is necessary is that the initial solution be composed of the positive slack and artificial variables. Thus, a problem can be formulated subject to mixed constraints; i.e., the defined objective function is to be optimized subject to equality conditions, "less than or equal to" conditions, and "greater than or equal to" conditions. With the equality an artificial variable is added to maintain the identity system necessary in the initial solution matrix. Given the initial solution matrix, the proper algorithm is employed for final solution.

For example, the linear system

$$\min f(x_1, x_2, x_3) = 2x_1 + 4x_2 + x_3$$

such that

$$\begin{aligned} x_1 + 2x_2 - x_3 &\leq 5 \\ 2x_1 - x_2 + 2x_3 &= 2 \\ -x_1 + 2x_2 + 2x_3 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

can be written

$$\min f(x_1, x_2, x_3, s_1, s_2, d_1, d_2) = 2x_1 + 4x_2 + x_3 + 0s_1 + 0s_2 + Md_1 + Md_2$$

such that

$$\begin{aligned} x_1 + 2x_2 - x_3 + s_1 &= 5 \\ 2x_1 - x_2 + 2x_3 + d_1 &= 2 \\ -x_1 + 2x_2 + 2x_3 - s_2 + d_2 &= 1 \\ x_1, x_2 &\geq 0. \end{aligned}$$

The initial solution matrix is shown in Table 3.12. The solution, if it exists, can be found by applying the simplex algorithm for minimization. This algorithm is required because the objective function is to be minimized.

TABLE 3.12

INITIAL SOLUTION, MIXED CONSTRAINTS

	c_j	2	4	1	0	0	M	M	
Cost	Solution Set	x_1	x_2	x_3	s_1	s_2	d_1	d_2	P_o
0	s_1	1	2	-1	1	0	0	0	5
M	d_1	2	-1	2	0	0	1	0	2
M	d_2	-1	2	2	0	-1	0	1	1
	z_j	M	M	4M	0	-M	M	M	3M
	$c_j - z_j$	2-M	4-M	1-4M	0	M	0	0	

Quadratic Programming

A natural extension of linear programming, quadratic programming represents a class of problems concerned with optimizing a quadratic function subject to linear constraints. As such, it is a type of problem exhibiting two essential characteristics: (1) there exists a quadratic function to be optimized, and (2) the function to be optimized is restricted by linear relationships.[27] These characteristics lead to the following definition.

Definition 3.8.--Let $f(x) = f(x_1, x_2, \ldots, x_n)$ be an n-variable function defined by

$$f(x_1, x_2, \ldots, x_n) = 1/2 \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}x_ix_j + \sum_{i=1}^{n} c_ix_i .$$

[27]John C. G. Boot, Quadratic Programming (Amsterdam, 1964), p. 5.

Let $\phi_j(x) = \phi_j(x_1, x_2,\ldots,x_n)$ be a system of linear inequalities, $j = 1, 2,\ldots,m$, defined by

$$\phi_j(x_1, x_2,\ldots,x_m) = a_{j1}x_1 + a_{j2}x_2+\ldots+a_{jn}x_n + a_j \leq 0.$$

The quadratic programming problem is defined as that problem involving the optimization of $f(x_1, x_2,\ldots,x_n)$ subject to $\phi_j(x_1, x_2,\ldots,x_n)$.

As a means of reducing the size of the problem, the quadratic programming problem can be written in matrix notation. This abbreviated description is given by

$$\text{optimize } f(x) = 1/2\ \underline{X}^T\underline{G}\underline{X} + \underline{C}^T\underline{X}$$

subject to

$$\underline{A}\underline{X} \leq \underline{b}$$

$$X \geq 0$$

where

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}, \quad \underline{G} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{bmatrix} = [\underline{g}_1, \underline{g}_2,\ldots,\underline{g}_n],$$

$$A = [\underline{a}_1, \underline{a}_2,\ldots,\underline{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \text{ and } \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}.$$

The general application of the quadratic programming has been that of minimizing a given objective function. This has led to the following set of assumptions relative to the formulation of the problem.

(1) The matrix defining G is both symmetric and positive definite. The symmetric property assures element $g_{ij} = g_{ji}$ and $\underline{G} = \underline{G}^T$. The positive definite property guarantees that all values of the quadratic form are positive except when the variables are all identically zero. A necessary and sufficient condition for G to be positive definite is for

$$g_{11} > 0, \quad \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} > 0, \ldots, \quad \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1} & g_{n2} & \cdots & g_{mn} \end{vmatrix} > 0$$

to be satisfied.[28]

(2) The restricting conditions define a region containing an interior point.

(3) The objective function defines a convex set. This requires G to be positive semidefinite.

From these assumptions the general nature of the quadratic programming problem can be derived as follows: although subject to linear constraints, the function to be optimized is nonlinear (quadratic), permitting optimal solutions within the feasible region. Kunzi, Tzschach, and Zehnder distinguished three problem formulations for the general quadratic programming problem, each of which yields solutions not necessarily on the boundary of the constraint set. These three formulations are

(1) min $\{\phi(x)\}$ such that $\underline{Ax} \leq \underline{b}$, $x \geq 0$;

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[28] Franklin A. Graybill, _An Introduction to Linear Statistical Models_ (New York, 1961), p. 3.

(2) min $\{\phi(x)\}$ such that $\underline{Ax} = \underline{b}$, $x \geq 0$;

(3) min $\{\phi(x)\}$ such that $\underline{Ax} \leq \underline{b}$.[29]

Although similar to linear programming (the only difference being the quadratic objective function), quadratic programming has several distinguishing characteristics: (1) the solution (optimal) to the problem can occur at a corner of the feasible set, on an edge (boundary) of the feasible set, or at a point interior to the feasible set; (2) with no degeneracy, at most n of the m + n inequalities are satisfied as equations at the solution point (in linear programming exactly n of the inequalities are satisfied as equalities); (3) nonnegativity is not necessary since it can be absorbed into the constraint functions;[30] and, (4) the quadratic formulation tends to be more realisitc in determining cost-profit relationships, especially when dealing with marginal concepts.

Another interesting feature of quadratic programming concerns the solution that is obtained. Although the objective function is quadratic subject to linear constraints, the solution that is obtained is exact; and, as in linear programming, linear methods can be used to obtain this exact solution.[31] Although there does not yet exist an algorithm similar to Dantzig's simplex method that can be applied to the general quadratic programming problem, there exist several different techniques that have been developed for solving this particular class of problems. A cursory selection of these techniques is presented here.

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[29] Hans P. Kunzi, H. G. Tzschach, and C. A. Zehnder, Numerical Methods of Mathematical Optimization (New York, 1968), p. 66.

[30] Boot, op. cit., p. 5.

[31] Philip Wolfe, Recent Developments in Nonlinear Programming, Report No. R-401-PP (Santa Monica, 1962), p. 21.

Although each of these techniques is taken from a documented source, the detail of the presentation in this study is unavailable elsewhere. This detail is provided as a means of facilitating the understanding of the particular computational technique under discussion and as a means of formally presenting the technique in an operational format.

Method of Wolfe and Frank.--The problem to be solved has the form

$$\text{minimize } f(\underline{x}) = \underline{px} + \underline{x}^T\underline{Cx}$$

subject to

$$\underline{Ax} = \underline{b}$$

and

$$\underline{x} \geq 0.$$

In this formulation $\underline{x}$ is a column vector with n components, $\underline{p}$ is a row vector with n components, and $\underline{b}$ is a column vector with m components. $\underline{C}$ and $\underline{A}$ are n x n and m x n matrices, respectively. The function $f(\underline{x})$ is a quadratic expression and is assumed convex. This assumption guarantees that the n x n matrix $\underline{C}$ is a positive semidefinite matrix.

The method of Wolfe and Frank utilizes the simplex algorithm from linear programming to solve the indicated quadratic expression. This is achieved by transforming the given quadratic programming problem into an equivalent linear programming problem. The solution that is obtained is then tested for optimality. If the solution is not optimal, a gradient vector is derived and its elements used to redefine the c_j coefficients of the simplex tableau. Given these new c_j values, the $(c_j - z_j)$ test of the simplex algorithm is applied to determine a new basis. If this new basis is not optimal, the general programming technique is reapplied.

The following algorithm was developed from a comprehensive analysis of sample problems. Because of limited problem availability, one of these problems has been recast and is used here to demonstrate in detail the application of this algorithm.[32]

Algorithm 3.4 (Algorithm for method of Wolfe and Frank).--Step 1. Form the matrix equation

$$\underline{B}\underline{w} = \underline{d} = \begin{bmatrix} b \\ -p^T \end{bmatrix},$$

where $\underline{p}^T$ is the transpose of the row vector $\underline{p}$; $\underline{B}$ is the matrix determined by

$$\underline{B} = \begin{bmatrix} \underline{A} & \underline{0} & \underline{I} & \underline{0} \\ 2\underline{C} & \underline{A}^T & \underline{0} & -\underline{I} \end{bmatrix};$$

$\underline{w}^T$ is the vector of variables defined by

$$\underline{w}^T = [\underline{x}^T, \underline{u}, \underline{y}^T, \underline{v}].$$

The components of $\underline{w}^T$ are defined as follows: $\underline{x}^T$ and $\underline{y}^T$ are the transposes of variables $\underline{x}$ and the slack vector $\underline{y}$, respectively;

$$\underline{Ax} + \underline{y} = \underline{b};$$

$$f(\underline{x}) = -\underline{uA} + \underline{y}; \text{ and,}$$

$$\underline{vx} + \underline{uy} = \underline{0}.$$

Step 2. Formulate the convex quadratic function

$$g(\underline{w}) = \underline{vx} + \underline{uy} = \underline{ub} + \underline{px} + 2\underline{x}^T\underline{Cx}.$$

The function $g(\underline{w})$ is to be minimized to zero (reduced in numerical value to zero), subject to the linear constraint $\underline{Bw} = \underline{d}$, where $\underline{d} = \begin{bmatrix} \underline{b} \\ -\underline{p}^T \end{bmatrix}$

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[32] Saaty, op. cit., pp. 197-203.

Step 3. Using the linear constraint $\underline{Bw} = \underline{d}$, select an arbitrary initial basic feasible solution. This is accomplished by arbitrarily equating k of the variables of the linear constraint to zero, placing the remaining variables into the solution set, and solving the reduced system for the values of the solution variables.

Step 4. Construct the solution matrix. This operation is composed of two steps: (1) introducing the values corresponding to the solution variables and (2) determining the row elements of the body matrix which correspond to the variables in the solution. The first step is accomplished when the reduced system of Step 3 is solved. The second step is accomplished by writing each column of the body matrix as linear combinations of the solution set variables.

Step 5. Using the solution set, a contracted form of the solution vector $\underline{w}_j$, write the total solution $\underline{w}_j{}^{*T}$, where $\underline{w}_j{}^{*T} = [\underline{x}^T, \underline{u}, \underline{y}^T, \underline{v}]$.

Step 6. Test $g(\underline{w}) = \underline{vx} + \underline{uy}$ to determine if $g(\underline{w}) = 0$ for $\underline{w} = \underline{w}_j{}^*$. If $g(\underline{w}) = 0$, the optimal solution has been found and is equal to $\underline{w}_j{}^*$. If $g(\underline{w}) \neq 0$, go to Step 7.

Step 7. Construct the cost-coefficient vector defined by the gradient vector of $g(\underline{w})$. Evaluate this gradient vector at the current solution. The elements of the gradient vector define the value of the corresponding contribution coefficients in the c_j row of the simplex algorithm.

Step 8. Construct a solution matrix that includes the c_j values. The result will be a standard simplex tableau.

Step 9. Apply the simplex algorithm to determine a new basis. Return to Step 5 and repeat Steps 5-9 as needed. At each iteration it is necessary to construct a new cost-coefficient vector as determined by the solution set at that iteration.

This algorithm will be demonstrated by solving the following modified problem. A new product, x_2, is to be added to an existing product line. Since the feasibility of adding the new product is subject to question, a study is made to determine if the minimal expected return brought about by the addition will exceed a predetermined value. The expected return function is given by

$$f(x_1, x_2) = 1000x_1^2 + 3000x_2^2 - 4000x_1 - 6000x_2,$$

where x_1 and x_2 define products x_1 and x_2 in units of 100 each. Because of limited funds, promotional effort will be expended in units of 1000 with a 1:2 proportion. Product x_2 is to receive the concentration of funds. The expression defining promotional effort is assumed to be linear and equal to at most \$4000; i.e.,

$$g(x_1, x_2) = 1000x_1 + 2000x_2 \leq 4000.$$

It is further assumed that x_1 and x_2 are both nonnegative.

The system defined by this problem can be written in the form

$$\text{minimize } f(x_1, x_2) = 1000x_1^2 + 3000x_2^2 - 4000x_1 - 6000x_2$$

subject to

$$g(x_1, x_2) = 1000x_1 + 2000x_2 \leq 4000$$

$$x_1 \geq 0$$

$$x_2 \geq 0.$$

This system can be further reduced to the form

$$\text{minimize } f(x_1, x_2) = x_1^2 + 3x_2^2 - 4x_1 - 6x_2$$

subject to

$$g(x_1, x_2) = x_1 + 2x_2 \leq 4$$
$$x_1 \geq 0$$
$$x_2 \geq 0.$$

This formulation corresponds to the quadratic form

$$\text{minimize } f(\underline{x}) = \underline{px} + \underline{x}^T\underline{C}\underline{x}$$

subject to

$$\underline{Ax} \leq \underline{b}$$
$$\underline{x} \geq 0.$$

From this correspondence, the following vectors and matrices can be written:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};\ \underline{x}^T = [x_1, x_2];\ \underline{p} = [-4, -6];\ \underline{c} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix};$$

$\underline{A} = [1, 2]$; and $\underline{b} = [4]$.

Since the objective function is a quadratic expression and is restricted by a linear function, the quadratic programming technique of Wolfe and Frank is applicable.

Iteration 1: Step 1. Form the matrix equation $\underline{Bw} = \begin{bmatrix} \underline{b} \\ -\underline{p}^T \end{bmatrix}$, where

$$\underline{B} = \begin{bmatrix} \underline{A} & \underline{0} & \underline{I} & \underline{0} \\ 2\underline{C} & \underline{A}^T & \underline{0} & -\underline{I} \end{bmatrix},$$

$\underline{w} = (\underline{w}^T)^T$, $\underline{w}^T = [\underline{x}^T, \underline{u}, \underline{y}^T, \underline{v}]$, and $\underline{p}^T$ is the transpose of $\underline{p}$. Substituting as indicated,

$$\underline{B} = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 & 0 \\ 0 & 6 & 2 & 0 & 0 & -1 \end{bmatrix};$$

$$\begin{bmatrix} \underline{b} \\ -\underline{p}^T \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

The vector $\underline{x}$ is given by $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Since there is only one constraint function, the slack vector $\underline{y}$ is given by $\underline{y} = [y]$. Since $\underline{x}$ is 1 x 2 and $\underline{y}$ consists of one element, $\underline{vx} + \underline{uy} = 0$ defines $\underline{v}$ as the row vector (v_1, v_2) and $\underline{u}$ as the vector $[u]$. From this development the vector $\underline{w}^T$ is written as follows:

$$\underline{w}^T = [x_1, x_2, u, y, v_1, v_2].$$

The necessary matrix equation is

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 & 0 \\ 0 & 6 & 2 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \\ y \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Expanding this system yields

$$x_1 + 2x_2 + 0u + y + 0v_1 + 0v_2 = 4$$

$$2x_1 + 0x_2 + u + 0y - v_1 + 0v_2 = 4$$

$$0x_1 + 6x_2 + 2u + 0y + 0v_1 - v_2 = 6.$$

Step 2. The convex quadratic function to be reduced to zero is $g(\underline{w}) = \underline{vx} = \underline{uy}$. This is given by $g(\underline{w}) = [v_1, v_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + uy = v_1x_1 + v_2x_2 + uy$.

Step 3. Utilizing $\underline{Bw} = d = \begin{bmatrix} \underline{b} \\ -\underline{p}^T \end{bmatrix}$, select an arbitrary initial solution. Let x_1, x_2, and v_1 all equal zero. This places u, y, and v_2 in the solution matrix. For $x_1 = x_2 = v_1 = 0$, the reduced system is given by

$$y = 4;$$

$$u = 4;$$

$$2u - v_2 = 6.$$

Substituting for u, v_2 is found to have a value of 2. Therefore, the first value of w_j^* is $\underline{w}_1^{T*} = [4, 4, 2]$.

Step 4. Construct the solution matrix. This matrix will have the form

	w_1^*	x_1	x_2	u	y	v_1	v_2
u	4						
y	4						
v_2	2						

with the elements of the body matrix to be determined. The elements in the body matrix are obtained by writing each column as a linear combination of the initial basis u, y, and v_2.

(1) Column x_1. Determine k_1, k_2, and k_3 such that

$$k_1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

The solution is given by $k_1 = 2$, $k_2 = 1$, and $k_3 = 4$. The elements in the body matrix under column x_1 are 2, 1, and 4, in that order.

(2) Column x_2. Determine k_1, k_2, and k_3 such that

$$k_1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}.$$

The solution is given by $k_1 = 0$, $k_2 = 2$, and $k_3 = -6$. The elements in the body matrix under column x_2 are 0, 2, and -6, in that order.

(3) Column u. Determine k_1, k_2, and k_3 such that

$$k_1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} .$$

The solution is given by $k_1 = 1$, $k_2 = 0 = k_3$. The elements in the body matrix under column u are 1, 0, and 0, in that order.

Repeated application of this process will produce values for the remaining columns. The results are shown in Table 3.13.

TABLE 3.13

INITIAL SOLUTION MATRIX

	w_1^*	x_1	x_2	u	y	v_1	v_2
u	4	2	0	1	0	-1	0
y	4	1	2	0	1	0	0
v_2	2	4	-6	0	0	-2	1

Step 5. Form the solution matrix, $u = 4$, $y = 4$, and $v_2 = 2$. The values of x_1, x_2, and v_1 are zero. Therefore,

$$\underline{w}_1^{*T} = [(x_1, x_2), u, y, (v_1, v_2)] = [x_1, x_2, u, y, v_1, v_2]$$
$$= [0, 0, 4, 4, 0, 2].$$

Step 6. Test $g(\underline{w}) = \underline{vx} + \underline{uy}$ to determine if $g(w) = 0$. From $\underline{w}_1^{*T}$, it is noted that $\underline{x}^T = (0, 0)$; $\underline{u} = 4$; $\underline{y} = 4$; and $\underline{v} = (0, 2)$. Thus,

$$g(\underline{w}) = \underline{vx} + \underline{uy} = (0, 2) \begin{matrix} 0 \\ 0 \end{matrix} + (4)(4)$$

$$= 0(0) + 2(0) + 16$$

$$= 16.$$

From these calculations, $g(\underline{w}) \neq 0$. The current solution is not optimal. Go to Step 7.

Step 7. The gradient vector for this problem is defined by

$$\nabla g(\underline{w}) = \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial u}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial v_1}, \frac{\partial g}{\partial v_2} .$$

The expanded form of $g(\underline{w}) = \underline{vx} + \underline{uy}$ is given by $g(\underline{w}) = v_1x_1 + v_2x_2 + uy$. Hence,

$$\nabla g(\underline{w}) = [v_1, v_2, y, u, x_1, x_2].$$

At $\underline{w}_1$*, $\nabla g(\underline{w}) = [0, 2, 4, 4, 0, 0]$. From this the corresponding c_j values are found to be $c_1 = 0$, $c_2 = 2$, $c_3 = 4$, $c_4 = 4$, $c_5 = 0$, and $c_6 = 0$.

Step 8. Using $\nabla g(\underline{w}_1^*)$ to determine the cost-coefficients, construct the standard simplex tableau by attaching the cost-coefficients to the solution matrix of Step 4. The results are shown in Table 3.14.

TABLE 3.14

SIMPLEX TABLEAU #1

C	$C = \nabla g(W_1^*)$: Solution	W_1^*	C_1 0 x_1	C_2 2 x_2	C_3 4 u	C_4 4 y	C_5 0 v_1	C_6 0 v_2
4	u	4	2	0	1	0	-1	0
4	y	4	1	2	0	1	0	0
0	v_2	2	4	-6	0	0	-2	1
	z_j	32	12	8	4	4	-4	0
	$c_j - z_j$		-12	-6	0	0	4	0

Step 9. Application of the $(c_j - z_j)$ rule for minimization requires $(c_j - z_j)$ be nonnegative for all values. Inspection of Table 3.14 reveals that $(c_j - z_j) \not\geq 0$ for all values. The variable to enter the solution is that variable which corresponds to the most negative value of $c_j - z_j$. Since -12 is the most negative value of $c_j - z_j$, variable x_1 will enter the solution. The variable to be replaced is the variable corresponding to the minimum positive value of θ, where

$$\theta = \frac{\text{component of } w_1^*}{\text{positive components of entering vector}} .$$

From this it is found that v_2 will leave the solution. The new basis can now be easily determined by application of the simplex algorithm. The results are shown in Table 3.15.

TABLE 3.15

SOLUTION MATRIX #2

Solution	w_2^*	x_1	x_2	u	y	v_1	v_2
u	3	0	3	1	0	0	$-\frac{1}{2}$
y	$\frac{7}{2}$	0	$\frac{7}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{4}$
x_1	$\frac{1}{2}$	1	$-\frac{3}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{4}$

Iteration 2: Step 5. From the solution matrix of Table 3.15, $u = 3$; $y = \frac{7}{2}$; $x_1 = \frac{1}{2}$. The values of x_2, v_1, and v_2 are all zero. Therefore,

$$\underline{w}_2^{*T} = [\tfrac{1}{2}, 0, 3, \tfrac{7}{2}, 0, 0].$$

Step 6. Test $g(\underline{w}) = \underline{vx} + \underline{uy}$ to determine of $g(\underline{w}) = 0$. From $\underline{w}_2^{*T}$, it is noted that $\underline{x}^T = (\frac{1}{2}, 0;$ $\underline{u} = 3$; $\underline{y} = \frac{7}{2}$; and $\underline{v} = (0, 0)$. Thus,

$$g(\underline{w}) = \underline{vx} + \underline{uy} = (0,\ 0)\begin{matrix}\frac{1}{2}\\ 0\end{matrix} + 3(\frac{7}{2})$$

$$= 0(\frac{1}{2}) + 0(0) + \frac{21}{2}$$

$$= \frac{21}{2}\ .$$

From these calculations $g(\underline{w}) \neq 0$. The current solution is not optimal. Go to Step 7.

Step 7. The gradient vector is given by $\nabla g(\underline{w}) = [v_1, v_2, y, u, x_1, x_2]$. At $\underline{w}_2^*$, $\nabla g(\underline{w}) = [0, 0, \frac{7}{2}, 3, \frac{1}{2}, 0]$. From this the corresponding c_j values are found to be $c_1 = 0$, $c_2 = 0$, $c_3 = \frac{7}{2}$, $c_4 = 3$, $c_5 = \frac{1}{2}$, and $c_6 = 0$.

Step 8. Using the cost-coefficients defined by $\nabla g(\underline{w}_2^*)$, the corresponding simplex tableau is constructed. The results are shown in Table 3.16.

Step 9. Reapplication of the simplex algorithm for minimization indicates that variable x_2 will enter the solution and replace either u or y. If x_2 replaces u, reapplication of the preceding process will yield an optimal solution. For $x_1 = 2$, $x_2 = 1$, and $y = 0$, $g(\underline{w}) = 0$. If x_2 replaces y, additional iterations will be needed. The iterations will terminate at $x_1 = 2$, $x_2 = 1$, and $y = 0$. For these values, $f(\underline{x}) = -7$.

If the firm chooses to add product x_2, the expected return function is minimized for 200 units of x_1 and 100 units of x_2. The minimal expected return is a loss of \$7,000.

TABLE 3.16

SIMPLEX TABLEAU #2

	$C = \nabla g(w_2^*)$		0	0	$\frac{7}{2}$	3	$\frac{1}{2}$	0
C	Solution	w_2^*	x_1	x_2	u	y	v_1	v_2
$\frac{7}{2}$	u	3	0	3	1	0	0	$-\frac{1}{2}$
3	y	$\frac{7}{2}$	0	$\frac{7}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{4}$
0	x_1	$\frac{1}{2}$	1	$-\frac{3}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{4}$
	z_j	21	0	21	$\frac{7}{2}$	3	$\frac{3}{2}$	$-\frac{5}{2}$
	$c_j - z_j$		0	-21	0	0	-1	$\frac{5}{2}$

It is to be noted that the method of Wolfe and Frank combines the essential ideas of an optimum on the boundary as a means of linearizing the problem. This permits the use of the simplex algorithm to obtain a convergent series which terminates in the final solution.

Method of Theil-Van de Panne.--The problem to be solved has the form

$$\text{maximize } f(\underline{x}) = \underline{a}^T\underline{x} - \frac{1}{2}\underline{x}^T\underline{G}\underline{x}$$

subject to

$$\underline{A}^T\underline{x} \leq \underline{b}.$$

In this formulation $\underline{x}$ is a column vector of n components. The matrix $\underline{G}$ is a positive definite symmetric matrix of order n x n. The matrix $\underline{A}$ is of order m x n.

The Theil-Van de Panne technique locates the solution vector $\underline{x}^*$ by finding the subset S out of the m constraints

$$\underline{A}^T\underline{x} \leq \underline{b}, \ \underline{A} \text{ of order } m \times n,$$

such that

$$f(\underline{x}) = \underline{a}^T\underline{x} - 1/2\underline{x}^T\underline{Gx}$$

is maximized with all constraints in S expressed as equalities. This optimal solution vector is given by $\underline{x}^S = \underline{x}^*$, where $\underline{x}^S$ is defined as the vector maximizing $f(\underline{x}) = \underline{a}^T\underline{x} - 1/2\underline{x}^T\underline{Gx}$ with all constraints of S expressed as equalities, S being any subset of the m constraints.[33]

This procedure is implemented by first optimizing the objective function without regard to any existing constraints. If the unconstrained optimum is not a feasible solution (i.e., the unconstrained optimum violates at least one of the constraints), the technique successively adds constraints until the iteration results in a feasible optimum. This is accomplished by an iterative process which determines on which side of the constraints being considered the solution lies; the technique then moves in the direction of an unsatisfied constraint. As each iteration produces its own optimum, the feasible solution is the final optimal solution. The optimal solution is identified as the solution vector which satisfies the following rule:

> The feasible vector x^S is optimal if and only if for all $h \epsilon S$ the vector x^{S-h} (s-h is the set S with the h^{th} constraint deleted) violates constraint h.[34]

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[33] H. Theil and C. Van de Panne, "Quadratic Programming as an Extension of Classical Quadratic Maximization," Mathematical Studies in Management Science, edited by Arthur F. Veinott, Jr. (New York, 1965), pp. 129-148.

[34] Boot, op. cit., p. 98.

The general procedure of the Theil-Van de Panne technique for maximizing a quadratic function subject to linear constraints can be summarized by a suitable algorithm. The algorithm to follow is the result of a detailed study of the work of Theil and Van de Panne and a thorough analysis of sample problems. Following this algorithm is a detailed example. Although the problem is of a type common to the literature, the detail of the presentation is not.

Algorithm 3.5 (Algorithm for method of Theil-Van de Panne).--
Step 1. Maximize the given quadratic function without considering any existing constraint. Denote the initial maximizing solution vector by $\underline{x}^{\circ}$. This initial solution vector can be obtained by straightforward differentiation of the objective function

$$f(\underline{x}) = \underline{a}^T\underline{x} - \tfrac{1}{2}\underline{x}^T\underline{G}\underline{x}$$

or by obtaining and using the inverse of the matrix $\underline{G}$.

Step 2. Test the initial maximal solution, $\underline{x}^{\circ}$, to determine whether or not $\underline{x}^{\circ}$ satisfies the constraints imposed on the objective function. If $\underline{x}^{\circ}$ satisfies the constraints of the problem, then $\underline{x}^{\circ}$ is the solution which maximizes $f(\underline{x})$. (A constrained maximum can never exceed the unconstrained maximum.)[35] If the initial solution $\underline{x}^{\circ}$ violates one or more constraints, then it is known that the optimal solution vector will be such that at least one of these violated constraints will be satisfied in equality form. If the solution $\underline{x}^{\circ}$ is not optimal, go to Step 3.

[35] Theil and Van de Panne, op. cit., p. 130.

Step 3. Given that the initial solution, $\underline{x}^{\circ}$, violates at least one of the constraints, it is necessary to determine the optimal solution vector subject to the defined linear constraints. This is accomplished by systematically considering $\underline{x}^{s}$ for all of the s subsets of the m constraints. ($\underline{x}^{s}$ has been defined as the vector maximizing $f(\underline{x})$ subject to s of the m constraints expressed as equalities.) The sets s to be considered are those containing at least one constraint that is violated by the initial solution $\underline{x}^{\circ}$. The procedure to be followed consists of the following set of operations:

(1) Let the i^{th} constraint be treated as an equality. Maximize $f(\underline{x})$ subject to the i^{th} constraint. Denote the solution vector by $\underline{x}^{i}$. If $\underline{x}^{i}$ is not feasible for some i, go to (2).

(2) Let constraints i and j be treated as equalities. Maximize $f(\underline{x})$ subject to the pair of equality constraints i and j. Denote the solution vector by $\underline{x}^{ij}$. If $\underline{x}^{ij}$ is not feasible for some pair (i, j), go to (3).

(3) Continue the process, adding one additional equality constraint each time, until an optimal solution is obtained. The optimal solution will be that solution vector which maximizes $f(\underline{x})$ subject to s-constraints treated as equalities. It is necessary to consider all possible combinations of the equality-treated constraints.[36] When a feasible solution is obtained, go to Step 4.

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[36] _Ibid._, p. 136.

Step 4. Test the feasible solution for optimality. A feasible solution is optimal if and only if

(1) $\underline{A}^T \underline{x}_{k+1} \leq \underline{b}$; and

(2) the optimal solution to the set of k+1 problems generated by deleting one at a time, one of the constraints violates this constraint.[37]

As a means of illustraing this technique, consider the problem of determining the amounts of product x_1 and x_2, respectively, that must be produced to maximize the continuous profit function

$$f(x_1, x_2) = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 - 4x_1x_2$$

subject to the demand functions

$$\begin{aligned} x_1 + x_2 &\leq 9; \\ x_1 + 2x_2 &\leq 10; \\ x_1 &\geq 0; \\ x_2 &\geq 0. \end{aligned}$$

This type of problem is similar to production activity problems where the constraints are those imposed by labor and/or resource availability. It is also similar to investment problems in which return on investment is to be maximized subject to the availability of investment funds.

Solution: Step 1. Maximize

$$f(x_1, x_2) = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 - 4x_1x_2$$

without considering any existing constraint. This is accomplished by $\frac{\partial f(x_1, x_2)}{\partial x_i} = 0$ for all x_i, $(i = 1, 2)$ and testing for a maximum solution

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[37] Ronald L. Gue and Michael E. Thomas, _Mathematical Models in Operations Research_ (London, 1968), p. 242.

by applying the criterion

$f_1(x_1, x_2) = f_2(x_1, x_2) = 0$ at the critical points (X_1, X_2);

$f_{12}^2(x_1, x_2) - f_{11}(x_1, x_2) \cdot f_{22}(x_1, x_2) < 0$ at (X_1, X_2); and,

$f_{11}(x_1, x_2) < 0$ at (X_1, X_2).

(1) Equating $\frac{\partial f(x_1, x_2)}{\partial x_i} = 0$ for i = 1, 2, yields

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = 10 - 20x_1 - 4x_2 = 0;$$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = 25 - 2x_2 - 4x_1 = 0.$$

The critical points for the given function are those points satisfying this linear system. Since the system is linear, the set of critical points is unique. Solving this system yields $X_1 = -\frac{40}{12}$ and $X_2 = \frac{115}{6}$.

(2) Testing $(X_1, X_2) = (-\frac{40}{12}, \frac{115}{6})$ for maximization yields the following:

(a) $\frac{\partial}{\partial x_1} f(x_1, x_2) = \frac{\partial}{\partial x_2} f(x_1, x_2) = 0$ at (X_1, X_2). This is evident since this condition was used to determine both X_1 and X_2.

(b) $f_{12}^2(x_1, x_2) - f_{11}(x_1, x_2) \cdot f_{22}(x_1, x_2) < 0$ at (X_1, X_2).

It is necessary to determine the appropriate partial derivatives and evaluate these at (X_1, X_2).

$$f_1(x_1, x_2) = \frac{\partial}{\partial x_1} f(x_1, x_2) = 10 - 20x_1 - 4x_2;$$

$$f_{11}^2(x_1, x_2) = \frac{\partial^2}{\partial x_1^2} f(x_1, x_2) = \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_1} f(x_1, x_2) = -20;$$

$$f_2(x_1, x_2) = \frac{\partial}{\partial x_2} f(x_1, x_2) = 25 - 2x_2 - 4x_1;$$

$$f_{22}^2(x_1, x_2) = \frac{\partial^2}{\partial x_2^2} f(x_1, x_2) = \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_2} f(x_1, x_2) \quad = -2;$$

$$f_{12}^2(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} f(x_1, x_2) = \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} f(x_1, x_2)$$

$$= \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_1} f(x_1, x_2) = -4.$$

Evaluating these partial derivatives at $X_1 = \frac{40}{12}$ and $X_2 = \frac{115}{6}$ yields

$$f_1(X_1, X_2) = f_2(X_1, X_2) = 0;$$

$$f_{12}^2(X_1, X_2) = -4;$$

$$f_{11}^2(X_1, X_2) = -20;$$

$$f_{22}^2(X_1, X_2) = -2.$$

Substituting these values into the relationships which identify (X_1, X_2) as a maximum, it is found that

$f_1(x_1, x_2) = f_2(x_1, x_2) = 0$ at (X_1, X_2);

$f_{12}^2(x_1, x_2) - f_{11}(x_1, x_2) \cdot f_{22}(x_1, x_2) = -36 < 0$ at (X_1, X_2); and

$f_{11}(x_1, x_2) = -20 < 0$ at (X_1, X_2).

The given function achieves an unconstrained maximum at $X_1 = -\frac{40}{12}$, $X_2 = \frac{115}{6}$. Define $\underline{x}^\circ$ as $\underline{x}^\circ = (X_1, X_2)^T = (x_1, x_2)^T = (-\frac{40}{12}, \frac{115}{6})^T$.

Step 2. Test the initial solution $\underline{x}^\circ$ for feasibility. This is accomplished by substituting $x_1 = -\frac{40}{12}$ and $x_2 = \frac{115}{6}$, into each of the constraints.

(1) at $\underline{x}^\circ$, $x_1 + x_2 \leq 9$ is violated: $x_1 + x_2 = 15\frac{5}{6} \nleq 9$;

(2) at $\underline{x}^\circ$, $x_1 + 2x_2 \leq 10$ is violated: $x_1 + 2x_2 = 35 \nleq 10$;

(3) at $\underline{x}^\circ$, $x_1 \geq 0$ is violated: $x_1 = -\frac{40}{12} \ngeq 0$;

(4) at $\underline{x}^\circ$, $x_2 \geq 0$ is satisfied: $x_2 = \frac{115}{6} > 0$.

From these calculations it is evident that the unconstrained solution is not optimal since it violates constraints (1), (2), and (3). At least one of these three constraints will be satisfied as an equality in the final solution.

Step 3. Given that the initial solution $\underline{x}^\circ$ is not optimal, it is necessary to determine $\underline{x}_s$ for all of the possible subsets of the three violated constraints. This is done by considering the violated constraints individually, in pairs, and then as a unit. If an optimal solution occurs at any point, the iterations cease.

(1) Subsets of one equality constraint

(a) max $f(x_1, x_2)$ subject to $x_1 + x_2 = 9$.

(b) max $f(x_1, x_2)$ subject to $x_1 + 2x_2 = 10$.

(c) max $f(x_1, x_2)$ subject to $x_1 = 0$.

Consider 1(a).

$$\max f(x_1, x_2) = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 - 4x_1x_2$$

subject to $x_1 + x_2 = 9$. Solving the constraint equality for x_2, $x_2 = 9 - x_1$, and substituting into $f(x_1, x_2)$ results in a one dimensional function, $f(x_1)$

$$f(x_1) = 144 - 33x_1 - 7x_1^2.$$

This function can be optimized by setting $f'(x_1)$ equal to zero to determine the critical point. This point can then be used to calculate x_2. The derivative of $f(x_1)$, $f'(x_1)$, is given by $f'(x_1) = -33 - 14x_1$. Equating $f'(x_1)$ to zero yields $x_1 = -\frac{33}{14}$. Since $f''(x_1) = -14 < 0$ at $x_1 = -\frac{33}{14}$, the given function is maximized at this point. At $x_1 = -\frac{33}{14}$, $x_2 = 9 - x_1$ yields a value of $\frac{159}{14}$ for x_2.

The solution set $\underline{x}' = (x_1, x_2) = (-\frac{33}{14}, \frac{159}{14})$ must be tested for feasibility. Substitution of these values into the constraint functions yields the following:

(1) at $\underline{x}'$, $x_1 + x_2 \leq 9$ is satisfied: $x_1 + x_2 = 9$;

(2) at $\underline{x}'$, $x_1 + 2x_2 \leq 10$ is violated: $x_1 + 2x_2 = 20\frac{5}{14} \not\leq 10$;

(3) at $\underline{x}'$, $x_1 \geq 0$ is violated: $x_1 = -\frac{33}{14} \not\geq 0$;

(4) at $\underline{x}'$, $x_2 \geq 0$ is satisfied: $x_2 = \frac{115}{6} > 0$.

The solution vector $\underline{x}'$ is not a feasible solution, hence not optimal.

Consider 1(b).

$$\max f(x_1, x_2) = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 - 4x_1x_2$$

subject to $x_1 + 2x_2 = 10$. Solving the constraint equality for x_1,

$x_1 = 10 - 2x_2$, and substituting for x_1 in $f(x_1, x_2)$ results in the one dimensional function, $f(x_2)$

$$f(x_2) = -900 + 365x_2 - 33x_2^2.$$

This function can be optimized by setting $f'(x_2)$ equal to zero to determine the critical point. This point can then be used to calculate x_1. The derivative of $f(x_2)$, $f'(x_2)$, is given by $f'(x_2) = 365 - 66x_2$. Equating $f'(x_2)$ to zero yields $x_2 - \frac{365}{66}$. Since $f''(x_2) = -66 < 0$ at $x_2 = \frac{365}{66}$, the given function is maximized at this point. At $x_2 = \frac{365}{66}$, $x_1 = 10 - 2x_2$ yields a value of $-\frac{35}{33}$ for x_1.

The solution set $\underline{x}^2 = (x_1, x_2) = (-\frac{35}{33}, \frac{365}{66})$ must be tested for feasibility. Substitution of these values into the constraint functions yields the following:

(1) at $\underline{x}^2$, $x_1 + x_2 \leq 9$ is satisfied: $x_1 + x_2 = 5 < 9$;

(2) at $\underline{x}^2$, $x_1 + 2x_2 \leq 10$ is satisfied: $x_1 + 2x_2 = 10$;

(3) at $\underline{x}^2$, $x_1 \geq 0$ is violated: $x_1 = -\frac{35}{33} \not\geq 0$;

(4) at $\underline{x}^2$, $x_2 \geq 0$ is satisfied: $x_2 = \frac{365}{66} > 0$.

The solution vector $\underline{x}^2$ is not a feasible solution, hence not optimal.

Consider 1(c).

$$\max f(x_1, x_2) = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 + 4x_1x_2$$

subject to $x_1 = 0$. With $x_1 = 0$ the function to be maximized is given by

$$f(x_2) = 25x_2 - x_2^2.$$

This function can be optimized by setting $f'(x)$ equal to zero to determine the critical point. The derivative of $f(x_2)$, $f'(x_2)$ is given by

$f'(x_2) = 25 - 2x_2$. Equating $f'(x_2)$ to zero yields $x_2 = \frac{25}{2}$. Since $f''(x_2) = -2 < 0$ at $x_2 = \frac{25}{2}$, the given function is maximized at this point.

The solution set $\underline{x}^3 = (x_1, x_2) = (0, \frac{25}{2})$ must be tested for feasibility. Substitution of these values into the constraint functions yields the following:

(1) at $\underline{x}^3$, $x_1 + x_2 \leq 9$ is violated: $x_1 + x_2 = 12\frac{1}{2} \nleq 9$;

(2) at $\underline{x}^3$, $x_1 + 2x_2 \leq 10$ is violated: $x_1 + 2x_2 = 25 \nleq 10$;

(3) at $\underline{x}^3$, $x_1 \geq 0$ is satisfied: $x_1 = 0$;

(4) at $\underline{x}^3$, $x_2 \geq 0$ is satisfied: $x_2 = \frac{25}{2} > 0$.

The solution vector $\underline{x}^3$ is not a feasible solution, hence not optimal.

At this point an optimal solution has been found. Each of the three subset possibilities of one equality resulted in solution vectors which violated at least one constraint.

1(a) $\underline{x}^1 = \frac{33}{14}, \frac{159}{14}$ violated constraints (2) and (3).

1(b) $\underline{x}^2 = \frac{35}{33}, \frac{365}{66}$ violated constraint (3).

1(c) $\underline{x}^3 = 0, \frac{25}{2}$ violated constraints (1) and (2).

Since no optimal solution has been found, it is necessary to consider maximizing $f(x_1, x_2)$ subject to all possible pairs of equality constraints. These pairs of equality constraints are then used to determine values of x_1 and x_2. These values are then tested for both feasibility and optimality.

(2) Subsets of two equality constraints

(a)
$$x_1 + x_2 = 9$$
$$x_1 + 2x_2 = 10$$

(b) $\left.\begin{aligned} x_1 + x_2 &= 9 \\ x_1 \quad &= 0 \end{aligned}\right\} x_2 = 9.$

(c) $\left.\begin{aligned} x_1 + 2x_2 &= 10 \\ x_1 \quad &= 0 \end{aligned}\right\} x_2 = 5.$

(d) $\left.\begin{aligned} x_1 + x_2 &= 9 \\ x_2 &= 0 \end{aligned}\right\} x_1 = 9.$

(e) $\left.\begin{aligned} x_1 + 2x_2 &= 10 \\ x_2 &= 0 \end{aligned}\right\} x_1 = 10.$

Before beginning, note that one of the equality conditions violates the original constraints of the original problem: $x_1 = 0$, $x_2 = 9$, violates constraint (1). Thus, 2(b) need not be considered.

Consider 2(a)

$$\max f(x_1, x_2) = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 + 4x_1x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 9 \\ x_1 + 2x_2 &= 10. \end{aligned}$$

Solving the linear system, $x_1 = 8$, and $x_2 = 1$, this solution, $\underline{x}^{12} = (8, 1)$, satisfies all the inequality constraints imposed on $f(x_1, x_2)$.

$$\begin{aligned} x_1 + x_2 &\leq 9 \\ x_1 + 2x_2 &\leq 10 \\ x_1 &\geq 0 \\ x_2 &\geq 0, \end{aligned}$$

and thus constitutes a feasible solution. Since $\underline{x}^{12}$ is feasible, go to Step 4.

Step 4. Test the feasible solution $\underline{x}^{12} = (8, 1)$ for optimality. The feasible solution is optimal if and only if the solution to the problems obtained by holding one constraint active violates the relaxed constraint; i.e., solving the problems generated by relaxing one constraint at a time must yield points that violate the constraint that has been relaxed.

In the first iteration, the relaxed constraints were written as

$$\begin{aligned} x_1 + x_2 &= 9; \\ x_1 + 2x_2 &= 10; \\ x_1 &= 0. \end{aligned}$$

The solution vectors achieved by relaxing these constraints one at a time were $\underline{x}^1 = (-\frac{33}{14}, \frac{159}{14})$; $\underline{x}^2 = (-\frac{35}{33}, \frac{365}{66})$; and $\underline{x}^3 = (0, \frac{25}{2})$. Solution vector $\underline{x}^1$ violated $x_1 + x_2 \leq 9$ and $x_1 \geq 0$. Solution vector $\underline{x}^2$ violated $x_1 \geq 0$. The solution given by $\underline{x}^2$ does not violate $x_1 + x_2 \leq 9$. Hence, the solution achieved by using constraints 1 and 2 simultaneously as an equality system, $\underline{x}^{12}$, is not an optimal solution. Return to Step 3 and continue the iteration process.

Step 3 (continued). Consider 2(c). (2(b) was discarded at the outset.)

$$\max f(x_1, x_2) = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 + 4x_1x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &= 10 \\ x_1 &= 0. \end{aligned}$$

Solving this linear system, $x_1 = 0$ and $x_2 = 5$, this solution, $\underline{x}^{23} = (0, 5)$, satisfies the inequality constraints, constituting a feasible solution. Since $\underline{x}^{23}$ is feasible, go to Step 4.

Step 4. Test the feasible solution $\underline{x}^{23} = (0, 5)$ for optimality. The feasible solution is optimal if and only if the solution to the problems obtained by holding one constraint active violates the relaxed constraint; i.e., solving the problems generated by relaxing one constraint at a time must yield points that violate the constraint that has been relaxed.

In the first iteration the relaxed constraints were written as

$$x_1 + x_2 = 9;$$
$$x_1 + 2x_2 = 10;$$
$$x_1 = 0.$$

The solution vectors achieved by relaxing these constraints one at a time were $\underline{x}^1 = (-\frac{33}{14}, \frac{159}{14})$; $\underline{x}^2 = (-\frac{35}{33}, \frac{365}{66})$; and $\underline{x}^3 = (0, \frac{25}{2})$. Since the current solution is $\underline{x}^{23}$, the maximum value of $f(x_1, x_2)$ subject to the paired equality constraints 2 and 3, the solutions to $\underline{x}^2$ and $\underline{x}^3$ are the only ones to be considered. Solution vector $\underline{x}^2 = (\frac{-35}{33}, \frac{365}{66})$ violated $x_1 \geq 0$. Solution vector $\underline{x}^3$ violated $x_1 + x_2 \leq 9$ and $x_1 + 2x_2 \leq 10$.

It is necessary to consider one of the constraints violated by $\underline{x}^2$. The only violated constraint was $x_1 \geq 0$. The test for optimality requires that $\underline{x}^3$ violate $x_1 + 2x_2 \leq 10$. Since $\underline{x}^2$ violates the third constraint, $x_1 \geq 0$, and $\underline{x}^3$ violates the second constraint, $x_1 + 2x_2 \leq 10$, the solution defined by $\underline{x}^{23}$ is optimal.

When $f(x_1, x_2)$ is maximized subject to the pair of equalities

$$x_1 + 2x_2 = 10$$
$$x_1 = 0,$$

the solution to the constraint equalities is optimal: $x_1 = 0, x_2 = 5$. Optimality is verified by examining the single equality solutions and their effect upon the constraints paired together. Since constraints (2) and (3) are paired, the unit solutions to the single equalities maximizing $f(x_1, x_2)$ will be examined. The solutions were

(2) max $f(x_1, x_2)$ subject to $x_1 + 2x_2 = 10$: $x_1 = -\frac{35}{33}$; $x_2 = \frac{159}{66}$;

(3) max $f(x_1, x_2)$ subject to $x_1 = 0$: $x_1 = 0$; $x_2 = \frac{25}{2}$.

Optimality exists if and only if the solution $\underline{x}^2$ violates constraint (3), $x_1 \geq 0$, and the solution $\underline{x}^3$ violates constraint (2), $x_1 + 2x_2 \leq 10$. Since this is the case, the solution to the pair ((2), (3)) optimizes $f(x_1, x_2)$.

Maximum profit will be achieved by concentrating all production on product x_2; i.e., produce 0 units of x_1, 5 units of x_2. Maximum profit, in dollars, will be \$100.

Differential algorithm of Wilde and Beightler.--The differential algorithm (developed by Candler and Townsend, and independently, by Beightler, Crawford, and Wilde) uses a special characteristic of quadratic programming, decision derivatives. Defined as the appropriate partial (or regular) derivative of a given set of independent variables, these derivatives are linear functions of the decision variables. These decision derivatives represent the rate of change in the objective function resulting from feasible (not arbitrary) changes in given (or selected) decision variables.

The use of a differential algorithm requires that the objective function be expressed in terms of the decision variables. With this expression the objective function can then be partially differentiated with respect to each of the decision variables; any other constrained variables are treated as constants. (A more detailed treatment of the differential algorithm as an optimization tool can be found in Wilde and Beightler: Ibid., pp. 58-78.)

Prior to the presentation of the differential algorithm as a tool for solving quadratic programming problems, certain terminologies and notations are necessary. These are given below.

Definition 3.9.--The m^{th} decision derivative relative to the current set of state variables (variables that adjust to changes in the decision variables), denoted v_m, indicates whether an increase or a decrease in the value of the corresponding decision variable, d_m, would be desirable.

As noted in the definition, the m^{th} decision derivative indicates the necessary change of value in the corresponding decision variable, d_m, i.e., increase or decrease. The necessary direction of change in the d_m value (increase or decrease) is given by the sign of the v_m.

(1) If $v_m < 0$ increasing the decision variable d_m (while holding all other decision variables constant) will decrease the value of the objective function.

(2) If $v_m > 0$ decreasing the decision variable d_m (while holding all other decision variables constant) will increase the value of the objective function.

The algorithm takes these values and, beginning with an initial solution, iteratively moves through the feasible region. At each iteration the value of the objective function is reduced (for minimization); however, a feasible solution is maintained that satisfies the constraints given by

$$\sum_{n=1}^{k} a_{kn}x_n - x_{n+k} = b_k.$$

In this manner the objective function is decreased (or increased) by adjusting one decision variable at a time. All other decision variables are held constant.

Applied to quadratic programming, the differential algorithm iteratively evaluates the value of each decision derivative at successive stages of the procedure. This algorithm is taken from Wilde and Beightler and is presented here in a manner similar to that of its authors.[38]

Algorithm 3.6 (Differential algorithm of Wilde and Beightler).--Step 1. Define v_i as the smallest decision derivative less than zero. Define v_h as the largest decision derivative such that the corresponding decision variable, d_h, exceeds zero (is positive). ($v_k = \partial y/\partial x_k$, the partial derivative of the objective function with respect to variable k.)

Step 2. If there does not exist some m^{th} decision derivative that is negative, let $v_i = 0$; i.e., if $v_m < 0$, let $v_i = 0$. For every positive decision derivative having decision variables equal to zero, let $v_h = 0$.

Step 3. If $v_i = v_h = 0$, an optimal solution has been found.

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[38] Wilde and Beightler, op. cit., pp. 63-66.

Step 4. Suppose v_i and v_h are not both equal to zero. Then it is necessary to compute a value V defined as follows:

$$V = v_i + v_h.$$

If V is $\begin{bmatrix} \text{nonpositive} \\ \text{positive} \end{bmatrix}$, then $\begin{bmatrix} \text{increase } d_i \\ \text{decrease } d_h \end{bmatrix}$, with all other decision variables held constant until:

I. some state variable, s_p, equals zero; or,

II. $v_r = 0$; if $V > 0$, $r = h$.

if $V \leq 0$, $r = i$.

III. $d_h = 0$.

Step 5. If a constraint is violated by a solution point, it is necessary to change the state set. This is accomplished by the following steps:

(1) solve the violated constraint for the variable leaving the solution;

(2) substitute the result of (1) into the objective function;

(3) determine the new decision derivatives;

(4) reapply Steps 1 - 4.

As a means of illustrating the differential algorithm, a sample problem which incorporates all of the computations is taken from Wilde and Beightler.[39] This example requires that a quadratic function be minimized subject to two linear constraints and nonnegativity of the variables. As a means of fully demonstrating the algorithm, the presentation here is in greater detail than that of Wilde and Beightler.

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[39] Ibid., pp. 73-78.

Determine the values of x_1 and x_2 for which the quadratic function

$$y = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 + 6$$

achieves its minimum value. It is assumed that y is differentiable. The function is restricted by

$$x_1 \geq 0,$$
$$x_2 \geq 0,$$
$$3x_1 + 4x_2 \leq 6,$$
$$-x_1 + 4x_2 \leq 2.$$

Iteration I: Let $\underline{x}^\circ = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ be the initial solution.

Step 1. Let v_i denote the smallest negative decision derivative, $v_i = \frac{\partial y}{\partial x_i}$. Let v_h denote the largest positive decision derivative such that the corresponding decision variable, d_h, is positive. Calculate $v_i = \frac{\partial y}{\partial x_i}$.

$$\min y = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1$$

subject to

$$6 = 3x_1 + 4x_2 + x_3,$$
$$2 = -x_1 + 4x_2 \qquad + x_4,$$

where x_3 and x_4 are arbitrary variables which change each inequality to an equality.

$$v_1 = \frac{\partial y}{\partial x_1} = 4x_1 - 2x_2 - 6;$$

$$v_2 = \frac{\partial y}{\partial x_2} = -2x_1 + 4x_2.$$

For $\underline{x}^\circ = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the initial solution, $v_1 = -6$ and $v_2 = 0$. At $\underline{x}^\circ$, v_3 and v_4, identified by x_3 and x_4, equal 6 and 2, respectively. (See Table 3.17.)

TABLE 3.17

INITIAL SOLUTION

Solution →	0	0			
State Set ↓	x_1	x_2	x_3	x_4	State Set Values
x_3	3	4	1	0	6
x_4	-1	4	0	1	2
y	-6	0	0	0	0
	v_1↑	v_2↑			

Step 2. Examine the results of Step 1. If $v_m < 0$, let $v_i = 0$. For all $v_m > 0$ having decision variables equal to zero, let $v_h = 0$. At $\underline{x}^\circ = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $v_1 = -6$ and $v_2 = 0$. Since v_1 is the smallest negative value, x_1 is the decision variable that is to be adjusted; i.e., $d_1 \equiv x_1$.

Step 3. Since v_i and v_h are not all zero, an optimal solution has not been found. Go to Step 4.

Step 4. Since v_i and v_h are not all zero, calculate V, where

$$V = v_i + v_h.$$

If V is $\begin{bmatrix}\text{nonpositive}\\\text{positive}\end{bmatrix}$, then $\begin{bmatrix}\text{increase } d_i\\\text{decrease } d_h\end{bmatrix}$ while holding all other decision variables constant until

I. some state variable, s_p, = 0; or

II. $v_r = 0$; if $V > 0$, $r = h$;

if $V \leq 0$, $r = i$.

III. $d_h = 0$.

For the first solution, $\underline{x}^{\circ}$,

$$v_i = v_1 = -6;$$

$$v_h = v_2 = 0;$$

$$V = -6.$$

Since $V \leq 0$, the decision variable d_1 must be increased. All other decision variables will be held constant.

For v_1, the initial rate of change with respect to x_1 is given by

$$\frac{\partial y}{\partial x_1} = 4x_1 - 2x_2 - 6.$$

With x_2 held constant at zero, $\underline{x}^{\circ} = \begin{bmatrix}0\\0\end{bmatrix}$, $\frac{\partial y}{\partial x_1}$ is linear in x_1 with rate of change -6. Thus, for $x_2 = 0$,

$$\left.\frac{\partial y}{\partial x_1}\right|_{x_2 = 0} = 4x_1 - 6.$$

Equating $\frac{\partial y}{\partial x_1}$ to zero when $x_2 = 0$, $4x_1 - 6 = 0$ requires that $x_1 - 3/2$. For $x_1 = 3/2$, $v_1 = 0$, and

$$3x_1 + 4x_2 \leq 6 \text{ is still satisfied;}$$

$$-x_1 + 4x_2 \leq 2 \text{ is still satisfied.}$$

When $x_1 = 3/2$ and $x_2 = 0$, the values of x_3 and x_4 are 3/2 and 7/2, respectively. The value of y is -9/2. Thus, at the second trial point, $\underline{x}_1$, where $\underline{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}$,

$$v_1 = 4x_1 - 2x_2 - 6 \qquad = 0;$$
$$v_2 = -2x_1 + 4x_2 \qquad = -3;$$
$$x_3 = 6 - 3x_1 - 4x_2 \qquad = 3/2;$$
$$x_4 = 2 + x_1 - 4x_2 \qquad = 7/2;$$
$$y = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 = -9/2.$$

These results are summarized in Table 3.18.

TABLE 3.18

FIRST ITERATION

Solution → / State Set ↓	3/2 x_1	0 x_2	x_3	x_4	State Set Values
x_3	3	4	1	0	3/2
x_4	-1	4	0	1	7/2
y	0	-3	0	0	-9/2
	v_1↑	v_2↑			

Iteration II: Step 1. Calculate $v_i = \frac{\partial y}{\partial x_i}$, where v_i is evaluated at $\underline{x}_1$.

$$v_1 = \frac{\partial y}{\partial x_1} = 4x_1 - 2x_2,$$

$$v_2 = \frac{\partial y}{\partial x_2} = -2x_1 + 4x_2.$$

At $\underline{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}$, $v_1 = 0$ and $v_2 = -3$.

Step 2. Examine the results of Step 1. If $v_m < 0$, let $v_i = 0$. For all $v_m > 0$ having decision derivatives equal to zero, let $v_h = 0$. At $\underline{x}_1 = \begin{matrix} \frac{3}{2} \\ 0 \end{matrix}$, $v_1 = 0$, $v_2 = -6$, $x_3 = 0$, and $x_4 = 0$. Since $v_2 = -6$ is the smallest negative value, x_2 is the decision variable that is to be adjusted; i.e., $d_2 \equiv x_2$.

Step 3. Since v_i and v_h are not all zero, an optimal solution has not been found. Go to Step 4.

Step 4. Since v_i and v_h are not all zero, calculate V, where

$$V = v_i + v_h.$$

If V is $\begin{bmatrix} \text{nonpositive} \\ \text{positive} \end{bmatrix}$, then $\begin{bmatrix} \text{increase } d_i \\ \text{decrease } d_h \end{bmatrix}$ while holding all other decision variables constant until

I. some state variable, $s_p = 0$; or

II. $v_r = 0$; if $V > 0$, $r = h$;

if $V \leq 0$, $r = i$.

III. $d_h = 0$.

For the second solution, $\underline{x}_1$,

$$v_i = v_2 = -3;$$

$$v_h = v_1 = 0;$$

$$V = -3.$$

Since $V \leq 0$, the decision variable d_2 must be increased. All other decision variables will be held constant.

For v_2, the initial rate of change with respect to x_2 is given by

$$\frac{\partial y}{\partial x_2} = -2x_1 + 4x_2.$$

With x_1 held constant at 3/2, $\underset{\sim}{x}_1 = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}$, $\frac{\partial y}{\partial x_2}$ is linear in x_2 with rate of change equal to -3. Thus, for $x_1 = 3/2$,

$$\left.\frac{\partial y}{\partial x_2}\right|_{x_1 = 3/2} = -3 + 4x_2.$$

Equating $\frac{\partial y}{\partial x_2}$ to zero when $x_1 = 3/2$, $-3 + 4x_2 = 0$ requires that $x_2 = 3/4$. For $x_1 = 3/2$, $x_2 = 3/4$, $v_1 = 0$, $v_2 = 0$, and $3x_1 + 4x_2 \leq 6$ is violated, $-x_1 + 4x_2 \leq 2$ is satisfied. When $x_1 = 3/2$ and $x_2 = 3/4$, the values of x_3 and x_4 are $-\frac{3}{2}$ and $\frac{3}{2}$, respectively. Thus, at the third trial point, $\underset{\sim}{x}_3$, where

$$\underset{\sim}{x}_3 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix},$$

$$v_1 = 4x_1 - 2x_2 - 6 \qquad = -\frac{3}{2};$$

$$v_2 = -2x_1 + 4x_2 \qquad = 0;$$

$$x_3 = 6 - 3x_1 - 4x_2 \qquad = \frac{3}{2};$$

$$x_4 = 2 + x_1 - 4x_2 \qquad = \frac{1}{2};$$

$$y = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 = -\frac{45}{8}.$$

Since a constraint is violated, it is necessary to change the state set. That is, since $x_2 = 3/4$ violates x_3, variable x_3 will enter as a decision variable. Variable x_2 will leave as a state variable. The required procedure is as follows.

(1) Solve the constraint containing x_3, the variable entering the solution, for x_2, the variable leaving the solution. Substitute this into the remaining constraint to eliminate x_2 from that constraint; i.e.,

$$3x_1 + 4x_2 + x_3 = 6$$

yields

$$x_2 = 1/4\ (6 - 3x_1 - x_3);$$

the constraint

$$-x_1 + 4x_2 + x_4 = 2$$

can then be written as

$$-x_1 + (6 - 3x_1 - x_3) + x_4 = 2$$

$$-4x_1 - x_3 + x_4 = -4.$$

(2) Substitute $x_2 = 1/4\ (6 - 3x_1 - x_3$ into the objective function.

$$y = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1$$

$$= 2x_1^2 - 2x_1[\tfrac{1}{4}\ (6 - 3x_1 - x_3)] + 2[\tfrac{1}{4}\ (6 - 3x_1 - x_3)]^2 - 6x_1$$

$$= 2x_1^2 - \frac{x_1}{2}\ (6 - 3x_1 - x_3) + 2[\tfrac{1}{16}\ (6 - 3x_1 - x_3)^2] - 6x_1$$

$$= 2x_1^2 - 3x_1 + \frac{3}{2}x_1^2 + \frac{x_1x_3}{2} + \frac{1}{8}(36 - 36x_1 - 12x_3 + 9x_1^2 + 6x_1x_3 + x_3^2)\ -6x_1$$

$$= \frac{7}{2}\ x_1^2 - 3x_1 + \frac{1}{2}(x_1x_3) + \frac{36}{8} - \frac{36}{8}\ x_1 - \frac{12}{8}\ x_3 + \frac{9}{8}\ x_1^2 + \frac{6}{8}\ x_1x_3$$

$$+ \frac{1}{8}\ x_3^2 - 6x_1$$

$$y = \frac{37}{8}\ x_1^2 - \frac{108}{8}\ x_1 + \frac{10}{8}\ x_1x_3 + \frac{1}{8}\ x_3^2 - \frac{12}{8}\ x_3 + \frac{36}{8}.$$

(3) The new decision derivatives are given by

$$v_1 = \frac{74}{8} x_1 - \frac{108}{8} + \frac{10}{8} x_3;$$

$$v_3 = \frac{10}{8} x_1 + \frac{2}{8} x_3 - \frac{12}{8};$$

$$x_2 = \frac{1}{4}(6 - 3x_1 - x_3); \text{ and}$$

$$x_4 = -4 + x_3 + 4x_1, \text{ where}$$

$$y = \frac{37}{8} x_1^2 - \frac{108}{8} x_1 + \frac{10}{8} x_1 x_3 + \frac{1}{8} x_3^2 + \frac{36}{8}.$$

Iteration III: Step 1. Let $\underline{x}_3 = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}$ be the initial solution for the constructed system. Calculate $v_i = \frac{y}{x_i}$, where v_i is evaluated at $\underline{x}_3$.

$$v_1 = \frac{y}{x_1} = \frac{1}{8}(74x_1 + 10x_3 - 108);$$

$$v_3 = \frac{y}{x_3} = \frac{1}{8}(10x_1 + 2x_3 - 12);$$

at $\underline{x}_3 = \begin{matrix} x_1 \\ x_3 \end{matrix} = \begin{matrix} \frac{3}{2} \\ 0 \end{matrix}$, $v_1 = \frac{3}{8}$ and $v_3 = \frac{3}{8}$.

Step 2. Examine the results of Step 1. If $v_m < 0$, let $v_i = 0$. For all $v_m > 0$ having decision derivatives equal to zero, let $v_h = 0$. At $\underline{x}_3 = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}$, $v_1 = \frac{3}{8}$, $v_3 = \frac{3}{8}$, $x_2 = \frac{3}{8}$, and $x_4 = 2$.

Step 3. Since v_i and v_h are not all zero, an optimal solution has not been found. Go to Step 4.

Step 4. Since v_i and v_h are not all zero, calculate V, where

$$V = v_i + v_h.$$

If V is $\begin{bmatrix}\text{nonpositive}\\ \text{positive}\end{bmatrix}$, then $\begin{bmatrix}\text{increase } d_i\\ \text{decrease } d_h\end{bmatrix}$, while holding all other decision variables constant until

I. some state variable, $s_p = 0$; or

II. $v_r = 0$; if $V > 0$, $r = h$;

if $V \leq 0$, $r = i$;

III. $d_h = 0$.

At the first adjusted solution, $\underline{x}_3$, v_1 and v_3 are both positive. Hence, $V > 0$. Since there is no $v_i < 0$, the variable to be decreased, x_1 or x_3, is arbitrary. However, since $x_3 = 0$, only x_1 can be decreased. The amount of decrease is determined by the ratio

$$\frac{v_1}{t_{11}},$$

where $t_{11} = \frac{\partial^2 y}{\partial x_1^2}$. The required calculations are summarized below.

(1) Compute $t_{11} = \frac{\partial^2 y}{\partial x_1^2}$.

$$t_{11} = \frac{\partial}{\partial x_1} \quad \frac{\partial y}{\partial x_1}$$

$$= \frac{\partial}{\partial x_1}\left[\frac{1}{8}(74x_1 + 10x_2 - 108)\right]$$

$$= \frac{74}{8}$$

$$= \frac{37}{4}.$$

(2) The amount of decrease is determined by the ratio $\frac{v_1}{t_{11}}$. The value of v_1 is $\frac{3}{8}$. From (1), $t_{11} = \frac{37}{4}$. Variable x_1 is to be decreased

by $(\frac{3}{8}) \div (\frac{37}{4})$. This ratio is equal to $\frac{3}{74}$. Therefore, x_1 will be decreased from $\frac{3}{2}$ to $\frac{54}{37}$.

(3) The fourth solution point, $\underset{\sim}{x}_4$, is defined by $\begin{bmatrix} x_1 - \Delta x_1 \\ x_3 \end{bmatrix}$. For $x_1 = \frac{3}{2}$, $x_3 = 0$, and $\Delta x_1 = \frac{3}{74}$,

$$\underset{\sim}{x}_4 = \begin{bmatrix} \frac{54}{37} \\ 0 \end{bmatrix}.$$

At $\underset{\sim}{x}_4$,

$$v_1 = \frac{1}{8}(74x_1 + 10x_3 - 108) = 0;$$

$$v_3 = \frac{1}{8}(10x_1 + 2x_3 - 12) = \frac{12}{37};$$

$$x_2 = \frac{1}{4}(6 - 3x_1 - x_3) = \frac{15}{37};$$

$$x_4 = -4 + x_3 + 4x_1 = \frac{68}{37};$$

$$y = -\frac{198}{37}.$$

The decrease in x_1 (from $\frac{3}{2}$ to $\frac{54}{37}$) has brought about the following:

(1) x_2 increased from 0 to $\frac{15}{37}$;

(2) x_4 decreased from 2 to $\frac{68}{37}$.

In general if a decision variable d_r is to be increased, it is sufficient to consider its effects on those state variables having positive entries in the d_r column; i.e., consider only the state variables s_k such that

$$kr = \frac{\partial f_k}{\partial x_r} > 0.$$

Conversely, if d_r is to be decreased, it is sufficient to consider the effects of this decrease on just those state variables having negative entries in the d_r column. It is necessary to test $\underline{x}_4$ for optimality. Go to Step 2.

Step 2. At $\underline{x}_4$, there is no decision derivative less than zero; i.e., $v_m \nless 0$. Hence, $v_1 = 0 = v_i$. For all $v_m > 0$ with decision variables equal to zero, let $v_h = 0$. At $\underline{x}_4$ $v_3 = \frac{12}{37}$, and its corresponding decision variable $x_3 = 0$. Thus $v_3 = v_h = 0$. Go to Step 3.

Step 3. Since $v_i = v_h = 0$, the solution defined by $\underline{x}_4$ is optimal. The minimum value of y is achieved at $x_1 = \frac{54}{37}$, $x_3 = 0$, $x_2 = \frac{15}{37}$, and $x_4 = \frac{68}{37}$. This minimum value equals $-\frac{198}{37}$. These results are shown in Table 3.19.

TABLE 3.19

FINAL SOLUTION

Solution →	$\frac{54}{37}$		0		
State Set ↓	x_1	x_2	x_3	x_4	State Set Values
x_2	$\frac{3}{4}$	1	$\frac{1}{4}$	0	$\frac{15}{37}$
x_4	-4	0	-1	1	$\frac{68}{37}$
y	0	0	$\frac{12}{37}$	0	$-\frac{198}{37}$
	v_1↑		v_3↑		

Geometric Programming

One of the most recent advances in the techniques of mathematical programming, geometric programming represents a technique by which a class of decision problems is solved by inspection of the defined objective function or by a "relentless exploitation of the properties of inequalities."[40] Developed by Richard J. Duffin and Clarence Zener in 1961, geometric programming is a technique by which problems formulated as posynomials can be solved for points of optima.[41]

As a means of describing the posynomial formulation of the geometric programming problem, consider the following theoretical cost function, $f(\underline{x})$:

$$f(\underline{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} \ldots x_m^{a_{1m}} + c_2 x_1^{a_{21}} x_2^{a_{22}} \ldots x_m^{a_{2m}} + \ldots + c_n x_1^{a_{n1}} x_2^{a_{n2}} \ldots x_m^{a_{nm}}.$$

This expression can be written as

$$f(\underline{x}) = \sum_{i=1}^{n} \left(c_i \prod_{j=1}^{m} x_j^{a}\right).$$

In this formulation $f(\underline{x})$ represents the sum of n component costs. The c_i, ($i = 1, 2, \ldots, n$), are all positive and represent the cost associated with variable x_j, ($j = 1, 2, \ldots, m$). The a_{ij} define the exponents attached to the m^{th} variable with cost c_i and are such that they can be positive

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[40] Richard J. Duffin, Elmor L. Peterson, and Clarence Zener, Geometric Programming--Theory and Application (New York, 1967), p. 1.

[41] The term "posynomial" defines a class of functions described by positive polynomials; i.e., a class of functions such that the coefficients of the terms are positive.

or negative.[42] Functions of the type described by $f(\underline{x})$ are said to be posynomials.

Although geometric programming is generally associated with multivariable, nonlinear functions, this is not its only application. Examination of $f(x)$ reveals that the objective function, for particular values of the a_{ij}, can be written as a linear function. For example,

$$f(x_1, x_2) = 2x_1 + 3x_2$$

can be written as

$$f(x_1, x_2) = 2x_1x_2^{\circ} + 3x_1^{\circ}x_2,$$

with $c_1 = 2$; $c_2 = 3$; $a_{11} = 1$; $a_{12} = a_{21} = 0$; and $a_{22} = 1$. In this form $f(x_1, x_2)$ satisfies the requirements of a posynomial function while maintaining the necessary characteristics of a linear function. The coefficients of the terms are positive, and the function $f(x_1, x_2)$ defines a power function in x_1 and x_2.

Although maximization problems can be solved using the technique of geometric programming, the primary application has been in solving problems such that the objective function is to be minimized. This is attributed to the fact that its origin and subsequent development stems from the minimization of posynomial cost functions. In this application (cost minimization), geometric programming distributes the totality of the summed components (costs) among the various terms of the objective

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[42]Negative values of the exponents (a_{ij}) indicate the existence of an inverse relationship. Examples of such functions include demand functions and income distribution functions, a particular one of which is Pareto's law of distribution function $N = a/x^b$. This function defines the number of individuals N from a given population of size a whose income exceeds x. The value of b is a known population parameter, generally assigned the value of 1.5.

function. Given these initial optimal allocations, generally obtained by inspection of a set of linear equations, the minimal (optimal) solution usually follows via some simple calculations.[43] These calculations involve the evaluation of the objective function at the initial solutions and either initiating the action necessary to implement the decision or modifying the objective in order to achieve workable (feasible) solutions. The first case represents the true optimal solution; the second case represents a sub-optimal solution.

Geometric programming can be used in the evaluation of designs, projects, products, etc., prior to the actual commitment of productive resources. This affords management the opportunity to find optimal solutions to problems (for example, budget allocations) when knowledge of the policy to be used is not yet known. This is accomplished whenever the objective function itself and all existing constraints are polynomials in the independent variables.[44] The necessary form is set forth in Definition 3.10.

Definition 3.10.--Let m denote the number of independent variables. Let n denote the number of terms in the cost function. Let p denote the number of constraints imposed on the objective function. Let n_p denote the number of terms in the objective function. Let x_j denote the j^{th} independent variable. Let a_{ij} denote the exponent of the independent variable x_j, with cost c_i. The geometric programming problem is defined by

[43] Wilde and Beightler, op. cit., p. 28.

[44] Ibid., p. 99.

$$\min f(x) = \sum_{i=1}^{n_o} \left(c_i \prod_{j=1}^{m} x_j^{a_{ij}}\right)$$

subject to

$$g_1(x) = \sum_{i=n_o+1}^{n_1} \left(c_i \prod_{j=1}^{m} x_j^{a_{ij}}\right) \leq 1,$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$g_p(x) = \sum_{i\,=\,n\,+1}^{n_p} \left(c_i \prod_{j=1}^{m} x_j^{a_{ij}}\right) \leq 1,$$

where

$$x_{ij} \geq 0, \quad j = 1, 2, \ldots, n_p,$$

$$c_i > 0, \quad i = 1, 2, \ldots, m, \text{ and}$$

$$a_{ij} = \text{any arbitrary power.}$$

Definition 3.10 defines the class of problem initially investigated by Zener and Duffin. Their investigation led to the conclusion that "the sum of component costs sometimes may be minimized almost by inspection when each cost depends on products of the design variables, each raised to arbitrary but known powers."[45]

The expanded form of Definition 3.10 can be used to point out some of the distinguishing features of the geometric programming problem. This expansion takes on the form

$$\min f(x_1, x_2, \ldots, x_m) = c_1 x_1^{a_{11}} x_2^{a_{12}} \ldots x_m^{a_{1m}} + c_2 x_1^{a_{21}} x_2^{a_{22}} \ldots x_m^{a_{2m}} + \ldots + c_{n_o} x_1^{a_{n_o1}} x_2^{a_{n_o2}} \ldots x_m^{a_{n_om}}$$

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[45] Ibid., p. 100.

subject to

$$g_1(x) = c_{n_0+1} x^{a_{n_0+1,1}} x_2^{a_{n_0+1,2}} \ldots x_m^{a_{n_0+1,m}} + c_{n_0+2} x_1^{a_{n_0+2,1}} x_2^{a_{n_0+2,2}} \ldots x_m^{a_{n_0+1,m}}$$

$$+ \ldots + c_{n_1} x_1^{a_{n_1,1}} x_2^{a_{n_1,2}} \ldots x_m^{a_{n_1,m}} \leq 1,$$

. . . .

$$g_p(x) = c_{n_{p-1}+1} x_1^{a_{n_{p-1}+1,1}} x_2^{a_{n_{p-1}+1,2}} \ldots x_m^{a_{n_{p-1}+1,m}} + c_{n_{p-1}+2} x_1^{a_{n_{p-1}+2,1}} x_2^{a_{n_{p-1}+2,2}} \ldots x_m^{a_{n_{p-1}+2,m}}$$

$$+ \ldots + c_{n_p} x_1^{a_{n_p,1}} x_2^{a_{n_p,2}} \ldots x_m^{a_{n_p,m}} \leq 1,$$

and is of such a nature that

(1) the objective function, $f(x_1, x_2, \ldots, x_m)$, is composed of a sum of values, each of which is given by a power function in the variables x_j;

(2) the constraint functions are of the same form as the objective function;

(3) the function need not be constrained in order to apply the solution techniques of geometric programming since the primary consideration centers on the optimal distribution of some commodity or resource among the various terms of the objective function; and,

(4) all constraint functions must be represented by inequalities rather than equalities, the constraint function having a maximum value of unity.

In addition to describing the makeup of the geometric programming problem, the preceding discussion has briefly outlined the basic methodology of the minimization process employed by geometric programming, has pointed to potential application and use, and has defined the basic geometric programming problem. Basic features of the geometric programming problem have been identified and the notational aspects demonstrated. This initial insight makes it feasible to begin an exploratory study of geometric programming. This study will be accomplished by discussing five basic topics, (1) the geometric-mean-inequality, (2) orthogonal conditions, (3) duality theory: minimization of posynomials and maximization of product functions, (4) computational techniques: unconstrained primal, and (5) computational techniques: constrained primal.

Geometric-mean-inequality.--The underlying basis of geometric programming is the geometric-arithmetic mean inequality which states that the arithmetic mean is at least as great as the geometric mean. This relationship permits the determining of lower bounds for posynomial functions and is defined as follows.

Definition 3.11.--Let x_i denote the i^{th} variable, i = 1, 2,...,N. Let $\prod_{i=1}^{N} x_i$ define the product of the N variables. Then the geometric-arithmetic mean inequality is given by

$$\left[\prod_{i=1}^{N} x_i\right]^{\frac{1}{N}} \leq \frac{1}{N} \sum_{i=1}^{N} x_i .$$

A cursory examination of Definition 3.11 reveals that the left side of the inequality, $\left[\prod_{i=1}^{N} x_i\right]^{\frac{1}{N}}$, defines the geometric mean. The right side of the inequality, $\frac{1}{N} \sum_{i=1}^{N} x_i$, defines the arithmetic mean. The combining of these two resulted in the term "geometric-mean-inequality."

When applied to a general summation (for example, $c_1u_1 + c_2u_2+\ldots+c_nu_n$), the use of the geometric mean inequality results in

$$c_1u_1 + c_2u_2+\ldots+c_nu_n \geq u_1^{c_1} u_2^{c_2} \ldots u_n^{c_n}.$$

The u_i values ($i = 1, 2,\ldots,n$) are defined as aribtrary, non-negative numbers. The c_i values ($i = 1, 2,\ldots,n$) are defined as arbitrary weights selected in such a way that $\sum_{i=1}^{n} c_i = 1$. The relationship

$$c_1u_1 + c_2u_2+\ldots+c_nu_n \geq u_1^{c_1} u_2^{c_2} \ldots u_n^{c_n}$$

can then be written in the form

$$\sum_{i=1}^{n} c_iu_i \geq \prod_{i=1}^{n} u_i^{c_i}.$$

Since the original function is to be minimized and the sum of the weights (the c_i values) must equal one, direct application of this relationship generally requires a change of variables. This change of variables is accomplished by defining

$$x_1 = c_1u_1;\ x_2 = c_2u_2;\ldots;x_n = c_nu_n.$$

Solving for the u_i and substituting the result into the geometric-mean inequality yields

$$x_1 + x_2+\ldots+x_n \geq \left(\frac{x_1}{c_1}\right)^{c_1} \left(\frac{x_2}{c_2}\right)^{c_2} \ldots \left(\frac{x_n}{c_n}\right)^{c_n}.$$

This relationship can then be written as

$$\sum_{i=1}^{n} x_i \geq \prod_{i=1}^{n} \left(\frac{x_i}{c_i}\right)^{c_i}.$$

In the solution techniques to follow, the result just obtained is of primary importance as it is the foundation upon which they are based. Recollection of the initial investigation of Zener and Duffin reveals the necessity for the posynomial formulation: positive coefficients are necessary because they are raised to fractional powers in the geometric inequality.

Orthogonality and normality.--Two stipulations placed upon the arbitrary weights (δ_j) of the geometric programming problem are such that these weights satisfy the following set of conditions:

$$\sum_{j=1}^{p} \delta_j = 1; \text{ and,}$$

$$\sum_{j=1}^{p} a_{kj}\delta_j = 0, \quad k = 1, 2, \ldots, N.$$

The first condition, $\sum_{j=1}^{p} \delta_j = 1$, is defined as the normality condition; the second condition, $\sum_{j=1}^{p} a_{kj}\delta_j = 0$ for $k = 1, 2, \ldots, N$, is defined as the orthogonality condition. Both of these conditions result from the classical max-min calculus approach to obtaining the maximum or minimum solution of a multivariable objective function (i.e., setting the n partial derivatives equal to zero and solving the resulting system of equations).[46]

[46] Duffin, Peterson, and Zener, op. cit., pp. 2-5; Gue and Thomas, op. cit., pp. 143-145.

In this formulation[47] the function to be optimized is given by

$$\min f(\underline{x}) = \sum_{j=1}^{P} c_j P_j(\underline{x}),$$

with $c_j > 0$ and $P_j(\underline{x}) = \prod_{i=1}^{N} x_i^{a_{ij}}$. The δ_j are defined by the transformation

$$\delta_j = \frac{c_j P_j}{f(\underline{x}^\circ)}, \quad j = 1, 2, \ldots, P,$$

with $f(\underline{x}^\circ)$ representing the optimal (minimal) solution. The result of the partial differentiation is a homogeneous system of equations (i.e., a system such that all equations are equated to zero) given by

$$\frac{1}{x_k} \sum_{j=1}^{P} a_{kj} c_j P_j = 0.$$

Each $c_j > 0$, and each $x_k > 0$. Because of this $f(\underline{x}^\circ)$, the value of the objective function at the vector $\underline{x}^\circ$, will be positive. Therefore,

$$\frac{1}{x_k} \sum_{j=1}^{P} a_{kj} c_j P_j = 0$$

can be written in the form

$$\sum_{j=1}^{P} a_{kj} c_j P_j = 0.$$

This in turn can be written as

$$\sum_{j=1}^{P} \frac{a_{kj} c_j P_j}{f(\underline{x}^\circ)} = 0.$$

For $\delta_j = \frac{c_j P_j}{f(\underline{x}^\circ)}$, $\sum_{j=1}^{P} \frac{a_{kj} c_j P_j}{f(\underline{x}^\circ)}$ reduces to $\sum_{j=1}^{P} a_{kj} \delta_j = 0$, the orthogonal requirement.

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[47] Gue and Thomas, op. cit., pp. 145-146.

If the δ_j are summed,

$$\sum_{j=1}^{p} \delta_j = \sum_{j=1}^{p} \frac{c_j P_j}{f(\underline{x}^o)} ,$$

the normality condition results; i.e., $\sum_{j=1}^{p} \delta_j = 1$. This is true because

$$\sum_{j=1}^{p} \frac{c_j P_j}{f(\underline{x}^o)} = \frac{1}{f(\underline{x}^o)} \cdot \sum_{j=1}^{p} c_j P_j$$

must equal unity at the optimal solution.[48]

Gue and Thomas[49] have reduced this dual requirement of orthogonality and normality to the point that direct use is made of matrix theory. Their work defines the coefficient matrix as

$$\underline{A} = \begin{bmatrix} 1 & \cdots & 1 \\ a_{11} & \cdots & a_{1P} \\ \vdots & & \vdots \\ a_{N1} & \cdots & a_{NP} \end{bmatrix} ,$$

the matrix of weights as

$$\underline{\delta} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_p \end{bmatrix} ,$$

and the matrix of constants as

$$\underline{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

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[48] Ibid., p. 146.

[49] Ibid., pp. 146-148.

In this form the a_{ij} values are given by the exponents of the product function

$$P_j(x) = \prod_{i=1}^{N} x_i^{a_{ij}}, \quad j = 1, 2, \ldots, P,$$

and the δ_j values ($j = 1, 2, \ldots, P$) are treated as unknowns. The values in the matrix of constants are those constants required to satisfy the conditions of orthogonality and normality. In order to solve the geometric programming problem, it is necessary to solve

$$\underline{A\delta} = \underline{b}$$

for δ. The set of δ values satisfying $\underline{A\delta} = \underline{b}$ is the set of δ values which solves the linear, nonhomogeneous equations defining the weights for the function to be minimized.

The set of simultaneous equations defined by

$$\underline{A\delta} = \underline{b}$$

is given by

$$\begin{aligned}
\delta_1 + \delta_2 + \ldots + \delta_P &= 1 \\
a_{11}\delta_1 + a_{12}\delta_2 + \ldots + a_{1P}\delta_P &= 0 \\
\ldots\ldots\ldots\ldots\ldots \\
a_{N1}\delta_1 + a_{N2}\delta_2 + \ldots + a_{NP}\delta_P &= 0
\end{aligned}$$

and is of order N x P; i.e., the system is composed of N equations with P variables. This system has as its augmented matrix

$$(\underline{A}, \underline{b}) = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ a_{11} & a_{12} & \ldots & a_{1P} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{N1} & a_{N2} & \ldots & a_{NP} & 0 \end{bmatrix}$$

and can be analyzed by the techniques of matrix theory. This analysis will yield one of three possibilities.

(1) The rank of the augmented matrix $(\underline{A}, \underline{b})$ exceeds the rank of the coefficient matrix $(\underline{A})$. In this case the system is inconsistent and has no solution. This indicates that the original function

$$f(\underline{x}) = \sum_{j=1}^{p} c_j P_j(x), \quad P_j(x) = \prod_{i=1}^{n} x_i^{a_{ij}},$$

has no vector $\underline{x} > 0$ for which it achieves a minimum.

(2) The rank of the augmented matrix $(\underline{A}, \underline{b})$ equals the rank of the coefficient matrix $(\underline{A})$, and the matrix $(\underline{A})$ is square. In this case the system is consistent and has a unique solution. When this results the solution vector δ is obtained by pre-multiplying $\underline{A\delta} = \underline{b}$ by A^{-1}; i.e.,

$$\begin{aligned} \underline{A\delta} &= \underline{b} \\ (\underline{A}^{-1}\underline{A})\underline{\delta} &= \underline{b} \\ \underline{I\delta} &= \underline{A}^{-1}\underline{b} \\ \underline{\delta} &= \underline{A}^{-1}\underline{b}. \end{aligned}$$

The optimal solution is expressed in terms of the δ_j values and is obtained following some algebraic manipulation.

(3) The rank of the coefficient matrix $(\underline{A})$ is less than the number of unknowns or the number of unknowns exceeds the number of equations plus one; i.e., $r(\underline{A}) < P$ or $P > N + 1$. In this case the given system of equations has an infinite number of solutions. This condition requires that additional work be done to locate the global minimum.

Duality.--Although the development of geometric programming is rooted to the minimizing of a restricted class of functional representations, the work of Duffin has provided a developed theory of duality for geometric programming. This duality theory, when applied to geometric programming, is based upon the observation that

> if the original "primal" function [y] were considered a weighted arithmetic mean. . .and the dual function [d] a weighted geometric mean of the same quantities. . . then Cauchy's inequality. . .gives $y \geq d$ with equality only when all quantities in parenthesis [the items being weighted] are equal.[50]

For example, consider the theoretical cost function

$$y = c_1x_1 + c_2x_1x_2^{-2} + c_3x_2^{-1}.$$

The primal form of this function is given by

$$y = w_1\left(\frac{c_1x_1}{w_1}\right) + w_2\left(\frac{c_2x_1x_2^{-1}}{w_2}\right) + w_3\left(\frac{c_3x_2^{-1}}{w_3}\right)$$

with w_1, w_2, and w_3 as arbitrary weights. This primal form can be written in the dual form

$$d = \left(\frac{c_1x_1}{w_1}\right)^{w_1}\left(\frac{c_2x_1x_2^{-1}}{w_2}\right)^{w_2}\left(\frac{c_3x_2^{-1}}{w_3}\right)^{w_3}.$$

The primal function is equal to the dual function only when

$$\frac{c_1x_1}{w_1} = \frac{c_2x_1}{w_2x_2} = \frac{c_3}{w_3x_2} .$$

Thus the primal function can be solved by maximizing the dual with respect to the weights. In addition any choice of weights in the dual can be used

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[50]Wilde and Beightler, *op. cit.*, pp. 100-101.

as a means of providing a lower bound on the optimum value of the primal.[51]

Examination of the dual function formulated in the preceding example reveals a major characteristic of the dual geometric programming problem: whereas the primal problem (minimization) requires a posynomial expression, the dual problem requires a product expression of the form

$$f(\underline{x}) = \prod_{i=1}^{n} c_i x_i^{a_i}.$$

The dual formulation of the primal problem, written as a function of the weights w_t and the coefficients c_t, takes on the form

$$d(\underline{w}) = \prod_{t=1}^{T} \left(\frac{c_t}{w_t}\right)^{w_t}.$$

Minimization is achieved by maximizing $d(\underline{w})$ with respect to the weights and requiring that the weights satisfy the conditions imposed by the requirements of orthogonality and normality.[52]

An important point to note is that the dual problem defined by $d(\underline{w})$ represents the dual of an unconstrained minimization problem. When the primal problem is constrained, the dual formulation is modified to account for the restrictive functions. This is accomplished by direct application of two basic programs developed by Duffin, Peterson, and Zener.[53]

Primal program.--Find the minimum value of a function $g_0(x)$ subject to the constraints $x_1 > 0$, $x_2 > 0, \ldots, x_m > 0$ and $g_1(x) \leq 1$, $g_2(x) \leq 1, \ldots,$

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[51] Ibid.

[52] The theoretical development, including proofs, is given by Duffin, Peterson, and Zener, op. cit., pp. 77-122, 164-226.

[53] Ibid., pp. 78-88.

$g_p(x) \leq 1$. The function $g_k(x)$ is defined by

$$g_k(x) = \sum_{i=J[k]} c_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_m^{a_{im}}, \quad k = 0, 1, \ldots, p.$$

$J[k]$ is the set $\{m_k, m_k+1, m_k+2, \ldots, n_k\}$, $k = 0, 1, 2, \ldots, p$, and $m_o = 1$, $m_1 = n_o + 1$, $m_2 = n_1 + 1, \ldots, m_p = n_{p-1} + 1$, $n_p = n$. The a_{ij} are arbitrary real numbers. The c_i are positive, thus defining the $g_i(x)$ to be posynomials.

In the primal program $g_o(x)$ is defined as the primal function. The variables x_j, $(j = 1, 2, \ldots, m)$, are defined as primal variables. The nature of the application which restricts the x_j to nonzero positive values is defined as a natural constraint. The requirement that the $g_k(x)$ be at most one (i.e., $g_k(x) \leq 1$) for all k is said to be a forced constraint. The expanded form of the primal program is identical to the form shown in Definition 3.10; i.e.,

$$\min g_o(x) = \sum_{i=J[0]} c_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_m^{a_{im}},$$

where $J[0] = \{m_0, m_0 + 1, m_0 + 2, \ldots, n_0\} = \{1, 2, 3, \ldots, n_0\}$, since $m_0 = 1$. This function is constrained by

$$g_1(x) = \sum_{i=J[1]} c_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_m^{a_{im}} \leq 1,$$

$$g_2(x) = \sum_{i=J[2]} c_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_m^{a_{im}} \leq 1,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$g_p(x) = \sum_{i=J[p]} c_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_m^{a_{im}} \leq 1,$$

where

$J[1] = \{m_1, m_1+1, \ldots, n_1\} = \{n_o+1, n_o+2, \ldots, n_1\}$, since $m_1 = n_o+1$;

$J[2] = \{m_2, m_2+1, \ldots, n_2\} = \{n_1+1, n_1+2, \ldots, n_2\}$, since $m_2 = n_1+1$;

. . . .

$J[p] = \{m_p, m_p+1, \ldots, n_p\} = \{n+1, n+2, \ldots, n\}$, since $m_p = n_{p-1}+1$, and $n_p = n$.

Dual program.--Find the maximum value of a product function $v(\delta)$ subject to the linear constraints $\delta_1 \geq 0$, $\delta_2 \geq 0, \ldots, \delta_n \geq 0$, $\sum_{i=J[0]} \delta_i = 1$, and $\sum_{i=1}^{n} a_{ij}\delta_i = 0$, $j = 1, 2, \ldots, m$. The function $v(\delta)$ is defined by

$$v(\underset{\sim}{\delta}) = \left[\prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i}\right] \prod_{k=1}^{p} \lambda_k(\underset{\sim}{\delta})^{\lambda_k(\underset{\sim}{\delta})},$$

where $\lambda_k(\underset{\sim}{\delta}) = \sum_{i=J[k]} \delta_i$, $k = 1, 2, \ldots, p$; $\underset{\sim}{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$. $J[k]$ is the set $\{m_k, m_k+1, m_k+2, \ldots, n_k\}$, $k = 0, 1, 2, \ldots, p$, and $m_o = 1$, $m_1 = n_o+1, \ldots, m_p = n_{p-1} + 1$, $n_p = n$. The c_i are positive and the coefficients a_{ij} are real numbers.

In the dual formulation, $v(\underset{\sim}{\delta})$ is defined as the dual function. The δ_i are defined as dual variables and represent the weight associated with the dual function. Inspection of the dual program reveals that the requirement $\sum_{i=J[0]} \delta_i = 1$ is the normality condition. The requirement that

$$\sum_{i=1}^{n} a_{ij}\delta_i = 0, \quad j = 1, 2, \ldots, m,$$

is the orthogonal condition. The expanded form of the dual program is given by

$$\max \gamma(\underset{\sim}{\delta}) = \left[\left(\frac{c_1}{\delta_1}\right)^{\delta_1}\left(\frac{c_2}{\delta_2}\right)^{\delta_2}\cdots\left(\frac{c_n}{\delta_n}\right)^{\delta_n}\right]\left[\lambda_1(\underset{\sim}{\delta})^{\lambda_1(\underset{\sim}{\delta})}\lambda_2(\underset{\sim}{\delta})^{\lambda_2(\underset{\sim}{\delta})}\cdots\lambda_p(\underset{\sim}{\delta})^{\lambda_p(\underset{\sim}{\delta})}\right]$$

where

$$\lambda_1(\underset{\sim}{\delta}) = \sum_{i=J[1]} \delta_i = \delta_{n_0+1} + \delta_{n_0+2}+\cdots+\delta n_1;$$

$$\lambda_2(\underset{\sim}{\delta}) = \sum_{i=J[2]} \delta_i = \delta_{n_1+1} + \delta_{n_1+2}+\cdots+\delta n_2;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda_p(\underset{\sim}{\delta}) = \sum_{i=J[p]} \delta_i = \delta_{n_{p-1}+1} + \delta_{n_{p-1}+2}+\cdots+\delta n_p.$$

The dual function is restricted by

$$\sum_{i=J[0]} \delta_i = \delta_1 + \delta_2+\cdots+\delta_{n_0} = 1,$$

and

$$\sum_{i=1}^{n} a_{ij}\delta_i = 0, \quad j = 1, 2, \ldots, m.$$

The expansion of $\sum_{i=1}^{n} a_{ij}\delta_i = 0$, $(j = 1, 2, \ldots, m)$, results in the following system of homogeneous linear equations:

$$a_{11}\delta_1 + a_{21}\delta_2 + a_{31}\delta_3+\cdots+a_{n1}\delta_n = 0$$

$$a_{12}\delta_1 + a_{22}\delta_2 + a_{32}\delta_3+\cdots+a_{n2}\delta_n = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{1m}\delta_1 + a_{2m}\delta_2 + a_{3m}\delta_3+\cdots+a_{mn}\delta_n = 0.$$

Simultaneous inspection of the primal and dual programs reveals the following: (1) the c_i in the dual function are taken from the primal function (for example, c_1 in the primal is also c_1 in the dual); (2) the i^{th} weight in the dual, δ_i, corresponds to the i^{th} term of the primal, $c_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_m^{a_{im}}$, a result which indicates that there will be as many weights in the dual program as there are terms in the primal program; (3) the correspondence between the weights of the dual and the terms of the primal is such that the correspondence is one-to-one: each term of the primal program corresponds to one and only one of the weights of the dual; (4) the factors defined by $\lambda_k(\underline{\delta})^{\lambda_k(\underline{\delta})}$ in the dual function are taken from the primal constraint functions: each $\lambda_k(\underline{\delta})^{\lambda_k(\underline{\delta})}$ factor is taken from each of the $g_k(\underline{x}) \leq 1$ constraints; and (5) the a_{ij} values for $\sum_{i=1}^{n} a_{ij}\delta_i = 0$, $(j = 1, 2, \ldots, m)$, are taken from the exponents of the primal program. Of particular note is the fact that the factor defined by $\lambda_k(\underline{\delta})^{\lambda_k(\underline{\delta})}$ is not contained in the primal function. This is due to the normality condition imposed on $\lambda_0(\underline{\delta})$. The normality condition is the only part of the dual program that distinguishes the primal function $g_0(\underline{x})$ from the posynomials $g_k(\underline{x})$, $k = 1, 2, \ldots, p$ appearing in the forced constraints. In addition $x^x = x^{-x} = 1$ for $x = 0$ is assumed.[54]

As an example of the relationship which exists between the primal program and the dual program, consider the following cost minimization

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[54] Ibid., p. 79.

problem:

$$\min g_0(\underline{x}) = 40x_1\ x_2\ x_3\ + 20x_1x_3 + 20x_1x_2x_3$$

subject to

$$g_1 = \frac{1}{3}\,x_1\ x_2\ + \frac{4}{3}\,x_2\ x_3\ \leq 1,$$

$$x_1,\ x_2,\ x_3 \geq 0.$$

According to the primal program, this problem has the form

$$\min g_0(\underline{x}) = c_1x_1^{a_{11}}\,x_2^{a_{12}}\,x_3^{a_{13}} + c_2x_1^{a_{21}}\,x_2^{a_{22}}\,x_3^{a_{23}} + c_3x_1^{a_{31}}\,x_2^{a_{32}}\,x_3^{a_{33}}$$

subject to

$$g_1(\underline{x}) = c_{n_0+1}x_1^{a_{n_0+1\,1}}\,x_2^{a_{n_0+1\,2}}\,x_3^{a_{n_0+1\,3}} + c_{n_0+2}x_1^{a_{n_0+2\,1}}\,x_2^{a_{n_0+2\,2}}\,x_3^{a_{n_0+2\,3}} + c_{n_0+3}x_1^{a_{n_0+3\,1}}\,x_2^{a_{n_0+3\,2}}\,x_3^{a_{n_0+3\,3}}$$
$$\leq 1.$$

The values of the c_i and the a_{ij} values can be obtained by equating like coefficients and exponents. This results in the following set of values:

$c_1 = 40;\ c_2 = 20;\ c_3 = c_{n_0} = 20;\ c_{n_0+1} = c_4 = \frac{1}{3};\ c_{n_0+2} = c_5 = \frac{4}{3};$

$a_{11} = -1;\ a_{12} = -\frac{1}{2};\ a_{13} = -1;\ a_{21} = 1;\ a_{22} = 0;\ a_{23} = 1;\ a_{31} = 1;\ a_{32} = 1;$

$a_{33} = 1;\ a_{n_0+1\,1} = a_{41} = -2;\ a_{n_0+1\,2} = a_{42} = -2;\ a_{n_0+1\,3} = a_{43} = 0;\ a_{n_0+2\,1} = a_{51} = 0;$

$a_{n_0+2\,2} = a_{52} = \frac{1}{2};\ a_{n_0+2\,3} = a_{53} = -1.$

With these values the dual function can be written by direct application of the dual program

$$\max v(\underline{\delta}) = \left[\prod_{i=1}^{n}\left(\frac{c_i}{\delta_i}\right)^{\delta_i}\right]\prod_{i=1}^{p}\lambda_k(\underline{\delta})^{\lambda_k(\underline{\delta})}$$

subject to

$$\sum_{i=J[0]}\delta_i = 1;$$

$$\sum_{i=1}^{n} a_{ij}\delta_i = 0, \quad j = 1, 2, 3, \ldots, m;$$

$$\lambda_k(\underset{\sim}{\delta}) = \sum_{i=J[k]} \delta_i .$$

Since the primal program has a total of five terms (three in the primal function and two in the constraint function), n = 5. Since there is but one $g_k(\underset{\sim}{x}) \leq 1$ constraint, p = 1. The dual program is then given by

$$\max\ v(\underset{\sim}{\delta}) = \left[\prod_{i=1}^{5}\left(\frac{c_i}{\delta_i}\right)^{\delta_i}\right]\lambda_1(\underset{\sim}{\delta})^{\lambda_1(\underset{\sim}{\delta})}$$

subject to

$$\sum_{i=J[0]} \delta_i = 1;$$

$$\sum_{i=1}^{5} a_{ij}\delta_i = 0, \quad j = 1, 2, \ldots, m.$$

$$\lambda_1(\underset{\sim}{\delta}) = \sum_{i=J[1]} \delta_i .$$

The set of values defined by J[0] is {1, 2, 3}. The set of values defined by J[1] is {4, 5}. Therefore, the expanded form of the dual problem is given by

$$\max\ v(\underset{\sim}{\delta}) = \left(\frac{c_1}{\delta_1}\right)^{\delta_1}\left(\frac{c_2}{\delta_2}\right)^{\delta_2}\left(\frac{c_3}{\delta_3}\right)^{\delta_3}\left(\frac{c_4}{\delta_4}\right)^{\delta_4}\left(\frac{c_5}{\delta_5}\right)^{\delta_5}(\delta_4 + \delta_5)^{\delta_4 + \delta_5}$$

subject to

$$\begin{aligned}
\delta_1 + \delta_2 + \delta_3 &= 1 \\
a_{11}\delta_1 + a_{21}\delta_2 + a_{31}\delta_3 + a_{41}\delta_4 + a_{51}\delta_5 &= 0 \\
a_{12}\delta_1 + a_{22}\delta_2 + a_{32}\delta_3 + a_{42}\delta_4 + a_{52}\delta_5 &= 0 \\
a_{13}\delta_1 + a_{23}\delta_2 + a_{33}\delta_3 + a_{43}\delta_4 + a_{53}\delta_5 &= 0.
\end{aligned}$$

For $k = 1$, $\lambda_k(\underline{\delta}) = \lambda_1(\underline{\delta}) = \sum_{i=J[1]} \delta_i = \sum_{i=\{4,\,5\}} \delta_i = (\delta_4 + \delta_5)$. With three variables, $j = 3$. Direct substitution of the c_k and a_{ij} values yields the following:

$$\max v(\underline{\delta}) = \left(\frac{40}{\delta_1}\right)^{\delta_1}\left(\frac{20}{\delta_2}\right)^{\delta_2}\left(\frac{20}{\delta_3}\right)^{\delta_3}\left(\frac{1/3}{\delta_4}\right)^{\delta_4}\left(\frac{4/3}{\delta_5}\right)^{\delta_5}(\delta_4 + \delta_5)^{(\delta_4 + \delta_5)}$$

subject to

$$\begin{aligned}
\delta_1 + \delta_2 + \delta_3 &= 1 \\
-\delta_1 + \delta_2 + \delta_3 - 2\delta_4 &= 0 \\
-\tfrac{1}{2}\delta_1 + \delta_3 - 2\delta_4 + \tfrac{1}{2}\delta_5 &= 0 \\
-\delta_1 + \delta_2 + \delta_3 - \delta_5 &= 0
\end{aligned}$$

In order to complete the problem and find the minimum value of the primal problem, it is necessary to find those values of δ_1, δ_2, δ_3, δ_4, and δ_5 which maximize $v(\underline{\delta})$ and, at the same time, satisfy the constructed linear system. This can be accomplished by performing a series of row operations on the given system in order to reduce it to row equivalent form. Since the system has more unknowns than equations, it will either be inconsistent (no solution) or it will have dependent solutions.

From a computational point of view, geometric programming has no specific solution technique such as the simplex algorithm for linear programming. However, certain characteristics can be attributed to existing methods of obtaining the optimal solution to a given problem. These characteristics include the following: (1) rewriting the primal function as the dual and solving the dual; (2) application of the calculus of maxima-minima to determine the optimal weights for the dual function; (3) application of matrix algebra to solve a linear system of simultaneous equations written as a function of the weights; and, given the optimal assignment of weights, (4) determining the policy assignment that satisfies those weights.

Although the calculus can be used to determine the weight assignment in both the constrained and the unconstrained case, the primary application is made when the function to be optimized is unconstrained. This is due to the fact that the dual of a minimizing posynomial is a product function and is suitable for the use of the partial derivative concept of maxima-minima. In this application the dual is maximized with respect to the weights. Once the weight assignment is known, the values of the primal variables can be suitably determined.

Computation technique: unconstrained primal.--In the unconstrained case, the primal function is defined by an m-variable function of the following type:

$$\min f(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{n} c_i \prod_{j=1}^{m} x_j^{a_{ij}}, \quad c_i, x_j \geq 0.$$

This function has as its dual

$$\max g(\underline{w}) = \prod_{t=1}^{T} (c_t/\delta_t)^{\delta_t}, \quad c_t, \delta_t > 0.$$

For $\delta_t = 0$, $(\delta_t)^{-1}$ is defined as unity, thus guaranteeing continuity.

Although it is possible to differentiate the unconstrained primal function and locate the values of x_j that minimize the primal,[55] the most feasible approach is to maximize the dual function subject to the conditions on the w_t values imposed by normality and orthogonality. This is because the direct application of the max-min calculus will normally require solving a system of simultaneous nonlinear equations for a vanishing first derivative.[56] Solving the dual requires solving a linear system of simultaneous equations for the appropriate weights. These weights are then used to evaluate the primal variables by applying the relation

$$\delta_j = c_j P_j / f(\underline{x}_\circ).$$

The $c_j P_j$ defines the j^{th} term of the primal function and is taken directly from the defined primal. The value of $f(\underline{x}_\circ)$ is the minimal value of the primal function.

As noted, the most feasible approach for solving the unconstrained primal function is to solve the dual. Since the dual is maximized by solving the simultaneous linear system for the weights, it is possible to achieve a solution set that is dependent. That is, n - k of the n weights are defined in terms of k independent weights. When this occurs,

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[55] $x^x = x^{-x} = 1$ for $x = 0$. This guarantees continuity.

[56] Wilde and Beightler, *op. cit.*, p. 28.

the dual function is maximized by the technique of the differential calculus.

As an example of this technique, consider a two product inventory problem in which total inventory cost per shipment is to be minimized. It is necessary to determine the quantity of each product to be produced per run if (1) the carrying cost per unit of time for product x_1 is \$10; (2) the carrying cost per unit of time for product x_2 is \$20; (3) the total number of pieces of product x_1 and x_2 to be supplied in a given time period is 500 units; and (4) production of product x_1 coincides with that of product x_2.

The problem to be minimized consists of three component costs: the average carrying cost of product x_1, the average carrying cost of product x_2, and the setup costs for products x_1 and x_2. It is assumed that the time period has a unit interval of 1 and that there is but one setup cost for both x_1 and x_2. With these assumptions the problem can be described by the following functional expression:

$$\min f(x_1, x_2) = \frac{c_1}{2} x_1 + \frac{c_2}{2} x_2 + \frac{Nc_{12}}{x_1 x_2} ,$$

where c_1 is the carrying cost per unit of time for product x_1, c_2 is the carrying cost per unit of time for product x_2, c_{12} is the setup cost per unit of time for products x_1 and x_2, and N is the total order quantity.

Substitution of the appropriate values yields

$$\min f(x_1, x_2) = 5x_1 + 10x_2 + 500x_1 \quad x_2 \quad .$$

This function corresponds to the primal program of the unconstrained geometric programming problem

$$\min f(x_1, x_2) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}} + c_3 x_1^{a_{31}} x_2^{a_{32}}.$$

Equating like parts yields the following set of values: $c_1 = 5$; $c_2 = 10$; $c_3 = 500$; $a_{11} = 1$; $a_{12} = 0$; $a_{21} = 0$; $a_{22} = 1$; $a_{31} = -1$; and $a_{32} = -1$. Letting δ_t represent the t^{th} weight ($t = 1, 2, 3$), the dual program is written as

$$\max v(\underline{\delta}) = \left(\frac{5}{\delta_1}\right)^{\delta_1} \left(\frac{10}{\delta_2}\right)^{\delta_2} \left(\frac{500}{\delta_3}\right)^{\delta_3}$$

subject to

$$\begin{aligned} \delta_1 + \delta_2 + \delta_3 &= 1 \\ \delta_1 \qquad - \delta_3 &= 0 \\ \delta_2 - \delta_3 &= 0. \end{aligned}$$

This system of linear equations is satisfied by $\delta_1 = \delta_2 = \delta_3 = 1/3$. Thus, the dual function has a maximum value of $[(15)(30)(1500)]^{1/3} = 87.72$. This value is also the minimum value of the primal program.[57] Hence, the minimum cost is \$87.72.

To determine the quantity of each product to be ordered, it is necessary to apply the relation

$$\delta_j = \frac{c_j P_j}{f(\underline{x}^\circ)} \quad \text{for } j = 1, 2, \ldots, n.$$

In this relation $f(\underline{x}^\circ)$ is the optimal value of the primal program. The relation defining P_j is $P_j = x_1^{a_{j1}} x_2^{a_{j2}} \ldots x_m^{a_{jm}}$. For the given problem,

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[57] Duffin, Peterson, and Zener, op. cit., p. 80. Theorem 1, part (ii).

$$\delta_1 = \frac{c_1 x_1^{a_{11}} x_2^{a_{12}}}{f(\underline{x}^o)} ;$$

$$\delta_2 = \frac{c_2 x_1^{a_{21}} x_2^{a_{22}}}{f(\underline{x}^o)} ; \text{ and}$$

$$\delta_3 = \frac{c_3 x_1^{a_{31}} x_2^{a_{32}}}{f(\underline{x}^o)} .$$

Substitution of the proper numerical values results in

$$\frac{1}{3} = \frac{5x_1}{87.72} ;$$

$$\frac{1}{3} = \frac{10x_2}{87.72} ;$$

$$\frac{1}{3} = \frac{500x_1 \quad x_2}{87.72} .$$

This system then yields the following values: $x_1 = 5.848$ and $x_2 = 2.924$. Thus, each production run should consist of 5.848 units of x_1 and 2.924 units of x_2. This will result in a minimum inventory cost of $87.72.

Computational technique: constrained primal.--In the constrained case the primal function is defined by an m-variable function of the following type:

$$\min f(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{n} c_i \prod_{j=1}^{m} x_j^{a_{ij}}, \quad c_i x_j \geq 0.$$

This function is to be minimized subject to the set of p constraints

$$g_k(x_1, x_2, \ldots, x_m) = \sum_{i=J[k]} c_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_m^{a_{im}}, \ k = 0, 1, 2, \ldots, p.$$

In this form J[k] is the set $\{m_k, m_k + 1, m_k + 2, \ldots, n_k\}$, $k = 0, 1, 2, \ldots, p$.

This problem has as its dual

$$\max v(\underline{\delta}) = \left[\prod_{i=1}^{n} (c_i/\delta_i)^{\delta_i}\right] \prod_{k=1}^{p} \lambda_k(\underline{\delta})^{\lambda_k(\underline{\delta})},$$

where $\lambda_k(\underline{\delta}) = \sum_{i=J[k]} \delta_i$, $k = 1, 2, \ldots, p$; $\underline{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$. The dual is restricted by the following set of equality constraints:

$$\sum_{i=J[0]} \delta_i = 1;$$

$$\sum_{i=1}^{n} a_{ij}\delta_j = 0, \quad j = 1, 2, \ldots, m.$$

Although the procedure for minimizing a constrained geometric programming problem is similar to that of the unconstrained problem, there are two basic differences. The first is the fact that the existence of the constraints in the primal program makes the use of the calculus impractical. The second is the manner in which the primal variables are determined. With these two exceptions, the two approaches are basically identical: (1) write the dual program and solve the linear system defined by the weights; (2) using these weights, maximize the dual, thus minimizing the primal;[58] and, (3) determine the primal variables.

Assume that the weights which maximize the dual program of a constrained primal program have been determined. The value of each of the primal variables is determined by applying the following:

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[58] *Ibid.*

$$c_j x_1^{a_{j1}} x_2^{a_{j2}} \dots x_m^{a_{jm}} = \begin{cases} \delta_j v(\underline{\delta}), & j \in J[0], \\ \dfrac{\delta_j}{\lambda_k(\underline{\delta})}, & j \in J[k], \end{cases}$$

where k is defined for those positive integers for which $\lambda_k(\underline{\delta}) > 0$.[59] This relation can be interpreted in the following manner: (1) for those weights corresponding to terms in the primal function,

$$\delta_j = \frac{c_j P_j}{v(\underline{\delta})} \quad \frac{c_j x_1^{a_{j1}} x_2^{a_{j2}} \dots x_m^{a_{jm}}}{v(\underline{\delta})}, \quad j \in J[0];$$

(2) for those weights corresponding to terms in the constraint functions,

$$\delta_j = c_j P_j [\lambda_k(\underline{\delta})] \equiv \lambda_k(\underline{\delta})\ c_j x_1^{a_{j1}} x_2^{a_{j2}} \dots x_m^{a_{jm}}, \quad j \in J[k].$$

As a means of demonstrating the constrained case, consider the previously mentioned constrained problem,

$$\min g_0(\underline{x}) = 40x_1\ x_2\ x_3 + 20x_1 x_3 + 20x_1 x_2 x_3$$

subject to

$$g_1(\underline{x}) = \frac{1}{3} x_1\ x_2 + \frac{4}{3} x_2\ x_3 \leq 1$$

$$x_1, x_2, x_3 \geq 0.$$

It has been shown that the dual program is defined by the following:

$$\max v(\underline{\delta}) = \left(\frac{40}{\delta_1}\right)^{\delta_1} \left(\frac{20}{\delta_2}\right)^{\delta_2} \left(\frac{20}{\delta_3}\right)^{\delta_3} \left(\frac{1/3}{\delta_4}\right)^{\delta_4} \left(\frac{4/3}{\delta_5}\right)^{\delta_5} (\delta_4 + \delta_5)^{(\delta_4 + \delta_5)}$$

subject to

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[59] Ibid., Theorem 1, part (iv).

$$\begin{aligned}
\delta_1 + \delta_2 + \delta_3 \qquad\qquad\quad &= 1 \\
-\delta_1 + \delta_2 + \delta_3 - 2\delta_4 \qquad &= 0 \\
-\tfrac{1}{2}\delta_1 \qquad + \delta_3 - 2\delta_4 + \tfrac{1}{2}\delta_5 &= 0 \\
-\delta_1 + \delta_2 + \delta_3 \qquad - \delta_5 &= 0.
\end{aligned}$$

The solution to the linear system defined by the weights δ_1, δ_2, δ_3, δ_4, and δ_5 is obtained by the following set of row operations, where $\mathcal{O}_{ij}(k)$ indicates that row j was multiplied by k and the result added to row i:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & -2 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & -2 & \frac{1}{2} & 0 \\ -1 & 1 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{matrix} \sim \mathcal{O}_{21}(1) \sim \\ \sim \mathcal{O}_{31}(\frac{1}{2}) \sim \\ \sim \mathcal{O}_{41}(1) \sim \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & -2 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} & -2 & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 2 & 0 & -1 & 1 \end{bmatrix} \begin{matrix} \sim \mathcal{O}_{12}(-\frac{1}{2}) \\ \sim \mathcal{O}_{32}(-\frac{1}{4}) \\ \sim \mathcal{O}_{42}(-1) \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 2 & 2 & -2 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 2 & -1 & 0 \end{bmatrix} \sim \mathcal{O}_{2}(\tfrac{1}{2}) \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 2 & -1 & 0 \end{bmatrix} \quad \mathcal{O}_{23}(-1)$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{matrix} \sim \mathcal{O}_{14}(-\frac{1}{2}) \sim \\ \sim \mathcal{O}_{24}(-\frac{1}{4}) \sim \\ \sim \mathcal{O}_{34}(3/4) \sim \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 2 & -1 & 0 \end{bmatrix}$$

This series of row operations reduces the original coefficient matrix to its row equivalent form. This form is shown by the solution matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 2 & -1 & 0 \end{bmatrix}$$

which has row rank equal to four. In this case the rank of the coefficient matrix (4) is less than the number of unknowns (5). Hence, the linear system has dependent solutions. These solutions are obtained directly from the solution matrix

$$\delta_1 + \frac{1}{2}\delta_5 = \frac{1}{2},$$

$$\delta_2 - \frac{1}{4}\delta_5 = \frac{1}{4},$$

$$\delta_3 - \frac{1}{4}\delta_5 = \frac{1}{4},$$

$$2\delta_4 - \delta_5 = 0.$$

This solution set can be written as

$$\delta_1 = \frac{1}{2}(1 - \delta_5),$$

$$\delta_2 = \frac{1}{4}(1 + \delta_5),$$

$$\delta_3 = \frac{1}{4}(1 + \delta_5),$$

$$\delta_4 = \frac{1}{2}\delta_5.$$

Since $\delta_i \geq 0$ for i = 1, 2, 3, 4, 5, it is possible to define limits on the values of the δ_i. The most critical restraint is the value of δ_1. Setting $\delta_1 \geq 0$, it is found that $\frac{1}{2}(1 - \delta_5) \geq 0$. This implies that $1 - \delta_5 \geq 0$, which indicates that $1 \geq \delta_5$. Thus δ_5 can assume only those values in the closed interval [0, 1]. The δ_5 weight is is the independent variable for the solution set.

With the given set of dependent solutions, it is necessary to arbitrarily assign values to δ_5 and calculate the value of the remaining weights which correspond to the selected value of δ_5. Suppose the random selection of values for δ_5 is $\delta_5 = 0$, $\delta_5 = 1/5$, $\delta_5 = 2/5$, $\delta_5 = 1/2$, and $\delta_5 = 3/4$. The corresponding set of weights, written as $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$ is then given by (1/2, 1/4, 1/4, 0, 0), (2/5, 3/10, 3/10, 1/10, 1/5), (3/10, 7/20, 7/20, 1/5, 2/5), (1/4, 3/8, 3/8, 1/4, 1/2), and (1/4, 3/8, 3/8, 1/4, 3/4). Evaluating the dual function at each of the given solutions yields the following:

$$v(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0) = \left(\frac{40}{1/2}\right)^{\frac{1}{2}}\left(\frac{20}{1/4}\right)^{\frac{1}{4}}\left(\frac{20}{1/4}\right)^{\frac{1}{4}}\left(\frac{1/3}{0}\right)^{0}\left(\frac{4/3}{0}\right)^{0}(0 + 0)^{0} = 80, \text{ since}$$

$x^x = x^{-x} = 1$ for $x = 0$, by definition;

$$v(\tfrac{2}{4}, \tfrac{3}{10}, \tfrac{3}{10}, \tfrac{1}{10}, \tfrac{1}{5}) = \left(\frac{40}{2/5}\right)^{3/5}\left(\frac{20}{3/10}\right)^{3/10}\left(\frac{20}{3/10}\right)^{3/10}\left(\frac{1/3}{1/10}\right)^{1/10}\left(\frac{4/3}{1/5}\right)^{\frac{1}{5}}\left(\frac{1}{10} + \frac{1}{5}\right)^{\frac{1}{10}+\frac{1}{5}} = 90.9;$$

$$v(\tfrac{3}{10}, \tfrac{7}{20}, \tfrac{7}{20}, \tfrac{1}{5}, \tfrac{2}{5}) = \left(\frac{40}{3/10}\right)^{3/10}\left(\frac{20}{7/20}\right)^{\frac{7}{20}}\left(\frac{20}{7/20}\right)^{\frac{7}{20}}\left(\frac{1/3}{1/5}\right)^{1/5}\left(\frac{4/3}{2/5}\right)^{2/5}\left(\frac{1}{5} + \frac{2}{5}\right)^{\frac{1}{5}+\frac{2}{5}} = 97.14;$$

$$v(\tfrac{1}{4}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{4}, \tfrac{1}{2}) = \left(\frac{40}{1/4}\right)^{\frac{1}{4}}\left(\frac{20}{3/8}\right)^{3/8}\left(\frac{20}{3/8}\right)^{3/8}\left(\frac{1/3}{1/4}\right)^{\frac{1}{4}}\left(\frac{4/3}{1/2}\right)^{\frac{1}{2}}\left(\frac{1}{4} + \frac{1}{2}\right)^{\frac{1}{4}+\frac{1}{2}} = 99.27;$$

$$v(\tfrac{1}{8}, \tfrac{7}{16}, \tfrac{7}{16}, \tfrac{3}{8}, \tfrac{3}{4}) = \left(\frac{40}{1/8}\right)^{\frac{1}{8}}\left(\frac{20}{7/16}\right)^{\frac{7}{16}}\left(\frac{20}{7/16}\right)^{\frac{7}{16}}\left(\frac{1/3}{3/8}\right)^{3/8}\left(\frac{4/3}{3/4}\right)^{3/4}\left(\frac{3}{8} + \frac{3}{4}\right)^{\frac{3}{8}+\frac{3}{4}} = 98.05.$$

Examination of the values of $v(\underline{\delta})$ reveals that the maximum value of $v(\underline{\delta})$ occurs for $\frac{1}{2} \leq \delta_5 < \frac{3}{4}$. This is because v(1/4, 3/8, 3/8, 1/4, 1/2) < v(1/83, 7/16, 7/16, 3/8, 3/4). Further evaluations are necessary to find the optimal weight for δ_5; i.e., the value of δ_5 that will be such that $v(\underline{\delta})$ is maximized. Since the value of δ_5 determines the value of each of the remaining weights, the optimal value of δ_5 will also yield optimal values for the other weights.

The general theory and computational aspects of geometric programming are summarized in Algorithm 3.7. Although this algorithm is written in a general form, it is applicable to the types of problems discussed in this study and does provide a synthesis suitable for practical application.

<u>Algorithm 3.7</u> (<u>geometric programming algorithm</u>).--Step 1. Transform the primal program into its corresponding dual program. This requires introduction of an appropriate set of weights, defined as δ_t, where $t = 1, 2, \ldots, T$.

Step 2. If the primal program is unconstrained, go to Step 3. If the primal program is constrained, go to Step 4.

Step 3. The system of equations defined by the δ_t values is linear. Solve this system by an appropriate algebraic or matrix technique.

(1) If the linear system is consistent, the δ_t values are uniquely determined. Go to Step 7.

(2) If the linear system is inconsistent, terminate the process. The problem, as given, has no optimal solution.

(3) If the linear system is dependent, express k of the δ values in terms of T - k of the δ values. T equals the total number of weights introduced into the problem. Go to Step 5.

Step 4. The system of equations defined by the δ_t values is linear. Solve this system by an appropriate algebraic or matrix technique.

(1) If the linear system is consistent, the δ_t values are uniquely determined. Go to Step 7.

(2) If the linear system is inconsistent, terminate the process. The problem, as given, has no optimal solution.

(3) If the linear system is dependent, express k of the δ values in terms of T - k of the δ values. T equals the total number of weights introduced into the problem. Go to Step 6.

Step 5. Substitute the k dependent values of δ into the dual program objective function. This will express the objective function of the dual program as a function of T - k of the δ's, where these remaining δ's are treated as independent variables. Apply the differential calculus to determine the set of δ's for which the modified objective function of the dual program is maximized. These values then uniquely determine the remaining k values of the δ. Given this optimal assignment, go to Step 7.

Step 6. Arbitrarily assign values to T - k of the δ's. This will determine the value of each of the remaining δ's for that assignment. Determine r such sets of the δ's. Evaluate the objective function of the dual program at each of the r sets, where r is a suitable number of these arbitrary assignments. Select that set of δ's for which the objective function of the dual program is maximal. Go to Step 7.

Step 7. Given values for the δ_t ($t = 1, 2, \ldots, T$), determine the maximum value of the dual program. The maximum value of the dual program is equal to the minimum value of the primal program. Denote this optimal value by $f(\underline{x}^\circ)$. Go to Step 9.

Step 8. Solve the system defined by $\delta_j = \dfrac{c_j x_1^{a_{j1}} x_2^{a_{j2}} \ldots x_m^{a_{jm}}}{f(\underline{x}^\circ)}$,

($j = 1, 2, \ldots, T$), for the m primal variables x_i ($i = 1, 2, \ldots, m$). Terminate the process.

Step 9. Solve the system defined by

$$\delta_j = \frac{c_j x_1^{a_{j1}} x_2^{a_{j2}} \ldots x_m^{a_{jm}}}{f(\underline{x}^\circ)}, \quad j \in J[0];$$

$$\delta_j = \lambda_k(\underline{\delta}) c_j x_1^{a_{j1}} x_2^{a_{j2}} \ldots x_m^{a_{jm}}, \; j \in J[k]$$

for the primal variables $x_1, x_2, \ldots, x_m$. In defining this system, the symbol $j \in J[0]$ indicates that the first relation is used only for those weights which correspond to terms in the primal function. The symbol $j \in J[k]$ indicates that the second relation is used only for those weights which correspond to terms in the k^{th} constraint function. $\lambda_k(\delta)$ is determined by applying the following:

$$\lambda_k(\delta) = \sum_{i \in J[k]} \delta_i; \text{ i.e.}$$

$\lambda_k(\delta)$ is the sum of the weights which correspond to terms in the k^{th} constraints where $\lambda_k(\delta) > 0$.

Duffin, Peterson, and Zener point out that the dual constraints are sometimes satisfied by unique values for each of the weights in the constraint system. This frequently occurs "when the number of terms in the primal program is one greater than the number of primal variables."[60] Such a system is one in the following form:

$$\text{Primal: minimize } f(x_1, x_2, x_3) = 40x_1x_2 + 20x_2x_3$$

subject to

$$x_1 > 0,\ x_2 > 0$$

$$f_1(x_1, x_2, x_3) = \frac{1}{5} x_1^{-1} x_2^{-\frac{1}{2}} + \frac{3}{5} x_2^{-1} x_3^{-2/3} \leq 1.$$

This problem will be used to demonstrate the algorithm that was developed in this study for geometric programming problems. In keeping with the established pattern, all calculations will be shown.

Step 1. Transform the primal program into its corresponding dual program. Let δ_t be the t^{th} weight. The primal program corresponds to the general form

$$\text{minimize } f(x_1, x_2, x_3) = c_1x_1^{a_{11}} x_2^{a_{12}} x_3^{a_{13}} + c_2x_1^{a_{21}} x_2^{a_{22}} x_3^{a_{23}}$$

subject to

$$f_1(x_1, x_2, x_3) = c_3x_1^{a_{31}} x_2^{a_{32}} x_3^{a_{33}} + c_4x_1^{a_{41}} x_2^{a_{42}} x_3^{a_{43}}.$$

Equating corresponding terms, the following set of values is obtained:

$c_1 = 40$	$a_{11} = 1$	$a_{21} = 0$	$a_{31} = -1$	$a_{41} = 0$
$c_2 = 20$	$a_{12} = 1$	$a_{22} = 1$	$a_{32} = -\frac{1}{2}$	$a_{42} = -1$
$c_3 = \frac{1}{5}$	$a_{13} = 0$	$a_{23} = 1$	$a_{33} = 0$	$a_{43} = -\frac{2}{3}$
$c_4 = \frac{3}{5}$				

With these values the dual program follows immediately:

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[60] Ibid., p. 92.

$$\text{maximize } v(\underline{\delta}) = \left(\frac{40}{\delta_1}\right)^{\delta_1} \left(\frac{20}{\delta_2}\right)^{\delta_2} \left(\frac{1/5}{\delta_3}\right)^{\delta_3} \left(\frac{3/5}{\delta_4}\right)^{\delta_4} (\delta_3 + \delta_4)^{(\delta_3 + \delta_4)}$$

subject to

$$\begin{aligned} \delta_1 + \delta_2 \phantom{- \tfrac{1}{2}\delta_3 - \delta_4} &= 1 \\ \delta_1 - \delta_3 &= 0 \\ \delta_1 + \delta_2 - \tfrac{1}{2}\delta_3 - \delta_4 &= 0 \\ \delta_2 - \tfrac{2}{3}\delta_4 &= 0. \end{aligned}$$

Step 2. Since the primal program is constrained, go to Step 4.

Step 4. The system defined by the δ_t values is linear. Solve this system by an appropriate algebraic or matrix technique. Inspection of the linear system reveals that it is 4 x 4, 4 rows and 4 columns. This system has a unique solution if the rank of the augmented matrix equals the rank of the coefficient matrix. The series of row operations necessary to reduce the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \end{bmatrix}$$

to its row equivalent form is shown below:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \end{bmatrix} \begin{matrix} \sim O_{21}(-1) \sim \\ \sim O_{31}(-1) \sim \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & -1 & -1 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \end{bmatrix} \begin{matrix} \sim O_{12}(1) \sim \\ \sim O_{42}(1) \sim \end{matrix} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & -1 & -1 \\ 0 & 0 & -1 & -\frac{2}{3} & -1 \end{bmatrix}$$

$$\begin{matrix}\mathcal{O}_2(-1)\sim\\ \mathcal{O}_3(-2)\sim\\ \mathcal{O}_4(-1)\sim\end{matrix}\begin{bmatrix}1 & 0 & -1 & 0 & 0\\ 0 & 1 & 1 & 0 & 1\\ 0 & 0 & 1 & 2 & 2\\ 0 & 0 & 1 & \frac{2}{3} & 1\end{bmatrix}\begin{matrix}\sim \mathcal{O}_{13}(1)\sim\\ \sim \mathcal{O}_{23}(-1)\sim\\ \sim \mathcal{O}_{43}(-1)\sim\end{matrix}\begin{bmatrix}1 & 0 & 0 & 2 & 2\\ 0 & 1 & 0 & -2 & -1\\ 0 & 0 & 1 & 2 & 2\\ 0 & 0 & 0 & -\frac{4}{3} & -1\end{bmatrix}\sim \mathcal{O}_4(-\tfrac{3}{4})$$

$$\begin{bmatrix}1 & 0 & 0 & 2 & 2\\ 0 & 1 & 0 & -2 & -1\\ 0 & 0 & 1 & 2 & 2\\ 0 & 0 & 0 & 1 & \frac{3}{4}\end{bmatrix}\begin{matrix}\sim \mathcal{O}_{14}(-2)\sim\\ \sim \mathcal{O}_{24}(2)\sim\\ \sim \mathcal{O}_{34}(-2)\sim\end{matrix}\begin{bmatrix}1 & 0 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & 1 & 0 & \frac{1}{2}\\ 0 & 0 & 0 & 1 & \frac{3}{4}\end{bmatrix}$$

The solution matrix for this system is given by

$$\begin{bmatrix}1 & 0 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & 1 & 0 & \frac{1}{2}\\ 0 & 0 & 0 & 1 & \frac{3}{4}\end{bmatrix}$$

The given linear system has a unique solution. The rank of the augmented matrix equals that of the coefficient matrix. This solution is given by $\delta_1 = 1/2$; $\delta_2 = 1/2$; $\delta_3 = 1/2$; and $\delta_4 = 3/4$. Go to Step 7.

Step 7. Determine the maximum value of the dual program for $\delta_1 = 1/2$; $\delta_2 = 1/2$; $\delta_3 = 1/2$; and $\delta_4 = 3/4$. Substituting these values into $v(\underline{\delta})$,

$$v(\underline{\delta}) = \left(\frac{40}{1/2}\right)^{1/2}\left(\frac{20}{1/2}\right)^{1/2}\left(\frac{1/5}{1/2}\right)^{1/2}\left(\frac{3/5}{3/4}\right)^{3/4}\left(\frac{1}{2}+\frac{3}{4}\right)^{(1/2+3/4)}$$

$$= (80)^{1/2}\ (40)^{1/2}\ (2/5)^{1/2}\ (4/5)^{3/4}\ (5/4)^{5/4}$$

$$= (1600)^{1/2}$$

$$= 40.$$

The maximum value of the dual program equals 40. Hence, the minimum value of the primal program equals 40. Let $f(\underline{x}°) = 40$. Go to Step 9.

Step 9. Determine the primal variables $x_1, x_2, \ldots, x_m$. This is accomplished by solving the system defined by

$$c_j x_1^{a_{j1}}\ x_2^{a_{j2}} \ldots x_m^{a_{jm}} = \begin{cases} \delta_j v(\underline{\delta}), & j \in J[0], \\ \dfrac{\delta_j}{\lambda_k(\underline{\delta})}, & j \in J[k], \end{cases}$$

where k is defined for those positive integers for which $\lambda_k(\underline{\delta}) > 0$. This relation can be interpreted in the following manner:

(1) for those weights corresponding to terms in the primal function,

$$\delta_j = \frac{c_j P_j}{v(\underline{\delta})} \equiv \frac{c_j x_1^{a_{j1}}\ x_2^{a_{j2}} \ldots x_m^{a_{jm}}}{v(\underline{\delta})}, \quad j \in J[0];$$

(2) for those weights corresponding to terms in the constraint function,

$$\delta_j = c_j P_j[\lambda_k(\underline{\delta})] \equiv [\lambda_k(\underline{\delta})]\ c_j x_1^{a_{j1}}\ x_2^{a_{j2}} \ldots x_m^{a_{jm}},\ j \in J[k].$$

Application of these forms to the current problem results in the following set of functional equalities:

$$\delta_1 = \frac{c_1 P_1}{v(\underline{\delta})} = \frac{c_1 x_1^{a_{11}}\ x_2^{a_{12}}\ x_3^{a_{13}}}{v(\underline{\delta})};$$

$$\delta_2 = \frac{c_2 P_2}{v(\underline{\delta})} = \frac{c_2 x_1^{a_{21}} x_2^{a_{22}} x_3^{a_{23}}}{v(\underline{\delta})} ;$$

$$\delta_3 = \lambda_1(\underline{\delta}) \; c_3 x_1^{a_{31}} x_2^{a_{32}} x_3^{a_{33}} ;$$

$$\delta_4 = \lambda_1(\underline{\delta}) \; c_4 x_1^{a_{41}} x_2^{a_{42}} x_3^{a_{43}} ;$$

$$\lambda_1(\underline{\delta}) = \sum_{i \varepsilon J[1]} \delta_i = (\delta_3 + \delta_4).$$

Substitution of the proper numerical values results in

$$\frac{1}{2} = \frac{40 x_1 x_2}{40} ;$$

$$\frac{1}{2} = \frac{20 x_2 x_3}{40} ;$$

$$\frac{1}{2} = (\frac{1}{2} + \frac{3}{4}) \cdot \frac{1}{5} x_1^{-1} \; x_2^{-\frac{1}{2}} \text{ ; and,}$$

$$\frac{3}{4} = (\frac{1}{2} + \frac{3}{4}) \cdot \frac{3}{5} x_2^{-1} \; x_3^{-\frac{2}{3}} .$$

The value of each of the primal variables is found by solving the simultaneous system

$$x_1 x_2 = \frac{1}{2};$$

$$x_2 x_3 = 1;$$

$$x_1 x_2 = \frac{1}{2};$$

$$x_2 x_3 = 1.$$

This system can be solved by (1) use of logarithms or (2) direct algebraic interpretation. The system has as its solution $x_1 = \frac{1}{2}$, $x_2 = 1$, and $x_3 = 1$.

In order to minimize $f(x_1, x_2, x_3) = 40x_1x_2 + 20x_2x_3$, it is necessary to let $x_1 = \frac{1}{2}$, $x_2 = 1$, and $x_3 = 1$. This solution set results in a minimum value of 40 for the primal function.

The characteristics of geometric programming are not readily apparent. This is possibly due to the lack of existing applications or the newness of the technique itself. However, certain advantages do exist and will be identified. This list of characteristic features is taken from the pioneering work of Duffin, Peterson, and Zener.[61]

(1) Geometric programming provides a systematic method for formulating a class of optimization problems. Although applicable to linear functions, geometric programming generally involves functions that are both nonlinear and convex. Suitable variables must be carefully selected and all constraints expressed as inequalities.

(2) For meaningful problems, either with or without constraints, geometric programming always produces a global minimum, not just a relative minimum. The minimum is equal to the maximum of a dual problem whose constraints are linear.

(3) The dual problem is essentially without constraints. This feature is important from a computational point of view. Even though the constraints of the primal program are nonlinear, the dual program, under constraints, is limited by a set of linear functions.

(4) Each value of the dual function provides a lower bound on the minimum value of the primal function. In addition, a maximizing sequence for the variable weights of the dual program produces a minimizing sequence for the variables of the primal program. This property provides an

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[61] Ibid., pp. 12-13

increasing sequence of lower bounds and a decreasing sequence of upper bounds on the minimum value of the primal function. Since the minimum value of the primal is equal to the maximum value of the dual, the increasing-decreasing sequence property can be used to estimate the optimum solution.

(5) The weights of the dual are in one-to-one correspondence with the posynomial terms of the primal problem. This property can be used to furnish information regarding the relative size of the posynomial terms.

Dynamic Programming

Unlike the preceding search techniques, dynamic programming represents a definite deviation from the restrictions of static analysis. Realizing that strict adherence to static analysis poses severe limitations to real-life problems (for example, sequential analysis of inventory problems), the feasibility of a suitable technique that incorporates time factors into the analysis is easily recognized. This awareness of the restrictive nature of static analysis techniques led to the investigation and development of dynamic programming.[62]

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[62] The recognized pioneer in dynamic programming is Richard Bellman. His text, Dynamic Programming (Princeton, 1957), represents the first of several compilations relative to the theoretical analysis and practical application of dynamic programming. Other writings include George Hadley's Nonlinear and Dynamic Programming (Reading, Mass., 1964), George Nemhauser's Introduction to Dynamic Programming (New York, 1966), and Harvey M. Wagner's Principles of Operations Research (Englewood Cliffs, 1969). Since these investigations, sample problems and industrial applications have been included as part of current literature and basic current textbooks (for example, J. William Gavett's Production and Operations Management (New York, 1968), Samuel B. Richmond's Operations Research for Management Decisions (New York, 1968), and Richard B. Maffei's "Planning Advertising Expenditures by Dynamic Programming," Decision Theory and Information Systems, edited by William T. Greenwood (Cincinnati, 1969), pp. 416-424.)

The practical importance of the investigative studies and textual discussions can be identified by considering some of the areas of application to which dynamic programming has been applied. These areas include the following:

(1) inventory analysis so as to determine when an item should be replenished and in what quantity;

(2) capital-budgeting procedures for the purpose of allocating scarce resources into potentially productive activities;

(3) selection of advertising media so as to achieve maximum exposure at minimum cost; and,

(4) long-range strategy selection for the optimum time for replacing assets subject to depreciation.

The analysis of these problems, and similar ones, has led to the realization that the dynamic programming model is of economic importance because it can make possible the taking of a wide range of actions via a routine approach that contains a minimum amount of human intervention.[63] Properly implemented, the models and techniques of dynamic programming can result in reductions of at least 25 per cent in terms of product costs without reducing the quality of service.[64]

The dynamic programming problem.--Basically, dynamic programming does not refer to a particular type of programming problem. Rather, it refers to a computational technique[65] that has been successfully applied

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[63]Harvey M. Wagner, op. cit., p. 255.

[64]Ibid.

[65]George Hadley, op. cit., p. 350.

to problems with the following characteristics:

1. The [problem] situation involves multistage processes containing a large number of variables.
2. The relationship between the stages is not a complex one.
3. At each stage, the state of the process is described by a small number of parameters.
4. The effect of a decision at any stage is to transform this set of parameters into a single set.[66]

Problems which have been solved and which exhibit these characteristics include scheduling and inventory problems, resource allocation problems, and dividend and investment problems.

Further inquiry into the nature of the dynamic programming problem reveals that its basic feature is a step-by-step approach to the optimum decision required in a given problem. The problem itself is not one in which all of its stages are considered simultaneously and, although time periods may be involved, they are not necessary. The approach to the solution can take one of the following two forms: (1) begin at stage one and proceed forward through the N stages of the problem, or (2) begin at the N^{th} stage and backtrack through the stages to the initial stage. Regardless of the approach taken, an optimum solution will be achieved.

Richmond has described the dynamic programming problem as one involving a sequence of decisions,

> each of which produces an outcome that influences the next. . .each decision must take into account its effect on the next decision and, indeed, its effect on the succeeding chain of decisions.[67]

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[66] Edward H. Bowman and Robert B. Fetter, Analysis for Production and Operations Management (Homewood, 1967), p. 136.

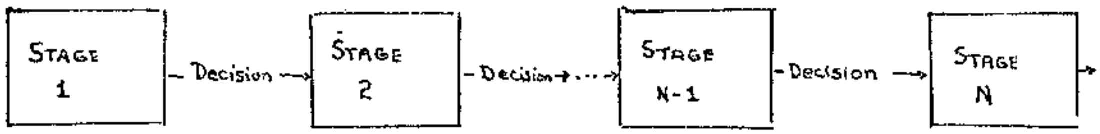
[67] Samuel B. Richmond, op. cit., p. 470.

In this context the dynamic programming problem is described as one which encourages step-by-step optimization. At the first decision point an optimum solution is achieved for that stage only. This optimum solution is then used as an input to the second decision point. Utilizing this input, the second decision point is optimized for that stage only, the only consideration being given to the inputs and parameters of the second stage. This optimum solution for the second decision point is then utilized as an input for the third decision point. At this third decision point, an optimum solution is achieved relative to the input from the second stage and the given parameters of the third stage. This process is repeated until all decision points have been examined and the optimum solution for each point obtained. The optimum solution to the initial problem is the solution obtained at the final decision point.

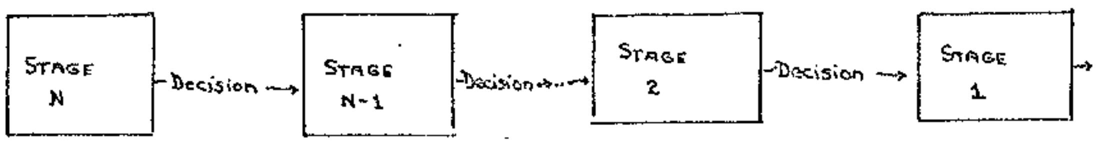
In terms of the two possible approaches to the step-by-step analysis of a given problem, this optimization technique takes on one of the following forms:

(1) forward optimization--the optimum solution at the k^{th} decision point is carried forward so as to be utilized at the $(k + 1)^{th}$ decision point;

(2) backward optimization--the solution is achieved which optimizes the effectiveness of the last decision point for each of the possible inputs to that last point (the output for the next-to-last point), the process being carried successively backward from the last decision point to the first; i.e., from point k to (k - 1), from (k - 1) to (k - 2), etc. These two approaches are shown in Figure 3.4.



Forward Recursion



Backward Recursion

Fig. 3.4--Recursive optimization

From this discussion it is easily seen that the dynamic programming problem is one in which the decision-stages are somewhat like a chain with each decision point linked to the next. The result is a problem consisting of N stages or N points of decision. This construction then transforms the original problem from one problem with N dimensions into N problems with one dimension per problem, each of which is interdependent with the other parts of the total problem.

As previously indicated, dynamic programming refers to a procedure for analyzing multi-stage decision problems in a step-by-step manner, a process that can be considered as indicative of sequential decision making. This process is summarized in the following definition.

Definition 3.12.--Dynamic programming is the technique (or theory) of multi-stage decision processes which are characterized by (1) a defined optimal return function, (2) a derived functional equation for the optimal return function, and (3) an optimal policy (any rule for making decisions which yields an allowable sequence of decisions which optimizes the preassigned function of the final parameters) derived from the optimal decision functions given by the use of the functional equation.

In application the definition provides the totality of the operations of the optimization process and its units of composition. However, it only describes a process which applies two basic ideas: recursive optimality and summarization in terms of the parameters of the problem.

Principle of optimality.--The crux on which the techniques of dynamic programming rest is the principle of optimality. Formulated by R. Bellman, this principle states that

> An optimal policy [set of decisions] has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.[68]

This principle is the key to the decision process identified as dynamic programming. It guarantees that an optimal decision in any state, say n, will lead to an optimal decision in the next stage, say n + 1. By successive application of the stage-by-stage optimal decisions, the process leads to the optimal decision for the entire sequence which represents the original decision problem. Whatever decision initiates the decision process, the remaining decisions will be optimal with respect to the outcome resulting from the initial decision.

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[68]Richard Bellman, op. cit., p. 83.

General theory.--The underlying theory to the formulation of the dynamic programming problem is that complex problems can often be decomposed (broken down) into a series of smaller problems. Each of these smaller problems is solved, and the solutions to the smaller problems are combined in such a way that the whole problem is solved. The transition from one small problem to another is accomplished by defining a function which transforms one state into another. The resulting system is a series of states (or stages) connected by functionally defined transformations. The determining of this transformation function is a crucial step in solving the problem.

Consider the problem of maximizing

$$R(x_1, x_2, \ldots, x_n) = g_1(x_1) + g_2(x_2) + \ldots + g_n(x_n)$$

subject to the following constraints: $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = \underline{x}$. The function defined by $R(x_1, x_2, \ldots, x_n)$ is such that it is separable, a requirement that is imposed upon all objective functions in dynamic programming.[69] It is required that allocations be made, one at a time, in such a way that a quantity of resources is assigned to the N^{th} activity, then to the $(N - 1)^{th}$ activity, etc. With the allocation required in this manner, the result is a dynamic allocation problem.

For the sequence of functions $\{f_n(\underline{x})\}$, n + 1, 2,...,$\underline{x} \geq 0$, let

$$f_n(\underline{x}) = \underset{\{x_i\}}{\text{Max}}\ R(x_1, x_2, \ldots, x_n)$$

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[69]The term separable defines a condition such that the function can be broken down into n individual return stages. According to Gue and Thomas, op. cit., p. 174, this requires that the objective be composed of the individual stage returns.

where $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = \underline{x}$. Then $f_n(\underline{x})$ represents the optimal return (maximum) from an allocation of the quantity of resources, x, to the N required activities. For N = 1, 2,... and for $g_i(0) = 0$ for each i (i = 1, 2,...,n), define $f_n(0) = 0$. For $x \geq 0$, let $f_1(x) = g_1(x)$.

Allowing N and $\underline{x}$ to be arbitrary, a recurrence relation can be obtained connecting $f_n(\underline{x})$ and $f_{n-1}(\underline{x})$. To the N^{th} activity, allocate resource x_n, $0 \leq x_n \leq x$. Then, regardless of the value of x_n, the remaining quantity of resources, $x - x_n$, will be used to achieve maximum allocation to the remaining (N - 1) activities. The optimal return for the (N - 1) remaining activities is a function of the resources remaining to be allocated, $x - x_n$. This function is defined by $f_{n-1}(x - x_n)$. Thus, the initial allocation of x_n units to the N^{th} activity yields a total return of $g_n(x_n) + f_{n-1}(x - x_n)$. This return is over the entire N-activity process, and the optimal choice for x_n is the one which maximizes the total return. Thus,

$$f_n(x) = \max_{0 \leq x \leq x_n} [g_n(x_n) + f_{n-1}(x - x_n)]$$

defines the basic functional equation for N = 2, 3,...,$x \geq 0$, where $f_1(x) = g_1(x)$. The function just derived is defined as the recursive relation by which the original problem will be maximized.

The essence of the recursive relation defined in the preceding paragraph can be explained as follows. Consider a limited resource, with x_j the quantity of resource allocated to activity j. The function $f_j(x_j)$ represents the return from activity j when x_j units are allocated to the j^{th} activity. With x_j being the optimum resource selection and $f_j(x_j)$

the optimal return from the allocation of x_j, the process moves to activity (j + 1). The solution from activity j is incorporated with activity (j + 1) to achieve optimality in stage (j + 1). In this way an optimum solution at each stage is carried into the next stage of the problem. The effect of this procedure is to transform "one n dimensional problem into n one (possibly two) one dimensional problems."[70] In addition, the result of stage (j + 1) is a function of the result obtained in stage j.

The heart of the dynamic programming approach to decision making is the recursive relation. Because of its importance, the term is formally defined in Definition 3.13.

Definition 3.13.--A definition in which a property is defined for the natural number n* whenever it is defined for n is called a recursive definition.[71] The n is defined as n* = n + 1. Thus, if a relation is defined for stage n in a recursive manner, it is also defined for stage n + 1.

Bellman explains the working of the recursive relation in the following manner.[72] Consider a quantity of resource that is to be allocated in such a way that the return from the total investment is to be maximized. Let x denote the total quantity of resource that is to be allocated. If y denotes the quantity allocated to the first of two investments, then

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[70] Bowman and Fetter, op. cit.

[71] Richard E. Johnson, First Course in Abstract Algebra (Englewood Cliffs, 1961), p. 15.

[72] Bellman, op. cit., pp. 3-9.

$x - y$ denotes the quantity of resource that remains for allocation to the second investment. Let $g(y)$ denote the return from the investment of the y units of resource and $h(x - y)$ the return from the investment of the remaining $x - y$ units of resource. The problem can be written in the functional form

$$\text{Max } R(x, y) = g(y) + h(x - y),$$

where $R(x, y)$ denotes the return function. The value of y can assume any value between 0 and x, inclusive.

Assuming that the allocation process is composed of N-stages, the maximizing of the total return is accomplished in the following manner. For the first stage of the allocation process, the maximum return, R_1, is obtained by maximizing $R_1(x, y) = g(y) + h(x - y)$. Since the return of $g(y)$ is achieved only at some expense (or price), let ay denote the amount that remains of resource y, $0 \leq a < 1$. Similarly, the return $h(x - y)$ is achieved only by paying some expense (or price); let $b(x - y)$, $0 \leq b < 1$, denote the amount of $x - y$ that remains after obtaining return $h(x - y)$. For the second stage of the N-stage allocation process, $ay + b(x - y)$ units of resource remain to be allocated. Since the initial quantity to be allocated has been symbolized by the letter x, let $x_1 = ay + b(x - y)$ denote the total quantity to be allocated at the second stage. Then, allocating y_1 units of the available x_1 units to the first investment leaves $(x_1 - y_1)$ units to be allocated to the second investment. The function to be maximized is the return realized from this second allocation process, $R_2(x_1, y_1)$. But, the objective is to maximize the total return from both stages; therefore, the function to be maximized is given by the total return function

$$R_2(x, y, y_1) = [g(y) + h(x - y)] + g(y_1) + h(x_1 - y_1),$$

where

(1) $0 \leq y \leq x$

(2) $x_1 = ay + b(y - x)$ and $0 \leq y_1 \leq x_1$.

The reallocation process is continued throughout the N-stages. At each new stage, the total quantity to be allocated, x_{N-1}, is the quantity of resource that remains, $ay_{N-2} + b(x_{N-2} - y_{N-2})$; i.e.,

$$x_{N-1} = ay_{N-2} + b(x_{N-2} - y_{N-2})$$

where

$$0 \leq y_{N-2} \leq x_{N-2};$$

$$0 \leq y_{N-1} \leq x_{N-1}.$$

Since the objective is to maximize the total return from all of the N stages, the total return function is given by

$$R_N(x, y, y_1, \ldots, y_{N-1}) = [g(y) + h(x - y)] + [g(y_1) + h(x_1 - y_1) + \ldots + [g(y_{N-1}) + h(x_{N-1} - y_{N-1})],$$

where the quantities of resource available for allocation at the end of the first, second,...,(N - 1)st stage are given by

$$x_1 = ay + b(x - y), \quad 0 \leq y \leq x$$

$$x_2 = ay_1 + b(x_1 - y_1), \quad 0 \leq y_1 \leq x_1$$

$$\vdots$$

$$x_{N-1} = ay_{N-2} + b(x_{N-2} - y_{N-2}), \quad 0 \leq y_{N-2} \leq x_{N-2}, \; 0 \leq y_{N-1} \leq x_{N-1}.$$

At the N^{th} stage, no further allocations are necessary as the final allocation, the quantities at the (N - 1)st stage, have been used to maximize the return of the N^{th} stage. The maximum value of the total return function is the sum of these individual maximal returns.[73]

It is important to note that the process just described is identified as the "forward approach" to solving a given problem. In this approach the input of the N^{th} stage is the output of the (N - 1)st stage. The optimal solution is obtained by proceeding forward in time from the initial stage (time 0) to the final stage, and the recursive relation utilized to accomplish the optimizing process is defined as the forward recursive relation.

As might be surmised from the preceding paragraph, another recursive relation exists for solving dynamic programming problems. This other recursive relation is defined as the backward recursive relation and achieves optimization by reversing the order of the sequential steps. That is, backward recursive optimization proceeds from stage N to stage 1. This is accomplished by (1) defining the successive stages of the problem, (2) optimizing the effectiveness of the last stage (say, N^{th} stage) for each of the possible inputs to that last stage (these inputs would have come from stage N - 1), and (3) solving the previous (N - 1)st stage in the same manner as the N^{th} stage was solved. In this manner, the original problem is solved by backing through the N stages of the N-stage process.

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[73]This is the result guaranteed by the principle of optimality. By maximizing at each stage of the stage-by-stage process, the maximum value of the total return is obtained.

As a means of demonstrating these concepts, consider the following example. A body shop offers three types of work situations: metal work, painting, and touchup. The shop is constructed in such a way that all work must proceed through these three departments in that order. In addition, each of the three departments has three parallel alternative stations; i.e., three metal working stations, three paint stations, and three touchup stations. Each of the nine individual stations has its own techniques, equipment, costs, and different output characteristics which affect the costs in the next department.

A work assignment is to be processed through the body shop in such a way that the total cost of the work is to be minimized. The work is to be scheduled in such a way that it will proceed through exactly one station in each department. As a means of initiating the work, estimates of $400, $420, and $395 are submitted from metal stations 1, 2, and 3, respectively. Table 3.20 summarizes the cost associated with work that passes from a given metal work station to a given paint station.

TABLE 3.20

COST SUMMARY: (m, p)

Paint Station / Metal Work Station	1	2	3
1	40	50	45
2	50	35	40
3	45	60	70

Table 3.21 summarizes the cost associated with work that passes from a given paint station to a given touchup station.

TABLE 3.21

COST SUMMARY: (p, t)

Touchup Station / Paint Station	1	2	3
1	18	24	15
2	21	18	20
3	12	15	5

The entries in the table are read as the pair (m, p) or (p, t) with (m, p) denoting from metal station m to paint station p and (p, t) denoting from paint station p to touchup station t.

Solution 1 (forward recursion).--The use of the forward recursive relation requires that the sequential stages of the decision-making process begin at the initial point of the N-stages; i.e., at the metal working station. At this initial point the station is chosen that minimizes the cost (station 3) and then selection of the paint station begins. The possible selections are shown in the decision trees shown in Figure 3.5. Each tree corresponds to the initial selection of one of the three metal work stations, M_i, i = 1, 2, 3.

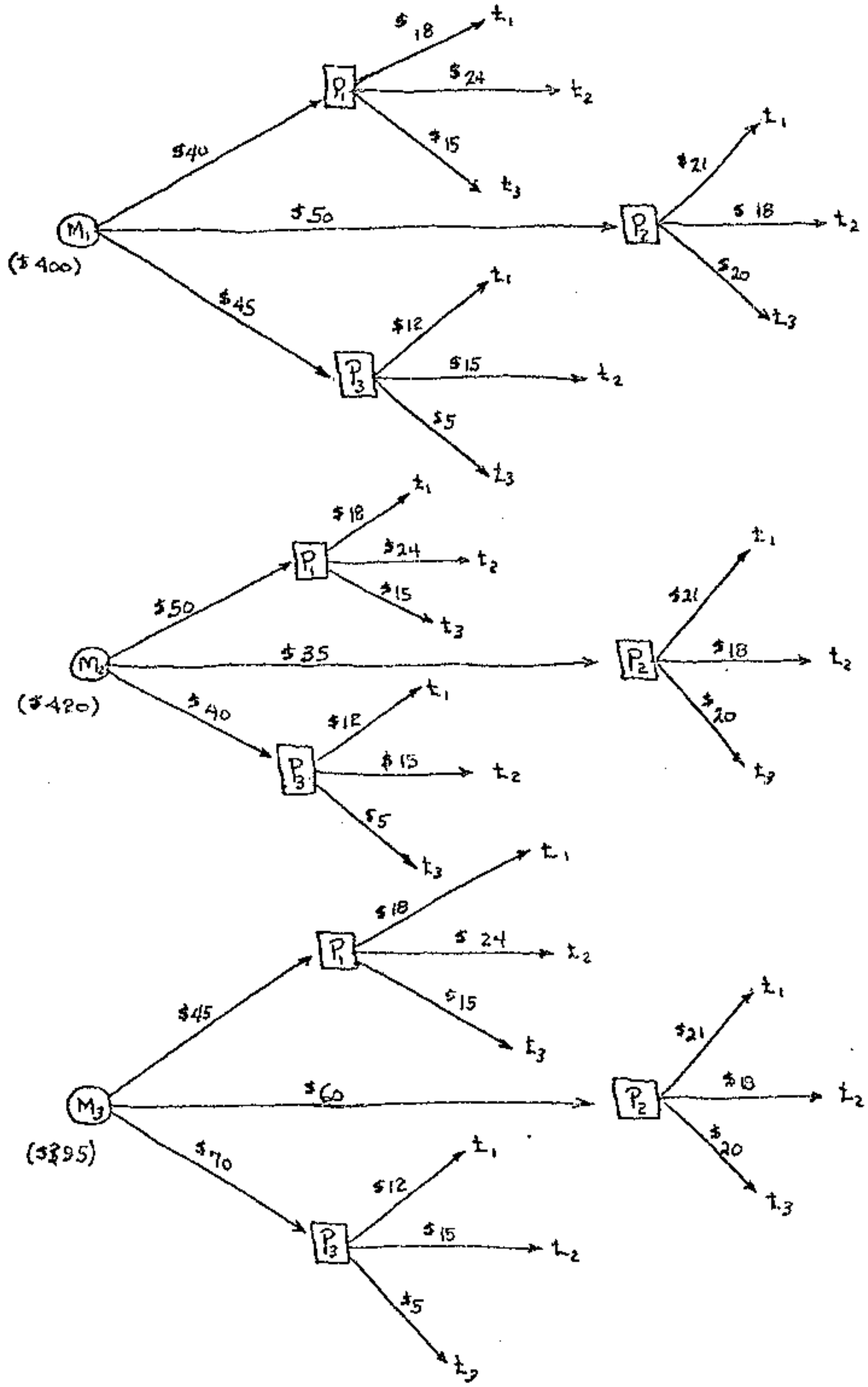

Fig. 3.5--Forward recursion

For example, if the initial assignment of the incoming work is M_1, the second stage assignment is between P_1, P_2, and P_3. Suppose the second stage assignment is P_3. Then the third and final stage assignment is between t_1, t_2, and t_3. Suppose the third stage assignment is t_2. Then the decision process is defined as the sequence (M_1, P_3, t_2). In this type of analysis, the output of each stage serves as an input to the next stage. Thus, the decision to utilize metal work station 1 serves as the input to the selection of the paint station, etc.

Since the objective of this problem is the minimization of the total cost of the process, the first decision will be to choose the metal work station which corresponds to minimum cost. This dictates metal work station 3 with a cost of \$395. Given this decision, the paint station with minimum cost will be selected. This results in paint station P_1 with a cost of \$45. Given this decision, the touchup station with minimum cost must be selected. This results in the selection of t_3 with a cost of \$15. This completes the process. The sequence of forward decisions necessary to minimize the total cost of the process is given by the decision set

$$\{m_1, (m_1, P_1), (m_1, P_1, t_3)\}.$$

The minimum cost for this set of decisions is \$455.

A functional description of this problem requires that the totality of three component costs be minimized. These three cost elements are the cost of optimizing at the initial stage (stage 1), the cost of optimizing at the second stage (stage 2), and the cost of optimizing at the final stage (stage 3). The function to be minimized is the function

$R(y, y_1, y_2)$, where $R(y_1, y_2, y_3) = y_1 + y_2 + y_3$, and y_1 denotes the cost associated with the decision to assign the job to metal work station m_i, $i = 1, 2, 3$; y_2 denotes the cost associated with the decision to assign the job to paint station p_j, $j = 1, 2, 3$, given the initial assignment of work to m_i; and, y_3 denotes the cost associated with the decision to assign the job to touchup station t_k, $k = 1, 2, 3$, given the assignment of work to p_j.

Since $R(y_1, y_2, y_3)$ is to be minimized, the minimum cost associated with each of the values y_1, y_2, and y_3 depends upon the work station assigned to the job. Thus, the value of y_1 depends upon the initial assignment m_i; i.e., $y_1 = h_i(m_i)$, $i = 1, 2, 3$. The value of y_2 depends upon the assignment p_j; i.e., $y_2 = h_j(p_j)$; but, since p_j is functionally dependent upon the initial assignment (i.e., $p_j = f_i(m_i)$), $y_2 = h[f(m_i)]$. Similarly, the value of y_3 depends upon the assignment t_k; i.e., $y_3 = h_k(t_k)$; but since t_k is functionally dependent upon p_j (i.e., $t_k = f_j(p_j)$), $y_3 = h_k[f_j(p_j)]$. This expression for y_3 can be written in the form $y_3 = h[f(p_j)] \equiv h\{f[f(m_i)]\}$, which defines y_3 in terms of the initial assignment m_i. Substitution of these expressions into $R(y_1, y_2, y_3)$ yields

$$\begin{aligned} \min R(y_1, y_2, y_3) &= y_1 + y_2 + y_3 \\ &= h_i(m_i) + h_j(p_j) + h_k(t_k) \\ &= h_i(m_i) + h_j[f_i(m_i)] \\ &\quad + h_k\{f_j[f_i(m_i)]\}, \\ &\qquad i, j, k = 1, 2, 3. \end{aligned}$$

This function defines the recursive relation connecting the decision at stage 2 with the decision at stage 1 and the decision at stage 3 with the decision at stage 2. However, since the decision at stage 2 is connected to the decision at stage 1, the decision at stage 3 is connected with the decision at stage 1. This relation carries the decision at stage 1 forward to the final decision at stage 3, a process which demonstrates the principle of optimality.

Solution 2 (backward recursion).--The use of the backward recursive relation requires that the sequential stages of the decision making begin at the last (N^{th}) stage and progressively move backward through the stages of the decision process. At this point the process requires selecting the optimal (minimal) cost and, using this selection, determine the optimal (minimal) cost at the second decision point. This second decision then leads to the final selection, that of choosing the station at which the work assignment will enter the process. The decision trees shown in Figure 3.5 can be used to trace a backward path, with the path selected being the one corresponding to the minimal cost; or, the decision trees shown in Figure 3.6 can be used to point out alternate decisions.

Proceeding in a manner similar to that of the forward recursive approach, the total minimum cost of the decision process can be calculated. This is accomplished by first selecting the t_i (i = 1, 2, 3) which corresponds to the minimum cost. Inspection of Figure 3.6 reveals that touchup station 3, receiving input from paint station p_3 has a minimal cost of $5. Thus, the backward path begins along (t_3, p_3). At p_3, evaluation

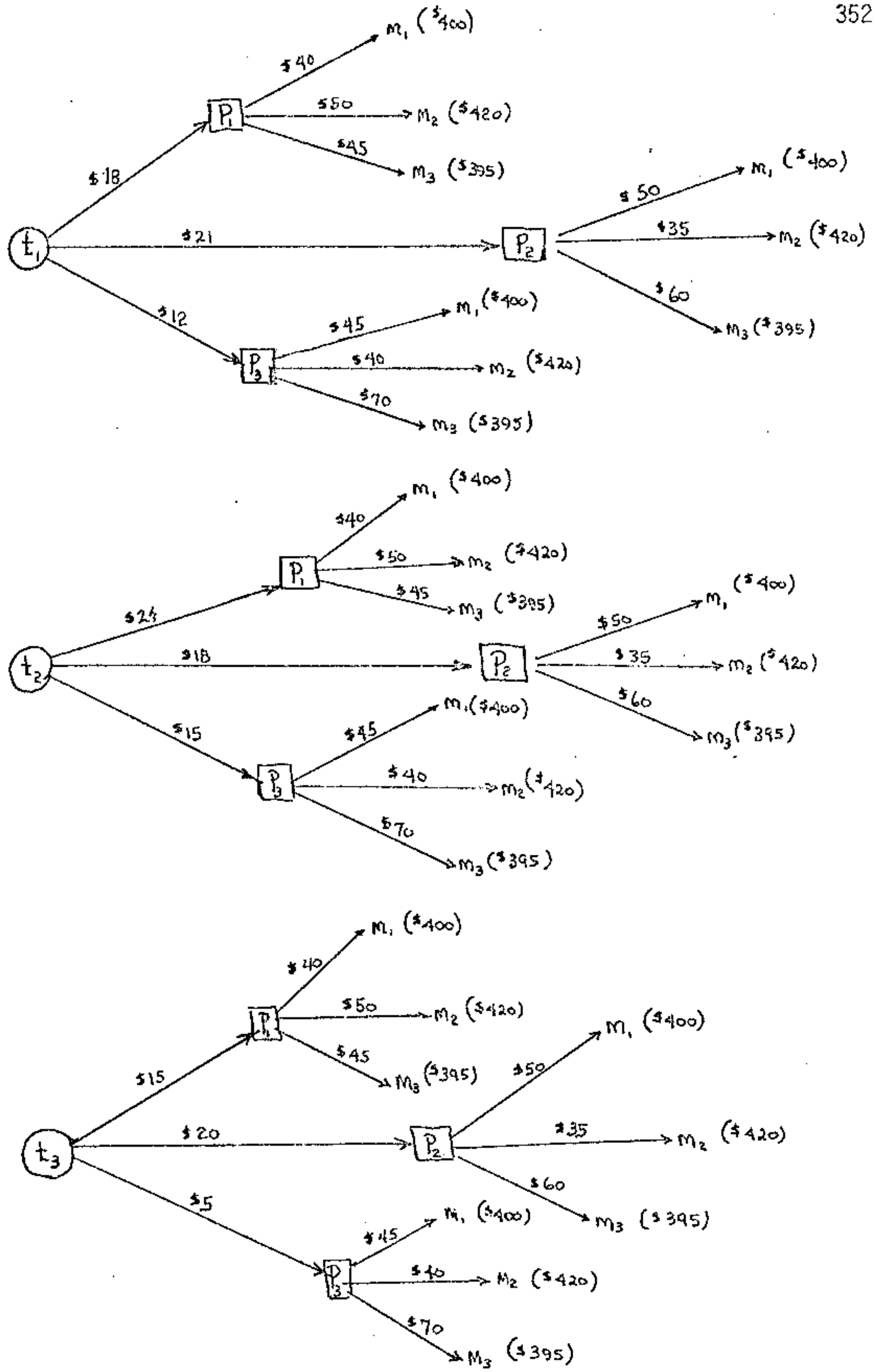

Fig. 3.6--Backward recursion

of the paths (p_3, m_1), (p_3, m_2), and (p_3, m_3) yield costs of \$445, \$460, and \$465, respectively, obtained by adding the cost of selecting a path and the cost at the end of that path. Thus, the minimum cost of \$450 is achieved by selecting the decision process defined by the decision set {(t_3, p_3), (t_3, p_3, m_1)}.

These two approaches to solving the same problem reveal that the result of the forward recursive relation does not yield the same result as the backward recursive relation. However, the results serve to point out the implications of the principle of optimality: given the conditions resulting from the decision made at the preceding stage, the remaining decisions at every stage constitute an optimal policy (set of decisions). In addition, these two approaches demonstrate the manner in which both the forward and backward recursive relations are utilized. According to Hadley,

> for deterministic sequential decision problems, one is able to use either. . . a forward solution or a backward solution, and one has the option of selecting one or the other. A forward solution works forward in time, with the first stage in the dynamic programming problem being that corresponding to the first decision to be made. The backward solution works backward in time so that the first stage of the dynamic programming [problem] corresponds to the last decision (in time) to be made. The criterion that determines whether a forward or a backward solution is to be preferred often depends on whether some parameter in the problem is specified to be a given value at the start of the process or at the end of the process (or both at the end and the beginning).[74]

Computational considerations.--Dynamic programming has been described as a technique for analyzing multi-stage problems. In this respect

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[74]Hadley, op. cit., p. 376.

dynamic programming does not utilize an algorithm of the same type as the simplex algorithm of linear programming. Instead, the manner in which a given problem is solved is totally dependent upon the nature of the problem being analyzed. This is readily apparent when consideration is given to the fact that there is no distinct defining class of problems categorized in the same manner as that for the previous techniques.[75] Dynamic programming is applicable to linear or nonlinear problems, stochastic or deterministic problems, and is equally suited so long as the problem being investigated is characterized as one of sequential analysis.

Nemhauser has described this solution technique as one in which the objective is the solving of a set of recursive relations of the form

$$f_n(X_n) = \underset{D_n}{\text{opt}}\ Q_n(X_n,\ D_n), \quad n = 1,\ 2, \ldots, N$$

where

$$Q_n(X_n,\ D_n) = r_n(X_n,\ D_n) \quad n = 1$$

and

$$Q_n(X_n,\ D_n) = r_n(X_n,\ D_n) \circ f_{n-1}(X_{n-1})$$

with

$$X_{n-1} = t_n(X_n,\ D_n) \quad n = 2, \ldots, N.$$[76]

The symbol "o" is used to indicate the relation that connects the N stages of the problem under study. For example, if

$$Q_n(X_n,\ D_n) = r_n(X_n,\ D_n) + r_{n-1}(X_n,\ D_n) + \ldots + r_1(X_1,\ D_1),$$

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[75] Ibid.

[76] Nemhauser, op. cit., p. 46.

the symbol "o" represents the additive property. Its primary function is to stipulate a condition described as separability, where separability is defined as the property whereby a given problem can be broken down into N individual stages.

In solving this set of recursive relations, Nemhauser utilizes the general algorithm described by the flow chart in Figure 3.7. This chart and following discussion is taken from Nemhauser.[77] In this flow chart, the circles, lines, and boxes serve the following functions:

(1) solid lines indicate the sequence in which the steps are followed;

(2) solid rectangular boxes contain information relative to calculations;

(3) solid circles contain information relative to settings or adjustments of the index n;

(4) diamond-shaped boxes relate to binary questions;

(5) dashed boxes indicate storage or saving of information, with the dashed lines indicating use.

The computational procedure begins at stage 1, the initial allocation; i.e., n = 1. At this state the function defined by $r_n(X_n, D_n)$ is evaluated (optimized) and found to be $r_1(X_1, D_1)$; i.e., for n = 1,

$$Q_n(X_n, D_n) = r_n(X_n, D_n)$$

is given by

$$Q_1(x_1, D_1) = r_1(X_1, D_1).$$

This result is utilized by applying

$$f_n(X_n, D_n) = \underset{D_n}{\text{opt.}}\ Q_n(X_n, D_n),$$

[77]Ibid., pp. 46-48.

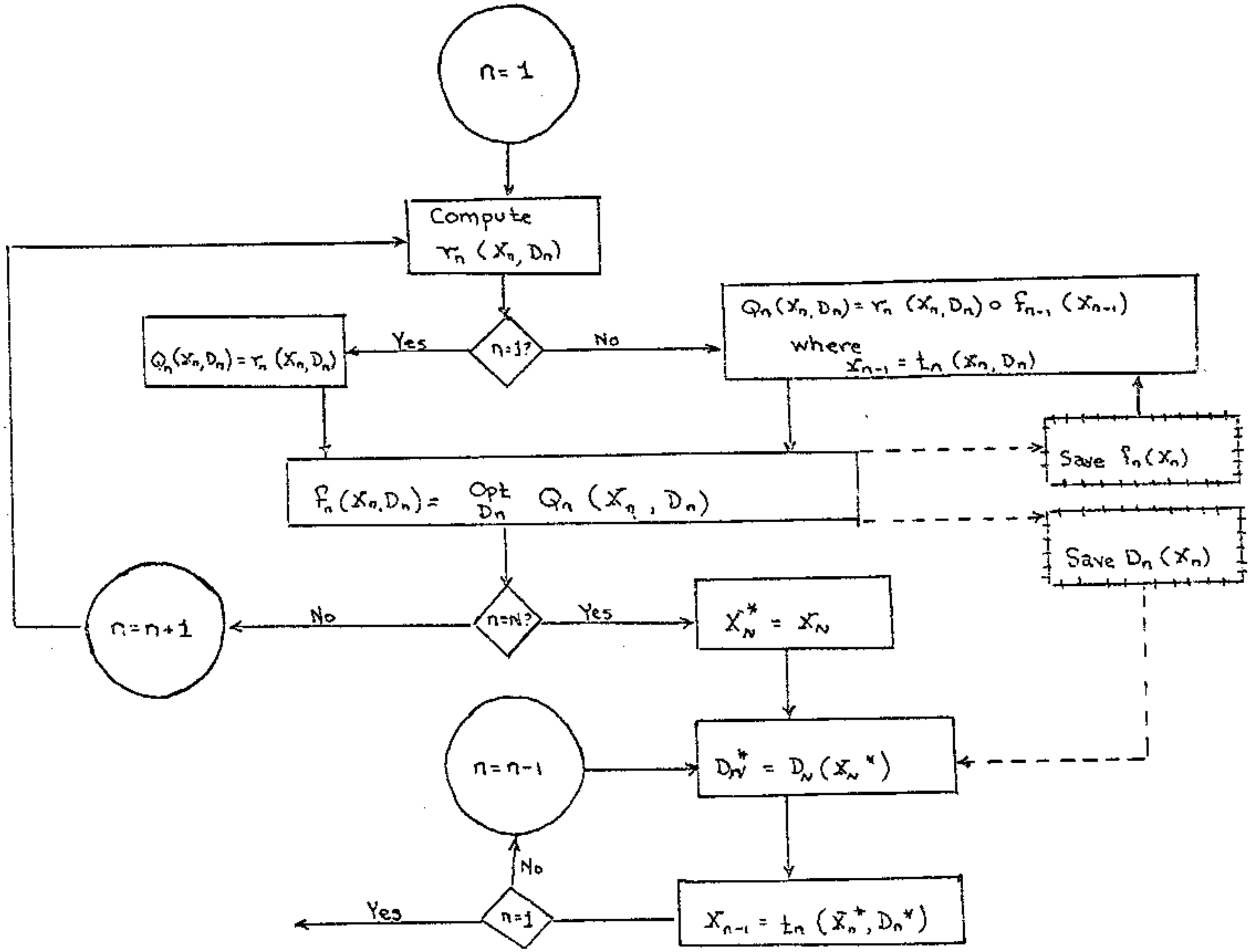

Fig. 3.7--Flow chart for dynamic programming

which, for n = 1, yields

$$f_1(X_1, D_1) = \underset{D_1}{\text{opt}}\; Q_1(X_1, D_1) = \underset{D_1}{\text{opt}}\; r_1(X_1, D_1).$$

At this point the problem under investigation is optimized with respect to the first decision, D_1, and the first allocation, X_1. As indicated, it is necessary to calculate $f_1(X_1)$ and $D_1(X_1)$ for storage. The value associated with $f_1(X_1)$ represents the optimal return for an input of X_1 units. The "value" associated with $D_1(X_1)$ represents the optimal decision policy. For $n = 1 \neq N$, the algorithm begins the recursive procedure by applying $n = n + 1 = 2$ for $n = 1$.

Since $n = 1 \neq N$, the recursive relation $n = n + 1$ reiterates the process for n = 2. This is the second stage of the computational process and requires that $r_n(X_n, D_n)$ be evaluated for n = 2; i.e., $r_2(X_2, D_2)$. Since $n \neq 1$, the iterative procedure requires the use of

$$Q_n(X_n, D_n) = r_n(X_n, D_n) \circ f_{n-1}(X_{n-1})$$

where $X_{n-1} = t_n(X_n, D_n)$. For n = 2, this yields

$$Q_2(X_2, D_2) = r_2(X_2, D_2) \circ f_1(X_1)$$

where $X_1 = t_2(X_2, D_2)$. At this point the stored value $f_1(X_1)$ is utilized subject to the property defined by the symbol "$\circ$." The process then moves to

$$f_n(X_n, D_n) = \underset{D_n}{\text{opt}}\; Q_n(X_n, D_n) = \underset{D_2}{\text{opt}}\; Q_2(X_2, D_2), \quad \text{for } n = 2.$$

This movement requires that

$$Q_2(X_2, D_2) = r_2(X_2, D_2) \circ f_1(X_1)$$

be optimized for the decision D_2, subject to the return $r_2(X_2, D_2)$ and the value of $f_1(X_1)$. It is here that the result of stage 1 is incorporated into the optimization at stage 2. If $n = 2 \neq N$, it is necessary to calculate $f_2(X_2)$, the optimal return at stage 2, and $D_2(X_2)$, the optimal decision policy for stage 2. These values are stored (retained) for use in the next iteration.

Repeated application of this process for $n = 3, 4, \ldots, N$ results in the optimal return for the N stage sequential process. The return at the $n = N$ stage yields

$$Q_n(X_n, D_n) = r_n(X_n, D_n) \circ f_{n-1}(X_{n-1})$$

where $X_{n-1} = t_n(X_n, D_n)$. The optimal return is found by evaluating

$$f_n(X_n, D_n) = \underset{D_n}{\text{opt}}\ Q_n(X_n, D_n) = \underset{D_n}{\text{opt}}\ r_n(X_n, D_n).$$

In this manner the problem is optimized for $n = N$ with respect to the $n = N^{th}$ decision, D_n, and the $n = N^{th}$ allocation, X_n. The value associated with $f_n(X_n)$ represents the optimal return for an input of X_n. The "value" associated with $D_n(X_n)$ represents the optimal policy decision for stage $n = N$.

Since $n = N$, it is necessary to determine the optimal set of inputs, X_n^*, for $n = 1, 2, \ldots, N-1$, and the optimal set of decisions, D_n^*, $n = 1, 2, \ldots, N$. This is accomplished by assuming the existence of a given (or prescribed) value of X_N, given by $X_N^* = X_N$, $n = N$. The value of D_N^* is obtained from storage, where $D_N^* = D_n(X_n) \equiv D_N(X_N)$ for $n = N$. In order to calculate X_{N-1}^*, it is necessary to apply the relation

$$X_{n-1} = t_n(X_n^*, D_n), \quad n = N.$$

Since $n = N \neq 1$, it is necessary to compute D_{N-1}^* from X_{N-1}^* and the stored "value" of $D_{N-1}(X_{N-1})$. When $n = 1$, the process terminates.

As previously noted, the functional connectors depends upon the problem being investigated. However, there are some characteristics that distinguish one problem from another. These major distinguishing characteristics are

(1) the return functions $r_n(X_n, D_n)$ and the transformations $t_n(X_n, D_n)$;

(2) the interpretation of the operator "o" which specifies how r_n and f_{n-1} are combined to obtain Q_n; and

(3) the technique used to [optimize] $Q_n(X_n, D_n)$ to obtain $f_n(X_n)$ and $D_n(X_n)$.[78]

Although the preceding discussion outlines a general methodology for solving a dynamic programming problem, it does not constitute an algorithm of the same magnitude as the simplex algorithm of linear programming or the algorithms for quadratic programming. According to W. Emory and P. Niland,

> other than the general statement that the total problem should be broken down into sequential, independent stages, [i.e., separable] there exists no set of instructions for dynamic programming formulation. Nor does there exist a general purpose algorithm. . .for solving a dynamic programming problem--this [the method of obtaining the solution] depends upon how it [the problem itself] has been formulated.[79]

This lack of a specific algorithm, however, does not negate the importance of the dynamic formulation. This formulation has two definite advantageous features relative to the solving of given problems:

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[78] Ibid., p. 48.

[79] William Emory and Powell Niland, Making Management Decisions (Boston, 1968), p. 240.

(1) [dynamic programming offers] a [problem] formulation which is [potentially] simpler to solve than an alternative formulation; and,

(2) [dynamic programming offers] a way of achieving an exact solution to a problem for which the calculus is not adequate [for example, a discontinuous cost function similar to one in which a new machine with cost factors different from the original must be purchased to expand the volume of production].[80]

From these features it is evident that dynamic programming refers to a decision-making process that is not characterized by a particular solution technique. Rather, dynamic programming refers to a conceptual method, not a computational one.

Applications of Basic Optimal Search

Introduction

The development of modern optimization theory as a tool of applied analysis begins with the utilization of the techniques of basic optimal search. Although these techniques can be described as extensions of classical optimization theory, the manner in which they are used and the types of problems to which they apply identify them as a distinct class of problem formulations and solution techniques.

Modern optimization theory utilizes a class of models that can be defined as optimum-seeking. The purpose of the modern optimization theory model is to determine the optimum level of activity for a given problem. This is accomplished by formulating the initial problem as one in which a defined objective function is to be optimized subject to a set of constraint functions. The objective function defines the relationship

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[80] Ibid.

which describes the activity that is to be optimized. The constraint functions define the relationships which describe the limitations within which the optimal activity is to occur.

The tools of basic optimal search have been identified as those which seek the optimum value of a defined objective function that is restricted by a set of linear constraints. With the exception of dynamic programming, the solution technique is dictated by the manner in which the problem is formulated. In each case, however, the objective is to select that level of activity from a set of alternative activities for which the objective function achieves its optimal value.

The solution techniques of basic optimal search are such that this optimum level of activity is located by a series of sequential iterations. At each iteration, the solution is tested for both feasibility and optimality. The optimal solution to the problem being investigated is only one of several possible solutions. However, the optimal solution is the solution which produces the best value of the objective function while satisfying all of the given constraints.

In the applications to follow, it is to be noted that the areas of applicability of the techniques of basic optimal search and classical optimization theory overlap. This is due to the fact that the implementation of basic optimal search techniques does not change the type of problem being investigated. Basic optimal search just provides a better means of formulating and solving problems that are unduly restricted when formulated as problems of classical optimization theory. This improved problem formulation is evidenced by the incorporation of the constraint functions in the mathematical model which is used to describe the given problem. The techniques of basic optimal search provide the means for efficiently solving these improved problem formulations.

Linear Programming

The development of linear programming evolved from the inability of simultaneous linear equations to adequately describe the real problem under consideration. Rather than locate the point (or points) at which a given system totally consumed all resources or commodities (money, machine-time, man-hours, etc.), the objective became the maximization (or minimization) of a defined function subject to a system of competing constraints. This introduced a multitude of possible solution points, only one of which was the maximum or the minimum. Since total consumption was not always possible (or feasible), the introduction of the inequality better represented the true situation.

As an administrative tool linear programming has been characterized as a method of analysis with certain distinct advantages. Among these are the following:

1. Insight and perspective into problem situations. Linear programming forces logical organization and study of information in the same way that the scientific approach to a problem requires. This generally results in a clearer picture of the true problem, which frequently is as valuable and revealing as the answer itself because it leads more surely to dealing with causes rather than effects--solutions rather than stop-gap expedients.

2. Consideration of all possible solutions to problems. Many management problems are so complex that difficulty is encountered in planning any feasible solution, let alone an optimum solution. By using linear programming, the manager makes sure he considers the best solution or solutions as well as any other that he might want to consider.

3. Better and more successful decisions. With linear programming the executive. . .builds into his planning a true reflection of the limitations and restrictions under which he must operate. . .when necessary to deviate from the best program [he] can evaluate the cost or penalty involved.

4. Better tools for adjusting to meet changing conditions. Once a basic plan is arrived at through linear programming, the basic plan can be reevaluated for changing conditions. Plans can be laid for several sets of conditions to find out how to prepare best for possible future changes. If conditions change when the plan is partly carried out, changes can be determined so as to adjust the remainder of the plan for best results.[81]

Other advantages include (1) improved use of productive factors by indicating the best use of existing facilities, (2) assistance in preparing future managers in analytical techniques, and (3) providing a base from which the allocation of scarce resources can be made.

The primary purpose of linear programming (as well as any mathematical programming technique) is the best allocation of some commodity or resource. The allocation process involves the best possible assignment of resources (e.g., man-hours, money, machines, raw materials, etc.) to specific activities in such a way that defined objectives are satisfied. Because of limited availability of productive resources, the resources must be assigned subject to restraining conditions, e.g.,

> The production manager must allocate the available machine time and labor hours in each department, along with the raw materials, to the activities of producing the different products which have been scheduled. He is limited by the availability of machines and labor, the amount of raw materials on hand, and the number of units required to be produced for each type of product.
>
> The shipping department must allocate the time and capacity of his trucks to deliver specific orders by a given time. He is limited by the number of trucks and the

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[81] Robert O. Ferguson and Lauren F. Sargent, Linear Programming (New York, 1958), pp. 11-14.

> capacities of each. Also he must consider the allowable delivery time associated with each order, and he must deliver the order within that interval.
>
> . . .[in capital budgeting] the company managers problem [may be that of] allocating a limited supply of capital to activities represented by the available investment opportunities.[82]

In each of these examples the objective must be obtained within the limits of the manager's capabilities. That is, he must select the optimum allocation of his resources (facilities) so as to achieve maximum return. In brief, this is the function of the linear programming model.

Although much of the work in the area of linear programming analysis has centered on the development of computational algorithms (e.g., the simplex algorithm of Dantzig, the revised simplex, the VAM (Vogel approximation method), algorithm, etc.), the problems to which they apply exhibit a basic, common characteristic: a defined linear objective function that is to be optimized (maximized or minimized) subject to a system of linear restraining functions. The restraining functions can be equalities or inequalities, and they are used to define the restriction placed upon the availability of each resource.

In addition to the defined linear objective function and the same requirements of linear analysis (with total consumption removed), problems amenable to linear programming exhibit some additional characteristics. Among these other characteristics are the following:

(1) The quantities of flow of various items into and out of the productive activity are proportional to the level of activity. (For

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[82] William R. Smythe, Jr. and Lynwood A. Johnson, Introduction to Linear Programming, with Applications (Englewood Cliffs, 1966), pp. 189-202.

example, doubling the activity level is accomplished by doubling all of the corresponding flows for the unit activity level.)

(2) There must be some way of describing the availability of resources for various combinations. (For example, in preparing an advertising budget the allocation of $X to market A reduces the amount of money available for use in market B.)

(3) The problem must be such that the decision maker has a choice of input combinations, or courses of action. One of these courses of action must achieve the desired objective.

(4) The problem under investigation must be such that it can be expressed in defined mathematical relationships. For linear programming these relationships are all linear.

Other considerations include the following:

(1) Raw material is available in unlimited quantities at a fixed cost per unit. (The firm may be buying in a perfectly competitive market or from a monopoly with a fixed price policy.)

(2) The output of each activity can be sold in unlimited quantity at a fixed price.

(3) The firm engages in "activities" to transform raw materials into products. These activities require the use of "internal resources" such as machine time or plant space.[83]

The applications that have been made to problems possessing these characteristics have been many. In addition to industrial and administrative applications, linear programming has been applied to problems involving agriculture, aircraft, and various military problems. For convenience these applications are summarized according to the general category.

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[83] Teichroew, op. cit., p. 416.

Industrial applications.--Linear programming has been used to solve a variety of industrial problems. Each of these applications is such that the general objective is to determine a plan for production and procurement in each of the time periods under consideration. It is necessary to satisfy all demand requirements without violating any of the constraints. The objective is to be optimized by maximizing a defined return or minimizing a defined cost. Examples of these applications follow.

(1) Production planning--the product mix problem. An industrial concern has available a certain productive capacity on various manufacturing processes and has the opportunity to utilize this capacity to manufacture various products. Typically, the different products will have different selling prices, will require different amounts of production capacity at the several processes, and therefore will have different unit profits. . .there may be minimum or maximum production levels set for given products. The problem is to determine the optimum, i.e., maximum profit, mix of products to produce for the capacities available. The solution would state how many units of each product to manufacture during the planning period.

(2) Production planning--the production-smoothing problem. An industrial concern has the problem of scheduling its production over a number of future time periods, with the total time span being considered called the "planning horizon."

(3) Production scheduling--alternate routings. In this type of problem the quantities of product to be manufactured during a period are fixed; however, there may be several alternate sequences (routings) of production processes by which a product can be manufactured. Each routing would have a different cost associated with it. Each manufacturing process would have a certain capacity, and the various products would have to compete for this capacity according to the particular routings selected for each product. The problem is to produce the required quantity of each product at minimum cost, subject to the constraints on process capacity. For each product, the solution would state how much of the required quantity of that product was to be produced by each alternative production routing.

(4) Distribution. There exist certain sources of supply (e.g., warehouses, factories, mines) geographically distributed and certain other locations (e.g., retail stores, warehouses) where a need exists for the materials available at the sources. Depending upon the relative geographical locations, freight rates, and possibly other considerations, there is a certain cost associated with transporting a unit of product from a given source to a given destination. Given these transportation costs, the resources at each source, and the requirements at each destination, the problem is to determine the minimum-cost shipping program. The solution would state what quantities are to be shipped from each source to each destination.

(5) Production-distribution problems. These problems occur when the products needed by the various destinations in a distribution problem do not exist in finished form, but rather must be manufactured at the sources before shipment. The sources may have different production costs. . .the solution would state what is to be produced at each source and where the goods are to be shipped.

(6) Blending problems. These products will occur when a product can be made from a variety of available raw materials of various composition and prices. The manufacturing process involves blending (mixing) some of these materials in varying quantities to make a product conforming to given specifications. The supply of raw materials and the specifications serve as constraints in obtaining the minimum-cost material blend. The solution would state the number of units of each raw material which are to be blended to make one unit of product.[84]

In each of these applications, the solution should specify the number of units of each product to be obtained from each production or procurement sources in each of the defined time periods.

Agricultural applications.--Linear programming has been used to determine the optimum allocation of a set of limited resources that maximizes the return from the allocation activity or minimizes some defined cost. Characteristic of these applications are the following:

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[84]Smythe and Johnson, op. cit., pp. 186-187.

(1) The diet (mixture) problem. Assume that the daily requirements of certain substances such as proteins, fat, carbohydrates, vitamins, etc. are given, the contents of these substances in available products are known, and the cost per unit of each product is also known. The problem is to compound a ration that will satisfy the daily requirements of the necessary substances and minimize costs. [An analogous application is the compounding of a mixture of petroleum products meeting minimum technical requirements with minimum cost.]

(2) Best use of arable land. Assume that n crops are to be grown on m plots with areas of $a_1, a_2, \ldots, a_m$, areas, respectively, where the average yield of the j^{th} crop from the i^{th} lot is a_{ij} units per acre. . .the return from one unit of the j^{th} crop is p_j dollars. [The problem] is to determine the area in each plot that should be sown in each crop so that a maximum return is obtained under a policy in which no less than b_j units are planted in the j^{th} crop ($j = 1, 2, \ldots, n$). [This particular problem is one involving mixed constraints.][85]

Flight-scheduling applications.--Linear programming has been effectively applied to problems of operational scheduling. This problem is similar to the transportation problem and is characterized by the flight-scheduling problem.

Assume that there are n different types of aircraft that must be scheduled for m routes, and assume that the monthly load transported by an aircraft of the i^{th} type on the j^{th} route is a_{ij} units, with an associated monthly operating cost of b_{ij} dollars. It is necessary to determine the number x_{ij} of aircraft of the i^{th} type that must be assigned to the j^{th} route to provide this line with a_{ij} units ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$) of carrying capacity with a minimal total operating cost when it is known that N_i aircraft of the i^{th} type ($i = 1, 2, \ldots, n$) are available (i.e., determine

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[85]S. I. Zukhovitskiy and L. I. Avdeyeva, Linear and Convex Programming (Philadelphia, 1966), pp. 101-103.

the optimal assignment of aircraft to given routes so as to provide the necessary carrying capcity with minimal total operating cost).[86]

Administrative applications.--The use of linear programming as a tool of administrative problem-solving is exampled by a variety of applications. Included in these areas of application are the following:

(1) Balancing of production and inventories. A company desires to schedule production according to a reliable sales forecast. The plant supervisor's interest is to stabilize his labor force and let inventories supply the slack, to which the treasurer objects since investing capital in inventories is not a very profitable enterprise. It is necessary to study the problem with a view to minimizing inventory costs and meeting sales forecasts.

(2) Personnel-assignment problem. Given a fixed number of persons and jobs, as well as the expected productivity of each person relative to each job, find an assignment of persons to jobs which maximizes the average productivity of the assigned personnel.

(3) Bid evaluation. Consider a problem with n depots and m separate bidders. Each of the m bidders wishes to produce an amount not exceeding $a_i (i = 1, 2, \ldots, m)$. The demands at n depots are b_j $(j = 1, 2, \ldots, n)$. It costs an amount c_{ij} to deliver a unit from the i^{th} bidder to the j^{th} depot. If x_{ij} denotes the quantity purchased from the i^{th} manufacturer for shipment to the j^{th} destination, then the problem is to minimize [delivery costs subject to a maximum acceptable shipment quantity and the condition that demand is met exactly].
i.e.,

$$\text{minimize} \sum_{i,\, j} c_{ij} x_{ij}$$

such that

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j;$$

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[86] Ibid., pp. 103-106.

$$\sum_j x_{ij} \leq a_i \text{ for each } i\text{; and}$$

$$\sum_i x_{ij} = b_j \text{ for each } j.$$

(4) Optimum estimation of executive compensation. The object is to determine a consistent plan of executive compensation. Salary, job rank, and the amounts of each factor required on the ranked job level are taken into consideration by the constraints of linear programming.

(5) Policy of a firm. Consider a firm which has access to certain factors of production whose supply, for one reason or another, cannot be increased in the time period in view. These resources limit the opportunities open to the firm. They may be utilized in various ways or not at all, and, depending on what is done with them, the resources, expenses, and profits of the firm will vary. The problem facing management is to find the productive program which will make the profits of the firm as great as possible, subject to the limitation that this program must not require more than the total available supply of any resource. The production problem becomes one of choosing which productive forces to use and the level at which to use each of them. [Application makes] it possible to state in advance how many different processes [need] to be used in order to maximize profit.[87]

Other applications.--It is easily seen that linear programming has been employed in many different areas. Other applications include investment decisions and

> problems of gasoline blending, structural design, scheduling of a military tanker fleet, minimizing the number of carriers to meet a fixed schedule, the least ballast shipping required to meet a specific shipping program, product distribution,. . . cost cutting in business, fabrication scheduling, profit scheduling,. . .computation of maximum flows in networks, steel-production scheduling, stocks and flows, the balancing of assembly lines, etc.[88]

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[87] Saaty, op. cit., pp. 168-174.

[88] Ibid., pp. 174-175.

In addition, linear programming has been used as an aid in maximizing portfolio selections, allocating advertising budgets between competing media for maximum penetration, and designing boxes for maximum storage space to minimize moving costs.[89]

The use of linear programming, as previously noted, has been quite extensive. Since its introduction as a tool of linear optimization analysis, linear programming theory has been well researched and extended with respect to both application and computational technique (e.g., the simplex method, the revised simplex, the MINIT method, integer programming, etc.).

Whatever the application, however, problems solvable by the techniques of linear programming possess certain distinct identifiable factors. These factors are (1) a defined function to be maximized or minimized, (2) resources fixed in terms of supply or availability, and (3) a competitive environment among alternative uses for available resources. All of these factors are assumed to be definable as linear expressions, i.e., a linear objective function subject to linear constraints.

Quadratic Programming

Quadratic programming represents a natural extension to the use of linear programming as a tool for administrative analysis. Whereas the linear programming problem restricts the analysis to total linear analysis, quadratic programming allows the use of a quadratic objective function

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[89] A. Channes and W. W. Cooper, "Optimizing Engineering Designs Under Inequality Constraints," Reprint No. 185 (Pittsburgh, 1962).

subject to linear constraints. For certain types of applications, this results in a better representation of the real situation; in fact, it is feasible that quadratic programming be applied to the same type of problems as linear programming, particularly those involving production planning (both product mix and production-smoothing), bid evaluation, investment selections, and cost-profit analysis.

However, the use of quadratic programming as an administrative tool of analysis has been somewhat limited. According to Beale there are three primary reasons for this lack of use.

> (1) In practical problems the nonlinearities in the constraints are often more important than the nonlinearities in the objective function;
>
> (2) When one has a problem with a nonlinear objective function there are generally rather few variables that enter the problem nonlinearly. But most methods for quadratic programming work no more simply in this case than with a completely general quadratic objective function; and,
>
> (3) Quadratic perturbations on a basically linear problem may well not be convex, although one may be fairly confident that, since they are perturbations, they will not introduce local minima. But most methods for quadratic programming cannot cope with such problems.[90]

Even with these drawbacks quadratic programming can be (and has been) applied to some classes of business problems, examples of which follow.

Minimum variance.--Quadratic programming has been applied to problems requiring the solution of a linear program with variable cost coefficients. These variable cost coefficients are such that they have been given

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[90] E. M. L. Beale, "Numerical Methods," Nonlinear Programming, edited by J. Abadie (New York, 1967), pp. 143-144.

expected costs and minimum variance. This application is characterized by Boot[91] in the following manner.

An investor has m dollars to invest in the stockmarket. His concern is the portfolio selection for his investments (i.e., the optimal combination of securities), a concern that will be reflected by the investor's attitude toward the rate of return on the individual investments and the amount of risk involved. As a means of quantifying his concern, it is assumed that the rate of return on investment in security i is normally distributed with expected rate of return μ_i and variance σ_{ii}.

As a measure of risk, the variance defines the amount by which the realized rate of return deviates from the expected rate of return. The greater the variance, the greater the risk. If investments are made in two different securities, say i and j, the covariance (σ_{ij}) of their returns defines the amount of correlation that exists between the rates of return on securities i and j. For $\sigma_{ij} > 0$ the interpretation is that an increase in the return on security i will be matched by an increase in security j; a decrease in the return on security i will also be matched by a decrease in the return on security j. (As a means of reducing risks, however, decreases in security i should be offset by increases in security j, a situation that is much desired.)

The investor is assumed to be one whose chief concern is the selection of an efficient portfolio, where an efficient portfolio is defined as one in which the following rules are observed: (1) given equal expected rates

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[91] John C. G. Boot, Quadratic Programming (Chicago, 1964), pp. 1-3.

of return, select the investment with minimum variance; and (2) given equal variances of returns, select the investment with the greatest expected rate of return. An inefficient portfolio can be similarly defined; i.e., an inefficient portfolio is one such that the construction of another portfolio results in one of the following: (1) given equal expected rates of return, a lower variance is observed in the newly constructed portfolio; or, (2) given equal variances of returns, a portfolio exists with a greater expected rate of return than the one selected.

The investor must now select the amount per investment to be made from the m dollars available for investment. The amount that will be invested in security i is given by π_i, the ratio between the dollar amount invested in security i and the m dollars available for investment. Given a risk aversion coefficient of $\rho \geq 0$, the investor's problem is summarized by

$$\max f(\pi, \mu, \sigma, \rho) = \sum_{i=1}^{n} \pi_i \mu_i - \rho \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \pi_i \pi_j$$

subject to $\sum_{i=1}^{n} \pi_i$, $\pi_i \geq 0$ for $i = 1, 2, \ldots, n$.

In the objective function, the term $\sum_{i=1}^{n} \pi_i \mu_i$ defines the total expected rate of return for the portfolio made up of n separate security investments. The term $\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \pi_i \pi_j$ defines the total variance. The value assigned to ρ defines the risk aversion determined by the investor. (For example, if $\rho = 0$ the investor has no risk aversion and will simply maximize his expected rate of return; if ρ is large the investor will minimize the variance of the rate of return, a case exemplified by an investor seeking a certain fixed amount of income.)

Given any value of ρ, maximizing $\sum_{i=1}^{n} \pi_i \mu_i - \rho \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}\pi_i\pi_j$ for known (or determined) values of π_i results in the most efficient portfolio: no other portfolio has a higher expected rate of return $(\sum_{i=1}^{n} \pi_i \mu_i)$ with the same variance $(\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}\pi_i\pi_j)$; nor does there exist another portfolio that has the same expected rate of return with a lower variance. If such did exist then the objective function was not maximal at the outset.

That this problem is one of quadratic programming is verified by examining the objective function $\sum_{i=1}^{n} \pi_i \mu_i - \rho \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}\pi_i\pi_j$ and the constraints $\sum_{i=1}^{n} \pi_i = 1$; $\pi_i \geq 0$ for i = 1, 2,...,n. The objective function is quadratic with respect to the decision variables π_i. This quadratic function is to be optimized subject to a set of nonnegative conditions and a linear equality.

Production analysis.--Production analysis via quadratic programming generally involves maximization of profits subject to linear production functions and linearly varying marginal costs.[92] This particular class of application is described in the following manner.[93]

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[92] George B. Dantzig, Linear Programming and Extensions (Princeton, 1963), p. 490.

[93] Boot, op. cit., p. 4.

A firm is capable of producing n commodities x_i $(i = 1, 2, \ldots, n)$. Each of the n products can be sold at a price of p_i $(i = 1, 2, \ldots, n)$. As the number of commodities produced increases, the profit per unit decreases linearly; i.e., $p_i = a_i - b_i x_i$, $(a_i > 0,\ b_i > 0\ \forall\ i)$. Thus, defining profit as the profit per unit, p_i, times the number of units of commodity x_i produced and sold, total profit is given by $\sum_{i=1}^{n} p_i x_i$, $(i = 1, 2, \ldots, n)$. Since the profit per unit is a function of the number of commodities produced, substitution for the p_i results in a total profit function defined in terms of the n commodities; i.e., total profit is given by

$$\sum_{i=1}^{n} (a_i - b_i x_i) x_i = \sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} b_i x_i^2 .$$

This total profit function is quadratic with respect to the units produced.

In addition to a decreasing profit per unit as the number of units produced increases, the production functions are assumed to be such that technical coefficients (denoted by C_{hi}) are fixed; resources (m in number) are scarce, where the resources are labor, capital, machine capacity, etc. As a means of describing the restrictions placed on the productive process, it is assumed that the restrictions are defined by linear functions and are such that at most d_h units of resource are available $(h = 1, 2, \ldots, m)$.

The problem confronting the firm is that of maximizing total profit subject to resource limitations and nonnegative units of production; i.e.,

$$\max \sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} b_i x_i^2$$

subject to $\sum_{i=1}^{n} C_{hi} x_i \leq d_h$ $(h = 1, 2, \ldots, m)$, and $x_i \geq 0$ for $i = 1, 2, \ldots, n$.

The profit function is a quadratic function. The constraint set is a system of linear equations. These two characteristics are the primary requirements of a quadratic programming problem.

Regression analysis.--Quadratic programming has also been used as a tool of regression analysis. In this use the objective is to find the optimal least-squares line fitting a given set of data, given that "certain parameters are known a priori to satisfy linear inequalities constraints."[94]

Statistical regression analysis is concerned with the development of a regression function, $y_j = \alpha + \beta x_j + \tau_j$, where y_j denotes the j^{th} observation of the dependent variable, x_j the j^{th} observation of a (nonrandom) independent variable, τ_j a random noise, and α and β the regression parameters. This regression function is such that the resulting curve (or line) minimizes the error of estimation (i.e., the resulting curve minimizes the difference between the value of the variable to be estimated, y_j, and the corresponding curve value, y_j^*). Minimal error is achieved by obtaining parameter estimates such that the regression equation is given by $y_j^* = a + bx_j$, where a and b minimize $\sum_{j=1}^{n} (y_j - y_j^*)^2 = \sum_{j=1}^{n} (y_j - a - bx_j)^2$.

Due to the fact that the minimizing process can lead to unrealistic a priori values, it is sometimes necessary to impose restrictions upon the parameter estimate b. This restriction takes the form of an interval

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[94]Dantzig, op. cit., p. 490.

within which the value of b might fall and transforms the regression analysis problem to one of the following form:

$$\min \sum_{j=1}^{n} (y_j - a - bx_j)^2$$

subject to $b_0 \leq b \leq b_1$.[95] The necessary requirements for quadratic programming are again present: a quadratic objective function subject to linear constraints.

Other applications.--Excluding the applications previously noted, and those in engineering design analysis, the primary use of quadratic programming has been in economic analysis. In fact, the uses most quoted as numerical examples in the literature relate to economic considerations: cost-profit analysis, demand analysis, and the study of utility functions. Consider the following two examples.

Example 1: A monopolist is confronted with m linear demand functions expressing demand as a function of the prices of the various m products produced by the monopolist. It is assumed that the quantities produced are sold (i.e., supply equals demand). The problem considered by the monopolist is that of maximizing total gross revenue subject to constraints that are linear representations of certain factors of production. It is further assumed that total gross revenue is quadratic, given that demand is linear in the prices; i.e., gross revenue is given by $\sum_{i=1}^{n} p_i x_i$, which is linear, where $x_i = b_i + \sum_{j=1}^{k} c_j p_j$, $i = 1, 2, \ldots, m$. The quadratic

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[95] Boot, op. cit., p. 3.

representation for gross revenue can be written in terms of the quantities demanded by solving the system of linear demand functions for the p_j's and substituting these into the gross revenue function. This is permissable so long as the coefficients of the functions are known with certainty.

The limiting factors confronting the monopolist are of two types: (1) all quantities produced are nonnegative, and (2) the factors of production are such that existing relationships are linearly defined. This condition allows representation of the limiting factors by a system of linear inequalities. Thus, the problem faced by the monopolist is given by max $\sum_{i=1}^{n} p_i x_i$, where $x_i = b_i + \sum_{j=1}^{k} c_j p_j$, i = 1, 2,...,m subject to $\sum_{i=1}^{n} d_i x_i \leq h_i$ and $x_i \geq 0$ for i = 1, 2,...,n.[96]

Example 2: Assume that the utility of an individual is a quadratic function of the quantities of goods available. The income of the individual and the prices paid for goods is known. Then, according to classical economic theory, the individual will seek to maximize his utility subject to the total consumption of income and nonnegativity of quantities purchased; i.e., letting x_i denote the quantities purchased, f(x) the utility function, B the income of the consumer, and p_i the price per unit of x_i, the problem is given by max $f(x) = f(x_1, x_2, \ldots, x_n)$ subject to $x_i \geq 0$ for i = 1, 2,...,n, and $\sum_{i=1}^{n} p_i x_i = B$.

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[96]H. Theil and C. Van de Panne, "Quadratic Programming as an Extension of Classical Quadratic Maximization," Mathematical Studies in Management Science, edited by Arthur F. Veinott, Jr. (New York, 1965), pp. 143-147.

The objective of this problem is to determine the combination of purchases that will maximize the utility function of the purchaser without exceeding his income. The budget constraint is linear, whereas the utility function is quadratic. The resulting system thus defines a quadratic objective function subject to linear restrictions, the necessary form for quadratic programming problems.

From these two examples it can be seen that there exist several problems exhibiting characteristics similar to those shown in the examples. In particular, the second can be extended to the theory of the firm with little effort. Consider a firm with the production function $f(x) = f(x_1, x_2, \ldots, x_n)$. With fixed output the x_i can vary in such a way that an isoquant (the locus of all combinations of the x_i yielding a fixed output) is constructed. Varying the value of $f(x)$ will yield a family of isoquant curves. Given a fixed outlay of cost, the firm will seek to maximize the production function subject to the fixed cost outlay (budget). A variation of this problem would be to minimize cost subject to a required level of output (budget). It is assumed that the restrictions are linearly related and the objective function quadratic.

As noted in the beginning, quadratic programming represents a natural extension to linear programming. The only difference with respect to problem formulation is the quadratic nature of the objective function and the lack of required nonnegativity. However, administrative application forces the nonnegative restriction with the result that the nature of the objective function is the primary difference.

Potential applications of quadratic programming relate to areas well entrenched by linear programming. In addition to those noted, other applications include (1) inventory problems structured in the form of quadratic programming so as to minimize inventory costs without exceeding storage capacity and maintaining a minimum order level (i.e., orders must exceed fixed quantities), (2) curvilinear (quadratic) analysis of growth and decay subject to a master planning budget, and (3) problems relative to aggregate planning models.[97] In each case there exists a quadratic function to be optimized subject to a set of linear constraints. Given these, the problem can be solved as a quadratic programming problem.

Geometric Programming

Although applicable to generalized polynomials, geometric programming has been restricted to strictly posynomial formulations. Wilde and Beightler indicate that this is due to the newness of the technique.[98] However, the posynomial case has been utilized in solving problems of the following types: (1) minimization of costs for a hauling operation, (2) design of electrical transformers for optimal performance, (3) minimization of costs for chemical processes, and (4) design of vapor condensers. Additional applications include the minimization of total cost for a given system in terms of the costs of the subsystem components, minimization of

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[97] Charles H. Kriebel, "Coefficient Estimation in Quadratic Programming Models," Reprint No. 315 (Pittsburgh, 1967), p. B-476.

[98] Wilde and Beightler, op. cit., p. 130.

total capital cost for a shipping operation so as to determine minimal cost, the optimal (minimal) number of ships, cargo weight, and speed, and minimization of costs of industrial operations.

As a means of describing the types of problems amenable to geometric programming, a selection of existing applications will be presented. Following this presentation, characteristics common to the demonstrated applications will be identified. These characteristics will then be extended to problems suitable for administrative analysis.

Shipping operations.--A long-term contract is to be let to the lowest bidder for the transport of a given quantity of two types of ore a given distance of nautical miles. The shipment is to be made between two ports and is such that the return trip will cost of like tonnage of similar products. Due to the tonnage that is to be transported, a successful bid will require purchasing a new fleet of completely automated cargo vessels. Of particular interest is the optimum number of ships necessary to perform the work, their tonnage, their speed when under sail, and the minimum cost incurred.

In solving this problem it is assumed that the capital cost of each cargo vessel is composed of two principal terms: the cost of the power plant and the cost of the vessel exclusive of the power plant. Consider first the cost of each vessel exclusive of power plant.

With the power plant excluded, each vessel is assumed to have a cost that is proportional to the cargo tonnage, q. The weight of each cargo vessle is then assumed to equal half the tonnage of the cargo; i.e., $\frac{1}{2}$ q. With a cost of $\$C_1$ per ton, the total cost of N new cargo vessels is given by $\frac{1}{2}(NC_1q)$.

The cost of the power plant is based upon the per shaft horsepower of the plant itself. Shaft horsepower is dependent upon displacement tonnage, q_D, the speed (velocity) of the vessel, V, and a fixed constant, K_1. With a cost of $\$C_2$ per shaft horsepower, the total cost of the N power plants is given by

$$\frac{C_2 N q_D^{2/3} V^3}{K_1} .$$

In addition to the capital cost of the new fleet, the bid of a given company must take into consideration the minimum amount of cost for the fuel. The cost of the fuel is to be considered as a capital cost and is calculated by applying the formula $\$f \frac{C_2}{K_1} q_D^{2/3} V^3$. The factor f defines the time duration during which each cargo vessel is under steam. For N vessels, the total fuel cost to be capitalized is given by

$$\frac{\$fNC_2 q_D^{2/3} V^3}{K_1} .$$

Assuming that the total cost is the sum of the three component costs, the total cost to be capitalized is given by the expression

$$\frac{NC_1q}{2} + \frac{C_2 N q_D^{2/3} V^3}{K_1} + \frac{fNC_2 q_D^{2/3} V^3}{K_1} .$$

The variables N, q, V, and f are subject to certain constraints. These constraints are given by the quantity to be transported per year and the docking facilities imposed on the time duration f. In defining these constraints it is assumed that K_2 hours are spent in travel per year, that unloading and reloading time is proportional to the cargo tonnage q and is known--say, K_3, and that loading and unloading operations are carried on 24 hours per day. Assuming a minimum yearly shipment of Q tons, a distance of T miles (one-way), and a minimum amount of travel time f, these constraints are given by the following:

(1) minimum tonnage per year: $NqV(\frac{K}{2T}) \geq Q$;

(2) limited docking facilities: $f \leq \frac{T/V}{(T/V) + (2q/K_3)}$.

The problem to be solved thus takes on the following form:

$$\text{minimize } H(N, q, V, f) = \frac{C_1 Nq}{2} + \frac{C_2 q_D^{2/3} NV^3}{K_1} + \frac{C_2 q_D^{2/3} fNV^3}{K_1}$$

subject to

$$NqV(K_2/2T) \geq Q;$$

$$f \leq \frac{(T/V)}{(T/V) + (2q/K_3)}$$

where K_1, K_2, K_3, T, C_1, C_2, and q_D are known. The values of N, q, V, and f are assumed to be nonnegative.[99]

Transformer design.--An air-cooled, two winding transformer with a fixed rating and fixed voltage is to be designed. The transformer has an expected load factor of 0.77 during a projected economic life, corrected for present-worth calculation of 20 years. The cost of electric power is

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[99]Duffin, Peterson, and Zener, *op. cit.*, pp. 161-163.

expected to average 0.03 monetary units per kilowatt-hour. The manufacturer desires to keep the variable production costs down. Sales literature will advertise low operating cost and low noise level. To meet competitors' claims of efficiency, the full load losses cannot exceed 28 watts.

The problem is to design a transformer meeting the specified requirements of rating and efficiency. This transformer is to be of such a design that the cost to manufacture and the cost to operate is minimal. In addition, the transformer's surface heat dissipation should not exceed 0.16 watts/cm^2.

The mathematical formulation of this problem is given by the following:

minimize $f(\underline{x}) = c_1 x_1 x_4 (x_1 + x_2 + x_3) + c_2 x_2 x_3 (x_1 + 1.57x_2 + x_4)$

$$+ c_3 x_1 x_4 (x_1 + x_2 + x_3) x_5^2 + c_4 x_2 x_3 (x_1 + 1.57x_2 + x_4) x_6^2$$

subject to

$$f_1(\underline{x}) = e x_1 x_2 x_3 x_4 x_5 x_6 - 1080 = 0$$

$$f_2(\underline{x}) = c_3 x_1 x_4 (x_1 + x_2 + x_3) x_5^2 + c_6 x_2 x_3 (x_1 + 1.57x_2 + x_4) x_6^2 - 28 \leq 0$$

$$f_3(\underline{x}) = \frac{c_3 x_1 x_4 (x_1 + x_2 + x_3) x_5^2 + c_6 x_2 x_3 (x_1 + 1.57x_2 + x_4) x_6^2}{2x_1(2x_1 + 4x_2 + 2x_3 + 3x_4) + 4x_2(1.57x_2 + 1.57x_3 + x_4) + 2x_3 x_4} - .16 \leq 0$$

$$\underline{x} \geq 0.$$

The x_i, i = 1, 2, 3, 4, define the dimensions of the transformer. The x_i, i = 5, 6 define the magnetic flux density and the current density, respectively. This problem is of special significance due to the quadratic nature of both the primal function and the primal constraints.[100]

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[100] R. Schinzinger, "Optimization in Electromagnetic System Design," Recent Advances in Optimization Techniques, edited by Abrahim Lavi and Thomas P. Vogl (New York, 1966), pp. 163-168.

<u>Design and transportation</u>.--Four hundred cubic yards of gravel must be ferried across a river. The gravel is to be shipped in an open box of length x_1, width x_2, and height x_3. The sides and bottom of the box cost \$10 per square yard and the ends of the box cost \$20 per square yard. The box will have no salvage value and each round trip of the box on the ferry will cost \$0.10. If it is required that the sides and bottom of the box be made from scrap material, and only four square yards are available, determine the minimum cost and the dimensions.

The solution to this problem is found by minimizing the total cost of the operation subject to the availability of material. This total cost is comprised of four separate cost figures: (1) cost of ferrying the 400 cubic feet of gravel, (2) cost of the two ends, (3) cost of the two sides, and (4) cost of the bottom. The constraint function is defined by the sum of the material needed for two sides and the bottom.

The mathematical description of the problem is given by

$$\min f(x_1, x_2, x_3) = \frac{40}{x_1 x_2 x_3} + 40x_2x_3 + 20x_1x_3 + 10x_1x_2$$

subject to $2x_1x_3 + x_1x_2 \leq 4$. The x_i, i = 1, 2, 3, values are nonnegative by definition.[101]

From this limited selection of known applications can be drawn several common characteristics. First, each of the defined objective functions is one of minimization. Second, in every case the objective function is nonlinear. Third, the problems are multivariable and are such that the existence of interrelationships is feasible. Fourth, the constraint

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[101] Duffin, Peterson, and Zener, <u>op. cit.</u>, pp. 5-11.

functions can be linear or nonlinear. The final, and perhaps most important common characteristic, is that the defined functions represent the sums of component costs and are constructed as posynomials.

A cursory examination of these examples and characteristics reveals some interesting points relative to geometric programming as a tool for administrative analysis. These points are as follows:

(1) Although a nonlinear technique, the class of problems to which geometric programming has been applied includes those of both linear and quadratic programming. Thus, geometric programming could be used to determine (a) the best product mix for cost minimization, (b) the best routing to minimize transportation costs, (c) minimum-cost shipping programs, (d) proper balance between production and inventories to minimize cost, and (e) minimum bids under competitive bidding situations.

(2) Unlike both linear and quadratic programming, the geometric program allows either linear or nonlinear constraint functions. In this sense it is truly a nonlinear technique.

(3) Geometric programming can be applied to a variety of mathematical functions: inverse functions of the type $x_1^{-n}x_2^{-m}$, general n^{th} degree functions, and general multivariable functions. With any type of function, either in the primal function or the primal constraint, the geometric program is equally competent.

(4) Each application is minimizing a defined posynomial cost function.

The usefulness of geometric programming as an administrative tool is twofold: (1) it affords the administrator with a somewhat simplified means of solving nonlinear minimization problems, and (2) it reduces a nonlinear problem to a linear one. The need for such a tool becomes apparent when linear analysis fails, as evidenced by the following two examples.

Example 1:

> Consider a company in the initial stages of developing a model for corporate-wide long-range planning. A management scientist usually knows that, even if he is an experienced businessman, it is difficult to give accurate, detailed forecasts of optimal production levels and market penetrations for his company for the next 10 or more years. Indeed, a major reason for an executive employing such a model is that he realizes how easily his intuition can be amiss when trying to fathom the influence of economic factors projected beyond the present. If production costs and revenues vary nonlinearly with the scale of operations, linearized guesses may not be good enough to give valid answers.[102]

Example 2:

> Consider a firm that schedules production by employing a dynamic, multi-item model that reflects the presence of significant machine setup times, limited machine-group capacities, and fluctuating demand requirements. . . the essence of the optimization problem is to modulate the various nonlinear effects of scheduling decisions. . . any simple linearization of the problem is likely to do violence to the fundamental nature of the optimization.[103]

Obviously, nonlinear assumptions can be made when suitable techniques are available for solving the problem thus formulated. As a tool of linear analysis, geometric programming is at best a check on some other technique. As a tool of nonlinear analysis, geometric programming represents a method by which strictly nonlinear, posynomial problems can be formulated and solved. Characteristic of these problems are the following:

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[102] Wagner, op. cit., pp. 514-515.

[103] Ibid.

(1) Gasoline blending. A model for blending gasoline from refinery raw stocks usually contains nonlinear constraints relating to each blend's octane rating, since this quality characteristic varies nonlinearly with the amount of tetraethyl lead added to the mix. The objective is to minimize the cost of the blend without exceeding or falling below the necessary octane rating.[104]

(2) Multi-item order quantities. A wholesaler frequently replenishes his inventory by ordering several items at one time from a given supplier. In this manner the wholesaler takes advantage of economies in shipping costs, paper work, and quantity discounts offered by the supplier. In this application the problem is to minimize the total cost when the associated costs of replenishment appear as nonlinear functions of the several order quantities.[105]

(3) Inventory analysis. The typical inventory problem is one in which the total cost of carrying inventory is to be minimized. The total cost function may or may not be restricted by a set of linear or nonlinear constraints. When constraints are imposed on the cost function, they generally define the maximum amount of space available for storing the inventory. Of particular interest is the inventory problem where the makeup of inventory includes at least two different products related in terms of a nonlinear-multivariable cost function. The geometric program reduces this nonlinear analysis to a linear case suitable for algebraic analysis.

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[104] Ibid., pp. 516-518.

[105] Ibid.

As an example of this type of analysis, consider the following problem.[106] Units of a certain model are in constant demand at a rate of h units per unit of time. It is assumed that stock shortages are not allowed. Additional assumptions include the following: (1) units are stocked in lots, (2) the set-up cost is independent of the number of pieces in the lot and is denoted by c_1, (3) the carrying cost for one piece per unit of time is c_s, (4) the total demand over a time interval θ is N, and (5) all lots contain an equal number of pieces, n. Determine the number of pieces in each lot that will minimize the total cost for set-up and carrying N pieces in inventory if: (1) there is but one product, (2) there are p different types of products, and (3) the multiple inventory is subject to a set of restrictive constraints.

Case 1: One product inventory. Assuming one common product carried in inventory, the mean inventory level during a period of time T is n/2. Carrying cost during that time period is $\frac{nc_sT}{2}$; i.e., mean inventory times the cost for the time period of length T. The total cost for one lot is then given by summing set-up cost and the carrying cost for time period T: $c_1 + \frac{1}{2} nc_sT$. At a demand rate of h units per unit of time, the number of pieces required will be given by $n = hT$. The number of lots to be run, r, is determined by the ratio between the total demand, N, and the number of pieces per lot, n: $r = N/n = \theta/T$. The total cost for the time interval θ is the product of the cost for setting up one run times the number of runs. Denoting total cost by F,

[106] Arnold Kaufman, Methods and Models of Operations Research (Englewood Cliffs, 1963), pp. 156-160.

$$F = (c_1 + \frac{nTc_s}{2})r$$

$$= (c_1 + \frac{nTc_s}{2})(\frac{N}{n})$$

$$= \frac{Nc_1}{n} + \frac{NT}{n}\ (\frac{c_s n}{2})$$

$$= \frac{Nc_1}{n} + \frac{\theta c_s}{2}\ n,\ \text{since}\ \frac{N}{n} = \frac{\theta}{T}.$$

Since the quantities N, θ, c_1, and c_s are known, F is a function of the single variable n:

$$F(n) = \frac{Nc_1}{n} + \frac{\theta c_s}{2}\ n.$$

The function F(n) is to be minimized and ca be written as

$$\min F(n) = C_1 n^{-1} + C_2 n,$$

where $C_1 = Nc_1$ and $C_2 = \frac{1}{2}\ \theta c_s$. Both C_1 and C_2 are constants.

Case 2: Multiple product inventory. The problem being considered here is one in which the inventory consists of p different types of products.[107] For this problem the following notation will be used:

c_i = carrying cost per piece per unit of time for product i;

γ_i = setup cost per lot or order for product i;

N_i = total number of pieces of product i;

n_i = number of pieces in a lot or order of product i;

T_i = time interval between two lots or orders;

θ = length of the supply period.

For this problem total inventory cost will be the sum of the individual costs of setting up each of the p product runs and producing r runs of

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[107] Ibid., pp. 396-399.

each product. Thus, since each of the p products has a cost function defined by

$$f_i(n_i) = \frac{N_i\gamma_i}{n_i} + \frac{\theta c_i}{2} n_i, \qquad i = 1, 2, \ldots, p,$$

the total cost for the multiple product inventory is given by

$$F(n_i) = \sum_{i=1}^{p} f_i(n_i)$$

$$= \sum_{i=1}^{p} \left[\frac{N_i\gamma_i}{n_i} + \frac{\theta c_i}{2} n_i \right].$$

The function to be minimized is the multivariable total cost function. This function can be written in the form

$$\text{minimize } F(n_i) = \sum_{i=1}^{p} C_i(n_i)^{-1} + C_i^* n_i \quad ,$$

where $C_i = N_i\gamma_i$ and $C_i^* = \frac{1}{2}\theta c_i$. Both C_i and C_i^* are nonnegative coefficients. The function defined by $F(n_i)$ is the sum of p separate component costs and is thus amenable to the technique of geometric programming.

Case 3: Multiple product inventory with constraints. The problem to be considered is an extension of the one discussed in Case 2. In Case 2 the total cost function is derived free from any type of restrictive constraint. When there are additional considerations other than just the order quantity to minimize inventory costs and still meet total demand during a given time period (for example, the amount of money available for carrying inventory), these considerations can be written into the mathematical program as constraint functions. These constraints can be either linear or nonlinear and are not restricted in number.

The multiple product inventory cost function must be modified to incorporate the addition of constraints into the problem.[108] In order to modify the results of Case 2, the following notation is necessary:

c_i = unit cost of raw material and labor for product i;

γ_i = all costs relating to one lot of product i;

n_i = number of units of product i in a lot;

ℓ_i = monthly sales of product i (a known constant);

α = monthly cost of inventory, expressed as a percentage of the average value of the inventory for some period.

The cost per unit of product i is the sum of the unit cost of raw material and labor for product i and the mean related cost per unit of product in each lot; i.e., $c_i + \gamma_i/n_i$. The cost for one month of productive activity is found by calculating the product of monthly sales (production must equal sales) and the cost per unit; i.e., $\ell_i[c_i + \gamma_i/n_i]$.

Assuming that demand is constant for each of the products and that shortage costs are negligible, the mean level of inventory maintained is $\frac{1}{2} n_i$. The inventory cost for the mean inventory level is given by the product of the monthly cost of inventory, α, the cost for one month's productive activity, $\ell_i[c_i + \gamma_i/n_i]$, and the mean inventory, $\frac{1}{2}n_i$; i.e.,

$$\alpha(\tfrac{1}{2}n_i)[\ell_i(c_i + \gamma_i/n_i)].$$

Since the total inventory cost for any one of the p products is the sum of the mean inventory cost and the cost for one month's productive activity, the total inventory cost for product i is given by

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[108] _Ibid._, pp. 399-413.

$$\ell_i[c_i + \gamma_i/n_i] + \frac{\alpha n_i}{2} [\ell_i(c_i + \gamma_i/n_i)].$$

The total inventory cost for the p products is the sum of their individual inventory costs. Letting total inventory cost be defined by $G(n_i)$,

$$G(n_i) = \sum_{i=1}^{p} \{\ell_i[c_i + \gamma_i/n_i] + \frac{\alpha n_i}{2} [\ell_i(c_i + \gamma_i/n_i)]\}$$

$$= \sum_{i=1}^{p} \{\ell_i c_i + \frac{\ell_i\gamma_i}{n_i} + \frac{\alpha n_i}{2} (\ell_i c_i + \frac{\ell_i\gamma_i}{n_i})\}$$

$$= \sum_{i=1}^{p} \{\ell_i c_i + \frac{\ell_i\gamma_i}{n_i} + \frac{\alpha n_i \ell_i c_i}{2} + \frac{\alpha n_i \ell_i \gamma_i}{2n_i}\}$$

$$= \sum_{i=1}^{p} [\frac{\alpha\ell_i c_i}{2} n_i + \ell_i\gamma_i(n_i)^{-1} + \ell_i c_i + \frac{\alpha\ell_i\gamma_i}{2}],$$

where α, ℓ_i, c_i, and γ_i are known constants. Letting $B_i = \alpha\ell_i c_i$, $A_i = \gamma_i\ell_i$, and $K = \sum_{i=1}^{p} [\ell_i c_i + \frac{\alpha\ell_i\gamma_i}{2}]$, this function can be written in the form

$$G(n_i) = K + \sum_{i=1}^{p} [\frac{B_i}{2} n_i + A_i(n_i)^{-1}].$$

Since inventory cost is to be minimized, $G(n_i)$ is written as

$$\text{minimize } G(n_i) = K + \sum_{i=1}^{p} [\frac{B_i}{2} n_i + A_i(n_i)^{-1}].$$

Possible constraints to the inventory level are costs, space for storage, and the time span allotted for inventory maintenance. These constraints can be expressed as either linear or nonlinear functions. The existence of these constraints yields an inventory cost function of the following form:

$$\text{minimize } G(n_i) = K + \sum_{i=1}^{p} [\frac{B_i}{2} n_i + A_i(n_i)^{-1}]$$

subject to

$$H_j(n_i) \leq M_j \qquad (i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, m.)$$

$$n_i \geq 0.$$

The function defined by $G(n_i)$ is a posynomial cost function that is to be minimized. This can be accomplished by the technique of geometric programming where $G(n_i)$ is the primal function and $H_j(n_i) \leq M_j$ is the set of primal constraints.

(4) Portfolio selection. In the portfolio problem an investor has a given amount of money he wishes to invest. There are n activities in which he may invest any amount. The return from the activities differ: some consistently pay a reasonable return, others fluctuate widely. Data are available for the past m periods. Let a_{ij} denote the return per dollar in the j^{th} activity during period i ($j = 1, 2, \ldots, n;\ i = 1, 2, \ldots, m.$) The problem is to determine the amount to be invested in each activity so as to achieve a minimum return of r and in such a way that the deviation from r is minimized.

The mathematical statement of this problem takes on the quadratic form

$$\text{minimize } \underline{Z}^T \underline{C}\underline{Z}$$

subject to

$$\underline{A}^T \underline{Z} \geq \underline{r}$$

$$1\underline{Z} \leq 1$$

$$\underline{Z} \geq 0,$$

where $\underline{Z}$ denotes the vector of investments Z_j, $\underline{C}$ denotes the matrix of coefficients defined by $c_{ij} = \frac{1}{M-1} \sum_{k=1}^{M} b_{ki}b_{kj}$, b_{ij} = the difference between the return per dollar, a_{ij}, and the average yield of the j^{th} activity, $a_j = \frac{1}{M} \sum_{i=1}^{M} a_{ij}$. The value of $\underline{r}$ is the minimum rate of return desired on investment. The problem to be solved then becomes one of minimizing a nonlinear objective function subject to a system of linear constraints.

Other applications.--Although additional applications and examples have not been presented, those discussed serve to demonstrate the types of problems to which geometric programming has and can be applied. Of particular note is the fact that each of these applications requires minimization of a function that is described as a posynomial. In addition, constraints, when they exist, are not restricted to linear functions. It is this characteristic that makes geometric programming an important addition to the administrator's kit of mathematical tools.

Potential applications for this new tool exist in areas of research and development of project designs, budget allocations, economic analysis for cost minimization, and investment analysis. It is possible to solve linear programming problems with geometric programming, a possibility that makes geometric programming applicable to minimization problems formulated as linear posynomials. However, the geometric programming technique is best utilized for nonlinear functions if its full usefulness is to be utilized. This usefulness lies in the fact that geometric programming reduces nonlinear analysis to the techniques of linear analysis.

Dynamic Programming

As a solution technique, dynamic programming has been applied to a variety of problems. Included in these applications are problems related to (1) the optimal assignment of machinery, (2) investment analysis, (3) the minimization of nonlinear cost functions, (4) multi-product production analysis, (5) the maximization of profit, (6) the maximization of expected demand, (7) network analysis, (8) replacement problems, (9) transportation analysis, and (10) the optimal allocation of productive resources. Other applications include (1) capacity design, (2) job sequencing, (3) budget analysis, (4) optimal timing for equipment replacement, (5) optimal selection of advertising media, (6) systematic search for locating vital resources, and (7) long-range strategy planning for replacement of depreciating assets.

It is evident that dynamic programming can be applied to a wide variety of problems. However, for a given application, there is no defined solution technique. Each problem is optimized on its own merit according to the manner in which the problem is formulated. This characteristic marks dynamic programming as the forerunner of true optimal search. It utilizes whatever solution technique is best applicable to the problem being investigated.

Unlike the previous solution techniques discussed, dynamic programming does not define a class (or family) of problems. It defines a computational technique whereby a multi-stage problem can be optimized as a series of one-dimensional problems. As such, any attempt to classify the problems to which dynamic programming can be applied must be somewhat general. With this in mind, the classification of problem types will be accomplished in the following series of steps: (1) describing the general characteristics of the dynamic programming formulation, (2) presenting a selective sampling of specific applications, (3) relating (1) and (2), and (4) generalizing from (1), (2), and (3). This approach will provide two basic ingredients: (1) a genuine feeling for the dynamic programming formulation and (2) a means whereby given problem types can be identified.

The formulation of a given problem as a dynamic programming problem requires the satisfying of a sequence of steps. If the problem under investigation is such that these requirements are satisfied, the problem is one that can be classified as belonging to the dynamic programming set. These requirements are as follows:

(1) the problem is of such a nature that it can be broken down into n-decision stages;

(2) given an n-stage process, the decision at the k^{th} stage involves the selection of at least one decision variable;

(3) the problem must be defined for any number of stages and, regardless of the number of stages, must be described by the same functional relationship at any of the n stages;

(4) given a problem consisting of k defined steps, there exists a set of parameters describing the state of the system (i.e., values on

which the decision variables and the value of the objective function for stage k depend);

(5) the set of parameters describing the state of the system at stage k are the same parameters describing the state of the system for all n stages; and,

(6) the selection of the decision variable(s) for stage n in an n-stage problem has no effect on the remaining n - 1 stages other than that of changing the parameters which describe the state with n stages into the set of parameters which describe the state with n - 1 stages.

Although formal documentation of problems to which dynamic programming is applicable has not been done in current literature, examination of some of the applications that have been made can be used to document, in a formal manner, the types of problems to which dynamic programming can be applied. For this examination process, applications of dynamic programming have been selected from the areas of design, allocation, inventory analysis, distribution analysis, equipment replacement, and production analysis.

Capacity design.--The term "capacity" is used in reference to the potential of a production or service facility to produce some rate of output. The manner in which capacity is measured and the manner in which it is changed is dependent upon the facility, the available technology, the product mix, the extent of the change, and the period of time for which the capacity is needed.

The dominant variable in capcity decisions is usually the rate of product demand expressed as capacity requirements for different time periods. This rate of demand tends to vary according to random, seasonal, general business, and long-term growth determinants. These factors necessitate the major decisions of when and how much capacity should be added.

Situations are often encountered in the long-term planning of production facilities in which the demand for output is an increasing function of time. The use of the S-curve, as shown in Figure 3.8, typifies the growth pattern of a product.

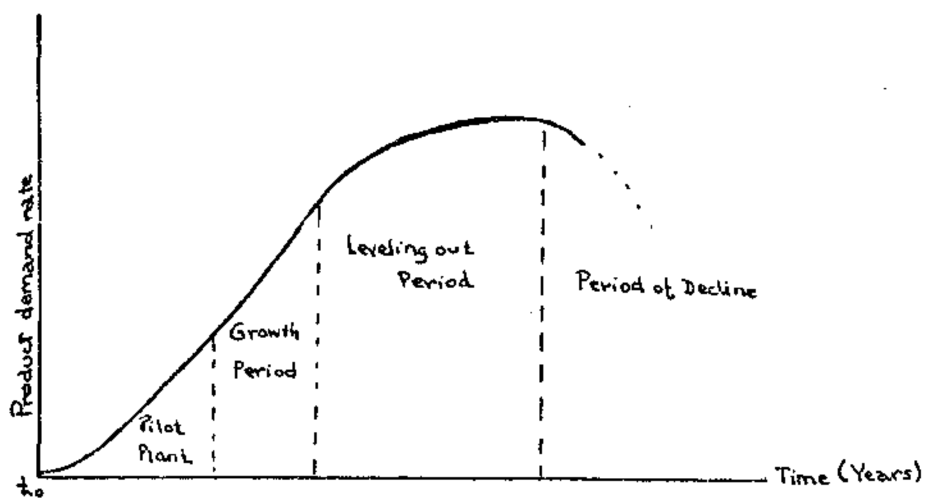

Fig. 3.8--Product growth pattern

A variety of policies exist for meeting the demand function. At one extreme, these policies include building capacity at the present time to meet all future requirements. At the other extreme, these policies include the adding on, in small increments, of capacity as demand increases. Between these two extremes lie a multitude of combinational possibilities.

In order to effect a choice, it is possible that cost tradeoffs must be made. The building of excess capacity offers the economies of scale in construction or initial capital costs. (For example, the larger a warehouse, the lower the cost per square foot of warehouse requirement.) Expanding the capacity at the current time to meet future requirements avoids the repeating of fixed costs of construction. Another consideration is that current expansion will reduce inflationary effects in later periods. On the other hand, increasing capcity in small increments avoids tying up capital in unused capacity. In addition, the commitment to smaller time intervals permits management to utilize improved technology and to avoid the risks inherent in long-term estimates.

In formulating this problem as one of dynamic programming, the following notation will be used:

${}_{t}A_{t'}$ = capital cost of building the requirements of period t in period t';

$$V_{t'} = \text{capacity built at time period } t = \begin{cases} v_t + v_{t-1} + \ldots + v_{t'}, & t > t' \\ v_t, & t = t' \end{cases}$$

$PWF(s)_i^{t'}$ = present worth of building costs at time t' and interest rate i.

It is assumed that capacity is built at the beginning of any period t and that the costs are incurred and discounted as of the end of the period to the present. If an economy of scale in capital costs is introduced such that ${}_{t}A_{t'} = \left(\frac{V_{t'}}{1 + cV_{t'}}\right) + k$, c, k constants, the problem involves a tradeoff between the economy of scale and the capital costs of unused capacity. The problem is to select the time period t for which

$$Z_t = \min_{t'} \{_tA_{t'}[PWF(s)_i^{t'} + Z_{t-1}]\}$$

is minimal.[109]

The problem described is one such that the expansion of capacity can take place at any one or more of n points in time. These points of expansion are located over the growth cycle of the product. (See Figure 3.9.) With n periods, there are $2^n - 1$ possible expansion policies.

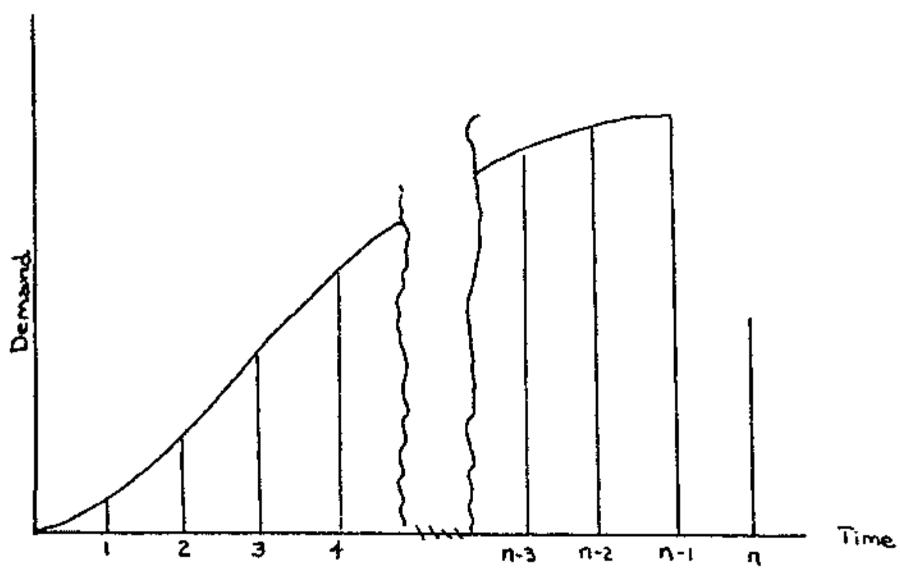

Fig. 3.9--Capacity design: expansion points

These policies range from building to meet future demand at the start of the growth period or building to meet increased demand at each of the n points (or stages).

[109]Gavett, op. cit., pp. 355-366.

Manpower loading.--A manufacturer is faced with the problem of determining the optimal size of his work force in each of the coming n months. Production requirements have been predetermined for the coming time period. It is assumed that in month j, m_j men would be the necessary number of men to complete the work during that period. If the manufacturer were not faced with layoff and hiring costs, he could use exactly m_j men during month j. It is further assumed that the work must be done in month j, but by working overtime it is possible to use less than m_j men.

Let x_j be the optimal number of men to do the work during month j. Let $f_j(x_j - x_{j-1})$ be the cost of laying off men or hiring men in month j; i.e., $f_j(x_j - x_{j-1})$ is the cost incurred when the size of the work force is changed from month x_{j-1} to month x_j, where

(1) $(x_j - x_{j-1}) > 0$ indicates a hiring cost and

(2) $(x_j - x_{j-1}) < 0$ indicates a layoff cost.

If there is no change in the work force from month j-1 to month j, $f_j(x_j - x_{j-1}) = 0$.

Let $g_j(x_j - m_j)$ be the cost incurred by not having the optimal number of men, x_j, on the job during month j; i.e., $g_j(x_j - m_j)$ is the cost incurred when the size of the work force is either inadequate $(x_j < m_j)$ or too large $(x_j > m_j)$. Thus,

(1) $(x_j - m_j) > 0$ indicates a cost of having men idle, and

(2) $(x_j - m_j) < 0$ indicates a cost of working men overtime.

If $x_j = m_j$, $x_j - m_j = 0$ and $g_j(x_j - m_j) = 0$.

The problem confronting the manufacturer is one of minimizing the total cost of both layoff and hiring, $f_j(x_j - x_{j-1})$, and the total cost of not having the optimal number of men on the job, $g_j(x_j - m_j)$; i.e., the minimal cost z is given by

$$z = \min \sum_{j=1}^{n} [f_j(x_j - x_{j-1}) + g_j(x_j - m_j)],$$

where $x_o = m_o$ defines the size of the work force at the beginning of the work period.[110] An important point to note is that no mention is made of the size of the work force in month $j = n + 1$. With $j = n + 1$ not specified the recursive relation utilized to solve this problem will be a backward relation. Had $j = n + 1$ been specified, with $x_o = m_o$ known, the recursive relation utilized would be a forward relation.[111]

<u>Distribution of effort</u>.--The distribution of effort problem is a modification of the traditional allocation of product problem in which management is seeking to allocate N products to s sources in such a way that total return, profit, is maximized. It is assumed that all N items available for distribution will be distributed to the s sources, a restriction that can be expressed linearly or nonlinearly.[112]

Let $R_j(x_j)$ be the return associated with the distribution of x_j units of product to store j. The x_j values are assumed to be nonnegative

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[110] Hadley, op. cit., pp. 376-379.

[111] Hadley provides the development of both the forward and backward solution to this problem in his work. The selection of a forward or backward relation is dependent upon the nature of the problem itself. For example, the forward solution indicates the influence of the work force in period n + 1; the backward solution indicates the influence of the work force at the start of the period.

[112] Wagner, op. cit., pp. 331-337.

integers. Then, the problem to be solved is given by

$$\max \sum_{j=1}^{s} R_j(x_j)$$

such that $\sum_{j=1}^{s} x_j = N$ and x_j is a nonnegative integer.

In order to convert this formulation into one of dynamic programming, the following are required:

$g_j(n)$ = profit when n items are optimally distributed to each of the s sources;

$x_j(n)$ = a distribution amount for source j yielding $g_j(n)$.

In this notation, the letter g indicates the profit received, n the number of items distributed, and j the source receiving the n items. A further consideration is the fact that this is not a time-oriented decision problem. The multistage property is that of considering one additional scurce at a time.

The dynamic programming problem is concerned with the maximization of the total return associated with the distribution of all N products to the s sources. Thus, the problem is described by

$$g_j(n) = \max_{x_j}[R_j(x_j) + g_{j-1}(n - x_j)], \quad \text{for } j = 1, 2,\ldots,s;$$

$$g_0(n) = 0 \quad \text{for } j = 0.$$

The maximization is for all nonnegative x_j such that $x_j \leq n$, $n = 0, 1,\ldots,N$. The objective of the problem is to find the value of $g_s(N)$. This is accomplished by evaluating $g_j(n)$ for $n = 0, 1, 2,\ldots,N$; $j = 1, 2,\ldots,s$. Given this set of values, the optimal distribution is found by tracing through to find those values of x_j that together yield $g_s(N)$.

Capital budgeting.--Each year a manufacturer of heavy equipment considers a large number s of independent proposals for major plant and machinery investments. Proposal j requires an outlay of K_j dollars from a fixed capital investment fund of M dollars. The expected return on the j^{th} proposal is R_j dollars. It is desired that total expected return be maximized.

Other factors to be considered include the limited availability of supervisors. Therefore, the total number of projects being considered in any one year cannot exceed N. Each proposal is unique, and the firm must decide to accept or reject the proposal for the entire year.[113]

Let x_j denote the j^{th} proposal. Then, $x_j = 0$ indicates that x_j was rejected; $x_j = 1$ indicates that x_j was accepted. Using this notation, the problem being studied can be written as

$$\max \sum_{j=1}^{s} R_j(x_j)$$

subject to

$$\sum_{j=1}^{s} x_j \leq N$$

$$\sum_{j=1}^{s} K_j(x_j) \leq M$$

$$x_j = \begin{cases} 0, \text{ reject, for each } j,\ j = 1, 2, \ldots, s \\ 1, \text{ accept.} \end{cases}$$

The dynamic programming formulation of this problem is accomplished by first defining $g_j(n, m)$ to be the total return when m dollars are available to invest optimally in n projects. An appropriate recursive

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[113] Ibid., p. 328.

relation is given by

$$g_j(n, m) = \max_{x_j} \{R_j(x_j) + g_{j-1}[n - x_j, m - K_j(x_j)]\}, \quad j = 1, 2, \ldots, s.$$

In this relation, n = 0, 1, 2,...,N, and m = 0, 1, 2,...,M. Maximization is over all nonnegative integers x_j such that $x_j \leq n$ and $K_j(x_j) \leq M$, where $K_j(x_j)$ denotes the dollar outlay associated with proposal x_j. At each stage consideration is given to the total number of proposals to be let and the amount of funds remaining for distribution.

Equipment replacement.--The equipment replacement problem is concerned with determining the optimal policy for replacing worn out or obsolete equipment.[114] Equipment in this context can be anything from tools to machinery to trucks, etc. For major pieces of equipment, neglecting such tools as hammers, pliers, etc., age tends to increase operating and maintenance costs, while decreasing productivity and salvage value.

The customary criterion used to determine an optimal replacement policy is that of maximizing the discounted expected profit or minimizing the discounted expected cost over some period of time. From a practical point of view, the usual approach is that of cost minimization. This is true for three primary reasons:

(1) when a product must undergo operations on a number of machines, it is difficult to determine what the contribution of any single machine is to the total profit;

(2) the output is frequently specified, and therefore the revenue received is a constant so that profit maximization is equivalent to cost minimization;

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[114] Hadley, op. cit., pp. 396-401.

(3) when known, lost profits from lower productivity can be included as a cost.[115]

It is assumed that decisions on equipment replacement are made periodically. It is also assumed that existing equipment can be maintained indefinitely if enough money and time is spent on repairs. The decisions relevant to equipment replacement will be made in light of the age of the equipment and its cost, with cost being a function of age. The analysis will be made on the basis of discounted expected cost. Thus, all that is necessary is the expected cost function.

Case 1: Replacement with no technological improvement. Under the condition of no technological improvement, it is assumed that replaced equipment is replaced by equipment of the same type. Costs to be considered are (1) installed costs, (2) expected operating and maintenance costs, and (3) salvage value, salvage value being defined as the sale price at time of salvage less dismantling costs. All of these costs are assumed to be the same for each new piece of equipment that is purchased. In addition, all operating and maintenance costs incurred during a given period of time are treated as though they were incurred at the end of the period.

For the analysis of this problem, suitable notation must be introduced. This requirement is met by the following:

I = the installed cost of the equipment;

$C(j)$ = the expected operating and maintenance costs for the j^{th} period;

$S(j)$ = the salvage value of the equipment after j periods of use;

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[115]Ibid., p. 397.

α = a discount factor for discounting costs known at the end of the period to the first of the period;

α^j = the discount factor which discounts costs known at the end of period j to the beginning of period 1;

N = the number of periods for which a machine is kept;

H = the cost of a piece of equipment over the period of time it is kept, discounted to the date of purchase; and,

K = the discounted cost of an infinite sequence of purchased equipment.

Since there is no technological improvement to be considered, an optimal replacement policy will be one such that the equipment is kept for N periods and is then replaced. With this arrangement, the value of N will be determined by assuming that equipment of the type being considered for replacement will be used for an infinite period of time and then minimizing the discounted cost over the infinite time period.

The cost of a piece of equipment over the time it is kept is calculated by summing the installed cost of the equipment and the expected cost of operating and maintaining the equipment for N periods, less salvage value. This cost is discounted to the date of purchase; i.e.,

$$H = I + \sum_{j=1}^{N} [C(j)]\alpha^j - [S(N)]\alpha^N,$$

where S(N) = the salvage value after N periods of use; α^N = the discount factor over N periods; α^j = the discount factor over j periods, $j \leq N$. The discounted cost for an infinite sequence of such units of equipment is found by discounting each unit in the sequence back to the date of purchase. Letting K(N) denote the discounted total cost of the equipment over N periods of time,

$$K(N) = H + H\alpha^{N} + H\alpha^{2N} + H\alpha^{3N} + \ldots$$

$$= H[1 + \alpha^{N} + \alpha^{2N} + \alpha^{3N} + \ldots].$$

Since the sequence $(1 + \alpha^{N} + \alpha^{2N} + \alpha^{3N} + \ldots)$ represents an infinite geometric sequence with first term equal to unity and common ratio α^{N},

$$K(N) = H[\frac{1}{1 - \alpha^{N}}],$$

where $\frac{1}{1 - \alpha^{N}}$ defines the sum of the sequence $(1 + \alpha^{N} + \alpha^{2N} + \alpha^{3N} + \ldots)$.

Substituting for H,

$$K(N) = \frac{1}{1 - \alpha^{N}} [I + \sum_{j=1}^{N} [C(j)]\alpha^{j} - [S(N)]\alpha^{N}].$$

The optimal replacement period is found by finding the nonnegative integer N for which K(N) is minimized. This is accomplished by evaluating K(N) for N = 1, N = 2,....

Given this straightforward approach to the problem being investigated, the dynamic programming problem can be developed. Using this formulation, it is assumed that at the beginning of each period a decision is made as to whether or not the equipment in current use will be replaced. This assumption defines a sequential decision problem.

Case 2: Replacement with technological improvement. With consideration being given to the influence of technological improvement, it is assumed that the total period of time under consideration is finite in length. It is also assumed that the total production of a piece of equipment is fixed over time. The assumption of constant productivity on the part of the equipment guarantees that increased productivity due to technological improvement will be reflected in lower operating costs. It is also assumed that the finite period under consideration consists of n one-year periods. All decisions relative to the equipment are made at the beginning of a given time period. The n one-year periods are assumed to begin with period 1. It is further assumed that the installed cost of a piece of equipment may depend upon the time period in which the equipment is purchased. Similarly, the operating and maintenance costs and the salvage value of a piece of equipment will depend not only on the age of the equipment, but also on the year in which the piece was purchased. It is further assumed that the state of technological improvement of the equipment is reflected by the year of purchase.

Inventory analysis.--Case 1: Nonlinear cost with linear constraints. The problem to be solved is one of determining an optimal production schedule and the cost associated with the optimal production schedule. It involves an analysis of both production-scheduling and inventory.[116] The assumptions made in formulating the problem are as follows:

(1) a fixed number of units, k, are to be produced and delivered over a given period of time consisting of n time periods;

(2) contract requirements require that k_1 units be delivered at the end of n_1 days, k_2 units at the end of n_2 days, and k_3 units at the end of n_3 days, etc., with the total time lapse not to exceed n days and total production not to exceed k.

(3) rather than expand existing facilities, overtime will be allowed, with the following nonlinear cost function being representative:

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[116] Richmond, op. cit., pp. 472-477.

$$y_i = a_i + bx_i(x_i - 1)$$

where y_i = the cost in dollars of manufacturing x_i units in the time interval $n_i - n_j$, $i > j$, and a_i and b_i are suitable cost constants.

(4) any excess units or parts of units produced and not utilized during a given time interval are carried over into the next period at a holding cost, h_i, of c_i per unit of carryover, v_i, and holding costs are assumed to be proportional for partial units held over; i.e., $h_i = c_i v_i$, where v_i = the starting inventory for period i;

(5) at the start of any given production cycle, there is no starting inventory, but units carried from one period to the next (for example, 10 units in time period 1 held over for time period 2) constitute the starting inventory for the new time period; and,

(6) units held as carry-over stock are not restricted to whole units (i.e., units are not defined as integer values only).

The problem to be solved is one of determining the number of units to be produced in a given period so that the sum of the manufacturing costs and the holding costs is minimized.

The cost function to be minimized is found by summing the holding costs for the periods being considered and adding this to the sum of the manufacturing costs for the periods involved; i.e.,

Total cost = total holding cost + total manufacturing cost.

This relationship is described by the nonlinear cost function

$$TC = \sum_{i=1}^{n} h_i + \sum_{i=1}^{n} y_i,$$

where $h_i = c_i v_i$ and $y_i = a_i + b_i x_i(x_i - 1)$, $i = 1, 2, \ldots, n$. This cost function is to be minimized subject to the restrictions imposed by the nonnegative values of x_i, the quantity produced in the first period, k_1, the total quantity produced in the first two periods, $k_1 + k_2, \ldots,$ and the total quantity that must be produced by the end of the time period, $k = \sum_{i=1}^{n} k_i$. Thus, the general problem to be solved takes on the following functional form:

$$\min (TC) = \sum_{i=1}^{n} c_i v_i + \sum_{i=1}^{n} [a_i + b_i x_i (x_i - 1)]$$

subject to the linear constraints

$$x_i \geq 0, \qquad i = 1, 2, \ldots, n;$$

$$x_1 \geq k_1;$$

$$x_1 + x_2 \geq (k_1 + k_2);$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$x_1 + x_2 + \ldots + x_n \geq (k_1 + k_2 + \ldots + k_n).$$

In expanded form, this problem is given by

$$\min (TC) = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n + [a_1 + b_1 x_1 (x_1 - 1)]$$

$$+ [a_2 + b_2 x_2 (x_2 - 1)] + \ldots + [a_n + b_n x_n (x_n - 1)]$$

subject to

$$x_i \geq 0, \quad i = 1, 2, \ldots, n;$$

$$x_1 \geq k_1;$$

$$x_1 + x_2 \geq k_1 + k_2;$$

$$\cdots\cdots\cdots$$

$$x_1 + x_2 + \ldots + x_n = (k_1 + k_2 + \ldots + k_n),$$

where x_i defines the number of units to be produced in the i^{th} period, n_i, and k_i defines the required units during the i^{th} period.

As a means of better illustrating this model, consider the case when the problem involves three time periods of equal duration. The problem can then be written as

$$\min\ (TC) = c_1v_1 + c_2v_2 + c_3v_3 + [a_1 + b_1x_1(x_1 - 1)] + [a_2 + b_2x_2(x_2 - 1)] + [a_3 + b_3x_3(x_3 - 1)]$$

subject to

$$(x_1, x_2, x_3) \geq 0;$$
$$x_1 \geq k_1;$$
$$x_1 + x_2 \geq (k_1 + k_2); \text{ and,}$$
$$x_1 + x_2 + x_3 = (k_1 + k_2 + k_3).$$

Since v_i defines the number of units to be carried forward into period i,

$$v_1 = 0;$$
$$v_2 = x_1 - k_1; \text{ and}$$
$$v_3 = (x_1 + x_2) - (k_1 + k_2).$$

These expressions for v_1, v_2, and v_3 can be substituted into the total cost function to yield an expression defining total cost in terms of the variables x_1, x_2, and x_3:

$$\min\ (TC) = c_2(x_1 - k_1) + c_3(x_1 + x_2 - k_1 - k_2) + [a_1 + b_1x_1(x_1 - 1)] + [a_2 + b_2x_2(x_2 - 1)] + [a_3 + b_3x_3(x_3 - 1)]$$

$$= c_2x_1 - c_2k_1 + c_3x_1 + c_3x_2 - c_3k_1 - c_3k_2 + a_1 + b_1x_1^2 - b_1x_1 + a_2 + b_2x_2^2 - b_2x_2 + a_3 + b_3x_3^2 - b_3x_3;$$

$$\min\ (TC) = [a_1 + a_2 + a_3 - (c_2 + c_3)k_1 - c_3k_2] + [(c_2 + c_3) - b_1]x_1 + [c_3 - b_2]x_2 - b_3x_3 + b_1x_1^2 + b_2x_3^2 + b_3x_3^2.$$

This function is to be minimized subject to the linear constraints

$$x_i \geq 0, \quad \text{for } i = 1, 2, 3;$$

$$x_1 \geq k_1;$$

$$x_1 + x_2 \geq (k_1 + k_2); \text{ and,}$$

$$x_1 + x_2 + x_3 = (k_1 + k_2 + k_2).$$

Although the problem can be solved by the use of the calculus and Lagrange multipliers, dynamic programming provides a simpler method of solution. In this respect, it is important to remember that a_i, b_i, c_i, and k_i are numerical constants for all i (i = 1, 2, 3). With this in mind, the dynamic programming formulation follows:

Let $f(v_i, x_i^*)$ = the optimum total cost for the remaining product-schedule starting at the start of period i. Let $f(v_i, x_i)$ = the total cost for period i plus the optimum total cost for the next period, i + 1. Then, $g(v_i, x_i) = (h_i + m_i) + f(v_{i+1}, x_{i+1}^*)$ defines the recursive relation.

Substituting for h_i and m_i,

$$g(v_i, x_i) = c_i v_i + a_i + b_i x_i (x_i - 1) + \sum_{j=i+1}^{n} c_j v_j + \sum_{j=i+1}^{n} [a_j + b_j x_j (x_j - 1)].$$

This relation best utilizes the backward solution technique of dynamic programming because all terms beginning with j = i + 1 vanish at the final stage. The iterative process is begun by assigning the optimal value to x_3^*. This optimal assignment is equal to the number of units required in the last period less the carryover from the preceding period; i.e.,

$$x_n^* = k_n - v_n \equiv k_n - [\sum_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} k_i].$$

The recursive relation at period n is given by

$$g(v_n, x_n) = c_n v_n + a_n + b_n x_n(x_n - 1).$$

Substitution of the optimal value of x_n, $x_n^* = k_n - v_n$, results in

$$g(v_n, x_n^*) = g(v_n) = c_n v_n + a_n + b_n(k_n - v_n)(k_n - v_n - 1).$$

Expansion of $g(v_n)$ yields $f(v_n, x_n^*)$

$$f(v_n, x_n^*) = b_n v_n^2 - 2b_n k_n v_n + b_n v_n + c_n v_n + b_n k_n^2 - b_n k_n + a_n,$$

which is nonlinear in v_n.

Had the objective been to minimize carryover into period n, $g(v_n)$ could be solved for v_n such that $g'(v_n) = 0$ and $g''(v_n) > 0$. However, since this is not the case, it is necessary to go back to stage n - 1 to derive the dynamic programming relation that can be utilized throughout the problem. At stage n - 1,

$$g(v_{n-1}, x_{n-1}) = (h_{n-1} + m_{n-1}) + f(x_n, x_n^*)$$

$$= c_{n-1}v_{n-1} + [a_{n-1} + b_{n-1}x_{n-1}(x_{n-1} - 1)] + f(v_n, x_n^*).$$

Application of the backward technique results in the optimal solution (least cost and production requirements) for the given problem. The solution set will be defined in such a way that

$v_1 = 0$, $x_1 = d_1$;

$v_2 = d_1 - k_1$, $x_2 = d_2$;

$v_3 = (d_1 + d_2) - (k_1 + k_2)$; $x_3 = d_3$;

. . . .

$v_n = (d_1 + d_2 + \ldots + d_{n-1}) - (k_1 + k_2 + \ldots + k_{n-1})$; $x_n = k_n - v_n$.

This backward process will result in a total cost function written in terms of the stage 1 production variable x_1. This function, which will be at most a quadratic function, can be minimized by applying the max-min calculus. Given this result, v_2 can be determined. At stage 2, the total cost function will have been expressed as a function of both v_2 and x_2, with x_2 written as a function of v_2. The process continues until it terminates at stage n.

Case 2: General inventory processes. The inventory problem discussed in Case 1 represents a specific application of inventory analysis subject to a set of restricting linear functions. This is a special application of the general inventory problem to be discussed in this section.

The problem to be analyzed is one in which a firm is confronted with the establishing of a production schedule for a given item during the next N time periods.[117] In this analysis, it is assumed that an accurate forecast of the inventory required to meet demand during each of the N periods has been presented to the management staff. Other assumptions include the following:

(1) production during time period i can be used to fill, completely or partially, the demand during period i;

(2) due to the varying nature of demand from one point to another, it is often economical for the firm to produce more of the item than is needed in one period, with the excess being stored for later use;

(3) the cost of storing excess production--attributed to such factors as interest on financial capital obtained through loans, rental fees for storage space, insurance, and maintenance--is lumped into a storage cost per unit of excess capacity; and,

(4) ending inventory will be zero.

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[117] Wagner, op. cit., pp. 261-271.

In order to establish a production schedule for the coming N periods, it is necessary to define a suitable criterion on which to base the optimal production schedule. It is assumed that the criterion of optimality will be the production schedule which minimizes total cost for the production-inventory period. As a means of formulating this problem, the following notation will be utilized:

x_i = the quantity produced in period i;

k_i = the inventory at the end of period i;

$c_i(x_i, k_i)$ = the cost function to be minimized;

D_i = the demand requirement in period i.

It is assumed that the demand requirement in period i is a nonnegative integer and is known at the beginning of period i.

Assuming that the cost in each period depends only on the production quantity and the ending inventory level, the cost function to be minimized can be written as

$$\min \sum_{i=1}^{N} c_i(x_i, k_i),$$

where the summation is taken over all of the N production periods. Restrictions placed on the x_i and k_i variables are as follows:

(1) for each period i, x_i is assumed to be an integer;

(2) inventory at the end of the N periods is zero;

(3) demand for each period is totally satisfied.

The third condition is met by defining two additional requirements:

(1) inventory at the end of period i = (inventory at the beginning of period i) + (production in period i) - (demand in period i); i.e.,

$$k_i = k_{i-1} + x_i - D_i; \text{ and,}$$

(2) each period's entering inventory and production must be large enough to guarantee that ending inventory for each period is a non-negative integer; i.e., $k_i = 0, 1, 2, \ldots$ for period i, $i = 1, 2, \ldots, N-1$.

As a matter of note, the equality $k_i = k_{i-1} + x_i - D_i$ can be written as $D_i = (k_{i-1} + x_i) - k_i$. This is a linear restriction. As such, if $c_i(x_i, k_i)$ is linear for all i, the problem is equivalent to a network problem and can be solved as one. However, if the cost function is such that it is necessary to batch the production, then the incremental unit cost of items in period i is less than that for items in period i - 1. Production in excess of the normal level for a given period may give rise to an increase in incremental cost because of overtime. Such problems are generally represented by nonlinear cost functions.

Problems of this general category can be solved by utilizing the dynamic programming formulation. However, the utilization of the dynamic programming requires that the following parameters be identified:

d_n = the demand requirement in the given period, with n periods remaining;

$c_n(x, j)$ = the cost of producing x units of product with j units of ending inventory for the period under consideration, n periods remaining;

$f_n(k_n)$ = the minimum policy cost when entering inventory is a level k with n periods remaining; and,

$x_n(k_n)$ = the production level yielding $f_n(k_n)$.

A cursory analysis of this problem reveals that the variable which determines the state of the production system at the start of the period is entering inventory. Another consideration is the amount of product demanded during the given period. Thus, it would be reasonable to assume that the minimum policy cost for a given period is a function of both the amount to be produced during the period and the inventory at the end of the period; i.e., for any given n,

$$f_n(k_n) = c_n(x_n, k_{n-1} + x_n - d_n), \quad i = 0, 1, 2, \ldots, d_n.$$

As an example of how this relation functions, consider the following general analytical approach. A value of n = 0 indicates that the production process is operating in the last stage. Thus, $f_o(k) = f_o(0) = 0$ since no periods remain. When n = 1, this indicates that the production process is into its next-to-the-last stage. At this point (n = 1), the productive capacity must be such that the total demand during the last stage is satisfied and ending inventory for the last period will be zero; i.e., $f_1(k_n) = c_1(x_1 = d_1 - k_1, 0)$. When n = 2, the indication is that two production periods remain. The minimal policy cost will be the cost defined by

$$f_2(k_n) = c_2(x_2, k_1 + x_2 - d_2) + f_1(x_1 + k_1 - d_2).$$

This formulation represents the cost of producing x units with two periods remaining with an ending inventory of $(k_1 + x_2 - d_2)$ units. This cost will be minimized subject to $d_2 - k_2 \leq x_2 \leq d_1 + d_2 - k_2$, where $d_2 - k_2$ equals the excess of demand over entering inventory in period 2 and $d_1 + d_2 - k_2$ equals the excess of total demand in the last two periods over inventory at the end of period 2. This simply defines the range within which x_2, the quantity produced with two periods remaining, can fall. A continuation of this process results in the following recursive relation:

$$f_n(k_n) = \min_x [c_n(x_n, k_{n-1} + x_n - d_n) + f_{n-1}(x_{n-1} + k_{n-1} - d_n)]$$

where $n = 0, 1, 2, \ldots, N$, $i = 0, 1, \ldots, d_1 + d_2 + \ldots + d_n$, and the value of x_n lies in the interval $d_n - k_n \leq x_n \leq d_1 + d_2 + \ldots + d_{n-1} - k_n$.

The recursive relation just developed developed defines the productive activity for a given period in terms of the entering inventory at the start of that period. It is applied by first determining the production level $x_n(k_o)$ which gave the minimal policy cost $f_n(k_o)$; i.e., how much production is necessary with N periods remaining to minimize policy cost with N periods remaining, given an initial period inventory of k units. At the next iteration, the entering inventory will be $k_o + x_n(k_o) - d_N$. Given this inventory, determine the production level $x_{N-1}(k_1)$ that minimizes $f_{N-1}(k_o + x_N(k_o) - d_N)$, etc.

From this development can be drawn two major points. These points are that (1) the level of entering inventory completely describes the state of the system when n periods remain, and (2) the quantity produced in one period affects the quantity of inventory remaining at the start of the next period.

Other applications.--Although the preceding selection has been limited, it does serve to illustrate some of the specific problems to which dynamic programming has been applied. In addition it demonstrates the wide range of problems to which dynamic programming is applicable and the manner in which given problems are formulated as dynamic programming problems.

Nemhauser indicates some general classes of problems suitable for formulation as dynamic programming problems.[118] These classes are divided into multistage inventory problems, nonlinear allocation problems, sequential processes, and combinational optimization problems. His work is summarized below.

Multistage inventory problems: The typical inventory problem is one in which the combined costs of production, inventory (both ordering and production inventory), and shortage are to be minimized in such a way that an optimal production scheduling-inventory level is established. The central thread in the structure of the dynamic inventory problem is that the optimal stock level in a given period, n, is that level which minimizes the cost in the n^{th} period and yields an optimal inventory at the beginning of period n - 1. The n^{th} period refers to the number of periods remaining.

Nonlinear allocation problems: The nonlinear allocation problem is defined by

$$\max \sum_{n=1}^{N} f_n(X_n)$$

subject to

$$\sum_{n=1}^{N} h_{ni}(X_n) \leq k_i, \qquad i = 1, 2, \ldots, m.$$

In this formulation X_n defines the quantity of resource, product, or effort allocated to the n^{th} source of return. The individual terms $f_n(X_n)$ define the return associated with the allocation of X_n. The $h_{ni}(X_n)$ define the constraints within which the optimal allocation is to

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[118] Nemhauser, <u>op. cit.</u>, pp. 244-247.

be achieved. By altering the terminology, this problem can be used to (1) analyze economic problems involving the allocation of productive resources, (2) analyze budget problems involving the allocation of advertising expenditures, and (3) analyze market areas for the optimal allocation of salesmen.

Sequential process problems: Sequential process problems involve the analysis of multistage systems. Applications in this area have centered on the minimization of cost of designing and building distillation towers, separation equipment, and chemical reactors. Other problem analysis has dealt with the optimization of both design and control processes.

Combinational optimization problems: Combinational optimization problems are problems involving sequencing, scheduling, or routing. Characteristic of such problems is the traveling salesman problem in which a territory consisting of n sites is to be covered in such a way that the total distance traveled in visiting these sites is minimal. This problem type can be extended to line balancing, machine scheduling, and job sequencing. The objective is to determine the multistage sequence or arrangement which optimizes the given objective function.

Analysis of these applications of dynamic programming reveals that in each case the basic requirements of the dynamic programming problem are satisfied as follows:

(1) the problem being analyzed is separated into a series of n stages;

(2) the decision to be made at any one stage involves the selection of at least one decision variable (for example, in the section on inventory

analysis, case 2, the decision variable is the quantity produced in the i^{th} period;

(3) each problem is defined for every stage by the same functional relationship;

(4) for each problem, the state of the system is described by a given set of parameters at each defined stage (for example, in the section on inventory analysis, case 2, each stage is defined, or set, by two known parameters, demand for that period and entering inventory);

(5) the set of parameters describing the state of the system at any given stage in the problem is the same for every stage; and,

(6) the selection of the decision variable at the n^{th} stage has no effect on the remaining (n - 1) stages other than that of changing the values of the parameters describing stage (n - 1).

Since each of the cited examples satisfy these basic requirements, each is a valid example of the dynamic programming formulation. Analysis of these examples reveals that each one groups the decision variables and associated restrictions into a series of sequential stages, summarizes the information from previous stages that is relevant to the selection of the optimal values for the current decision variables, provides a forecastable influence on the state of the system at the next stage (the current decision, given the present state of the system), and determines the optimality of the current decision in light of its potential impact on the present stage and on all following stages.

As previously noted, dynamic programming currently does not refer to a specific mathematical model as does, say, linear programming. Rather, the term refers to a manner in which problems are viewed. In this way, dynamic programming is applicable to a variety of mathematical models, not to just one class of models. This characteristic alone is the main contributing factor to the lack of a general dynamic programming model.

From a practical point of view, the general dynamic programming problem is one in which a given set of resources (time, money, labor, etc.) is to be allocated in such a way that a defined criterion is optimized. It is this classification into which the problems confronting the management analyst fall. Administrative problems involve allocations of jobs to machines, dollars to projects, capacity expansion to points in time, workers to time periods, products to warehouses or sales areas, production to points in time, etc.; and, given the satisfaction of the six requirements for dynamic programming formulation, these allocations are such that they are dynamic and respond to the environment at the time of decision. With these thoughts in mind, the administrative dynamic programming problem is defined in the following manner.

Definition 3.14.--Administrative dynamic programming is defined as an allocation problem in which a given resource or set of resources is to be distributed in a series of n sequential stages over a defined period of time in such a way that a defined criterion of effectiveness is optimized. The allocation is made in such a way that any constraints are satisfied and the following set of conditions are met:

(1) the problem consists of n defined stages;

(2) each stage is defined by at least one decision variable;

(3) the defining parameters remain the same for all stages;

(4) the objective function is defined in such a way that it is the same at each of the n stages;

(5) the values of the defining parameters are known at each stage; and,

(6) with the exception of the parameter values, the decision values at each stage are selected in an independent manner.

An important point to note in this definition is that no mention is made of the form of the objective function or its constraints. This is attributed to the fact that the criterion of effectiveness can be linear or nonlinear. In addition, it can be restricted by linear or nonlinear constraints. At each stage in the optimization process, dynamic programming permits the function to be optimized in the manner best suited to the problem being analyzed. Regardless of the number of stages, optimization at each stage in the multistage problem yields the optimal solution for the total problem.

CHAPTER IV

MODERN OPTIMIZATION THEORY: ADVANCED TECHNIQUES OF OPTIMAL SEARCH

Introduction

The presentation of optimization theory has moved from the classic techniques of linear and nonlinear functional analysis, systems of linear equations, the max-min calculus, the Lagrange multiplier, and queueing theory to the basic techniques of modern optimization theory. The techniques of modern optimization theory considered as basic optimal search have been identified as linear programming, quadratic programming, geometric programming, and dynamic programming. The discussion has traced administrative application of optimization theory from simple breakeven analysis to dynamic inventory analysis and constrained nonlinear analysis for cost minimization and/or profit maximization. Related problems have been identified. From this development it is evident that a wide range of optimization-oriented administrative problems exist. It is also evident that these problems can be solved by a variety of techniques.

The development of the advanced techniques of modern optimization theory begins with the realization that the term "modern optimization theory" refers to the conceptual development of a given problem and the implementation of the solution technique best suited to solving the particular problem. The technique does not dictate the formulation of the problem. Rather, the formulation of the problem dictates the selection of a suitable solution technique. In this way modern optimization theory can be considered as having two central characteristics: (1) the problem under investigation is described by the most appropriate mathematical formulation, and (2) given this formulation the most appropriate solution technique is utilized to find the optimal solution.

It is important to remember that the optimal solution may be one that is suboptimal in theory but optimal in application. This concept can be explained in the following manner. Consider the nonlinear profit function shown in Figure 4.1. It is assumed that the firm is capable of achieving a profit of y_i dollars when x_i units are available for sale. Sales are such that they may range between any values of x_i. Suppose the firm is

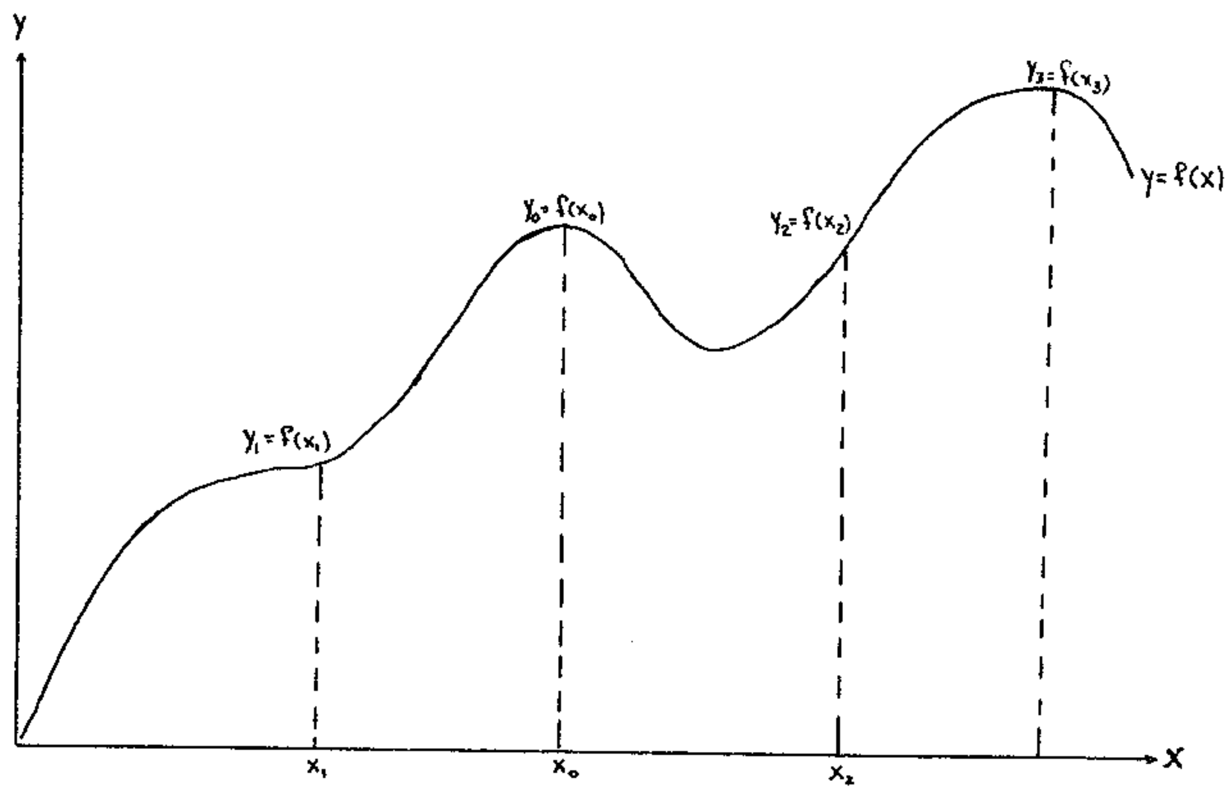

Fig. 4.1--Nonlinear profit function

attempting to maximize profit. This maximization is to be achieved subject only to the number of units sold. If the firm has only x_1 units available for sale, then the maximum profit will be y_1 dollars, $y_1 = f(x_1)$. If the firm has $x_1 \leq x \leq x_2$ units available for sale, the profit will range between $y_1 = f(x_1)$ and $y_2 = f(x_2)$ dollars, with the maximum profit occurring at some point x_o, $y_o = f(x_o)$, lying in the interval $x_1 \leq x \leq x_2$. However, neither of these "optimal" solutions is the point at which maximum profit is achieved. The point of maximum profit is x_3, with a profit of $y_3 = f(x_3)$ dollars. If it is assumed that attempting to sell x_3 units is not feasible, then the decision to strive for a profit of $y_o = f(x_o)$ or $y_1 = f(x_1)$ or some other figure is one that is theoretically suboptimal but practically optimal. The analysis of the problem is then described as optimum-seeking.

Modern optimization theory is a combination of both optimization and optimum-seeking techniques and approaches to problem solving. It is a tool of optimization in that its basic techniques select the optimal solution from among a set of defined feasible solutions. It is optimum-seeking in that it provides means whereby problems that are not as clearly defined and constructed as those of mathematical programming can be solved. As an optimum-seeking technique, modern optimization theory seeks the optimum solution without the confining limitations of strictly static analysis. As such, the term optimum-seeking relates more to the techniques of modern optimization theory and the manner in which each technique seeks the optimal solution to a given problem.

The framework within which the advanced techniques of modern optimization theory will be discussed will have two major headings: indirect search and direct search. Although the discussion will not include all possible subdivisions, the discussion will be sufficient to point out the major characteristics of the optimum search classification. In this way it is hoped that a "feel" for the techniques will be achieved, as well as an appreciation for the technique itself.

The Optimum-Seeking Problem

Kunzi, Tzschach, and Zehnder have described the optimum-seeking problem as one in which both the objective function and its constraints are nonlinear. Their formulation is one in which the objective function is to be minimized subject to a set of m constraints.[1] For example,

$$\text{minimize } G(\underline{x}) = G(x_1, x_2, \ldots, x_n)$$

subject to

$$g_j(\underline{x}) \leq 0 \quad (j = 1, 2, \ldots, m), \text{ and}$$

$$x_i \geq 0 \quad (i = 1, 2, \ldots, n).$$

Such a problem, in the two dimensional case, is shown in Figure 4.2.

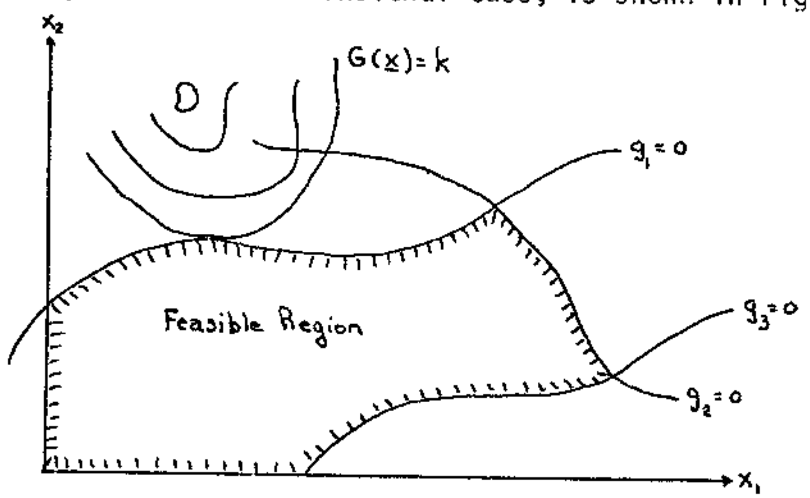

Fig. 4.2--Nonlinear optimization

[1] H. P. Kunzi, H. G. Tzschach, and C. A. Zehnder, Numerical Methods of Mathematical Optimization (New York, 1968), pp. 62-63.

This problem can be described in the following manner. Suppose $G(x) = k$ defines a cost function that has a fixed value of k; i.e., x_1 and x_2 can vary freely, but the combined mix of x_1 and x_2 is such that the cost is k. The function defined by $g_j(x_1, x_2) = 0$ ($j = 1, 2, 3$) represents the constraints forced on the possible combinations of x_1 and x_2. The problem is to find the optimum mix of x_1 and x_2 with a cost of k that does not violate the constraints.

Baumol[2] describes the optimum-seeking problem as one which seeks to optimize an *n*-variable objective function subject to a set of m constraints. He also assumes the n-variable to be nonnegative. This description takes on the form

$$\text{maximize (or minimize) } f(x_1, x_2, \ldots, x_n)$$

subject to

$$\begin{aligned} g_1(x_1, x_2, \ldots, x_n) &\leq c_1 \\ g_2(x_1, x_2, \ldots, x_n) &\leq c_2 \\ \cdots\cdots\cdots \\ g_m(x_1, x_2, \ldots, x_n) &\leq c_m \\ (x_1, x_2, \ldots, x_n) &\geq 0. \end{aligned}$$

In providing a geometric representation, Baumol relates his analysis to the iso-profit curve, or profit indifference curves. His work is presented here in summary form. Reference will be made to Figures 4.3, 4.4, and 4.5.

Figure 4.3(a) represents a nonlinear profit function defined in terms of the two units x_1 and x_2. Because of the curvature of the surface of the

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[2] William J. Baumol, Economic Theory and Operations Analysis, 2nd ed. (Englewood Cliffs, 1968), pp. 129-133.

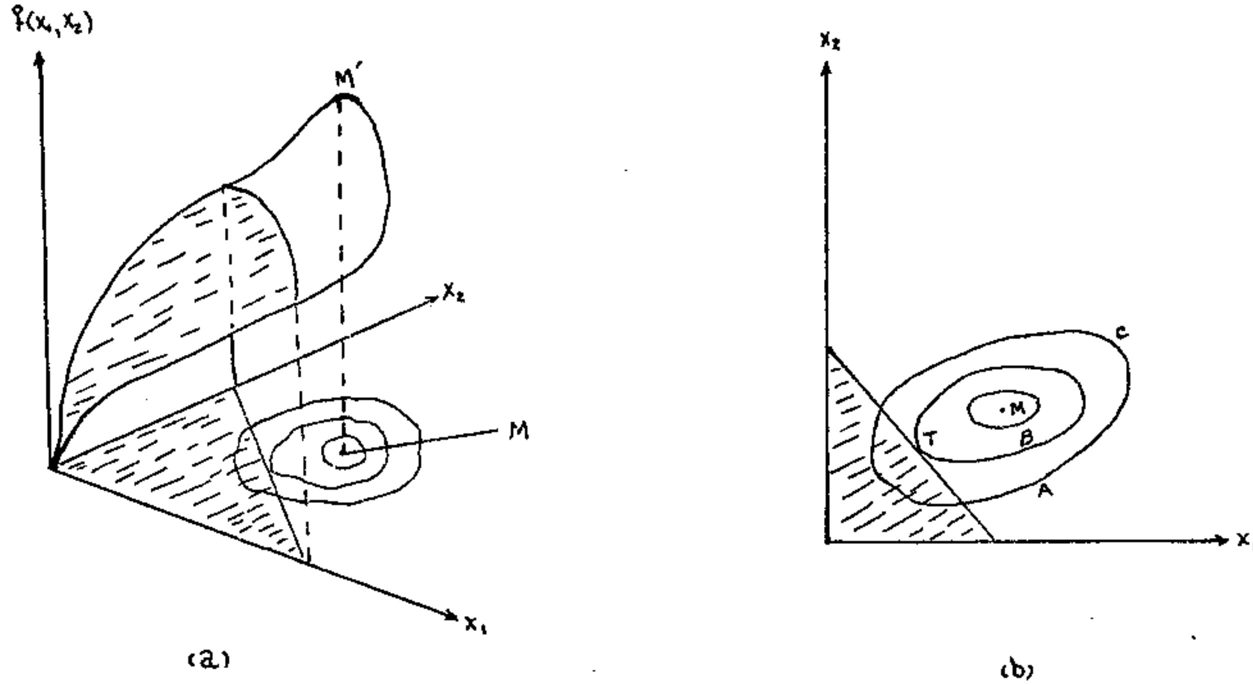

Fig. 4.3--Nonlinear profit analysis

function,[3] the problem is one of diminishing returns; i.e., increases in outputs reach a point such that additional increases yield decreased return per unit of increase (marginal return). Examination of Figure 4.3(a) reveals that $f(x_1, x_2)$ will increase for every increase in x_1 and x_2 until the point M' on the curve is reached. At this point the combination of x_1 and x_2 which maximizes $f(x_1, x_2)$ is that combination $M = (x_1, x_2)$ with a maximum value of M'.

Figure 4.3(b) represents the linear representation of the same problem. The use of a functional approach such as that shown in Figure 4.3(a) allows greater freedom in describing the iso-profit curves corresponding to various mixtures of the input variables. The linear case does not

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[3]The term "curvature of the surface of the function" refers to the U-shape of the cross-sections.

permit the direction of profitable movement to change as does the nonlinear application.

Figure 4.4(a) represents a set of iso-profit curves such that moving from curve H to J to K to L increases the amount of profit defined by $f(x_1, x_2)$.

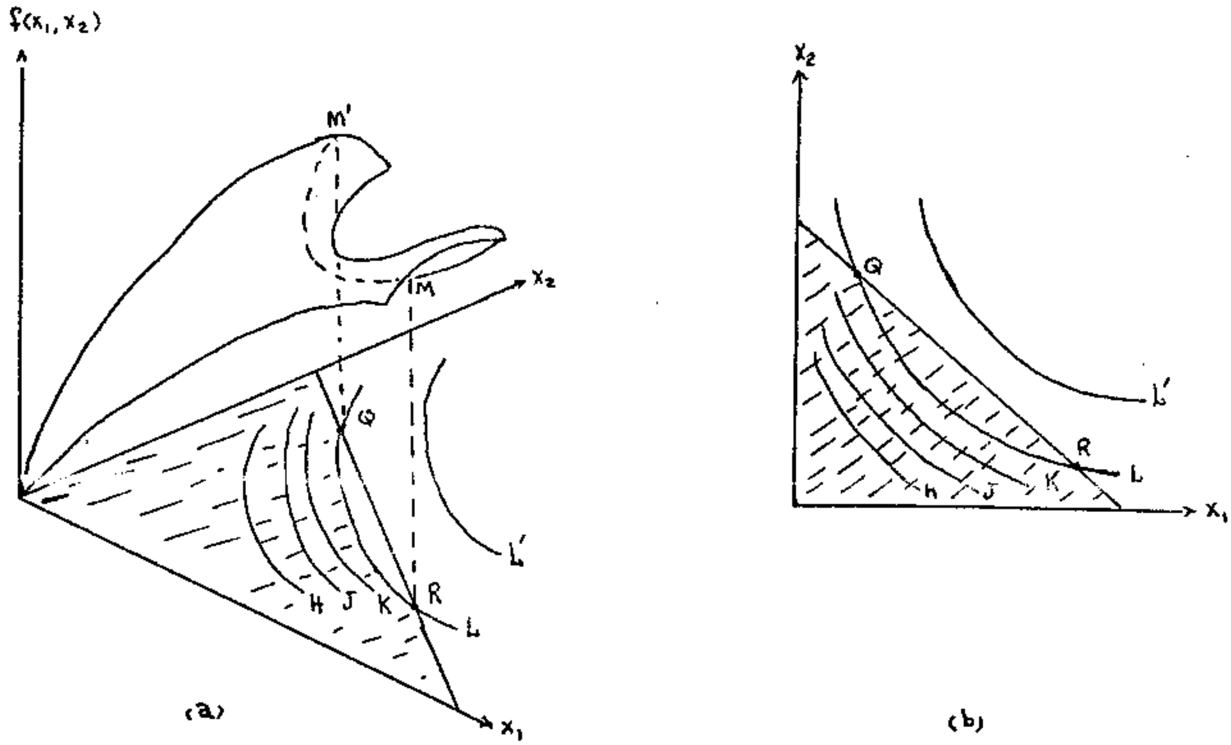

Fig. 4.4--Iso-profit analysis

Curve L corresponds to the ridge that yields the maximum profit, shown by MM'. If a move is made to curve L', the amount of profit will fall. Figure 4.4(b) represents the linear formulation of the problem.

Figure 4.5 represents the concept of increasing returns due to specialization. As one of the outputs, say X_1, increases, the profit defined by $f(x_1, x_2)$ increases along the upward curvature of OCT'. Such functions result in increasing marginal return.

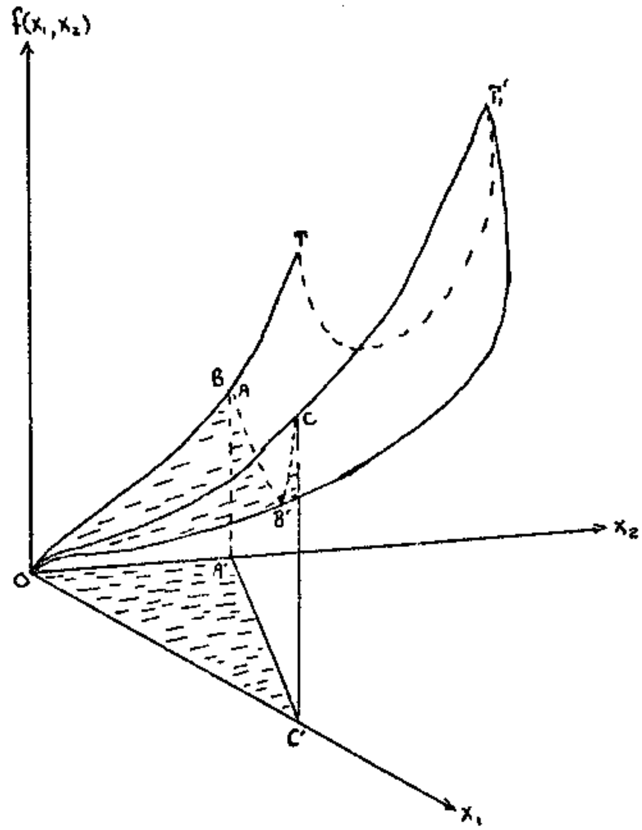

Fig. 4.5--Increasing returns due to specialization

A cursory examination of Figures 4.2 to 4.5 reveals that the techniques of classical optimization theory are ill-suited for solving such problems. In addition, the pronounced nonlinearity of the problems prohibits ready application of either classical techniques or basic optimal search techniques. Thus, it is prudent that consideration be given to the availability and utilization of more suitable techniques of optimization, particularly nonlinear optimization.

> . . .the practical value of the solution of these problems is high. Almost no real problem is linear; linearity represents our compromise between reality and the limitations of our tools for dealing with it. The user who has only linear techniques. . .must accept the results of a linearization of his problem

> and bear the expense of their deviation from reality, or undertake the imprecise and laborious task of making the results better by heuristic methods [i.e., applying selected routines to reduce the size of the problem].[4]

With such concepts as marginal productivity, marginal cost, marginal product, and marginal profit, and such problems as inventory control, portfolio selection, pricing, and cost analysis to be considered, it is fitting that administrative analysis concern itself with the suitability of advanced optimization techniques. It is more fitting that these techniques be integrated into the "kit of tools" available for administrative problem solving. Such an integration will provide a greater variability of problem formulations due to the availability of suitable solution techniques.

In the context just presented, the administrator is confronted with the problem of optimizing a given objective function subject to a defined set of constraints. The objective function and any existing constraints can be linear or nonlinear, univariable or multivariable. The problem to be solved is one which satisfies Definition 4.1.

Definition 4.1.--Given the n-variabled objective function $f(\underline{x}) = f(x_1, x_2,\ldots,x_n)$ and the m constraint functions $g_i(\underline{x}) = g_i(x_1, x_2,\ldots,x_n)$, $i = 1, 2,\ldots,m$, determine $\underline{x} = (x_1, x_2,\ldots,x_n)$ so that $f(\underline{x})$ is optimized without violating $g_i(\underline{x})$ for any i and $\underline{x} \geq 0$ for all $\underline{x} = (x_1, x_2,\ldots,x_n)$.

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[4]Philip Wolfe, "Recent Developments in Nonlinear Programming," _A Report Prepared for United States Air Force Project Rand_, Report No. R-401-PP (Santa Monica, 1963), pp. 1-2.

This definition can be further clarified by considering Figure 4.6.

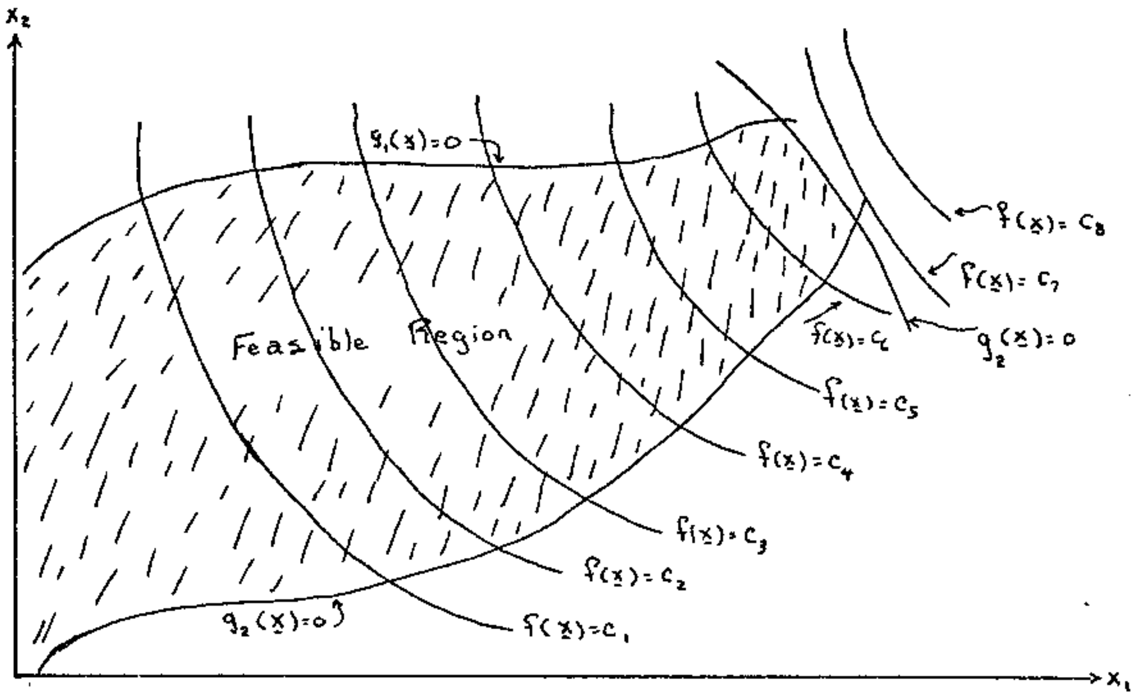

Fig. 4.6--Nonlinear optimization problem

Suppose $f(\underline{x}) = f(x_1, x_2)$ defines a profit function and $g_i(\underline{x}) = g_i(x_1, x_2)$, $(i = 1, 2, 3)$, defines the constraint functions. By defining $f(\underline{x}) = c_j$, $(j = 1, 2, \ldots)$, it is possible to construct a set of contours which define the profit, c_j, obtained by the indicated combination of x_1 and x_2. Assuming that profit is to be maximized, the solution is that combination $\underline{x}^* = (x_1^*, x_2^*)$ that yields the highest value contour without violating the constraint set, i.e., remains within the feasible region. Unlike the linear programming case, the optimal solution is not required to lie at one of the corner points (intersections).

It is important to note that the term "optimal solution" depends upon the nature of the objective function itself. If $f(\underline{x})$ is to be maximized, the optimal solution is that set of values x_1 and x_2 that yields the highest valued contour without violating any of the constraints. If $f(\underline{x})$ is to be minimized, the optimal solution is that set of values x_1 and x_2 that yields the lowest valued contour without violating any of the constraints. In either case the optimal solution is free to lie anywhere within the constraint set. In addition the function $f(\underline{x})$, although illustrated as being nonlinear, can be linear. Finally, it is important to realize that the techniques to follow describe techniques for searching out this optimal solution.

Advanced Techniques of Optimal Search

Indirect Search

Indirect search has been described as an optimization technique which

> involves constructing conditions which, if satisfied, assure. . .that a given point is indeed a local or relative maximum. These conditions are derived analytically from the function under consideration.[5]

This approach makes use of some derived property of the function rather than the function itself. The most common tool of indirect search is the traditional max-min calculus.

A common characteristic of indirect search techniques is that they tend to identify optimal solutions without considering nonoptimal ones.[6] For this reason, these methods, when applicable, are extremely effective. Their application to a given problem tends to reduce the initial problem

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[5] Ronald Gue and Michael E. Thomas, Mathematical Models in Operations Research (London, 1968), p. 26.

[6] Douglas J. Wilde and Charles S. Beightler, Foundations of Optimization (Englewood Cliffs, 1967), p. 18.

to one requiring the solving of a set of equations (at least one) for the roots. These roots correspond to the location of optimum solutions and can be tested to determine the nature of the optimum point achieved.

The techniques of indirect search have been investigated by various authorities in the fields of mathematics and engineering. A common characteristic of these investigations is the manner in which indirect search techniques are categorized: functions of one variable and functions of more than one variable. These two topics are then further divided into unconstrained and constrained problems.[7] It is the intent here to utilize the same approach in order to better develop the transition from one stage to the next. This approach illustrates the conceptual linkage between classical techniques and modern techniques.

The topic of indirect search covers a wide variety of techniques; and, consequently, the discussion to follow represents a selected portion of indirect search techniques, in particular those techniques that are best suited to the reduction of the initial problem to one of solving for a set of defined roots. The topics to be discussed are the differential approach, the Newton-Raphson technique, and the technique of constrained derivatives under both equality and inequality constraints.

The differential approach.--The differential approach to solving optimization problems has been traced back to the work of Johannes Kepler

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[7] See Gue and Thomas, _op. cit._, pp. 15-53; Wilde and Beightler, _Ibid._, pp. 18-95; Douglass J. Wilde, _Optimum Seeking Methods_ (Englewood Cliffs, 1964), pp. 10-92; and James M. Dobbie, "A Survey of Search Theory," _Operations Research_, Vol. XVI (May-June, 1968), 525-537.

in the seventeenth century.[8] The crux of this approach is the derivative concept of the differential calculus and the fact that the derivative is defined as a limiting function; i.e., for any continuous function f(x), the derivative of f(x), f'(x), is defined as

$$\lim_{\Delta x \to 0} \frac{f(x + \ x) - f(x)}{\Delta x},$$

provided the limit exists. Utilizing this concept, the differential approach to optimization theory proceeds according to whether or not the function under investigation is defined by a single variable or n-variables.

Single-variable analysis: The use of the calculus as a tool of modern optimization theory follows the same basis as that of classical optimization theory. It is employed to determine points of maximal and/or minimal values. However, whereas classical optimization tends to limit the function under investigation to at most cubic functions, modern optimization permits the function to assume whatever degree is best suited for describing the given problem. At the same time provision is made for determining whether or not the function achieves a maximum or a minimum at a defined critical point.

Consider the n^{th} degree polynomial f(x). It is assumed that f(x) is continuous and has a continuous first derivative in a defined interval. The necessary condition for f(x) to have an optimal value within the given interval is that the first derivative vanish at some point within this interval. This "optimal point," however, is one of two types: a maximum point or a minimum point. If two or more points satisfy the condition that the first derivative be zero at that point, there may be more than

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[8]Wilde and Beightler, op. cit., p. 16.

one point of maximization, minimization, or inflection. The existence of two or more critical points for f(x) forces consideration for points of relative optima as well as points of absolute optima.

In order to determine whether or not an n^{th} degree function, $n \geq 2$, has maximum points, minimum points, or inflection points, it is necessary to first set the first derivative equal to zero. The resulting $(n - 1)^{st}$ degree function is then solved for its critical point. Given these critical points, the following theorem is applied.

Theorem 4.1.--Let f(x) be continuous and differentiable through the first m derivatives, f(x) an n^{th} degree function. Let x^o be any x such that $f'(x) = 0$. Then f(x) has an optimum point at x^o if and only if m is even, where m is the order of the first nonvanishing derivative at x^o. The optimum value of f(x) is determined by applying the following:

(1) if $f^m(x_o) < 0$, $f(x_o)$ is a maximum point;

(2) if $f^m(x_o) > 0$, $f(x_o)$ is a minimum point.[9]

As an example of this theorem, consider the following equation: $f(x) = (x - 4)^6$. Setting $f'(x) = 0$ yields $f'(x) = 6(x - 4)^5 = 0$, which has five roots, all of which are $x = 4$. Thus, $x = 4$ is a critical point for $f(x) = (x - 4)^6$. Continuing the differentiation process,

$$f''(x) = 30(x - 4)^4 = 0 \text{ at } x = 4;$$

$$f'''(x) = 120(x - 4)^3 = 0 \text{ at } x = 4;$$

$$f^{iv}(x) = 360(x - 4)^2 = 0 \text{ at } x = 4;$$

$$f^{v}(x) = 720(x - 4) = 0 \text{ at } x = 4;$$

$$f^{vi}(x) = 720.$$

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[9]Gue and Thomas, op. cit., p. 29.

Since the first nonvanishing derivative is of order $m = 6$ and is even, $f(x)$ achieves an optimum value at $x = 4$. Since $f^{vi}(x_o) = f^{vi}(4) = 720 > 0$, the point $x = 4$ is a point at which $f(x)$ achieves a minimum. Had m been odd, the point $x = 4$ would not have been an optimum point. It would have indicated a change in the curvature of $f(x)$.[10]

Multivariable analysis: The introduction of more than one variable into the defining function requires the use of the partial derivative. This concept is the same as that described in the section on classical optimization theory. However, unlike classical optimization, the degree of the multivariable function is not restricted. It is in the multivariable case that modern optimization theory offers the greatest improvement over the classical use of the differential calculus.

As a means of re-establishing the base from which multivariable differentiation stems, consider again the definition of the partial derivative. It is assumed that $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$ is continuous and differentiable with respect to the variable being considered. Then, the partial derivative of $f(\underline{x})$ with respect to x_j, $\frac{\partial}{\partial x_j} f(x_1, x_2, \ldots, x_n)$, is defined as

$$\frac{\partial}{\partial x_j} f(x_1, x_2, \ldots, x_n) = \lim_{\partial x_j \to 0} \frac{f(x_1, x_2, \ldots, x_{j-1}, x_j + \partial x_j, \ldots, x_n) - f(x_1, \ldots, x_n)}{\partial x_j}$$

provided the limit exists. This procedure is accomplished by differentiating with respect to the particular x_j under consideration, treating all other variables as though they were constants.

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[10] Ibid., pp. 30-31.

The partial differentiation process reduces the multivariable function $f(\underline{x}) = f(x_1,\ldots,x_n)$ to a problem consisting of n simultaneous equations. In the majority of applications, the simultaneous system is nonlinear. As in the case of classical optimization, this simultaneous system is solved for all possible solutions, each set of solutions being a critical point for $f(\underline{x})$.

Theorem 4.2.--Let $f(\underline{x})$ be a continuous function with continuous first and second partial derivatives. A necessary condition for $f(\underline{x})$ to have an extreme point at $\underline{x}_o$ is for the first partial derivative of $f(\underline{x})$ to be equal to zero when evaluated at $\underline{x}_o$; i.e.,

$$\frac{\partial}{\partial x_i}[f(x)]\Bigg|_{\underline{x}_o} = 0, \qquad (i = 1, 2,\ldots,n).$$

The vector $\underline{x}_o$ is the solution set satisfying $\frac{\partial}{\partial x_i}[f(\underline{x})] = 0$, $(i = 1, 2,\ldots,n)$.

Theorem 4.2 states the condition under which $f(\underline{x}) = f(x_1, x_2,\ldots,x_n)$ has an extreme point, or set of extreme points. If the function defined by $f(\underline{x})$ has nonlinear partial derivatives, there exists the possibility that $f(\underline{x})$ has more than one extreme point. Although the vanishing of the first partial derivative is a necessary condition for $f(\underline{x})$ to have an extreme point, it is not a sufficient condition. Sufficiency is established at a later point in the study and is related to the concept of the Hessian matrix.

Definition 4.2.--Let $f(\underline{x}) = f(x_1, x_2,\ldots,x_n)$ be a continuous function with continuous first and second partial derivatives. The matrix formed by the second partial derivatives of $f(\underline{x})$ is defined as the Hessian matrix.

As an example of the Hessian matrix concept, consider the following nonlinear, multivariable function. It is assumed that this function defines the cost associated with the development of a hypothetical chemical process plant.

$$f(x_1, x_2) = 100x_1 + 40{,}000x_1^{-1}x_2^{-1} + 25{,}000\ x_2.$$

The partial derivatives are as follows:

$$\frac{\partial}{\partial x_1}[f(x_1, x_2)] = 100 - 40{,}000\ x_1^{-2}x_2^{-1};$$

$$\frac{\partial}{\partial x_2}[f(x_1, x_2)] = -40{,}000x_1^{-1}x_2^{-2} + 25{,}000;$$

$$\frac{\partial^2}{\partial x_1^2}[f(x_1, x_2)] = \frac{\partial}{\partial x_1}\left[\frac{\partial}{\partial x_1}[f(x_1, x_2)]\right] = 80{,}000\ x_1^{-3}x_2^{-1};$$

$$\frac{\partial^2}{\partial x_1 \partial x_2}[f(x_1, x_2)] = \frac{\partial}{\partial x_1}\left[\frac{\partial}{\partial x_2}[f(x_1, x_2)]\right] = 40{,}000x_1^{-2}x_2^{-2};$$

$$\frac{\partial^2}{\partial x_2 \partial x_1}[f(x_1, x_2)] = \frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial x_1}[f(x_1, x_2)]\right] = 40{,}000x_1^{-2}x_2^{-2};$$

$$\frac{\partial^2}{\partial x_2^2}[f(x_1, x_2)] = \frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial x_2}[f(x_1, x_2)]\right] = 80{,}000x_1^{-1}x_2^{-3}.$$

The Hessian matrix is formed by positioning these second partial derivatives as follows:

$$\begin{bmatrix} \frac{\partial^2}{\partial x_1^2}[f(x_1, x_2)] & \frac{\partial^2}{\partial x_1 \partial x_2}[f(x_1, x_2)] \\ \frac{\partial^2}{\partial x_2 \partial x_1}[f(x_1, x_2)] & \frac{\partial^2}{\partial x_2^2}[f(x_1, x_2)] \end{bmatrix}.$$

The Hessian matrix for the given problem is then written as

$$\begin{bmatrix} 80{,}000x_1^{-3}x_2^{-1} & 40{,}000x_1^{-2}x_2^{-2} \\ 40{,}000x_1^{-2}x_2^{-2} & 80{,}000x_1^{-1}x_2^{-3} \end{bmatrix} .$$

The numerical values for the given Hessian matrix are obtained by substituting values for x_1 and x_2.

From this example it is seen that the elements of the Hessian matrix are obtained by determining the various partial derivatives of a given n-variable function. In addition the general form of the Hessian matrix can be easily generated. For this form define $y''_{ij} = \frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_j} f(\underline{x})\right]$, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n)$, where $\underline{x}$ is an n-component vector. Let $\underline{H}$ denote the Hessian matrix. Then

$$\underline{H} = \begin{bmatrix} y''_{11} & y''_{12} & y''_{13} \cdot & \cdot & \cdot y''_{1n} \\ y''_{21} & y''_{22} & y''_{23} \cdot & \cdot & \cdot y''_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y''_{n1} & y''_{n2} & y''_{n2} \cdot & \cdot & \cdot y''_{nn} \end{bmatrix} .$$

Definition 4.3.--The Hessian matrix $\underline{H}$ is said to be positive definite if the form defined by $\underline{H}$ has the property that all of its values are positive except when all of the variables are zero.

The form defined by the Hessian matrix $\underline{H}$ is $(\partial \underline{x})^T \underline{H} (\partial \underline{x})$. As an example, consider the form

$$[\partial x_1,\ \partial x_2] \begin{bmatrix} y''_{11} & y''_{12} \\ y''_{21} & y''_{22} \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix} ,$$

where $y = f(x_1, x_2)$. The expanded form of $(\partial \underline{x})^T \underline{H}(\partial \underline{x})$ is given by

$$[\partial x_1, \partial x_2] \begin{bmatrix} y''_{11} & y''_{12} \\ y''_{21} & y''_{22} \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix} = (y''_{11}(\partial x_1) + y''_{21}(\partial x_2),\ y''_{12}(\partial x_1) + y''_{22}(\partial x_2)) \begin{matrix} \partial x_1 \\ \partial x_2 \end{matrix}$$

$$= (\partial x_1)[y''_{11}(\partial x_1) + y''_{21}(\partial x_2)] + \partial x_2[y''_{12}(\partial x_1) + y''_{22}(\partial x_2)]$$

$$= y''_{11}(\partial x_1)^2 + y''_{21}(\partial x_1)(\partial x_2) + y''_{12}(\partial x_1)(\partial x_2) + y''_{22}(\partial x_2)^2$$

$$= y''_{11}(\partial x_1)^2 + [y''_{21} + y''_{12}](\partial x_1)(\partial x_2) + y''_{22}(\partial x_2)^2.$$

The y''_{ij} values are constants. They were obtained by evaluating the respective second partial derivatives of $y = f(x_1, x_2)$ at a given value of $\ddot{\underline{x}} = (x_1, x_2)$. The unknown quantities, or variables, are the (∂x_i), $(i = 1, 2,)$. If this quadratic form is positive for all $\partial \underline{x} = \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then the Hessian matrix is positive definite.

Definition 4.4.--Let $(\partial \underline{x})^T \underline{H}(\partial \underline{x})$ be the quadratic form of first partial derivatives associated with the Hessian matrix $\underline{H}$. The Hessian matrix $\underline{H}$ is said to be negative definite if the form defined by $\underline{H}$ has the property that all of its values are negative except when all of the variables are zero.

This concept can be better understood by considering again the quadratic form defined by $(\partial \underline{x})^T \underline{H}(\partial \underline{x}) = y''_{11}(\partial x_1)^2 + [y''_{21} + y''_{12}](\partial x_1)(\partial x_2) + y''_{22}(\partial x_2)^2$. If this quadratic form is negative for all $(\partial \underline{x}) = \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then the Hessian matrix is negative definite.

The importance of the Hessian matrix concept is readily established when consideration is given to the sufficiency condition for multivariable optimization. This condition is contained in the following theorem.

Theorem 4.3.--Let $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$ be a continuous function with continuous first and second partial derivatives. A sufficient condition for $f(\underline{x})$ to have a relative optima at $\underline{x}_o$, that is, where

$$\frac{\partial}{\partial x_i} f(\underline{x}) \Bigg|_{\underline{x} = \underline{x}_o} = 0, \quad (i = 1, 2, \ldots, n),$$

is that the Hessian matrix be positive definite for a relative minimum, negative definite for a relative maximum.

Further use of the Hessian matrix results when consideration is given to the existence of valleys, ridges, and/or saddlepoints. If the quadratic form of the Hessian matrix is greater than or equal to zero, it is said to be positive semidefinite. Such a condition indicates that $\underline{x}_o$, the set of critical points for $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$, lies in a valley, with the quadratic form being equal to zero along the valley. If the quadratic form of the Hessian matrix is less than or equal to zero, it is said to be negative semidefinite. This condition indicates that $\underline{x}_o$, the set of critical points for $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$, is located on a ridge, with the quadratic form being equal to zero along the ridge. Finally, if the quadratic form of the Hessian matrix is positive, negative, or zero, depending on the value of $\partial\underline{x}$, it is said to be indefinite. This condition indicates that $\underline{x}_o$ is located at a saddlepoint.

The nature of the Hessian matrix can also be investigated by considering the row equivalent form of the derived Hessian matrix. Utilization of this tool requires the introduction of some additional properties.

Theorem 4.4.--A real symmetric matrix $\underline{A}$ of rank r is congruent to matrix

$$\underline{B} = \begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_{r-p} & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

The integer p is uniquely determined by $\underline{A}$.[11]

Definition 4.5.--Two matrices are congruent if and only if one is obtainable from the other by a succession of pairs of elementary operations, each pair consisting of a column operation and the corresponding row operation. In each pair either operation may be performed first.[12]

Definition 4.6.--The integer p of Theorem 4.4 is defined as the index of the symmetric, real matrix $\underline{A}$. This integer p equals the number of positive diagonal elements.[13]

At this point an example is in order. Consider the following 3 by 3 real, symmetric matrix,

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 8 \end{bmatrix} .$$

In order to determine the index of the given matrix, it is necessary to reduce the matrix to the form shown in Theorem 4.4. This will be accomplished

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[11] Sam Perlis, Theory of Matrices (Reading, Mass., 1952), p. 91.

[12] Ibid., p. 89.

[13] Ibid., p. 92.

by performing the rwo-column pair operations of Definition 4.5. Let $\mathcal{O}_{ij}(k)$ denote the multiplying of row i by k and adding the result to row j. Let $c\mathcal{O}_{ij}(k)$ denote the multiplying of column i by k and adding the result to column j.

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 8 \end{bmatrix} \sim \mathcal{O}_{12}(-\tfrac{1}{2}) \sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & \frac{5}{2} & \frac{1}{2} \\ 3 & 2 & 8 \end{bmatrix} \sim c\mathcal{O}_{12}(-\tfrac{1}{2}) \sim \begin{bmatrix} 2 & 0 & 3 \\ 0 & \frac{5}{2} & \frac{1}{2} \\ 3 & \frac{1}{2} & 8 \end{bmatrix} \sim \mathcal{O}_{13}(-\tfrac{3}{2})$$

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & \frac{5}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{7}{2} \end{bmatrix} \sim c\mathcal{O}_{13}(-\tfrac{3}{2}) \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{5}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{7}{2} \end{bmatrix} \sim \mathcal{O}_{23}(-\tfrac{1}{5}) \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{5}{2} & \frac{1}{2} \\ 0 & 0 & \frac{17}{5} \end{bmatrix} \sim c\mathcal{O}_{23}(-\tfrac{1}{5})$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{17}{5} \end{bmatrix} \begin{matrix} \sim \mathcal{O}_1(1/\sqrt{2}) \sim \\ \sim \mathcal{O}_2(\sqrt{2/5}) \sim \\ \sim \mathcal{O}_3(\sqrt{5/17}) \sim \end{matrix} \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{17}{5}} \end{bmatrix} \begin{matrix} \sim c\mathcal{O}_1(1/\sqrt{2}) \sim \\ \sim c\mathcal{O}_2(\sqrt{2/5}) \sim \\ \sim c\mathcal{O}_3(\sqrt{5/17}) \sim \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Comparing $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to $\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_{r-p} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ indicates that r = 3 and p = 3. (There are three positive diagonal elements.) Therefore, the given matrix has rank equal to three and index equal to three.

This information is used to determine the nature of the matrix. In

Theorem 4.5.--Let A be any n x n real, symmetric matrix. Let r be the rank of the matrix A and p its index. The matrix A is positive definite or positive semidefinite according to the following:

(1) $p = r < n$: A is positive semidefinite;

(2) $p = 4 = n$: A is positive definite.[14]

Returning to the previous example, the matrix $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ is positive definite. The canonical form resulted in $r = p = n = 3$.

Although this theorem is restricted to positive definite and/or positive semidefinite matrices, an analogous criterion can be derived for negative definite and/or negative semidefinite matrices. Such an extension is provided by the following theorem.

Theorem 4.6.--Let A be any n x n real, symmetric matrix. Let r be the rank of the matrix A and p its index. Then, the matrix A is negative definite or negative semidefinite according to the following:

(1) $p = 0$ and $r < n$: A is negative semidefinite;

(2) $p = 0$ and $r = n$: A is negative definite.

Theorem 4.6 indicates that a given n x n real, symmetric matrix is negative semidefinite only when its index equals 0 and its rank is less than the number of rows or columns. A given n x n real, symmetric matrix is negative semidefinite only when its index equals 0 and its rank equals the number of rows or columns.

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[14] Ibid., p. 94.

As an example of the preceding discussion, consider the problem of determining the optimal value of the cost function

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 8x_2 - 12x_3 + 56.$$

The critical points for $f(x_1, x_2, x_3)$ are those values of x_1, x_2, and x_3 for which

$$\frac{\partial}{\partial x_i} [f(\underline{x})] \Big|_{\underline{x}_o} = 0, \quad (i = 1, 2, 3).$$

$$\frac{\partial}{\partial x_1} f(x_1, x_2, x_3) = 2x_1 - 4$$

$$\frac{\partial}{\partial x_2} f(x_1, x_2, x_3) = 2x_2 - 8$$

$$\frac{\partial}{\partial x_3} f(x_1, x_2, x_3) = 2x_3 - 12.$$

Equating the system of partial derivatives to zero,

$$2x_1 - 4 = 0,$$

$$2x_2 - 8 = 0,$$

$$2x_3 - 12 = 0.$$

The critical points for $f(x_1, x_2, x_3)$ are found to be $x_1 = 2$, $x_2 = 4$, $x_3 = 6$. The Hessian matrix

$$\underline{H} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

requires evaluation of the following second partial derivatives:

$$f_{11} = \frac{\partial}{\partial x_1}\left[\frac{\partial}{\partial x_1} f(x_1, x_2, x_3)\right] = 2$$

$$f_{12} = \frac{\partial}{\partial x_1}\left[\frac{\partial}{\partial x_2} f(x_1, x_2, x_3)\right] = 0 \equiv f_{21}$$

$$f_{13} = \frac{\partial}{\partial x_1}\left[\frac{\partial}{\partial x_3} f(x_1, x_2, x_3)\right] = 0 \equiv f_{31}$$

$$f_{23} = \frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial x_3} f(x_1, x_2, x_3)\right] = 0 \equiv f_{32}$$

$$f_{22} = \frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial x_2} f(x_1, x_2, x_3)\right] = 2$$

$$f_{33} = \frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial x_3} f(x_1, x_2, x_3)\right] = 2.$$

Thus,

$$\underline{H} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \equiv 2(\partial x_1)^2 + 2(\partial x_2)^2 + 2(\partial x_3)^2.$$

The form defined by $\underline{H}$ is positive for all $\partial \underline{x} = \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence, $\underline{H}$ is positive definite and $\underline{x}_o = (2, 4, 6)$ defines a minimum point.

Another approach to determining the nature of the Hessian matrix is that afforded by the row rank concept. The Hessian matrix defined by

$$\underline{H} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

has row rank equal to 3. The index of the Hessian matrix is also equal to 3. The Hessian is a 3 x 3 real, symmetric matrix, n = 3. Since $p = r = n = 3$, Theorem 4.5 identifies the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} .$$

as being positive definite.

An important point to note is that the types of problems particularly suited to the differential approach are those for which side conditions (restrictions) are not given. This is due to the fact that the introduction of side restrictions increases the computational difficulty of the problem being investigated. In this regard, the primary use made of the differential approach is in solving nonlinear optimization problems with no side conditions. Such functions generally consist of n variables and require solving of a nonlinear system of first-partial derivatives.[15]

Newton-Raphson formula.--The Newton-Raphson formula is a computational technique for solving nonlinear equations or systems of nonlinear equations. The formula is applicable to either functions of one variable or functions of more than one variable.

Single-variable case: Consider the nonlinear function $y = F(x)$. The function defined by $F(x)$ may be of any degree. Suppose $y = F(x)$ is the curve shown in Figure 4.7. It is desired that $y = f(x)$ be solved for all x such that $F(x) = 0$.

The solution to $y = F(x) = 0$ is that set of values of x at which the curve defined by $y = F(x)$ crosses the x axis. If a solution (or solutions) exist, it can be obtained by (1) initiating an initial solution, x_o,

[15] Wilde and Beightler, op. cit., p. 22.

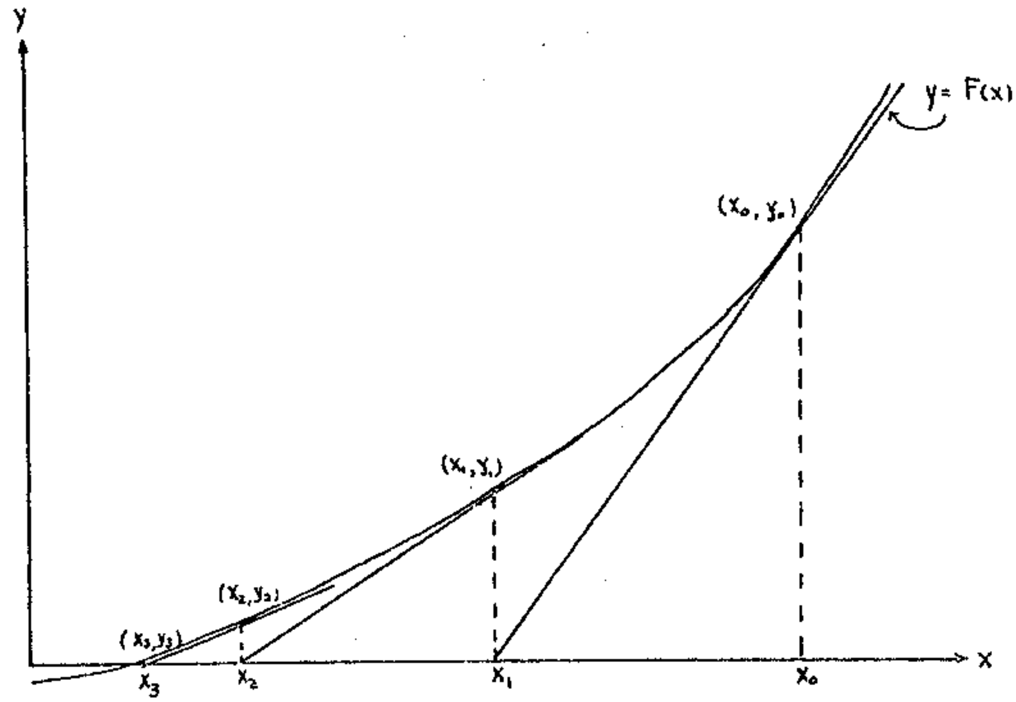

Fig. 4.7--Newton-Raphson technique

(2) drawing a line tangent to the curve at (x_o, y_o), (3) using the point where the constructed tangent crosses the x axis as an improved solution, x_1, and (4) repeating steps (1) - (3) until the solution is sufficiently accurate.

This same result can be achieved by direct application of a computational formula which yields the improved solution point. This formula is given by

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}.$$

This formula is the Newton-Raphson formula for non-multiple roots. The validity of the formula can be readily seen when consideration is given to the fact that the tangent line at any point (x_n, y_n) is given by

$$y - y_n = (x - x_n)F'(x_n).$$

The improved value of x_n, x_{n+1}, is the value that makes the y of the tangent line equal to zero. For $y_n = F(x_n)$,

$$0 - F(x_n) = (x_{n+1} - x_n)F'(x_n)$$

$$- F(x_n) = (x_{n+1} - x_n)F'(x_n)$$

$$-\frac{F(x_n)}{F'(x_n)} = x_{n+1} - x_n$$

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} .$$

Although the derivation of the Newton-Raphson formula considers $y = F(x_n)$ to be an n^{th} degree polynomial, the formula is suitable for solving derivative functions for critical points. This is accomplished by simply defining the derivative function as $F(x_n)$ and applying the given formula.

In application the Newton-Raphson formula is particularly useful when solutions converge rapidly. Of particular importance is the selection of a suitable starting solution. The location of starting solutions should be made after inspection of the given function by some suitable technique such as Descartes rule of signs. This inspection process can be used to determine values within which solutions lie.

<u>Algorithm 4.1</u> (<u>Newton-Raphson technique for univariable functions</u>).--

Step 1. Select a suitable starting solution, x_n.

Step 2. Calculate $F(x_n)$ and $F'(x_n)$.

Step 3. Apply the Newton-Raphson formula $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$, where $F'(x_n) \neq 0$.

Step 4. Utilizing x_{n+1}, reiterate on steps 2 and 3 until Step 3 produces no change in x_{n+1}.

As a means of demonstrating the Newton-Raphson formula, consider the problem of solving $y = x^2 - 4x + 3$ for those values of x for which y = 0.

Step 1. Suppose an initial solution is $x_o = 3\frac{1}{2}$.

Step 2. $y = F(x) = x^2 - 4x + 3$: $F(3\frac{1}{2}) = \frac{5}{4}$.

$y' - F'(x) = 2x - 4$: $F'(3\frac{1}{2}) = 3$.

Step 3. Applying the Newton-Raphson formula,

$$x_1 = x_o - \frac{F(x_o)}{F'(x_o)}$$

$$= 3\frac{1}{2} - \frac{5}{12}$$

$$x_1 = \frac{37}{12} = 3\frac{1}{12} .$$

Step 4. Reapplying the Newton-Raphson formula,

$$x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}$$

$$= \frac{37}{12} - (\frac{25}{144})/(\frac{26}{12})$$

$$= \frac{37}{12} - \frac{25}{312}$$

$$x_2 = \frac{937}{312} = 3\frac{1}{312}.$$

If the iteration process is continued, the solution will be found to be x = 3. However, since $y = F(x) = x^2 - 4x + 3$ is a quadratic function, there are two solutions. The second solution, x = 1, can be found by reassigning an initial starting solution and reapplying the Newton-Raphson technique until the process terminates. As a matter of note, the process terminates when x_{n+1} does not improve over x_n.

The Newton-Raphson formula, although not readily suited for hand calculations, is easily programmed for computer use. The selection of suitable starting solutions is of particular importance when the cost of the computer run is considered. However, the usefulness and applicability of the technique justifies its implementation as a tool of functional analysis.

Multivariable case: Analysis of a given multivariable, nonlinear function for points of optima is accomplished by first obtaining the critical points defined by equating the system of first partial derivatives to zero and then examining these points for maxima or minima. The Newton-Raphson technique provides a method for solving the given function in a direct manner. This is accomplished by a suitable modification of the Newton-Raphson technique for the single-variable case.

Let $y(\underline{x}) = F(x_1, x_2, \ldots, x_n)$ be the given objective function. Let $\frac{\partial y}{\partial x_j} = y_j'(\underline{x})$, $(j = 1, 2, \ldots, n)$, denote the first partial derivative of $y(\underline{x})$ with respect to variable x_j. Let $\underline{x}_k$ be the k^{th} trial point. Let $\left[\frac{\partial y_j}{\partial x_p}\right]_k$ be the n partial derivatives of the function y_j' evaluated at point $\underline{x}_k$, where

$$\left[\frac{\partial y_j'}{\partial x_p}\right]_k \equiv \left[\frac{\partial^2 y}{\partial x_j \partial x_p}\right]_k \equiv y_{jp}''(\underline{x}_k).$$

The Newton-Raphson formula for solving a continuous, multivariable, nonlinear function is given by

$$y_j'(\underline{x}_{k+1}) = 0 = y_j'(\underline{x}_k) + \sum_{p=1}^{n} y_{jp}''(\underline{x}_k)(x_{p,\ k+1} - x_{pk}),$$

where $x_{p,\ k+1}$ is the value of x_p at iteration k + 1, and j = 1, 2,...,n.

The Newton-Raphson formula uses $y_{jp}''(\underline{x}_k)$ to estimate the values of the first derivatives in the neighborhood of trial point $\underline{x}_k$. The new point $\underline{x}_{k+1}$, where the predicted values all vanish, is selected as a better approximation to the solution of $y(\underline{x}) = 0$. The iteration process is repeated until all y_j' become acceptably small, at which time the iteration process terminates. The trial point defined by $\underline{x}_k$ is the set of values of $(x_1, x_2,\ldots,x_n)$ for which $\partial y/\partial x_j = 0$.

The steps required in the Newton-Raphson method for multivariable, nonlinear analysis can be summarized in the following manner.

Algorithm 4.2 (Newton-Raphson technique for multivariable functions).--

Step 1. Set $\partial y/\partial x_j = 0$ for j = 1, 2,...,n. The solution to this system of simultaneous equations constitutes the initial trial point defined by $\underline{x}_k = (x_1, x_2,\ldots,x_n)^T$. The vector $\underline{x}_k$ may consist of more than one set of values.

Step 2. Determine $\frac{\partial^2 y}{\partial x_j \partial x_p} = \frac{\partial}{\partial x_p}\left(\frac{\partial}{\partial x_j}\right)$ for all (j, p = 1, 2,...,n).

Step 3. Evaluate all first and second order partial derivatives at $\underline{x}_k$. Retain these values for use in the Newton-Raphson technique.

Step 4. Apply the Newton-Raphson formula

$$y_j'(\underline{x}_{k+1}) = 0 = y_j'(\underline{x}_k) + \sum_{p=1}^{n} y_{jp}''(\underline{x}_k)(x_{p,\ k+1} - x_{pk}),$$

where $y_j'(\underline{x}_k) = (\partial y/\partial x_j)\big|_{\underline{x}_k}$, $(j = 1, 2, \ldots, n)$, $y_{jp}''(\underline{x}_k) = (\frac{\partial}{\partial x_p}(\partial y/\partial x_j))\big|_{\underline{x}_k}$, and $x_{p,\ k+1}$ denotes the p^{th} variable at iteration k + 1.

Step 5. Repeat iterations 3 and 4, using $\underline{x}_{k+1}$ for $\underline{x}_k$, until no improvement in the solution vector results. When there is no further improvement, terminate the process. The solution vector defined by $\underline{x}_{k+1}$ is the solution to the given problem.

As a means of demonstrating this computational technique, consider the problem of minimizing the cost function $y(x_1, x_2) = 1000x_1 + (4\times10^9)x_1^{-1}x_2^{-2} + (2.5\times10^5)x_2$ subject to the solution requirements

$$0 \leq x_1 \leq 2200$$

$$0 \leq x_2 \leq 8.$$

This type of problem is similar to inventory control problems where stock levels of certain products must not exceed a fixed value.

Solution: Step 1. Set $\frac{\partial y}{\partial x_j} = 0$ for j = 1, 2. Solve this simultaneous system for the critical points. Define $\underline{x}_k$, the trial point, as being equal to this set of critical points.

$$\frac{\partial y}{\partial x_1} = 1000 - (4\times10^9)x_1^{-2}x_2^{-1} = 0$$

$$\frac{\partial y}{\partial x_2} = (2.5\times10^5) - (4\times10^9)x_1^{-1}x_2^{-2} = 0.$$

The solution set is given by $x_1 = 1000$ and $x_2 = 4$. Both of these points lie within the range of values imposed on x_1 and x_2. Therefore, for k = I,

$$\underline{x}_I = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1000 \\ 4 \end{bmatrix}.$$

Step 2. Determine $\frac{\partial^2 y}{\partial x_p \partial x_j} = \frac{\partial}{\partial x_p}\left(\frac{\partial y}{\partial x_j}\right)$, (p, j = 1, 2).

$$\frac{\partial y}{\partial x_1} = y_1' = 1000 - (4\text{x}10^9)x_1^{-2}x_2^{-1}$$

$$\frac{\partial y}{\partial x_2} = y_2' = (2.5\text{x}10^5) - (4\text{x}10^9)x_1^{-1}x_2^{-2}$$

$$\frac{\partial^2 y}{\partial x_1 \partial x_1} = \frac{\partial}{\partial x_1}(y_1') \equiv y_{11}'' = (8\text{x}10^9)x_1^{-3}x_2^{-1}$$

$$\frac{\partial^2 y}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}(y_2') = y_{12}'' = (4\text{x}10^9)x_1^{-2}x_2^{-2} \equiv y_{21}''$$

$$\frac{\partial^2 y}{\partial x_2 \partial x_2} = \frac{\partial}{\partial x_2}(y_2') = y_{22}'' = (8\text{x}10^9)x_1^{-1}x_2^{-3}.$$

Step 3. Evaluate all first and second order partial derivatives at $\underline{x}_I = \begin{matrix} x_1 \\ x_2 \end{matrix} = \begin{matrix} 1000 \\ 4 \end{matrix}$. Retain these values for later use.

$$y_1' = y_1'(\underline{x}_I) = 0$$

$$y_2' = y_2'(\underline{x}_I) = 0$$

$$y_{11}'' = y_{11}''(\underline{x}_I) = 2$$

$$y_{12}'' = y_{12}''(\underline{x}_I) = y_{21}''(\underline{x}_I) = 250$$

$$y_{22}'' = y_{22}''(\underline{x}_I) = 125{,}000.$$

Step 4. Apply the Newton-Raphson formula

$$y_j'(\underline{x}_{k+1}) = 0 = y_j'(\underline{x}_k) + \sum_{p=1}^{n} y_{jp}''(\underline{x}_k)(x_{p,\ k+1} - x_{pk}).$$

For j = 1, 2, this formula expands into

$$y_1'(\underline{x}_{k+1}) = y_1'(\underline{x}_k) + \sum_{p=1}^{n} y_{1p}''(\underline{x}_k)(x_{p,\ k+1} - x_{pk})$$

$$y_2'(\underline{x}_{k+1}) = y_2'(\underline{x}_k) + \sum_{p=1}^{n} y_{2p}''(\underline{x}_k)(x_{p,\ k+1} - x_{pk}).$$

Since k = I and n = 2, the final form of the expanded Newton-Raphson formula is given by

$$y_1'(\underline{x}_{II}) = y_1'(\underline{x}_I) + \sum_{p=1}^{2} y_{1p}''(\underline{x}_I)(x_{p,\ II} - x_{pI})$$

$$y_2'(\underline{x}_{II}) = y_2'(\underline{x}_I) + \sum_{p=1}^{2} y_{2p}''(\underline{x}_I)(x_{p,\ II} - x_{pI}).$$

This form expands into

$$y_1'(\underline{x}_{II}) = y_1'(\underline{x}_I) + y_{11}''(\underline{x}_I)(x_{1,\ II} - x_{1,\ I}) + y_{12}''(\underline{x}_I)(x_{2,\ II} - x_{2,\ I})$$

$$y_2'(\underline{x}_{II}) = y_2'(\underline{x}_I) + y_{21}''(\underline{x}_I)(x_{1,\ II} - x_{1,\ I}) + y_{22}''(\underline{x}_I)(x_{2,\ II} - x_{2,\ I}).$$

Substituting for the first and second order partial derivatives and for $x_{1,\ I}$ and $x_{2,\ I}$,

$$y_1'(\underline{x}_{II}) = 0 + 2(x_1 - 1000) + 250(x_2 - 4)$$

$$y_2'(\underline{x}_{II}) = 0 + 250(x_1 - 1000) + 125{,}000(x_2 - 4).$$

Setting $y_1'(\underline{x}_{II}) = y_2'(\underline{x}_{II}) = 0$,

$$2(x_1 - 1000) + 250(x_2 - 4) = 0$$

$$250(x_1 - 1000) + 125{,}000(x_2 - 4) = 0.$$

The solution to this 2 x 2 system is $x_1 = 1000$, $x_2 = 4$. Therefore, the solution vector for the second iteration is given by

$$\underline{X}_{II} = \begin{bmatrix} x_{1,\ II} \\ x_{2,\ II} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1000 \\ 4 \end{bmatrix}.$$

Step 5. Since $\underline{X}_I = \underline{X}_{II}$, no better solution exists. The iteration process terminates. The function

$$y(x_1, x_2) = 1000x_1 + (4x10^9)x_1^{-1}x_2^{-1} + (2.5x10^5)x_2$$

subject to

$$0 \leq x_1 \leq 2200$$

$$0 \leq x_2 \leq \quad 8$$

achieves a minimum at $x_1 = 1000$, $x_2 = 4$. The value of the given function at $x_1 = 1000$, $x_2 = 4$ is $y(1000, 4) = 3{,}000{,}000$.

From this example it can be seen that the Newton-Raphson technique provides a means for iteratively solving a system of simultaneous equations. The technique is particularly useful when the system of simultaneous equations is nonlinear. Although the initial trial point can be the set of $(x_1, x_2, \ldots, x_n)$ values for which $\partial y/\partial x_j = 0$, $(j = 1, 2, \ldots, n)$, it is not necessary. For trial points located within a suitable neighborhood of the true solution, convergence is fairly rapid.

In application the Newton-Raphson technique is used to find the critical points for a given function. The critical points can then be used to evaluate the Hessian matrix associated with the given function. As noted previously, the value of the Hessian matrix defines the conditions under which a set of points defines a maximum or minimum value. Thus, incorporation of the Hessian matrix concept enables a given set of solutions to $\partial y/\partial x_j = 0$ ($j = 1, 2, \ldots, n$) to be readily examined for max-min. The Newton-Raphson technique provides a means for locating these points.

Constrained derivatives.--In the study of optimization theory, it is necessary to consider the possible existence of side conditions which must be satisfied by the optimal solution. As noted in the discussion of classical optimization theory, these side conditions can be either equalities or inequalities. Although the underlying theory for the optimization process is the same as that for unconstrained analysis, the introduction of side conditions does increase the difficulty of the computations.

Examples of constrained problems include the amount of stock carried in inventory with a limited amount of storage space available, allocation of advertising expenses subject to departmental budget restrictions, and minimization of costs subject to defined demand levels. Unlike the problems of classical optimization theory and basic modern optimization theory, the function to be optimized and the side conditions are free to assume any degree. Thus, the objective function and the side conditions can be linear or nonlinear. For the techniques of modern optimization theory, the functional expressions are generally nonlinear.

In Foundations of Optimization, Wilde and Beightler consider constrained indirect search in terms of state and decision variables. These concepts are then incorporated into a set of derivations defined as constrained derivatives.[16] Since the previously noted work is the major

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[16] Ibid., pp. 30-37.

publication in the area of modern optimization theory, the discussion to follow is the result of a comprehensive study of their work. This discussion is such that the work of Wilde and Beightler is presented in a format conducive to practical application. Terms are defined as needed, and a computational algorithm is presented and demonstrated for each discussed technique.

Equality constraints: The use of state and decision variables is a technique for dealing with constraints that are expressed as equalities. The purpose of this solution technique is to permit

> partitioning the independent variables into decision variables and state variables, then solving the linear equations for the state differentials as linear functions of the decision differentials. The coefficients of these linear expressions [are called] decision derivatives.[17]

The technique utilized in solving the resulting linear system is one that expresses the state differentials in terms of the decision differentials. An arbitrary assignment of numerical value to each of the decision differentials uniquely determines the value of the associated state differential.

As a means of establishing a common terminology, the concepts of state and decision variables are defined formally in the following manner.

<u>Definition 4.7</u>.--The state variable in a decision problem is defined as the variable that is used to describe the system at any given point in time.

<u>Definition 4.8</u>.--The decision variable in a decision problem is defined as the variable that is allowed to assume any value permitted by the problem.

The assignment of state and decision variables is arbitrary. However, once the assignment is made, and the decision differentials calculated, the state differential serves to keep the solution within the defined feasible region.

In a given decision problem consisting of N independent variables, there will be M state variables and P decision variables, where M + P = N. When the M state variables are assigned, there are P = N - M decision variables. The value of P has been called the degree of freedom and indicates the number of variables that can be manipulated without regard to feasibility.

The notation utilized in the state and decision variable approach to equality constrained analysis is summarized below. These notational symbols are written as definitions as a means of stressing their importance.

Definition 4.9.--Let m, n = 1, 2,...,M. Then, $s_m = x_n$ denotes the state variable of the problem.

Definition 4.10.--Let p = 1, 2,...,P, where p = n - M and n = M + 1, M + 2,...,N. Then, $d_p = x_p$ denotes the decision variable of the problem.

The system of equations relating the state and decision differentials is given by

$$-\partial y + \sum_{m=1}^{M} \left(\frac{\partial y}{\partial s_m}\right)\partial s_m = -\sum_{p=1}^{P} \left(\frac{\partial y}{\partial x_p}\right)\partial d_p$$

$$\sum_{m=1}^{M} \left(\frac{\partial f_k}{\partial s_m}\right)\partial s_m = -\sum_{p=1}^{P} \left(\frac{\partial f_k}{\partial d_p}\right)\partial d_p,$$

where k = 1, 2,...,M. and f_k denotes the function relating the state and decision variables. The value associated with s_m and d_p represents the change effected in both the decision variable and the state variable and is identified as the appropriate decision differential. In formulating the system of equations which relates the state and decision differentials, it is assumed that linear independence is maintained.

The incorporation of the state and decision differential concept into the analysis of a given problem is achieved via the use of the Jacobian. In this analysis the Jacobian of coefficients of the state variables is given by

$$\frac{\partial \underline{f}}{\partial \underline{s}} = \begin{vmatrix} \frac{\partial f_1}{\partial s_1} & \frac{\partial f_1}{\partial s_2} & \cdots & \frac{\partial f_1}{\partial s_m} \\ \frac{\partial f_2}{\partial s_1} & \frac{\partial f_2}{\partial s_2} & \cdots & \frac{\partial f_2}{\partial s_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial s_1} & \frac{\partial f_m}{\partial s_2} & \cdots & \frac{\partial f_m}{\partial s_m} \end{vmatrix}$$

and has a determinant defined by

$$|\underline{J}| = \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)}.$$

Two other necessary Jacobians are obtained by (1) replacing f_k $(k = 1, 2, \ldots, M)$ by the objective function y wherever it appears and (2) adding to $|\underline{J}|$ a row and column involving the objective function y and a decision variable d_p. The resulting Jacobians are given by

$$\frac{\partial(f_1, f_2,\ldots,f_{k-1}, y, f_{k+1},\ldots,f_m)}{\partial(s_1, s_2,\ldots,s_m)} = \begin{vmatrix} \frac{\partial f_1}{\partial s_1} & \frac{\partial f_1}{\partial s_2} & \cdots & \frac{\partial f_1}{\partial s_m} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial f_{k-1}}{\partial s_1} & \frac{\partial f_{k-1}}{\partial s_2} & \cdots & \frac{\partial f_{k-1}}{\partial s_m} \\ \frac{\partial y}{\partial s_1} & \frac{\partial y}{\partial s_2} & \cdots & \frac{\partial y}{\partial s_m} \\ \frac{\partial f_{k+1}}{\partial s_1} & \frac{\partial f_{k+1}}{\partial s_2} & \cdots & \frac{\partial f_{k+1}}{\partial s_m} \\ \frac{\partial f_m}{\partial s_1} & \frac{\partial f_m}{\partial s_2} & \cdots & \frac{\partial f_m}{\partial s_m} \end{vmatrix} \; ; \; k = 1, 2,\ldots,M,$$

and

$$\frac{\partial(y, f_1, f_2,\ldots,f_m)}{\partial(d_p, s_1, s_2,\ldots,s_m)} = \begin{vmatrix} \frac{\partial y}{\partial d_p} & \frac{\partial y}{\partial s_1} & \cdots & \frac{\partial y}{\partial s_m} \\ \frac{\partial f_1}{\partial d_p} & \frac{\partial f_1}{\partial s_1} & \cdots & \frac{\partial f_1}{\partial s_m} \\ \frac{\partial f_2}{\partial d_p} & \frac{\partial f_2}{\partial s_1} & \cdots & \frac{\partial f_2}{\partial s_m} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial f_m}{\partial d_p} & \frac{\partial f_m}{\partial s_1} & \cdots & \frac{\partial f_m}{\partial s_m} \end{vmatrix} \; ; \; p = 1, 2,\ldots,P.$$

In addition to the Jacobians, use is made of the constrained derivative $\frac{\delta y}{\underline{\delta d}_p}$, p = 1, 2,...,P. The interpretation associated with $\frac{\delta y}{\underline{\delta d}_p}$ is that $\frac{\delta y}{\underline{\delta d}_p}$ defines the constrained derivative of the function y with respect to the decision variable d_p. $\frac{\delta y}{\underline{\delta d}_p}$ is defined as the p^{th} decision derivative.

Definition 4.11.--The constrained derivative of the objective function y with respect to decision derivative d_p is given by

$$\frac{\delta y}{\delta d_p} = \frac{\partial(y,\ f_1,\ldots,f_m)}{\partial(d_p,\ s_1,\ldots,s_m)} \Big/ \frac{\partial(f_1,\ f_2,\ldots,f_m)}{\partial(s_1,\ s_2,\ldots,s_m)} .$$

The value associated with $\delta y/\delta dp$ represents the change in the objective function while holding P - 1 of the decision derivatives constant. The result is the rate of change in the objective function that is attributed to feasible changes in the p^{th} decision variable. Any change in the p^{th} decision variable will bring about a change in the corresponding state variable.

One of the shortcomings of indirect search techniques is that they can be applied only when the optimal solution lies within the feasible region and not on a boundary. As such, equality constrained problems must be modified if indirect techniques are to be applicable. The use of the decision derivative modifies the equality constrained problem in such a way that the classical theory for interior optima applies. That is, the system defined by the decision derivatives is equal to zero at a feasible optimum.

Theorem 4.7.--Let $y = F(x_1,\ x_2,\ldots,x_N)$ be any N-variable objective function subject to a set of equality constraints. Let the N variables be partitioned in such a way that there are M state variables and P = N - M decision variables. Let $\frac{\delta y}{\delta d_p}$ (p = 1, 2,...,P) denote the constrained derivative of y with respect to the decision variable d_p. A necessary condition for a set of points to be a point of optima is that the system defined by the P decision derivatives equal zero at that point; i.e., if $\underline{x}^*$ denotes a feasible optima,

$$\left.\frac{\delta y}{\underline{\delta d}_p}\right|_{\underline{x}^*} = 0; \quad p = 1, 2, \ldots, P.$$

These concepts are synthesized into an operational format in Algorithm 4.3. Although the construction of this algorithm is based upon the work of Wilde and Beightler, the presentation shown here is the result of their work and not a part of it.

Algorithm 4.3 (differential algorithm for equality constrained optimization).--Step 1. Formulate the function which relates the state and decision variables. This is accomplished by defining $f_k(s_1, s_2, \ldots, s_m, d_1, d_2, \ldots, d_p)$ to be equal to the k^{th} equality constraint.

Step 2. Determine the state and decision differentials.

Step 3. Evaluate the Jacobian for the given problem.

Step 4. Determine the p^{th} decision derivative $\frac{\delta y}{\underline{\delta d}_p}$.

Step 5. Set the system defined by $\frac{\delta y}{\underline{\delta d}_p}$ equal to zero. Solve this system for the state and decision variables. Using these values, determine the corresponding values of the original variables.

As a means of illustrating Algorithm 4.3, consider the problem of minimizing

$$y = 1000x_1 + (4x10^9)x_1^{-1}x_2^{-1} + (2.5x10^5)x_2$$

subject to

$$x_1 x_2 = 9000$$

$$0 < x_2 < 2000$$

In this problem, N = 2. Therefore, there will be one decision variable and one state variable: M = 1 and P = N - M = 1. Since the selection of the state and decision variable is arbitrary, suppose the state variable is x_2 and the decision variable is x_1; i.e., $s_1 = x_2$ and $d_1 = x_1$. The operations leading to the optimal solution are summarized below.

Solution: Step 1. Formulate the function which relates the state and decision variables. This is accomplished by defining $f_k(s_1, s_2, \ldots, s_m, d_1, d_2, \ldots, d_p)$ to be equal to the k^{th} equality constraint. Therefore, for k = 1, $f_1(s_1 d_1) = d_1 s_1 - 9000 = 0$.

Step 2. Determine the state and decision differentials.

(1) the state differential: $\partial f_1/\partial s_1 = d_1$.

(2) the decision differential: $\partial f_1/\partial d_1 = s_1$.

Step 3. Determine the Jacobian. For this problem the Jacobian is given by

$$\frac{\partial(y, f_1)}{\partial(d_1, s_1)} = \begin{vmatrix} \frac{\partial y}{\partial d_1} & \frac{\partial y}{\partial s_1} \\ \frac{\partial f_1}{\partial d_1} & \frac{\partial f_1}{\partial s_1} \end{vmatrix} .$$

Before $\partial y/\partial d_1$ and $\partial y/\partial s_1$ can be determined, it is necessary to express the objective function y in terms of both d_1 and s_1. This is accomplished by substituting for x_1 and x_2 in the original y. The result of this substitution is the nonlinear expression

$$y = 1000 d_1 + (4\times10^9) d_1^{-1} s_1^{-1} + (2.5\times10^5) s_1 .$$

Then,

$$\frac{\partial y}{\partial d_1} = 1000 - (4\times10^9) d_1^{-2} s_1^{-1};$$

$$\frac{\partial y}{\partial s_1} = (2.5\text{x}10^5) - (4\text{x}10^9)d_1^{-1}s_1^{-2}.$$

With these values, $\frac{\partial(y,\ f_1)}{\partial(d_1,\ s_1)}$ can be written as

$$\begin{vmatrix} 1000 - (4\text{x}10^9)d_1^{-2}s_1^{-1} & (2.5\text{x}10^5) - (4\text{x}10^9)d_1^{-1}s_1^{-2} \\ s_1 & d_1 \end{vmatrix}$$

Evaluation of this determinant yields

$$\frac{\partial(y,\ f_1)}{\partial(d_1,\ s_1)} = d_1[1000 = (4\text{x}10^9)d_1^{-2}s_1^{-1}] - s_1[(2.5\text{x}10^5) - (4\text{x}10^9)d_1^{-1}s_1^{-2}]$$

$$= 1000d_1 - (4\text{x}10^9)d_1^{-1}s_1^{-1} - (2.5\text{x}10^5)s_1 + (4\text{x}10^9)d_1^{-1}s_1^{-1}$$

$$= 1000d_1 - (2.5\text{x}10^5)s_1.$$

Step 4. Determine the p^{th} decision derivative. Since there is only one decision derivative, this requires only the determining of $\frac{\delta y}{\underline{\delta} d_1}$.

$$\frac{\delta y}{\underline{\delta} d_p} = \frac{\partial(y,\ f_1,\ldots,f_m)}{\partial(d_p,\ s_1,\ldots,s_m)} \Bigg/ \frac{\partial(f_1,\ldots,f_m)}{\partial(s_1,\ldots,s_m)}.$$

For the given problem,

$$\frac{\delta y}{\delta d_1} = \frac{\partial(y,\ f_1)}{\partial(d_1,\ s_1)} \Bigg/ \frac{\partial f_1}{\partial s_1}$$

$$= [1000d_1 - (2.5\text{x}10^5)]/d_1$$

$$= 1000 - (2.5\text{x}10^5)\ \frac{s_1}{d_1}\ .$$

Step 5. Apply Theorem 4.7 to determine the values for d_1 and s_1. This theorem requires $\frac{\delta y}{\underline{\delta} d_p} = 0$ at optimal points for all $\underline{p}$. Since there is but one decision derivative, this step requires only the solving of the resulting equation.

$$\frac{\delta y}{\delta d_1} = 1000 - (2.5\text{x}10^5)\frac{s_1}{d_1} = 0$$

$$1000 = (2.5\text{x}10^5)\frac{s_1}{d_1}$$

$$1000d_1 = (2.5\text{x}10^5)s_1$$

$$s_1 = \frac{1000}{2.5\text{x}10^5}\, d_1$$

$$= \frac{1000}{250{,}000}\, d_1$$

$$s_1 = \frac{1}{250}\, d_1 .$$

Since $d_1 s_1 - 9000 = 0$ can be written as $d_1 s_1 = 9000$,

$$d_1 = \frac{9000}{s_1}.$$

Substituting into $\frac{\delta y}{\delta d_1} = 0$,

$$s_1 = \frac{1}{250}\left(\frac{9000}{s_1}\right)$$

$$s_1^2 = 36$$

$$s_1 = \pm 6.$$

Since negative values are not permitted, $s_1 \equiv x_2$ and $0 \leq x_2 \leq 8$, the value of s_1 will be 6. For $s_1 = 6$, $d_1 = 1500$. The optimal point for the given problem is $x_1 = 1500$ and $x_2 = 6$. Both of these values are within the imposed limits. The value of y is 3,444,444.44, to the nearest hundredth.

Inequality constraints: The decision derivative concept can be extended to include optimization problems in which the function to be optimized is restricted by a set of inequality restrictions. Such problems are characterized by one of the following types:

(1) minimize $y(\underline{x})$ subject to the conditions that $\underline{x} = (x_1, x_2,\ldots,x_n) \geq 0$ for every x_i $(i = 1, 2,\ldots,n)$ and $f_k(\underline{x}) \geq 0$, $(k = 1, 2,\ldots,K)$.

(2) maximize $y(\underline{x})$ subject to the conditions that $\underline{x} = (x_1, x_2,\ldots,x_n) \geq 0$ for every $x_i(i = 1, 2,\ldots,n)$ and $f_k(\underline{x}) \leq 0$, $(k = 1, 2,\ldots,K)$.

In both types of problems the function to be optimized is permitted to be nonlinear. In addition, the constraint functions can be either linear or nonlinear.

In extending the decision derivative concept to include inequality constraints, it is necessary to express the inequalities as equalities. This is accomplished by introducing an appropriate slack variable which measures the difference between the value of the function and zero.

In the discussion to follow, it is assumed that the optimization problem is one which requires the minimization of a given objective function subject to the conditions imposed by requiring $x_n \geq 0$ for all n variables and the set of functional restrictions. The techniques and general theory can be extended to maximization problems by simply redefining the conditions under which optimality is achieved.

The principle indirect search technique for general nonlinear optimization is the differential algorithm. This technique utilizes the constrained derivative in a manner similar to that of the preceding section. In addition, the concept of the Jacobian is expanded to include the constrained derivatives of the differential algorithm. Although other nonlinear optimization techniques exist, this study of indirect search techniques will be confined to the differential algorithm. The reason for this is the fact that the differential algorithm can be applied to any type of continuous, nonlinear, differentiable problem.

The theoretical development of the differential algorithm is due to Wilde and Beightler.[18] The technique moves toward the optimum feasible solution of a given problem by systematically moving from one trial solution to another. At each trial point, the surrounding neighborhood is investigated to determine whether or not another point provides an improvement on the current solution. Since it is assumed that the critical constraints at any given trial point can be identified, indirect search is applicable. The technique employed is that of the differential algorithm.

The use of the differential algorithm requires that the trial point be examined for sensitivity to changes in the state and decision variables. This is accomplished by determining the rate of change of the optimum value with respect to changes in the constraint function. With this approach, the constraint functions are forced to a value of zero.

In the analysis to follow, $f_k(\underline{x})$ denotes the k^{th} constraint function and f_k denotes the numerical value of $f_k(\underline{x})$. The values attributed to each of the f_k represent M new variables and are free to assume arbitrary values. The vector of values associated with each f_k $(k = 1, 2, \ldots, M)$ is denoted by $\underline{f}$.

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[18] Ibid., pp. 44-95.

Definition 4.12.--Let y be the defined objective function. Let $\underline{f}$ be the column vector of M arbitrary f_k values. Let $\frac{\partial f_k}{\partial x_n} \equiv \frac{\partial}{\partial x_n} f_k(\underline{x})$. The constrained derivative of y with respect to $f_k(\underline{x})$, denoted $\frac{\underline{\delta} y}{\underline{\delta} f_k}$, is given by

$$\frac{\underline{\delta} y}{\underline{\delta} f_k} = \frac{\partial(f_1, f_2, \ldots, f_{k-1}, y, f_{k+1}, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)} \Big/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)}$$

where s_m is the m^{th} state variable.

Definition 4.13.--Let s_m denote the m^{th} state variable. Let d_p denote the p^{th} decision variable. The constrained derivative of the s_m with respect to d_p, denoted $\frac{\underline{\delta} s_m}{\underline{\delta} d_p}$, is given by

$$\frac{\underline{\delta} s_m}{\underline{\delta} d_p} = - \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_{m-1}, d_p, s_{m+1}, \ldots, s_m)} \Big/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)}.$$

In both definitions it is assumed that the constraint functions have been written in terms of both the state and decision variables. The value associated with $\frac{\underline{\delta} y}{\underline{\delta} f_k}$ indicates the change in the optimum value resulting from a change of one unit in the value of the associated constraint function.

Consider again the problem of minimizing

$$y = 1000x_1 + (4x10^9)x_1^{-1}x_2^{-1} + (2.5x10^5)x_2$$

subject to

$$x_1x_2 = 9000$$

$$0 \leq x_1 \leq 2000$$

$$0 \leq x_2 \leq 8.$$

This problem was found to have a minimum value at $x_1 = 1500$ and $x_2 = 6$, where $x_1 \equiv d_1$ and $x_2 \equiv s_1$. In terms of the preceding discussion, the problem would have the form

$$y = 1000d_1 + (4\text{x}10^9)d_1^{-1}s_1^{-1} + (2.5\text{x}10^5)s_1$$

subject to $f_1(s_1, d_1) = d_1 s_1 - 9000 = 0$. The only value for s_i $(i = 1, 2, \ldots, M)$ is s_1. Therefore, according to Definition 4.12,

$$\frac{\partial y}{\partial s_1} = -(4\text{x}10^9)d_1^{-1}s_1^{-2} + (2.5\text{x}10^5)$$

and

$$\frac{\partial f_1}{\partial s_1} = d_1$$

give

$$\frac{\delta y}{\underline{\delta f}_1} = [\frac{-(4\text{x}10^9)}{d_1 s_1^2} + 2.5\text{x}10^5] \div d_1$$

$$= \frac{2.5\text{x}10^5}{d_1} - \frac{4\text{x}10^9}{d_1^2 s_1^2} .$$

For $d_1 = 1500$ and $s_1 = 6$,

$$\frac{\delta y}{\underline{\delta f}_1} = \frac{250{,}000}{1500} - \frac{4{,}000{,}000{,}000}{2{,}250{,}000(36)}$$

$$= 166.67 - 49.43$$

$$\frac{\delta y}{\underline{\delta f}_1} = \$117.24.$$

This means that if the constraint were $x_1 x_2 = 9001$, the increase in cost would be \$117.24.

Optimization of a given objective function subject to K constraint functions is achieved by first equating each of the K inequalities to zero.

This is accomplished by subtracting the slack variable f_k $(k = 1, 2,\ldots,K)$ from each of the K inequalities $f_k(\underline{x}) \geq 0$. This transforms the optimization problem (assumed to be one of minimization) into the following:

$$\text{minimize } y(\underline{x})$$

subject to

$$f_k(\underline{x}) - f_k = 0;$$
$$\underline{x} \geq 0;$$
$$f_k \geq 0.$$

The vector defined by $\underline{x}$ is assumed to be contained within the feasible region defined by the constraint set.

Application of the differential algorithm requires that M of the K slack variables equal zero at the vector $\underline{x}$; i.e., for M of the K variables f_k,

$$f_m = 0, \quad m = 1, 2,\ldots,M,$$
$$f_k > 0, \quad k = M+1, M+2,\ldots,K.$$

This requirement modifies the optimization problem and expresses it in the form

$$\text{minimize } y(\underline{x})$$

subject to

$$f_m(\underline{x}) = 0, \quad m = 1, 2,\ldots,M,$$
$$f_k(x) - f_k = 0, \quad k = M+1, M+2,\ldots,K,$$
$$\underline{x} \geq 0.$$

The constraint $f_m(\underline{x}) = 0$ is said to be a tight constraint. The constraint $f_k(\underline{x}) - f_k = 0$, for $f_k(\underline{x}) > 0$, is said to be a loose constraint. The M tight constraints will be considered as independent variables. In addition, with each tight variable $f_m = 0$, there is an associated constraint equation,

$$f_m(\underline{x}) - f_m = 0.$$

As in the case of the equality constrained problem, it is necessary to arbitrarily select state and decision variables. However, this selection is better controlled under inequality constraints than under equality constraints.

> The state variables vary uncontrollably when the decisions are [changed] and may therefore increase or decrease, but no variable is permitted to become negative. Therefore, it is unwise to designate a variable as a state [variable] if its value at $\underline{x}$ is already zero. . .any variable having the value zero at $\underline{x}$ must be treated as a decision [variable] and never decreased.[19]

In developing the differential algorithm,[20] it is assumed that M of the constraints have been made equal to zero by equating f_m $(m = 1, 2,\ldots,M)$ to zero and that $M \leq N$, N being the number of variables defined for the problem. Of the N variables, N - M of the variables x_n have been selected as decision variables. The decision variables will be designated d_R, $R = 1, 2,\ldots,N\text{-}M$. The M state variables will be designated s_m, $m = 1, 2,\ldots,M$. It is assumed that $s_m > 0$ for all m and $d_R \geq 0$ for all R.

The differential algorithm utilizes the constrained derivative of s_m with respect to the p^{th} decision variable d_p concept of Definition 4.13 and the constrained derivative of the objective function y with respect to the k^{th} constraint function concept of Definition 4.12. In addition, use is made of individual slack derivatives.

Definition 4.14.--Let f_t denote the t^{th} constraint written as a function of the state and decision variables $(t = 1, 2,\ldots,M)$. Let $\frac{\delta s_m}{\delta f_t}$ and $\frac{\delta y}{\delta f_t}$

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[19] Ibid., p. 48.

[20] The general development of the differential algorithm is credited to Wilde and Beightler. The discussion and algorithmic development presented here is an elaboration of their work for the purpose of understanding and formalization. See ibid., pp. 44-62.

denote the constrained slack derivative of s_m with respect to f_t and the constrained slack derivative of y with respect to f_t, respectively. Then,

$$\frac{\delta s_m}{\delta f_t} = \frac{\partial(f_{t+1}, f_{t+2},\ldots,f_m, f_1, f_2,\ldots,f_{t-1})}{\partial(s_{m+1}, s_{M+2},\ldots,s_m, s_1, s_2,\ldots,s_{m-1})} \Big/ \frac{\partial(f_1, f_2,\ldots,f_m)}{\partial(s_1, s_2,\ldots,s_m)}; \text{ and,}$$

$$\frac{\delta y}{\delta f_t} = \frac{\partial(f_1, f_2,\ldots,f_{t-1}, y, f_{t+1},\ldots,f_m)}{\partial(s_1, s_2,\ldots,s_m)} \Big/ \frac{\partial(f_1, f_2,\ldots,f_m)}{\partial(s_1, s_2,\ldots,s_m)}.$$

The slack derivative $\frac{\delta y}{\delta f_t}$ measures the rate of change of the objective function with respect to changes in the t^{th} constraint, with all other decisions and tight constraints held constant. At all points other than the optimum, the value of $\frac{\delta y}{\delta f_t}$ depends upon the variables which are designated as state variables.

Necessary conditions for a given point to be an optimum point can be established by considering the constrained derivative of the objective function with respect to both the decision variables and the slack variables. These conditions are given in Definition 4.15.

Definition 4.15.--Let y be any constrained function that is to be minimized. Let $\frac{\delta y}{\delta d}$ denote the constrained derivative of y with respect to the set of decision variables, $\frac{\delta y}{\delta d}$ being identified as the decision derivative. Let $\frac{\delta y}{\delta f}$ denote the constrained derivative of y with respect to the set of slack variables, $\frac{\delta y}{\delta f}$ being identified as the slack derivative.

Then, a necessary condition for y to have a minimum value at a given solution vector $\underline{x}$ is for

$$\left.\frac{\delta y}{\underline{\delta d}}\right|_{\underline{x}} \geq 0 \text{ and } \left.\frac{\delta y}{\underline{\delta f}}\right|_{\underline{x}} \geq 0.$$

The condition described here is defined as the nonnegative conditions for a minimum.

Definition 4.16.--Let y be any constrained function that is to be minimized. Let d_R and f_t denote the R^{th} decision variable and the t^{th} constraint, respectively. Let $\frac{\delta y}{\underline{\delta d}_R}$ denote the constrained derivative of y with respect to the R^{th} decision variable. Let $\frac{\delta y}{\underline{\delta f}_t}$ denote the constrained derivative of y with respect to the t^{th} tight constraint. At a minimum point $\underline{x}$,

$$\left(\frac{\delta y}{\underline{\delta d}_R}\right) d_R = 0 \quad (R = 1, 2, \ldots, N-M)$$

and

$$\left(\frac{\delta y}{\underline{\delta f}_t}\right) f_t = 0 \quad (t = 1, 2, \ldots, M).$$

The condition described here is defined as the complementary slackness condition.

Definition 4.17.--Let y be any constrained function that is to be minimized. Let $\frac{\partial y}{\partial d_R}$ denote the partial derivative of y with respect to the decision variable d_R. Let $\frac{\delta y}{\underline{\delta d}_R}$ denote the constrained derivative of y with respect to the R^{th} decision variable. Let $\frac{\delta y}{\delta \underline{f}}$ denote the slack derivative. Let $\frac{\partial \underline{f}}{\partial d_R}$ denote the partial derivative of the constraint set with respect to the decision variable d_R. Let $\underline{x}$ be any trial point. A necessary condition for $\underline{x}$ to be a minimum point is for

$$\frac{\partial y}{\partial d_R} = \left(\frac{\delta y}{\delta d_R}\right) + \left(\frac{\delta y}{\delta \underline{f}}\right)\left(\frac{\partial \underline{f}}{\partial d_R}\right), \quad (R = 1, 2, \ldots, N\text{-}M)$$

when evaluated at $\underline{x}$.

The value defined by $\frac{\partial y}{\partial d_R}$ is obtained by partially differentiating the objective function with respect to the R^{th} decision variable. This is accomplished while ignoring the set of constraint functions. The partial derivatives required for $\frac{\partial \underline{f}}{\partial d_R}$ are obtained by direct differentiation of the t^{th} constraint, where f_t is written as a function of the state and decision variables and the derivatives of $\frac{\partial \underline{f}}{\partial d_R}$ are given by $\frac{\partial f_t}{\partial d_R}$.

The condition defined by Definition 4.17 can be rearranged into the form

$$\frac{\delta y}{\delta \underline{d}} = \left(\frac{\partial y}{\partial \underline{d}}\right) - \left(\frac{\delta y}{\delta \underline{f}}\right)\left(\frac{\partial \underline{f}}{\partial \underline{d}}\right).$$

This form expresses the decision derivative vector in terms of the unconstrained gradient $\frac{\partial y}{\partial \underline{d}}$ and the correction factor $(\frac{\delta y}{\delta \underline{f}})(\frac{\partial \underline{f}}{\partial \underline{d}})$. The correction factor serves to keep the constraints tight.

Sufficient conditions for a solution vector $\underline{x}$ to be a minimum value are given by the conditions of nonnegativity and complementary slackness. For a nonsingular, nondegenerate solution, the conditions set forth in

Definitions 4.15 and 4.16 are sufficient to guarantee a local minimum. Global minimization is achieved by requiring the objective function y and the feasible region to be convex.

The differential algorithm for solving constrained nonlinear optimization problems utilizes two more tools. The tools are the loose constraint derivatives $\frac{\delta f_k}{\delta d_m}$ and $\frac{\delta f_k}{\delta f_t}$.

Definition 4.18.--Let $f_k(\underline{s}, \underline{d})$ denote the k^{th} loose constraint of a guven optimization problem, where the constraint is written as a function of the vector of state variables $\underline{s}$ and the vector of decision variables $\underline{d}$, $k = M+1, M+2, \ldots, K$. Let $\frac{\delta f_k}{\delta d_m}$ denote the constrained derivative of the k^{th} loose constraint with respect to the m^{th} decision variable. Let $\frac{\delta f_k}{\delta f_t}$ denote the constrained derivative of the k^{th} loose constraint with respect to the t_{th} tight constraint. Then, for $k = M+1, M+2, \ldots, K$, and $t = 1, 2, \ldots, M$,

$$\frac{\delta f_k}{\delta d_m} = \frac{\partial(f_k, f_1, \ldots, f_m)}{\partial(d_m, s_1, \ldots, s_m)} \Bigg/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)}; \text{ and,}$$

$$\frac{\delta f_k}{\delta f_t} = \frac{\partial(f_1, f_2, \ldots, f_{t-1}, f_k, f_{t+1}, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)} \Bigg/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)}.$$

At this point a brief review of the initial optimization problem and of the necessary modifications is in order. The purpose of this review is to bring into focus all that has been done in formulating the problem into one amenable to the differential algorithm.

The problem being investigated has the form

$$\text{minimize } y(x_1, x_2, \ldots, x_N)$$

subject to the K inequalities

$$f_k(\underline{x}) = f_k(x_1, x_2, \ldots, x_N) \geq 0, \qquad k = 1, 2, \ldots, K;$$

$$\underline{x} = (x_1, x_2, \ldots, x_N)^T \geq 0.$$

The K inequalities have been transformed into equalities by introducing the K slack variables f_k, $k = 1, 2, \ldots, K$. Of these K slack variables, M have been set equal to zero at the solution vector $\underline{x}$. With this adjustment, the initial problem takes on the form

$$\text{minimize } y(\underline{x})$$

subject to

$$f_m(\underline{x}) = 0 \qquad (m = 1, 2, \ldots, M)$$

$$f_k(\underline{x}) - f_k = 0 \qquad (k = M+1, \ldots, K)$$

$$\underline{x} \geq 0,$$

where $f_m(\underline{x})$ is a tight constraint, $f_k(\underline{x}) - f_k$ a loose constraint.

Upon introduction of the state and decision variables, the problem is formulated as

$$\text{minimize } y(s_1, s_2, \ldots, s_m, d_1, d_2, \ldots, d_{N-M})$$

subject to

$$f_t(s_1, s_2, \ldots, s_m, d_1, d_2, \ldots, d_{N-M}) = 0$$

$$f_k(s_1, s_2, \ldots, s_m, d_1, d_2, \ldots, d_{N-M}) - f_k = 0$$

for $t = 1, 2, \ldots, M$ and $k = M+1, M+2, \ldots, K$. An additional requirement is that $\underline{s}$ and $\underline{d}$ both be nonnegative. It is this formulation that is solved by the technique of the differential algorithm. The values for $s_1, s_2, \ldots, s_m$, $d_1, d_2, \ldots, d_{N-M}$ will be the values of the x_i $(i = 1, 2, \ldots, N)$ to which the

The general approach of the differential algorithm requires that the decision and slack derivatives ($\frac{\delta y}{\delta \underline{d}_m}$ and $\frac{\delta y}{\delta \underline{f}_t}$, respectively) be evaluated at any feasible nonsingular, nondegenerate point. This evaluation can be accomplished through the use of the appropriate Jacobian or system of matrix equations. The signs of both $\frac{\delta y}{\delta \underline{d}_m}$ and $\frac{\delta y}{\delta \underline{f}_t}$ can then be examined for non-negativity. If any decision or slack derivative is negative, the value of the objective function y can be improved by increasing the d_m or f_t value, whichever applies. If the nonnegative requirement for $\frac{\delta y}{\delta \underline{d}_m}$ and $\frac{\delta y}{\delta \underline{f}_t}$ is satisfied, the complementary slackness conditions are examined. The complementary slackness conditions can be violated only if (1) a decision variable, d_m, and its corresponding derivative $\frac{\delta y}{\delta \underline{d}_m}$, are both positive or (2) a tight constraint, f_t, and its corresponding derivative $\frac{\delta y}{\delta \underline{f}_t}$ are both positive. If the complementary slackness conditions are violated, the value of y can be improved by decreasing the value of d_m or f_t. The appropriate decrease is determined according to the violator. Decreases are permissable so long as nonnegativity is maintained. The algorithm terminates when both nonnegativity and complementary slackness are satisfied, as this determines the existence of a local minimum. If the objective function is convex, the minimum is a global minimum.

The differential algorithm with inequality constraints can be written as a series of sequential computations. This series of computational steps is the result of a detailed analysis of both theory and examples and is presented here as a means of formalizing the computational procedure.

Algorithm 4.4 (differential algorithm with inequality constraints).--

Step 1. Begin the search at any feasible point $\underline{x}_I = (x_1, x_2, \ldots, x_N)^T$.

Step 2. Determine the number of tight constraints. Denote this number by M. With K initial constraints, this will leave K - M loose constraints.

Step 3. Determine the state and decision variables. If any variable has a value equal to zero, it must be a decision variable. If there are no variables with an initial value equal to zero, the selection of state and decision variables is arbitrary. Reformulate the problem in terms of the state and decision variables.

Step 4. Obtain the decision derivative by evaluating the Jacobian

$$\frac{\delta y}{\underline{\delta} d_R} = \frac{\partial(y, f_1, \ldots, f_m)}{\partial(d_R, s_1, \ldots, s_m)} \Bigg/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)}$$

at the solution point. The decision derivative can then be examined to guarantee that the nonnegativity condition is satisfied ($\frac{\delta y}{\underline{\delta} d_R} \geq 0$ at $\underline{x}_I$) and that the complementary slackness condition is satisfied ($(\frac{\delta y}{\underline{\delta} d_R}) d_r = 0$).

Step 5. Obtain the slack derivative by evaluating the Jacobian

$$\frac{\delta y}{\underline{\delta} f_t} = \frac{\partial(f_1, f_2, \ldots, f_{t-1}, y, f_{t+1}, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)} \Bigg/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)}$$

at the solution point. The slack derivative can then be examined to guarantee that the nonnegativity condition is satisfied ($\frac{\delta y}{\underline{\delta f}_t} \geq 0$ at $\underline{x}_I$) and that the complementary slackness condition is satisfied ($(\frac{\delta y}{\underline{\delta f}_t})f_t = 0$).

Step 6. If either $\frac{\delta y}{\underline{\delta d}_R}$ or $\frac{\delta y}{\underline{\delta f}_t}$ violates the nonnegativity requirement or complementary slackness, adjust the appropriate decision variable or tight constraint as indicated by the violator. In some cases, it may be feasible to adjust both the decision variable and the tight constraint. Determine the direction of adjustment by evaluating $\frac{\partial y}{\partial x_i}$ $(i = 1, 2,\ldots,N)$ at $\underline{x}_I$. The adjustment will result in a new trial point, $\underline{x}_{II}$.

Step 7. Using the new trial point $\underline{x}_{II}$, determine any change in the assignment of tight constraints resulting from the change in solutions. Reapply Steps 2 through 5. If an optimum solution does not result, proceed to Step 8.

Step 8. Determine the constrained derivative of each of the k constraint functions with respect to the mth decision variable, $\frac{\delta f_k}{\underline{\delta d}_m}$, and the constrained derivatives of the mth state variable with respect to the pth decision variable $\frac{\delta s_m}{\underline{\delta d}_p}$. This is accomplished by evaluating

$$\frac{\delta f_k}{\underline{\delta d}_m} = \frac{\partial(f_k, f_1, f_2,\ldots,f_m)}{\partial(d_m, f_1, f_2,\ldots,f_m)} \Big/ \frac{\partial(f_1, f_2,\ldots,f_m)}{\partial(s_1, s_2,\ldots,s_m)}$$

and

$$\frac{\delta s_m}{\underline{\delta d}_p} = \frac{\partial(f_1, f_2,\ldots,f_m)}{\partial(s_1, s_2,\ldots,s_{m-1}, d_p, s_{m+1},\ldots,s_m)} \Big/ \frac{\partial(f_1, f_2,\ldots,f_m)}{\partial(s_1, s_2,\ldots,s_m)}$$

at the trial point. The values of these constrained derivatives are used to indicate how far along the tight constraint the solution is allowed to move.

Step 9. Reapply the test for satisfaction of both nonnegativity and complementary slackness (Steps 4 and 5). If an optimum has not yet been found, reapply Steps 2-9. Reiterate Steps 2-9 until nonnegativity and complementary slackness is satisfied. Then, proceed to Step 10.

Step 10. An optimal solution (a local minimum) has been found when

$$\frac{\delta y}{\delta d_R} \geq 0;$$

$$\frac{\delta y}{\delta f_t} \geq 0;$$

$$\left(\frac{\delta y}{\delta d_R}\right) d_R = 0; \text{ and,}$$

$$\left(\frac{\delta y}{\delta f_t}\right) f_t = 0.$$

It is assumed that $t = 1, 2, \ldots, M$ and $R = 1, 2, \ldots, N-M$.

The following numerical example, taken from Wilde and Beightler, will serve to demonstrate the sequential processes of the differential algorithm.[21] The problem under investigation is one of minimizing a nonlinear function subject to a set of nonlinear constraints. Analogous administrative problems can be found in cost minimization problems, nonlinear inventory problems, and budget allocation problems.

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[21] Ibid., pp. 59-62.

$$\text{minimize } y = -e^{[(x_1-1)^2 + (x_2-2)^2]} = -\exp[(x-1)^2 + (x_2-2)^2],$$

subject to

$$f_1(x_1, x_2) = x_1 - x_2^2 \geq 0$$

$$f_2(x_1, x_2) = -e^{-x_1} + x_2 \geq 0$$

$$f_3(x_1, x_2) = -2(x_1-1)^2 + x_2 \geq 0$$

$$x_1 \geq 0$$

$$x_2 \geq 0.$$

Iteration I: Step 1. Let $\underline{X}_I = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step 2. At $\underline{X}_I$, there is but one tight constraint, $f_1(x_1, x_2)$. Therefore, M = 1. Since the problem contains three initial constraints, there will be two loose constraints, $f_2(x_1, x_2)$ and $f_3(x_1, x_2)$.

Step 3. Determine the state and decision variables. Since there are two independent variables, N = 2. With M = 1, there will be N - M = 2 - 1 = 1 decision variable and one state variable. Since neither x_1 nor x_2 have a value of zero in the initial trial point, the selection of the state and decision variables is arbitrary. Therefore, let $x_1 = d_1$ be the decision variable and $x_2 = s_1$ be the state variable.

$$\text{minimize } y = -e^{[(d_1-1)^2 + (s_1-2)^2]}$$

subject to

$$f_1(d_1, s_1) = d_1 - s_1^2 = 0$$

$$f_2(d_1, s_1) = e^{-d_1} + s_1 \geq 0$$

$$f_3(d_1, s_1) = -2(d_1-1)^2 + s_1 \geq 0$$

$$\underline{s} = (s_1) \geq 0$$

Step 4. Obtain the decision derivative by evaluating the Jacobian

$\frac{\delta y}{\delta d_R}$ at $d_1 = 1$, $s_1 = 1$.

$$\frac{\delta y}{\delta d_R} = \frac{\partial(y, f_1, \ldots, f_m)}{\partial(d_R, s_1, \ldots, s_m)} \bigg/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)}.$$

For the given problem,

$$\frac{\delta y}{\delta d_R} = \frac{\delta y}{\delta d_1}$$

$$= \frac{\partial(y, f_1)}{\partial(d_1, s_1)} \bigg/ \frac{\partial(f_1)}{\partial(s_1)}$$

$$= \begin{vmatrix} \frac{\partial y}{\partial d_1} & \frac{\partial y}{\partial s_1} \\ \frac{\partial f_1}{\partial d_1} & \frac{\partial f_1}{\partial s_1} \end{vmatrix} \div \frac{\partial f_1}{\partial s_1} .$$

Since $y = -e^{[(d_1-1)^2 + (s_2-2)^2]}$ and $f_1 = d_1 - s_1^2$, the required partial derivatives are

$$\frac{\partial y}{\partial d_1} = -2(d_1-1)e^{[(d_1-1)^2 + (s_1-2)^2]};$$

$$\frac{\partial y}{\partial s_1} = -2(s_1-2)e^{[(d_1-1)^2 + (s_1-2)^2]};$$

$$\frac{\partial f_1}{\partial d_1} = 1; \text{ and,}$$

$$\frac{\partial f_1}{\partial s_1} = -2s_1.$$

Evaluating these partial derivatives at $d_1 = 1$ and $s_1 = 1$, $\frac{\partial y}{\partial d_1} = 0$; $\frac{\partial y}{\partial s_1} = -2y$; $\frac{\partial f_1}{\partial d_1} = 1$; and, $\frac{\partial f_1}{\partial s_1} = -2$. Therefore,

$$\frac{\delta y}{\underline{\delta} d_1} = \begin{vmatrix} 0 & -2y \\ 1 & -2 \end{vmatrix} \div (-2)$$

$$= \frac{2y}{-2}$$

$$= -y$$

$$= -[-e^{(d_1-1)^2 + (s_1-2)^2}]$$

$$= e \text{ at } d_1 = 1 \text{ and } s_1 = 1.$$

Since $\frac{\delta y}{\underline{\delta} d_1} = e > 0$, the nonnegativity requirement for the decision derivative is satisfied. However, $(\frac{\delta y}{\underline{\delta} d_1})d_1 = e(1) \neq 0$. This violates the condition of complementary slackness. The value of d_1 can be reduced as a means of improving (further minimizing) the value of y at $x_1 = d_1 = 1$ and $x_2 = s_1 = 1$.

Step 5. Obtain the slack derivative by evaluating the Jacobian $\frac{\delta y}{\underline{\delta} f_t}$ at $d_1 = 1$, $s_1 = 1$.

$$\frac{\delta y}{\underline{\delta} f_t} = \frac{\partial(f_1, f_2, \ldots, f_{t-1}, y, f_{t+1}, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)} \Bigg/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)} .$$

For the given problem there is only one tight constraint, f_1. Therefore, $t = 1$. In addition there is only one state variable ($M = 1$).

$$\frac{\delta y}{\delta f_t} \equiv \frac{\delta y}{\underline{\delta} f_1} = \frac{\partial y}{\partial s_1} \Bigg/ \frac{\partial f_1}{\partial s_1}$$

The required partial derivatives are

$$\frac{\partial y}{\partial s_1} = -2(s_1-2)e^{[(d_1-1)^2 + (s_1-2)^2]};$$

$$\frac{\partial f}{\partial s_1} = -2s_1.$$

At $d_1 = 1$ and $s_1 = 1$, $\frac{\partial y}{\partial s_1} = -2y$ and $\frac{\partial f}{\partial s_1} = -2$. Therefore,

$$\frac{\underline{\delta y}}{\underline{\delta f_1}} = \frac{-2y}{-2} = y = -e^{[(d_1-1)^2 + (s_1-2)^2]}.$$

At $d_1 = 1$ and $s_1 = 1$, $\frac{\underline{\delta y}}{\underline{\delta f_1}} = -e < 0$. This violates the requirement that the slack derivative be nonnegative. This indicates that the value of y can be improved by relaxing the constraint $f_1(d_1, s_1)$; i.e., by letting $f_1(d_1, s_1)$ become a loose constraint. Since this results in a system free of tight constraints, it is more feasible to do this than adjust d_1.

Step 6. Since the nonnegativity conditions are violated by both the decision derivative and the slack derivative, it is necessary to determine the direction in which any adjustment is to take place. This is accomplished by evaluating the unconstrained derivatives at $\underline{x}_I$. The unconstrained derivative is given by partially differentiating the objective function with respect to each of the N independent variables.

$$y = -e^{(x_1-1)^2 + (x_2-2)^2}$$

$$\frac{\partial y}{\partial x_1} = -2(x_1-1)e^{(x_1-1)^2 + (x_2-2)^2}$$

$$\frac{\partial y}{\partial x_2} = -2(x_2-2)e^{(x_1-1)^2 + (x_2-2)^2}.$$

At $\underline{x}_I = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\frac{\partial y}{\partial x_1} = 0$ and $\frac{\partial y}{\partial x_2} = 2e$. The direction of change will be made by holding x_1 constant while varying x_2. Since $f_1(d_1, s_1)$ is to become a loose constraint as indicated in Step 6, it is necessary to examine the value of $\frac{\partial f_1}{\partial s_1}$. Since $s_1 = x_2$, this will be accomplished by evaluating $\frac{\partial f}{\partial x_2}$ for possible adjustments to the value of x_2.

$$\frac{\partial f_1}{\partial x_2} = -2x_2.$$

At $x_1 = 1$, $x_2 = 1$, $\frac{\partial f_1}{\partial x_2} = -2$. Thus, a decrease in x_2 will increase $f_1(x_1, x_2)$ while decreasing the value of y. The minimum feasible point on the line $x_1 = 1$ (x_1 was held constant) is reached at $x_2 = e^{-1}$, the point at which $f_2(x_1, x_2) = 0$.

Step 7. Define the second trial point, $\underline{x}_{II}$, by $\underline{x}_{II} = (1, e^{-1})^T$. From a table of exponential functions, $e^{-1} = .37$. At the second point,

$$y(\underline{x}_{II}) = y(1, .37) = -14.30.$$

Since $y(\underline{x}_I) = -2.72$, this second trial point is an improvement over the first.

Reapplication of Steps 2 through 6 is presented in summary form.

Step 2. At $\underline{x}_{II}$, $f_1(x_1, x_2) > 0$, $f_2(x_1, x_2) = 0$, and $f_3(x_1, x_2) > 0$. Thus, $f_2(x_1, x_2)$ is the tight constraint. Since $f_2(x_1, x_2)$ is the only tight constraint, M = 1. There are two loose constraints, $f_1(x_1, x_2)$ and $f_3(x_1, x_2)$.

Step 3. Let $x_1 = d_1$ and $x_2 = s_1$. The problem is then expressed as

$$\text{minimize } y = -\exp[d_1-1)^2 + (s_1-2)^2]$$

subject to

$$f_1(d_1, s_1) = d_1 - s_1^2 \geq 0$$

$$f_2(d_1, s_1) = -e^{-d_1} + s_1 = 0$$

$$f_3(d_1, s_1) = -2(d_1-1)^2 + s_2 \geq 0$$

$$s_1 \geq 0$$

$$d_1 \geq 0.$$

Step 4. The decision derivative $\frac{\delta y}{\underline{\delta d_1}}$ is given by $\frac{\delta y}{\underline{\delta d_1}} = -2e^{-1}(e^{-1}-2)y$. At $\underline{x}_{II}$, $\frac{\delta y}{\underline{\delta d_1}}$ is negative, as is $(\frac{\delta y}{\underline{\delta d_1}})d_1$.

Step 5. The slack derivative $\frac{\delta y}{\underline{\delta f_2}}$ is given by $\frac{\delta y}{\underline{\delta f_2}} = 2(e^{-1}-2)y$. At $\underline{x}_{II}$, $\frac{\delta y}{\underline{\delta f_2}}$ is positive, as is $(\frac{\delta y}{\underline{\delta f_2}})f_2$.

Step 6. Examination of the decision derivative $\frac{\delta y}{\underline{\delta d_1}}$ and the slack derivative $\frac{\delta y}{\underline{\delta f_2}}$ reveals that $\frac{\delta y}{\underline{\delta d_1}}$ violates nonnegativity. Thus, the objective function y can be further decreased by increasing d_1. As d_1 is allowed to increase, f_2 will be held at zero since $\frac{\delta y}{\underline{\delta f_2}}$ and $(\frac{\delta y}{\underline{\delta f_2}})f_2$ are both nonnegative.

An immediate result is that the second trial point is not an optimal solution. It is now necessary to evaluate the required constrained derivatives $\frac{\delta f_k}{\underline{\delta d_m}}$ and $\frac{\delta s_m}{\underline{\delta d_p}}$.

Step 8. Determine $\frac{\delta f_k}{\delta d_m}$ and $\frac{\delta s_m}{\delta d_p}$ for the loose constraints, where

$$\frac{\delta f_k}{\delta d_m} = \frac{\partial(f_k, f_1, \ldots, f_m)}{\partial(d_m, s_1, \ldots, s_m)} \Bigg/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)}$$

and

$$\frac{\delta s_m}{\delta d_p} = \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_{m-1}, d_p, s_{m+1}, \ldots, s_m)} \Bigg/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)} .$$

From Step 7, the loose constraints are

$$f_1(d_1, s_1) = d_1 - s_1^2;$$

$$f_3(d_1, s_1) = -2(d_1-1)^2 + s_1.$$

Since $f_2(d_1, s_1)$ is the tight constraint, $f_m \equiv f_2$. Thus, k = 1 and 3. There is only one decision variable, d_1. In addition, there is only one state variable s_1. The required decision derivatives are thus defined by $\frac{\delta f_1}{\delta d_1}$, $\frac{\delta f_3}{\delta d_1}$, and $\frac{\delta s_1}{\delta d_1}$.

$$\frac{\delta f_1}{\delta d_1} = \frac{\partial(f_1, f_2)}{\partial(d_1, s_1)} \Bigg/ \frac{\partial f_2}{\partial s_1} = \begin{vmatrix} \frac{\partial f_1}{\partial d_1} & \frac{\partial f_1}{\partial s_1} \\ \frac{\partial f_2}{d_1} & \frac{\partial f_2}{s_1} \end{vmatrix} \div \frac{\partial f_2}{\partial s_1}$$

$$\frac{\delta f_3}{\delta d_1} = \frac{\partial(f_3, f_2)}{\partial(d_1, s_1)} \Bigg/ \frac{\partial f_2}{\partial s_1} = \begin{vmatrix} \frac{\partial f_3}{\partial d_1} & \frac{\partial f_3}{\partial s_1} \\ \frac{\partial f_2}{\partial d_1} & \frac{\partial f_2}{\partial s_1} \end{vmatrix} \div \frac{\partial f_2}{\partial s_1}$$

$$\frac{\delta s_1}{\delta d_1} = -\frac{\partial f_2}{\partial d_1} \Bigg/ \frac{\partial f_2}{\partial s_1}.$$

$$\frac{\partial f_1}{\partial d_1} = 1;$$

$$\frac{\partial f_1}{\partial s_1} = -2s_1;$$

$$\frac{\partial f_3}{\partial d_1} = -4(d_1-1)$$

$$\frac{\partial f_3}{\partial s_1} = 1;$$

$$\frac{\partial f_2}{\partial d_1} = 1;$$

$$\frac{\partial f_2}{\partial s_1} = e^{-s_1}.$$

Evaluating the constrained derivatives,

$$\frac{\underline{\delta} f_1}{\underline{\delta} d_1} = 1 + 2e^{-2} > 0;$$

$$\frac{\underline{\delta} f_3}{\underline{\delta} d_1} = -e^{-1} < 0;$$

$$\frac{\underline{\delta} s_1}{\underline{\delta} d_1} = -e^{-1} < 0.$$

These results indicate that by moving in the direction tangent to $f_2(x_1, x_2) = 0$ in such a way that x_1 increases, the first constraint loosens, the third tightens, and x_2 will decrease in value.

Step 9. Since an optimal solution has not yet been located, it is necessary to reapply the procedure outlined in Steps 1 - 8. However, since $f_3(x_1, x_2)$ tightens at $x_1 = 1$, $x_2 = e^{-1}$, and x_1 is to be increased, a third trial solution can be obtained and evaluated in the same manner as $\underline{x}_I$ and $\underline{x}_{II}$. The interest in this third iteration will be minimization of y with the first constraint loose and the second and third constraints tight. Optimality in this problem occurs at $x_1 = 1.3586$, $x_2 = 0.2571$. In this case, M = 2, indicating that both variables are to be treated as state variables. Applying

$$\frac{\delta y}{\underline{\delta} f_t} = \frac{\partial(f_1, \ldots, f_{t-1}, y, f_{t+1}, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)} \Bigg/ \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(s_1, s_2, \ldots, s_m)},$$

the two slack derivatives are

$$\frac{\delta y}{\underline{\delta} f_2} = \frac{\partial(y, f_3)}{\partial(s_1, s_2)} \Bigg/ \frac{\partial(f_2, f_3)}{\partial(s_1, s_2)}$$

and

$$\frac{\delta y}{\underline{\delta} f_3} = \frac{\partial(f_2, y)}{\partial(s_1, s_2)} \Bigg/ \frac{\partial(f_2, f_3)}{\partial(s_1, s_2)}.$$

Step 10. Evaluation of these slack derivatives at $\underline{x}_{III} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.3586 \\ 0.2571 \end{bmatrix}$ yields

$$\frac{\delta y}{\underline{\delta} f_2} = \frac{-4.29}{1.69}\, y > 0, \text{ and } \frac{\delta y}{\underline{\delta} f_3} = \frac{-1.625}{1.69}\, y > 0.$$

Since $f_2 = f_3 = 0$ at $\underline{x}_{III}$,

$$\left(\frac{\delta y}{\underline{\delta} f_2}\right) f_2 = \left(\frac{\delta y}{\underline{\delta} f_3}\right) f_3 = 0.$$

Hence, the conditions of complementary slackness are satisfied. With no decision variables, there are no decision derivatives to test for non-negativity. The necessary and sufficient conditions for an optimal solution are those of complementary slackness and are satisfied. The optimal solution is that given by $\underline{x}_{III}$. The minimum value of y is -23.8.

A graphic illustration of this problem is shown in Figure 4.8. The graph is obtained by constructing contours for the objective function and by equating the constraints to zero. With the constraints equated to zero, one of the variables can be arbitrarily assigned numerical values. The other variable is then uniquely determined.

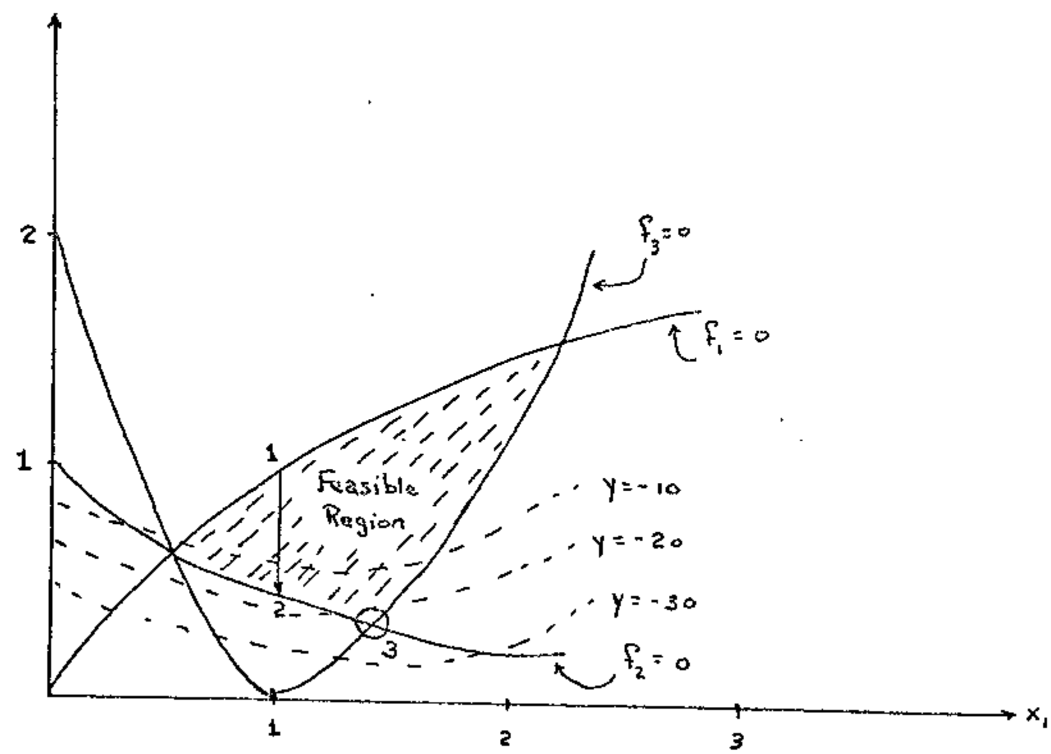

Fig. 4.8--Graphic representation of the differential algorithm

As shown in Figure 4.8, the initial solution is at 1. The second solution is at 2. The optimal solution is at 3. In moving from 1 to 2, the adjustment in x_2 was made to the point where $f_2 = 0$ was met. The third solution adjustment moved along $f_2 = 0$ until $f_3 = 0$ was contacted. This point was the optimal solution.

From this discussion of technique and the detailed illustrations, it is possible to derive some specific characteristics of indirect search. The main characteristics are the following:

(1) necessary conditions for the existence of an optimum solution are established in the form of equations;

(2) iterative techniques are generally required to solve the problem being investigated;

(3) while iterative techniques yield numerical results, the nature of the side conditions at the optimum solution reveals information not available via any other technique; and,

(4) the use of indirect search techniques requires a complete mathematical formulation of the function to be optimized.

Direct Search

Direct search has been described as an optimization technique which "depends upon direct comparison of the values of the function at two or more points."[22] Direct search is generally utilized when the objective function is either too complicated for indirect search techniques or is unknown. The techniques of direct search fall into two distinct categories: (1) elimination techniques and (2) direct climbing. Elimination techniques seek to reduce the size of the region in which the optimal solution lies by continually shrinking the interval of uncertainty. This interval can be given, or it can be assumed. Direct climbing techniques

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[22]Theodore N. Edelbaum, "Theory of Maxima and Minima," Optimization Techniques with Applications to Aerospace Systems, edited by George Leitmann (New York, 1962), p. 16.

attempt to drive directly toward the optimum solution by utilizing information that is generated along the way.

A characteristic of direct search techniques is that they identify nonoptimal solutions as well as optimal ones. The general procedure is to examine the neighborhood of a trial point for improvement. Any adjustment in the solution is made in the direction of the greatest improvement.

As in the case of indirect search, direct search techniques are generally applied under the assumption of unimodality. This assumption guarantees that a given function has but one peak (point of maximum or minimum). Gue and Thomas stress the fact that, when unimodality is assumed, it is important that the assumption specify whether the function is unimodal at a maximum or unimodal at a minimum.[23] It is permissable for a fucntion to be unimodal over the total range of the function or unimodal over a defined interval. In the first case, the function is parabolic; and the optimum is a global optimum. In the second case, the function may contain several points of local optima but only one global optimum. The importance of the unimodal assumption is explained by the following:

> If we are assured that a given function is unimodal, we can derive direct methods that guarantee convergence to [an optimal] point <u>x</u>* and establish that the point <u>x</u>* to which we converge yields the optimal solution. . . . If a function has more than one peak, then the point to which we converge will depend on the starting point. . . .[24]

The direct search techniques to be presented in this study will be restricted to a select group. The group of techniques will be presented

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[23]Gue and Thomas, <u>op. cit.</u>, pp. 101-102.

[24]<u>Ibid.</u>, p. 102.

under the two major categories of direct search, direct elimination and direct climbing. The individual techniques to be discussed are the following:

(1) direct elimination: interval elimination, sequential search, golden search, and contour tangents;

(2) direct climbing: response surfaces, gradient techniques, and parallel tangents.

As noted previously, the intent in this study is not to develop or provide a broad theoretical development. Rather, the intent is to explain, define, and demonstrate the technique being investigated. In this way, the intent centers on developing an intuitive feel for, and appreciation of, the particular technique involved.

Direct elimination.--The use of a particular technique for direct elimination depends upon the manner in which the function is written. By this is meant that the selection of a particular technique is directly related to the number of independent variables defined for a given problem. As such, the discussion to follow is divided into two distinct cases: univariable functions and multivariable functions. With the exception of the method of contour tangents, all of the techniques of direct elimination to be discussed in this study belong to the single variable case.

Univariable functions: interval elimination: The primary purpose of interval elimination is to minimize, after n evaluations, the maximum interval within which the optimum solution lies. The general procedure is to (1) determine the value of the independent variable for the next function evaluation, (2) compare the result of the new evaluation with the preceding one, (3) eliminate the interval which does not contain the optimum value, and (4) repeat steps (1) through (3) until the optimum is achieved. This procedure has been described as minimizing the interval of uncertainty.[25]

The interval of uncertainty is defined as that interval which lies between the end points of the interval within which an optimum solution lies. In final form, it is that interval which lies between the experiments on either side of the one producing the optimum value of the objective. This concept is formally defined in Definition 4.19.

Definition 4.19.--Let K equal the index of experiment x_k, where x_k produces the optimum outcome for the defined objective. Let n equal the number of experiments to be conducted. Let $x_o = 0$ and $x_{n+1} = 1$ be the left and right ends, respectively, of the original interval. For a set of n experiments, the interval of uncertainty is that interval containing the optimum value x* and for which

$$x_{k-1} \leq x^* \leq x_{k+1}$$

defines the interval. If ℓ_n denotes the length of the interval of uncertainty,

$$\ell_n = x_{k+1} - x_{k-1}.$$

The manner in which interval elimination is accomplished depends upon the plan by which the solution search is carried out. The general scheme is to determine in advance the number of experiments that will be allowed

[25] Douglas J. Wilde, Optimum-Seeking Methods (Englewood Cliffs, 1964), pp. 15-18.

in locating the optimum interval. Proceeding from x_o, a run of experiments is made to initiate the search process. The point used is labeled x_k and its outcome is labeled y_k. The interval within which to concentrate the next series of runs is that interval for which a given x_k produces an optimum outcome, y_k. For example, suppose three experiments are run and are labeled x_1, x_2, x_3; i.e., n = 3. Suppose, further, that the results correspond to those shown in Figure 4.9.

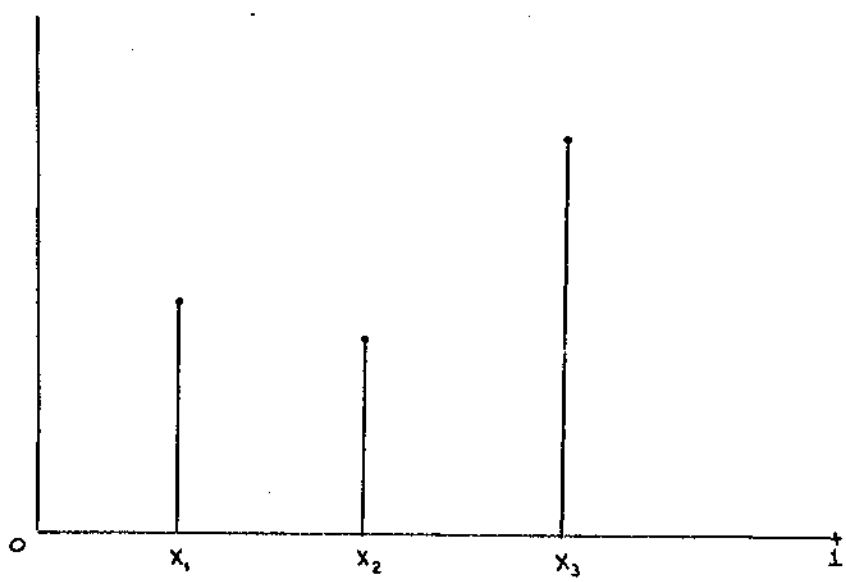

Fig. 4.9--Interval elimination with three experiments

If the objective is one of minimization and k = 2, then $x_1 \leq x^* \leq x_3$. The optimum value of x lies between the points x_1 and x_3. If the objective is one of maximization, and k = 3, then $x_2 \leq x^* \leq 1$. The optimum value of x lies between x_2 and 1. Given this information, the additional experiments can be run within the appropriate interval. Reapplication of the reduction process will further reduce the interval size.

The use of interval elimination requires that additional terminology be introduced. The appropriate terminology, from Wilde and Beightler, is presented here as a set of definitions.[26]

Definition 4.20.--Let $y(x)$ be any unimodal function of one variable defined over a fixed closed interval. Let x^* be the value of the independent variable for which $y(x)$ achieves the maximum value $y(x^*)$ in the closed unit interval $0 \leq x \leq 1$. Let x_1 and x_2 be any set of two points for which $y(x)$ has been evaluated. Let $y(x)$ have no intervals of finite length for which $y(x)$ is horizontal. Then, the function $y(x)$ is said to be strictly unimodal in the maximum sense if the following conditions are satisfied:

(1) $x_1 < x_2 < x^* \rightarrow y(x_1) < y(x_2) < y(x^*)$;

(2) $x^* < x_1 < x_2 \rightarrow y(x^*) > y(x_1) > y(x_2)$.

Definition 4.21.--Let $y(x)$ be any unimodal function of one variable defined over a fixed closed interval. Let x^* be any value of the independent variable for which $y(x)$ achieves the minimum value $y\ (x^*)$ in the closed unit interval $0 \leq x \leq 1$. Let x_1 and x_2 be any set of two points for which $y(x)$ has been evaluated. Let $y(x)$ have no intervals of finite length for which $y(x)$ is horizontal. Then, the function $y(x)$ is said to be strictly unimodal in the minimum sense if the following conditions are satisfied:

(1) $x_1 < x_2 < x^* \rightarrow y(x_1) > y(x_2) > y(x^*)$;

(2) $x^* < x_1 < x_2 \rightarrow y(x^*) < y(x_1) < y(x_2)$.

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[26]Wilde and Beightler, op. cit., pp. 219-222.

Neither of these two definitions requires that $y(x)$ be a continuous function. However, both of them guarantee that for any pair of evaluations of $y(x)$ the optimum value is contained in a shorter interval than before.

The objective of the interval elimination centers on reducing the size of the interval within which the optimal solution lies. In this manner, the process tends to converge on the point for which the function is optimal. These results can be written in an explicit form. This is done as a means of summarizing the process. Let L_n be the maximum length of the interval being investigated. Let ℓ_n be the length of the interval of uncertainty. Let x_k be the value of x at the K^{th} evaluation. Then

$$L_n(x_k) = \max_{1 \leq k \leq n} \{\ell_n(x_k, K)\}.$$

The purpose of the search is to minimize the maximum value of L_n. This is accomplished by performing a search plan x_k^* that yields the optimum value of L_n, L_n^*:

$$L_n^* = \min_{x_k} \max_{1 \leq k \leq n} \{\ell_n(x_k, K)\}.$$

If the function to be optimized is assumed to be differentiable (and the functional expression is known), the interval elimination process can be achieved by considering the first derivative. The necessary definitions are given below.

<u>Definition</u> <u>4.22</u>.--Let $y(x)$ be a defined, implicit function of one variable. Let $y(x)$ be continuous within a given interval. Let $y'(x)$ be the derivative of $y(x)$. Let x^* be the optimal value of the independent variable. Let x be any trial point taken within the interval for which $y(x)$ is defined. The function $y(x)$ is said to be differentiably unimodal

in the maximal sense if the following conditions are satisfied:

(1) if $x < x^*$, $y'(x) > 0$;

(2) if $x > x^*$, $y'(x) < 0$.

From these conditions can be made the following inferences:

(1) $y'(x) > 0$ implies $x < x^*$;

(2) $y'(x) < 0$ implies $x > x^*$.

Definition 4.23.--Let $y(x)$ be a defined, implicit function of one variable. Let $y(x)$ be continuous within a given interval. Let $y'(x)$ be the derivative of $y(x)$. Let x^* be the optimal value of the independent variable. Let x be any trial point taken within the interval for which $y(x)$ is defined. The function $y(x)$ is said to be differentiably unimodal in the minimal sense if the following conditions are satisfied:

(1) if $x < x^*$, $y'(x) < 0$;

(2) if $x > x^*$, $y'(x) > 0$.

From these conditions can be made the following inferences:

(1) $y'(x) < 0$ implies $x < x^*$;

(2) $y'(x) > 0$ implies $x > x^*$.

Application of Definition 4.22 results in three points, ℓ_k, r_k, and m_k. The point defined by m_k is the location of the best point from among the three. The point defined by ℓ_k is the one lying immediately to the left of m_k. The point defined by r_k is the one lying immediately to the right of m_k. For $y'(m_k) > 0$, $m_k < x^* < r_k$. For $y'(m_k) < 0$, $\ell_k < x^* < m_k$. The sign of the derivative thus indicates the direction of x^* in reference to m_k.

An application analysis similar to that of Definition 4.22 can be made for Definition 4.23. Using the points ℓ_k, r_k, and m_k as previously defined, $y'(m_k) < 0$ implies $m_k < x^* < r_k$. For $y'(m_k) > 0$, $\ell_k < x^* < m_k$.

Sequential search: A sequential search plan is one in which the results of previous experiments are used as a means of reducing the interval of uncertainty.[27] There are several techniques by which sequential search plans can be implemented: Bolzano's method, even-block search, odd-block search, uniblock (Fibonacci) search, adaptive search, etc. Of these techniques, this study will be limited to the Bolzano method and the technique of uniblock (Fibonacci) search.

(1) The Bolzano method. The Bolzano search technique is related to the Bolzano technique for locating the root (solution) of a monotonically decreasing function over a defined interval. Bolzano's root-finding technique requires that the function being investigated be evaluated in the center of the interval being considered. As the interval is reduced, the function is reevaluated at the midpoint of the remaining interval. If the evaluation of the function at the midpoint yields a negative value, the portion of the interval to the right of the midpoint is eliminated. If the evaluation of the function at the midpoint yields a positive value, the portion of the interval to the left of the midpoint is eliminated. For maximization, the procedure is reversed.

The Bolzano technique for sequential optimal search requires that the objective function be implicit and differentiable. Each experiment is then used to evaluate both the defined objective function and the first

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[27] Ibid., p. 230.

derivative of the objective function. Since the peak of a differentiably unimodal function is the root of the monotonic function, the Bolzano technique for sequential search is a modification of the root-finding method. The difference between the two is that the procedure for sequential optimal search utilizes the first derivative. This is accomplished by using the derivative in the same manner as the function in the root-finding method is used. The interest, however, is the solution to the derivative function. When the derivative equals zero, the optimal solution has been located.

The Bolzano technique for optimal search can be used to determine the number of experiments required to produce a given interval reduction. For example, Wilde and Beightler point out that for a 1 per cent reduction in the original length of a given interval the Bolzano technique requires but seven observations. With a technique similar to that of interval elimination, 99 observations are required.[28]

The criterion under differentiability will be altered from that of the regular Bolzano root-finding technique. Let x^1 be the midpoint of the interval $x_1 \leq x \leq x_2$ over which the function $y(x)$ is continuous and differentiable. The superscript denotes the number of the sequential observation. Let $y'(x)$ be the derivative of $y(x)$. If $y(x)$ is a minimizing function, $y'(x = x^1) > 0$ indicates that the portion of the interval to the right of x^1 can be eliminated from consideration. If $y'(x = x^1) < 0$, the portion of the interval to the left of x^1 can be eliminated from consideration. The second trial point, x^2, will be the midpoint of the

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[28] *Ibid.*

interval $x_1 \leq x \leq x^1$ or $x^1 \leq x \leq x_2$ depending upon the sign of the first derivative. If y(x) is a maximizing function, $y'(x = x^1) > 0$ indicates that the portion of the interval to the left of x^1 can be eliminated from consideration. If $y'(x = x^1) < 0$, the portion of the interval to the right of x^1 can be eliminated from consideration.

Consider the quadratic profit function $y(x) = 1000 + 7.2x - .06x^2$. Assume that y(x) is defined over the productivity range $0 \leq x \leq 100$; i.e., the profit function is defined for a maximum output of 100 units. Determine the point of maximum profit using the Bolzano technique. The function is assumed to be differentiable.

The Bolzano technique requires the derivative, y'(x). For the given function, $y'(x) = 7.2 - .12x$. The demonstration is broken down into a series of iterations.

Step 1. Let $x^1 = \frac{1}{2}(x_1 + x_2) = \frac{1}{2}(0 + 100) = 50$. Evaluation of y'(x) at x = 50 yields a value of 1.2 for $y'(x = x^1)$. Since $y'(x^1) > 0$ and y(x) is to be maximized, the second trial point will be the midpoint of the interval lying to the right of x^1; i.e., $x^2 = \frac{1}{2}(x^1 + x_2)$.

Step 2. Let $x^2 = \frac{1}{2}(x^1 + x_2) = \frac{1}{2}(50 + 100) = 75$. Evaluation of y'(x) at x = 75 yields a value of -1.8 for $y'(x = x^2)$. Since $y'(x^2) < 0$ and y(x) is to be maximized, the third trial point will be the midpoint of the interval lying to the left of x^2; i.e., $x^3 = \frac{1}{2}(x^1 + x^2)$.

Step 3. Let $x^3 = \frac{1}{2}(x^1 + x^2) = \frac{1}{2}(50 + 75) = 62.5$. Evaluation of y'(x) at x = 62.5 yields a value of -0.3 for $y'(x = x^3)$. Since $y'(x^3) < 0$ and y(x) is to be maximized, the fourth trial point will be the midpoint of the interval lying to the left of x^3; i.e., $x^4 = \frac{1}{2}(x^1 + x^3)$.

Step 4. Let $x^4 = \frac{1}{2}(x^1 + x^3) = \frac{1}{2}(50 + 62.5) = 56.25$. Evaluation of $y'(x)$ at $x = 56.25$ yields a value of +0.45 for $y'(x = x^4)$. Since $y'(x^4) > 0$ and $y(x)$ is to be maximized, the fifth trial point will be the midpoint of the interval lying to the right of x^4; i.e., $x^5 = \frac{1}{2}(x^4 + x^3)$.

Step 5. Let $x^5 = \frac{1}{2}(x^4 + x^3) = \frac{1}{2}(56.25 + 62.50) = 59.375$. Evaluation of $y'(x)$ at $x = 59.375$ yields a value of + 0.075 for $y'(x = x^5)$. Since $y'(x^5) > 0$ and $y(x)$ is to be maximized, the sixth trial point will be the midpoint of the interval lying to the right of x^5; i.e., $x^6 = \frac{1}{2}(x^5 + x^3)$.

Step 6. Let $x^6 = \frac{1}{2}(x^5 + x^3) = \frac{1}{2}(59.375 + 62.500) = 60.9375$. Evaluation of $y'(x)$ at $x = 60.9375$ yields a value of -0.1125 for $y'(x = x^6)$. Since $y'(x^6) < 0$ and $y(x)$ is to be maximized, the seventh trial point will be the midpoint of the interval lying to the left of x^6; i.e., $x^7 = \frac{1}{2}(x^5 + x^6)$.

Step 7. Let $x^7 = \frac{1}{2}(x^5 + x^6) = \frac{1}{2}(59.375 + 60.9375) = 60.15625$. Evaluation of $y'(x)$ at $x = 60.15625$ yields a value of -0.01875 for $y'(x = x^7)$. Since $y'(x^7) < 0$ and $y(x)$ is to be maximized, the eighth trial point will be the midpoint of the interval lying to the left of x^7; i.e., $x^8 = \frac{1}{2}(x^5 + x^7)$.

Step 8. Let $x^8 = \frac{1}{2}(x^5 + x^7) = \frac{1}{2}(59.375 + 60.15625) = 59.260625$. Evaluation of $y'(x)$ at $x = 59.260625$ yields a value of +0.088725 for $y'(x = x^8)$. Since $y'(x^8) > 0$ and $y(x)$ is to be maximized, the ninth trial point will be the midpoint of the interval lying to the right of x^8; i.e., $x^9 = \frac{1}{2}(x^8 + x^7)$.

Step 9. Let $x^9 = \frac{1}{2}(x^8 + x^7) = \frac{1}{2}(59.260625 + 60.15625) = 59.7084375$. Evaluation of $y'(x)$ at $x = 59.7084375$ yields a value of +0.0349875 for $y'(x = x^9)$. Since $y'(x^9) > 0$ and $y(x)$ is to be maximized, the tenth trial point will be the midpoint of the interval lying to the right of x^9; i.e., $x^{10} = \frac{1}{2}(x^9 + x^7)$.

Step 10. Let $x^{10} = \frac{1}{2}(x^9 + x^7) = \frac{1}{2}(60.15625 + 59.7084375) = 59.83234475$. Evaluation of y'(x) at x = 59.83234475 yields a value of +0.02011863 for $y'(x = x^{10})$. Since $y'(x^{10}) > 0$ and y(x) is to be maximized, the eleventh trial point will be the midpoint of the interval lying to the right of x^{10}; i.e., $x^{11} = \frac{1}{2}(x^{10} + x^9)$.

Step 11. Let $x^{11} = \frac{1}{2}(x^{10} + x^9) = \frac{1}{2}(59.83234475 + 60.15625) = 59.994297375$. Evaluation of y'(x) at x = 59.994297375 yields a value of +0.000784315 for $y'(x = x^{11})$. Since $y'(x^{11}) > 0$ and y(x) is to be maximized, the twelfth trial point will be the midpoint of the interval lying to the right of x^{11}; i.e., $x^{12} = \frac{1}{2}(x^{11} + x^9)$.

Step 12. Let $x^{12} = \frac{1}{2}(x^{11} + x^9) = \frac{1}{2}(59.994297375 + 60.15625) = 60.0752736875$. Evaluation of y'(x) at x = 60.0752736875 yields a value of -0.0090328425 for $y'(x = x^{12})$. Since $y'(x^{12}) < 0$ and y(x) is to be maximized, the thirteenth trial point will be the midpoint of the interval lying to the left of x^{12}; i.e., $x^{13} = \frac{1}{2}(x^{11} + x^{12})$.

Inspection of the convergence pattern indicates that the solution point is x = 60. Using x = 60 as the trial point, y'(x) evaluated at x = 60 yields a value of zero for y'(x = 60). This indicates that the line tangent to y(x) is horizontal at x = 60, the required condition for an optimal solution. Therefore, the profit function is maximized at a sales level of 60 units.

This example demonstrates the manner in which the Bolzano technique converges to a finite solution in a series of sequential iterations.

Although tedious and lengthy when hand calculations are required, the technique is well suited for computer use. The iterations can be reduced by noting the indicated point of convergence and adjusting the search interval accordingly. For example, an adjustment could have been made at Step 7 of the illustration to zero the solution on x = 60.

(2) Uniblock (Fibonacci) search. The most efficient technique of block search is that of uniblock search. As implied by the term uniblock, the technique involves the selection of a single block and observing one measurement per block.

Uniblock search is based upon a defined relationship which utilizes the concept of Fibonacci numbers.[29] This relationship defines the n^{th} Fibonacci number in terms of the recursive relation

$$F_n = F_{n-1} + F_{n-2}, \quad n = 2, 3, \ldots$$

where $F_o = F_1 = 1$ and F_k denotes the k^{th} Fibonacci number. This relationship can be used to develop a series of numbers defining the n^{th} Fibonacci number.

$$F_2 = F_1 + F_o = 1 + 1 \rightarrow F_2 = 2;$$
$$F_3 = F_2 + F_1 = 2 + 1 \rightarrow F_3 = 3;$$
$$F_4 = F_3 + F_2 = 3 + 2 \rightarrow F_4 = 5;$$
$$F_5 = F_4 + F_3 = 5 + 3 \rightarrow F_5 = 8;$$
$$F_6 = F_5 + F_4 = 8 + 5 \rightarrow F_6 = 13;$$
$$F_7 = F_6 + F_5 = 13 + 8 \rightarrow F_7 = 21;$$

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[29]Wilde, op. cit., pp. 24-30.

$$F_8 = F_7 + F_6 = 21 + 13 \rightarrow F_8 = 34;$$
$$F_9 = F_8 + F_7 = 34 + 21 \rightarrow F_9 = 55;$$
$$\vdots$$

Utilization of uniblock search requires that the number of experiments be known in advance. This number can either be established subjectively or by application of a formula. The use of the formula requires that the desired interval of uncertainty (I.U.) be known. This formula is given by

$$\text{I.U.} = \frac{1}{F_n} + \varepsilon\left(\frac{F_{n-2}}{F_n}\right),$$

where ε equals the minimum separation between two points, and I.U. identifies the proportional length of the n^{th} interval of uncertainty. The approximate number of observations required can be obtained by requiring ε to equal zero for the n^{th} interval. With this assumption, the number of experiments required is that value of n such that I.U. is at most equal to $\frac{1}{F_n}$. The proportion associated with I.U. is set as the percentage of the original interval that is to remain after the completion of the n sequential experiments.[30] It is generally assumed that the original interval is written as having a length equal to unity.

Given a starting point for a Fibonacci search, each succeeding stage is determined. Within each remaining interval of uncertainty will be a previous experiment. The search for a solution point is extended from stage k to stage (k + 1) by locating experiment (k + 1) in such a way that it is symmetric to the experiment already located within the given interval.

According to Wilde, the first Fibonacci experiment must be placed L_2^* units from one end of the original unit interval of uncertainty.[31] Because of the symmetry of the technique, the end at which the process starts is immaterial. Gue and Thomas have shown that with n experiments the first should be placed at $x = F_{n-1}/F_n$.[32] This is borne out when consideration is given to the fact that Wilde initiates the first experiment at

$$L_2^* = \frac{F_{n-1}}{F_n} + \frac{(-1)^n \varepsilon}{F_n} \text{ .}^{33}$$

However, since ε is forced to zero, this reduces L_2^* to

$$L_2^* = \frac{F_{n-1}}{F_n}.$$

Given this placement for the first test, the second test is placed x (or L_2^*) units from the unused end. The results of these two tests are then compared and the unpromising portion of the original interval is eliminated. This leaves an interval containing one experiment with a known result. The third test is then placed symmetrically to the remaining experiment within the remaining interval, and the elimination process is repeated. This process is then repeated for the n required experiments.[34]

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[31] Ibid., p. 30.

[32] Gue and Thomas, op. cit., p. 109.

[33] Wilde, op. cit., p. 30.

[34] As a matter of note, the location of L_2^* can be written in terms of the units of the interval length. This is accomplished by multiplying the ratio $\frac{F_{n-1}}{F_n}$ by the length of the original interval L_o: $L_2^* = L_o(\frac{F_{n-1}}{F_n})$. As initially written, L_2^* defines the percentage distance from either end

For example, if n = 11, F_{11} = 144 and F_{10} = 89. The first observation will be placed at

$$x = \frac{89}{144} = .62 = 62\%$$

of the total length of the interval from either end. That is, for an initial interval of length 10, the first observation will be placed 6.2 units from either end. The second point will be placed at a distance of 62% of the length of the remaining interval from either end. This process is repeated until the solution point is reached.

Golden section search: In many instances, it may not be possible for a researcher to know in advance the number of observations required for locating an optimal solution. In such cases, the technique of Fibonacci search is not applicable since its use requires that the number of observations be predetermined. In these situations, there is a need for a search plan that converges rapidly to the optimum solution. The method of golden search provides such a search plan as it is applicable when the number of experiments to be run is not known. Since the number of experiments to be run is independent of the number of experiments available, golden search represents an improvement over that of Fibonacci search.

Let j denote the number of experiments already run. Let n denote the number of available experiments. Let I_k^n denote the n^{th} interval of k experiments (i.e., I_1^2 denotes the second interval containing one experiment). Wilde and Beightler indicate that the experimental plan should locate successive experiments in such a way that

$$I_1^{n-j} = I_1^{n-(j-1)} + I_1^{n-(j-2)}.$$ [35]

[35]Wilde and Beightler, op. cit., p. 243.

By holding the ratio of successive lengths, τ, constant,

$$\frac{I_1^{n-j}}{I_1^{n-(j-1)}} = \tau = \frac{I_1^{n-(j+1)}}{I_1^{n-j}} .$$

Utilizing

$$\tau^2 = \frac{I_1^{n-(j+1)}}{I_1^{n-(j-1)}}$$

and dividing τ by $I_1^{n-(j-2)}$, τ^2 can be written in the form $\tau^2 = \tau + 1$. The positive root to $\tau^2 = \tau + 1$ is given by

$$\tau = \frac{1+\sqrt{5}}{2} = 1.618033989. . . .$$

After n experiments, the remaining I_1^n is given by

$$I_1^n = \frac{1}{\tau^{n-1}}.$$

The method of golden search utilizes the values of both τ and τ^2 to determine which segment of the given interval is to receive further consideration. The interval that is selected for further exploration will contain a result from a previous experiment. The search for the optimum solution is continued by simply placing the next set of experiments symmetrically within the interval to be explored. The process can be repeated as often as deemed necessary.

The placement of the first test is based upon the ratio that exists between two unequal parts of a line segment. This ratio is defined as

$$\frac{\text{length of smaller part}}{\text{length of larger part}} = \frac{\text{length of the larger part}}{\text{length of original line segment}}.$$

The length of the original line segment is defined by the length of the interval being considered. The manner in which the line segment is divided is arbitrary. The distance the first test is placed from either end is given by $1/\tau$. In terms of the original units, this distance is $L_o(1/\tau)$, where L_o equals the length of the original interval.

Multivariable functions: The multivariable search problem is to find, after a limited number of experiments, a set of operating conditions that optimizes a defined objective function. Of the direct elimination techniques for multivariable functions, the most feasible technique is that of contour tangent elimination.[36]

Definition 4.24.--Let $y(x_1, x_2, \ldots, x_n) = y(\underline{x})$ be any defined multivariable function. Let $\underline{x}^o$ be any set of points for which $y(\underline{x})$ is defined. The contour of $y(\underline{x})$ is defined as the collection of values of $\underline{x}$ which yield a fixed value of y.

For example, a cost curve can be defined in terms of two independent variables, x_1 and x_2: $y(\underline{x}) = y(x_1, x_2)$. With $y(\underline{x})$ fixed in value, a

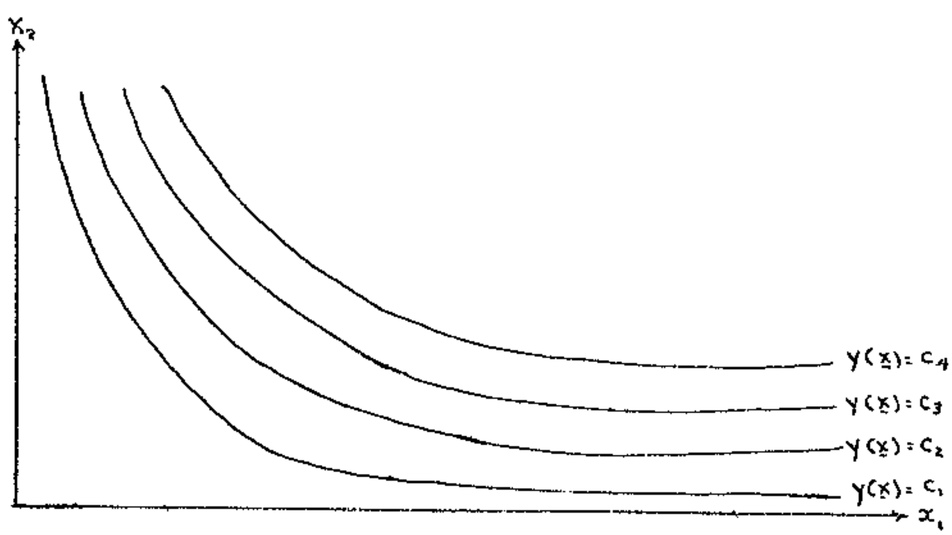

Fig. 4.10--Contours for $y(x_1, x_2) = c_k$, $k = 1, 2, 3, 4$

bivariate graph can be constructed similar to that of Figure 4.10. The collection of curves is defined as the contour of $y(\underline{x})$.

Definition 4.25.--Let $y(\underline{x}) = y(x_1, x_2, \ldots, x_n)$ be any defined multivariable function. Let $\underline{x}^o$ be any point on the contour of $y(\underline{x})$. The contour tangent to $y(\underline{x})$ is defined as the line or N - 1 dimensional hyperplane passing through $\underline{x}^o$ tangent to $y(\underline{x})$. The contour tangent at $\underline{x}^o$ is such that $y(\underline{x}) = y(\underline{x}^o)$.

Application of the contour tangent elimination technique requires that the given function be differentiable in a neighborhood of a point $\underline{x}^o$. The point $\underline{x}^o$ defines a vector of feasible values of the independent variables. The requirement of differentiability is necessary because use will be made of the gradient vector of the defined objective function.

Definition 4.26.--Let $y(x_1, x_2, \ldots, x_n) = y(\underline{x})$ be any differentiable objective function. Let $\underline{x}^o$ be any vector of feasible values of the independent variables. Let $\frac{\partial y}{\partial x_i}$ be the partial derivative of $y(\underline{x})$ with respect to the i^{th} variable. The gradient vector of $y(\underline{x})$, denoted $\underline{\nabla y}$, is defined as the vector of first partial derivatives; i.e.,

$$\underline{\nabla y} = \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_n}\right).$$

The respective partial derivatives are evaluated at $\underline{x}^o$.

The purpose of the gradient vector $\underline{\nabla y}$ is to provide a means of determining changes in the value of the objective function corresponding to small changes in the values of the solution vector. The vector of changes in $\underline{x}^o$, denoted by $\underline{\Delta x}$, is given by

$$\underline{\Delta x} = (\Delta x_1, \Delta x_2, \ldots, \Delta x_n) = (x_1 - x_1^o, x_2 - x_2^o, \ldots, x_n - x_n^o).$$

This is accomplished by defining the change in $y(\underline{x})$ by the linear relation

$$\underline{\Delta y} = \underline{\nabla y \Delta x}^T = 0.$$

The components of $\underline{\nabla}y$ are known constants, having been evaluated at $\underline{x}^o$. The linear relation defining $\underline{\Delta}y$ has dimension N - 1 and passes through $\underline{x}^o$.

Another requirement of the contour tangent elimination technique is that the objective function, in addition to being differentiable, be strongly unimodal. This concept refers to the direction in which a line can be drawn from any point in the experimental region to the peak.

Definition 4.27.--Let $y(x_1, x_2, \ldots, x_n) = y(\underline{x})$ be any defined objective function. Let $\underline{x}^o$ be any feasible trial solution vector. The function $y(\underline{x})$ is said to be strongly unimodal if a straight line drawn from $\underline{x}^o$ to the peak is rising if $y(\underline{x})$ is to be maximized and falling if $y(\underline{x})$ is to be minimized. A straight line is rising (falling) if it has positive (negative) slope.

When a given function is strongly unimodal, two major results are realized: (1) if $y(\underline{x})$ is to be maximized, the region of exploration lies above the contour tangent; (2) if $y(\underline{x})$ is to be minimized, the region of exploration lies below the contour tangent. In this way, substantial portions of the original feasible region can be systematically eliminated from consideration.

The method of contour tangents thus refers to an elimination technique that utilizes successive local explorations of strongly unimodal surfaces to eliminate parts of the feasible region from further consideration. This technique uses the contour tangent as a lower bound for the optimal solution of a maximizing function and as an upper bound for the optimal solution of a minimizing function. The following algorithm, developed from a study of theory and examples, summarizes the computational aspects of the technique.

Algorithm 4.5 (algorithm for contour tangent elimination).--Step 1. Let $\underline{x}^o$ be an initial solution vector contained in the feasible region for a defined objective function $y(\underline{x})$. Determine the contour tangent at $\underline{x}^o$, $\underline{x}^o$ being interior to the feasible region. Using this contour tangent, eliminate the appropriate portion of the feasible region.

Step 2. Let $\underline{x}^1$ be an initial solution vector contained in the reduced feasible region of $y(\underline{x})$. Determine the contour tangent at $\underline{x}^1$, $\underline{x}^1$ being interior to the reduced region of feasibility. Using this second contour tangent, eliminate the appropriate portion of the reduced feasible region.

Step 3. Repeat the process of Steps 1 - 2 through k iterations. At the k^{th} iteration, the remaining region of feasibility will be of whatever size is desired since the number of experiments is treated as an independent value.

At each stage of the contour tangent elimination process, it becomes necessary to select a new feasible solution. This "new" feasible solution is then used to determine the contour tangent passing through the solution point. Although there is no best way to select the feasible solution at each stage, four possibilities are worthy of consideration: the midpoint, the minimax point, the median (center of volume), and the mean (centroid). Each of these possibilities represents an attempt to locate the feasible solution vector in the center of the remaining feasible region.

The validity of these four measures as locators of solution vectors is discussed by Wilde.

> Whenever the experimental region is convex, that is, such that any straight line connecting two boundary points will lie entirely within the region, any of the four points [midpoint, minimax point, median, or mean] will be inside the region. Since any region bounded entirely by hyperplanes is convex, the region of uncertainty will tend to be convex after a few blocks of experiments have been performed. . .even when the region is not convex. . .the four points more often than not will be in the interior of the region.[37]

In the absence of any more suitable technique for selecting an initial solution point, one of these four will suffice.[38]

Definition 4.28.--Let s_i be the minimum value of the variable x_i in the region of uncertainty, i = 1, 2,...,N. Let t_i be the maximum value of the variable x_i in the region of uncertainty, i = 1, 2,...,N. Let $x_i^{\cdot}$ be the midpoint for the i^{th} region of uncertainty, i = 1, 2,...,N. Then,

$$x_i^{\cdot} = \frac{1}{2}(s_i + t_i) \text{ for all } i,\ i = 1, 2,\ldots,N,$$

where $s_i = \min x_i$ and $t_i = \max x_i$.

Definition 4.29.--Let $\underline{x}^m$ be the minimax point for a defined region of uncertainty. Let $\underline{x}$ be any point contained within a suitable feasible region. Let $v(\underline{x})$ be the hypervolume of a defined region of uncertainty. The minimax point $\underline{x}^m$ is defined as that point which minimizes the maximum possible hypervolume in an explored region of uncertainty; i.e.,

$$v(\underline{x}^m) = \min_{\underline{x}}\{v(\underline{x})\}.$$

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[37] Wilde, op. cit., p. 98.

[38] Ibid., pp. 99-104.

Definition 4.30.--Let $A(\underline{x})$ be the cross-sectional area of a given experimental region. Let x_i be constant within this region. Let $\underline{x}'$ be any point in the experimental region of distance $\hat{d}_i(\underline{x}')$ from x_i, where

$$\hat{d}_i(\underline{x}') \equiv \frac{\int_a^b |x_i - x_i'| A_i(\underline{x}) dx_i}{\int_a^b A_i(\underline{x}) dx_i}$$

and a and b are the limits of integration. The median is defined as that set of values of x_i' for which

$$\int_a^{x_i} A_i(x_i) dx_i \equiv \int_{\hat{x}_i}^b A_i(x_i) dx_i$$

and which minimizes the mean distances $\hat{d}_i(\underline{x}')$.

Definition 4.31.--Let $r_i(\underline{x})$ be the root mean square distance defined by

$$r_i(\underline{x}) = \frac{\int_a^b (x_i - x_i')^2 A_i(x_i) dx_i}{\int_a^b A_i(x_i) dx_i},$$

where x_i' equals the i^{th} value in the point $\underline{x}'$ and $A_i(x_i)$ equals the cross-sectional area around x_i. Let $\overline{x}_i$ be the centroid (mean) of the experimental region with radius $r_i(\overline{x})$. Then,

$$\overline{x}_i = \frac{\int_a^b x_i A_i(x_i) dx_i}{\int_a^b A_i(x_i) dx_i}.$$

The values of a and b define the endpoints of the interval (limits of integration) associated with x_i.

The manner in which the chosen mid-region point is utilized depends upon its relationship to the previous trial point. Given that a mid-region point has been selected within a defined feasible region, there are three possibilities: (1) the value of the objective function at the new mid-region point exceeds the value of the objective function at the old point of exploration; (2) the value of the objective function at the new mid-region point equals the value of the objective function at the old point of exploration; or, (3) the value of the objective function at the new mid-region point is less than the value of the objective function at the old point of exploration. If the new mid-region point, $\underline{x}'$, is an improvement over the original trial point, $\underline{x}$, a new contour tangent can be determined and additional experiments performed. If the new mid-region point $\underline{x}'$ is not an improvement over the original trial point $\underline{x}$, there are two alternatives available: (1) a new contour tangent can be determined, the intent of which is the further reduction of the region of uncertainty, or (2) a new contour tangent can be determined in such a way that the contour tangent passes through the point $\underline{x}'$ and the original trial point $\underline{x}$.

From a practical point of view, so long as $\underline{x}'$ defines an improvement over $\underline{x}$, it is feasible to continue the exploration of a given region of uncertainty. When $\underline{x}'$ fails to improve over $\underline{x}$, two major considerations must be evaluated.

> (1) Optimization explorations are rarely conducted for their own sakes--there is usually a manager involved who has requested the study and, although knowing little about search technique, is vitally interested in the results of each experiment. A good way to lose such a man's confidence is to carry out a set of experiments which, giving a lower value [of the objective function] than at the original location, would from his viewpoint be considered "failures."

(2) [Optimization] studies often are made on plants actually in production where decreases in [the value of the objective function] represent actual financial loss. Although a company could tolerate a loss during one or two isolated experiments for the sake of research, it would be expecting too much to ask it to sustain continued losses [for a series of such experiments].[39]

As a means of illustrating the method of contour tangents, consider the problem of maximizing $y = 3 + 6x - 4x^2$ over the closed interval $0 \leq x \leq 1$. This function can be identified with a profit function where the optimal solution is required to fall in the closed interval $0 \leq x \leq N$. This interval can be scaled to the required closed interval by dividing through by N. The solution will then be identified as x/N. The quadratic example is used so that the result of each iteration can be graphically demonstrated. The procedure outlined by the algorithm for contour tangents is demonstrated in detail.

Step 1. See Figure 4.11(a). Let $\underline{x}_o'$ be the initial trial point, arbitrarily selected as being equal to 1/2. For $y = 3 + 6x - 4x^2$, the

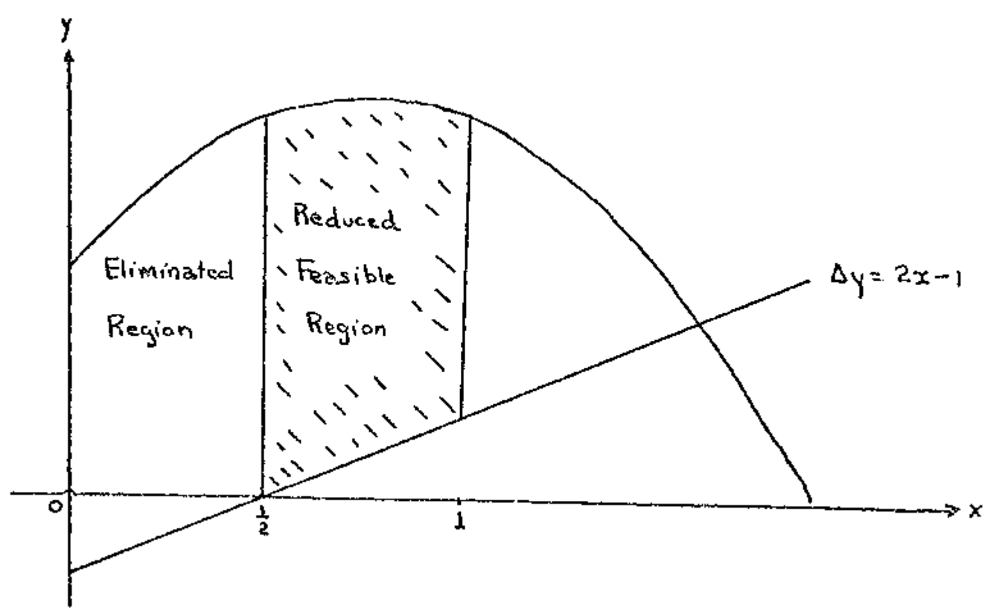

Fig. 4.11(a)--First contour tangent to $y = 3 + 6x - 4x^2$

gradient vector ∇y is given by $\nabla y = 6 - 8x$. At $\underset{\sim}{x}_0' = x = 1/2$, $\nabla y = 2$. The contour tangent defined by $\Delta y = \nabla y \Delta x^T$ is given by the linear relation $\Delta y = 2(x - 1/2) = 2x - 1$. As shown in Figure 4.11(a), the reduced feasible region is bounded by the lines $x = 1/2$, $x = 1$, and $\Delta y = 2x - 1$. Any value of x contained within this region is a potential "new" solution.

Step 2. See Figure 4.11(b). Let $\underset{\sim}{x}_1'$ be the second trial point, arbitrarily selected as being equal to 2/3. At $\underset{\sim}{x}_1' = 2/3$, the gradient

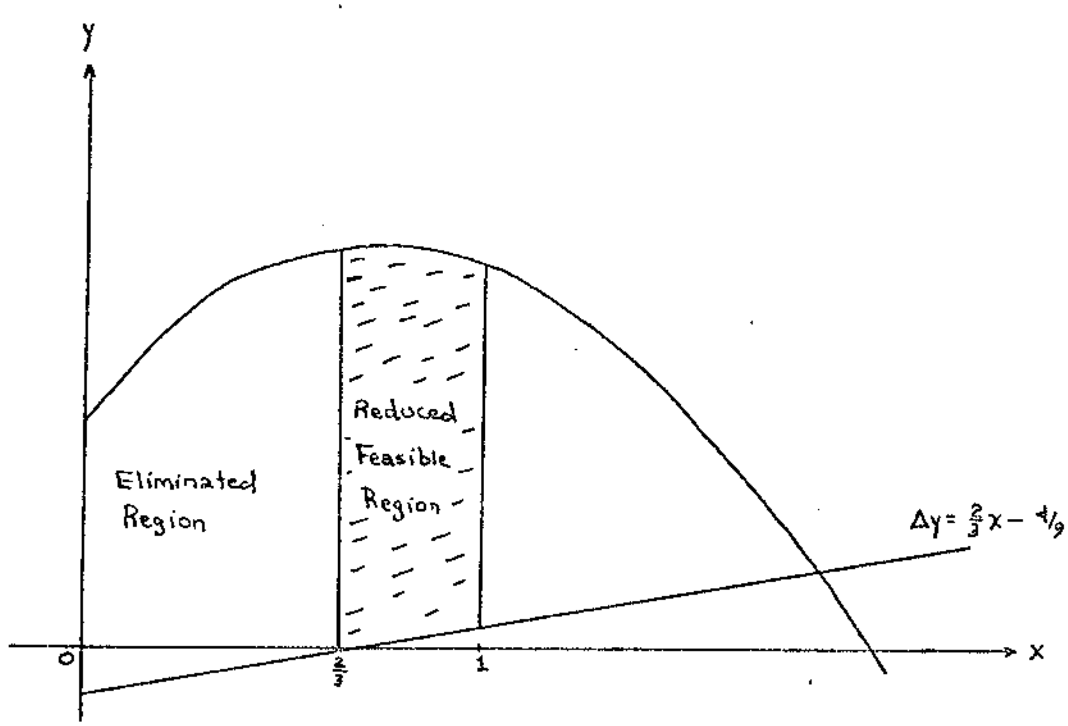

Fig. 4.11(b)--Second contour tangent to $y = 3 + 6x - 4x^2$

vector ∇y equals 2/3. The contour tangent is given by the linear relation $\Delta y = \frac{2}{3}(x - \frac{2}{3}) = \frac{2}{3}x - \frac{4}{9}$. As shown in Figure 4.11(b), the reduced feasible region is bounded by lines $x = \frac{2}{3}$, $x = 1$, and $\Delta y = \frac{2}{3}x - \frac{4}{9}$. Any value of x contained within this region is a potential "new" solution.

Step 3. See Figure 4.11(c). Let $\underline{x}_2'$ be the third trial point, arbitrarily selected as being equal to 3/4. At $\underline{x}_2' = 3/4$ the gradient vector ∇y equals 0. The contour tangent is given by the linear relation $\Delta y = 0(x - 3/4) = 0$.

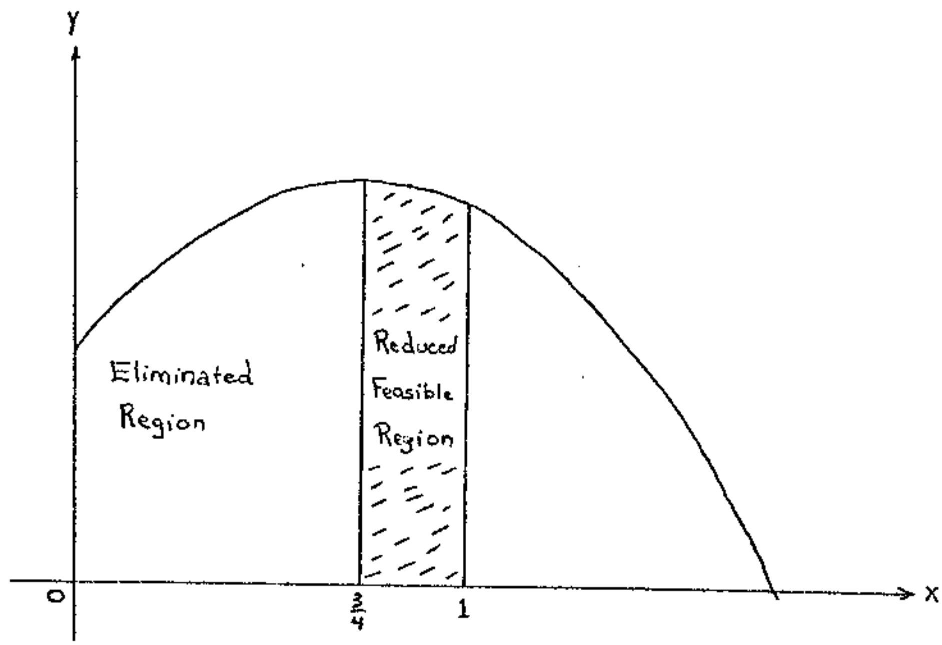

Fig. 4.11(c)--Third contour tangent to $y = 3 + 6x - 4x^2$

As shown in Figure 4.11(c), the reduced feasible region is bounded by the lines $x = 3/4$, $x = 1$, and $\Delta y = 0$. Any value of x contained within this region is a potential "new" solution.

Step 4. See Figure 4.11(d). Let $\underline{x}_3'$ be the fourth trial point, arbitrarily selected as being equal to $\frac{3}{4} + \varepsilon$, where ε is any small positive number. At $x_3' = \frac{3}{4} + \varepsilon$, the gradient vector equals $-\varepsilon$. The contour tangent is given by the linear relation $\Delta y = -\varepsilon(x - \frac{3}{4} - \varepsilon) = -\varepsilon x + \varepsilon(\frac{3}{4} + \varepsilon)$. As shown in Figure 4.11(d), the reduced feasible region is bounded by the lines $x = \frac{3}{4} + \varepsilon$, $x = 1$, and $\Delta y = -\varepsilon x + \varepsilon(\frac{3}{4} + \varepsilon)$. However, the slope of

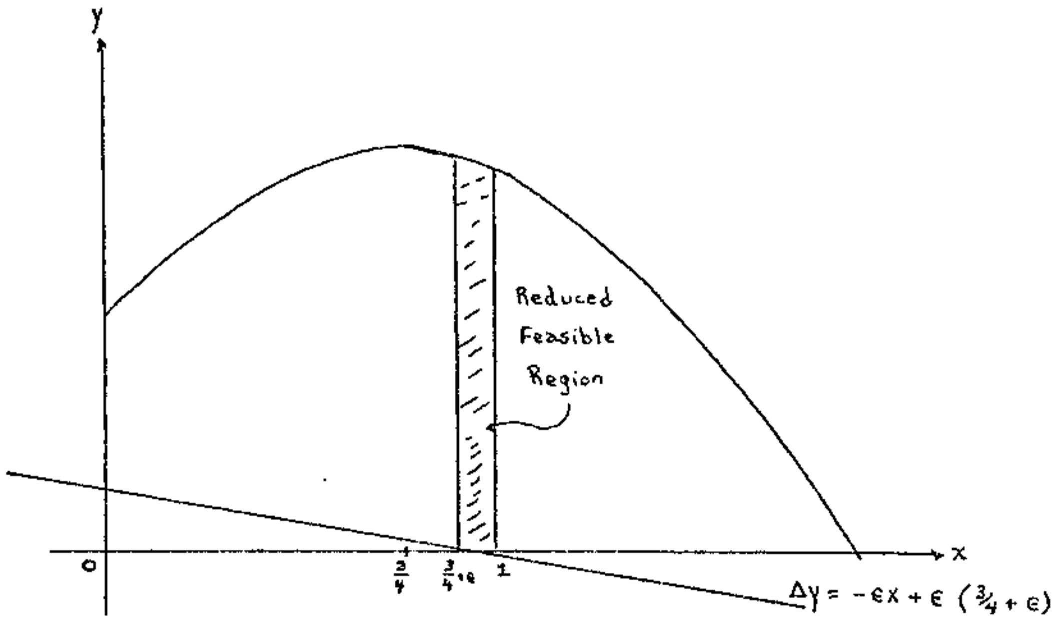

Fig. 4.11(d)--Fourth contour tangent to $y = 3 + 6x - 4x^2$

the contour tangent has changed from positive to zero to negative as x took on values on either side of x = 3/4. Since the criterion for a maximum is that the slope of the tangent line change from positive to negative within an ε-neighborhood of the optimal value $\underline{x}^*$, the optimal value of $\underline{x}$ is $\underline{x}^* = 3/4$. This result can be readily verified by the classical max-min calculus.

From this example, it is possible to infer the criterion by which the optimal solution is identified when the contour tangent technique is used. This criterion can be attached as Step 4 in the general procedure previously outlined and is as follows:

(1) if the function under consideration is to be maximized, the optimal solution is that set of values of the indepenent variables for which the gradient vector equals zero and for which the slope of the contour

tangent changes from positive to negative within an ε-neighborhood of the optimal point;

(2) if the function under consideration is to be minimized, the optimal solution is that set of values of the independent variables for which the gradient vector equals zero and for which the slope of the contour tangent changes from negative to positive within an ε-neighborhood of the optimal point.

The use of the contour tangent elimination technique is particularly well suited to those problems for which defined functional expressions do not exist. In such cases the limits within which the optimal solution must fall are either known in advance or assumed. For example, a manager is interested in allocating a fixed budget among four departments. The budget function is not known, nor are the functional expressions relating the budget expenditures. It is known, however, that both the budget allocation function and the restrictive relations may be nonlinear. In order to guarantee a minimum cost allocation, the budget function is assumed to be strictly unimodal. In solving this problem by the method of contour tangent elimination, the function would be approximated by a tangent plane. The procedure previously outlined would then be followed. Each trial solution would be located within the region of feasibility, with adjustments made as needed to improve the solution set. Since the function is to be minimized, $\underline{\nabla y} < 0$ will indicate that the values in the solution can be increased; $\underline{\nabla y} > 0$ will indicate that a decrease is necessary.

<u>Direct climbing</u>.--The term "direct climbing" is used as a means of describing any direct search technique that uses past information to locate

better solution points. In this manner the technique climbs from one point to the next when the function under investigation is being maximized and descends from one point to the next when the function under investigation is being minimized.

The use of climbing techniques in direct search problems serves two basic purposes:

(1) it provides a means of locating improved solutions which increase or decrease the defined objective as needed;

(2) it provides information which can be used to determine the location of future trial solutions.

In application, the search process is one of determining whether or not to continue the climbing process or to further explore the region containing the current solution.

Although the use of climbing techniques (either ascent or descent) can be used for analyzing functions of one variable, the most common application is the solving of multivariable functions. This is attributed to the fact that univariable functions can be reasonably approximated by the technique of least squares and the resulting functions analyzed by some other technique. As a result, the discussion of direct climbing techniques will be restricted to multivariable functions and will include the following topics: response surfaces, gradient techniques, and the technique of parallel tangents. Of the available multivariable search techniques in the literature, these are the most promising. From the viewpoint of practical application, these techniques are the most feasible.

The use of direct search techniques for solving multivariable functions suggests that a common terminology be established for the multivariable search problem. For the discussion to follow, the multivariable search problem will be that of Definition 4.32.

Definition 4.32.--Let y be a defined criterion which is defined in terms of n independent variables $x_1, x_2, \ldots, x_N$, where the functional expression for y is not known. The multivariable search problem is defined as a problem in which the optimum value of y is sought by systematically varying the values of the N-independent variables. With each set of values of the N-independent variables there corresponds a defined value for y. It is not necessary that the relationship defining y be one-to-one.

Response surfaces: The use of the response surface[40] represents an attempt to measure the outcome of a given problem without the benefit of a defined functional relation. In this sense the response curve can be compared to the set of results corresponding to the experimental processes of chemistry, physics, or biology. Such a comparison leads to the following definition.

Definition 4.33.--The map of a set of outcomes of a given multivariable search problem is defined as the response surface of that problem. For problems of more than three dimensions, this mapping will be called a hypersurface of dimension (n + 1) and will be written $(x_1, x_2, \ldots, x_n; y)$.

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[40]The context of this discussion was taken from the following three sources: Acheson J. Duncan, Quality Control and Industrial Statistics (Homewood, 1959), pp. 765-787; Wilde, op. cit., pp. 53-62; Wilde and Beightler, op. cit., pp. 273-287.

The use of multivariable functions begins with functions of two variables. It is assumed that the response surface is continuous within the region of feasibility and has continuous derivatives. Additional assumptions include the following: (1) the unknown function is assumed to be unimodal within the region of feasibility; (2) the response surface can be sufficiently approximated by a plane; and, (3) in the region of the optimum, the true function can be sufficiently approximated by a second-degree polynomial.

The response surface for two independent variables can take on several forms: planes, elliptical paraboloids, hyperbolic paraboloids, straight ridges, parabolic ridges, or elliptical ridges. Regardless of the surface, the response curve describes the various combinations of the two independent variables that yield a fixed outcome. This concept can be readily recognized as that corresponding to a set of indifference curves or utility functions. The mapping which results is called a contour map similar to that of a topographical map. The outcome value defines the elevation with closed contour lines indicating sharp rises and separated contour lines indicating gradual slopes.

Analytical considerations relative to the study of response surfaces defined by two independent variables center on describing the type of surface which results and on describing the contours. Such considerations require that some description be given of the possible surfaces.[41]

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[41]The basis for this discussion was taken from Duncan, op. cit., pp. 773-779.

(1) Planes. If the response surface for a given problem is a plane, the contours will be a set of parallel straight lines. These lines will be the projections of the intersections of horizontal planes with the response plane onto the base plane. Such contour maps are seen when a set of contours is constructed for fixed levels of demand, where demand is defined in terms of two independent variables. See Figure 4.12(a). If the response plane

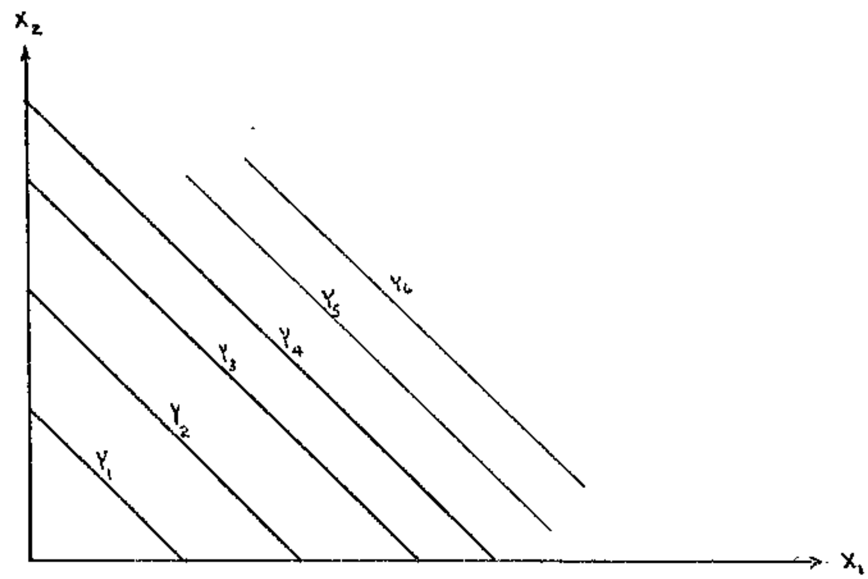

Fig. 4.12(a)--Demand contour: $Y = f(x_1, x_2)$, Y fixed

is defined by the linear relation $Y = a + bx_1 + cx_2$, the contour at Y_c is defined by the linear relation $Y_c = a + bx_1 + cx_2$. This contour function can be written in the form $bx_1 + cx_2 + (a - Y_c) = 0$ and defines a straight line with slope -c/b on the x_2-axis.

(2) Paraboloids. If the response surface for a given problem is a paraboloid, it will be defined by a second-degree equation of the form

$$Y = a + b_1x_1 + b_2x_2 + b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2.$$

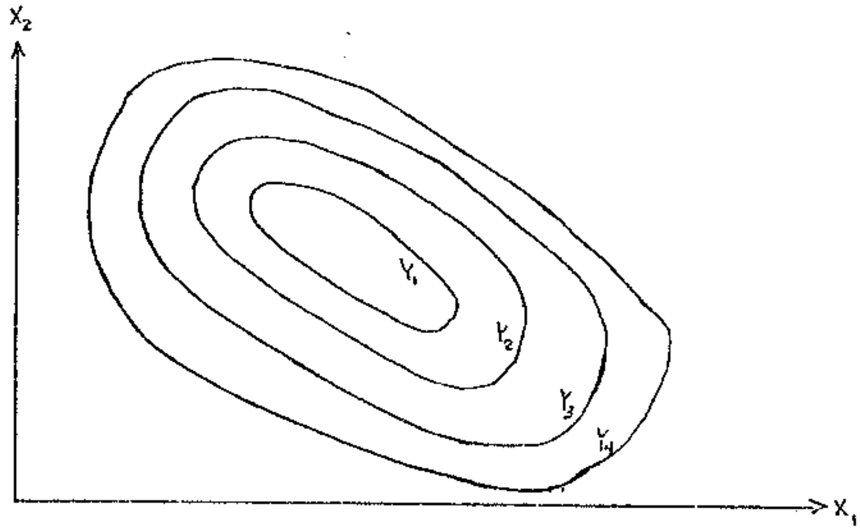

Fig. 4.12(b)--Contour map: elliptical paraboloid, $Y = f(x_1, x_2)$, Y fixed

Vertial sections of the response surface are defined by parabolas while projections of horizontal sections onto the x_1x_2-plane are generally defined by ellipses or hyperbolas. See Figures 4.12(b) and 4.12(c).

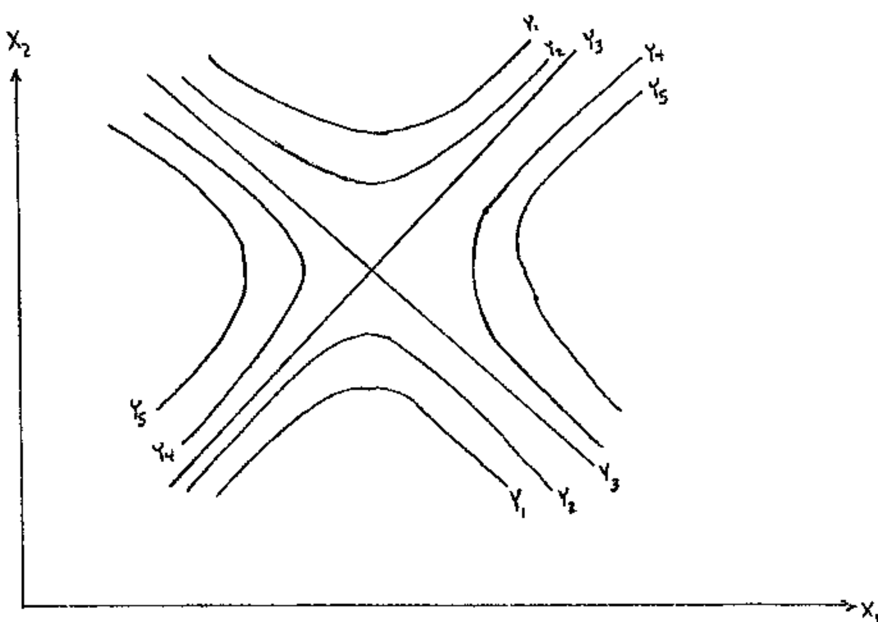

Fig. 4.12(c)--Contour map: hyperbolic paraboloid, $Y = f(x_1, x_2)$, Y fixed

The general equation for the contour of the paraboloid at Y_c is given by

$$Y_c = a + b_1x_1 + b_2x_2 + b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2.$$

This equation can be written in the form

$$b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2 + b_1x_1 + b_2x_2 + (a - Y_c) = 0.$$

The analysis of such functions can be enhanced if the determinant

$$\begin{vmatrix} b_{11} & \frac{b_{12}}{2} & \frac{b_1}{2} \\ \frac{b_{12}}{2} & b_{22} & \frac{b_2}{2} \\ \frac{b_1}{2} & \frac{b_2}{2} & (a - Y_c) \end{vmatrix} \neq 0$$

and if

$$\begin{vmatrix} b_{11} & \frac{b_{12}}{2} \\ \frac{b_{12}}{2} & b_{22} \end{vmatrix} = b_{11}b_{12} - \frac{1}{4}(b_{12})^2 \neq 0.$$

If the preceding two conditions are satisfied, the general equation can be reduced to a simplified form by shifting the origin for x_1 and x_2 to the center of the conic and rotating the coordinate axes until they coincide with the axes of the conic. The result of this operation is the reduced equation

$$B_{11}X_1^2 + B_{22}X_2^2 + A - Y_c = 0.$$

The X_1 and X_2 now identify the new axes. This form can then be analyzed according to the following criteria:

(1) if $B_{11} < 0$ and $B_{22} < 0$, $B_{11}X_1^2 + B_{22}X_2^2 + A - Y_c = 0$ defines an ellipse and the stationary point which occurs at the center is a maximum;

(2) if $B_{11} > 0$ and $B_{22} > 0$, $B_{11}X_1^2 + B_{22}X_2^2 + A - Y_c = 0$ defines an ellipse and the stationary point which occurs at the center is a minimum;

(3) if $B_{11} < 0$ and $B_{22} < 0$, $B_{11}X_1^2 + B_{22}X_2^2 + A - Y_c = 0$ defines a hyperbola with a saddle-point that is a maximum for X_1 and a minimum for X_2;

(4) if $B_{11} > 0$ and $B_{22} < 0$, $B_{11}X_1^2 + B_{22}X_2^2 + A - Y_c = 0$ defines a hyperbola with a saddle-point that is a maximum for X_2 and a minimum for X_1.

(3) Ridges. The occurrence of a ridge is a special case of the paraboloid. As a paraboloid, the ridge is defined by the reduced contour function

$$B_{11}X_1^2 + B_{22}X_2^2 + A - Y_c = 0.$$

Its form is determined by the values of B_{11}, B_{22}, and the determinants of the original conic function

$$b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2 + b_1x_1 + b_2x_2 + a - Y_c = 0.$$

The determinant of this quadratic form is given by

$$D = \begin{vmatrix} b_{11} & \frac{b_{12}}{2} & \frac{b_1}{2} \\ \frac{b_{12}}{2} & b_{22} & \frac{b_2}{2} \\ \frac{b_1}{2} & \frac{b_2}{2} & (a - Y_c) \end{vmatrix} .$$

(a) If B_{11} is small, the surface defined by $B_{11}X_1^2 + B_{22}X_2^2 + A - Y_c = 0$ begins to become narrow in the direction of X_1. If B_{22} is small, the surface defined by $B_{11}X_1^2 + B_{22}X_2^2 + A - Y_c = 0$ begins to become narrow in the direction of X_2. The resulting experimental surface is an elliptical ridge. See

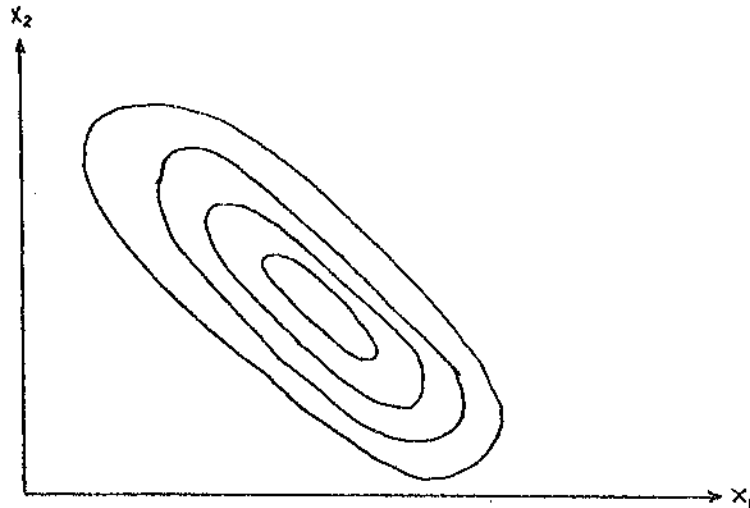

Fig. 4.13(a)--Elliptical ridge

(b) If $B_{11} = 0$, the contours are straight lines parallel to the X_1 axis. The result is an infinite ridge in the direction of X_1. See Figure 4.13(b).

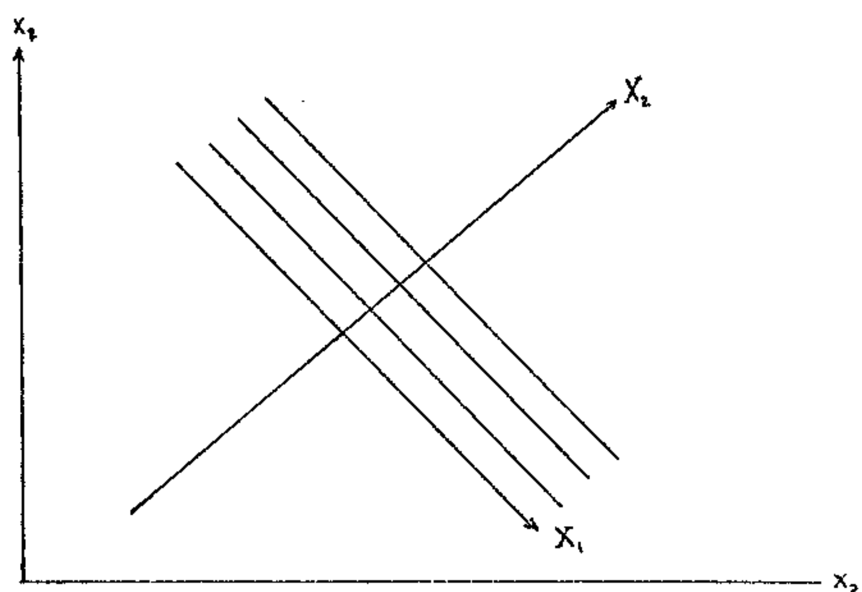

Fig. 4.12(b)--Straight ridge

(c) If $B_{22} = 0$, the contours are straight lines parallel to the X_2 axis. The result is an infinite ridge in the direction of X_2. See Figure 4.13(c).

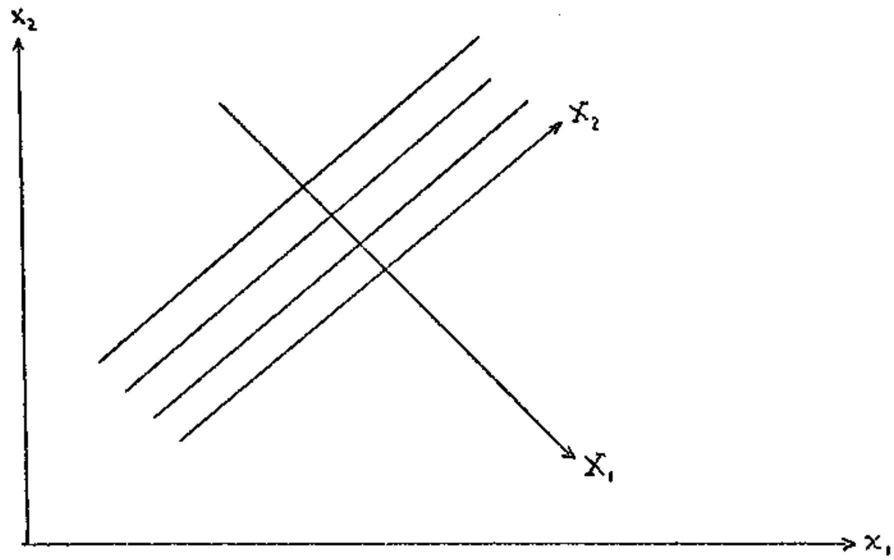

Fig. 4.13(c)--Straight ridge

(d) If the determinant of the original conic function equals zero, the contours are described by a set of parabolic ridges. See Figure 4.13(d).

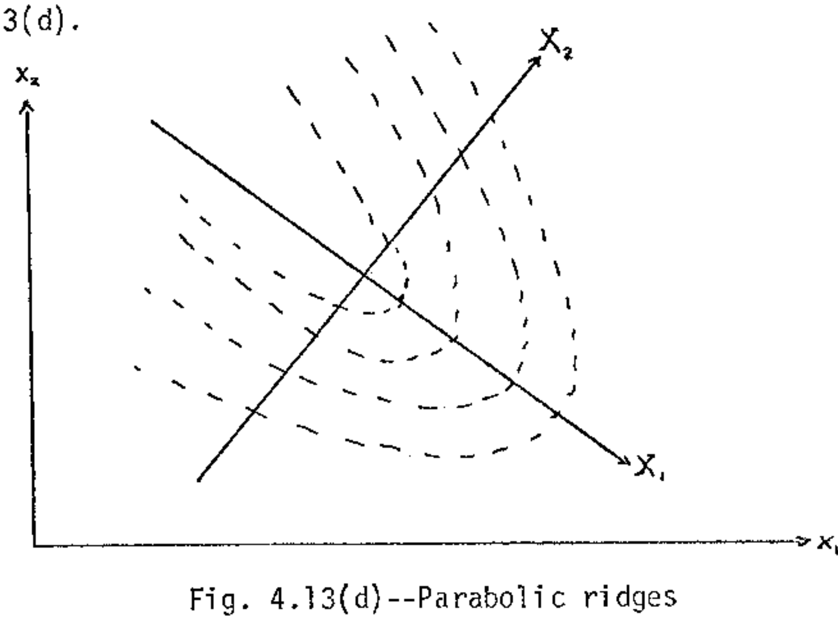

Fig. 4.13(d)--Parabolic ridges

The introduction of more than two independent variables into the problem compounds the difficulty of graphically representing a response surface. However, the surfaces that are generated can still be described by one of the preceding types of surfaces. The functions under investigation are assumed to be continuous within the region of feasibility and are assumed to have continuous derivatives. In addition, the function is assumed to be unimodal within the region of optimality and amenable to nonlinear approximations. The desired result of response surface analysis is summarized by the following:

(1) to find, after a minimum number of experiments, a set of operating conditions which optimize a given objective;

(2) to reach, in as few experiments as possible, a minimum level of acceptable performance; and,

(3) to generate information useful for locating future experiments that will lead to the optimum value of the defined objective.

The analysis of a given response surface is generally achieved by fitting suitable surface approximations to the trial region. These fitted surfaces are then examined to determine whether or not to continue the climbing process or to further explore the immediate region. In this analysis use is made of two major concepts, linear explorations and parallel tangents.

At the outset of the analysis of a given response surface, there is no information as to where the search should begin. Thus, any starting solution will be one of exploration. The only information conveyed will be the location of the output value on the response surface. Wilde has indicated that this analysis should be initiated by the technique of linear explorations. These should then be used to construct a representative tangent plane.[42]

The linear exploration-tangent plane technique is begun by arbitrarily selecting an initial solution $\underline{x}_o$. The trial point $\underline{x}_o$ is then used to determine an initial value of the objective function y_o. Since $(\underline{x}_o; y_o)$ conveys no information other than the value of the objective at $\underline{x}_o$, additional experiments are needed if the objective is to be improved. Thus, a second trial point, $\underline{x}_1$, is needed. This second trial point can be obtained in a variety of ways, the most common of which is to hold all values of the independent variables constant but one. This second trial point is then used to obtain a second value of the objective, y_1. If a straight line is constructed between y_o and y_1, this line will be approximately tangent to the response surface.

As a means of better determining the slope of this line of tangency, a third trial point, $\underline{x}_2$, is selected and its corresponding value on the surface, y_2, noted. A straight line is then constructed between y_o and y_1. This line will be approximately tangent to the response surface. The three points y_o, y_1, and y_2 can then be used to construct a plane tangent to the response surface at y_o. This tangent plane can then be used as an approximation of the response surface and examined for directional improvement.

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[42]Wilde, op. cit., pp. 65-71.

The values of the solution vector $\underline{x}_k$ (k = 0, 1,...,m) are arbitrary. With each $\underline{x}_k$ there is a corresponding y_k which identifies a point on the response surface. Given three such y_k values, a tangent plane can be constructed as an approximation of the response surface in the neighborhood of the initial value y_o. This tangent plane is defined by the linear relation

$$y(x_1, x_2,\ldots,x_N) = m + m_1x_1 + m_2x_2+\ldots+m_Nx_N$$

where m , $m_1,\ldots,m_N$ are constants satisfying

$$y_0 = m_0 + m_1x_{01} + m_2x_{02}+\ldots+m_Nx_{0N}$$
$$y_1 = m_0 + m_1x_{11} + m_2x_{12}+\ldots+m_Nx_{1N}$$
$$y_2 = m_0 + m_1x_{21} + m_2x_{22}+\ldots+m_Nx_{2N}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$y_m = m_0 + m_1x_{m1} + m_2x_{m2}+\ldots+m_Nx_{mN}$$

and x_{mN} identifies the m^{th} value of the N^{th} variable; i.e., x_{01} equals the initial value of variable x_1, and x_{m1} equals the m^{th} value of the variable x_1.

From this discussion, it is readily seen that the use of the response surface does not require that the mathematical expression for the objective function be known in advance. In addition, the mathematical expressions for any constraints are not required. Trial solutions are set within the region of feasibility, and tangent planes are used to approximate the functional expression for the objective function. Response surfaces can be constructed from experimental data derived from statistical experimental design, from accounting records, from production records, etc. The use of the response surface allows any number of variables, with the resulting plane being defined as a hyperplane. As a tangent plane is constructed, it is explored for better solution points lying within the plane. If better solutions exist, they are used to construct additional tangent planes and the process is repeated. The primary objective of each set of experiments is to construct a tangent plane that leads to the optimum solution as rapidly as possible. As additional planes are derived, the contour tangent concept can be utilized to reduce the region of feasibility. This will help in locating the solutions to be used as experimental values.

The procedure to be followed in linear exploration-tangent plane analysis can be written as a series of sequential operations. The operations can be repeated as often as necessary and terminate when a suitable functional expression has been derived or an optimum identified.

Algorithm 4.6 (linear exploration-tangent plane analysis).--Step 1. Let $\underline{x}_o = (x_1, x_2,\ldots,x_N)$ be the initial trial solution. Using $\underline{x}_o$, estimate the value of the objective function. Denote this initial value by y_o.

Step 2. Let $\underline{x}_1 = (x_1, x_2,\ldots,x_N)$ be the second trial solution, arbitrarily selected in the neighborhood of the initial trial solution $\underline{x}_o$. Using $\underline{x}_1$, estimate the value of the objective function. Denote this second estimate by y_1.

Step 3. Let $\underline{x}_2 = (x_1, x_2,\ldots,x_N)$ be the third trial solution, arbitrarily selected in the neighborhood of the initial trial solution $\underline{x}_o$ but different from the second trial solution $\underline{x}_1$. Using $\underline{x}_2$, estimate the value of the objective function. Denote this third estimate by y_2.

Step 4. Using the values y_0, y_1, y_2 and the corresponding solution vectors, determine the equation of the plane tangent to the response surface which contains y_0, y_1, and y_2. This tangent plane is defined by the relation

$$y(x_1, x_2, \ldots, x_N) = m_0 + m_1 x_1 + m_2 x_2 + \ldots + m_N x_N$$

where the values of the m_k $(k = 0, 1, 2, \ldots, N)$ are solutions to the linear system

$$y_0 = m_0 + m_1 x_{01} + m_2 x_{02} + \ldots + m_N x_{0N}$$
$$y_1 = m_0 + m_1 x_{11} + m_2 x_{12} + \ldots + m_N x_{1N}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$y_m = m_0 + m_1 x_{m1} + m_2 x_{m2} + \ldots + m_N x_{mN}.$$

The values of y_k $(k = 0, 1, \ldots, m)$ are the values of the objective function at the k^{th} trial solution. The values of x_{kN} $(k = 0, 1, \ldots, m)$ are the values of the respective variables in the k^{th} trial point and are known. This tangent plane will be used as an approximation of the response surface in the neighborhood of y_0.

Step 5. Examine the tangent plane for directional improvement. This can be done by redefining $\underline{x}_0$ and reiterating Steps 1 - 4.

Gradient techniques: The use of the gradient in direct search has been described as a "creeping" approach to optimization.[43] This terminology has been employed because of the manner in which gradient search techniques seek out the optimum solution to a given problem. The manner employed involves sliding around the feasible region in such a way that any move is a continuous motion and in the direction that improves the objective function. If the function is to be maximized, the movement will be in

[43] William J. Baumol, Economic Theory and Operations Analysis (Englewood-Cliffs, 1965), p. 142.

the direction of steepest ascent. If the function is to be minimized, the movement will be in the direction of steepest descent. Baumol has described this approach in the following manner.

> Suppose it is desired to find the outputs which maximize profit, $R = f(X_1, X_2, \ldots, X_n)$, where the X_i are the outputs of the firm's different products. A gradient method sets up the differential equation (in which t represents computing time elapsed):
>
> $$\frac{dX_i}{dt} = \frac{\partial R}{\partial X_i}.$$
>
> This states that we increase the quantity X_i of commodity i ($dX_i/dt > 0$) in the trial solution so long. . .as this increase in X_i results in a rise in the firm's profits. . .we make this time rate of increase in output, X_i, proportionate (equal for an appropriate time unit) to its marginal profitability ($\partial R/\partial X_i$). In other words, we increase (decrease) all quantities whose rise leads to higher (lower) profits, and, in effect, give a priority ordering to the changes in the different quantities in proportion to their profit contribution. Moreover, we impose the condition that any quantity which falls to zero be stopped at that point:
>
> $$\frac{\partial X_k}{\partial t} = 0 \text{ if } X_k = 0 \text{ and } \frac{\partial R}{\partial X_k} < 0.$$[44]

From this description, it is seen that gradient techniques utilize the property that the direction of the gradient is the one providing the greatest response of the objective function per unit length of the independent variable. This direction is determined by the proportional relationship between the incremental change in each variable and its partial derivative.

The use of gradient techniques provides a means whereby any unimodal function can be solved. When the defining function is convex or concave,

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[44] Ibid., p. 143.

the use of gradient search will result in the global optimum. The use of gradient search is further enhanced by the fact that gradient techniques work when experimental error is a consideration and by the fact that gradient techniques avoid saddle-points.

The gradient search techniques to be discussed in this study serve to represent the variety of gradient search techniques available and their application. The gradient techniques to be examined are those described as the differential gradient, steepest ascent-steepest descent (utilizing the differential gradient), and deflected gradients.

(1) The differential gradient. Gradient search via the differential gradient technique can be applied to defined constrained or unconstrained functions or to response surfaces. If $y = f(x_1, x_2,\ldots,x_n)$ is any continuous multivariable objective function, the gradient ∇y is defined by the set of n first partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2},\ldots,\frac{\partial f}{\partial x_n}$.

$$\nabla y = [\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2},\ldots,\frac{\partial f}{\partial x_n}].$$

The gradient at any solution point $\underline{x} = (x_1, x_2,\ldots,x_n)$ is perpendicular to the contour of $f(x_1, x_2,\ldots,x_n)$ which passes through $\underline{x} = (x_1, x_2,\ldots,x_n)$ and points in the direction of optimal improvement. If the objective function is to be maximized, ∇y points in the direction of steepest ascent. If the objective function is to be minimized, ∇y points in the direction of steepest descent.

In the development of the differential gradient, it is assumed that the problem being investigated is defined by

$$\text{maximize } y = f(x_1, x_2, \ldots, x_n)$$

subject to

$$g_i(x_1, x_2, \ldots, x_n) \leq 0 \qquad (i = 1, 2, \ldots, k)$$

$$x_i \geq 0 \text{ for all } i.$$

The objective function and the constraints are assumed to be continuous and differentiable. In addition, both the objective function and the constraint functions are unrestricted as to degree.

Direct application of the differential gradient technique utilizes the formula

$$\frac{dx}{dt} = \nabla y - \sum_{i=1}^{k} \delta_i(\underline{x}) \nabla g_i(\underline{x}),$$

where

$$\delta_i(\underline{x}) = \begin{cases} 0 & \text{if } g_i(\underline{x}) \leq 0, \\ K & \text{if } g_i(\underline{x}) > 0. \end{cases}$$

At each iteration the trial point $\underline{x} = (x_1, x_2, \ldots, x_n)$ is moved in the direction of greatest increase in $f(x_1, x_2, \ldots, x_n)$. The term defined by

$$\sum_{i=1}^{k} \delta_i(\underline{x}) \nabla g_i(\underline{x})$$

serves to keep the solution inside the constraint set. The value assigned to K is selected in such a way that it keeps all x_i from leaving the constraint set.

Successive application of the differential gradient requires that each new trial point be determined according to

$$\underline{x}_{k+1} = \underline{x}_k + \rho \underline{\nabla y}_k,$$

where

$$\rho \equiv \frac{r}{|\underline{\nabla y}|}$$

and r equals the radius of the n-dimensional hypersphere centered at $\underline{x}_k$. The optimal value of ρ can be found by a suitable elimination technique (such as Fibonacci search) or by substituting

$$\underline{\Delta x}_k \equiv \rho \underline{\nabla y}_k$$

into the objective function and evaluating

$$\left. \frac{\partial y(\underline{x}_k + \rho \underline{\nabla y}_k)}{\partial \rho} \right|_{\rho = \rho^*} = 0.$$[45]

It has been shown that if $\underline{\nabla} y = \underline{\nabla} f(x)$ is continuous, then any limit point of a sequence of points given by the $\underline{x}_{k+1} = \underline{x}_k + \rho \underline{\nabla y}_k$ is a stationary point. This provides sufficient proof for convergence of the iterative process.[46]

An interesting property of successive gradients is that they are perpendicular to each other. This property results in a series of right angle iterations which result in a stairway leading to the optimal solution. A graphic comparison of these two uses of the differential gradient is shown in Figure 4.14.

The differential gradient technique is incorporated into optimization theory through the use of the terms steepest ascent and steepest descent. If the function under investigation is to be maximized, the climbing procedure is one of steepest ascent. If the function under investigation is to be minimized, the climbing procedure (in a downhill manner) is one

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[45] Wilde and Beightler, op. cit., pp. 288-289.

[46] P. Wolfe, "Methods of Nonlinear Programming," Nonlinear Programming, edited by J. Abadie (Amsterdam, 1967), pp. 112-113.

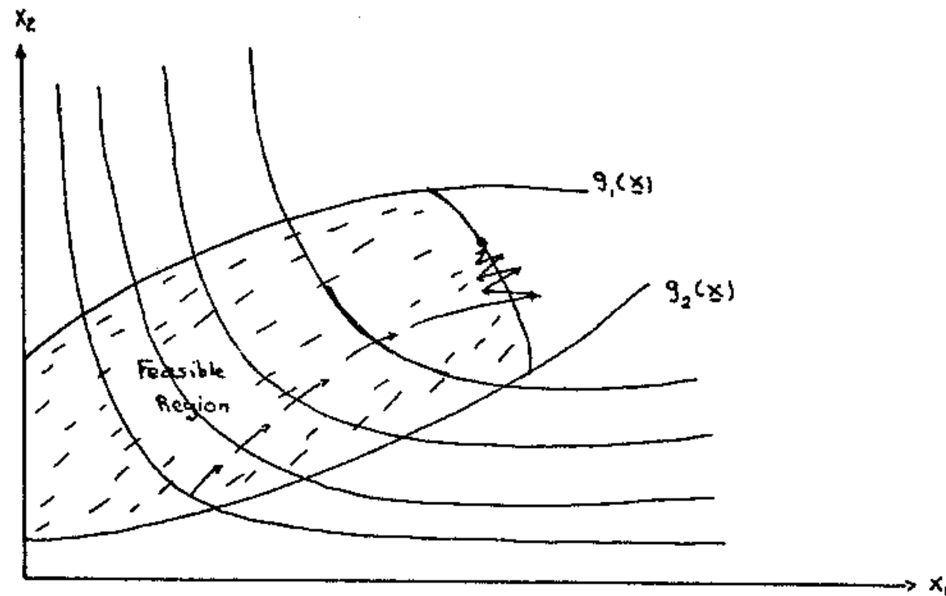

Fig. 4.14(a)--Direct differential gradient: $\frac{dx}{dt} = \nabla y - \sum_i \delta_i(\underline{x})\nabla g_i(\underline{x})$[47]

of steepest descent. Optimization via steepest ascent or steepest descent is accomplished by evaluating the gradient at a given point, searching for the optimum along the calculated gradient, and then repeating the process until the gradient has been reduced to a suitable size ($\nabla y = 0$).

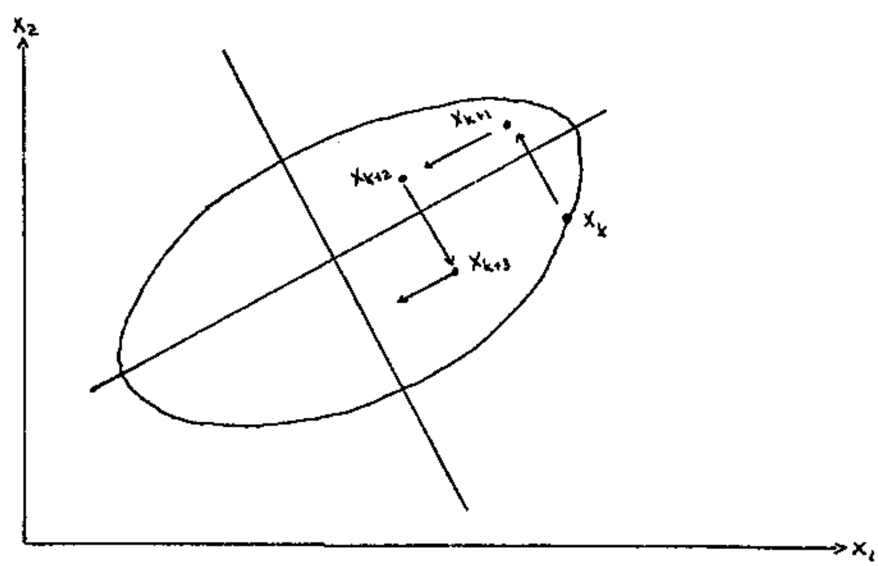

Fig. 4.14(b)--Stepwise minimization: $\underline{x}_{k+1} = \underline{x}_k + \rho\nabla y_k$

[47]Wolfe, "Recent Developments in Nonlinear Programming," op. cit., p. 8.

(a) Steepest ascent. If $f(x_1, x_2,\ldots,x_N)$ is any multivariable objective function, the direction in which a move is to be made is given by the square-root of the sum of the squared values of the first partial derivatives. This requirement is defined by the relation

$$\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2$$

As successive iterations are made, it will be found that directional derivatives are perpendicular to each other, as noted in Figure 4.14(b). This is due to the fact that the initial gradient is the tangent to the contour at the second trial point, and the new gradient is perpendicular to the tangent of the contour at this second point. This property has resulted in gradient techniques being classified as stepwise optimization techniques, with the orthogonality condition defined as

$$\sum_{i=1}^{N} \frac{\partial f(x^k)}{\partial x_j} \cdot \frac{\partial f(x^{k+1})}{\partial x_i} = 0.$$

The notation $\frac{\partial f(x^k)}{\partial x_i}$ indicates that the partial derivative has been evaluated at trial solution x^k.

Gradient search via steepest ascent requires that the step size be selected either arbitrarily or by some defined technique. Wagner has shown that the optimal step size is that value of $t^k \geq 0$ which maximizes $f(x_1^k + td_1^k,\ldots,x_N^k + td_N^k)$. In this relation, d_i^k denotes the value of the first partial derivative with respect to x_i at the k^{th} trial solution.

The application of steepest ascent gradient search has been described in algorithmic form. This algorithm, taken from Wagner,[48] is reproduced here in the form most applicable to a given problem.

Algorithm 4.7 (gradient search by steepest ascent).--Step 1. Select an arbitrary, feasible trial solution. Denote this initial trial solution by $\underline{x}^o$.

Step 2. Determine $\frac{\partial}{\partial x_i} f(x_1, x_2,\ldots,x_N)$ for i = 1, 2,...,N. Evaluate $\frac{\partial f}{\partial x_i}$ at the trial solution. If $\frac{\partial f}{\partial x_i} = 0$ at $\underline{x}^k$ for all i, terminate the algorithm since there is no further improvement. If $\frac{\partial f}{\partial x_i} \neq 0$ for all i, determine y_i^k for i = 1, 2,...,N, and go to Step 3. The value y_i^k equals the value of the objective function at the k^{th} iteration.

Step 3. Calculate a new trial point by applying

$$x_i^{k+1} = x_i^k + t^k d_i^k,$$

where d_i^k equals the first partial derivative of $f(x_1, x_2,\ldots,x_N)$ with respect to x_i when evaluated at trial solution $\underline{x}^k$ and t^k is that value of t which maximizes $f(x_1^k + td_1^k, x_2^k + td_2^k,\ldots,x_N^k + td_N^k)$.

Step 4. The solution vector $\underline{x}^k$ is an optimal solution if and only if the gradient vanishes at that point; i.e., $\frac{\partial f}{\partial x_i} = 0$ for all i. Repeat Steps 1 - 3 as needed.

Consider the problem of maximizing $f(x_1, x_2) = -(x_1 - 3)^2 - 4(x_2 - 2)^2$. Let the initial trial solution be (1, 1). The given function is assumed to be continuous and differentiable.[49]

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[48]Harvey M. Wagner, Principles of Operations Research (Englewood Cliffs. 1969), p. 520.

Step 1. Let $\underline{x}^0 = (x_1^0, x_2^0) = (1, 1)$.

Step 2. Determine $\frac{\partial f}{\partial x_i}$ for $i = 1, 2$.

$$\frac{\partial f}{\partial x_1} = -2(x_1 - 3),$$

$$\frac{\partial f}{\partial x_2} = -8(x_2 - 2).$$

At $\underline{x}^0 = (1, 1)$, $d_1^0 = \frac{\partial f}{\partial x_1} = 4$ and $d_2^0 = \frac{\partial f}{\partial x_2} = 8$. Since $\frac{\partial f}{\partial x_i} \neq 0$ for all x_i, the solution is not optimal. Go to Step 3.

Step 3. Calculate a new trial point by applying

$$x_i^{k+1} = x_i^k + t^k d_i^k.$$

It is necessary to determine the value of t for which $f(x_1^k + td_1^k, \ldots, x_N^k + td_N^k)$ is maximized,

$$f(x_1^0 + td_1^0, x_2^0 + td_2^0) = f(td_1^0, td_2^0); \quad x_1^0 = x_2^0 = 0.$$

For $\underline{x}^0 = (0, 0)$, the function to be maximized is defined by

$$\begin{aligned} f(4t, 8t) &= -(4t - 3)^2 - 4(8t - 2)^2 \\ f'(4t, 8t) &= -2(4t - 3)(4) - 8(8t - 2)(8) \\ &= -8(4t - 3) - 64(8t - 2) \\ &= -32t + 24 - 512t + 128 \\ &= -544t + 152. \end{aligned}$$

Setting $f'(4t, 8t) = 0$ yields $t^0 = \frac{19}{68}$, which maximizes $f(4t, 8t)$. Therefore,

$$x_1^1 = x_1^0 + (\tfrac{19}{68})(4) = 1.12;$$

$$x_2^1 = x_1^0 + (\tfrac{19}{68})(8) = 2.24.$$

The procedure just completed will be reapplied using $\underline{x}^1 = (1.12, 2.24)$. So long as the gradient is nonzero, the optimum has not been reached. So long as the gradient is positive, the solution values must be increased. If the gradient is negative, the solution values must be decreased. As indicated, the iterations terminate when the gradient vector vanishes. At this point, no further improvement can be made, regardless of step size.

Iteration II: Step 1. Let $\underline{x}^1 = (x_1^1, x_2^1) = (1.12, 2.24)$.

Step 2. Determine $\frac{\partial f}{\partial x_i}$ for $i = 1, 2$.

$$\frac{\partial f}{\partial x_1} = -2(x_1 - 3),$$

$$\frac{\partial f}{\partial x_2} = -8(x_2 - 2).$$

At $\underline{x}^1 = (1.12, 2.24)$, $d_1^1 = \frac{\partial f}{\partial x_1} = 3.76$ and $d_2^1 = \frac{\partial f}{\partial x_2} = -1.92$. Since $\frac{\partial f}{\partial x_i} \neq 0$ for all x_i, the solution is not optimal. Go to Step 3.

Step 3. Calculate a new trial point by applying

$$x_i^{k+1} = x_i^k + t^k d_i^k, \text{ where } k = 1.$$

It is necessary to determine the value of $t^k = t^1$ for which $f(x_1^k + td_1^k, \ldots, x_N^k + td_N^k)$ is maximized,

$$f(x_1^1 + td_1^1,\ x_2^1 + td_2^1) = f(1.12 + td_1^1,\ 2.24 + td_2^1).$$

Since $\underline{x}^1 = (x_1, x_2) = (1.12, 2.24)$, the function to be maximized is defined by

$$f(1.12 + 3.76t,\ 2.24 - 1.92t) = -(1.12 + 3.76t - 3)^2 - 4(2.24 - 1.92t - 2)^2$$

$$f(1.12 + 3.76t,\ 2.24 - 1.92t) = -(-1.88 + 3.76t)^2 - 4(.24 - 1.92t)^2.$$

$$f'(1.12 + 3.76t, 2.24 - 1.92t) = -2(-1.88 + 3.76t)(3.76) - 8(.24 - 1.92t)(-1.92)$$
$$= -7.52(-1.88 + 3.76t) + 15.36(.24 - 1.92t)$$
$$= 14.1376 - 28.2752t + 3.6864 - 29.4912t$$
$$= 17.8240 - 57.7664t.$$

Setting $f'(1.12 + 3.76t, 2.24 - 1.92t) = 0$ yields $t = .31$, which maximizes the function $f(1.12 + 3.76t, 2.24 - 1.92t)$. Therefore,

$$x_1^2 = x_1^1 + (.31)(3.76) = 1.12 + 1.1656 = 2.1856 = 2.29$$
$$x_2^2 = x_2^1 + (.31)(-1.92) = 2.24 - 0.5952 = 1.6448 = 1.64$$

The new trial point can then be utilized in the same manner as $\underline{x}^1$ to determine a third trial solution, $\underline{x}^3$. The reapplication process will continue until the gradient vanishes, an indication that an optimal solution has been achieved.

(b) Steepest descent. Steepest descent can be described as steepest ascent in reverse. Whereas steepest ascent moves in the direction of greatest increase, steepest descent moves in the direction of greatest decrease. This can be interpreted as moving along the gradient in a negative direction.

The process of steepest descent can be approached from either the continuous process of the differential gradient or the stepwise process discussed under steepest ascent. The discussion here will parallel that of steepest ascent. In this way, the technique of steepest descent will be seen to be basically identical to that of steepest ascent.

Steepest descent requires a modification of the stepwise iterative formula of steepest ascent. Whereas steepest ascent iterations utilize

$$x_i^{k+1} = x_i^k + td_i^k,$$

steepest descent iterations utilize

$$x_i^{k+1} = x_i^k - td_i^k.$$

The value of t is determined by maximizing $f(x_1 + t_1^1 d_1, \ldots, x_N + t^k d_i^k)$. As in the case for steepest ascent, d_i^k equals the value of $\frac{\partial f}{\partial x_i}$ evaluated at trial point x^k. This modification indicates that the trial solution is moved along the gradient in a negative manner. Other than requiring the use of the modified iterative formula, the technique is the same as that for steepest ascent.

Algorithm 4.8 (gradient search by steepest descent).--Step 1. Select an arbitrary, feasible trial solution. Denote this initial trial solution by $\underline{x}^0$.

Step 2. Determine $\frac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_N)$ for $i = 1, 2, \ldots, N$. Evaluate $\frac{\partial f}{\partial x_i}$ at the trial solution. If $\frac{\partial f}{\partial x_i} = 0$ at $\underline{x}^k$ for all i, terminate the algorithm since there is no further improvement. If $\frac{\partial f}{\partial x_i} \neq 0$ for all i, determine y_i^k for $i = 1, 2, \ldots, N$, and go to Step 3. The value y_i^k equals the value of the objective function at the k^{th} iteration.

Step 3. Calculate the new trial point by applying

$$x_i^{k+1} = x_i^k - t^k d_i^k,$$

where d_i^k equals the first partial derivative of $f(x_1, x_2, \ldots, x_N)$ with respect to x_i when evaluated at trial solution $\underline{x}^k$ and t^k equals the value of t which maximizes $f(x_1 + td_1, \ldots, x_N + td_N)$ at the k^{th} iteration.

Step 4. The solution vector $\underline{x}^k$ is an optimal solution if and only if the gradient vanishes at that point; i.e., $\frac{\partial f}{\partial x_i} = 0$ for all i. Repeat Steps 1 - 3 as needed.

The use of steepest ascent-steepest descent in optimization problems requires repetitious evaluation of the partial derivatives defined by $\frac{\partial f}{\partial x_i}$. For large N, this is expensive in terms of both time and effort. Hence, it is desirable to take as large a step as feasibly possible without violating any constraints. The procedure previously outlined is such a technique, as it utilizes the optimal step size at each iteration. The result is the reduction of an N-dimensional optimization problem to a sequence of one-dimensional problems.

(2) Deflected gradient technique. The deflected gradient technique is an iterative technique developed by R. Fletcher and M. J. D. Powell. It has also been identified as the Fletcher-Powell technique.

The deflected gradient technique utilizes information that is generated at each iteration. This information is used to construct the Hessian matrix of the defined objective function. If the function is quadratic, unimodal, and differentiable through at least second derivatives, this Hessian matrix is constructed after N iterations. For functions more complex than a quadratic, the deflected gradient technique provides an adequate approximation to the Hessian in the neighborhood of the optimum solution.[50]

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[50]Gue and Thomas, <u>op. cit.</u>, p. 117.

Definition 4.34.--The term deflected gradient technique is used in reference to optimization analysis which utilizes the gradient of the defined function as a means of improving the value of the objective function without moving along the gradient.

The deflected gradient technique requires that suitable information be generated from the gradient at a number of points. This information is obtained by (1) determining the gradient at $\underline{x}_n$, $\nabla y(\underline{x}_n)$, (2) constructing a direction, $\Delta \underline{x}$, in which to move, and (3) moving along the direction $\Delta \underline{x}$ to some new point $\underline{x}_{n+1}$. This procedure is repeated at each iteration until the gradient has been reduced to a predetermined acceptable size.

(a) Minimization technique. The problem to be minimized is defined by the quadratic form

$$y = y_o + \underline{c}^T\underline{x} + \frac{1}{2}\underline{x}^T\underline{C}\underline{x}.$$

The general procedure is to move from point $\underline{x}_{k-1}$ to the optimum point $\underline{x}^*$ when the gradient $\nabla y(\underline{x}_{k-1})$ is known. The move from point $\underline{x}_{k-1}$ to $\underline{x}^*$ is defined by

$$\Delta \underline{x}^* = (\underline{x}^* = \underline{x}_{k-1}) = -\underline{Q}^{-1}\nabla y(\underline{x}_{k-1}),$$

where $\underline{Q}$ is a rarely known matrix. The deflected gradient technique provides a means of determining $\underline{Q}$.[51]

Let $\underline{H}_{k-1}$ be an N x N matrix satisfying

$$\underline{x}_k - \underline{x}_{k-1} = \Delta \underline{x}_k = -\mu_k \underline{H}_{k-1}\nabla y(\underline{x}_{k-1}).$$

Since the function is to be minimized, $\underline{H}_{k-1}$ will be a positive definite matrix. The value of μ_k is the value of the search parameter which

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[51] Wilde and Beightler, op. cit., pp. 331-336.

optimizes $y(\underline{\Delta x}_k)$ along the line of search defined by

$$\underline{\Delta x}_k = \underline{x}_k - \underline{x}_{k-1}.$$

If the direction of the optimum solution is known, the matrix $\underline{H}_{k-1}$ should be selected accordingly. If the direction of the optimum solution is not known and no information is available which might indicate the direction, the N x N identity matrix can be used as an estimate of $\underline{H}_{k-1}$.

Application of the deflected gradient technique for minimizing a given function generates N sequential solution points $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ such that the gradient $\underline{\nabla} y(\underline{x}_n)$, (n - 1, 2,...,N), is perpendicular to all preceding steps $\underline{\Delta x}_1, \underline{\Delta x}_2, \ldots, \underline{\Delta x}_n$. This condition is expressed by the relation

$$(\underline{\nabla} y(\underline{x}_N))^T \underline{\Delta x}_i = 0; \quad i = 1, 2, \ldots, n; \; n = 1, 2, \ldots, N.$$

The N^{th} gradient vector, $\underline{\nabla} y(\underline{x}_N)$, is perpendicular to N vectors $\underline{\Delta x}$. These N vectors have been constructed in such a way that they are linearly independent. This forces $\underline{\nabla} y(\underline{x}_N)$ to equal zero at an optimum. In addition, the sequence of iterations has generated at the N^{th} iteration the N^{th} Hessian matrix $\underline{H}_N$. This N^{th} positive definite matrix is the inverse of the original unknown Hessian matrix $\underline{Q}$; i.e., $\underline{H}_N \equiv \underline{Q}^{-1}$.

Practical application and the introduction of round-off error generally results in a nonvanishing N^{th} gradient vector. This requires an additional iteration to verify optimality. This additional iteration utilizes the Hessian matrix of the N^{th} iteration. The step for iteration (N + 1) is defined by

$$\underline{\Delta x}_{N+1} = \underline{x}_{N+1} - \underline{x}_N \equiv -\mu_{N+1}\underline{H}_N \underline{\nabla} y(\underline{x}_N).$$

Since $\underline{H}_N = \underline{Q}^{-1}$, this relation can be written as

$$\underline{\Delta x}_{N+1} = \underline{x}_{N+1} - \underline{x}_N = -\mu_{N+1}\underline{Q}^{-1}\underline{\nabla} y(\underline{x}_N).$$

However, $\underline{Q}^{-1}\underline{\nabla y}(\underline{x}_N) \equiv (\underline{x}^* - \underline{x}_N)$. Therefore,

$$\underline{\Delta x}_{N+1} = \underline{x}_{N+1} - \underline{x}_N = -\mu_{N+1}(\underline{x}^* - \underline{x}_N).$$

For $\mu_{N+1} = 1$, $\underline{x}_{N+1} = \underline{x}^*$. Further correction for round-off error can be made by minimizing with respect to μ_{N+1}.

Having briefly outlined the process by which a function is minimized by the deflected gradient technique, the construction of an application-oriented algorithm is in order. This algorithm will then be demonstrated by application to a quadratic function in three variables. Because of lengthy calculations, only one iteration will be demonstrated.

Algorithm 4.9 (minimization by deflected gradients).--Step 1. Determine the initial solution, $\underline{x}_o$. Determine the initial positive definite Hessian matrix $\underline{H}_o$ by defining $\underline{H}_o \equiv \underline{I}_N$.

Step 2. Determine the gradient vector $\underline{\nabla y}$, where $\nabla y = (\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_N})^T$. Using the solution point, determine $\underline{\nabla y}(\underline{x}_n)$ by evaluating $\underline{\nabla y}$ at the solution point. For $\underline{x}_o$, this will be $\underline{\nabla y}(\underline{x}_o)$.

Step 3. Determine $\underline{\Delta x}_1$. Utilizing the relation $\underline{\Delta x}_1 = \underline{x}_1 - \underline{x}_o = -\mu_1\underline{H}_o\underline{\nabla y}(\underline{x}_o)$, the new trial point $\underline{x}_1$ can be written as a function of the search parameter μ_1.

Step 4. Determine the value of μ_1 by minimizing $y(\underline{x}_1)$ with respect to μ_1.

Step 5. Substitute the value of μ_1 from Step 4 into the relation defining $\underline{x}_1$:

$$\underline{x}_1 = \underline{x}_o - \mu_1\underline{H}_o\underline{\nabla y}(\underline{x}_o).$$

Step 6. Calculate $\underline{\nabla y}(\underline{x}_1)$. If $\underline{\nabla y}(\underline{x}_1) = 0$, terminate the process. The solution defined by $\underline{x}_1$ is the optimal solution. If $\underline{\nabla y}(\underline{x}_1) \neq 0$, go to Step 7.

Step 7. Replace $\underline{x}_o$ by $\underline{x}_1$. Replace $\underline{H}_o$ by $\underline{H}_1$, where $H_1 = \underline{H}_o + \underline{A}_1 + \underline{B}_1$. The relation defining $\underline{A}_n$ (n = 1, 2,...,N) is given by

$$\underline{A}_n = \frac{\underline{\Delta x}_n \, \underline{\Delta x}_n^T}{\underline{\Delta x}_n^T \, \underline{g}_n}$$

where

$$\underline{g}_n = \underline{\nabla} y(\underline{x}_n) - \underline{\nabla} y(\underline{x}_{n-1}).$$

The relationship defining $\underline{B}_n$ (n = 1, 2,...,N) is given by

$$\underline{B}_n = - \frac{\underline{H}_{n-1}\underline{g}_n\underline{g}_n^T \, \underline{H}_{n-1}^T}{\underline{g}_n^T \, \underline{H}_{n-1} \, \underline{g}_n} .$$

Increase all subscripts by one for each additional iteration and repeat Steps 1 - 7 until the gradient at the N^{th} iteration is suitably small.

Consider the problem of minimizing $y(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + 3x_3^2$. A suggested starting solution is the set of values (-1, 1, -1).[52]

Solution: Step 1. Let $\underline{x}_o = (-1, 1, -1)^T$. Let $\underline{H}_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Step 2. Determine $\underline{\nabla} y$, where $\underline{\nabla} y = (\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial y}{\partial x_3})^T$. Applying partial derivatives to $y(x_1, x_2, x_3)$, $\frac{\partial y}{\partial x_1} = 4x_1$; $\frac{\partial y}{\partial x_2} = 2x_2$; $\frac{\partial y}{\partial x_3} = 6x_3$. At the initial solution point, $\underline{\nabla} y(\underline{x}_o) = (-4, 2, 6)^T$.

Step 3. Determine $\underline{\Delta x}_1$, where $\underline{\Delta x}_1 = -\mu_1 \underline{H}_o \underline{\nabla} y(\underline{x}_o)$.

$$\underline{\Delta x}_1 = -\mu_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$$

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[52] Ibid., p. 336.

$$= -\mu_1 \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} 4\mu_1 \\ -2\mu_1 \\ 6\mu_1 \end{bmatrix} .$$

Utilizing the relation $\underline{\Delta x}_1 = \underline{x}_1 - \underline{x}_o$, the new trial solution can be written

$$\underline{x}_1 = \underline{x}_o + \underline{\Delta x}_1$$

$$= \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 4\mu_1 \\ -2\mu_1 \\ 6\mu_1 \end{bmatrix}$$

$$= \begin{bmatrix} 4\mu_1 - 1 \\ 1 - 2\mu_1 \\ 6\mu_1 - 1 \end{bmatrix} .$$

Step 4. Determine the value of μ_1 by minimizing $y(\underline{x}_1)$ with respect to μ_1. At $\underline{x}_1$, $x_1 = 4\mu_1 - 1$, $x_2 = 1 - 2\mu_1$, and $x_3 = 6\mu_1 - 1$. Therefore,

$$y(\underline{x}_1) = 2(4\mu_1 - 1)^2 + (1 - 2\mu_1)^2 + 3(6\mu_1 - 1)^2$$

$$= 2(16\mu_1^2 - 8\mu_1 + 1) + (1 - 4\mu_1 + 4\mu_1^2)^2 + 3(36\mu_1^2 - 12\mu_1 + 1)$$

$$= 32\mu_1^2 - 16\mu_1 + 2 + 1 - 4\mu_1 + 4\mu_1^2 + 108\mu_1^2 - 36\mu_1 + 3$$

$$= 144\mu_1^2 - 56\mu_1 + 6.$$

Differentiating with respect to μ_1 and equating the result to zero yields $288\mu_1 - 56 = 0$. Solving for μ_1, $\mu_1 = \frac{56}{288} \doteq 0.1944$. This value of μ_1 minimizes $y(\underline{x}_1)$ since the second derivative of $y(\underline{x}_1)$ with respect to μ_1 is positive for $\mu_1 \doteq 0.1944$.

Step 5. Since $\mu_1 = 0.1944$, the solution point $\underline{x}_1 = \begin{bmatrix} 4\mu_1 - 1 \\ 1 - 2\mu_1 \\ 6\mu_1 - 1 \end{bmatrix}$

is given by

$$\underline{x}_1 = \begin{bmatrix} -0.2224 \\ 0.6112 \\ 0.1664 \end{bmatrix}, \text{ and } \underline{\Delta x}_1 = \begin{bmatrix} .7776 \\ -.3888 \\ 1.1664 \end{bmatrix}.$$

Step 6. Calculate $\underline{\nabla}y(\underline{x}_1)$. At $\underline{x}_1 = \begin{bmatrix} -0.2224 \\ 0.6112 \\ 0.1664 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, the gradient

for the first iteration is given by $\underline{\nabla}y(\underline{x}_1) = [-.8896, 1.2224, 0.9984]^T$. Since $\_y(\underline{x}_1) \neq 0$, the optimal solution has not been found. Go to Step 7.

Step 7. Replace $\underline{x}_o$ by $\underline{x}_1$. Replace $\underline{H}_o$ by $\underline{H}_1$, where $\underline{H}_1 = \underline{H}_o + \underline{A}_1 + \underline{B}_1$. Calculate $\underline{A}_1$ and $\underline{B}_1$ by applying the following relations:

$$\underline{A}_1 = \frac{\underline{\Delta x}_1 \ \underline{\Delta x}_1^T}{\underline{\Delta x}_1^T \ \underline{g}_1};$$

$$\underline{g}_1 = \underline{\nabla}y(\underline{x}_1) - \underline{\nabla}y(\underline{x}_o);$$

$$\underline{B}_1 = -\frac{\underline{H}_o \underline{g}_1 \underline{g}_1^T \ \underline{H}_o^T}{\underline{g}_1^T \ \underline{H}_o \ \underline{g}_1}$$

(a) Since $\underline{g}_1$ is required for both $\underline{A}_1$ and $\underline{B}_1$, determine $\underline{g}_1$ first. As a means of simplifying the calculations, the values of x_1, x_2, and x_3 in $\underline{x}_1$ have been rounded to the nearest tenth; i.e., $\underline{x}_1^T = (-0.2, 0.6, 0.2)$.

$$\underline{g}_1 - \underline{\nabla}y(\underline{x}_1)^T - \underline{\nabla}y(\underline{x}_o)^T = (-.9, 1.2, 1.0)^T - (-4, 2, -6)^T$$

$$= (-.9 + 4, 1.2 - 2, 1.0 + 6)^T$$

$$= (3.1, -.8, 7)^T.$$

(b) Since $\underline{\Delta x}_1 = \begin{bmatrix} .7776 \\ -.3888 \\ 1.1664 \end{bmatrix} = \begin{bmatrix} .8 \\ -.4 \\ 1.2 \end{bmatrix}$ and $\underline{g}_1 = [3.1, -.8, 7]$,

$$\underline{A}_1 = \frac{\begin{bmatrix} .8 \\ -.4 \\ 1.2 \end{bmatrix} [.8, -.4, 1.2]}{[.8, -.4, 1.2] \begin{bmatrix} 3.1 \\ -.8 \\ 7 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} .64 & -.32 & .96 \\ -.32 & .16 & -.48 \\ .96 & -.48 & 1.44 \end{bmatrix}}{11.2}$$

$$= \begin{bmatrix} .06 & -.03 & .09 \\ -.03 & .01 & -.04 \\ .09 & -.04 & .13 \end{bmatrix}.$$

(c) Since $\underline{H}_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\underline{g}_1 = \begin{bmatrix} 3.1 \\ -.8 \\ 7 \end{bmatrix}$,

$$\underline{B}_1 = -\frac{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.1 \\ -.8 \\ 7 \end{bmatrix} [3.1, -.8, 7] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}{[3.1, -.8, 7] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.1 \\ -.8 \\ 7 \end{bmatrix}}$$

$$= -\frac{\begin{bmatrix} 3.1 \\ -.8 \\ 7 \end{bmatrix} [3.1, -.8, 7]}{[3.1, -.8, 7] \begin{bmatrix} 3.1 \\ -.8 \\ 7 \end{bmatrix}}$$

$$= - \frac{\begin{bmatrix} 9.61 & -2.48 & 21.7 \\ -2.48 & .64 & -5.6 \\ 21.7 & -5.6 & 49 \end{bmatrix}}{59.25}$$

$$= - \begin{bmatrix} .16 & -.04 & .37 \\ -.04 & .01 & -.09 \\ .37 & -.09 & .83 \end{bmatrix}$$

$$= \begin{bmatrix} -.16 & .04 & -.37 \\ .04 & -.01 & .09 \\ -.37 & .09 & -.83 \end{bmatrix}$$

Combining these results as required, $\underline{H}_1 = \underline{H}_o + \underline{A}_1 + \underline{B}_1$,

$$\underline{H}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} .06 & -.03 & .09 \\ -.03 & .01 & -.04 \\ .09 & -.04 & .13 \end{bmatrix} + \begin{bmatrix} -.16 & .04 & -.37 \\ .04 & -.01 & .09 \\ -.37 & .09 & -.83 \end{bmatrix}$$

$$= \begin{bmatrix} .90 & .01 & -.28 \\ .01 & 1.00 & .05 \\ -.28 & .05 & .30 \end{bmatrix}.$$

Thus, the new positive definite Hessian matrix has been determined. This matrix can be utilized with $\underline{x}_1$ to determine $\underline{x}_2$ by reapplying Steps 1 - 6.

(b) Maximization technique. The problem to be maximized is defined by the quadratic form

$$y = y_o + \underline{p}^T\underline{x} + \underline{x}^T\underline{P}\underline{x}.$$

As in the minimization technique, the use of the deflected gradient defines the move from point $\underline{x}_{k-1}$ to the optimal point $\underline{x}^*$ in terms of the rarely known matrix $\underline{Q}$ and the gradient $\underline{\nabla}y(\underline{x}_{k-1})$; i.e.,

$$\underline{\Delta x}^* = (\underline{x}^* - \underline{x}_{k-1}) = -\underline{Q}^{-1}\underline{\nabla}y(\underline{x}_{k-1}).$$

Since $\underline{Q}$ is rarely known, this step is made by defining it in the form

$$\underline{x}_k - \underline{x}_{k-1} = \underline{\Delta x}_k = -\mu_k \underline{H}_{k-1} \underline{\nabla} y(\underline{x}_{k-1}),$$

where $\underline{H}_{k-1}$ is a suitably chosen negative definite matrix. The value of μ_k is the value of the search parameter which optimizes $y(\underline{\Delta x}_k)$ along the line of search defined by

$$\underline{\Delta x}_k = \underline{x}_k - \underline{x}_{k-1}.$$

If the direction of the optimum solution is known, the matrix $\underline{H}_{k-1}$ should be selected accordingly. If the direction of the optimum solution is not known and no information is available which might indicate the direction, the negative of the N x N identity matrix can be used as an estimate of $\underline{H}_{k-1}$.

Reference to the description of functional minimization by deflected gradients reveals that the maximization procedure is basically the same. The differences to be noted are two: (1) the matrix $\underline{H}_{k-1}$ is approximated by the N x N negative identity matrix, $-\underline{I}$; (2) the N^{th} Hessian matrix is negative definite instead of positive definite. With these considerations in mind, the following algorithm can be used to maximize a given quadratic function by the technique of deflected gradients.

Algorithm 4.10 (maximization by deflected gradients).--Step 1. Determine the initial solution $\underline{x}_o$. Determine the initial negative definite Hessian matrix $\underline{H}_o$ by defining $\underline{H}_o \equiv -I_N$.

Step 2. Determine the gradient vector $\underline{\nabla} y$, where $\underline{\nabla} y = (\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_N})^T$. Using the solution point, determine $\underline{\nabla} y(\underline{x}_n)$ by evaluating $\underline{\nabla} y$ at the solution point. For $\underline{x}_o$, this will be $\underline{\nabla} y(\underline{x}_o)$.

Step 3. Determine $\underline{\Delta x}_1$. Utilizing the relation $\underline{\Delta x}_1 = x_1 - x_o = -\mu_1 \underline{H}_o \underline{\nabla y}(\underline{x}_o)$, the new trial point $\underline{x}_1$ can be written as a function of the search parameter μ_1.

Step 4. Determine the value of μ_1 by maximizing $y(\underline{x}_1)$ with respect to μ_1.

Step 5. Substitute the value of μ_1 from Step 4 into the relation defining $\underline{x}_1$:

$$\underline{x}_1 = \underline{x}_o - \mu_1 \underline{H}_o \underline{\nabla y}(\underline{x}_o).$$

Step 6. Calculate $\underline{\nabla y}(\underline{x}_1)$. If $\underline{\nabla y}(\underline{x}_1) = 0$, terminate the process. The solution defined by $\underline{x}_1$ is the optimal solution. If $\underline{\nabla y}(\underline{x}_1) \neq 0$, go to Step 7.

Step 7. Replace $\underline{x}_o$ by $\underline{x}_1$. Replace $\underline{H}_o$ by $\underline{H}_1$, where $\underline{H}_1 = \underline{H}_o + \underline{A}_1 + \underline{B}_1$. The relation defining $\underline{A}_n$ $(n = 1, 2, \ldots, N)$ is given by

$$\underline{A}_n = \frac{\underline{\Delta x}_n \, \underline{\Delta x}_n^T}{\underline{\Delta x}_n^T \, \underline{g}_n}$$

where

$$\underline{g}_n = \underline{\nabla y}(\underline{x}_n) - \underline{\nabla y}(\underline{x}_{n-1}).$$

The relation defining $\underline{B}_n$ $(n = 1, 2, \ldots, N)$ is given by

$$\underline{B}_n = - \frac{\underline{H}_{n-1} \underline{g}_n \underline{g}_n^T \, \underline{H}_{n-1}^T}{\underline{g}_n^T \, \underline{H}_{n-1} \underline{g}_n} .$$

Increase all subscripts by one for each additional iteration and repeat Steps 1 - 7 until the gradient at the N^{th} iteration is suitably small.

The computations necessary for maximizing a given function are identical to those required for minimizing a given function. As noted, the only significant changes in the computational process are the use of a negative definite Hessian matrix for functional maximization and the use of $-\underline{I}_N$ for the estimate of the initial negative definite matrix $\underline{H}_o$.

Parallel tangents: The method of parallel tangents, the third general category of direct search techniques, is both a climbing technique and an elimination technique. It has been used to solve a variety of multivariable maximization-minimization problems and consists of two basic techniques: (1) gradient partan and (2) general partan. Whereas gradient partan uses the gradient to define directional moves, general partan does not. Of the two techniques, gradient partan is considered the most advantageous.[53]

(1) General partan. The general partan procedure can be described in terms of both two-space and three-space. The description of two-space general partan will follow that given by Wilde.[54] The description of three-space general partan will follow that given by Wilde and Beightler.[55]

(a) Two-space. Let $\underline{P}_o$ be any starting solution on the contours of Figure 4.15. Let $\underline{x}^*$ be the optimal solution. Let π_o be the line tangent to the contour surface at $\underline{P}_o$. Let $\underline{P}_2$ be an arbitrary point and $\underline{P}_3$ any tangent to the inner contour. Given the contour tangent π_o, any line (π_1) parallel to π_o is suitable. This parallel line is then explored for its highest point. For elliptical contours, this high point will be in a colinear relationship with $\underline{P}_o$ and the optimum point $\underline{P}_4$. The search for the optimum solution is conducted along the line connecting $\underline{P}_3$ and $\underline{P}_o$. Any one dimensional search technique will suffice.

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[53] Wilde, op. cit., p. 135.

[54] Ibid., pp. 130-133.

[55] Wilde and Beightler, op. cit., pp. 323-325.

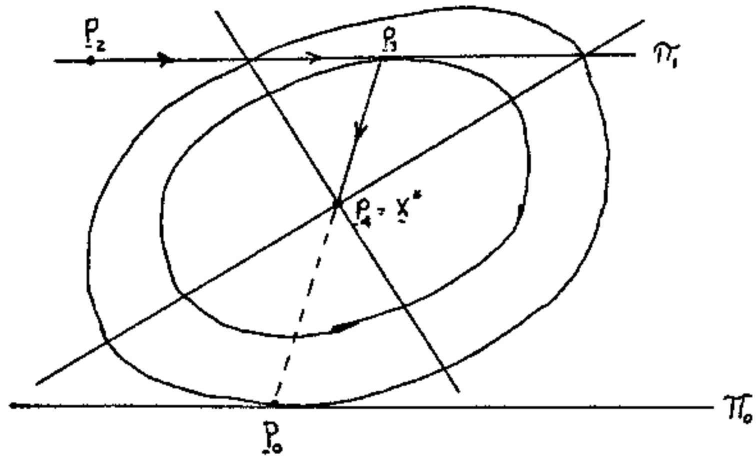

Fig. 4.15--Two-space general partan

(b) Three-space. Let $\underline{P}_0$ be any starting solution in the space shown in Figure 4.16. Let $\underline{x}^*$ be the optimal solution. Let π_0 be the tangent plane at $\underline{P}_0$. Let $\underline{P}_2$ be the high point on any line from $\underline{P}_0$ where

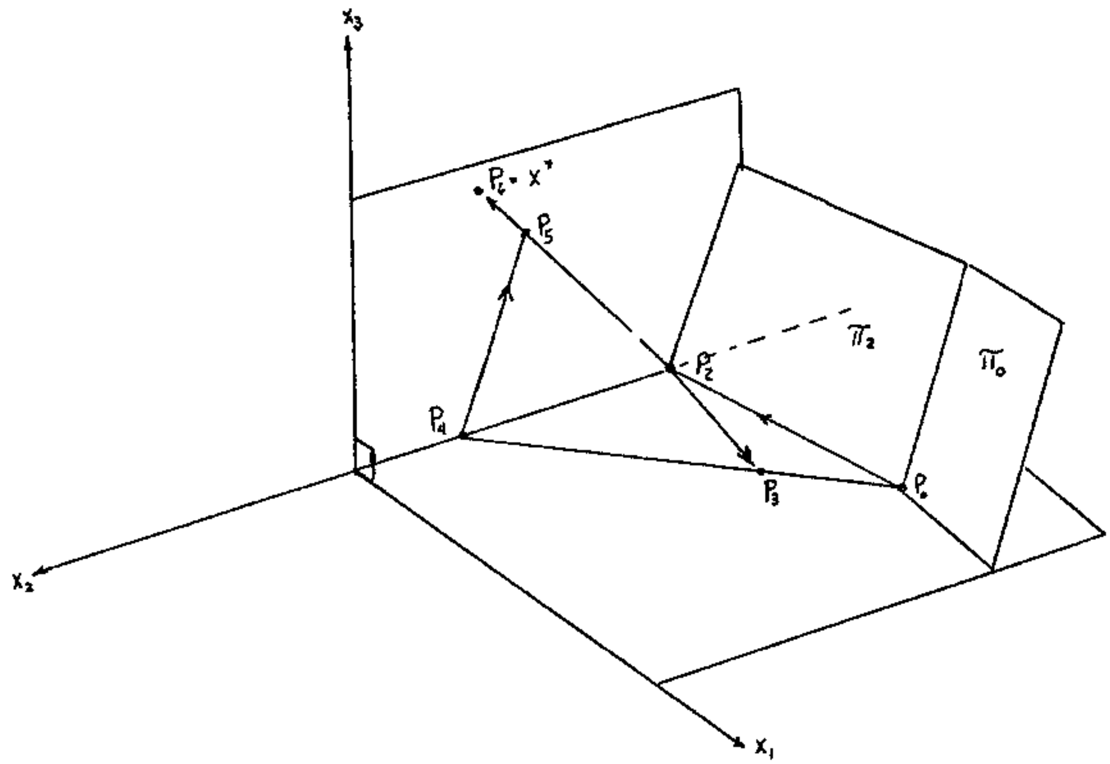

Fig. 4.16--Three-space general partan

the line is not contained in the tanget plane π_o. Let $\underline{P}_3$ be the high point on any line from $\underline{P}_2$ parallel to the tangent plane π_o, but not contained in the tangent plane π_2. Construct a line passing through $\underline{P}_o$ and $\underline{P}_3$. Let $\underline{P}_4$ be the summit of the line passing through $\underline{P}_o$ and $\underline{P}_3$. Construct a line parallel to the intersection of π_o and π_2. This line will be unique. Let $\underline{P}_5$ be its summit. Construct a line through $\underline{P}_2$ and $\underline{P}_5$. Let $\underline{P}_6$ be the high point of the line passing through $\underline{P}_2$ and $\underline{P}_5$. The point defined by $\underline{P}_6$ will be at the center of the system of three-dimensional ellipsoids and will be equal to the optimal solution.[56] As in the two-dimensional case, the search along this final line can be accomplished by univariate search.

The study of parallel tangents to this point has been descriptive. Since the gradient partan technique is a more suitable optimization technique, the computational aspects of parallel tangents will be demonstrated in that discussion.

(2) Gradient partan. As in the discussion for general partan, gradient partan will be described in both two-space and three-space. Following these descriptions, a general computational algorithm for gradient partan will be given. This computational algorithm will then be demonstrated by a numerical example.

(a) Two-space. The functions to which gradient partan is generally applied are described as elliptical contours. (See Figure 4.17.) The optimum solution is located by a series of one-dimensional searches along the line of steepest ascent or steepest descent. The direction of the search is dictated by the gradient vector.

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[56] _Ibid._, pp. 325-326.

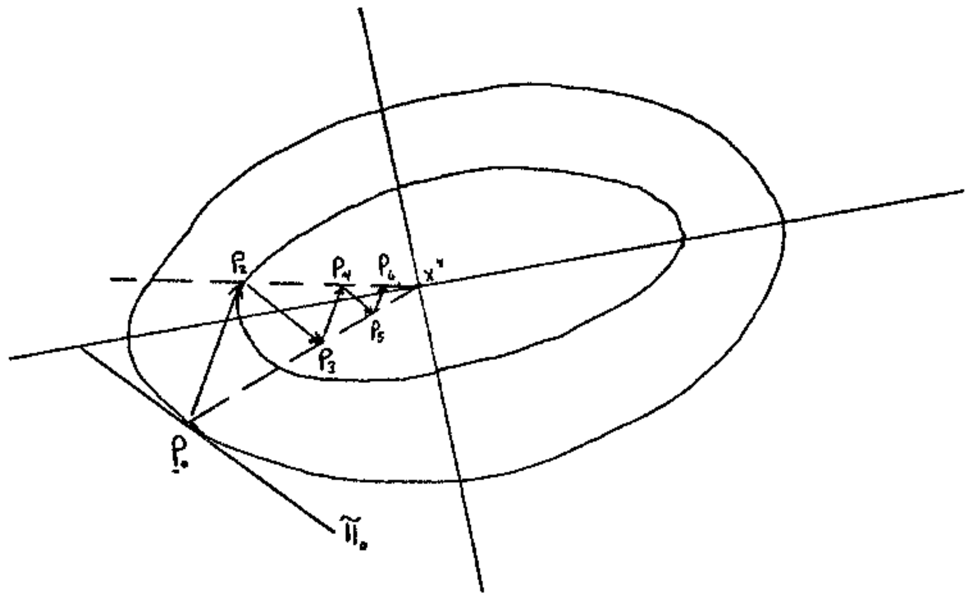

Fig. 4.17--Two-space gradient partan

Let $\underline{P}_o$ be the initial starting point and π_o the contour tangent at $\underline{P}_o$. Application of the gradient search will produce the zig-zag course shown in passing from $\underline{P}_o$ to $\underline{P}_2$ to $\underline{P}_3$ to $\underline{P}_4$ to $\underline{P}_5$ to $\underline{P}_6$ to the optimum $\underline{x}^*$. This zig-zag pattern is bounded by two lines which intersect at the optimum solution. This property permits the search for $\underline{x}^*$ to be concluded after three one-dimensional searches: (1) along the gradient from $\underline{P}_o$ to $\underline{P}_2$, (2) along the gradient from $\underline{P}_2$ to $\underline{P}_3$, and then (3) from $\underline{P}_3$ along the line passing through $\underline{P}_o$ and $\underline{P}_3$.

(b) Three-space. Gradient partan in three-space first locates a plane containing the center of the three-dimensional ellipsoids. General partan is then applied to locate the center in as accurate a manner as possible. As in the case for general partan, the optimum solution is located in six iterations. (See Figure 4.18.)

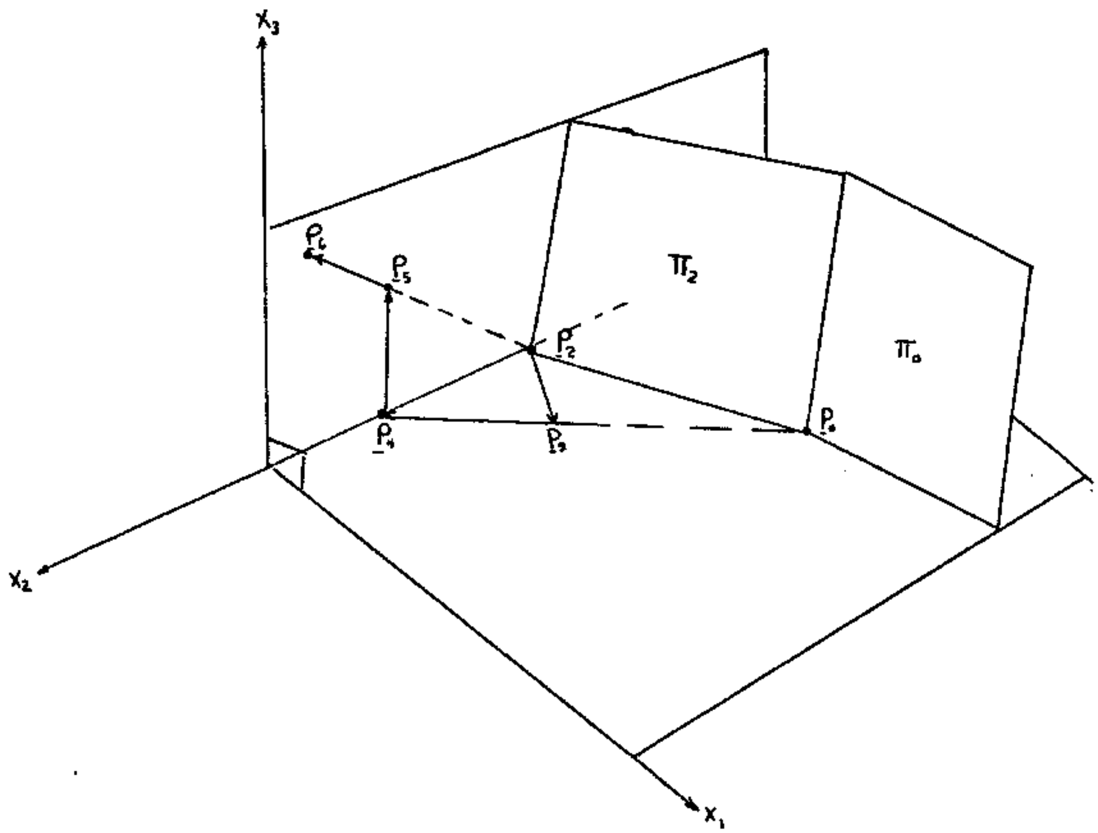

Fig. 4.18--Three-space gradient partan

Let $\underline{P}_o$ be any starting solution and π_o the tangent plane at $\underline{P}_o$. Let $\underline{P}_2$ be the high point of the line determined by the gradient vector at $\underline{P}_o$. Let π_2 be the tangent plane at $\underline{P}_2$. Successive application of the gradient vector produces the zig-zag pattern shown in moving from $\underline{P}_o$ to $\underline{P}_2$ to $\underline{P}_3$ to $\underline{P}_4$ to $\underline{P}_5$. If a one dimensional search is conducted along the line passing through $\underline{P}_2$ and $\underline{P}_5$, the search will terminate at the point $\underline{P}_6$. This terminating point is the optimum solution.

It is easily seen that the gradient partan technique closely follows that of general partan. However, unlike general partan, the lines of search are dictated by the gradient. This is the basic difference between the two partan techniques.

The discussion of gradient partan is summarized by Algorithm 4.11. The construction of this algorithm is an outgrowth of this study and summarizes the presentation of gradient partan.

Algorithm 4.11 (gradient partan).--Step 1. Select any arbitrary starting solution. Denote this initial solution by $\underline{P}_o$.

Step 2. Determine the equation of the tangent plane at $\underline{P}_o$. The equation of the tangent plane at any point $\underline{a}$ is defined by

$$\sum_{i=1}^{N} m_i(x_i - a_i) = 0,$$

where $m_i = \frac{\partial y}{\partial x_i}$, $(i = 1, 2, \ldots, N)$, evaluated at $\underline{a}$.

Step 3. Determine the high point for the gradient line at the current solution. This is accomplished by defining each variable x_i $(i = 1, 2, \ldots, N)$ in terms of the common parameter ρ, expressing the objective function in terms of ρ, and then solving for that value of ρ for which $\frac{d}{d\rho} Y(\rho) = 0$. The necessary transformation is defined by

$$m_i\rho = x_i - a_i \quad (i = 1, 2, \ldots, N).$$

This transformation can be written as

$$x_i = m_i\rho + a_i \quad (i = 1, 2, \ldots, N).$$

Step 4. Determine $\underline{P}_2$, the next solution point. The coordinates of $\underline{P}_2$ are obtained by substituting the optimal value of ρ into

$$x_i = m_i\rho + a_i \quad (i = 1, 2, \ldots, N).$$

Step 5. Determine the gradient at $\underline{P}_2$. If $\underline{\nabla}y(\underline{P}_2) = 0$, the process terminates and $\underline{P}_2$ is the optimal solution. If $\underline{\nabla}y(\underline{P}_2) \neq 0$, replace $\underline{P}_o$ by $\underline{P}_2$ and reapply Steps 1 - 5. Increase the subscript of $\underline{P}$ by one to indicate the new solution defined in Step 4. Repeat as needed.

This algorithm will be demonstrated by performing one iteration on the function

$$y(x_1, x_2, x_3) = -2x_1^2 - x_2^2 - x_3^2.$$

This function is to be minimized.[57]

Iteration I: Step 1. Let $\underline{P}_o$ be any arbitrary starting solution. For convenience, let $\underline{P}_o = (-1, 1, -1)$.

Step 2. Determine the equation of the tangent plane at $\underline{P}_o$. The equation of the tangent plane at any point $\underline{a}$ is defined by

$$m_1(x_1 - a_1) + m_2(x_2 - a_2) + m_3(x_3 - a_3) = 0,$$

where $m_i = \frac{\partial y}{\partial x_i}$ (i = 1, 2, 3) evaluated at $\underline{a}$. Thus,

$$m_1 = \frac{\partial y}{\partial x_1} = -4x_1;$$

$$m_2 = \frac{\partial y}{\partial x_2} = -2x_2;$$

$$m_3 = \frac{\partial y}{\partial x_3} = -2x_3.$$

At $\underline{P}_o = (-1, 1 -1)$, $m_1 = 4$, $m_2 = -2$, and $m_3 = 2$. Therefore, the equation of the tangent plane is

$$4(x_1 + 1) - 2(x_2 - 1) + 2(x_3 + 1) = 0.$$

Step 3. Determine the high point for the gradient line at the current solution. This is accomplished by defining each variable x_i (i = 1, 2, 3) in terms of the common parameter ρ, expressing the objective function in terms of ρ, and then solving for the value of ρ for which $\frac{dy(\rho)}{d\rho} = 0$. The necessary transformation is defined by

$$m_i\rho = x_i - a_i \quad (i = 1, 2, \ldots, N),$$

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[57]This problem was taken from <u>ibid.</u>, pp. 327-330. The selection was based upon the need of a solved problem to provide a check on the validity of the algorithm.

where m_i equals $\frac{\partial y}{\partial x_i}$ evaluated at the current solution. Therefore,

$$4\rho = x_1 + 1;$$

$$-2\rho = x_2 - 1;$$

$$2\rho = x_3 + 1.$$

Solving for x_1, x_2, and x_3, respectively,

$$x_1 = 4\rho - 1;$$

$$x_2 = -2\rho + 1;$$

$$x_3 = 2\rho - 1.$$

(a) As a function of ρ, the objective function is given by

$$y(\rho) = -2(4\rho - 1)^2 - (-2\rho + 1)^2 - (2\rho - 1)^2.$$

(b) The derivative of $y(\rho)$, $\frac{dy(\rho)}{d\rho}$, is given by

$$\frac{dy(\rho)}{d\rho} = -4(4\rho - 1)(4) - 2(-2\rho + 1)(-2) - 2(2\rho - 1)(2)$$

$$= -16(4\rho - 1) + 4(-2\rho + 1) - 4(2\rho - 1)$$

$$= -64\rho + 16 - 8\rho + 4 - 8\rho + 4$$

$$= -80\rho + 24.$$

Setting $\frac{d}{d\rho} y(\rho) = 0$ and solving, $\rho^* = 0.3$.

Step 4. Determine $\underline{P}_2$. The coordinates of $\underline{P}_2$ are obtained by substituting the optimal value of ρ into the transformations defined by

$$m_i\rho = x_i - a_i \quad (i = 1, 2, \ldots, N).$$

For the current problem, $\rho^* = 0.3$, and

$$x_1 = 4\rho - 1 = 0.2;$$

$$x_2 = -2\rho + 1 = 0.4;$$

$$x_3 = 2\rho - 1 = -0.4.$$

The second trial point, $\underline{P}_2$, is given by $\underline{P}_2 = (0.2, 0.4, -0.4)$.

Step 5. Determine the gradient at $\underline{P}_2$. If $\underline{\nabla}y(\underline{P}_2) = 0$, the process terminates and $\underline{P}_2$ is the optimal solution. If $\underline{\nabla}y(\underline{P}_2) \neq 0$, replace $\underline{P}_o$ by $\underline{P}_2$ and reapply Steps 1 - 5. Repeat as needed.

$$\underline{\nabla}y(x_1, x_2, x_3) = \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial y}{\partial x_3}\right)$$
$$= (-4x_1, -2x_2, -2x_3).$$

At $\underline{P}_2 = (0.2, 0.4, -0.4)$, $\underline{\nabla}y(\underline{P}_2) = (-0.8, -0.8, +0.8) \neq 0$. The second iteration will find $\underline{P}_o$ replaced by $\underline{P}_2$.

From the discussion of technique and the illustrations, it is possible to derive some specific characteristics of direct search. The main characteristics are the following:

(1) direct search techniques yield numerical answers rather than analytical solutions;

(2) direct techniques are basically iterative search techniques;

(3) direct search can be used to optimize functions whose structure is not known but is being explored in a step by step manner; and,

(4) direct search techniques take on the character of an equation search and are useful for fitting of surfaces.

In addition, the general nature of the problems to which direct search is applied can be categorized by the following:

(1) the functions are continuous and differentiable;

(2) nonlinearity is readily amenable to direct search;

(3) the functions are such that the problem can be described in terms of one or n variables;

(4) although the availability of the functional expression simplifies the work, it is not a necessity since a response surface can be utilized to approximate the objective function; and,

(5) functional expressions for constraints are not required.

Applications of Advanced Optimal Search

The incorporation of indirect and direct search techniques into the problem solving activity provides a more flexible approach to problem formulations than previously available. If a given problem is formulated in such a way that the techniques of classical optimization theory are not applicable, the problem can be examined for possible application of one of the basic techniques of modern optimization theory. If the problem is such that it cannot be classified as one amenable to the solution techniques of basic optimal search, then the problem can be examined for applicability of one of the techniques of advanced optimal search. This process typifies the contribution of modern optimization theory to administrative analysis. That is, the problem is formulated in a manner which best describes the real situation. Given this formulation, the problem is correlated with a suitable technique for final solution. Modern optimization provides techniques for solving more complex formulations than that allowed by classical optimization theory.

Indirect Search

The use of indirect search in applied analysis extends the applicability of the classical max-min calculus and its use in optimizing continuous, differentiable functions. However, unlike the applications shown in Chapter II and Chapter III, indirect search does not restrict the mathematical formulation of the problem to a particular type. Total nonlinearity is permitted in both the objective function and the set of constraint functions. All that is required is that the mathematical formulation be continuous, differentiable and suited to iterative techniques.

Administrative problems amenable to the techniques of indirect search include the analysis of production functions, the study of pricing decisions, cost analysis, profit analysis, inventory analysis, and investment analysis. The specific technique which is used to solve a given problem is determined by the manner in which the problem is mathematically formulated. For example, an unconstrained, continuous and differentiable nonlinear function defined in terms of n variables can be effectively optimized by utilizing the Hessian matrix to identify the relative optima associated with the points at which the system of first partial derivatives vanishes. This system can be solved by the Newton-Raphson technique for functions of n variables. If this function is to be optimized subject to a set of constraints, the concepts of the constrained derivative can be utilized to locate the optimal solution. The use of the constrained derivative incorporates the Jacobian. The solution to the problem is obtained following a series of iterations which were begun by arbitrarily assigning values to a set of state and decision variables.

This example identifies several points relative to the applicability of indirect search in administrative analysis. First, the techniques require continuity and differentiability. Second, the particular technique employed to locate the optimum solution is determined by the manner in which the mathematical expression is written; i.e., with or without constraints. Third, the Hessian matrix, in the univariable case, is used to identify the relative optima associated with a given point. Fourth, the Jacobian, in the multivariable case, is used as a means of evaluating the appropriate constrained derivatives at a given solution point. Fifth, the Newton-Raphson formulas are used as a means of solving functions (generally differential functions) or solving the simultaneous, homogeneous systems defined by the first partial derivatives. These solutions are then used in conjunction with some other technique, such as the Hessian matrix, to determine the type of optima they define.

It is apparent that indirect search is applicable to the same class of administrative problems as both classical optimization theory and basic optimal search. However, the applicability of indirect search to highly nonlinear functions permits greater flexibility in the construction of the mathematical formulation representing the real problem. In this sense the techniques of indirect search extend the applicability of both classical optimization and basic optimal serach. Although the solution techniques have changed, the areas of application remain the same.

The need for nonlinear representations has been noted by various writers. In a study of allocation problems, M. H. J. Webb has shown that under suitable circumstances the costs to be minimized are best represented by nonlinear functions.

> Mathematical methods of locating depots utilize simple functions of delivery data, e.g., weight and distance from the depot, to measure the "cost"; the total "cost" is minimized to find the depot location.
>
> . . .the cost of a delivery is influenced by the occurrence of other deliveries. . .simple functions of the delivery data are not always good measures of

> variable cost. . .the minimum points of the simple functions rarely coincide with the point of minimum variable cost.
>
> Published road transport operating cost information supports the general assertion that transport cost is nonlinear with regard to both distance and quantity.[58]

In such a nonlinear case, the use of linear programming does not provide a suitable representation of reality. However, the use of the constrained derivative concept of indirect search provides a means of solving the problem when the nonlinear nature of the problem is incorporated into the problem formulation. The constraints can be written as linear or nonlinear equalities or inequalities, whichever best describe the real situation.

As a means of further validating the applicability of indirect search, specific examples of nonlinear administrative problem areas will be identified. Each of these problems could be solved by a suitable technique of indirect search. These problems are described below.

Multi-item order quantities.--The typical problem involves a distribution source where current inventory is maintained by ordering all items at one time from a single supplier. This practice enables the distribution source to take advantage of economies in shipping costs, paper work, and quantity discounts.[59] The objective is to minimize total cost. The total cost function may or may not be constrained. Typical constraints include storage space, demand, and minimum level order requirements.

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[58] M. H. J. Webb, "Cost Functions in the Location of Depots for Multiple-Delivery Journeys," Operational Research Quarterly, XIX (September, 1968), p. 311.

[59] Wagner, op. cit., p. 516.

Safety-stock inventory levels.--Multiperiod corporate planning models usually provide for inventory control based upon a given safety-stock level. In providing for this safety-stock, it is possible for the safety-stock level to be a function of both the forecasted sales level and the fraction of capacity utilization implied by this forecast. Such a problem can be formulated in the following manner. Let x equal the forecasted mean weekly sales and b the weekly capacity available for production. Let n equal the number of weeks' sales which depend on the capacity utilization factor x/b. The required safety stock level is equal to nx. Let $n = d + m(x/b)$ be the regression function for determining n as a function of capacity utilization. The safety stock level, nx, is then given by the quadratic function $nx = dx + (m/b)x^2$. This safety-stock level may appear in the constraints as well as the function to be optimized.[60]

Economic production lot size.--The economic order quantity problem is one in which costs are to be minimized subject to such constraints as the amount of space available, the availability of men and machines, and the number of hours available for setup. Multiproduct, nonlinear cost functions can be formulated. Unlike the problems of classical optimization, the constraints can be nonlinear and of such a nature that interdependent relationships exist.

Cost-profit analysis.--Cost-profit analysis has been identified as an area of administrative activity that is readily suited to nonlinear

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[60] Ibid.

representation.[61] If the function to be optimized is unconstrained, continuous, and differentiable, the differential approach can be used to locate the optimal solution. If the function to be optimized is continuous, differentiable, and constrained, the constrained derivative can be used to locate the optimal solution. The cost or profit involved may or may not be dollars. For example, if the problem under investigation is one of minimizing total time on a project, the "cost" identified in the objective function is time. Constraints, if they exist, can be deadlines, dollar expenses, etc.

Production activity analysis.--Clough points out that in practice production activity analysis generally results in situations where "the objective function or the constraints or both might be nonlinear."[62] Such problems can be written in the form

$$\text{optimize } f(x_1, x_2, \ldots, x_N)$$

subject to the constraints

$$g_i(x_1, x_2, \ldots, x_N) = b_i = 0, \quad i = 1, 2, \ldots, K,$$

and

$$x_j \geq 0, \quad j = 1, 2, \ldots, N.$$

The function to be optimized may be one of profit maximization or cost minimization, where the x_j $(j = 1, 2, \ldots, N)$ denote the various products produced. The end result is the optimal combination for the given production problem.

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[61] Baumol, op. cit., 2nd ed., pp. 138-141.

[62] Donald Clough, Concepts in Management Science (Englewood Cliffs, 1963), p. 165.

Inventory control.--The inventory problem is one in which the total cost of inventory is to be minimized. This problem is typical of cost minimization problems. It can be written as a univariable function or a function with n variables. Constraints, if they exist, can be linear or nonlinear. (This problem, in the multivariable, constrained case, was identified in Chapter III as one solvable by geometric programming.) The problem is given by the mathematical expression

$$\min G(n_i) = K + \sum_{i=1}^{p} [\frac{B_i}{2} n_i + A_i(n_i)^{-1}]$$

subject to

$$H_j(n_i) \leq M_j, \quad (i = 1, 2,\ldots,p;\ j = 1, 2,\ldots,m).$$

In this formulation, $B_i = \alpha \ell_i c_i$, where c_i = unit cost of raw material and labor for product i, ℓ_i = monthly sales of product i, and α = monthly cost of inventory. The value of K is defined by

$$K = \sum_{i=1}^{p} [\ell_i c_i + \frac{\alpha \ell_i \gamma_i}{2}],$$

where c_i, ℓ_i, and α are as before; γ_i = all costs relating to one lot of product i. The value of A_i is determined by $A_i = \ell_i \gamma_i$. The objective is to find the optimal value of n_i for i = 1, 2,...,p which minimizes the cost function subject to the restrictions imposed by the m constraints.

Kaufman utilizes the Lagrange multiplier in his analysis of this class of problems. He comments that "there is no known general method for the case in which there are three or more variables."[63] Since this statement was made, the technique of the constrained derivative was developed. If

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[63]Arnold Kaufman, Methods and Models of Operations Research Englewood Cliffs, 1963), p. 412.

the function is unconstrained, the differential approach, used in conjunction with the Newton-Raphson technique, can be implemented.

From this limited listing of areas of applicability, it is evident that indirect search can be applied to a variety of administrative problems. Although these areas of applicability tend to overlap those of both classical optimization and basic optimal search, indirect search provides the techniques for solving problems ill-suited to these latter techniques. In addition, indirect search provides the tools for working with problems that, because of improved formulations, better fit the real problem being investigated.

Indirect search, as such, has not been applied to problems of an administrative nature. A survey of the literature has shown that the primary application of indirect search has been in the physical sciences and mathematical test problems. In particular, application has been made to such problems as the minimization of costs in engineering design[64] and the determination of minimal fuel requirements for orbital transfer.[65] An analysis of these known applications and of sample problems has produced a set of characteristics common to problem suited to analysis by indirect search.

(1) The functional expressions of both the objective function and the constraint set are known, and unimodality within the interval of uncertainty is assumed;

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[64] R. Schinzinger, "Optimization in Electromagnetic System Design," Recent Advances in Optimization Techniques, edited by Abraham Lavi and Thomas P. Vogl (New York, 1966), pp. 163-213.

[65] P. Kenneth and G. E. Taylor, "Solution of Variational Problems with Bounded Control Variables by Means of the Generalized Newton-Raphson Method," Lavi and Vogl, op. cit., pp. 471-483.

(2) the objective function is nonlinear and consists of N independent variables;

(3) the constraint set is described by a set of mixed linear and nonlinear equalities and/or inequalities;

(4) final solutions are achieved by a series of iterations, the iterations consisting of a suitable iterative technique such as the Newton-Raphson technique.

In correlating technique with application, an important point to note is that the selection of the method of solution is dictated by the manner in which the problem itself is written. If the function to be optimized is an unconstrained, continuous, univariable function, the differential technique can be used to determine the optimal point. Should the derivative be nonlinear, the Newton-Raphson technique can be utilized to determine critical points. If the function to be optimized is an unconstrained, continuous function in N variables, the partial derivative can be utilized to describe the system of equations which determines the critical points. This system can then be solved algebraically or by a suitable modification of the Newton-Raphson technique. If the constraints are not expressed as functions but simply limits within which the optimal solution set must lie, the Newton-Raphson technique can be used to find the optimum. The use of the partial derivative approach is facilitated by the utilization of the Hessian matrix for defining points of maxima or minima.

If the function to be optimized is a constrained, continuous nonlinear function with continuous linear and/or nonlinear constraints, the constrained derivative technique should be used to determine the optimal solution. This requires utilization of the Jacobian and the constrained derivatives as shown in the examples. The particular approach taken in optimizing the given function is dictated by the manner in which the constraints are formulated. Equality constraints are handled in a somewhat direct manner. Inequality constraints necessitate the use of slack variables and conditions of complementary slackness.

This discussion restates again the manner in which indirect search is utilized as a tool of modern optimization theory. In this it is readily seen that the decision to utilize one of the techniques of indirect search is made after the problem is formulated. After the problem is formulated, it can be examined to determine whether or not an indirect search technique is appropriate for locating the optimal solution.

Direct Search

The utilization of direct search in administrative analysis provides a means of optimizing an objective function even though its exact functional expression is not known. If the functional expression is known, the function may be discontinuous, hence nondifferentiable, or of such a form that the use of the calculus or any related technique is infeasible. In such situations, it is necessary to utilize an optimization technique that can locate the optimal solution in a direct manner.

Direct search has been identified as consisting of two basic categories, direct elimination and direct climbing. The elimination techniques are designed to reduce the size of the interval within which the optimal solution lies. Direct climbing uses local measurements as a means of indicating the direction in which a move should be made so that the optimal solution can be located.

Application of direct search presupposes some knowledge about the objective function and the values within which the optimal solution lies. It is also assumed that "given a specific set of values of the independent variables, one can compute the corresponding value of the objective function."[66]

Direct elimination.--Direct elimination is utilized as a means of reducing the size of the interval within which the optimal solution lies. When economic costs are a factor, the proper use of direct elimination can result in less trial points, hence, less cost.

As an example of the use that can be made of direct elimination, consider the following replacement problem.[67] A company finds that a piece of capital equipment, with initial purchase price of K dollars, has linear operating costs. These operating costs are linear with respect to time. If V_i equals the annual operating cost in dollars in the i^{th} year and i equals the age (in years) at the end of the i^{th} year, the operating cost can be written as $V_i = b + ci$, where b and c are known constants. Assuming that the salvage value of the machine equals its price on the used machine market, the resale price after n years if given by

$$\text{Resale price} = K(r)^n,$$

[66]Wilde and Beightler, op. cit., p. 215.

[67]The operational context of this problem is taken from Samuel B. Richmond, Operations Research for Management Decisions (New York, 1968), pp. 88-96.

where r equals the proportionate value of the initial cost which remains after the year's use. For example, if the value of the machine decreases by one-third of its residual value each year, r = 2/3. The total capital cost over n years, denoted C_n, is given by

$$C_n = K - K(r)^n.$$

The average annual cost over n years, denoted AC, is given by the sum of the mean of total operating cost V_i for n years and the mean total capital cost; i.e.,

$$AC = \frac{1}{n}\sum_{i=1}^{n} V_i + \frac{C_n}{n}$$

$$= \frac{1}{n}[\sum_{i=1}^{n} V_i + C_n].$$

Substituting for V_i and C_n,

$$AC = \frac{1}{n}[\sum_{i=1}^{n} (b + ci) + K - K(r)^n].$$

The objective in the replacement problem is to determine the value of n for which average annual cost over n years is minimized. This is equivalent to finding that value of n for which the derivative of the average annual cost function vanishes; i.e., find n such that

$$\frac{d}{dn}(AC) = 0.$$

For the given function,

$$\frac{d}{dn}(AC) = c\frac{d}{dn}[\frac{1}{n}\sum_{i=1}^{n} i] - \frac{K}{n^2} - \frac{K}{n}(r)^n \ln r + (r)^n(\frac{K}{n^2}).$$

Summing the arithmetic series defined by $\sum_{i=1}^{n} i$, the derivative function can be written as

$$\frac{d}{dn}(AC) = c\frac{d}{dn}\left[\frac{1}{n} \cdot \frac{n(n+1)}{2}\right] - \frac{K}{n^2} - \frac{K}{n}(r)^n \ln r + (r)^n\left(\frac{K}{n^2}\right).$$

Since $\frac{d}{dn}\left[\frac{1}{n} \cdot \frac{n(n+1)}{2}\right] = \frac{1}{2}$, the expression defining $\frac{d}{dn}(AC)$ reduces to

$$\frac{d}{dn}(AC) = \frac{c}{2} - \frac{K}{n^2} - \frac{K}{n}(r)^n \ln r + (r)^n\left(\frac{K}{n^2}\right).$$

It is necessary to equate $\frac{d}{dn}(AC)$ to zero and solve for n.

The optimal solution to this problem can be determined without resorting to an expression of the type defined by $\frac{d}{dn}(AC)$. In fact, the optimal solution can be determined by direct examination of the annual average cost function

$$AC = \frac{1}{n}\left[\sum_{i=1}^{n} (b + ci) + K - K(r)^n\right].$$

Since b, c, K, and r are known constants, AC can be evaluated for various values of n. These evaluations can be used to determine the direction in which future evaluations are to be made. For example, either interval elimination or the Bolzano technique can be used to search for the value of n for which average annual cost is minimized. Since the piece of equipment will have a known (or estimated) useful life, k, the starting solution could be the midpoint of the closed interval [0, k].

There are two points to note concerning the application of either interval elimination or the Bolzano technique to problems of this type. First, the use of interval elimination does not require the derivative of the objective function. Second, the use of the Bolzano technique does require the use of the derivative. Hence, differentiability of the objective function enters in as a factor to be considered when a solution technique is being chosen.

The method of contour tangents could be used to locate the value of n for which average annual cost is minimized. However, this requires the use of the derivative, a use that strict interval elimination avoids. As noted in the discussion of direct elimination techniques, the method of contour tangents is most useful when the objective function is differentiable and consists of more than one variable. Thus, if the replacement problem consisted of more than one capital asset, the problem would be to determine the optimal replacement time n_j for the j^{th} machine. The optimal replacement time is that value of n_j which maximizes the total average annual cost for m capital assets; i.e., determine n_j such that

$$\sum_{j=1}^{m} (AC)_j = \sum_{j=1}^{m} \frac{1}{n_j} [\sum_{i=1}^{n_j} (b_j + c_j i) + K_j - K_j (r_j)^{n_j}]$$

is minimized. For this function,

$$\frac{\partial}{\partial n_j} (AC) = \frac{c_j}{2} - \frac{K_j}{n_j^2} - \frac{K_j}{n_j} (r_j)^{n_j} \ln r + (r)^{n_j} \frac{K_j}{n_j},$$

where j denotes the j^{th} machine.

Problems in administrative analysis which are similar to this problem include the determining of the optimal time to sell an asset, replacement of a depreciating asset, group replacements, and investment analysis. As a means of formally indicating the applicability of direct elimination to problems in administrative analysis, a selected set of specific application areas is included in this study.

Investment analysis: The problem is to determine the optimal length of an investment. In this analysis an investment should be continued so long as the percentage yield (the percentage marginal yield, i.e., the percentage return per unit time change) exceeds the level of percentage return of some other alternative such as interest.[68] The following formulation, taken from Baumol,[69] will be used in demonstration.

Let V equal the value of the total product, where V is assumed to be a function of the amount invested, I, and the length of time, t, for which the investment runs; i.e.,

$$V = f(I, t).$$

Let P equal the anticipated profit at the date of the investment and equal the difference between the present value of the value of the total product, V, and the cost of the investment, I. If r equals the current interest rate, V will be discounted at the rate r for t periods. Hence,

$$P = Ve^{-rt} - I \equiv [f(I, t)]e^{-rt} - I.$$

The optimal value of t is that value which maximizes P. The techniques of direct search can be utilized by defining the total time period during which it is feasible for the investment to run. For the technique of interval elimination, the function

$$P = [f(I, t)]e^{-rt} - I$$

will be utilized in a direct manner. For the Bolzano technique or the

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[68]Given the level of investment, a rise in the interest rate must reduce the optimal length of a profit making point-input, point-output solution. (Baumol, _op. cit._, p. 429.)

[69]_Ibid._, p. 428.

method of contour tangents, the derivative, given by

$$\frac{\partial P}{\partial t} = e^{-rt} \frac{\partial}{\partial t} [f(I, t)] - re^{-rt}[f(I, t)]$$

$$= e^{-rt} \frac{\partial}{\partial t}[V] - re^{-rt}[V],$$

will be required.

Portfolio analysis: Miller and Starr consider the determining of the optimal portfolio in terms of the investor's indifference curves for expected return and variance.[70] The appropriate indifference curve is assumed to be equivalent to the investor's utility function for portfolios. This linear utility function is given by

$$U = r_p - sV_p,$$

where r_p equals the return on the p^{th} investment, V_p is the variance of the portfolio, and s is the parameter chosen by the investor as a quantitative measure of his aversion to risk.

Let w_j equal the proportion of the total investment placed in portfolio j. Let I equal the total investment. Let $\overline{r}_p$ equal the expected return on the portfolio. Let C_{ij} be the covariance between the i^{th} and j^{th} variables,

$$C_{ij} = \mathcal{E}[(x_i - \overline{x}_i)(x_j - \overline{x}_j)],$$

where the $\mathcal{E}$ means "expected value."

Consider the case for two possible investments. Then,

$$\overline{r}_p = w\overline{r}_1 + (1 - w)\overline{r}_2;$$

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[70]David W. Miller and Martin K. Starr, Executive Decisions and Operations Research, 2nd ed. (Englewood Cliffs, 1969), pp. 475-487.

$$V_p = w^2V_1 + (1 - w)^2V_2 + 2w(1 - w)C_{12}.$$

The investor's utility function,

$$U = r_p - sV_p,$$

can then be written as

$$U = w\bar{r}_1 + (1 - w)\bar{r}_2 - s[w^2V_1 + (1 - w)^2V_2 + 2w(1 - w)C_{12}].$$

The problem is to determine the value of w for which U is maximized.

Application of the derivative yields

$$\frac{dU}{dw} = \bar{r}_1 - \bar{r}_2 - 2swV_1 + 2sV_2 - 2wsV_2 - 2sC_{12} + 4swC_{12}.$$

Setting $\frac{dU}{dw} = 0$ and solving for w yields

$$w = \frac{\bar{r}_1 - \bar{r}_2 + 2s(V_2 - C_{12})}{2s(V_1 + V_2 - 2C_{12})} .$$

Since r_1, r_2, V_1, V_2, and C_{12} are known (or can be readily determined), the problem resolves into that of determining the optimal value of s. By defining the interval within which the value of s lies, interval search by one of the direct elimination techniques can be used to determine the value of w which maximizes the investor's utility function.

Inventory management: The problem to be solved is that of determining the optimal number of orders per year. This problem has been discussed by Goetz and involves a fixed-period, target-inventory plan.[71] Goetz indicates that the fixed-period, target-inventory problem is analogous to branch warehousing problems. The manner of solution is direct enumeration. The mathematical formulation of the fixed-period, target-inventory problem follows.

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[71] Billy E. Goetz, Quantitative Methods: A Survey and Guide for Managers (New York, 1965), pp. 375-382.

Let N equal the number of orders placed per year. Let S equal the total expected costs of stockout. Let $(\overline{b}C)$ equal the total expected costs of carrying buffer stocks per year. Let Q equal the annual sales in dollars. Let B equal the cost per order. Let C equal the annual cost per dollar in inventory. The problem is to determine N such that the total cost of the inventory policy, given by

$$N(B + \sum \hat{S}) + \frac{QC}{2N} + \sum(\overline{b}C),$$

is minimized. $\hat{S}$ equals the expected cost of stockout per cycle. The value of $\sum \hat{S}$ is given by

$$\sum \hat{S} = \frac{S_i}{k},$$

where k equals the ratio between expected daily sales (or average daily sales) for a given sales period and the aggregate daily sales.[72] S_i is the expected cost of stockout for the i^{th} cycle. The value of $\overline{b}C$ is given by the ratio

$$\frac{\text{buffer cost per year}}{k}.$$

Expected daily sales, k, equals the product of the expected units sold per day and the selling price per unit.

Since B, $\sum \hat{S}$, Q, C, and $\sum(bC)$ are known constants the inventory policy is a function of one variable, N. Given a relevant range of values for N, the technique of interval elimination can be used to reduce the number of iterations required to locate the N for which the cost of the inventory policy is minimal. For multiproduct functions, the method of contour

[72]Aggregate daily sales is calculated by dividing annual sales by the number of days worked in a year.

tangents can be utilized to locate the optimal value of each N_j, where N_j is the order quantity for the j^{th} product.

From this limited survey of applications, it is to be noted that each one is amenable to a series of numerical iterations. This is a characteristic of direct elimination. Each iteration is used to improve upon the preceding result. At each iteration the resulting solution is utilized to determine the most feasible portion of the remaining interval that is to be explored. In this analysis, the use of Fibonacci search or the method of golden section can be invaluable in minimizing the number of trial points.

Potential areas of applicability in administrative analysis include economic analysis (the study of cost and profit functions), financial analysis (determining the optimal rate of return), and market analysis (measuring the effect of experiments on sales).[73] In each of these areas, a defined objective is to be optimized. Constraints are not written in functional form. Rather, the constraint is imposed by defining the interval within which the optimal solution lies. This interval of feasibility can then be explored by one of the techniques of direct elimination. If the exact interval of feasibility is unknown, the use of Fibonacci numbers or golden section search can be used to locate the interval of optimality. For multivariable, continuous functions, the method of contour tangents can be utilized. If the functional expression for the objective function is not known, it can be approximated by a tangent plane and then analyzed by the method of contour tangents.

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[73]John D. C. Little, "A Model of Adaptive Control of Promotional Spending," Applications of Management Science in Marketing, edited by David B. Montgomery and Glen L. Urban (Englewood Cliffs, 1970), pp. 123-124.

Direct climbing.--This study has considered three primary types of direct climbing techniques. These three techniques were response surface analysis, gradient techniques, and parallel tangents. Because of the diversity in application of these three techniques, each of these will be considered on its own merit. In actual practice, however, it is feasible that each of these be used in conjunction with the other as a means of optimizing a given problem.

Response surface analysis: The use of the response surface provides a means of estimating the functional expression of the objective function when the precise functional relationship is not known. This is accomplished by first fitting suitable surface approximations to the trial region. This fitted surface, generally a tangent plane, is then used as an approximation of the true objective function.

Response surface analysis has been used in the conducting of market experiments. In this application, the objective is to determine how sales and net profits respond to changes in a variety of marketing-effort variables or factors. These marketing-effort variables or factors include such controllable variables as radio advertisements, newspaper advertisements, magazine advertisements, sales promotions, personal selling, etc. These surfaces, defined in terms of two independent variables, would be similar to the surface shown in Figure 4.19.

The response surface can be generated with two or three carefully selected experiments. This surface can then be examined for optimality and the result used as an estimate of the true optimal value. Sevin

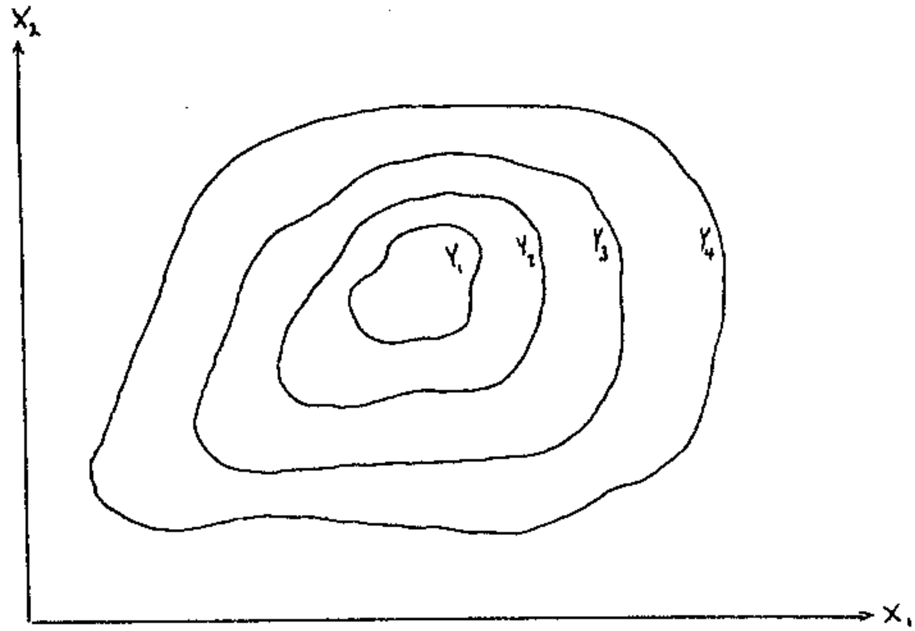

Fig. 4.19--Two-factor response surface: $Y = f(X_1, X_2)$

indicates that

> . . .the very concept of a response surface itself is of the greatest value. By suggesting the form of the underlying marketing-effort (marketing-mix) combinations that are of most significance for the marketing system of each product, the concept of the response surface provides indispensable guidance in performing market experiments most efficiently with the object of improving productivity.[74]

The applicability of response surface analysis has been shown to be an effective tool for analyzing interactions between natural-expense variables (for example, television advertising and point-of-purchase displays).[75] The information gained from the experiments required to generate the surface may be such that the influence of unconsidered factors is noted. This, in turn, could lead to the generation of a more realistic surface.

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[74]Charles H. Sevin, Marketing Productivity Analysis (New York, 1960), p. 123.

[75]Ibid., pp. 126-127.

The use of the response surface in administrative analysis follows that of the use described in the market experiment application. As such, the concept is applicable to such problems as production analysis, inventory control, pricing decisions, investment analysis, stock decisions, cost and profit analysis, and exploratory analysis for selection of future projects. All of these cases represent situations which can be described in terms of at least two independent variables and situations for which the exact function to be optimized is not necessarily known. If there is no historical data available, simulation can be used to generate the output point on the response surface. Varying the input values will produce different levels of output. Given these input-output values, a response surface can be constructed. As noted, the functional expression for this surface will be used as an estimate of the true objective function.

The use of the response surface to generate functional expressions for the objective function is especially appealing when consideration is given to the fact that such surfaces can be constructed when the functional relationships of any existing constraints are not known. All that is required is some knowledge of the domain of definition for the objective function. Given this information, values within the domain of definition can be used to locate points on the response surface. These points can then be used to derive an appropriate functional expression that can be used as an estimate of the true objective function.

As a means of further illustrating the use of the response surface, consider a production process that maximizes its profit on the strength of two major products X_1 and X_2. With each combination of X_1 and X_2

there is an associated level of profit. Given a suitable number of profit-product combinations, a surface similar to that of Figure 4.20 can be approximated. This surface can then be explored directly or its approximating functional expression examined for the optimal combination of products X_1 and X_2.

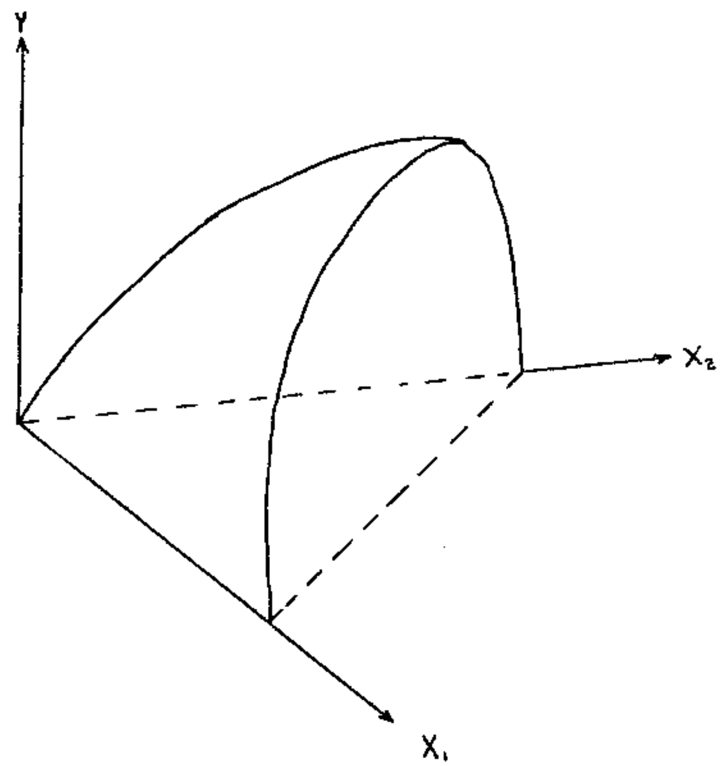

Fig. 4.20: Two-factor production surface: $Y = f(X_1, X_2)$

Of particular importance to the administrator is the potential offered by the use of response curves in determining functional expressions for activities for which there is no precedence. For example, a series of simulation runs can be used to determine the output corresponding to different values of the input variables. These various points can then be used to determine the functional expression which describes the anticipated activity. This function can then be analyzed to determine the feasibility of the activity.

As noted previously, the use of the response surface provides a means of determining a suitable approximation of an unknown objective function. In addition, the functional relationships which define existing constraints need not be known as the response surface can be constructed within the limits imposed on the variables of the problem. For example, although the cost function for a given activity is not known, the limits within which feasible solutions lie may be known. Values within these limits can be used to generate the approximating surface. All that is required is that the output for each input be measurable.

It is evident from this discussion that the concept of the response surface can be utilized as an effective tool of administrative analysis. This viewpoint is supported by Clough. In his text, Concepts in Management Science, Clough utilizes the response surface concept in discussing demand analysis and production activity analysis.[76] It is pointed out that historical data can be used to derive an appropriate response surface. If such data are to be used, it is necessary that the data be valid and of such a nature that the approximating surface describes the real problem.

Gradient techniques: The use of gradient techniques in solving administrative problems presupposes the existence of a continuous, differentiable function. The particular gradient technique to be utilized in solving a given problem depends upon the manner in which the problem itself is formulated. For example, suppose a nonlinear cost function is to be minimized subject to the conditions of nonnegativity and a set of defined

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[76] Clough, op. cit., pp. 128-142, 146-153.

nonlinear constraints. Since the function is constrained, the most appropriate direct search technique is that of the differential gradient. The search process defined by the differential gradient contains a correction factor that will keep the solution points from violating the constraint set.

If a given function is unconstrained, continuous, and differentiable, the Fletcher-Powell technique of deflected gradients is applicable. The general application is the optimization of multivariable quadratic functions. Unlike the differential gradient technique which identifies optima by the vanishing of the gradient, the deflected gradient technique utilizes the Hessian matrix to test a given trial point for optima characteristics. In addition, the deflected gradient technique searches for the optimum solution by evaluating the neighborhood along the gradient. It has been noted that

> when gradients are relatively easy to measure, or when the objective function is particularly difficult to measure, the deflected gradient procedure with its quadratic convergence seems best.[77]

Application of gradient search permits the administrator to locate points of optima for problems ill-suited to indirect search techniques. This improvement is evidenced by the fact that gradient search can be initiated from any potential solution point. The techniques of gradient search utilize information generated from this trial solution to determine the direction in which the optimal search is to move. At each iteration the trial solution is corrected, and a new solution point is generated. This process continues until the optimal solution is derived or its

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[77]Wilde and Beightler, op. cit., p. 339.

interval of uncertainty is reduced to such a length that direct elimination can be used to locate the final solution.

Although the use of gradient search has received little attention as a tool of administrative analysis, potential applications do exist. For example, in the July, 1967, issue of Management Science, Tillman and Liittschwager investigate the minimization of nonlinear cost functions subject to nonlinear restraints and a minimum level of acceptability. The problem they investigate is given by the following: determine m_j, the number of redundant units at stage j such that

$$z = [c_1 m_1 e^{-\frac{m_1}{2}}] + [c_2 m_2 e^{-\frac{m_2}{2}}]$$

is minimized subject to

$$g_1 = [a_{11} m_1 + m_1^2] + [a_{12} m_2 + a_{13} m_2^2 + a_{14}] \leq b_1$$

$$g_2 = [a_{21}(m_1 + e^{-m_1})] + [a_{22}(m_2 + e^{-m_2}) - a_{23}] \geq b_2$$

$$g_3 = [a_{31} m_1 e^{-\frac{m_1}{4}}] + [a_{32} m_2 e^{-\frac{m_2}{4}}] \geq b_3$$

and satisfies the restraint

$$d_1 \leq \sum_{j=1}^{n_1} \sum_{k=0}^{n_2} \Delta \ln R_{jk} m_{jk},$$

where $\Delta \ln R_{jk}$ equals the change in $\ln R_j$ by adding the k^{th} redundancy at stage j. The m_{jk} identifies the k^{th} redundancy at stage j, where $m_{jk} = 1$ for $k \leq m_j$ and $m_{jk} = 0$ for $m_j < k \leq m_j'$.[78]

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[78]F. A. Tillman and J. M. Liittschwager, "Integer Programming Formulation of Constrained Reliability Problems," Management Science, XIII (July, 1967), 892-894.

Tillman and Liittschwager indicate that problems of this type can be solved by existing techniques if there are but a few linear constraints. However, "it seems [that] these [existing] methods are inadequate for solving this [problem] which includes multiple nonlinear restraints."[79] This type of problem, assuming continuity and differentiability, is amenable to gradient search. Given the general form of the gradient vector, a trial solution can be arbitrarily selected which satisfies the restraints and the minimum reliability requirement. From this point a series of solutions can be generated which move toward the optimal solution.

Another potential area of application is the determining of the optimal number of warehouses and their optimal location. This problem has been described as one of

> finding the location of a warehouse which is optimum with respect to the total cost of transporting known quantities of goods from the warehouse to each of n destinations, given that the latter is a linear function of the warehouse-to-destination distances.[80]

The formulation of this problem is as follows:

$$\text{minimize } c = k \sum_{i=1}^{n} D_i[x_i - x)^2 + (y_i - y)^2]^{\frac{1}{2}}$$

where k equals the unit cost of transporting the known quantity D_i to destination i. The warehouse is located at (x, y), and destination i is located at (x_i, y_i).[81]

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[79] Ibid., p. 893.

[80] Roger T. Eddison, "Warehousing, Distribution, and Finished Goods Management," Progress in Operations Research, II, edited by David B. Hertz and Roger T. Eddison (New York, 1964), p. 110.

[81] Ibid.

The traditional approach to the problem of determining the optimal number of warehouses and their optimal location has been to solve the system given by

$$\frac{\partial c}{\partial x} = k \sum_{i=1}^{n} D_i \frac{x_i - x}{[(x_i - x)^2 + (y_i - y)^2]^{\frac{1}{2}}} = 0;$$

$$\frac{\partial c}{\partial y} = k \sum_{i=1}^{n} D_i \frac{y_i - y}{[(x_i - x)^2 + (y_i - y)^2]^{\frac{1}{2}}} = 0.$$

This solution is determined by resorting to the trigonometric relations corresponding to $\frac{\partial c}{\partial x} = 0$ and $\frac{\partial c}{\partial y} = 0$, $\sum_{i=1}^{n} D_i \cos \theta_i = 0$ and $\sum_{i=1}^{n} D_i \sin\theta = 0$, respectively. If θ_i is used to represent the angle formed by the line joining the warehouse and the destination, the slope of this line is defined by tan θ_i. "The strategy to be adopted for two warehouses. . .would be to pick two likely locations and move each according to a prescribed pattern in two dimensions. . .to use a form of steepest ascent."[82]

Other areas of administrative analysis suited to gradient technique include economic analysis (cost-profit analysis), market analysis (response surface examination), production analysis (inventory control), and budget analysis (allocation of departmental budgets). As a means of enhancing this presentation of applications of gradient search techniques, a limited selection of other types of problems suited to gradient search is presented.

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[82] Ibid., pp. 111-112.

Maximization of salesman's compensation: The problem being investigated is one of maximizing a salesman's compensation when sales commissions are based on gross margins rather than sales commissions.[83] The problem is formulated on the basis of the following two considerations: maximum profits for the firm and maximum commission for the individual salesman.

In the first case, maximum profits for the firm, the mathematical expression is given by

$$\pi = \sum_{i=1}^{n} Q_i[(P_i - K_i)(1 - B_i)].$$

This function is to be maximized subject to the salesman's time constraint

$$C = \sum_{i=1}^{n} t_i .$$

In the profit function π equals the company's gross profit in dollars, Q_i equals the unit quantity of product i sold, P_i equals the unit selling price of product i, K_i equals the unit variable non-selling cost of product i, and B_i equals the percentage commission rate paid on product i. In the constraint function C equals the total time the salesman spends selling during a given time period, and t_i equals the time spent selling product i.

In the second case, maximum commission for the individual salesman, the mathematical expression is given by

$$S = \sum_{i=1}^{n} M_i Q_i B_i .$$

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[83] John U. Farley, "An Optimal Plan for Salesmen's Compensation," <u>Applications of Management Science in Marketing</u>, <u>op. cit.</u>, pp. 380-390.

This function is to be maximized subject to the time constraint

$$C = \sum_{i=1}^{n} t_i .$$

In the commission function M_i equals the gross margin to the company from product i, Q_i equals the unit quantity of product i, and B_i equals the percentage commission rate. The values of C and t_i are the same as in the company's constraint function.

In both expressions the unit quantity sold of product i, Q_i, is assumed to be an increasing function of the time spent selling that one product. That is,

$$Q_i = f_i(t_i).$$

The expression $f_i(t_i)$ is also assumed to be continuous and differentiable. The exact functional expression for $f_i(t_i)$ is not necessarily known. However, it is assumed that $f_i(t_i)$ can be fitted with a reasonably valid function.

Farley indicates that this problem, given the assumptions indicated, is amenable to the use of partial derivatives.[84] The technique he utilizes is that of the Lagrange multiplier. Incorporation of such items as price incentives and "loss leaders" tends to make the problem too complex for the technique utilized by Farley. However, gradient search provides a means of optimizing the problem by providing an iterative technique for locating the optimal combination of t_i values.

Product analysis and decision models: The use of decision models in product analysis has been investigated on the basis of four sub-models.

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[84] Ibid., pp. 383-384.

> The demand model is structured to consider life cycle, industry, competitive and product interdependency effects and will admit non-linear and discontinuous functions. A cost minimization model is joined to the demand model to formulate a constrained profit maximization problem. . . . The final decision is based on the businessman's criterion in combining uncertainty and the rate of return on investment.[85]

In describing the demand, cost, and profit models, Urban indicates the feasibility of utilizing the response surface concept to derive a functional expression for each model. These expressions can then be optimized by an appropriate selection of technique.[86] Suitable techniques include the differential gradient and the deflected gradient.

From this selection, it is to be noted that in every application the use of gradient techniques requires that the function be known or of such a nature that it can be derived. It is also required that the function be continuous and differentiable. Unlike indirect search, it is not necessary that a simultaneous system of partial derivatives (multivariable case) be solved and the critical points tested for optimality. All that is required is that a trial point be selected. The gradient search techniques of steepest ascent, steepest descent, and deflected gradients improve on this trial solution until the optimal solution is located.

Parallel tangents: As noted in the presentation, the technique of gradient partan is the more efficient of the two parallel tangent techniques discussed in this study. This is attributed to the fact that the technique of parallel tangents is both a climbing technique and an elimination

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[85]Glen L. Urban, "A New Product Analysis and Decision Model," Applications of Management Science in Marketing, op. cit., p. 410.

[86]Ibid., pp. 413-415, 418-421.

technique. In this capacity, gradient partan provides a means of moving toward an optimal solution by a series of sequential iterations. At each iteration the trial solution is modified in such a way that the next trial solution is closer to the optimal solution.

Gradient partan has been identified as being especially suited to the analysis of elliptical contours. In administrative analysis, these contours are generally described by such concepts as indifference curves, cost contours, and profit contours. Since gradient partan provides a means for analyzing such contour surfaces, its applicability in administrative analysis is readily established because of the current use of such contour surfaces.[87]

As a means of further supporting the applicability of gradient partan, some specific examples of problems suited to contour surface representation will be presented. Although contour surface representation is not a requirement for implementation of gradient partan, it does provide a means of graphic representation.

Cost minimization: The problem under investigation is that of minimizing the total cost of inventory policy.[88] Minimization is to be accomplished relative to the economic reorder level and the economic lot size. Both of these quantities are to be determined so that total cost is minimized.

The inventory cost function to be minimized is given by

$$TC = \frac{Q}{L}(B + \hat{S}) + \left(\frac{L}{2} + \bar{b}\right)VC.$$

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[87]This is supported by the use of such contours by Goetz, op. cit., pp. 384-392, Baumol, op. cit., pp. 258-261, and A. H. Boas, "Special Mathematical Techniques," Cost and Optimization Engineering, edited by F. C. Jelen (New York, 1970), pp. 288-292.

[88]The context of this example is taken from Goetz, ibid., pp. 385-391.

In this expression, Q equals the annual rate of use in units, L equals the economic lot size in units, S equals the expected costs of stockout, and b equals average buffer stock (set at a minimum) in units. V equals the incremental cost per unit, and C equals carrying costs per dollar per year in inventory. Batch costs, B, include clerical costs, setup costs, and loss caused by learning time.

The reorder level is denoted by N. It is equal to the ratio between the annual rate of use in units, Q, and the economic lot size in units, L; i.e.,

$$N = \frac{Q}{L}.$$

The relationship between L and N can be graphically represented as shown in Figure 4.21. This figure is constructed by holding total cost constant, assigning a value to L, determining N, and plotting the resulting points. The result of this plot of points, for different cost figures, is a cost contour representation of the optimal inventory policy problem.

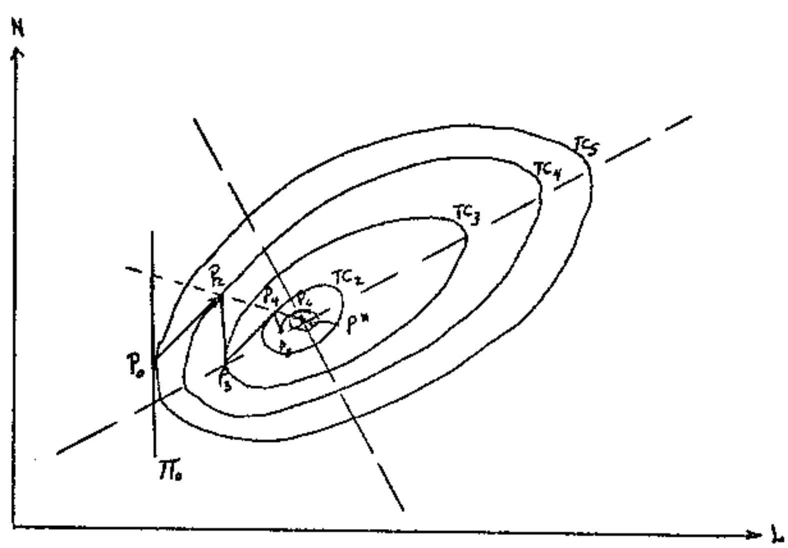

Fig. 4.21--Inventory policy cost contour

The search for the optimal combination of lot size and reorder level utilizes a contour tangent approximation at an initial starting point, $\underline{P}_o$ = (L, N). This contour tangent is given by $\underline{\pi}_o$. Applications of the gradient partan technique produces the series of points $\underline{P}_2$, $\underline{P}_3$, $\underline{P}_4$, $\underline{P}_5$, and $\underline{P}_6$. The technique, as shown in the algorithm, terminates at $\underline{P}$*, the optimal combination of lot size and reorder level.

Profit maximization: The problem under investigation is one of maximizing profit by achieving an optimal mix between sales promotion expenditures, S_e, and price, p.[89] The profit contour is obtained by constructing a set of isoprofit curves. The resulting figure is shown in Figure 4.22.

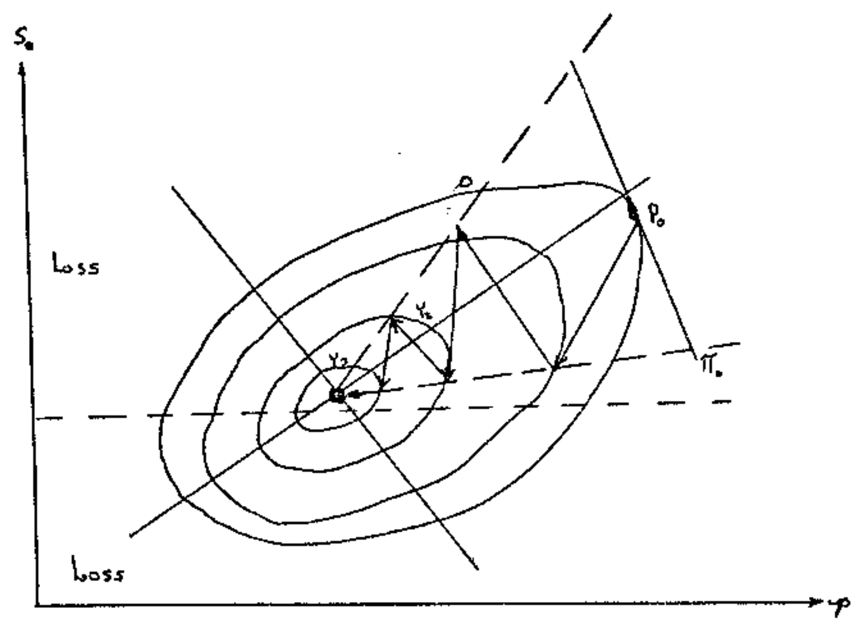

Fig. 4.22--Profit contour: Y = f(p, S_e)

[89] Ibid., pp. 25-26.

Profit maximization by gradient partan requires that the profit function be known. As is the case for all applications of gradient partan, it is assumed that the function being investigated is continuous and differentiable. The search is begun at any trial point $\underline{P}_o$, through which the tangent plane $\underline{\pi}_o$ is constructed. From this initial solution, gradient partan moves, as shown in Figure 4.22, to the optimal combination of the input variables. For the profit function of Figure 4.22, this optimal solution is that combination of price and sales promotion expenditures which maximizes profit.

Analogous applications of these cost-profit examples can be readily identified. For example, the profit maximization example can be extended to one of determining the optimal combination of investments for the purpose of maximizing rate of return. The cost minimization problem can be extended to one of determining the optimal number of men and machines required to minimize the total time spent on a project.

This presentation has served to indicate areas of administrative analysis that are amenable to the techniques of direct search. Although the presentation has been limited to a select group of application areas, the importance of this study has not been lessened. All that remains is for these techniques to be incorporated into the collection of tools utilized in everyday administrative work.

With this thought in mind, it is feasible to reconsider the relevance of direct search as a technique for administrative analysis. This can be accomplished by answering the question, "Why use direct search?"

(1) With highly nonlinear, univariable functions (degree greater than or equal to four), evaluation of the derivative to locate critical points

can be complicated. The use of such theorems as those of Sturm and Budan[90] or the Newton-Raphson technique can be used to locate the point intervals within which solutions lie. Given this interval, or set of intervals, the utilization of a suitable elimination technique can provide the necessary solution or set of solutions.

(2) With multivariable, nonlinear functions of degree greater than or equal to three, the condition $\frac{\partial f}{\partial x_i} = 0$, $i = 1, 2, \ldots, N$, results in a homogeneous system of nonlinear functions. Solving this system can prove to be more difficult than solving the original function by direct search. In this case the contour tangent can be used, as can gradient techniques or parallel tangents. It is assumed that the function is continuous for the N^{th} derivative.

(3) The use of direct search in multivariable analysis requires only that the gradient be obtained and evaluated at each iteration. This is generally not too complicated since the initial solution can be arbitrarily selected and additional solution points are obtained by systematic computations using information that is made available by the iterative process.

(4) Direct search can be used to solve complex, nonlinear functions subject to nonlinear constraints. In this application, the most feasible technique is that of the differential gradient. However, it has been noted that greater efficiency can sometimes be achieved by reducing the constrained problem to one that is unconstrained and then selecting some other direct search technique.

As in the case of indirect search, applications of direct search duplicate those of both classical optimization and basic optimal search.

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[90] Nelson Bush Conkwright, Introduction to the Theory of Equations (New York, 1957), pp. 87-96.

The use of these techniques makes feasible the use of nonlinear functions in representing reality. For multiproduct cases, multivariable nonlinear expressions can be used with the knowledge that solution techniques are available for locating optimal solutions.

The feasibility of the response surface has been established through the use of simulation or market experimentation. Through the use of this technique, and the incorporation of a suitable search method, new areas of activity can be explored without commiting the full resources of the firm.

CHAPTER V

SUMMARY AND CONCLUSION

Summary

The purpose of this investigation is to provide an interpretive study of optimization theory and its use as a tool of administrative analysis. In this respect, this study is divided into three general topical areas. These topical areas are (1) classical optimization theory, (2) basic techniques of modern optimization theory, identified as basic optimal search, and (3) advanced techniques of modern optimization theory, identified as advanced optimal search. Each of these three topics is further divided into the areas of technique and application. In addition to providing for the presentation of the general characteristics of the problems that can be solved by classical and/or modern optimization theory, this format provides for a comprehensive review of classical optimization theory, a comprehensive study of modern optimization theory, and the elaboration or development and demonstration of computational algorithms that enhance the application of modern optimization theory to problems of an administrative nature.

The techniques of classical optimization theory are identified in Chapter II. The techniques include methods for solving algebraic equations and systems of linear equations, the max-min calculus up to and including the Lagrange multiplier, and queueing theory. As a means of facilitating the understanding of each of these techniques, a general discussion of underlying theory is presented. This discussion includes conditions for identifying points of optima and the characteristics that a problem is required to have if a particular solution technique is applicable. The presentation in this chapter provides the technical and conceptual foundation from which the techniques of modern optimization theory are discussed.

Applications of classical optimization theory are presented in the latter part of Chapter II. The presentation of these applications is made on the basis of the use that is made of a particular mathematical technique. For example, administrative problems requiring breakeven analysis are classified under the heading "algebraic equations." Administrative problems requiring linear application are classified under linear systems. Similarly, administrative problems requiring points of maxima or minima are discussed under the heading of "max-min calculus" or "Lagrange multiplier." This manner of presentation permits ready correlation between the solution technique and the types of problems to which these techniques can be applied. In addition, the general characteristics of the problem formulations are identified.

The emphasis in Chapter III is upon the basic techniques of modern optimization theory. The study is limited to the areas of linear, quadratic, geometric, and dynamic programming. This limitation is based upon the conclusion that, of the basic techniques of modern optimization theory, these are the ones most applicable to general administrative analysis. In the case of linear programming, the existence of modified problem formulations such as the network problem and the transportation problem is noted. In addition, it is also noted that special computational techniques have been derived for these special cases of linear programming.

The presentation of these basic techniques is made in such a way that the classes of problems to which the techniques apply are discussed and the characteristics of the problem formulations are identified. This presentation is followed by a general description of the solution technique itself. With the exception of dynamic programming, computational algorithms are constructed from a study of published theory and technique development. Each algorithm is demonstrated in detail by application to a representative problem. Except as noted, these algorithms and the demonstrations are of an original nature and are not available in the literature.

In the latter part of Chapter III, the administrative applications of these basic techniques of modern optimization theory are identified. The presentation of these applications is made on the basis of the mathematical formulation of the problem itself. For example, a linear objective function with linear constraints is classified as a linear programming problem. A quadratic objective function with linear constraints is classified as a quadratic programming problem. The presentation of these applications includes general areas of current applicability as well as areas of potential applicability.

The emphasis in Chapter IV is upon the advanced techniques of modern optimization theory. Prior to the presentation of these techniques, however, the optimum-seeking problem is examined and defined. This provides a base for the discussion of various optimal search techniques and a brief review of the calculus.

Chapter IV is divided into two basic topical areas, indirect search and direct search. Each of these areas is further divided as a means of facilitating the presentation of the material. In the case of indirect search, the basic techniques discussed are (1) the differential approach, (2) the Newton-Raphson formula, and (3) the constrained derivative. In the discussion of each of these indirect search techniques, terms and concepts are identified by presenting them as definitions or theorems. This manner of presentation is used to facilitate the discussion of the underlying theory. This discussion is then used as a basis from which a computational algorithm for each technique is constructed. Each algorithm is then demonstrated by application to a representative problem. These algorithms serve to summarize the computational aspects of the respective techniques and, except as noted, are not available elsewhere.

In the case of direct search, the basic techniques presented are those of (1) direct elimination and (2) direct climbing. The direct elimination techniques presented are interval elimination, sequential search, golden search, and contour tangents. It is noted that direct elimination techniques are used to evaluate the problem at several points. If necessary, these evaluations can be used to construct a functional expression of the problem being investigated. The direct climbing techniques discussed are the response surface, gradient techniques, and parallel tangents. It is noted that direct climbing techniques optimize a given function by utilizing information that is generated along the search route.

As a means of facilitating the discussion and enhancing the application of direct search, direct elimination and direct climbing are subdivided into topical areas. This permits an improved presentation of some of the techniques that are classified as techniques of direct search. For example, gradient techniques are discussed under the headings of the differential gradient and the deflected gradient. Each discussion is followed by a computational algorithm that has been constructed from a study of the literature, and each algorithm is demonstrated by applying it to a representative problem. From the discussion and the computational demonstrations, characteristics peculiar to a particular technique are identified.

The latter portion of Chapter IV is concerned with administrative applications of advanced optimal search. However, as noted, there is a dearth of documented administrative application. This is attributed to the newness of the solution techniques and the lack of a sufficient data base from which empirical studies can be made. As a result of these limitations, this portion of the study of modern optimization theory and its use as a tool of administrative analysis is developed by describing areas of administrative analysis that have problems similar to those of classical optimization theory and basic optimal search. This development is such that specific problems are identified, described, and correlated with a specific solution technique. For example, administrative problems amenable to indirect search are presented under indirect search. Administrative problems amenable to direct search are presented under direct search.

Conclusion

An interpretive study, this investigation provides (1) an explanation of the general theoretical development of the techniques of modern optimization theory, (2) computational algorithms for implementing the techniques of modern optimization theory, (3) detailed demonstrations of the computational aspects of each technique and its related algorithm, and (4) a means of identifying the types of problems to which these techniques are applicable. Although the manner of presentation distinguishes between classical optimization theory and modern optimization theory, it supports the thesis that modern optimization theory is a natural extension of classical optimization theory by relating the techniques and applications of modern optimization theory to those of classical optimization theory.

As a part of the transition that must be made from the abstract conceptualization of a technique to its application, this study documents in a formal manner the techniques and administrative applications of modern optimization theory. This documentation is evidenced by the following: (1) classes of administrative problems amenable to the techniques of modern optimization theory are identified; (2) general computational algorithms suitable for administrative application are developed; (3) specific solution techniques are related to classes of problems; and (4) detailed computational examples are provided which demonstrate the use of a particular technique. In this manner, the presentation provides a basis for correlating a given problem with a suitable solution technique. This correlation is determined by the manner in which the problem is formulated. For example, a continuous and differentiable nonlinear profit function by the differential algorithm of indirect search. If the mathematical expression is quadratic with linear constraints, the problem is classified as one of quadratic programming. It can then be solved by one of the quadratic programming techniques presented in this study.

The effective utilization of techniques becomes commonplace only after sufficient data analysis. This study has shown that the techniques of modern optimization theory are feasible for use as tools of administrative optimization. A more extensive data base is now needed to provide for empirical studies that further demonstrate the validity of this claim.

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