# A NEW GRAVITATIONAL APPROACH
# TO LEAST TRANSPORTATION COST
# WAREHOUSE LOCATION

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# A NEW GRAVITATIONAL APPROACH TO LEAST TRANSPORTATION COST WAREHOUSE LOCATION

DISSERTATION

Presented to the Graduate Council of the
North Texas State University in Partial
Fulfillment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

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Denton, Texas

May, 1970

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# CHAPTER I

# INTRODUCTION

One of the most significant remaining areas for cost reduction in marketing is physical distribution. In most industries transportation and distribution costs have become one of the largest expenses of doing business, and the increase in these costs in recent years has outstripped corresponding costs in production. As a result, increasing attention has been focused on cost reduction in the many varied areas of physical distribution. One of these areas refers to warehouse location, especially least transportation cost warehouse location. While attention in the literature has been focused on this conception additional investigation may well be a profitable area of endeavor, as continued analysis is fundamental to all research.

## Statement of the Problem

Basically, formal study is needed to develop other warehouse locational techniques to facilitate the determination of a least transportation cost location. This presentation will essentially undertake such a study in an effort to develop accurate warehouse locational methodology which may be very easily implemented. Therefore, this study proposes to supplement the existing state of the art through the development of a new potentially optimal single facility warehouse location model.

The need for such a study is primarily two-fold. First, single facility warehouse location models which determine an alleged optimum location through a coordinate system have been developed. However, preliminary studies have either only assumed optimality or presented semblances of proof attesting to model optimality, even though such proof is viewed as absolute. Moreover, the existing state of the literature has questioned the optimality of the approach. Thus, continued research is needed to reenforce the indicated optimality of the method.

Secondly, the need for additional research is necessary because the approaches involving linear programming, simulation, or heuristic programming do not by definition generate an optimal location. Rather, they just determine a warehouse location or locations from an arbitrary series of tentative warehouse sites. Most apparent is the fact that the true optimum location could well exist external of the alternative arrays of potential sites.

Based upon these existing limitations a new approach to single facility warehouse location will be developed, which may be formulated without the need of a coordinate system. The approach itself, like the existing coordinate system, should determine a centroid location or a point of equilibrium for a series of weighted customer and supplier points spread over a geographical plane or market, the centroid location being significant because it has been viewed in the literature as the point of least transportation cost.

## Significance of the Study

The study and development of the new noncoordinate centroid model is significant in that a new tool or approach may be placed in the arsenal of the locations researcher. Additionally, the attempt to generate proof of the optimality of a centroid location may help to reenforce the validity of its coordinate counterpart, and in the event the centroid proves nonoptimal a methodological approach will be formulated to generate a least transportation cost location. Possibly the results of this investigation may also help to bridge the gap involving the arbitrary selection of warehouse points which now pervades some of the more expensive available approaches, and the model should assume an identity of its own when management's goal is centered on a guide to a least transportation cost warehouse location, particularly through inexpensive manual means.

With this background significance having been presented the focus will now concentrate on generating the hypotheses which will be attempted to be proven through the results of this study.

## Hypotheses

The hypotheses which will govern the research framework of this dissertation are presented as follows:

1. That a new noncoordinate centroid determining model may be formulated for warehouse location. The inputs of the model will consist of distance, demand (tonnage), and cost (per ton mile freight rates).

2. That a warehouse location at a scientifically determined centroid site will result in an optimum or near optimum warehouse location for a designated series of customer and supplier points.

This hypothesis is generated because most preliminary surveys of the literature have only assumed the optimality of a centroid location. Yet, as will be seen in the survey of the literature presented in Chapter II, cost comparisons attesting to the absolute validity of the location have been presented by the developers of the coordinate methodology.

However, there has been some question in the literature as to the optimality of such a centroid location, but as will be seen in Chapter II, the bulk of the published approaches attest to a centroid location. Therefore, further analysis is needed. Lastly, the reference to near optimal implies a slight margin of error and for all practical purposes such a margin may be viewed as an optimal location.

3. That a methodology may be developed which will allow for the inclusion of nonlinear freight rates as a model input. This is significant because the existing state of the art assumes that freight rates are linear or directly proportional to distance. This implies that rates per mile per hundred weight remain constant over expanding distances. However, in actuality freight rates are nonlinear or nonproportional to distance.

This hypothesis also assumes additional significance because the linear assumption produces freight rates as model inputs which generally understate and overstate the rates associated with given points. However, by including nonlinear freight rates as model inputs, the rate which most closely associates shipping to a given customer or from a given point of supply may be identified.

## Definition of Key Terms

To facilitate an understanding of the terms permeating hypotheses construction the following definitions are presented.

1. Centroid location, or equilibrium point--the absolute point of balance for a weighted series of customer and supplier points spread over a geographical plane.

2. Coordinate centroid determining model--a model which encompasses all customer and supplier points within the positive quadrant of a Cartesian coordinate system and determines the coordinates of a centroid location. A complete discussion of the precise methodology associated with this approach is presented in Chapter II.

3. Customer points--a series of customers or demand points who are to be supplied from the to-be-ascertained warehouse location.

4. Supplier points--the reference made to the supplier or a series of suppliers who will ship to the to-be-ascertained warehouse location.

5. Optimum location--the point at which transportation costs will be minimized based upon the model inputs of tonnage, distance, and assigned freight rates per ton mile.

6. Near optimal location--a centroid location which may be viewed as being optimal within an acceptable range of tolerance limits.

## Model Orientation

To orient the reader a brief description of the type of model to be developed is generic to the task. Briefly described, the model isolates a specific site within a designated market area which, based upon the prevailing information inputs, may produce a least transportation cost location. The site so isolated is viewed as a centroid or equilibrium point.

The approach to a new model will be based on the systematic determination of balance points between two designated points. These points are weighted by the combined impact of tonnages and per ton mile freight rates. The approach is significant because it tends to open new vistas to locational analysis. For example, the approach may be utilized to systematically produce two significant centroids. One centroid is based on weighted consumer points and a second centroid which additionally considers the impact of supply and, therefore, produces the overall centroid of the system. A knowledge of both centroids may possibly prove beneficial if the overall system centroid is compromised. For example, it may be physically impossible to generate a location at the indicated centroid site.

## Delimitations

To enhance the development of a workable framework for model presentation and analysis the following delimitations are presented:

1. The model application will be restricted to single facility warehouse location.

2. The model orientation will be concerned only with least transportation costs per se.

3. The products or tonnages to be handled through the to-be-determined warehouse are restricted to homogeneous staples. The inclusion of perishables in the analysis would imply a time priority rather than a cost priority.

4. The analysis of freight rates will be delimited to rail and motor common carriers, the predominant forms of warehouse movements.

5. The analysis of freight rates will also be restricted to viewing inbound warehouse shipments in car or truckload volume and outbound warehouse shipments in less than car or truckload volume. Indeed, these are the basic forms of inbound and outbound warehouse shipments.

## Method of Research

The basic method of research generated in this presentation will involve primary research in the form of mathematical proofs. The bulk of these proofs for those concerned with model mathematics are seen in Chapters III and IV, and the following types of proof will be applied to the designated hypotheses.

### The First Hypothesis

In regard to the first hypothesis, two types of proof will be generated which will attest to validity of the new centroid determination model, i.e., the ability to produce a centroid location. First, the mathematical development of the new model will be self-proving. That is, the derived mathematical formulations attest to the precise validity of the resulting model's ability to generate a centroid location. Additional proof is also forthcoming in the form of a comparison between the resulting centroid location produced by the model and the centroid of a known configuration, in this case an isosceles triangle. Thus, the resulting corollary proof is viewed as a supplemental affirmation to the self-proving abilities of the model itself.

## The Second Hypothesis

The application of proof to the second hypothesis, which is concerned with the optimality or near optimality of the centroid location, will be researched through mathematical confirmation. It should be evident that cost comparisons could be run between the centroid and several other points to depict the optimality of such a centroid, yet such results would not be absolutely conclusive. Therefore, a mathematical confirmation model will be developed through calculus and the application of partial differentiation. As a result, this model may be utilized to prove the existence of the optimality or near optimality of a centroid location. This confirmation model may reveal such optimality because the confirmation model is mathematically related to the determination of minimum values of distance through the use of the first partial derivative of a ton-mile formula equated to zero and solving for the $\overline{X}$ and $\overline{Y}$ coordinate parameters. This will, therefore, determine whether or not the centroid location is indicative of minimum movement costs.

In the event the centroid is viewed as being non-optimal per se a new methodological approach will be attempted to be developed to arrive at the point of least cost. This approach will still require the usage of a centroid location, and the application of a confirmation model.

## The Third Hypothesis

Hypothesis number three, which is concerned with the possibility of creating a methodological framework for generating nonlinear freight rates as model inputs, will be developed and proven from an extensive analysis of the existing freight rate literature. Thus, conclusions will primarily be derived from secondary research.

## Chapter Overview

To help develop an understanding of the methodological framework which will govern this dissertation the following chapter overview has been developed.

Chapter II, Survey of the Literature.--The emphasis of Chapter II involves an historical survey of the centroid determining literature. This involves the presentation of the various coordinate centroid determining techniques with pertinent comments as to discernible limitations. By presenting these approaches a framework for value assessment is generated and the uniqueness of the to-be-developed new approach may be identified.

Chapter III, Theoretical Model Development.--This particular chapter encompasses the conceptual foundation for the new centroid determining approach to the location problem. Here the new balance point model will be developed and proven. To facilitate this the emphasis will center on generating a ton-mile centroid with the freight rate applications to the model being introduced in Chapter IV. Such a consideration serves to simplify the presentation, but by no means impedes the theorizing behind the model. Thus, ton mileage is the objective for minimization rather than cost considerations.

Additionally, a confirmation model will be developed to determine the degree of optimality of a centroid location. In the event any limitations as to a centroid location are discovered these will, likewise, be presented.

Chapter IV, Freight Rate Application to the Model.--The focus of Chapter IV is primarily concerned with building on the conceptual base presented in Chapter III. Here the pure cost considerations of the model will be considered. Now the objective centers on minimizing the distances associated with the per ton mile cost weightings which will be assigned to each customer and supplier point due to the impact of freight rates.

To facilitate a further understanding of the freight rate ramifications of the model the parameters of freight rate application which will pervade the model will be identified, along with the refinement of both the balance point and confirmation models through rate inclusion. Additionally, a general freight rate orientation will be provided.

Chapter V, Procedural Steps and Secondary Locational Considerations.--The concern of this chapter is to synthesize the results generated in Chapters III and IV and to also present the major procedural steps that the locations researcher should follow in ascertaining the inputs to be included in the model.

Emphasis is also placed on viewing the secondary factors which could compromise a scientifically determined least cost location. Lastly, a complete presentation of the conclusions derived from the study will be presented in Chapter VI.

# CHAPTER II

# SURVEY OF LITERATURE

Basic to the development of a new gravitational approach to least transportation cost warehouse location is a historical survey of the centroid determining literature. To facilitate the presentation this chapter will impart emphasis on two related warehouse locational techniques which are discernible from the literature. The first refers to the determination model, which precisely determines the center of gravity for a series of points through mathematical calculations. The second refers to the trial and error approach which does not reveal a least cost location per se, but merely checks a potential least cost location through trial. By presenting these approaches a basis for value assessment is generated and the uniqueness of the new, manual approach, which is introduced in Chapter III, may be identified.

## The Determination Model

In the determination model, the most common presentation found in the literature, emphasis centers on the usage of applied mechanics to precisely ascertain the centroid of a series of weighted points spread over a physical plane. The theory of utilizing the centroid as a means of determining an optimum warehouse location was initially introduced by

K. B. Keefer (6) in 1934, yet such centroid location was accomplished by nonmathematical means. In the Keefer approach emphasis centered on determining an optimum food distribution outlet. To accomplish this objective retail food store locations were scaled on a piece of cardboard and BB shot were glued to each retail location to reflect the sales volume importance of each location. A pencil was then moved under the cardboard until the centroid or point of balance was determined. The point so determined was then viewed as the optimum warehouse location, based upon the assigned weights and mileage.

However, rather than utilize such a cumbersome approach to warehouse location, mathematical techniques involving the usage of Cartesian coordinates have been borrowed from engineering, more precisely the field of mechanics, to arrive at a centroid location based upon various types of information inputs. The method involves placing all customer and supplier points in the positive quadrant of a Cartesian coordinate system with a vertical Y axis denoting the ordinate and the horizontal X axis denoting the abscissa. As a result all X and Y values for each customer or supplier will be expressed in positive terms. This is easily discernible by viewing Figure 1, which depicts a complete Cartesian coordinate system.

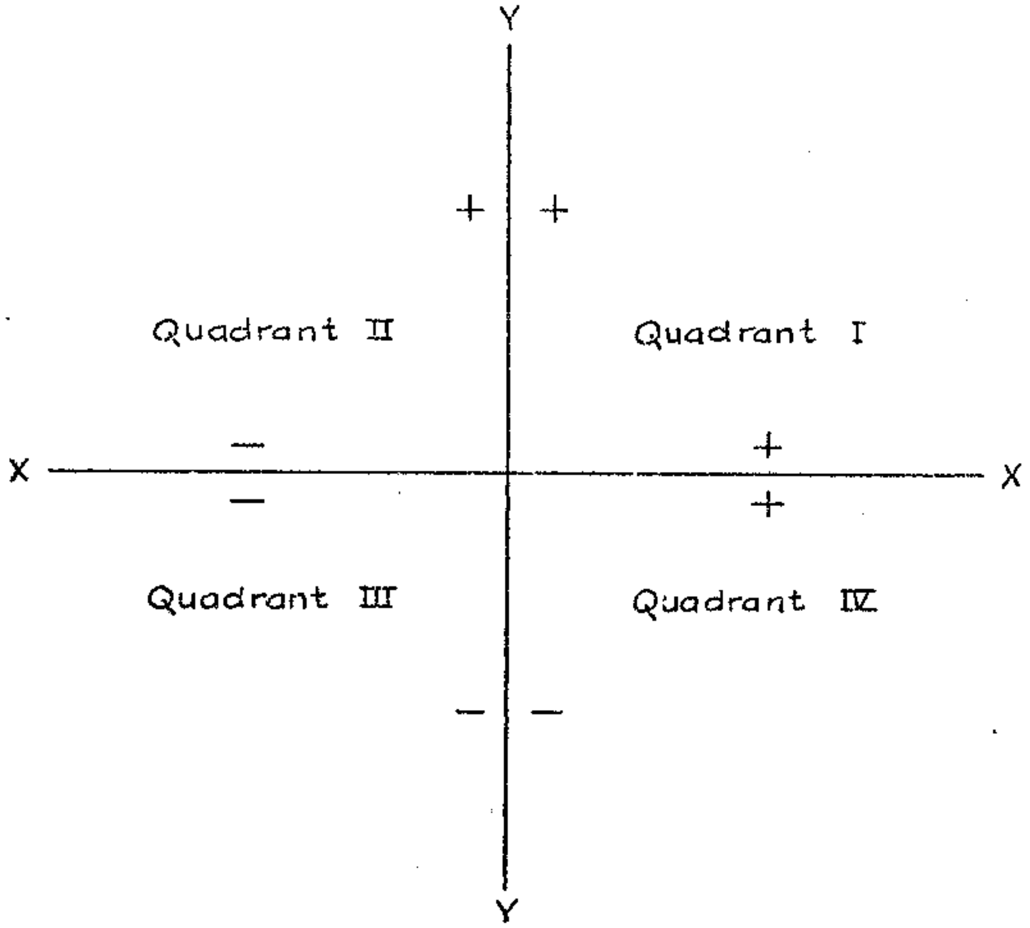

Fig. 1--Basic Cartesian coordinate system

Note the positive values in quadrant I. The centroid may then be determined for a series of weighted points by using the following engineering formulation:

$$\overline{X} = \frac{W_1W_1 + X_2W_2 + . . . + X_nW_n}{W_1 + W_2 + . . . + W_n} ,$$

$$\overline{Y} = \frac{Y_1W_1 + Y_2W_2 + \ . \ . \ . + Y_nW_n}{W_1 + W_2 + \ . \ . \ . + W_n} ,$$

where,

$\overline{X}$ = coordinate of centroid on X axis

$\overline{Y}$ = coordinate of centroid on Y axis

X = coordinate for customer and supplier points on X axis

Y = coordinate for customer and supplier points on Y axis

W = weights associated with each customer and supplier.

With this brief construct now in view attention will center on presenting the chronological development of the coordinate approach to centroid location as applied to warehouse location.

## 1958 Eneborg Model

The first evident mathematical publication involving the implicit usage of Cartesian coordinates, as applied to the warehouse location problem, was introduced by the industrial and mechanical engineer, Carl G. Eneborg (4). In this initial publication it was implicitly assumed that location at a centroid or center of gravity would produce the optimal mathematical warehouse location. As a result of this particular assumption, no proof or theoretical structuring attesting to the validity of this premise was presented. However, it was indicated that the method had been used by Eneborg in helping prominent firms establish distribution centers (4, p. 53).

Methodological approach.--Basically, the Eneborg publication is indicative of a how-to-do-it approach or the methodological procedures to be followed in mathematically locating a warehouse at a centroid location. In this regard Eneborg utilizes a hypothetical construct consisting of five sales outlets located in a geographical plane and within the constraints of a horizontal and vertical axis.

According to Eneborg (4, p. 53), to determine the ideal location requires indicating the sales per calendar year associated with each point. In turn this figure may be expressed in any common unit--prices, measured quantity or money value. Eneborg's (4, p. 52) presentation of these outlets weighted by units appears in Figure 2.

After identifying these outlets and the units sold per year weightings, the Eneborg model requires measuring the distance from a predetermined vertical axis (Y) to each distribution point and likewise the distance from a predetermined horizontal axis (X) to each point. The resulting, implicitly designated, coordinates are then substituted in the Eneborg table calculation, presented as Table I on page 19 (4, p. 53).

Eneborg (4, p. 53) explains his table by indicating that it is necessary to multiply the units sold per year at each outlet by the outlet's distance from the vertical axis and to tabulate the finding in column 4 for each outlet.

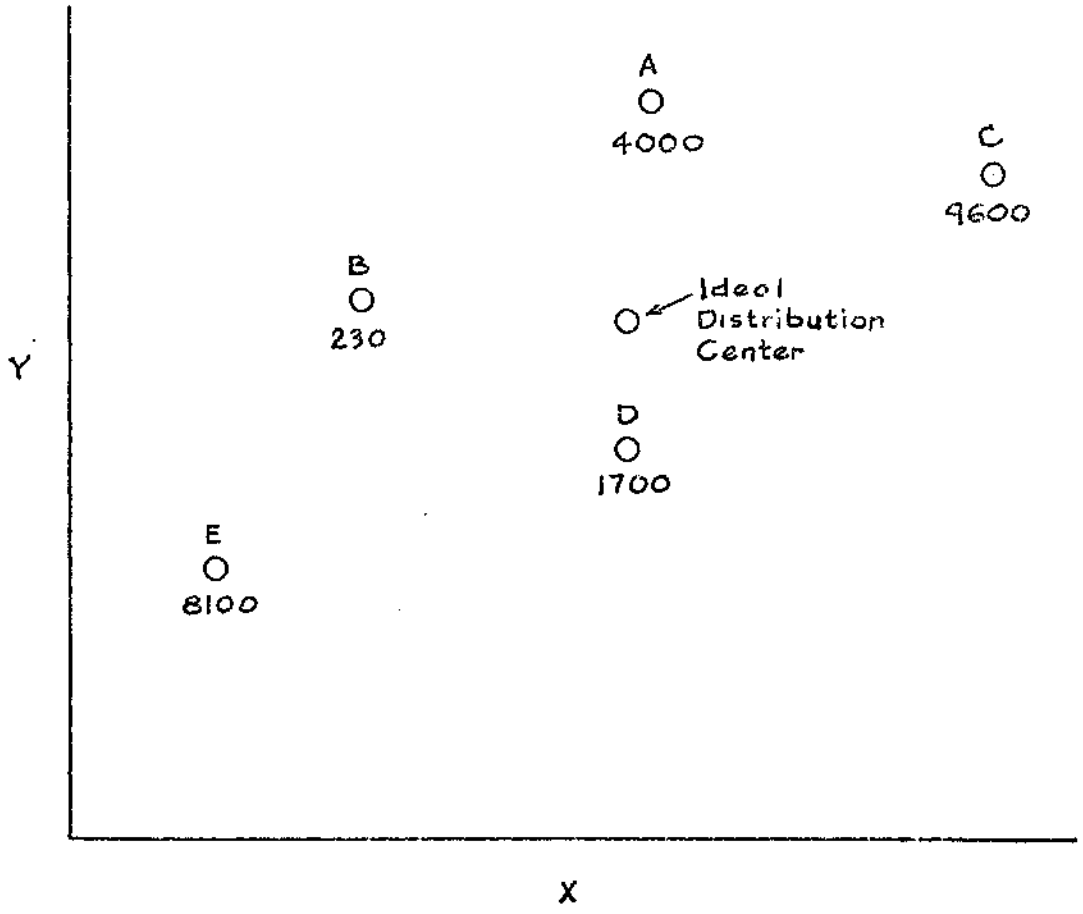

Fig. 2--Eneborg's example of sales outlets and associated weightings

Likewise, the same procedure is followed with regard to the horizontal axis.

Totals are then ascertained for columns 2, 4, and 6, with the total of column 2 being divided into the total of column 4 to determine the distance of the warehouse from the vertical axis (Y). The total of column 2 is then

TABLE I

ENEBORG CALCULATION TABLE

| Sales Outlet (1) | Units Sold Per Year (2) | Measured Distance From Vertical Axis (3) | Vertical Quantity-Distance Value (4) | Measured Distance From Horizontal Axis (5) | Horizontal Quantity-Distance Value (6) |
|---|---|---|---|---|---|
| A | 4,000 | 3" | 12,000 | 3 3/4" | 15,000 |
| B | 230 | 1 1/2" | 345 | 2 3/4" | 632.5 |
| C | 9,600 | 4 3/4" | 45,600 | 3 3/8" | 32,400 |
| D | 1,700 | 2 7/8" | 4,887.5 | 2" | 3,400 |
| E | 8,100 | 3/4" | 6,075 | 1 3/8" | 11,137.5 |
| Total | 23,630 | . . . | 68,907.5 | . . . | 62,570 |

divided into the total of column 6 to determine the distance of the warehouse from the horizontal axis (X). Using Table I as a basis produces a vertical coordinate value of 2 7/8", and a horizontal coordinate value of 2 5/8". The intersection of these coordinates is denoted in Figure 2.

<u>Survey of approach</u>.--Noteworthy in the Eneborg presentation is the complete lack of reference to the basic Cartesian coordinate approach. For example, all sales outlets were implicitly encompassed within the positive quadrant of a Cartesian coordinate system with the vertical axis implicitly denoting the ordinate and with the horizontal axis implicitly denoting the abscissa, nor was any

reference made to the mathematical technique used. Nevertheless, the approach demonstrates an integration of disciplines and lays forth a framework for further thought on warehouse optimization through a centroid location.

## 1959 Smykay Model

The 1958 Eneborg model was unique in that it purported to obtain a mathematical ideal location based only on the assigned information inputs of units sold per year and distance. Cost considerations were not included in the model and were viewed only as compromising forces.

However, the 1959 Smykay model (8) added a new dimension to the Cartesian coordinate approach to warehouse location through the inclusion of cost considerations. Thus, the weightings associated with each point covertly assumed the form of transportation costs. Again the implicit assumption is that a centroid location will result in the mathematical ideal based upon the assigned inputs, or a location which would produce the lowest total transportation expense. Such an assumption is obvious, mainly because the model follows the Cartesian coordinate approach for arriving at a centroid location, even though there is no reference made to this engineering principle. Moreover, the presentation of Smykay presents a semblance of proof that a mathematically determined location following an implicitly designated Cartesian coordinate system will result in the optimal location for a plant or warehouse. Additionally, the Smykay presentation gives consideration to supply points as well as customer points, hence both outbound and inbound shipments to the warehouse which is to be determined.

Methodological approach.--Turning to the methodological presentation of Smykay, it is necessary to present a map which includes the sales and purchasing territories of the selected firm. Also, it is generic to the task to superimpose upon this map a grid system so that tons of each shipment may be entered in the appropriate square. In this regard, Smykay (8, p. 32) indicates that too many squares tend to unduly complicate the analysis and that too few causes a lack of analytical detail. Ideally, Smykay (8, p. 32) indicates that the selection of square size should be coordinated with sales and purchasing data collection. Yet, no further elaboration is presented. The presentation of this map and superimposed grid system along with accompanying inputs is seen in Figure 3 (8, p. 33). By presenting this map and grid, Smykay has provided a framework for proving the potential optimality of a centroid location. Additionally, the map and superimposed grid serve as the basis for the development of the procedural steps to be followed in model implementation. In this regard Figure 3 denotes four weighted consumer and two supplier points, which make up the warehouse network.

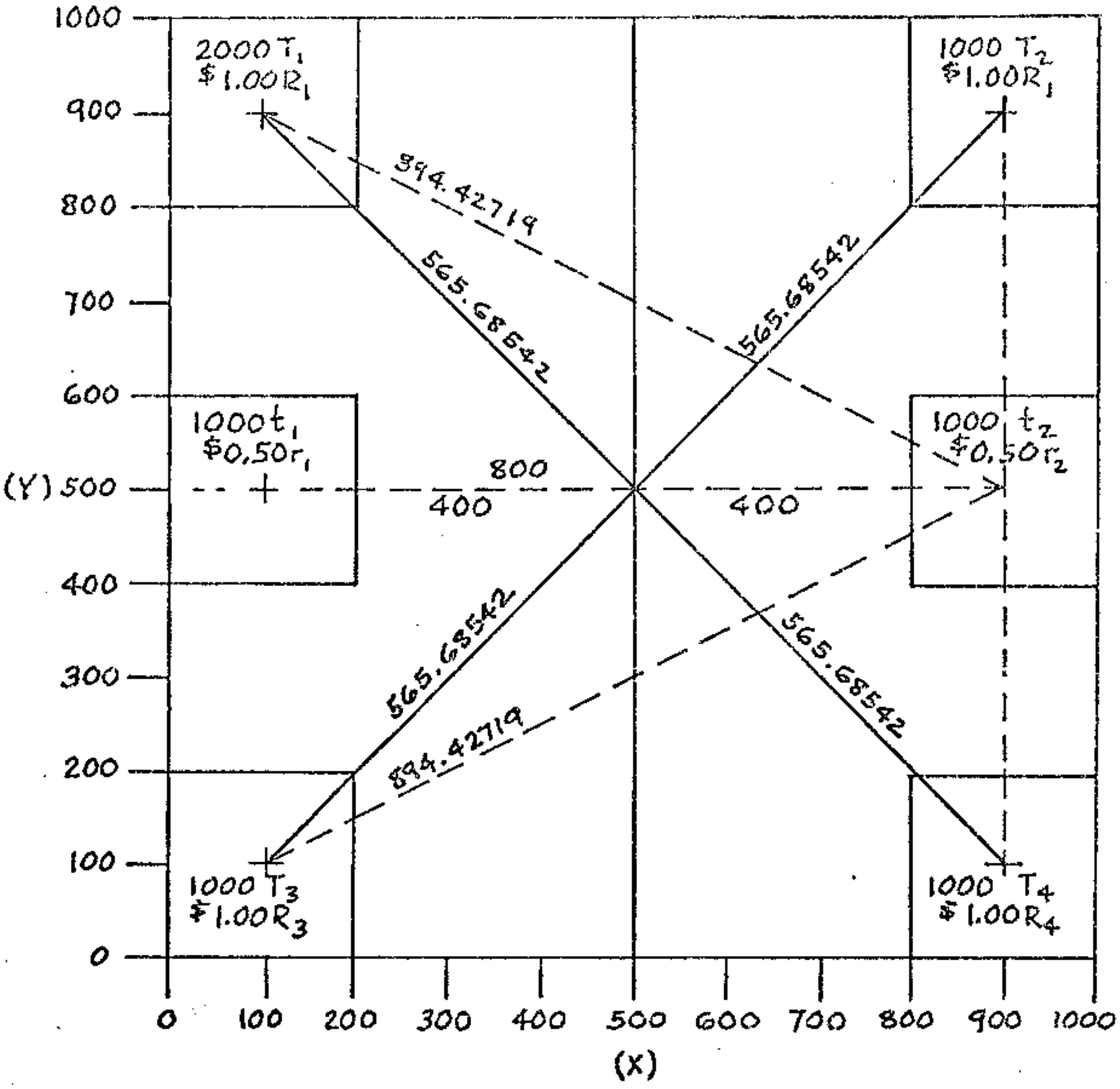

Fig. 3--Smykay map and grid presentation

In Figure 3,

TR = outbound weight and rate

tr = inbound weight and rate

_____ = mileage for calculated point X = 500, Y = 500

- - - = mileage for assumed point X = 900, Y = 500.

The additional steps presented by Smykay are as follows:

> 1. Code each block by some numerical system to enhance data collection by automatic means.
> 2. Lay out appropriate mileage scales which will be adequate for both horizontal and vertical axes.
> 3. Enter tonnages outbound (sales) and inbound (purchases) for each block.
> 4. Enter the average transportation cost per ton mile of individual commodities by inbound and outbound shipments. This will be a combination of class, exception, and commodity rates, which will reflect the transport bargaining power of the firm. These costs are not equal to the applicable rate, but include all costs on the freight bill.
> 5. Express transport costs per ton mile in each block by commodities as a ratio with the highest cost per ton mile having the base value of 1.00 (8, p. 32).

Smykay (8, p. 32) then applies the following formulas to determine the least transportation cost warehouse or plant location for the sales and purchases outlets in Figure 3:

$$\mathrm{Av} = \frac{[(DT_1+DT_3)R_1 + dt_1r_2 + (DT_2 + DT_4)R_1 + dt_2r_2]}{[(T_1 + T_3)R_1 + tr_1 + (T_2 + T_4)R_1 + tr_2]},$$

$$\mathrm{Ah} = \frac{[(DT_3 + DT_4)R_1 + (dt_1 + dt_2)r^2 + (DT_1 + DT_2)R_1]}{[(T_3 + T_4)(R_1 + (t_1 + t_2)r_2 + (T_1 + T_2)R_1]},$$

where,

Av = formula for vertical axis (produces vertical coordinate of centroid location)

Ah = formula for horizontal axis (produces horizontal coordinate of centroid location)

T = tons outbound

t = tons inbound

R = transport cost outbound

r = transport cost inbound

D = distance in miles outbound

d = distance in miles inbound.

Note: for purposes of simplification Smykay uses only one outbound and one inbound commodity. With this in view the usage of the information inputs presented in Figure 3 through substitution in the previously denoted formulas produces (8, p. 32):

$$Av = \frac{(100 \text{ X } 1000 + 100 \text{ X } 1000)1 + 100\ (1000).5}{(1000) + 1000)1 + (1000).5}$$

$$\frac{+\ (900 \text{ X } 1000 + 900 \text{ X } 1000)1 + 900\ (1000).5}{+\ (1000 + 1000)1 + 1000 \text{ X } .5}$$

$$Av = 500 \text{ miles,}$$

$$Ah = \frac{100(1000 + 1000)1 + 500\ (1000 + 1000).5}{(1000 + 1000)1 + (1000 + 1000).5}$$

$$\frac{+\ 900(1000 + 1000)1}{+\ (1000 + 1000)\ 1}$$

$$Ah = 500 \text{ miles.}$$

The coordinates (of the centroid) for Figure 3, according to Smykay (8, p. 33), reflect the lowest sum of the products of distance, weight and rate for all movements. This occurs on the vertical axis at the 500 mile point and on the horizontal axis at the 500 mile point for Figure 3.

Naturally, since the system in Figure 3 has been constructed symmetrically, the midpoint consisting of the previously delineated coordinates would be expected to produce the lowest total transport cost location. However, Smykay (8, p. 33) indicates that "even if the system were nonsymmetrical the same method of analysis would yield lowest transportation cost." Yet no proof of this assertion is presented.

Cost considerations.--Sighting in on the calculation of costs Smykay (8, p. 33) indicates the following:

> Calculation of the actual costs are found by multiplying the tons of haul in each block by the applicable rate and the distance of that block from the least cost point. In this case it is assumed that the rate relationships employed in the analysis are equal to the actual rates. This means that the transport costs per ton on outbound are $1.00 per ton mile and the costs inbound are 50¢ per ton mile.
>
> Distances from the calculated point for $T_1$, $T_2$, $T_3$, $T_4$, may be found by the Pythagorean Theorem and are found to be 565.6854 miles. Distances for $t_1$ and $t_2$ may be found by reading directly from the horizontal scale. These are found to be 400 miles (500 - 100 and 900 - 500) (8, p. 33).

With this in view, the calculation of costs involving coordinates 500 and 500 is depicted by Smykay (8, p. 33) in Table II.

TABLE II

SMYKAY COST CALCULATION FROM COORDINATES
(X)500 AND (Y)500

| Points (1) | Tonnages (2) | Rates (3) | Distances (4) | Costs (2) X (3) X (4) |
|---|---|---|---|---|
| $T_1$ | 1,000 | $1.00 | 565.68542 | $ 565,685.42 |
| $T_2$ | 1,000 | 1.00 | 565.68542 | 565,685.42 |
| $T_3$ | 1,000 | 1.00 | 565.68542 | 565,685.42 |
| $T_4$ | 1,000 | 1.00 | 565.68542 | 565,685.42 |
| $t_1$ | 1,000 | .50 | 400.00000 | 200,000.00 |
| Total | | . . . | | $2,662,741.68 |

Likewise, cost calculations for the arbitrary location with coordinates (X)900 and (Y)500 is presented in Table III.

According to Smykay (8, p. 33), the selection of any point other than 500 and 500 will always yield a differential in favor of 500 and 500. As a result of this analysis a semblance of proof attesting to the optimizing ability of a centroid location has been presented.

TABLE III

SMYKAY COST CALCULATION FROM COORDINATES
(X)900 AND (Y)500

| Points (1) | Tonnages (2) | Rates (3) | Distances (4) | Costs (2) X (3) X (4) |
|---|---|---|---|---|
| $T_1$ | 1,000 | $1.00 | 894.42719 | $ 894,427.19 |
| $T_2$ | 1,000 | 1.00 | 400.00000 | 400,000.00 |
| $T_3$ | 1,000 | 1.00 | 894.42719 | 894,427.19 |
| $T_4$ | 1,000 | 1.00 | 400.00000 | 400,000.00 |
| $t_1$ | 1,000 | .50 | 800,00000 | 400,000.00 |
| $t_2$ | 1,000 | .50 | 000.00000 | 000,000.00 |
| Total* | | . . . | | $2,988,854.38 |
| Total** | | . . . | | $2,662,741.68 |

*X = 900, Y = 500. **X = 500, Y = 500.

Survey of approach.--It is now evident that the Smykay presentation has supplemented the state of the art for warehouse location. However, certain elements of the presentation remain cloudy. For example, the assumptions behind and the significance of the grid system are not immediately clear. Further, the assumptions underlying freight rate determination have not been presented, nor has the methodology for freight rate determination. Also, no reference is made to the time period over which the inclusion of outbound and inbound tonnages in the model is manifest. And most significant is the lack of proof or documentation attesting to his assertion that a model determined (centroid) location would yield the lowest transportation cost even if the system were asymmetrical. Yet in spite of these discernible limitations a base for further Smykay elaboration has emerged.

### 1961 Smykay Elaboration

Edward W. Smykay's publications dealing with plant and warehouse location were not restricted to his "Formula to Check for a Plant Site" (8) article. In 1961 Smykay, Bowersox, and Mossman authored the pioneering text *Physical Distribution Management* (9) (pioneering in that it marked the initial attempt to develop the integration of corporate physical distribution activities), which elaborated upon the earlier Smykay work. As a result, there is need to give further attention to the Smykay model.

*Clarification of Keefer*.--A key particle of the Smykay elaboration centered on making explicit that which was implicit in the earlier Keefer (6) article. In this regard it was indicated that "the fulcrum at which the balance was achieved was the center of gravity of the system (9, p. 177)." This fulcrum or center of gravity, according to Smykay, Bowersox, and Mossman (9, p. 181), yields the ton mile center, which is the same as least ton miles.

Reference to Cartesian coordinate system.--The 1961 Smykay elaboration also clarifies the usage of the Cartesian coordinate system, which was implicit in the earlier Smykay writing. This was accomplished by reference to the positive quadrant of a Cartesian coordinate system, within which all customer and supplier points are encompassed (9, p. 180).

The implicit Cartesian coordinate formulation is presented as follows (9, p. 182):

$$A = \frac{\Sigma DT + \Sigma dt}{\Sigma T + \Sigma t},$$

where

$A$ = axis

$Ah$ = horizontal axis

$Av$ = vertical axis

$\Sigma DT$ = summation of product sums of distance and outbound tonnage

$\Sigma T$ = summation of outbound tonnages

$\Sigma dt$ = summation of product sums of distance and inbound tonnages

$\Sigma t$ = summation of inbound tonnages.

This particular formula is viewed by Smykay, Bowersox, and Mossman (9, p. 182) as the general formulation employed in determining the least ton mile center. However, the formula is implicitly viewed as a least cost determining point based only on the information inputs of tonnage and mileage.

Asymmetric system.--Smykay, Bowersox and Mossman also point out the applicability of the coordinate system when applied to an asymmetrical market situation. In this regard the discussion revolves around the grid and information inputs presented in Figure 4.

Based upon Figure 4, it is denoted "that the value of the vertical axis will probably be something less than 500, and for the horizontal axis, it is likely to be more than 500 (9, p. 185)." The verification of this location is presented as follows (9, p. 185):

$$Av = \frac{100\ (4{,}000) + 900\ (3{,}000)}{4{,}000 + 3{,}000}$$

$$Av = 448.6 \text{ miles}$$

where Av = vertical axis coordinate,

$$Ah = \frac{100\ (2{,}000) + 500\ (2{,}000) + 900\ (3{,}000)}{2{,}000 + 2{,}000 + 3{,}000}$$

$$Ah = 500 \text{ miles}$$

where Ah = horizontal axis coordinate.

Smykay, Bowersox, and Mossman (9, p. 185) then indicate that the results are in agreement with the conclusion as to the general location of the least ton mile center, when $T_1$ is equal to 2,000 tons. Moreover, the resulting location is

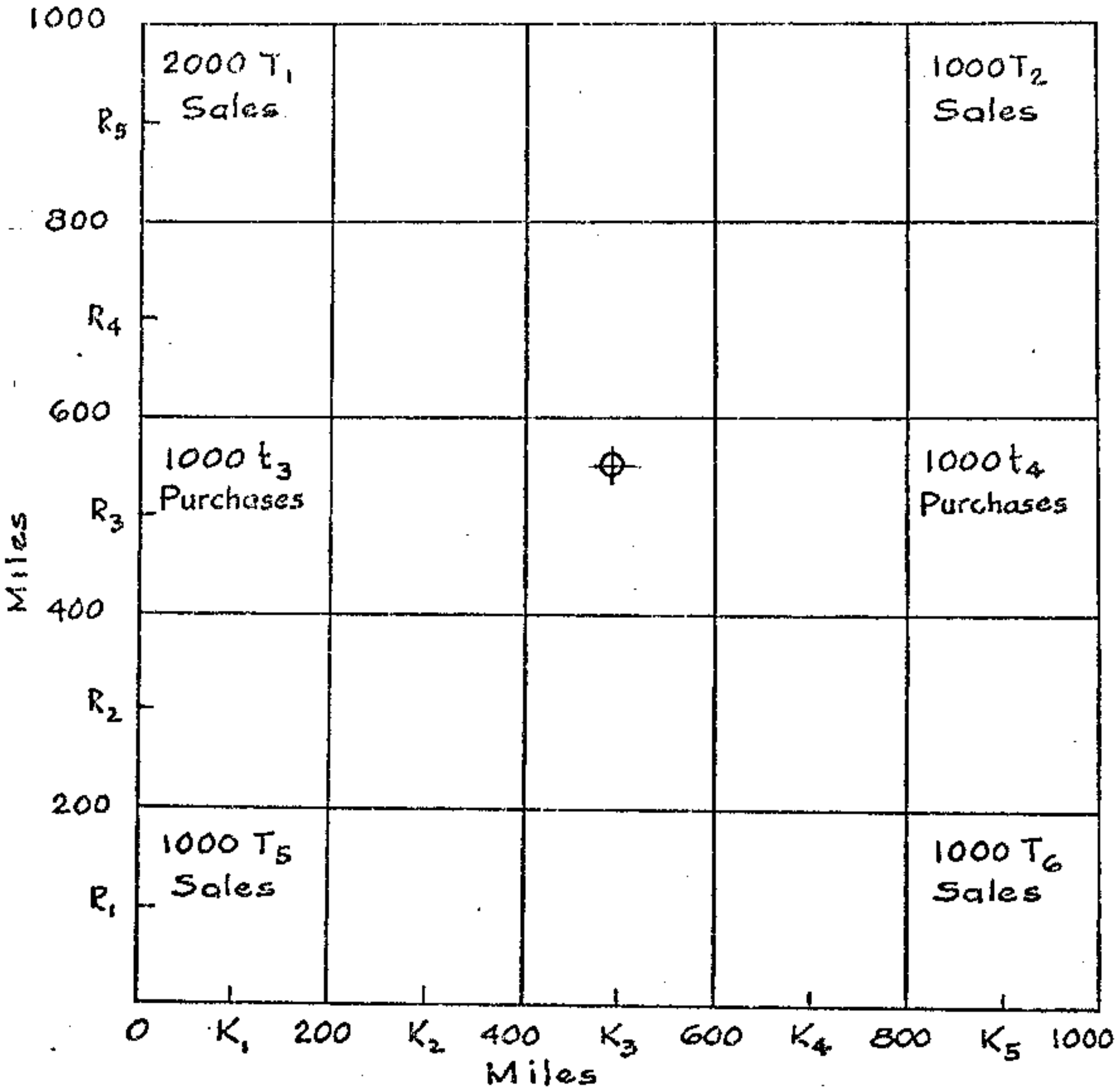

Fig. 4--Smykay grid and accompanying information inputs

viewed as the least ton mile center (9, p. 185). Thus, in the 1961 Smykay elaboration, methodological application of the coordinate system to what is viewed as an asymmetric system is presented. Yet no proof of the optimizing abilities of the new location (centroid) is presented, other than the earlier denoted assertion that the ton mile center is the same as least ton miles (9, p. 181).

Cost elements.--In the analysis of warehouse location, the information necessary for the resolution of this problem is weight rate and distance (9, p. 178). Smykay, Bowersox, and Mossman (9, p. 186) also indicate that rates would involve no consideration in the model framework if transport facilities were completely homogeneously distributed and if rates were linear with distance. If this were the case the ton mile center would be the least cost center. However, since this is not the case, it is denoted that the least ton mile center calculation is the starting point of the next stage of analysis (9, p. 183).

This next stage refers to the determination of freight costs from the least ton mile center. Here, Smykay, Bowersox, and Mossman (9, p. 187) indicate that the traffic department can determine the probable freight rates that will apply on inbound and outbound traffic to and from the least cost ton mile center (centroid). The assertion is also made that the probabilities are very high that the final least cost location will be in the general area of the ton mile center (9, p. 187). However, no proof is presented attesting to this statement, nor is any freight rate analysis provided.

Once freight rates have been determined, the general Smykay, Bowersox, and Mossman least ton mile formula would appear as follows:

$$A = \frac{\Sigma DTC + \Sigma dtc}{\Sigma TC + \Sigma tc} ,$$

where

$A$ = axis

$Ah$ = horizontal axis

$Av$ = vertical axis

$\Sigma DTC$ = summation of product sums of distance, outbound tonnages and costs

$\Sigma TC$ = summation of product sums of outbound tonnages and costs

$\Sigma dtc$ = summation of product sums of distance, outbound tonnages and costs

$\Sigma tc$ = summation of product sums of inbound tonnages and costs.

After developing the freight rate inclusions in the least ton mile center formulation, Smykay, Bowersox, and Mossman utilized the same grid and inputs depicted in the earlier Smykay writing (Figure 3), to reflect the usage of the model. Proof of the model's accuracy was then presented by denoting total transportation costs from an arbitrary point other than the least cost point and comparing these with the transportation costs emanating from the ton mile cost center (centroid). The same virtual proof was depicted in the 1959 Smykay model.

Grid development.--In the 1959 Smykay presentation the significance of the grid system was at best understood. However, in the 1961 Smykay elaboration it was indicated that the grid served to reduce the number of items in the final working equation (9, p. 182). Turning to Figure 4 gives evidence to a series of rows (R) and columns (K). Therefore, by summing the weights associated with each respective row and column, the multiplication of mileage distances from the vertical and horizontal axes to the points in question are reduced. These points refer to the midpoint in each square, and each grid square assumes that tonnages are distributed homogeneously.

Smykay contribution.--With the basic Smykay oriented elaboration now in view, the first comprehensive presentation of least-cost center analysis is easily discernible. As a result of this elaboration the optimizing role of a centroid location has been further attested. For example, the ton mile center (centroid) is viewed as being synonymous with least ton miles. Proof of the conception was presented through cost comparisons. And the centroid determining model is viewed as the formula for the least ton mile center. Also, the application of the model was applied to an asymmetrical system, yet no explicit proof was presented, nor were any cost comparisons attesting to the optimizing abilities of such a centroid location. However, a deeper base of analysis has been presented, and this base will serve to facilitate future developments in the field.

## 1964 Heskett, Ivie, Glaskowsky Model

Another form of the Cartesian coordinate approach to least transportation cost analysis appeared in the 1964 physical distribution text, Business Logistics (5). The approach utilized borrows from the Smykay methodology for least-cost center analysis as evidenced by a direct reference to the text Physical Distribution Management. Again the implicit assumption is that location at a centroid, based upon the assigned inputs, will produce the actual least cost location. It is also indicated that least cost center analysis may well offer the best opportunity for use by management (5, p. 182).

Basically, the approach is presented in the framework of plant location and relocation. However, it is indicated that "the same procedure by which a single plant location is determined on the basis of incoming and outgoing logistics movements applies equally well to a single warehouse (5, p. 203)."

Model methodology.--The methodological approach itself centers on the case of the Easthampton Comclean Corporation, a mythical manufacturer of an industrial cleaning compound. This particular corporation is located in Easthampton, Massachusetts and is supplied raw materials in the form of sawdust and chemicals from Pittsfield, Massachusetts. With this in view the objective is the determination of the site where the Comclean Corporation should be located based on transportation costs. Figure 5 presents these locations and a plant location grid (5, p. 183).

Once the grid system has been superimposed over the locations in question, the determination of the least cost location for Comclean involves a weighting of relative costs to move raw materials and finished products to and from sources and markets, respectively (5, p. 183). The information needed to complete this ultimate weighting is shown in Table IV (5, p. 184).

TABLE IV

SUMMARY OF MATERIALS USED AND SHIPPED, TRANSPORTATION COSTS, AND SUPPLY SOURCE AND CUSTOMER LOCATION

| Product | Material Used or Finished Product Shipped (cwt.) | Transportation Cost Per Distance Unit (5 miles) Per cwt. | Location Cost Factor | Supply Source and Customer Location | |
|---|---|---|---|---|---|
| | | | | Grid Row Number | Grid Column Number |
| Sawdust | 2,000 | $.025 | $50.00 | 2 | 8 |
| Chemicals | 500 | .075 | 37.50 | 11 | 3 |
| Comclean | 2,000 | .031 | 62.00 | 6 | 2 |

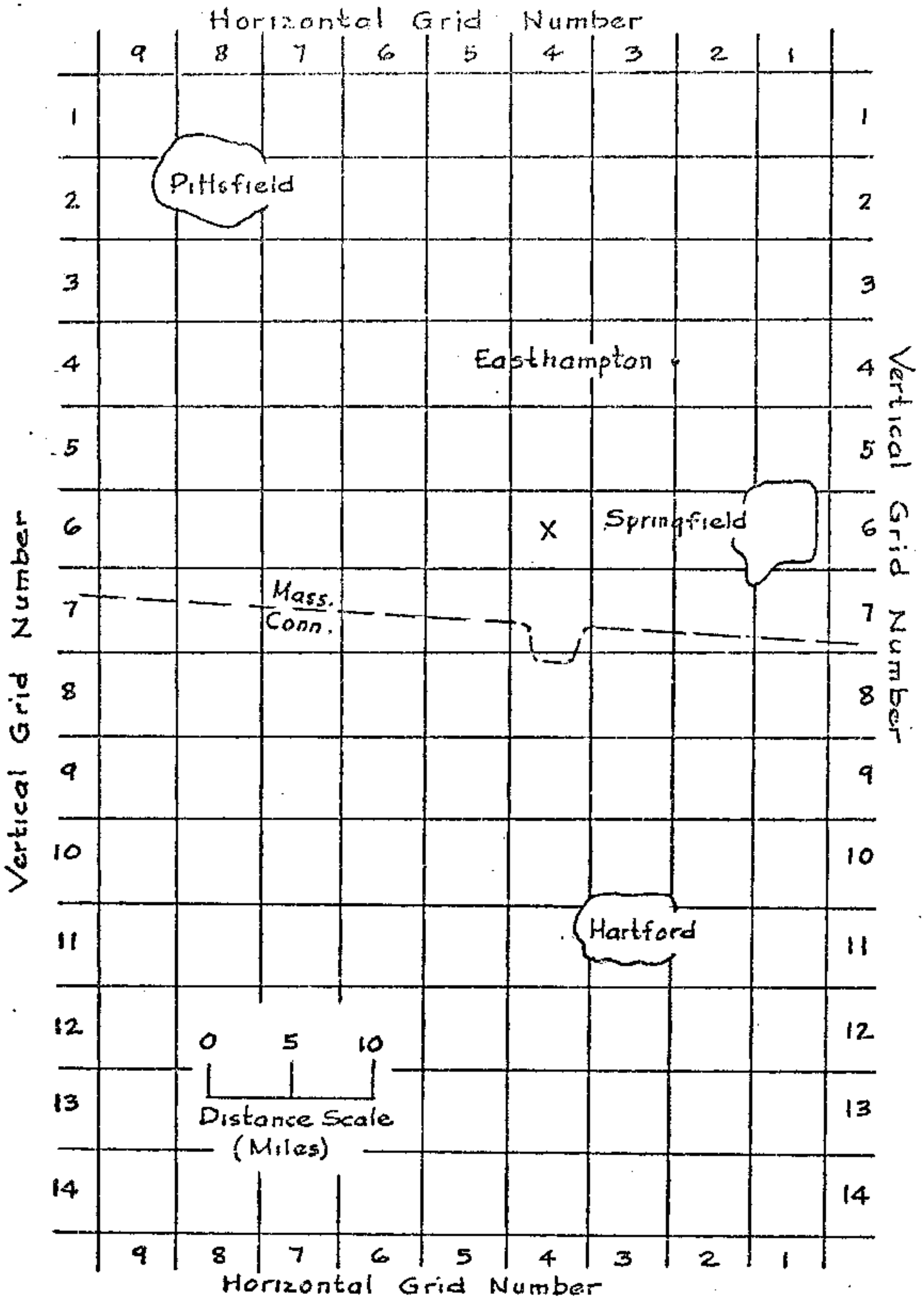

Fig. 5--Plant location grid, Comclean Corporation

Note, Table IV presents a location cost factor which serves as a basis for the final coordinate formulas.

The next step in this weighting is the one for distance (5, p. 184). To accomplish this end all squares in Figure 4 have been numbered both vertically and horizontally to find the point at which all weightings are in balance (5, p. 184). Such a weighting involves multiplying each location cost factor by each row and column number, and summing the results. Or what amounts to the determination of the numerator of the Cartesian coordinate formulation for both the X and Y axis. The grid row numerator and grid column numerator are then divided by the summation of location cost factors to determine the respective grid row and column coordinates. The inputs necessary to determine the grid row coordinate are presented in Table V (5, p. 185).

TABLE V

INPUTS FOR GRID ROW DETERMINATION

| Location Cost Factor | Grid Row Number | Weight |
|---|---|---|
| $50.00 | 2 | 100.0 |
| 37.50 | 11 | 412.5 |
| 62.00 | 6 | 372.0 |
| $149.50 | . . . | 884.5 |

Based upon these inputs the calculation of the grid row midpoint is calculated as follows:

$$\text{Grid Row Midpoint} = \frac{884.5}{149.5} = 5.9.$$

The inputs necessary to determine the grid column candidate are presented in Table VI (5, p. 185).

TABLE VI

INPUTS FOR GRID COLUMN DETERMINATION

| Location Cost Factor | Grid Column Number | Weight |
|---|---|---|
| $50.00 | 8 | 400.0 |
| 37.50 | 3 | 112.5 |
| 62.00 | 2 | 124.0 |
| $149.5 | . . . | 636.5 |

Based upon these inputs the calculation of the grid column midpoint appears as follows:

$$\text{Grid Column Midpoint} = \frac{636.5}{149.5} = 4.2.$$

The results of these calculations thus produce the theoretical optimum location for the Comclean plant at the midpoint of row 6 and column 4. Naturally, based upon the designated inputs, the application of any Cartesian coordinate approach would have produced the same initial location.

It should now be apparent that the presented methodological approach has been primarily descriptive rather than theoretical, and that new terminology has been applied to the basic Cartesian coordinate system.

*Limitations*.--The 1964 publication does present some limitations of the grid system. First, it is indicated that it assumes a linear relation of transportation cost with distance, which is not the case in reality (5, p. 189). Also, unless large numbers of squares are drawn in the grid, or a large detailed map is used, results will not be specific in nature (5, p. 189).

It is also indicated that all supply and market points are given the value of the square in which they are located, thereby assuming that they are located in the center of the square (5, p. 189). Of course, such an assumption is questionable. Additionally, no information is presented concerning freight rate determination. However, the presented method serves to further reinforce the centroid determining approach.

### *The 1965 Mossman and Morton Model*

The next key publication concerning the methodological approach for warehouse location at a point where movement costs are minimized appeared in the Mossman and Morton text, *Logistics of Distribution Systems* (7). The approach depicted is an explicit Cartesian coordinate system which determines an implicit centroid location by taking into consideration the elements of cost, tonnage, and mileage. Such a methodology is referenced as originally appearing in Smykay's article, "Formula to Check for a Plant Site."

Methodological Approach.--Turning to methodology per se it is denoted that "a Cartesian coordinate system will be utilized to determine the point at which the cost-ton-mile movement between given volume and less than volume destinations will be at a minimum (7, p. 239)." The methodology involved is simply presented as follows:

> First, make sure that all points of origin and destination are included (within the X and Y axis); second, assign mileage scales to both the X and Y axes; third, locate the points of origin and distribution with respect to the zero value; fourth, indicate for each point the amount of tonnage to be shipped from or to that point; fifth, determine the applicable cost per ton-mile for movement; sixth, compute a weighted average of the cost-ton-mile forces for both the X and Y axes. The intersection of the coordinates erected for these weighted averages will indicate the point of least cost-ton-mile movement (7, p. 240).

The application of these steps begins with Figure 6, which depicts the positive coordinate of a Cartesian coordinate system, and accompanying weighted points of origin and distribution. This presentation appears on the following page (7, p. 241).

In computing the weighted average of the inputs in Figure 6, the object is to determine the coordinates for

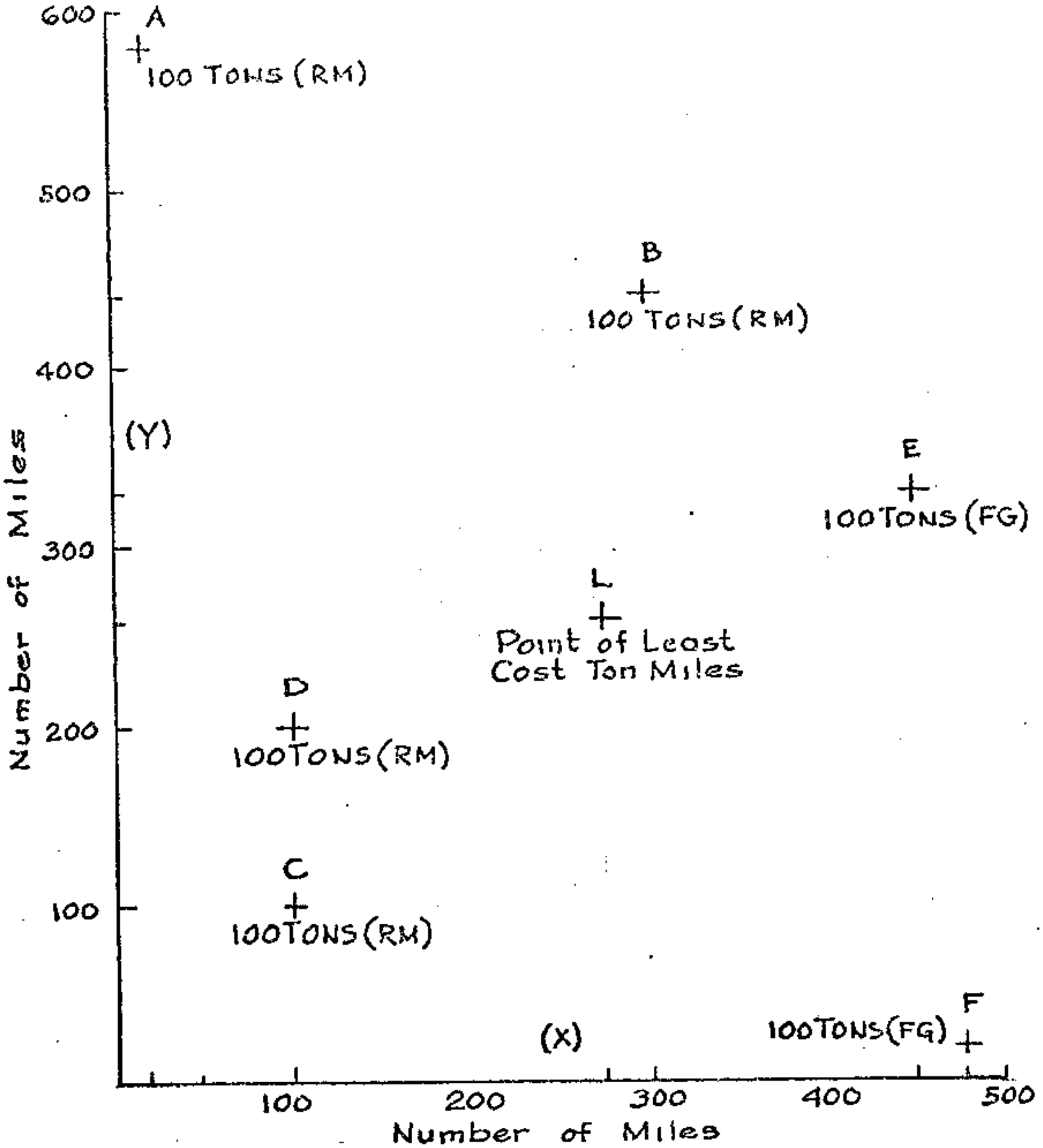

Fig. 6--Illustration of data for computation of least cost ton miles

In Figure 6, volume shipment cost--$.03 per ton mile. Less than volume shipment cost--$.06 per ton mile. ABC are points of origin. DEF are markets and points of destination.

each of the X and Y axes. Presentation of this methodology for the X axis is as follows (7, p. 242):

$$X = \frac{\text{the sum of (cost x weight x miles)}}{\text{the sum of (cost x weight)}} ,$$

$$X = \frac{\text{the sum of cost weight miles}}{\text{the sum of cost weights}} ,$$

$$X = \text{a weighted average in miles on the X axis.}$$

Likewise, a similar computation would be made for the weighted average on the Y axis.

Substituting in the respective equations produces the following (7, p. 242):

$$X = \frac{(.03 \times 100 \times 20) + (.03 \times 100 \times 300) + (.03 \times 100 \times 100)}{(.03 \times 100) + (.03 \times 100) + (.03 \times 100)}$$

$$\frac{+ (.06 \times 100 \times 100) + (.06 \times 100 \times 450)}{+ (.06 \times 100) + (.06 \times 100)}$$

$$\frac{+ (.06 \times 100 \times 480)}{+ (.06 \times 100)}$$

$$X = \frac{7440}{27} = 275.6$$

$$Y = \frac{(.03 \times 100 \times 580) + (.03 \times 100 \times 540) + (.03 \times 100 \times 100)}{(.03 \times 100) + (.03 \times 100) + (.03 \times 100)}$$

$$\frac{+ (.06 \times 100 \times 200) + (.06 \times 100 \times 330)}{(.06 \times 100) + (.06 \times 100)}$$

$$\frac{+ (.06 \times 100 \times 20)}{+ (.06 \times 100)}$$

$$Y = \frac{6960}{27} = 257.8.$$

The point of least cost ton miles will be at the intersection of the 275.6 mile point on the X axis and at the

257.8 mile point on the Y axis. This location depicted as Point "L" is the location where movement costs will be minimized with respect to the cost per ton mile, the tonnages moved, and the mileages involved (7, p. 243).

*Survey of the approach.*--Note that the presented methodology has abandoned the grid network utilized by both Smykay and Heskett in their presentations. As a result the model assumes a higher degree of simplicity. However, no information relating to freight rate determination is presented, nor are any limitations concerning the model's ability to generate a least cost location. Yet, the presentation of the model has also served to further attest to the validity of the approach.

## *1966 Constantin Model*

A further presentation of an implicit Cartesian coördinate approach appeared in Constantin's text, *Principles of Logistics Management* (3). Accordingly, the presented system was denoted as being a modification of the earlier Smykay approach (3, p. 540). However, any distinctions between the two centers squarely on terminology. Further, for the sake of simplicity, the Constantin model assumes all transportation costs are the same per ton mile. As a result, the ton-mile center (centroid) is viewed as being the actual point of least cost.

Nature of the approach.--The system presented by Constantin is viewed as a grid system approach, yet no grids are utilized in the implicit determination of a centroid location. However, the following implicitly designated positive quadrant of a Cartesian coordinate system is presented in Figure 7 along with accompanying information inputs (3, p. 541).

In this particular illustration Constantin (3, p. 541) indicates that only outbound warehouse shipments are utilized to keep the illustration simple. Hence, points A-I are designated as customers or recipients of warehouse shipments. To facilitate the calculation of the ton-mile center for these customers the following information is presented by Constantin (3, p. 542) in Table VII on page 47. Based upon this information the calculation of the optimum location is as follows (3, p. 542):

$$\frac{\text{Ton-Miles to the North}}{\text{Tons to the North}} = \frac{113{,}000}{440} = 256.8 \text{ miles north}$$

$$\frac{\text{Ton-miles to the East}}{\text{Tons to the East}} = \frac{106{,}000}{440} = 240.9 \text{ miles east}$$

Clarifying the calculations, Constantin depicts the weighting of tonnages by multiplying sold tonnages by the distance "north" of the origin (horizontal axis) for each customer, then summing them and dividing by the summation of total tonnages sold (3, p. 540). The same procedure is repeated, only sold tonnages are multiplied by the distance "east" of

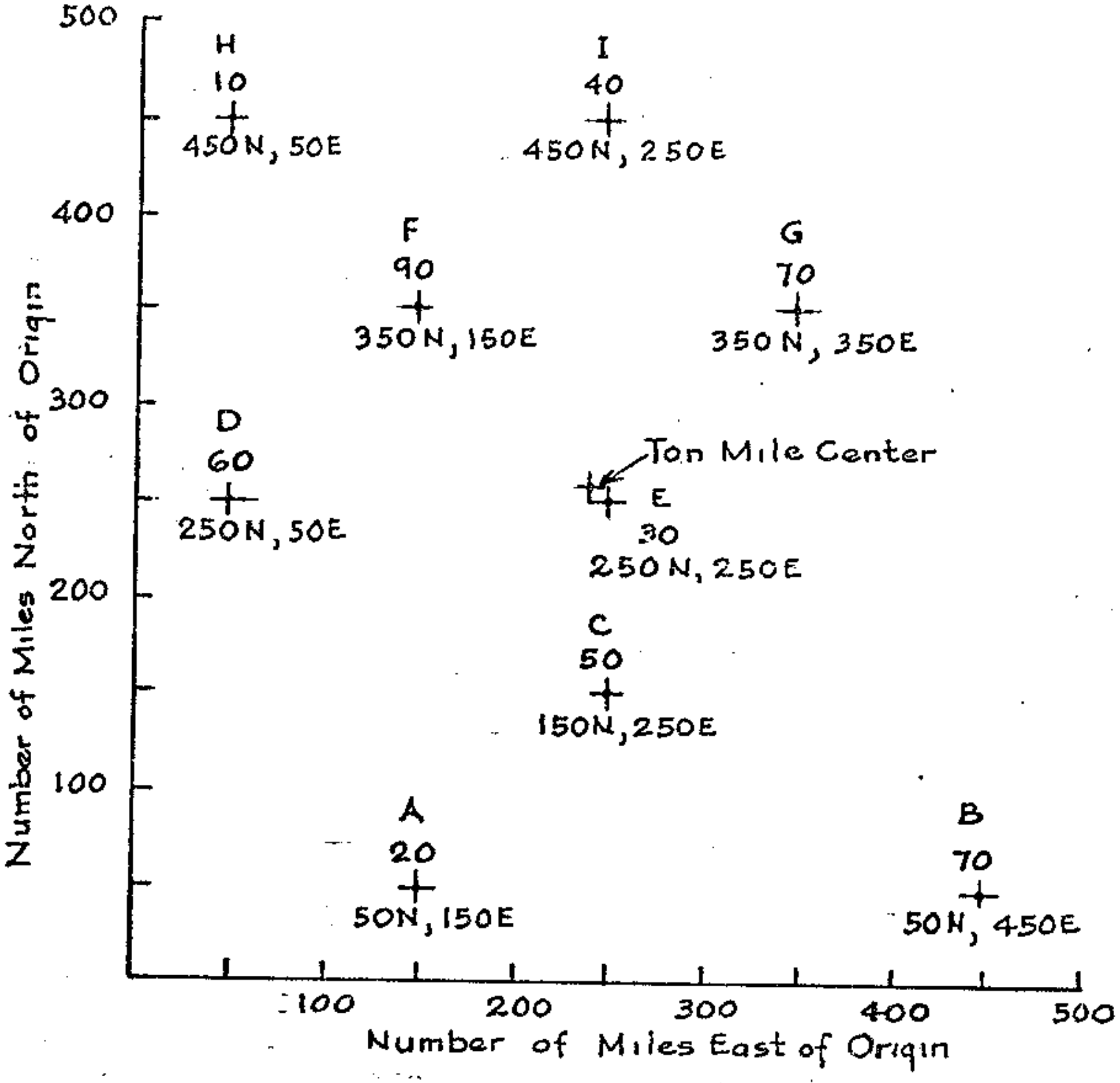

Fig. 7--Geographic location of customers and number of tons shipped to each location

TABLE VII

SUMMARY CALCULATIONS OF LEAST COST TERMINAL LOCATION

| (1) Customer | (2) Number of Tons Sold | (3) Number Miles North of Origin | (4) Number Miles East of Origin | (5) Weighted Pull to North (3) x (2) | (6) Weighted Pull to East (4) x (2) |
|---|---|---|---|---|---|
| A | 20 | 50 | 150 | 1,000 | 3,000 |
| B | 70 | 50 | 450 | 3,500 | 31,500 |
| C | 50 | 150 | 250 | 7,500 | 12,500 |
| D | 60 | 250 | 50 | 15,000 | 3,000 |
| E | 30 | 250 | 250 | 7,500 | 7,500 |
| F | 90 | 350 | 150 | 31,500 | 13,500 |
| G | 70 | 350 | 350 | 24,500 | 24,500 |
| H | 10 | 450 | 50 | 4,500 | 500 |
| I | 40 | 450 | 250 | 18,000 | 10,000 |
| Total | 440 | . . . | . . . | 113,000 | 106,000 |

the vertical axis. The resulting miles north and east indicate the coordinates of the ton-mile centroid.

<u>Survey of approach</u>.--While the Constantin system is essentially the same as any depicted Cartesian coordinate approach the introduction of directions represents new angles of terminology. However, the approach is primarily a descriptive one concerning methodology, and is characterized by a lack of limitations or strengths, or other

theoretical assertions. Yet, its inclusion in Constantin's text helps to depict the approach as a traditional tool of the physical distribution scientist.

## 1966 Lewis and McBean Model

Another development involving the usage of a Cartesian coordinate system was submitted by the Chicago and North Western Railway Company's Industrial Development Department to Industrial Marketing's marketing research competition. As a result of this presentation a Silver Medal was awarded to this department for "creating a new and utilitarian concept in this area (2, p. 61)." However, in the light of the earlier presented models the newness and the utilitarian conceptual uniqueness of the approach is open to question.

Basically, the model presentation of the coordinate approach was attributed to Eugene M. Lewis, Industrial Development Agent, and Donald A. McBean, Industrial Development Analyst. According to Mr. Lewis:

> The only significant remaining area for cost reduction in marketing is physical distribution. Little or nothing has been done in the area of precisely and scientifically determining the center of any given market. This has been left to "feel," "judgment," "intuition," and "educated guess" (2, p. 61).

Again the implicit premise is that transportation costs from a mathematically determined centroid location will result in the minimization of transportation costs. Basic to this premise is the Lewis and McBean (2, p. 61) indication that "very broadly, transportation costs are directly proportional to the distance a product is carried, (therefore) it can be theoretically assumed that freight costs are minimized by distribution from the exact center of the market." The location of this precise equilibrium point, according to Lewis and McBean (2, p. 61), is determined through the usage of an engineering principle: determination of the center of gravity of a bi-planer nonhomogeneous body. Bi-planer refers to the earth's surface and nonhomogeneous means the varying sizes of consuming points.

Methodological explanation.--According to Mr. Lewis (2, pp. 61-62), the determination of the center of gravity is applied by a grid-coordinate method using longitude and latitude figures to come up with a mathematically related identification number for each consuming point. Once these coordinates are determined, they are electronically matched with the variable, for example sales volume, through the alphabetical spelling of each consuming point. The computer then multiplies units of volume for each consumer point by the matching coordinates, totals them, and divides by the total units of volume. Lastly, the end figures make up the coordinates of the market's sales equilibrium point or center of gravity.

To facilitate further calculations of centroid locations, Lewis and McBean (2, p. 62) have also developed their own coordinate system. This was accomplished by using Dun and Bradstreet lists of cities and towns, and assigning each longitude and latitude coordinate. Additionally, coordinates were assigned to every identifiable town, crossroad, community, and railroad station in an eleven-state area. Finally, rather than concentrating the reference system in a limited geographical area, plans for expansion included referencing the entire United States and Canada. As a result, a permanent classification system will evolve.

Survey of approach.--The methodological procedure is now denoted as being precisely the same as the other depicted approaches. However, the usage of longitude and latitude as coordinates for centroid determination adds sophistication to the approach. Yet, it is also evident that the approach is primarily concerned with consuming points and does not reflect the importance of incoming shipments on plant and warehouse locations. However, a consideration of incoming shipments would not warrant the premise that transportation costs are directly proportional to distance, as a distinction would commonly have to be made between volume incoming shipments and less than volume outgoing shipments. It is now apparent that Lewis and McBean have not directly considered the impact of freight rates, and that, therefore, they are focusing their emphasis on minimizing the distances associated with the weight assignments to each consuming point.

It is also apparent that earlier centroid determining models have been overlooked in the Lewis and McBean presentation. As a result the technique has been introduced without reliance on any of the earlier presented bases. Thus, the assumption of locational optimization remains with Lewis and McBean, yet the tenets behind the assumption are not presented.

## The Trial and Error Model

While the centroid determination models have been most evident in the literature of optimum warehouse location, the trial and error model has only made a brief appearance. Basically, the trial and error model involves the Cartesian coordinate approach of the centroid determination models, yet this model arrives at an optimum site location through the arbitrary selection of a potential optimum warehouse location, which is either approved or disapproved. Most likely, the selected location will be disapproved, hence another point must be chosen and the approach repeated. Finally, through the continuation of this procedure the optimum site becomes discernible.

With this brief construct now in view, emphasis will center on presenting this approach.

## The 1955 Trial and Error Publication

The trial and error approach to optimum warehouse location was presented, anonymously, in *Materials Handling Manual, Number One* (10). This publication was primarily concerned with the presentation of a methodology where total ton miles for a group of chain stores would be at a minimum. No consideration was given to the inclusion of freight rates in the model due to the assumption that transportation costs were constant or linear. Moreover, no consideration was given to inbound shipments to the to-be-determined warehouse location.

Rather than placing reliance on a centroid location to generate ton mile minimization, it was indicated that "the center of moments (centroid) does not result in minimum ton miles (10, p. 84)." The alleged proof of this statement was presented as follows:

> Assume two stores A and B are located 10 miles apart on a straight highway. Assume store A needs 30 tons per week and store B needs 70 tons per week. The Center of Moments CM for these two stores would be 7 miles from A and 3 miles from B resulting in a total weekly ton miles of 30 x 7 + 70 x 3 = 420 miles.
>
> Now assume that the warehouse was located at store B. The resulting ton miles per week would be 30 x 10 + 70 x 0 = 300 ton miles (10, p. 84).

It was then held that "the above example should prove without doubt that locating a warehouse at the center of moments does not result in minimum ton miles (10, p. 84)." Other than the example, no proof was presented attesting to this statement. However, it was indicated that the amount of error produced by locating at a centroid location will be quite serious where the distribution of stores and tonnages was asymmetrical (10, p. 85).

In the 1961 Smykay elaboration reference to this 1955 trial and error publication was included in the selected bibliography. Therefore, apparently in response to the presented example of the lack of applicability of the centroid location, Smykay, Bowersox, and Mossman indicate the following:

> If, for example, the source of raw material were located at a single point and the market were located at another single point, then the economic location would generally tend to be at one of these two points. However, in operational circumstances, markets are seldom, if ever, located at a single point (9, p. 176).

It is, therefore, connotated that when such a situation exists the center of gravity would fail to produce the optimum location. However, it is also implied that such an example is not represetnative of the plant or warehouse location problem.

Methodological explanation.--Although no mention is made of the positive quadrant of a Cartesian coordinate system, the implicit methodological procedure involves determining the coordinates of an arbitrarily selected potential optimum warehouse location. The procedure then centers on attempting to confirm these coordinates through the usage of the following formulations (10, p. 85):

$$x = \frac{\frac{x_1T_1}{d_1} + \frac{x_2T_2}{d_2} + \frac{x_3T_3}{d_3} + \ldots \frac{x_nT_n}{d_n}}{\frac{T_1}{d_1} + \frac{T_2}{d_2} + \frac{T_3}{d_3} + \ldots \frac{T_n}{d_n}},$$

$$y = \frac{\frac{y_1T_1}{d_1} + \frac{y_2T_2}{d_2} + \frac{y_3T_3}{d_3} + \ldots \frac{y_nT_n}{d_n}}{\frac{T_1}{d_1} + \frac{T_2}{d_2} + \frac{T_3}{d_3} + \ldots \frac{T_n}{d_n}},$$

where

$x$, $y$ = coordinates of potential warehouse location

$x_n$ = coordinate for store location on x axis

$y_n$ = coordinate for store location on y axis

$T_n$ = the tonnage of the shipment in question

$d_n$ = distance that each store location is separated from the warehouse.

In the event the coordinates do not check, the location is viewed as being nonoptimal and another potential warehouse location is selected and the formulas are applied again. Therefore, through a continuous trial and error procedure, the coordinates of the optimum warehouse location are finally determined. The article also indicates that it will be found easier to determine actual total ton miles for three or four locations than to attempt to solve the above equations by trial and error (10, p. 85).

Survey of approach.--The presented trial and error approach which holds that the centroid is nonoptimal is based on rather dubious proof. Moreover, the mathematical calculations behind the development of the trial and error formulations are missing, hence the proof of the model's ability to confirm a least cost location is lacking.

Additionally, the trial and error approach has been ignored in the literature, until being revived by Donald J. Bowersox in 1962. Attention will now proceed to the Bowersox presentation.

## The 1962 Bowersox Model

The second reference to trial and error methodology appeared in Donald J. Bowersox's (1) paper, Food Distribution Center Location. This presentation referenced the 1955 trial and error publication and built upon the approach by including time as an information input, the objective now being to minimize the inputs of time, tonnage, and mileage for a food distribution center. As a result of this, freight rate considerations involved no part of the study.

Unlike the 1955 publication (10), Bowersox (1) makes no explicit attack on a centroid location, other than to indicate that the trial and error approach is superior to the centroid method. In this regard, it was purported that the trial and error method gives consideration to distance while the centroid approach does not (1, p. 17). However, contrary to this assertion in the first of two presented examples, Bowersox (1, p. 37) depicts the centroid approach as actually being superior to the trial and error approach based upon ton mileage. Other than the second presented example which depicted a superior trial and error location based upon ton mileage (1, p. 45), no explicit reasoning was presented as to why a centroid location might prove inferior. Moreover, according to Bowersox (1, p. 49), both examples contained a considerable amount of total asymmetry.

With this background having been presented, the attention will focus on the trial and error approach which considers time as an information input.

Methodological explanation.--The modified trial and error approach which considers time as an input requires the usage of the following formulations (1, p. 18):

$$X = \frac{\sum_{i=1}^{n} \frac{x_1 F_1}{M_1}}{\sum_{i=1}^{n} \frac{F_1}{M_1}}$$

$$Y = \frac{\sum_{i=1}^{n} \frac{y_1 F_1}{M_1}}{\sum_{i=1}^{n} \frac{F_1}{M_1}}$$

where

$X, Y$ = unknown coordinate values of the distribution center

$x_n, y_n$ = supermarket locations designated by appropriate subscripts

$F_n$ = annual tonnage to each supermarket expressed as standard trailers identified by appropriate subscript

$M_n$ = supermarket location differentiated in terms of miles per minute from the initial distribution center location and sequentially from each new location until the trial and error procedure is completed.

Accordingly, the value of $M_n$ is determined by selecting the coordinates of a potential optimum warehouse location and ascertaining the distances from this location to each supermarket location. Next, the total time to each supermarket from this suspected warehouse location is derived from a time estimation table, and the result is divided into the distance from the suspected warehouse location to the supermarket in question. The quotient indicates the necessary miles per minute or $M_n$ value (1, p. 18). The approach is then trial and error until the selected coordinates check with the coordinates produced by the model.

Survey of approach.--In regard to the trial and error approach, no proof in the form of mathematical documentation was presented, other than the reference to the 1955 trial and error publication. However, a lack of centroid optimality was implied in the study when it was indicated that the trial and error approach was superior to a center of gravity location. Yet, such an implication was not specifically supported. In fact, the implication was discounted in the literature. In this regard, the 1965 Mossman and Morton study made reference to the 1962 Bowersox study (7, p. 243); however, this 1965 presentation still attested to centroid optimality.

## State of the Art

The presented survey of the literature has revealed two schools of thought--least cost determination through a centroid location per se and least cost location through a trial and error procedure. As has been seen, the greatest emphasis in the literature has squarely involved a centroid location. However, the centroid conception is not universely held. Therefore, in Chapter III, when the new gravitational centroid determination model is presented, the role of the centroid in generating a least cost location will be assessed and any limitations involving a centroid's ability to generate a least cost location will be revealed. Based upon these conclusions the state of the art should be advanced and any ambiguity concerning a centroid location will be clarified.

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# CHAPTER III

# THEORETICAL MODEL DEVELOPMENT

Now that the current status of the art of warehouse location through gravitational methodology has been presented, emphasis will center on developing a conceptual foundation for a new gravitational approach to the location problem. This foundation will, therefore, encompass model development, subsequent proof of the concept, and the basic limits of application. For explanatory purposes, the model will be somewhat simplified, but this by no means impedes the theorizing behind the model. In essence, Chapter IV, which is primarily oriented in terms of freight rate application to the model, builds upon the conceptual framework laid forth in this chapter. With this construct now in view attention will center on the model hypothesis and the overall framework for theoretical model development.

## The Framework for Theoretical Model Development

A key hypothesis of this presentation centers on the development and subsequent proof of a completely original gravitational model as applied to the warehouse locational problem. Therefore, it may be shown that the model is a representative application to the gravitational locational problem.

However, it is generic to the task to lay out a framework which will govern the theoretical development of the model. The framework so chosen centers squarely on the scientific method of analysis. This methodology is essentially the approach of the physical scientist, yet it has been adopted by virtually all successful researchers in the relatively new sciences involving operations research and management science. Such a scientific approach to analysis is generally broken into the following four steps.

1. Observation of the problem situation and the establishment of an objective for decision making.

2. Development of a model that explains the relationship of all relevant factors in the problem to each other and to the objective.

3. Performance of a model test to determine if the model is an appropriate representation of the problem situation, and making revisions when necessary.

4. Determination of model limitations and developing quantifiable and nonquantifiable rules for recognition.

The scientific method of analysis is used because there are numerous alternatives for a least transportation cost warehouse location. Therefore, in problems where there are numerous alternatives, making it undesirable to examine them all, the appropriate approach is to establish and prove the validity of a model that describes the variables in the problem as they would exist in the best alternative. This readily involves steps two and three of the scientific method. Moreover, the approach isolates those situations which are nonrepresentative in terms of absolute model application in step four.

The study will now shift to the practical implementation of the scientific method.

## Model Objective

Given an array of points scattered over a geographical plane and with each point characterized by spatial independence and associated weightings the objective of the model, from the standpoint of theoretical model development, is to determine a centroid location or an equilibrium point for the entire system of points. The purpose of obtaining this point of equilibrium centers on its significance as a potential optimum warehouse location. For based upon the assigned information inputs, a location at this point may result in the determination of an optimum or near optimum location.

The points involved are viewed as customers and the associated weightings for these points refer to tonnages to be shipped from a warehouse to these customers over a stipulated period of time. Thus, the primary orientation of the model centers on the determination of a centroid location for these customers. Outside of certain limitations which will be mathematically identified and resolved and based upon the assigned information inputs, location at a centroid for a series of customers should result in the minimization or near minimization of ton mileage to be shipped from the warehouse to these customers. And, also by including freight rates as an information input, transportation costs may be ultimately minimized.

However, rather than delving into the least cost considerations of a centroid location at this particular point of presentation, the focus will center on a centroid location which may produce a minimization of ton mileage. Figure 8 is an illustration of such a conception.

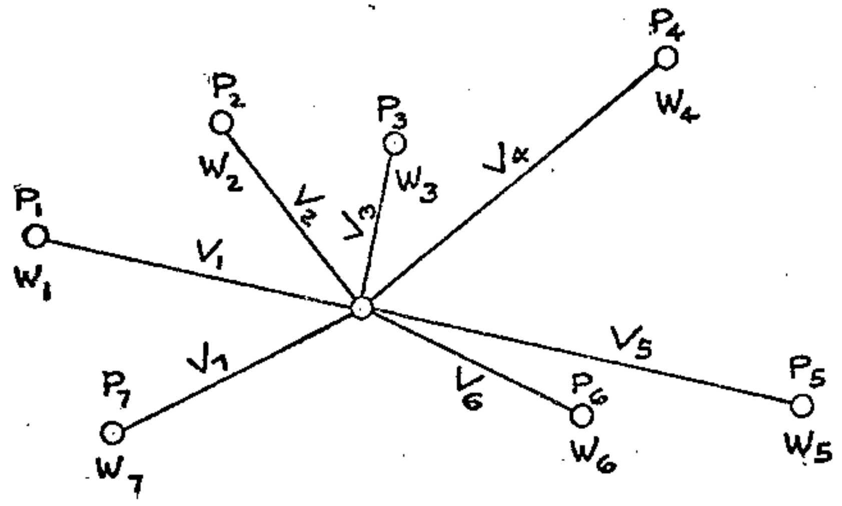

Fig. 8--Minimization of ton mileage

In Figure 8,

P's = customers

W's = tonnages associated with customers

V's = vector radii, denoting distances from centroid to customers

$W_1 = W_2 = W_3 = \ldots = W_n$.

By locating a warehouse at the centroid depicted in Figure 8, total ton mileage to be shipped from this warehouse to these indicated customer points may be minimized. Symbolically, the following least ton mile relationship between any centroid and any noncentroid location may prevail:

$$TM = W_1V_1 + W_2V_2 + \ldots + W_nV_n \quad t_m = W_1V'_1 + W_2V'_2 + \ldots + W_nV'_n ,$$

where

TM = centroid location

tm = noncentroid location.

From the symbolic relationship it is indicated that ton mileage from a centroid location to a customer designated P is the product of the weight or tonnage associated with that point multiplied times its respective radius vector or the distance from the centroid to that point. By summing the ton mileages associated with each customer, total ton mileage may be at a minimum or near minimum. Moreover, outside of the limitations which will be mathematically identified and resolved in the last section of this chapter, the symbolic relationship presented will either indicate optimality or near optimality.

The preceding mentioned limitations refer to those spatial arrangements of points or associated weightings which are nonrepresentative of ton mile minimization or near minimization through a centroid location. However, the appearance of these limitations should not be construed as minimizing the significance of a centroid location, as even when the limitations do appear, the model still plays an active role in analysis.

## In Support of the Model Objective

Additional support for the model objective may be presented at this particular stage of analysis. To facilitate this, Figure 9 presents an optimum and a nonoptimum location based on total ton mileage.

In this illustration, points $P_1$, $P_2$, $P_3$, $P_4$ are laid out on the quadrants of a circle. From plane geometry the point of balance or center of gravity for these four points is at point 0. Moreover, the distances from the center of gravity to each of these points, expressed as V's for vector radii, are equal. Therefore by letting $V = 1$ and $W = 1$, or again assuming equal weights, the following formula for ton mileage may be expressed:

$$W_1V_1 + W_2V_2 + W_3V_3 + W_4V_4 = 4\ WV \text{ or ton miles.}$$

This formula is indicative of the total ton mileage emanating from the centroid in Figure 9. In this regard any other location in Figure 9 will be characterized by a greater amount of ton mileage.

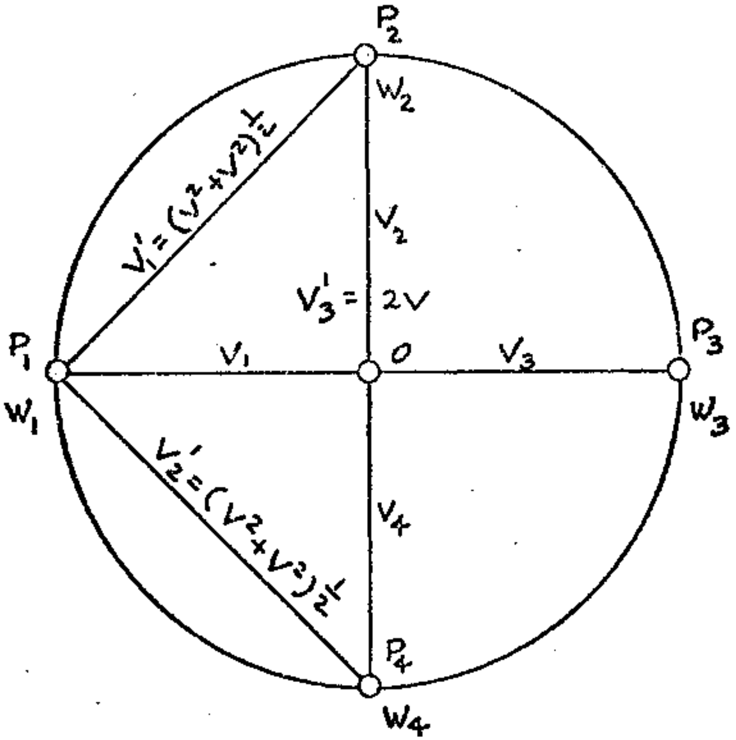

**Fig.** 9--Optimum and nonoptimum location

For example, by locating at point $P_1$, a nonoptimal point, total ton mileage or $\Sigma$ WV is not minimized. Again assuming that V = 1, $V'_1$ or the distance from $P_1$ to $P_2$ is computed by the following formula:

$$V'_1 = \sqrt{V^2 + V^2} \quad \text{or } 1.414 ,$$

and since $V'_1 = V'_2$

$$V'_2 = \sqrt{V^2 + V^2} \quad \text{or } 1.414.$$

Therefore, the total ton mileage from point $P_1$ would appear as:

$$V'_1W_2 + V'_2W_4 + V'_3W_3 = W\left(\sqrt{V^2 + V^2} + \sqrt{V^2 + V^2} + 2V\right) ,$$

substituting

$$1.414 \cdot 1 + 1.414 \cdot 1 + 2 \cdot 1 = 4.828 \text{ ton miles,}$$

and $4.828 > 4.000$ Q.E.D.

However, the generation of such proof does not necessarily connotate that a centroid location is always optimal or at least near optimal. In this regard, a confirmation model needs to be developed which will confirm the actual point of least cost for a given array of weighted points. Such a model is developed and applied to three separate centroid locations (Figures 33, 34, 35) in Appendix A. The purpose of this is to show that a centroid location may be both optimal and near optimal.

## Model Development

With the proof of the optimizing or near optimizing abilities of a centroid location now in view, emphasis will turn to the development of the model through a balance point formulation, its symbolic presentation and tabulation, coupled with key locational applications.

In determining the centroid location of a spatial array or arrangement of points or nodes with either equal or unequal weightings, the solution will not be based on the Cartesian coordinate approach. Rather, the approach will be based on the systematic determination of balance points or centers of gravity between two designated points. The approach being significant in that it opens new vistas to locational analysis, which have otherwise been ignored to this time. Also, to avoid confusion, the terms center of gravity, balance point, equilibrium point, and centroid are viewed as being synonymous.

## Balance Point Determination

Given a spatial array of points or customers with the necessary assigned weightings or tonnages to be shipped from a warehouse, it is generic to the problem to identify these points or nodes as $P_1$, $P_2$, $P_3$, . . . $P_n$. However, the final solution or ultimate center of gravity determination is simplified if the node or point with the largest and next to largest weights are identified. The point with the largest weight is then designated $P_1$ and the next to largest $P_2$. After this has been accomplished any sequence of point designation is permissible. Also, the weights involved should be designated with W's corresponding to their respective points, such as $P_3$ and $W_3$. Ultimately the point and weight assignments should always appear as follows:

$$P_1, W_1 \geq P_2, W_2 \geq P_3 W_3 .$$

After the proper symbols and numerals have been assigned the respective points, the computation of the center of gravity or point of balance between the two points designated $P_1$ and $P_2$ is necessary. Further, no more than two nodes separated by a finite distance will be considered during any single computation. The balance point of the two point system will then be determined and by concentrating the weights or the sum of the magnitudes of the points already considered at this balance point and by connecting a line from this balance point to a third node, a second two node problem has been created and a new balance point may be determined. This systematic process is continued for any number of nodes until an ultimate center of gravity is determined. The procedure is as follows using Figure 10 as a basis.

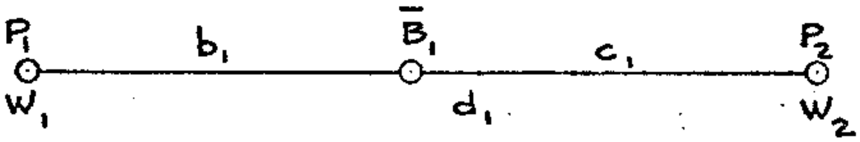

Fig. 10--Balancing a two point system

In Figure 10,

$P_1, P_2$ = spatial points or customers

$W_1, W_2$ = magnitudes or tonnages associated with spatial points

$\overline{B}_1$ = hypothetical center of gravity or balance point

$b_1$ = distance from $P_1$ to $\overline{B}_1$

$c_1$ = distance from $\overline{B}_1$ to $P_2$

$d_1$ = distance from $P_1$ to $P_2$

and where $W_1 \geq W_2$.

According to mechanics and the theory of moments (1, p. 37) the system shown in Figure 10 will be in balance at the centroid $\overline{B}_1$ if the value of $W_1$ multiplied times $b_1$ is equal to $W_2$ multiplied times $c_1$. For further clarification, the product of $W_1$ and $b_1$ may be referred to as the first moment and the product of $W_2$ and $c_1$ as the second moment. Therefore, when these moments are equal the system must be in balance. Now by assuming that Figure 10 is in balance at $\overline{B}_1$, a basis for delineating this point is shown as follows: from Figure 10, it follows that

$$W_1 \cdot b_1 = W_2 \cdot c_1, \text{ but } c_1 = d_1 - b_1,$$

substituting $$W_1 \cdot b_1 = W_2 (d_1 - b_1) ,$$

and $$W_1 \cdot b_1 = W_2 d_1 - W_2 b_1 ,$$

rearrange $$W_1 \cdot b_1 + W_2 \cdot b_1 = W_2 d_1 ,$$

and $$b_1 (W_1 + W_2) = W_2 d_1 ,$$

then $$b_1 = \frac{d_1 W_2}{W_1 + W_2} . \qquad (1)$$

Once $b_1$ has been determined it is a simple matter to identify $\overline{B}_1$ by plotting the distance of $b_1$ from $P_1$ to the center of gravity or $\overline{B}_1$. The above makes it clear that the centroid of a system of two weights lies on the line between them and it divides the segment inversely as the weights (1, p. 40).

Now extend the two point problem shown in Figure 10 to include three points as shown in Figure 11. The objective is to identify an overall center of gravity for a three point system.

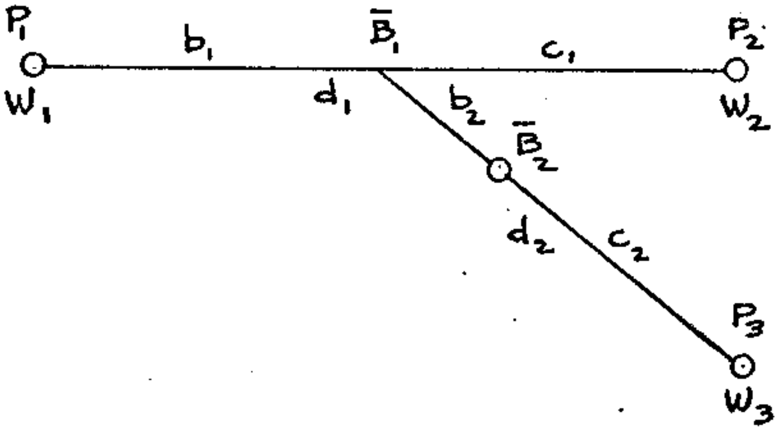

Fig. 11--Balancing a three point system

In Figure 11,

$P_1$, $P_2$, $P_3$ = spatial points or customers

$W_1$, $W_2$, $W_3$ = magnitudes or tonnages

$\bar{B}_2$ = assumed overall centroid of system

$b_2$ = distance from $\bar{B}_1$ to $\bar{B}_2$

$c_2$ = distance from $\bar{B}_2$ to $P_3$

$d_2$ = distance from $\bar{B}_1$ to $P_3$

and where $W_1 + W_2$ = total weight of first 2 node system applied at $\bar{B}_1$.

By concentrating the weights or tonnages associated with $P_1$ and $P_2$ at $\bar{B}_1$, a second two node problem involving $\bar{B}_1$ and $P_3$ has evolved. Therefore, by either scaling or computing the distance between these two points, a basis for a new center of gravity determination has emerged. This new center of gravity which lies on a line between $\bar{B}_1$ and $P_3$ is the centroid of the entire three point system. By assuming that $\bar{B}_2$ is the centroid of the system, a basis for delineating this point is shown as follows: from Figure 11, it follows that

$$(W_1 + W_2)b_2 = W_3 \cdot c_2 \text{ , but } c_2 = d_2 - b_2 \text{ ,}$$

substituting $$(W_1 + W_2)b_2 = W_3(d_2 - b_2) \text{ ,}$$

and $$b_2W_1 + b_2W_2 = W_3d_2 - W_3b_2 \text{ ,}$$

rearrange $$b_2W_1 + b_2W_2 + b_2W_3 = d_2W_3 \text{ ,}$$

and $$b_2(W_1 + W_2 + W_3) = d_2W_3 \ ,$$

then $$b_2 = \frac{d_2W_3}{W_1 + W_2 + W_3} \ . \tag{2}$$

Now that $b_2$ has been determined, the centroid $\overline{B}_2$ may be identified by plotting the distance of $b_2$ from $\overline{B}_1$ on the line connecting $\overline{B}_1$ with $P_3$. Moreover, it is clear that the centroid of a three point system can be determined by the successive use of a pair of two node problems.

Continuing the expansion of the system, consider the four point or four customer system in Figure 12. Again it is assumed that the entire system is in balance or that the product of the first moment is equal to the product of the second moment.

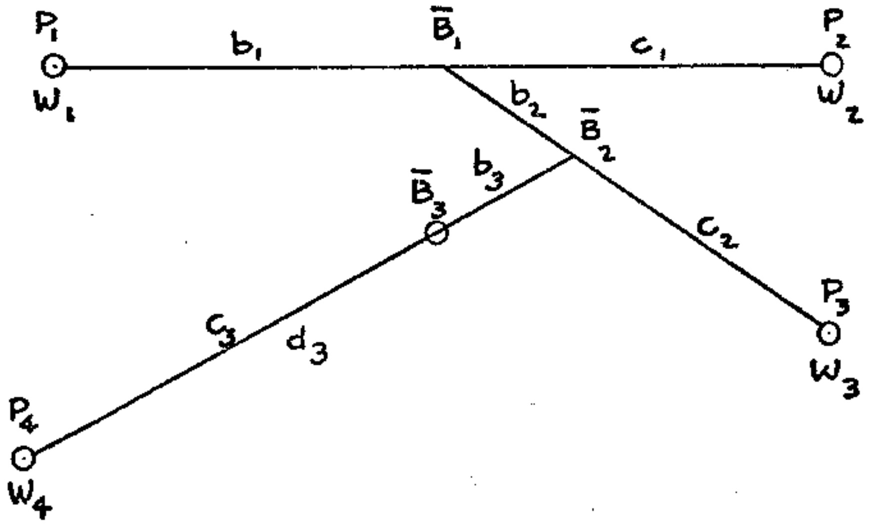

Fig. 12--Balancing a four point system

In Figure 12,

$P_1, P_2, P_3, P_4$ = spatial points or customers

$W_1, W_2, W_3, W_4$ = magnitudes or tonnages

$\overline{B}_3$ = assumed new overall centroid of system

$b_3$ = distance from $\overline{B}_2$ to $\overline{B}_3$

$c_3$ = distance from $\overline{B}_3$ to $P_4$

$d_3$ = distance from $\overline{B}_2$ to $P_4$

and where $W_1 + W_2 + W_3$ = total weight of the 3 node system $P_1, P_2, P_3$, applied at $\overline{B}_2$.

The new overall centroid of the four point system will lie on line $d_3$ which connects $\overline{B}_2$ with $P_4$. Moreover, a third two node problem has evolved by concentrating the weights of the first three point system at $\overline{B}_2$. And by assuming that the entire system is in balance at $\overline{B}_3$ or that the sum of weights of the three point system multiplied times $b_3$ equals $W_4$ multiplied times $c_3$, a basis for identifying this overall centroid is shown as follows: again beginning with the assumption of balanced moments in Figure 11, it follows that

$$(W_1 + W_2 + W_3)b_3 = W_4 \cdot c_3, \text{ but } c_3 = d_3 - b_3 ,$$

substituting $$(W_1 + W_2 + W_3)b_3 = W_4(d_3 - b_3) ,$$

and $$b_3W_1 + b_3W_2 + b_3W_3 = d_3W_4 - b_3W_4 ,$$

rearrange $$b_3W_1 + b_3W_2 + b_3W_3 + b_3W_4 = d_3W_4 ,$$

and $$b_3(W_1 + W_2 + W_3 + W_4) = d_3W_4 ,$$

then $$b_3 = \frac{d_3W_4}{W_1 + W_2 + W_3 + W_4} . \qquad (3)$$

Once given $b_3$ the overall centroid is determined by plotting the distance of $b_3$ from $\overline{B}_2$ on line $d_3$. Further, for purposes of clarification it will be convenient to rearrange the solution for the centroid of the four point system shown in Figure 12 into a vertical plane.

From Figure 13 it is noted that each pair of associated weights is in balance and that the entire system is in equilibrium with the center of gravity at $\overline{B}_3$. It is now evident that the systematic determination of balance points leading to such an ultimate center of gravity determination may be applied to a large number of points or greatly enlarged systems.

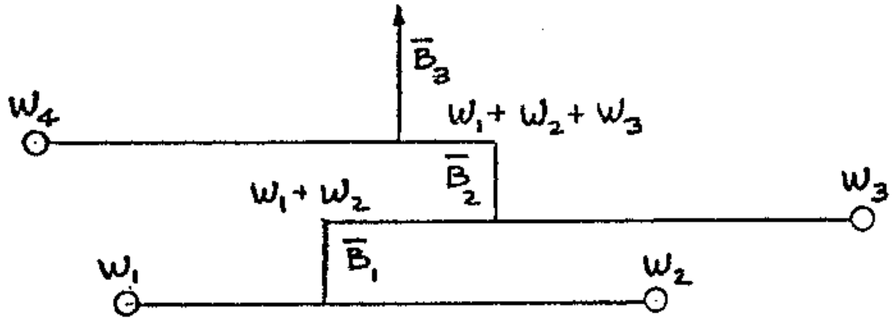

Fig. 13--Vertical plane arrangement of system

## Symbolic Model Presentation

Now that the theoretical construct of balance point determination has been placed in its proper perspective, additional emphasis will be placed on the actual formulas to be used for overall center of gravity determination. After having identified the respective points involved and applying the proper symbols to these points, being sure to consider that

$$P_1, W_1 \geqq P_2, W_2 \geqq P_3 W_3 ,$$

where

P's = points or customers

W's = tonnages or weights associated with the respective points,

use the following formula to determine the center of gravity between points $P_1$ and $P_2$:

$$b_1 = \frac{d_1 W_2}{W_1 + W_2} , \qquad (1)$$

where

$\overline{B}_1$ = balance point of 2 node system

$d_1$ = scaled or computed distance between $P_1$ and $P_2$

$b_1 = P_1 \sim \overline{B}_1$ or the distance from $P_1$ to $\overline{B}_1$ on line $d_1$.

For a three point system apply formula one and then formula two as follows:

$$b_2 = \frac{d_2 W_3}{W_1 + W_2 + W_3} , \qquad (2)$$

where

$\overline{B}_2$ = balance point of 3 node system

$d_2$ = scaled or computed distance between $\overline{B}_1$ and $P_3$

$b_2 = \overline{B}_1 \sim \overline{B}_2$ or the distance from $\overline{B}_1$ to $\overline{B}_2$ on lines $d_2$.

For a four point system apply formulas one and two, then formula three as follows:

$$b_3 = \frac{d_3 W_4}{W_1 + W_2 + W_3 + W_4} , \tag{3}$$

where

$\overline{B}_3$ = balance point of 4 node system

$d_3$ = scaled or computed distance between $\overline{B}_2$ and $P_4$

$b_3 = \overline{B}_2 \sim \overline{B}_3$ or the distance from $\overline{B}_2$ to $\overline{B}_3$ on line $d_3$.

With the above in view a general equation for centroid determination or for determining the point which divides the line connecting the weights in inverse ratio is

$$b_n = \frac{d_n W_{n+1}}{W_1 + \ . \ . \ . + W_{n+1}} , \tag{4}$$

where

$\overline{B}_n$ = centroid of any 2 node system

$d_n = \overline{B}_{n-1} \sim P_{n+1}$ or the scaled or computed distance between $\overline{B}_{n-1}$ and $P_{n+1}$

$b_n = \overline{B}_{n-1} \sim \overline{B}_n$ or the distance from $\overline{B}_{n-1}$ to $\overline{B}_n$ .

Substituting in the general formula for a five point system would produce the following result:

$$b_4 = \frac{d_4 \ W_{4+1}}{W_1 + \ldots + W_{4+1}} \quad \text{or} \quad b_4 = \frac{d_4 \ W_5}{W_1 + W_2 + W_3 + W_4 + W_5},$$

where

$$d_4 = \overline{B}_{4-1} \sim P_{4+1}$$

$$b_4 = B_{4-1} \sim \overline{B}_4 \quad .$$

For the purpose of clarifying the general equation, n will always be one less than the highest numbered point or node involved in a particular balance point calculation. Also, equation four (4) requires that all preceding centroids for each pair of nodes be computed prior to computing the nth balance point. Moreover, the balance point which produces the overall center of gravity will always be one less than the total number of points in the system under consideration.

It may also be noticed that the ordering of the largest and next to largest weights for a system where

$$P_1W_1 \geqq P_2W_2 \geqq P_3W_3$$

simplifies formula presentation, for the lesser weight associated with a pair of nodes must always be the weight included in the formula's numerator. By ordering the two heaviest points initially the weight associated with a given customer location or point must always be smaller than the sum of weights associated with a given balance point. If ordering were not involved an "either-or priori" would result involving the weights concentrated at a balance point and the weight or tonnage assigned a particular point or customer.

## Tabulation

To facilitate the ease of calculation of the presented formulas a tabular arrangement may be useful for computational purposes. Table VIII presents such a tabulation for formulas (1) and (2).

TABLE VIII

TABULAR ARRANGEMENT OF FORMULA DATA

| (1) Node | (2) Value | (3) $\Sigma W$'s | (4) Distance | (5) (2) x (4) | (6) (5) ÷ (3) | (7) b |
|---|---|---|---|---|---|---|
| $P_1$ | $W_1$ | $W_1$ | | | | |
| $P_2$ | $W_2$ | $W_1 + W_2$ | $d_1$ | $W_2 \times d_1$ | $\frac{W_2 \times d_1}{\Sigma W's}$ | $b_1$ |
| $P_3$ | $W_3$ | $W_1 + W_2 + W_3$ | $d_2$ | $W_3 \times d_2$ | $\frac{W_3 \times d_2}{\Sigma W's}$ | $b_2$ |

This tabular arrangement makes it convenient to assign values and lays forth a basic framework for computation. To illustrate, Table IX will encompass actual values derived from the three point or three customer system shown in

Figure 14, the purpose being to generate the overall centroid of the system based on tabular headings.

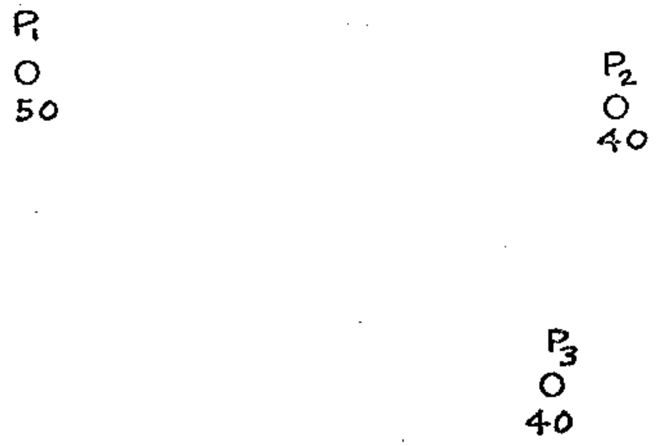

Fig. 14--Spatial array of points and associated weightings

Given the system shown in Figure 14, a basis for substituting actual values into Table IX has now been generated.

TABLE IX

TABULAR ARRANGEMENT OF NUMERICAL VALUES

| (1) Node | (2) Value | (3) $\Sigma W$'s | (4) Distance | (5) (2)x(4) | (6) (5)÷(3) | (7) b |
|---|---|---|---|---|---|---|
| $P_1$ | 50 | 50 | | | | |
| $P_2$ | 40 | 90 | 3 | 120 | $\frac{120}{90}$ | 1.33 |
| $P_3$ | 40 | 130 | 2 | 80 | $\frac{80}{130}$ | 0.62 |

The first object of the tabular arrangement presented in Table IX is to determine the balance point between points $P_1$ and $P_2$. To accomplish this end the distance between $P_1$ and $P_2$ must be obtained. As the tabular arrangement indicates, a value of 3 units has been assigned. This distance may be expressed in miles or any other unit of measure. By following the tabular calculations the balance point $\overline{B}_1$ between points $P_1$ and $P_2$ lies at a distance of 1.33 units from $P_1$, the heavier weight. Once this balance point or $\overline{B}_1$ has been obtained a new two node problem involving $\overline{B}_1$ and $P_3$ has evolved. As Figure 15 will indicate, the distance from $\overline{B}_1$ to $P_3$ has been assigned a distance of 2 units, and by following the tabular computations the overall center of gravity lies at 0.62 units from $\overline{B}_1$.

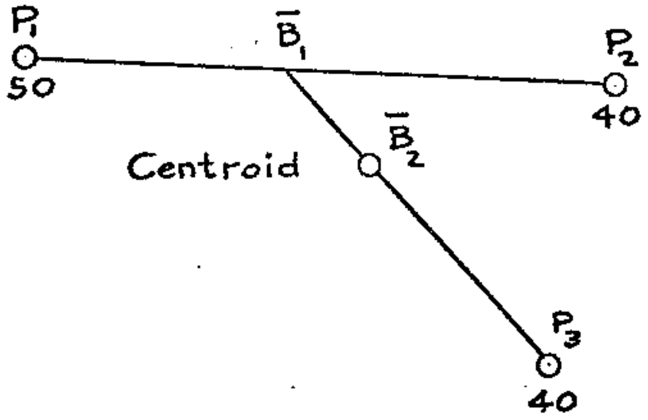

Fig. 15--Results of tabular computation

For the purpose of emphasis column (7) in Tables VIII and IX always indicates the distance from the heavier mode in any balance point calculation to the point of balance between two weighted points.

## Supply and Demand Considerations

For the purpose of theoretical model development the points spread over a geographical plane have been viewed as customers or recipients of warehouse shipments, and the primary orientation has centered on determining the center of gravity of these particular points. However, a major consideration as applied to the warehouse locational problem is the minimization of total ton mileage involving both inbound shipments to the warehouse from suppliers and outbound shipments from the warehouse to customers. Therefore, consideration must be given to supply as well as demand when determining the points to be encompassed for a center of gravity formulation. This involves identifying the supply points which will ship to the projected new warehouse and weighting them with the tonnages which will be shipped to this warehouse over a stipulated period of time. Thus, with the addition of supply considerations to an array of demand or customer points a new overall center of gravity will result, which may minimize or nearly minimize total ton mileage, both inbound and outbound.

However, from a realistic standpoint ton mile minimization per se should not be the primary goal with the addition of supply considerations to an array of demand nodes. Rather, the primary objective should center on the minimization of ton mileage within the constraints of the existing freight rate structure. To illustrate this, consider that shipments into a warehouse are generally carload or truckload and that shipments from a warehouse are generally less than carload or less than truckload. As a result of this, a significant freight rate differential exists between inbound warehouse shipments and outbound warehouse shipments. For the purpose of further model development, a realistic assumption centers on viewing inbound warehouse shipments as being twice as low as outbound shipments in terms of freight costs. Of course, the percentage differential between inbound and outbound costs may vary from the 50 per cent norm in given instances, but from the standpoint of general application the 50 per cent differential warrants consideration. Moreover, most of the widely accepted transportation models use such a differential. With this conception in view it will be necessary to weight all inbound tonnages associated with supply points by one-half. As a result ton mile minimization will occur within the constraints of the existing freight rate structure. Emphasis will now proceed to a consideration of supply factors in the overall center of gravity formulation.

Overall center of gravity formulation.--With the addition of supply considerations to a spatial configuration of demand nodes, it follows that the summation of tonnage associated with the newly added supply points should equal the tonnages associated with the existing demand nodes. However, these supply point tonnages should be weighted by one half to reflect their lessened impact or else they would assume the same significance as demand point weights. Once such a reduction has occurred, the points and associated weights are ordered so that

$$W_1 \geq W_2 \geq W_3 ,$$

without any concern over whether the point is either demand or supply. An example of such a formulation and balance point calculation is seen in Figure 16.

In Figure 16,

$P_1, P_2, P_3, P_4, P_7$ = demand points

$S_5, S_6$ = supply points

$W_1, W_2, W_3, W_4, W_7$ = tonnages associated with demand points

$Q_5, Q_6$ = tonnages associated with supply points weighted by 0.5

$\overline{B}_n$ = balance points between nodes and balance points

$\overline{B}_6$ = balance point for the entire system.

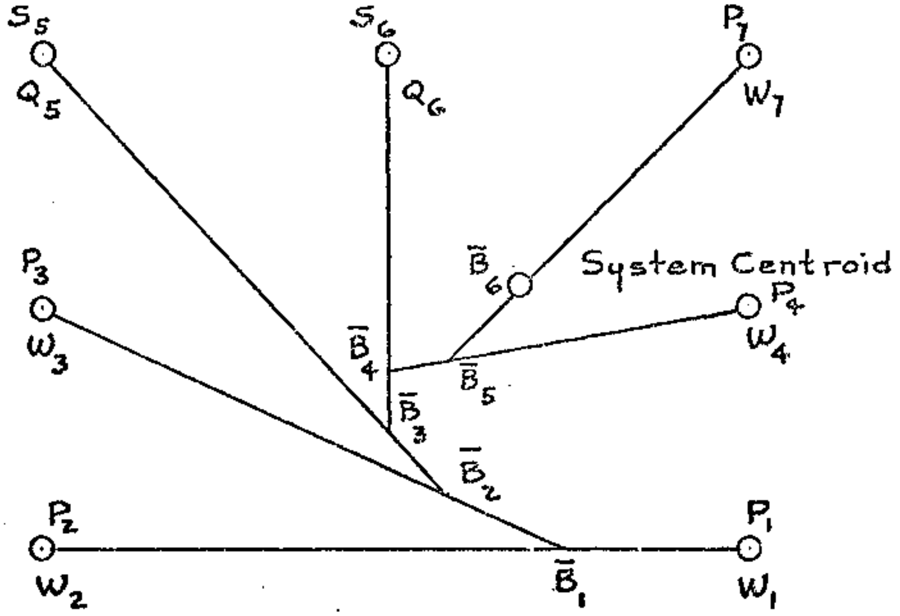

Fig. 16--Overall center of gravity with supply points included

It may be noticed that the general formula (4) may be utilized to determine the balance point between each pair of nodes regardless of whether the node is a supply point or a demand point. In the case of Figure 16, the ultimate center of gravity is determined by the following formula:

$$b_6 = \frac{d_6 W_7}{W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7},$$

where $W_5$ and $W_6$ are substituted for $Q_5$ and $Q_6$. However, instead of emphasizing an overall centroid formulation which encompasses both supply and demand considerations, special attention will be given to the concept of multiple centers of gravity involving a systematic view of both supply and demand. In essence, all future presentations of both supply and demand considerations will be concerned with subsets or subsystems.

Two subsets and overall center of gravity formulation. Rather than resorting to a single overall center of gravity approach to the locational problem, decision making may be enhanced by giving cognizance to a multiple center of gravity system. This refers to the separate determination of a center of gravity for both demand and supply, and then linking them both in an overall centroid formulation. Thus, three centers of gravity are ascertainable, one for a subset of demand points, another for a subset of supply points, and finally an overall centroid of the system.

By engaging in a two subset and overall gravitational analysis, a basis for a nonoptimal warehouse location has been delineated in the form of a "line of force" extending from the centroid of the overall system to the centroid of the demand point subset. Also by attacking the locational problem from the standpoint of the two subsets involving demand and supply, a basis for expanding these subsets has readily emerged without reevaluating the entire procedure used in computing the center of gravity of each subset. Commonly, supply considerations may be considered as fixed while demand considerations may vary through time, hence additional demand points may be added to the demand subset to determine their effect on the centroid of the subset and the overall center of gravity of the system.

An illustration of the solution of the two subset and overall centroid formulation appears in Figure 17.

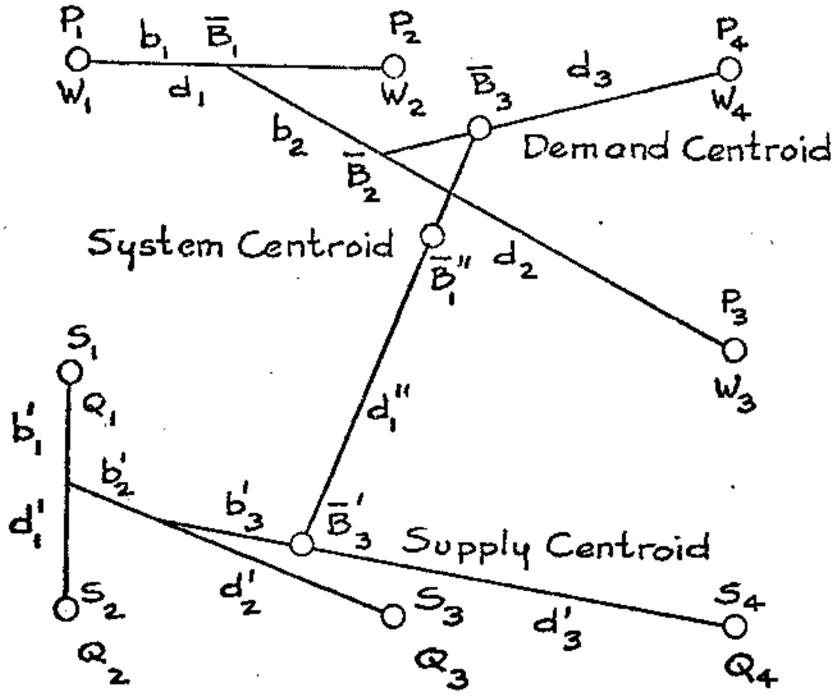

Fig. 17--Two subsets and overall centroid solution

In Figure 17,

Subset (1) = $P_1$, $P_2$, $P_3$, $P_4$ or demand nodes

Subset (2) = $S_1$, $S_2$, $S_3$, $S_4$ or supply nodes

$\overline{B}_3$ = center of gravity of subset (1)

$\overline{B}_3{}'$ = center of gravity of subset (2)

$\overline{B}_1{}''$ = center of gravity of the system

$W_1, W_2, W_3, W_4$ = weights of subset (1)

$Q_1, Q_2, Q_3, Q_4$ = weights of subset (2)

$d_1, d_2, d_3$ = distances between nodes or balance points and nodes for subset (1)

$d_1', d_2', d_3'$ = distances between nodes or balance points and nodes for subset (2)

$b_1', b_2', b_3'$ = distances from nodes or balance points to centroids for subset (2)

$d_1''$ = distance between subset (1) and subset (2)

$b_1''$ = distance from $\overline{B}_3$ to $\overline{B}_1''$

and where $W_1 + W_2 + W_3 + W_4 = 2K$

and $Q_1 + Q_2 + Q_3 + Q_4 = K$

express the weighted differential between inbound and outbound warehouse shipments.

Solving the center of gravity for subset (1) in Figure 17 required formula (4) expressed as

$$b_n = \frac{d_n \; W_{n+1}}{W_1 + \ldots + W_{n+1}} \tag{4}$$

and

$$b_3 = \frac{d_3 W_4}{W_1 + W_2 + W_3 + W_4} .$$

Solving the center of gravity for subset (2) also required formula (4) in which Q was substituted for W as follows,

$$b_n' = \frac{d_n \; Q_{n+1}}{Q_1 + \ldots + Q_{n+1}}$$

and $$b_3' = \frac{d_3' \; Q_4}{Q_1 + Q_2 + Q_3 + Q_4} \, . \qquad (4)$$

The solution for the overall centroid of the system was engendered by the following new formula,

$$b_1" = \frac{d_1" \quad \Sigma(Q_1 + . \; . \; . + Q_n)}{\Sigma[(W_1 + . \; . \; . + W_n) + (Q_1 + . \; . \; . + W_n)]} \, , \qquad (5)$$

or $$b_1" = \frac{d" \; K}{2K + K} = \frac{d"}{3} \, ,$$

which was calculated by substituting K values for both Q's and W's.

It is now discernible that once the centroids of both systems have been determined, the overall centroid may be ascertained by measuring the distance from the centroid of subset (1) to the centroid of subset (2) and dividing this distance by 3. Naturally, the plotting of this resulting distance on a line running from subset (1) to subset (2) will delineate the precise center of gravity.

Moreover, the entire array of supply and demand points in Figure 17 may be arranged in a vertical plane to illustrate the balance point principle and to demonstrate how equilibrium of the system is obtained.

Rather than always resorting to the calculation of a center of gravity for both supply and demand and then linking them together as shown in Figure 18, a third approach to supply and demand considerations serves as a ready alternative.

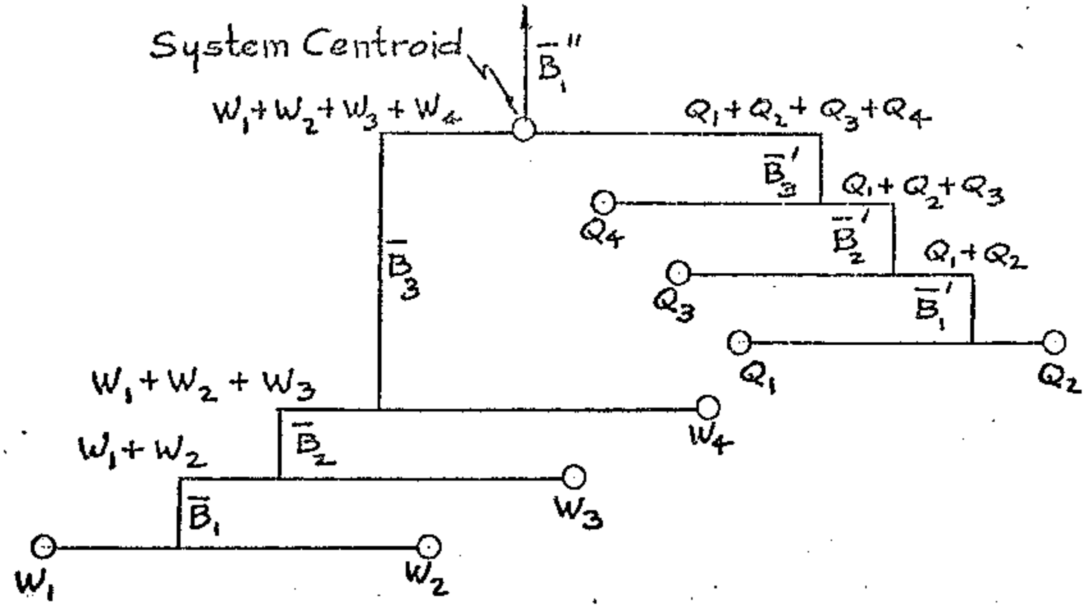

Fig. 18--Vertical plane arrangement of two subset systems and the overall center of gravity.

<u>One subset and overall center of gravity formulation</u>.--By resorting to the calculation of one subset and an overall centroid for the system, formula (5) may be dispensed with completely. To accomplish this end all demand points are viewed as belonging to a subset of the entire system and a centroid is calculated for these points. Next, by joining a line from the center of gravity of the demand subset to a supply point and by computing its balance point along with other following supply points, the last balance point so computed represents the centroid of the system. Figure 19 illustrates such a solution.

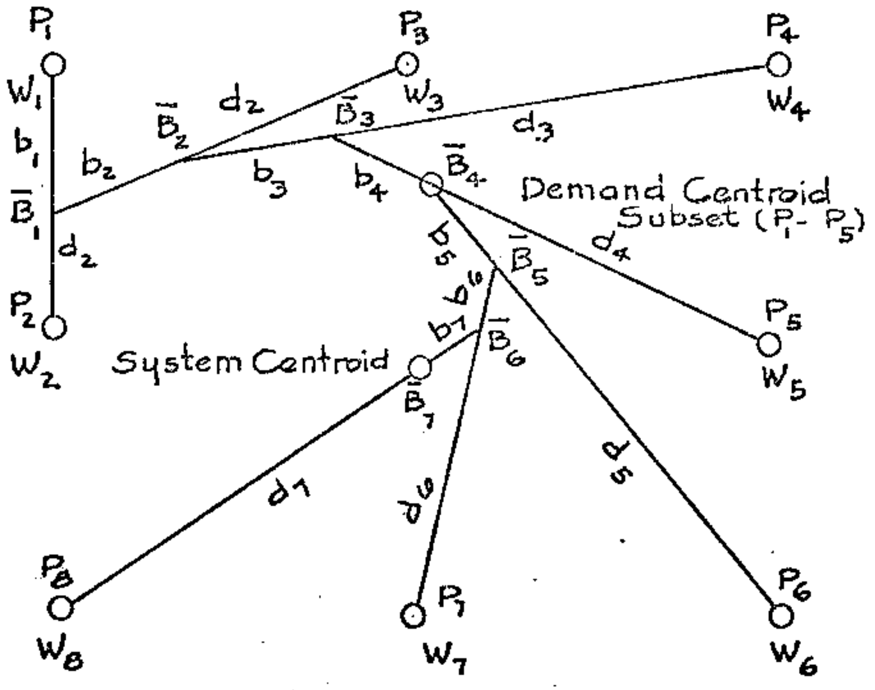

Fig. 19--One subset and overall centroid solution

In Figure 19,

$P_1$, $P_2$, $P_3$, $P_4$, $P_5$ = demand nodes (subset 1)

$P_6$, $P_7$, $P_8$ = supply points

$W_1$, $W_2$, $W_3$, $W_4$, $W_5$ = demand weights

$W_6$, $W_7$, $W_8$ = supply tonnages weighted by 0.5

d's = distances between nodes or balance points and nodes

b's = distances from balance points to centroids.

Solving for the center of gravity for the demand subset requires formula (4) for each pair of points as follows:

$$b_n = \frac{d_n W_{n+1}}{W_1 + \ . \ . \ . + W_{n+1}} \tag{4}$$

with the centroid of the demand subsystem being determined by

$$b_4 = \frac{d_4 W_5}{W_1 + W_2 + W_3 + W_4 + W_5} \ .$$

By continuing the general formula supply points are encompassed, and the centroid of the entire system is ultimately determined by this last formula:

$$b_7 = \frac{d_7 W_8}{W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8} \ .$$

Naturally, the one subset and overall center of gravity approach also gives rise to the concept of a "line of force" for nonoptimal warehouse location. The approach is also readily advantageous when only a few supply points are to be considered, and of course the approach is amenable to relatively large arrays of supply points. However, in the latter case it follows that the entire summation of weights for the entire system is involved in the nth calculation. While in the two subset and overall gravitational approach only the summation of weights associated with the supply subset are involved in the nth calculation.

## Model Proof

Although the preceding analysis of the balance point procedure and its determination was self-proving due to the reiterative balancing of moments, it will be convenient to illustrate a corollary proof to the conception. To accomplish this end the centroid as determined by the balance point model will be compared to the centroid of a known configuration, in this case an isosceles triangle. Figure 20 illustrates such a triangle and its centroid.

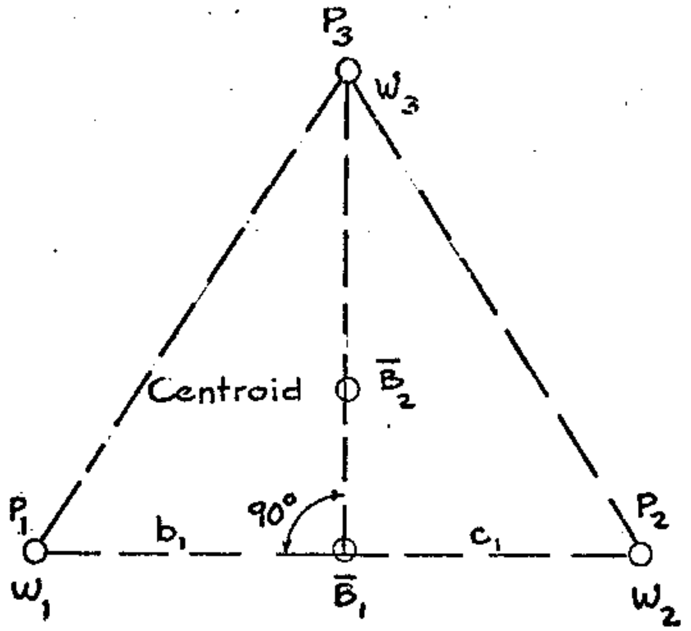

Fig. 20--Centroid of isosceles triangle

In the above illustration all weights $W_1$, $W_2$, and $W_3$ are assumed equal, and $b_1 = c_1$. Since $\overline{B}_1$ is the midpoint of line $P_1$, $P_2$, the centroid for the triangle $P_1$, $P_2$ and

$P_3$ by plane geometry is one third the altitude or one-third the distance of line $\overline{B}_1$, $P_3$.

Now by taking the same three points and their weights and by placing them in the identical form of the isosceles triangle, a basis for model application has been delineated. Figure 21 illustrates this centroid determination.

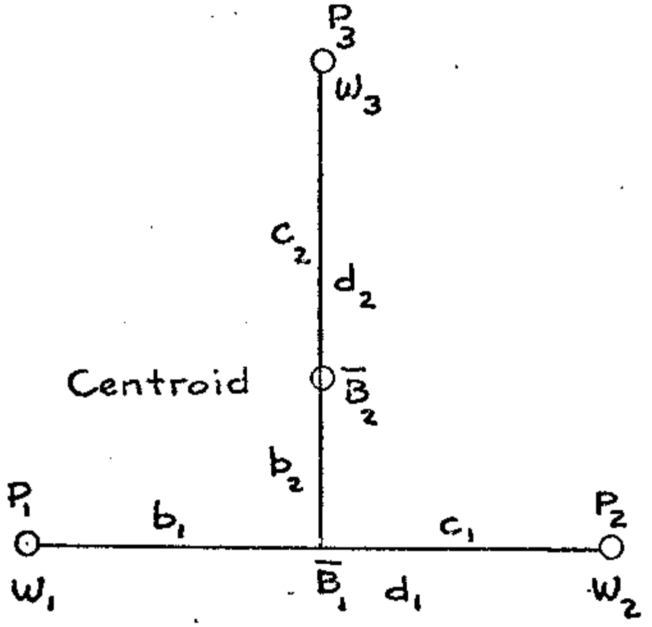

Fig. 21--Model application to an array of points in the form of an isosceles triangle

In Figure 21,

$d_1$ = distance from point $P_1$ to point $P_2$

$b_1$ = distance from $P_1$ to balance point $\overline{B}_1$

$d_2$ = distance from $\overline{B}_1$ to point $P_3$

$b_2$ = distance from $\overline{B}_1$ to overall centroid $\overline{B}_2$.

By applying formula (1) in the following form

$$b_1 = \frac{d_1 W_2}{W_1 + W_2} \quad \text{or} \quad \frac{d_1}{2} ,$$

and since all weights are assumed equal, the midpoint of line $d_1$ at $\overline{B}_1$ has been determined.

Next, by considering $W_1 + W_2$ to be applied at balance point $\overline{B}_1$, a new two node problem involving $\overline{B}_1$ and $P_3$ has been identified, therefore setting up formula (2) for application as follows:

$$b_2 = \frac{d_2 W_2}{W_1 + W_2 + W_3} \quad \text{or} \quad \frac{d_2}{3} .$$

Due to the assumption of equal weights, it follows that the centroid of the system lies on a point one third of the distance from $B_1$ to $P_3$, or the absolute identical point for the centroid of the isosceles triangle.

With this background in view, it may be seen that the model locates a center of gravity for any spatial arrangement of points and their associated weightings, thus the model is a representative application to the gravitational problem.

Naturally, the significance of the model centers on the need for ton-mile minimization, and as indicated in the section on the Model Objective, the centroid may produce such an orientation. However, there are certain limitations on the model's ability to produce a centroid location which will invariably result in a minimization or near minimization of ton mileage. These limitations will now be presented along with the mathematical techniques for tentatively recognizing them. Moreover, the model's role in these nonrepresentative arrays of both points and weightings will be developed.

## Model Limitations

Of course, it would be ideal if the centroid of any spatial array of points and associated weightings would always produce a definite minimization of ton mileage or a minimization fo cost considerations. However, while there are no limitations as applied to the model's ability to generate a centroid location, there are, nevertheless, limitations on the centroid's ability to generate a minimization of ton mileage. In this regard the centroid as a point of optimality or near optimality may become nonoptimal per se due to the existence of either extreme variations in weightings or due to extreme asymmetry in the spatial array of points. For example, a system with a disproportionately heavy weight or cluster of weights, as it relates to the entire system of weights, may offset the optimality or near optimality of a centroid location. In these situations a ton mile center or point of minimization exists which is superior to the ton mile centroid, even though the system is in perfect equilibrium or balance in the latter case. Naturally, a series of points and associated weightings characterized by a lack of these limitations producesa near optimal or optimal centroid location.

With these limitations in view the problem for the locations researcher centers on identifying those situations in which the centroid is actually compromised. For example, there may be systems of points which are clearly optimal and nonoptimal as they relate to a centroid locatin. In other cases a grey area may be discernible in which the centroid is marginal as it relates to optimization. Therefore, a check model must be utilized which will either confirm or disprove a centroid location. In the event that the check model fails to confirm the centroid or its near optimality, a ton mile minimal point other than the centroid exists which must be identified. Such identification involves ton mile comparisons in the geographic vicinity of the least cost center with the lowest ton mile point produced subject to confirmation through the check model.

In the event the centroid is nearly optimal, the researcher can engage in cost comparisons in the immediate vicinity of the centroid and a confirmation of the lowest cost location obtained may be engendered. However, in the event the centroid is clearly nonoptimal, some useful generalizations will be presented which will help the researcher locate the general area where cost comparisons should be made.

With the development of the above structure, emphasis will be placed on presenting the check model and the general limitations which serve to offset the optimality or near optimality of a centroid location. Finally, the generalizations concerning the areas in which cost comparisons should be made will be indicated.

## The Confirmation Model

The confirmation or check model is based on determining coordinates of a potential warehouse location and confirming these coordinates by substituting actual vectors or distances radiating from the warehouse, to both customers and suppliers, into the model per se. The conception was initially developed as a means to scientifically determine a point of least cost (3, p. E85), instead of now serving as a confirming model. Yet, the presented limitation of this method centered on the inability of the model to substitute actual distances or vectors from the warehouse into the model as the warehouse was the point to be determined. Therefore, while the model cannot identify a least cost point per se, it looms paramount as a confirming device.

By working from this conceptual base the necessary calculations for confirming the model's ability to substantiate a least cost location are presented in Appendix A. However, the formulas were presented for specific cases and not in general form. Presentation of the confirmation model in general terms appears as follows:

$$\overline{X} = \frac{\frac{X_1 \cdot W_1}{V_1} + \frac{X_2 \cdot W_2}{V_2} + \frac{X_3 \cdot W_3}{V_3} + \ldots + \frac{X_n \cdot W_n}{V_n}}{\frac{W_1}{V_1} + \frac{W_2}{V_2} + \frac{W_3}{V_3} + \ldots + \frac{W_n}{V_n}}, \qquad (5a)$$

$$\overline{Y} = \frac{\frac{Y_1 \cdot W_1}{V_1} + \frac{Y_2 \cdot W_2}{V_2} + \frac{Y_3 \cdot W_3}{V_3} + \ldots + \frac{Y_n \cdot W_n}{V_n}}{\frac{W_1}{V_1} + \frac{W_2}{V_2} + \frac{W_3}{V_3} + \ldots + \frac{W_n}{V_n}}, \qquad (5b)$$

where

$\overline{X}$ = coordinate of centroid or potential least cost site on X axis

$\overline{Y}$ = coordinate of centroid or potential least cost site on Y axis

$X_n$ = coordinate for customer or supplier point on X axis

$Y_n$ = coordinate for customer or supplier point on Y axis

$V_n$ = vector or distance of customer or supplier from warehouse at centroid or potential least cost location

$W_n$ = weight or tonnage associated with a given point.

To facilitate the handling of coordinates all customer and supplier points should be placed in the positive plane of the coordinate system with the vertical Y axis denoting the ordinate and the horizontal X axis denoting the abscissa.

As a result, all X and Y values for each customer or supplier will be expressed in positive terms.

Moreover, confirmation of a centroid location is forthcoming when the assigned $\overline{X}$ and $\overline{Y}$ values of the centroid are identical with the resulting $\overline{X}$ and $\overline{Y}$ values produced by the check model. Consideration should also be given to a check with a high degree of physical proximity to the assigned $\overline{X}$ and $\overline{Y}$ values, rather than an absolute identical check, as rounding may cause a slight discrepancy. Also the coordinates produced by the confirmation model may not be identical with the centroid's coordinates, yet the coordinates may be so close as to denote near centroid optimality. In this regard the researcher may wish to establish acceptable tolerance limits. For example, if an acceptable tolerance of ± .5 miles were selected, this would indicate that if the coordinates of the confirmation model fall within a one mile square area of the centroid's coordinates the centroid would be viewed as being optimal. Naturally, the selection of such limits is arbitrary with the researcher. If no tolerance limits are established and the centroid is confirmed as being near optimal cost comparisons within the immediate vicinity of the centroid location should be undertaken with the tentative point of lowest cost subject to the confirmation of its coordinates.

In the event the centroid location is nonoptimal per se (as indicated by the confirmation model) or what may be viewed as a lack of near optimality, this is indicative of the limitations of a centroid location coming into existence. Emphasis will now proceed to the general limitations of a centroid location and the role of the confirmation model in either proving or disproving the optimality or near optimality of a centroid location.

## The Impact of a Dominant Point or Cluster

Given an array of points with an optimal centroid, varying the weights associated with these points will still produce centroid optimality or near optimality unless these variations in weightings are extreme. In this regard, a point or a cluster of points may become disproportionately heavy as they relate to the entire system of weights, thus serving to offset the centroid as the point of ton mile minimization or near minimization. Such a disproportionately heavy point or cluster of points may therefore be referred to as a dominant point or dominant cluster. Thus, the task of the locations researcher may present a confrontation with a potentially dominant point and the confirmation model will have to be implemented to either confirm or deny its existence. Figure 25 presents such a confrontation. Note supply considerations have been disregarded to simplify the discussion, yet this in no manner impedes the theorizing behind the concept.

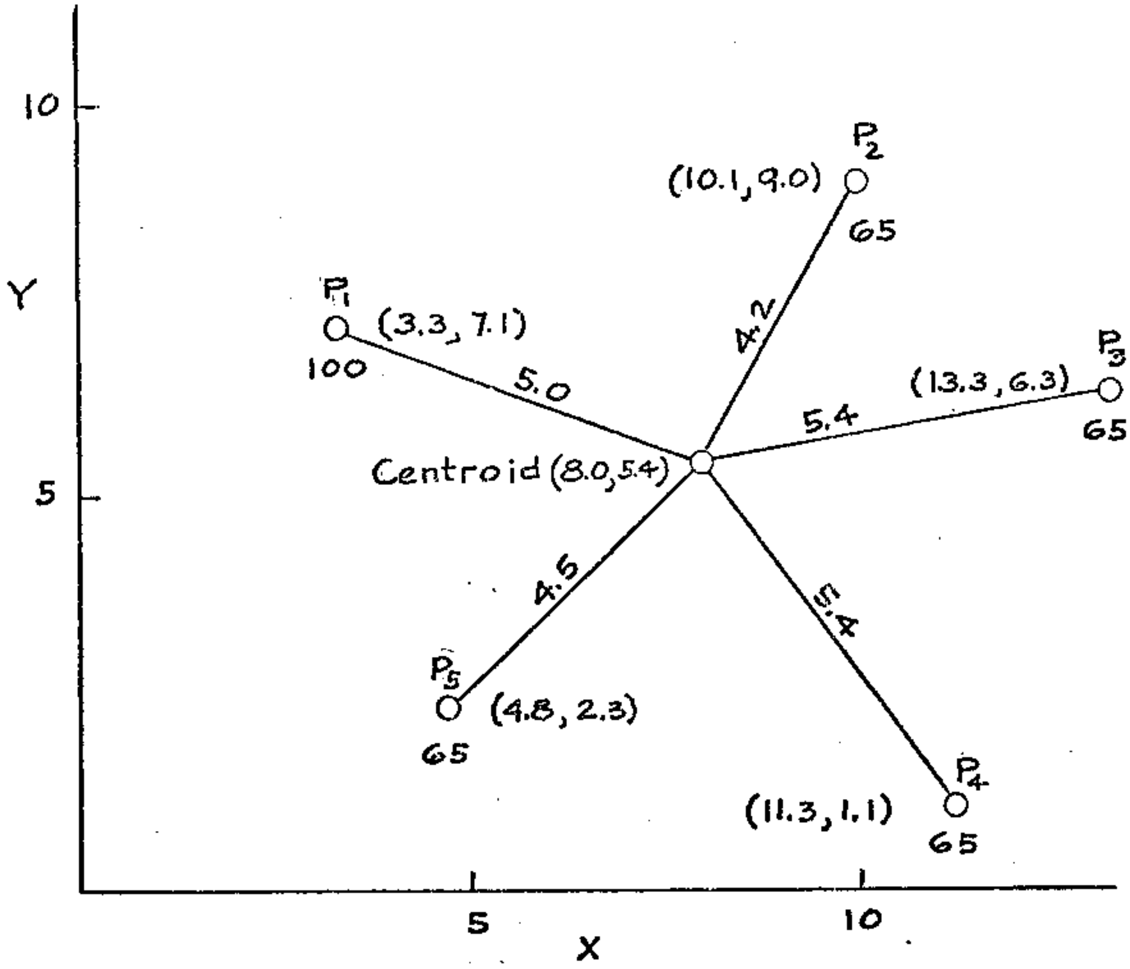

Fig. 22--Potential dominant point array

In Figure 22 a potentially dominant point is apparent at Location $P_1$, hence the possibility exists that the size of this point may offset the centroid (as determined by the balance point method) as the point of ton mile minimization or near minimization. Therefore, the coordinates of the centroid, which are 8.0 and 5.4 for $\overline{X}$ and $\overline{Y}$ respectively, must be confirmed or disproved.

Using the confirmation model requires the determination of vectors or the distances from the centroid to each point. In this particular case the vectors were computed rather than scaled, although the approach is arbitrary with the individual researcher. To accomplish this end the following formula was utilized to determine the distance between two points:

$$V_n = \left[(X_n - \bar{X})^2 + (Y_n - \bar{Y})^2\right]^{1/2} , \qquad (6)$$

where

$V_n$ = distance from the centroid to any point

$X_n$ = the coordinate associated with each point on the X axis

$Y_n$ = the coordinate associated with each point on the Y axis

$\bar{X}$ = the coordinate of the centroid on the X axis

$\bar{Y}$ = the coordinate of the centroid on the Y axis.

Substituting in equation 6 from Figure 22 generates

$$V_1 = \left[(4.7)^2 + (1.7)^2\right]^{1/2} = 5.0$$
$$V_2 = \left[(2.1)^2 + (3.6)^2\right]^{1/2} = 4.2$$
$$V_3 = \left[(5.3)^2 + (0.9)^2\right]^{1/2} = 5.4$$
$$V_4 = \left[(3.3)^2 + (4.3)^2\right]^{1/2} = 5.4$$
$$V_5 = \left[(3.2)^2 + (3.1)^2\right]^{1/2} = 4.5$$

and substituting these vectors and the coordinates and weights derived from Figure 22 into the confirmation model produces

$$\overline{X} = \frac{\frac{3.3 \cdot 100}{5.0} + \frac{10.1 \cdot 65}{4.2} + \frac{13.3 \cdot 65}{5.4} + \frac{11.3 \cdot 65}{5.4} + \frac{4.8 \cdot 65}{4.5}}{\frac{100}{5.0} + \frac{65}{4.2} + \frac{65}{5.4} + \frac{65}{5.4} + \frac{65}{4.5}}$$

$\overline{X} = 8.0$ ,

$$\overline{Y} = \frac{\frac{7.1 \cdot 100}{5.0} + \frac{9.0 \cdot 65}{4.2} + \frac{6.3 \cdot 65}{5.4} + \frac{1.1 \cdot 65}{5.4} + \frac{2.3 \cdot 65}{4.5}}{\frac{100}{5.0} + \frac{65}{4.2} + \frac{65}{5.4} + \frac{65}{5.4} + \frac{65}{4.5}}$$

$\overline{Y} = 5.4$ .

The confirmation model has produced coordinates 8.0 and 5.4 which represent a perfect check on the centroid coordinates of 8.0 and 5.4 for $\overline{X}$ and $\overline{Y}$, respectively. As a result of this confirmation the centroid is the true point of ton mile minimization as the potentially dominant point at $P_1$ was not sufficiently heavy to offset the centroid.

The dominant point.--While Figure 22 depicted a potentially dominant point, Figure 23 presents the impact of an actual dominant point.

The centroid of Figure 23 lies at the intersection of coordinates 7.1 and 5.9 for $\overline{X}$ and $\overline{Y}$ respectively. By implementing the confirmation model as follows, the optimality of the centroid may be determined:

$$\overline{X} = \frac{\frac{3.3 \cdot 100}{4.20} + \frac{10.0 \cdot 40}{4.35} + \frac{13.2 \cdot 40}{6.13} + \frac{11.0 \cdot 40}{6.10} + \frac{4.5 \cdot 40}{4.15}}{\frac{100}{4.20} + \frac{40}{4.35} + \frac{40}{6.13} + \frac{40}{6.10} + \frac{40}{4.15}}$$

$\overline{X} = 6.8$,

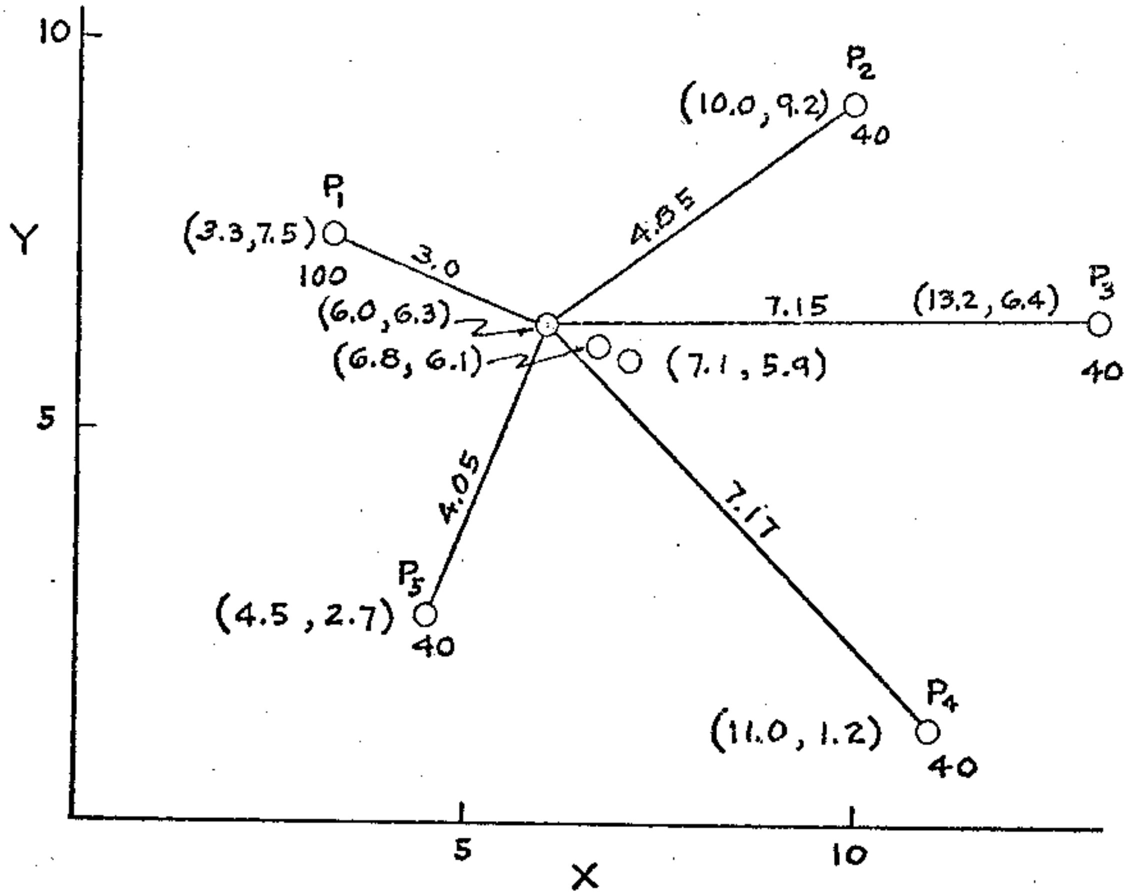

Fig. 23--The impact of a dominant point

$$\overline{Y} = \frac{\frac{7.5 \cdot 100}{4.2} + \frac{9.2 \cdot 40}{4.35} + \frac{6.4 \cdot 40}{6.13} + \frac{1.2 \cdot 40}{6.10} + \frac{2.7 \cdot 40}{4.15}}{\frac{100}{4.2} + \frac{40}{4.35} + \frac{40}{6.13} + \frac{40}{6.10} + \frac{40}{4.15}}$$

$\overline{Y} = 6.1.$

As a result of scaling vectors from the centroid and implementing the model, the centroids' coordinates do not check with the confirmation model, hence the centroid is nonoptimal. Therefore, a location exists external of a centroid location, which will produce a minimization of ton mileage. In Figure 23 the weight associated with point $P_1$ was disproportionately heavy, and thus offsets a location at the equilibrium point or center of gravity. Moreover, the actual least cost location in this situation should not be construed as a new equilibrium location, as there will be no balancing of moments.

In the final analysis the least cost location will bear a closer degree of physical proximity to the dominant point than the relationship of the centroid to the dominant point. In essence, the tentative least cost location for Figure 23 occurs at the intersection of X coordinate 6.0 and Y coordinate 6.3. Confirmation of this location using scaled vectors and the weights and coordinates provided by Figure 23 appears as follows:

$$X = \frac{\frac{3.3 \cdot 100}{3.0} + \frac{10.0 \cdot 40}{4.85} + \frac{13.2 \cdot 40}{7.15} + \frac{11.0 \cdot 40}{7.17} + \frac{4.5 \cdot 40}{4.05}}{\frac{100}{3.0} + \frac{40}{4.85} + \frac{40}{7.15} + \frac{40}{7.17} + \frac{40}{4.05}}$$

$X = 6.0,$

$$Y = \frac{\frac{7.5 \cdot 100}{3.0} + \frac{9.2 \cdot 40}{4.85} + \frac{6.4 \cdot 40}{7.15} + \frac{1.2 \cdot 40}{7.17} + \frac{2.7 \cdot 40}{4.05}}{\frac{100}{3.0} + \frac{40}{4.85} + \frac{40}{7.15} + \frac{40}{7.17} + \frac{40}{4.05}}$$

$Y = 6.3.$

From the above confirmation the optimal location for a warehouse occurs at the intersection of coordinates 6.0 and 6.3 for X and Y, respectively. However, concern may be engendered on the determination of this point for confirmation. From Figure 23 it is apparent that relationships exist between the dominant point, the centroid, and the confirmation point. These relationships will be later explored in the form of a line of force.

The dominant cluster.--Since a single point may be characterized by a disproportionately heavy weighting, a pair or series of points may be clustered so that their combined effect or weighting produces the same effect of a single dominant point. To develop this conception Figure 24 presents an array of points denoted by the lack of a dominant cluster. Similarly, there is no dominant point in this figure which serves to offset the centroid, as will be demonstrated through the confirmation model.

In Figure 24 the centroid of the system (as determined by the balance point method) lies at the intersection of coordinates 9.8 and 6.4 for both $\overline{X}$ and $\overline{Y}$. Proof that this centroid is optimal is depicted as follows:

$$\overline{X} = \frac{\frac{6.8 \cdot 100}{5.7} + \frac{14.8 \cdot 90}{6.0} + \frac{4.7 \cdot 30}{6.0} + \frac{9.7 \cdot 20}{5.2}}{\frac{100}{5.7} + \frac{90}{6.0} + \frac{30}{6.0} + \frac{20}{5.2}}$$

$\overline{X} = 9.8,$

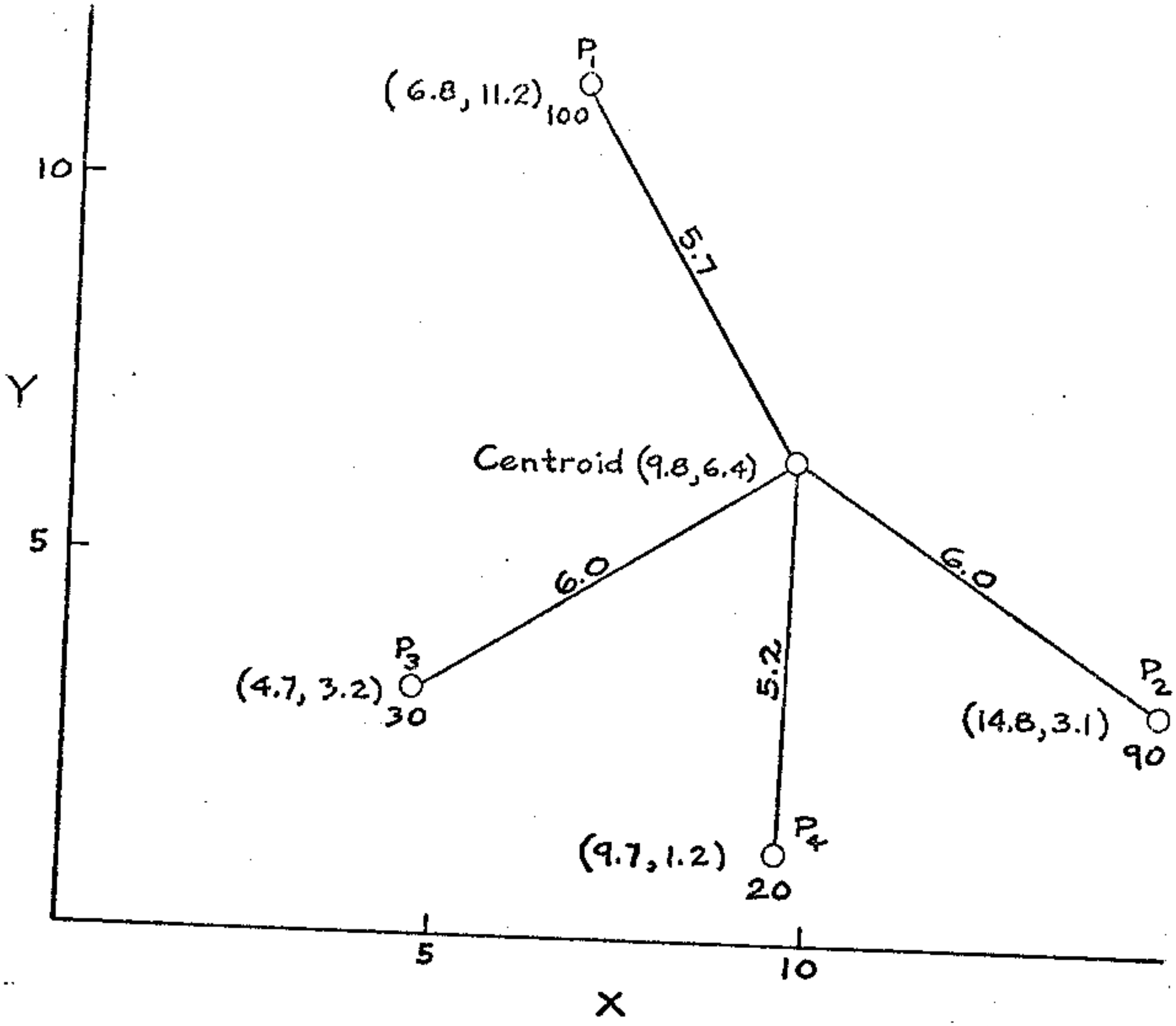

Fig. 24--Conceptual foundation for a dominant point cluster

$$\bar{Y} = \frac{\frac{11.2 \cdot 100}{5.7} + \frac{3.1 \cdot 90}{6.0} + \frac{3.2 \cdot 30}{6.0} + \frac{1.2 \cdot 20}{5.2}}{\frac{100}{5.7} + \frac{90}{6.0} + \frac{30}{6.0} + \frac{20}{5.2}}$$

$\bar{Y} = 6.4.$

However, for the same array of points depicted in Figure 24, a dominant cluster will be created. Figure 25 presents the addition of another point characterized by a weight of 40 to the former array. Note that the addition

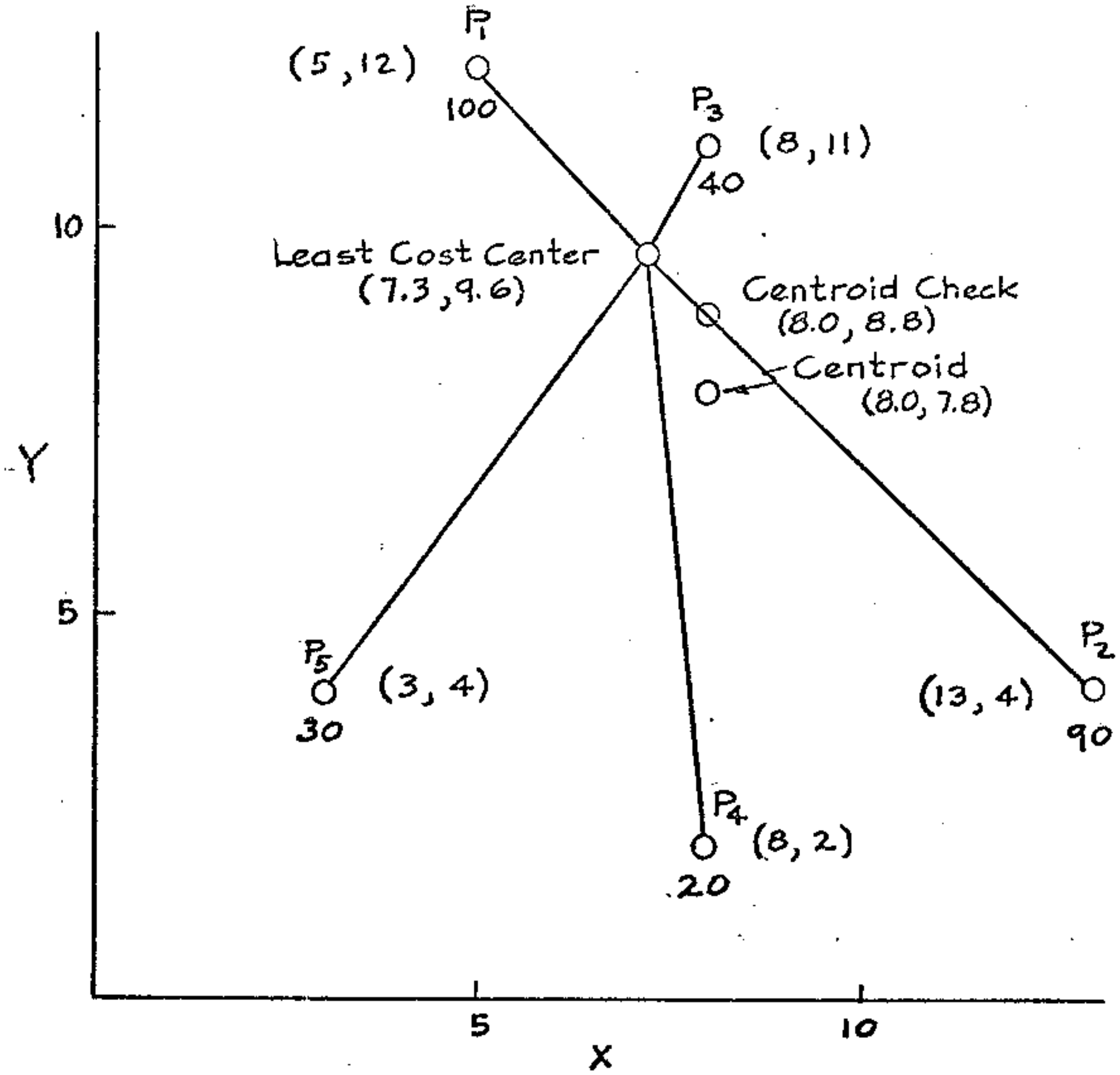

Fig. 25--The impact of a dominant cluster

of this point is in effect clustered in close physical proximity to point $P_1$, thus creating a combined tonnage of 140. It is also evident that a series of additional points could have been utilized.

With the addition of the new weighting the overall centroid of the system has slightly shifted. However, the centroid is now nonoptimal per se as its coordinates of 8.0 and 7.8 for both $\bar{X}$ and $\bar{Y}$ are not confirmed by the following check model:

$$\bar{X} = \frac{\frac{5.0 \cdot 100}{5.1} + \frac{13.1 \cdot 90}{6.3} + \frac{8.0 \cdot 40}{3.2} + \frac{8.0 \cdot 20}{5.8} + \frac{3.0 \cdot 30}{6.3}}{\frac{100}{5.1} + \frac{90}{6.3} + \frac{40}{3.2} + \frac{20}{5.8} + \frac{30}{6.3}}$$

$\bar{X} = 8.0,$

$$\bar{Y} = \frac{\frac{12.0 \cdot 100}{5.1} + \frac{4.0 \cdot 90}{6.3} + \frac{11.0 \cdot 40}{3.2} + \frac{2.0 \cdot 20}{5.8} + \frac{4.0 \cdot 30}{6.3}}{\frac{100}{5.1} + \frac{90}{6.3} + \frac{40}{3.2} + \frac{20}{5.8} + \frac{30}{6.3}}$$

$\bar{Y} = 8.8.$

Plainly, the effect of the dominant cluster (points $P_1$ and $P_3$) has served to offset centroid optimality. It will now be found that the true point of ton mile minimization occurs at coordinates 7.3 and 9.6 for X and Y respectively. Confirmation of these coordinates is now presented:

$$X = \frac{\frac{5.0 \cdot 100}{3.3} + \frac{13.1 \cdot 90}{8.1} + \frac{8.0 \cdot 40}{1.6} + \frac{8.0 \cdot 20}{7.7} + \frac{3.0 \cdot 30}{7.1}}{\frac{100}{3.3} + \frac{90}{8.1} + \frac{40}{1.6} + \frac{20}{7.7} + \frac{30}{7.1}}$$

$X = 7.3,$

$$Y = \frac{\frac{12.0 \cdot 100}{3.3} + \frac{4.0 \cdot 90}{8.1} + \frac{11.0 \cdot 40}{1.6} + \frac{2.0 \cdot 20}{7.7} + \frac{4.0 \cdot 30}{7.1}}{\frac{100}{3.3} + \frac{90}{8.1} + \frac{40}{1.6} + \frac{20}{7.7} + \frac{30}{7.1}}$$

$Y = 9.6.$

## The Impact of Extreme Asymmetry

While the centroid may generate ton mile minimization or near minimization for an asymmetrical array of points, a series of points characterized by extreme asymmetry serves to offset the centroid as the location of least cost or near least cost. Figure 26 depicts an example of extreme asymmetry.

In this figure the highest order of extreme asymmetry has been generated, and as a result the centroids' coordinates of 9.6 and 9.6 prove to be nonoptimal. The following presentation of the confirmation model readily indicates the centroids' lack of optimality:

$$\overline{X} = \frac{\frac{4.7 \cdot 10}{6.8} + \frac{12.4 \cdot 10}{4.7} + \frac{11.6 \cdot 10}{8.8}}{\frac{10}{6.8} + \frac{10}{4.7} + \frac{10}{8.8}} = 9.9,$$

$$\overline{Y} = \frac{\frac{14.4 \cdot 10}{6.8} + \frac{13.5 \cdot 10}{4.7} + \frac{1.1 \cdot 10}{8.8}}{\frac{10}{6.8} + \frac{10}{4.7} + \frac{10}{8.8}} = 10.9.$$

As a result, the point of least cost tentatively occurs at coordinates 10.3 and 12.1 for both X and Y respectively.

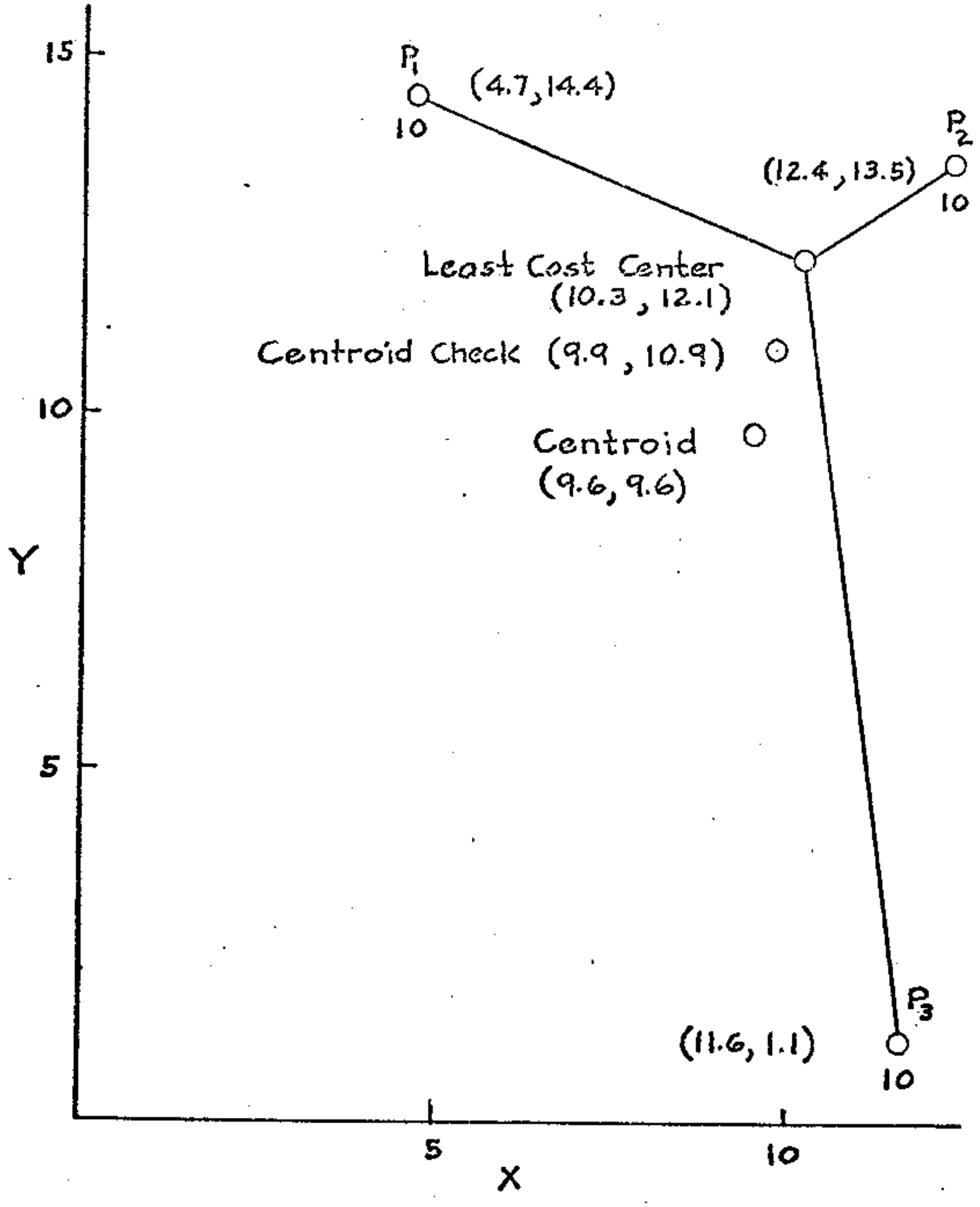

Fig. 26--A spatial array characterized by extreme asymmetry

Confirmation of the optimality of this location is now presented:

$$X = \frac{\frac{4.7 \cdot 10}{6.0} + \frac{12.4 \cdot 10}{2.5} + \frac{11.6 \cdot 10}{11.1}}{\frac{10}{6.0} + \frac{10}{2.5} + \frac{10}{11.1}} = 10.3,$$

$$Y = \frac{\frac{14.4 \cdot 10}{6.0} + \frac{13.5 \cdot 10}{2.5} + \frac{1.1 \cdot 10}{11.1}}{\frac{10}{6.0} + \frac{10}{2.5} + \frac{10}{11.1}} = 12.1.$$

It is also interesting to note that if the equal weights associated with each point were eliminated in Figure 26 that a mile centroid would likewise produce coordinates equal to 9.6 and 9.6, respectively, and that minimal mileage would result from X coordinate 10.3 and Y coordinate 12.1. Thus, for any situation characterized by equal weightings of points, the ton mile centroid and mile centroid will be synonymous.

While Figure 26 generated a theoretical presentation of extreme asymmetry, such patterns of points are not readily common to the warehouse location problem. Therefore, emphasis will center on the conception of extreme asymmetry, which is perhaps most representative of the problem which will confront the location's researcher.

_The conception of distant markets_.--Until the present, the development of supply and demand considerations had depicted spatial interdependence. In other words, supply and demand points were both grouped in the same general market area. Yet, supply and demand factors may be characterized by spatial independence, or a supply market or point denoted by a distant relationship to the demand market which it serves.

Therefore, when supply considerations bear an exogenous relationship to the demand market, which is to be served through a warehouse location, the overall centroid of the system becomes nonoptimal per se. Thus, when the researcher is confronted with a supply point or supply market characterized by a distant relationship to its demand market, a spatial array of points denoted by extreme asymmetry has been generated. Figure 27 on the following page demonstrates the conception using a single distant supply point.

In Figure 27, the overall centroid of the system lies at coordinates 7.8 for $\overline{X}$ and 13.2 for $\overline{Y}$. To reflect the effect of extreme asymmetry on this centroid, the confirmation model produced the following coordinates:

$$\overline{X} = \frac{\frac{4.2 \cdot 100}{9.0} + \frac{12.2 \cdot 90}{5.9} + \frac{2.9 \cdot 80}{6.4} + \frac{12.8 \cdot 70}{8.9} + \frac{8.2 \cdot 20}{7.2} + \frac{7.6 \cdot 180}{12.0}}{\frac{100}{9.0} + \frac{90}{5.9} + \frac{80}{6.4} + \frac{70}{8.9} + \frac{20}{7.2} + \frac{180}{12.0}} = 7.8,$$

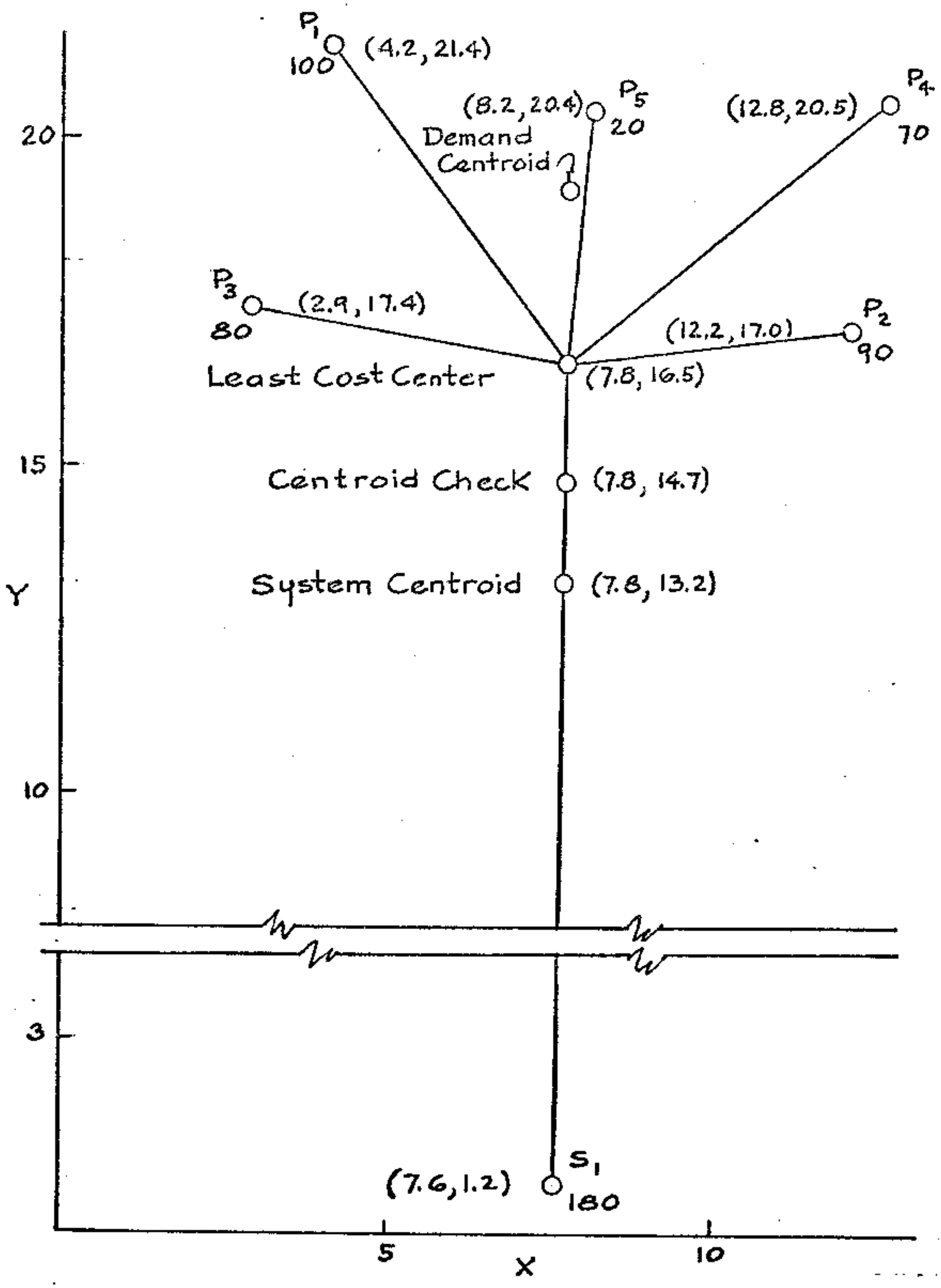

Fig. 27--Extreme asymmetry through distant markets

$$\overline{Y} = \frac{\frac{21.4 \cdot 100}{9.0} + \frac{17.0 \cdot 90}{5.9} + \frac{17.4 \cdot 80}{6.4} + \frac{20.5 \cdot 70}{8.9} + \frac{20.4 \cdot 20}{7.2} + \frac{1.2 \cdot 180}{12.0}}{\frac{100}{9.0} + \frac{90}{5.9} + \frac{80}{6.4} + \frac{70}{8.9} + \frac{20}{7.2} + \frac{180}{12.0}} = 14.7.$$

In this situation, the point of least cost lies in closer proximity to the series of demand points than the overall centroid of the system. Tentatively, the least cost location appears at coordinates 7.8 and 16.5 for X and Y respectively. Confirming the optimization of this point is as follows:

$$X = \frac{\frac{4.2 \cdot 100}{6.0} + \frac{12.2 \cdot 90}{4.4} + \frac{2.9 \cdot 80}{4.9} + \frac{12.8 \cdot 70}{6.4} + \frac{8.2 \cdot 20}{3.9} + \frac{7.6 \cdot 180}{15.3}}{\frac{100}{6.0} + \frac{90}{4.4} + \frac{80}{4.9} + \frac{70}{6.4} + \frac{20}{3.9} + \frac{180}{15.3}} = 7.8.$$

$$Y = \frac{\frac{21.4 \cdot 100}{6.0} + \frac{17.0 \cdot 90}{4.4} + \frac{17.4 \cdot 80}{4.9} + \frac{20.5 \cdot 70}{6.4} + \frac{20.4 \cdot 20}{3.9} + \frac{1.2 \cdot 180}{15.3}}{\frac{100}{6.0} + \frac{90}{4.4} + \frac{80}{4.9} + \frac{70}{6.4} + \frac{20}{3.9} + \frac{180}{15.3}} = 16.5.$$

However, it must be indicated that if the distant supply point in Figure 27 were to be moved in closer proximity to the series of demand points the centroid of the system would approach optimality. Moreover, while only a single supply point was used for explanatory purposes a series of supply points could have been utilized. In this situation, as long as the centroid of the supply subset bears a distant relationship to the demand subset, the same effect occurs as in the case of a single distant supply point.

Additionally, the determination of the noncentroid least cost location for Figure 27 was facilitated by the utilization of a line of force. Concern will now center on this conception and other generalizations which will facilitate the determination of a least cost site for those situations in which the centroid of a system has been confirmed as being nonoptimal.

## General Guides to Optimum Site Determination

When the centroid of a system of points has proven near optimal as indicated by a close relationship between the coordinates produced by the confirmation model and the coordinates of the centroid, the determination of a new tentative least cost site which will be subject to confirmation is relatively easy. All that has to be accomplished is to engage in cost comparisons in the immediate vicinity of the centroid with the lowest cost site so obtained subject to confirmation. However, when the centroid is indicated as being clearly nonoptimal the determination of the area in which cost comparisons are to be made is not as easy. Therefore, some useful generalizations will be presented to help the researcher in this respect. To facilitate the presentation the limiting factors on a centroid location will serve as the framework for discussion.

The dominant point or cluster of weights.--In the event that a centroid location has been offset as an optimal or near optimal point (within designated limits) due to the impact of a dominant point or cluster, the true point of cost minimization will lie between the centroid and the dominant point or cluster. The heavier this weighting in relation to the entire system of weights the closer the proximity of the true point of least cost to the heavier weighting than to the centroid. Conversely, the lighter the weighting the closer the proximity to the centroid location.

Also in the event that a centroid location has been offset by a single dominant point a "line of force" may be established. This refers to a line upon which the point of least cost may actually lie. To develop this conception a line is drawn from the centroid of the system to the dominant point. Figure 28 depicts this line using the illustration in Figure 22 as a workable base.

Note in Figure 28 the straight line relationship between the dominant point, the confirmation check on the centroid, and the centroid of the system. In this regard it may be generalized that when such a relationship is in

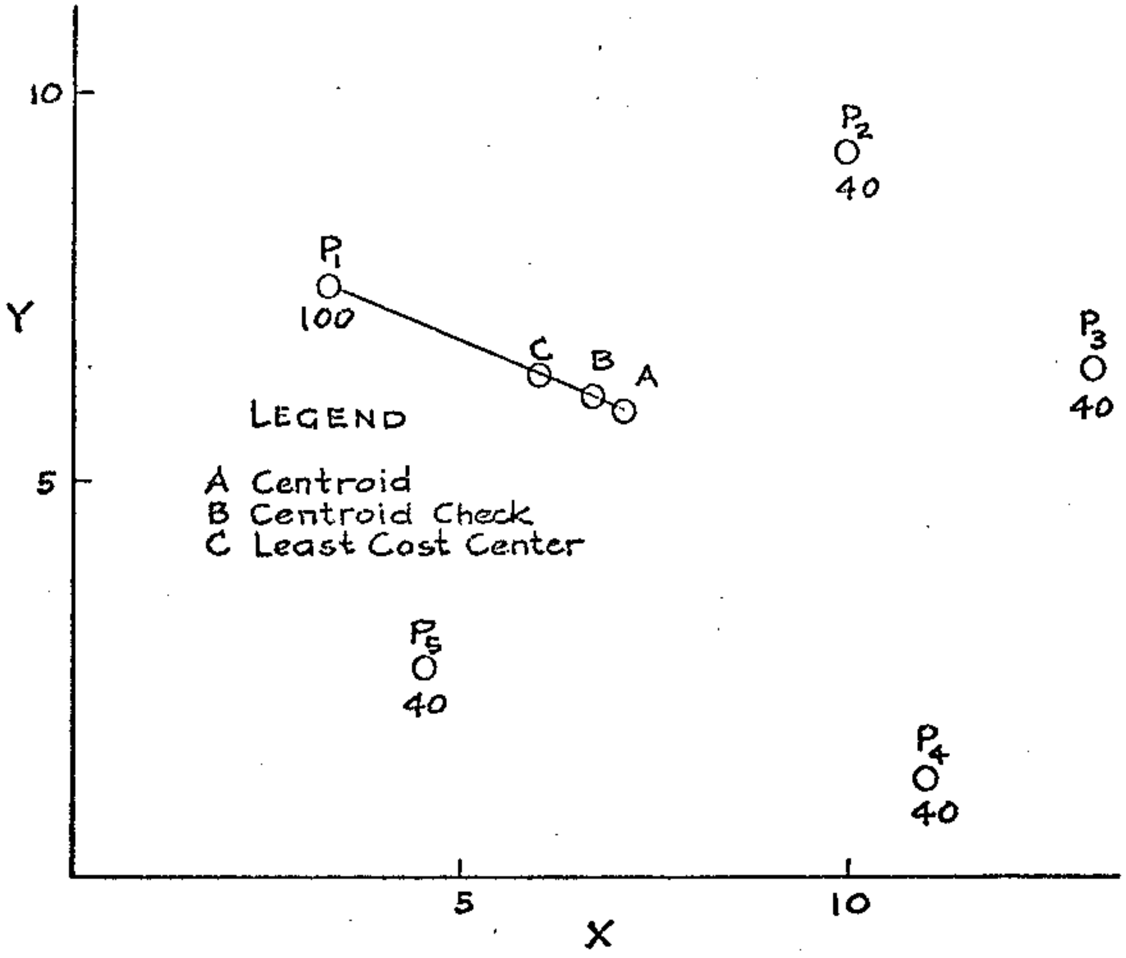

Fig. 28--The development of a line of force for a single dominant point

existence the point of least cost (c) will fall on this line. In the event that the confirmation check on a centroid location produces coordinates which do not fall upon this line, some factor other than a single dominant point may be viewed as compromising the centroid, and the point of least cost may or may not lie on this line.

Extreme asymmetry in the spatial array.--When confronted with extreme asymmetry in the spatial array of points due to a distant relationship between a single supply point (or cluster of supply points) and the demand points which it serves, the centroid becomes clearly nonoptimal. In such a situation the true point of least cost will lie between the centroid of the system and the series of demand points. Thus, cost comparisons should be undertaken only within these constraints.

In the event that the impact of a single distant supply point has offset the centroid as the point of optimality or near optimality, the potentiality for the establishment of another line of force comes into existence. All that has to be accomplished is to extend a line from the single distant supply point to the centroid of the series of demand points. In the event the coordinates produced by the confirmation model check on the centroid fall on this line, the actual point of least cost should lie on this line. If these coordinates do not fall on this line, some factor other than a single distant supply point may be viewed as compromising the centroid, and the point of least cost may or may not lie on this line.

## General Conclusions

The first hypothesis which was concerned with the development of a new noncoordinate centroid determining model was generated and proven. Moreover, the ability of this model to produce a site which would result in an optimum or near optimal warehouse location was proven. However, this second hypothesis does not have universal application due to the indicated limitations on the optimizing abilities of the resultant centroid location. Yet, as has been seen, the centroid location when clearly nonoptimal may be used as a guide to facilitate cost comparisons with the lowest potentially optimum point so obtained subject to the confirmation of its coordinates.

It may now be seen that the determination of a centroid location has definite merit as long as it is coupled with the usage of a confirmation model and acceptable tolerance limits. Thus, the approach is in the virtual reach of any locations researcher.

Also, a small portion of this investigation was confirmed by a part of another publication which made its appearance in 1967 (2). This study briefly indicated that a center of gravity approach could produce answers which were far from optimal if there was a considerable disparity in weights, as this presentation has independently shown. However, the publication gave no recognition to the impact of a dominant cluster in which there may be no actual considerable disparity in weights. Also, no mention was made of the effect of extreme asymmetry in the spatial arrangement of points which may offset the optimality or near optimality of a centroid location. Additionally, the publication advocated the usage of a costly computerized trial and error methodology which arrives at a point of least cost. However, the approach did not indicate the possibility of the usage and confirmation of the centroid's coordinates within designated tolerance limits as the objective centered on an absolute precise point of least cost. Moreover, this absolute degree of preciseness is not needed as the resulting location will be no more valuable than the accuracy of the designated tonnage estimates.

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# CHAPTER IV

## FREIGHT RATE APPLICATION TO THE MODEL

With the development of the conceptual base for ton mile minimization in Chapter III now having been presented, emphasis will shift to the pure cost considerations of the model, or the implementation of freight rates as an added weighting factor. In its basic form this weighting centers on converting the tonnage weightings associated with each point to total per ton mile cost weightings by including freight rates as an additional factor input. Thus, while in Chapter III the objective centered on minimizing the distances associated with tonnage weights assigned to each point, the new orientation of this chapter centers on minimizing the distances associated with per ton mile cost weightings, which will now be assigned to each point due to the impact of freight rates.

Basically, the conceptual foundation laid forth in Chapter III could have easily included freight rates as a weighting factor coupled with tonnages for each customer and supplier, thus the entire orientation of the chapter would have centered on pure cost minimization rather than ton mile minimization. Yet, due to the many nuances and subtleties which pervade the current structure of freight rates it is felt that individualized attention should be focused on this rate conception. To develop this end the parameters of freight rate application which pervade the model will be identified, along with the refinement of both the balance point and confirmation models through rate inclusion. Yet, an understanding of the types of rates which will be substituted into the models is necessary, hence a general freight rate orientation will be provided. Additionally, this rate orientation provides a basis for a discussion of linear and nonlinear rates as information inputs. Lastly, this latter construct provides a conceptual base for depicting a realistic example of rate substitution in the models.

## Model Refinement through Rate Inclusion

By utilizing the conceptual base presented in Chapter III freight rates may now be incorporated into the model to give cognizance to the third and final element of transportation expense. Thus, ton mile minimization per se provides a base for the further inclusion of per ton mile freight rates as an information input, as total transportation expense is a product of: tonnage times mileage times rate per ton mile. Therefore, the ultimate orientation of the study is to determine a location which generates the following relationship:

$TM = W_1R_1V_1 + W_2R_2V_2 + \ldots W_nR_nV_n$

$tm = W_1R_1V_1' + W_2R_2V_2' + \ldots W_nR_nV_n'$

where

TM = confirmed centroid or confirmed alternate location

tm = nonconfirmed centroid or nonconfirmed alternate location

and where

W's = tonnages associated with customers and suppliers

R's = freight rates per ton mile associated with customers and suppliers

V's = vector radii or distances from C to customers and suppliers

V' 's = vector radii or distances from D to customers and suppliers.

The objective depicted in Chapter III has now shifted from ton mile minimization to cost minimization with the addition of freight rates as an added information input. However, since rates and tonnages are viewed as constants, the inclusion of rates produces an additional weighting factor which does not alter the model theory or the proof generated in Chapter III. In this regard, the utilization of rates converts tonnages to total per ton mile costs, which now serves as the weighting associated with each customer and supplier. Thus, costs serve as the weighting for each point and the objective centers on minimizing the distances associated with these total per ton mile costs, hence transportation expense.

An example of such minimization through a centroid location is depicted in Figure 29.

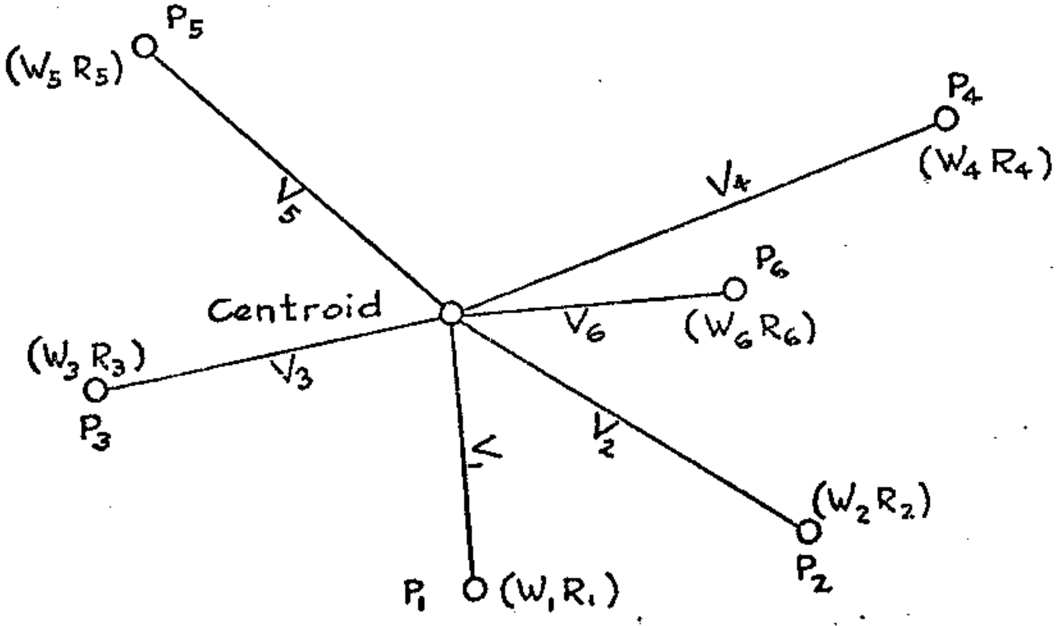


Fig. 29--Cost minimization through a centroid location

In Figure 29,

P's = customers

W's = tonnages associated with customers

R's = per ton mile freight rate assigned each customer

V's = vector radii, denoting distances from centroid to customers.

By locating a warehouse commensurate with the centroid in Figure 29, and assuming the optimality of such a centroid, the distances associated with the newly assigned weights for each point have been minimized. Basically, outside of the limitations delineated in Chapter III, and based upon the assigned information inputs, a centroid location will result in definite cost minimization or near minimization.

With the new orientation through the inclusion of rates now in view, emphasis will proceed to the inclusion of these rates in the balance point formulation.

### The Ton Mile Rate Balance Point Model

With the addition of freight rates as information inputs the product of the tonnages assigned each point multiplied times the respective per ton mile rates results in the assignment of weightings to these points. Once such weightings have been determined the points for the systems involved should be numerically ordered as follows:

$$W_1R_1 \geqq W_2R_2 \geqq W_3R_3 .$$

Once such ordering has been completed the same procedural steps involving the ton mile balance point formulation are implemented, except the new weightings are utilized. This involves the utilization of the following new formulation for ascertaining the point of balance $\bar{B}_1$ between the first two points designated $P_1$ and $P_2$:

$$b_1 = \frac{d_1(W_2R_2)}{W_1R_1 + W_2R_2} \tag{6a}$$

where

$\overline{B}_1$ = balance point between points $P_1$ and $P_2$

$W_1, W_2$ = tonnages associated with $P_1$ and $P_2$

$R_1, R_2$ = freight rates per ton mile associated with points $P_1$ and $P_2$

$d_1$ = scaled or computed distances between $P_1$ and $P_2$

$b_1 = P_1 \sim \overline{B}_1$ or the distance from $P_1$ to $\overline{B}_1$ on line $d_1$.

With the determination of the balance point formulation for points $P_1$ and $P_2$ now in view, the following general formula may be implemented for points $P_3$ through $P_n$:

$$b_n = \frac{d_n(W_{n+1}R_{n+1})}{W_1R_1 + \ldots + W_{n+1}R_{n+1}} \tag{6b}$$

where

$\overline{B}_n$ = balance point of any 2 node system

$W_n$ = tonnages associated with respective points

$R_n$ = freight rates per ton mile associated with respective points

$d_n = \overline{B}_{n-1} \sim P_{n+1}$ or the scaled or computed distance between $\overline{B}_{n-1}$ and $P_{n+1}$

$b_n = \overline{B}_{n-1} \sim \overline{B}_n$ or the distance from $\overline{B}_{n-1}$ to $\overline{B}_n$ on line $d_n$.

Additionally, it will be recalled that n in the general equation will always be one less than the highest numbered point involved in a particular balance point calculation. Also, equation (6b) requires that all preceding centroids for each pair of nodes be computed prior to computing the nth balance point. Therefore, by assuming that $\overline{B}_5$ has been determined for a 7 point system, the final centroid or balance point $\overline{B}_6$ may be determined by substituting into the general equation as follows:

$$b_6 = \frac{d_6 (W_7 R_7)}{W_1R_1 + W_2R_2 + W_3R_3 + W_4R_4 + W_5R_5 + W_6R_6 + W_7R_7}$$

where

$$d_6 = \overline{B}_5 \sim P_7$$
$$b_6 = \overline{B}_5 \sim \overline{B}_6.$$

It is now evident that with the addition of freight rates as information inputs, the weightings associated with each customer and supplier have been converted from tonnages per se to total per ton mile costs. As a result of the conversion the center of gravity assumes the nomenclature of a ton mile rate centroid, as all of the elements of transportation expense have been included in the balance point model.

Plainly, the ton mile rate formulation is subject to the same limitations which confronted the ton mile balance point model. However, the impact of a dominant point or cluster now refers to extreme cost weightings rather than extreme tonnage weightings per se. Yet, the possibility of such limitations essentially indicates the need for rate inclusion in the confirmation model. The orientation will now, therefore, center on the ton mile rate confirmation model.

## The Ton Mile Rate Confirmation Model

The conversion of the weightings associated with customers and suppliers from tonnages to total per ton mile costs carries ramifications for the confirmation model, as freight rates must also be viewed as an additional weighting factor. Yet from a theory standpoint the inclusion of rates into the confirmation model does not alter the model's ability to confirm a least cost location. In this regard the structure of the model remains constant with only the weightings associated with the respective points of a system assuming a new dimension. In essence, formulations 5a and 5b depicted in Chapter III could retain the same symbolic format with $W_n$ now referring to the product of tonnages times per ton mile rates, instead of tonnages per se. However, freight rates may be implemented in the model to give cognizance to the rate factor in symbolic terms. Such an implementation appears as follows:

$$\overline{X} = \frac{\frac{X_1(W_1R_1)}{V_1} + \frac{X_2(W_2R_2)}{V_2} + \ldots + \frac{X_n(W_nR_n}{V_n}}{\frac{W_1R_1}{V_1} + \frac{W_2R_2}{V_2} + \ldots + \frac{W_nR_n}{V_n}} \qquad (7a)$$

$$\overline{Y} = \frac{\frac{Y_1(W_1R_1)}{V_1} + \frac{Y_2(W_2R_2)}{V_2} + \ldots + \frac{Y_n(W_nR_n)}{V_n}}{\frac{W_1R_1}{V_1} + \frac{W_2R_2}{V_2} + \ldots + \frac{W_nR_n}{V_n}} \qquad (7b)$$

where

$\overline{X}$ = coordinate of centroid or potential least cost site on X axis

$\overline{Y}$ = coordinate of centroid or potential least cost site on Y axis

$X_n$ = coordinate for customer or supplier point on X axis

$Y_n$ = coordinate for customer or supplier point on Y axis

$V_n$ = vector or distance of customer or supplier from warehouse at centroid or potential least cost location

$W_n$ = tonnage associated with a given point

$R_n$ = per ton mile freight rate associated with a given point.

With the symbolic addition of the rate factor to the confirmation model now intact, the ton mile rate centroid formulation may be either confirmed or disproved. In the event that the substitution of the necessary information inputs into equations (7a) and (7b) produces $\overline{X}$ and $\overline{Y}$ coordinates noncommensurate with the coordinates of the centroid or external of an acceptable range of tolerance, a point of least cost exists outside of this location. Similarly, in Chapter III when the ton mile centroid per se was nonconfirmed, an optimum point of ton mile minimization existed external of a centroid location. Therefore, when the ton mile rate centroid is disproved cost comparisons are also made for a series of potential locations with the tentative point of potentially lowest cost subject to the confirmation of its coordinates.

To facilitate the development of cost comparisons the same generalizations presented in Chapter III are likewise applicable and obviously the comparison of costs from alternative locations is based upon holding the product of tonnage times per ton mile rate constant for each point and by varying distances or mileage.

With the presentation of both the ton mile rate balance point and confirmation models now having been completed, a complete reiteration of both the determination models (balance point and coordinate) and the confirmation models is succinctly depicted in Appendix B.

## Parameters of Freight Rate Application

Since the application of freight rates to the model will squarely focus on the common carrier as the primary transport agency it will be generic to the task to develop the parameters of such rate application to the model. This involves developing a framework which consists of the key environmental factors which will be encompassed by the model. Thus, such a framework centers on such factors as the nature of the product, the general modes of applicable transport, volume of shipment considerations, and the types of rate application. Emphasis will now center on the development of each parameter.

### Nature of the Product

For the purpose of model development it is assumed that the nature of the product refers to homogeneous, nonperishable, staple products. This, of course, is in line with the freight rate orientation of the model, as when dealing with perishable products the criteria of speed of delivery, or the temporal aspects of the problem, tend to take precedence over transport cost considerations.

### Modes of Transport

To facilitate the development of the environmental constraints of the location problem it is also assumed that both rail and motor common carriers serve as the basic modes of transport. This refers to either rail and/or motor carriers for both inbound and outbound warehouse shipments. Basically, these two modes of transport handle the predonderance of ton mile movements, hence warehouse shipments, thus water carriers, pipelines and air transport are readily delimited.

## Volume of Shipment Considerations

Since the primary orientation of the study centers on the delineation of a least cost warehouse location, it is assumed for the purpose of analysis that all inbound warehouse shipments are characterized by volume shipments, while all outbound warehouse shipments are evidenced by less than volume transport movements. Indeed, this is the normal channel pattern of warehousing operation (1, p. 167).

## Types of Rate Application

Based upon the typical pattern of volume differences between inbound and outbound shipments it is assumed that all outbound less than car load and/or less than truckload warehouse shipments will be characterized by class rates. And as will be evidenced in the rate orientation section of this chapter this is the predominant rate application to less than volume shipments.

In regard to inbound warehouse shipments, in volume quantities, it is assumed that commodity rates and/or class rates generally apply. Traditionally, commodity rates per se apply to volume shipments while class rates apply to less than volume shipments.

## Freight Rate Orientation

To facilitate an understanding of the types of rates which will confront the locations researcher a general freight rate orientation involving class, commodity, and exception rates will be developed. Such an orientation is, nonetheless, significant, as it sets the structure for a discussion of both linear rates and nonlinear rates as information inputs. Moreover, since the parameters of the study have been delimited to include only rail and motor common carrier transport, the discussion of rates will likewise be so delimited. Yet for explanatory purposes the rate orientation will make no distinction between rail and motor carrier rates as the bases upon which they are structured are so similar. Even the formation of rail and motor carrier tariffs, or rate books, follows the same framework. Based upon these considerations a discussion of rail and motor carrier rates as distinct entities would involve needless repetition.

## Class Rates

To enhance the determination and recognition of freight rates, commodities are readily grouped in classes based upon their respective characteristics. By engaging in such grouping the assignment of rates for each of the countless thousands of commodities comprised in transport activities is eliminated. As a result, the entire freight rate structure is simplified as each class has its own less than volume and volume ratings. These ratings when linked with the conception of distance serve to identify the freight rate in question.

Since class ratings and distance both serve as the basic elements in the class rate structure, attention will now center on the factors which determine the assignment of ratings to each class. These factors center on the supply and demand characteristics of the class in question. Additionally, the study will also focus on the impact of distance.

Supply or cost considerations.--Arbitrarily, there are five general supply considerations which facilitate the assignment of ratings to each other, or whether a class will be characterized by a high or low rating, hence freight rate. Also the following supply characteristics are useful in identifying the assignment of commodities to particular classes.

1. Commodity density and weight.--Carload and truckload minimum weights, hence volume ratings are in part derived from the combined conceptions of density and weight. Essentially, both of these factors are influential in determining the volume or capacity of a rail car or motor carrier. Moreover, a commodities' or classes' density in relation to weight facilitates the determination of a high or low rating. For example, a class characterized by considerable bulk or density coupled with a small concentration of weight results in the assignment of a high rating for this particular characteristic.

2. Susceptibility to damage and pilferage.--Since the common carrier is liable in most shipments for damage and pilferage, those classes which are most susceptible to such an end are assigned higher ratings.

3. Product value.--The value of a product per pound is also a useful rating device. Generally, the higher the value of a class the greater the pressure for the assignment of higher ratings based on this characteristic.

4. Special handling characteristics of a commodity.--Plainly, the need for specialized facilities for handling, loading, and transporting a particular class generates additional costs, hence higher ratings.

5. Characteristics of movement.--Lastly, the regularity and volume of movement is useful in helping to delineate a rating for individual classes. For example, the lesser the degree of regularity in movement the greater the pressure for the assignment of a high rating. Moreover, carload and truckload minimum weights are influenced by such a conception.

Generally, characteristics such as the above when taken in composite form result in a partial determination of a rating for the class in question.

<u>Demand considerations</u>.--While supply considerations provide the general framework for the determination of ratings, demand considerations serve to temper, solidify,

or offset supply characteristics which point to a given rating. In this regard two primary demand considerations stand manifest:

1. The degree and extent of competition among carriers for the product.

2. The extent of competition of the commodity in question with other commodities.

The nature of such demand factors serves to interact with the composite effect of supply considerations and the result is seen in the respective volume and less than volume ratings for each designated class of commodity. In this regard, using 100 as a base, class ratings commonly range from class 25 to class 400, and are expressed as a percentage of class 100. Therefore, given a class rating for a commodity, all that is needed is the distance over which the commodity moves from origin to destination to determine the freight rate in question.

With this in view the study will now concentrate on the rudiments of distance.

The conception of distance.--Basically, the cost of transporting freight increases with an increasing length of movement, hence class rates are depicted on a distance basis. However, such a distance basis is unique because freight rates increase over distance in terms of mileage blocks and not directly proportional to such distance.

Such a distance basis is therefore constructed on the tapering principle, which indicates that the total transportation cost is greater for the longer than for the shorter distances, yet the rate per ton mile is less for the longer distances (2, p. 177).

There are several reasons for not constructing freight rates which increase in exact proportion to distance, or for not establishing linear rates. The first reason centers on terminal costs which are obviously the same regardless of the length of haul; therefore, the longer the transport distance the greater the distance over which the constant terminal cost can be spread. Hence rates per ton mile begin to taper off over increasing distances.

Another significant reason for tapering rates or making rates nonlinear with respect to distance centers on the desirability of tapping distant markets, as if rates were linear over distance they would soon become prohibitive, thus impeding the movement of traffic over lengthy distances.

Moreover, the development of rates of progression over distance are not readily amenable to precise formula calculation, as the Interstate Commerce Commission has placed more reliance on the practical necessities of fitting the scale into existing rate levels and of joining the scales with the rate structures in bordering territories (4, p. 181).

With this tapering principle now having been presented, it is apparent that the distance between the point of origin and the point of destination must be determined in order to identify the rate associated with a particular class rating. To facilitate the determination of this distance each class rate tariff or rate book for the geographical locale in question contains an alphabetical index of points and the distance between these points in the form of a rate basis number.

## Commodity Rates

While outbound warehouse shipments with some exceptions move on class rates, inbound warehouse shipments are commonly characterized by commodity rates. In this regard, commodity rates account for more than 90 per cent of rail carload tonnage (2, p. 96), and a healthy percentage of motor common carriage shipping in truckload quantities, regularly, between particular points or areas. These commodity rates are not characterized by commodity classifications, as rates are quoted directly on the article in question between designated points. However, in the event that a shipment involves a point not formally designated, rulings may be formulated in the tariff for commodity rates applicable to intermediate points (5, p. 48).

The primary reasoning behind direct rate quotes centers on the inability of class rates to meet the needs of carriers, shippers, and other parties interested in rate levels. Moreover, when commodity rates are developed in response to these needs, these rates take precedence over the class rating published on the same commodity.

However, while the class rate structure is characterized by both standardization and uniformity of determination, such consistency is not evident in the formulation of commodity rates, as three forms of determination are in evidence (4, p. 173). First, there are commodity rates which are directly tied to the class rate structure of a given territory. This relationship results in the establishment of commodity rates expressed as a percentage of first class or class 100 for the designated points in question.

A second type of commodity rate is developed irrespective of class rates yet is constructed according to a special distance scale. Here the grouping of origin and destination points may prevail to reduce the number of specific rates to be published.

Lastly, commodity rates may apply only between two points on a point to point basis per se. In this instance rates are not based on a systematic determination through distance scales, as they are oriented in the needs of some particular shipper or community, or to meet some competitive condition.

## Exceptions to the Classification

When dealing with class rates, exceptions to the general classifications are provided to insure more flexibility in the structuring of rates. Such exceptions tend to center on changes in minimum weights, rules, descriptions, or changes in ratings. For example, if motor carrier competition threatens rail transportation on a given commodity in a given territory, the railroads may meet this competition through an exception to the classification without disturbing the level of rates on that commodity in the remainder of the territory.

Further, exceptions to the classification when they are apparent must take precedence over the standard ratings appearing in the classifications. However, it is significant to note that less than volume exceptions have almost completely disappeared (1, p. 57). Therefore, when dealing with outbound warehouse shipments and the prevailing less than volume class rate structure, exceptions should not confront the locations researcher.

With the confrontation of the types of rates which will involve the locations researcher now complete, it is apparent that the substitution of per ton mile rates into the balance point and confirmation models is no easy task. In this regard, every conceivable locations model involving rates is confronted with the same spectrum of rates, yet guidelines for rate identification are commonly absent.

Therefore, to facilitate the determination of rates as information inputs separate attention will next be given to the assumption of linear freight rates and an accompanying example of model application utilizing these rates.

## Linear Freight Rates as Information Inputs

To briefly repostulate, distance is vital to a large proportion of rate determination. For example, when dealing with class rates as distance expands in terms of mileage blocks freight rates increase. Yet due to the principle of tapering distance, rates per mile per hundred weight for each block decline with such an expansion. Similarly, the same is true for those commodity rates tied to class rates and generally for those commodity rates with their own distance scales.

Such nonlinearity in the prevailing structure of rates complicates the determination of a least cost warehouse location. Clearly, if freight rates were linear or directly proportional to distance rates per mile per hundred weight would remain constant over expanding distances and the substitution of freight rates in both the gravitational and confirmation models would be facilitated. However, since rates are nonlinear, or do not remain constant over distance when expressed in terms of rates per ton mile, the substitution of rates into the model is in fact complicated, For example, since the warehouse is the point to be determined the rates associated with shipping to and from this warehouse are not overtly evident; therefore, to supplant this limitation the assumption of linear freight rates for all types of warehouse location models is common.

### Average Per Ton Mile Rates as Information Inputs

Utilizing this assumption of linear freight rates the locations researcher may actually substitute rates into the model by developing an average or weighted average per ton mile rate for each customer and supplier point. All that need to be determined is an average outbound rate for each class and an average inbound rate for each commodity and/or class. Thus a weighted average per ton mile rate may be developed based upon the tonnages associated with each point. This, in turn, reflects the impact of multiple classes or commodities which may be associated with a given point. Moreover, when both rail and motor carriers are utilized for inbound shipments an average rail and motor carrier rate may be developed for the commodities in question, although it may be assumed that all inbound shipments move via rail. Typically, less than volume rail and motor carrier rates are extremely competitive, hence the type of transport should be of no concern for outbound warehouse shipments.

To accomplish the determination of the average per ton mile freight rate associated with a given class or commodity, the firm may turn to its existing warehouse operations and bills of lading. Thus, an average inbound and outbound distance factor is developed and by utilizing the firm's bills of lading an average rate for each class and commodity under consideration is delineated. Therefore, by utilizing these average distance factors, the average class and commodity rates are ultimately converted to an average per ton mile rate.

In the event that warehousing operations are being initially developed, an obvious lack of historical perspective prevails; hence for the purpose of rate identification it may be assumed that both inbound and outbound shipments move via class rates. Therefore, class rate distance scales may be utilized to determine average rates per ton mile over a series of distance intervals. All that is required is the determination of an approximate minimum and maximum distance of both customers and suppliers from a tentative warehouse location area, and to develop incremental mileage distance intervals within this range. Based upon the rates associated with these intervals an average per ton mile rate may be developed for each less than truckload class rate and for each carload and/or truckload class rate. An example of this determination is depicted in Table X.

In essence, Table X hypothetically represents inbound class 100 rates per ton mile for the mileage increments ranging from 400 to 700 miles. This range is arbitrarily

## TABLE X

## INCREMENTAL TRUCKLOAD RATES PER TON MILE

| Incremental Mileages | Rate Per Hundred Weight | Rate Per Mile Per Hundred Weight | Rate Per Ton Mile |
|---|---|---|---|
| 400 | 2.99 | .0075 | .150 |
| 450 | 3.19 | .0071 | .142 |
| 500 | 3.35 | .0067 | .134 |
| 550 | 3.56 | .0065 | .130 |
| 600 | 3.69 | .0062 | .124 |
| 650 | 3.93 | .0061 | .122 |
| 700 | 4.06 | .0058 | .116 |

representative of the closest supplier existing within 400 miles of the warehouse and of the furthermost supplier within a 700 mile range of the same general location. As a result of developing this range of weights the rate per ton mile which most closely corresponds with this series is 0.131¢ per ton mile.

It should now be apparent that once the determination of average per ton mile rates has been completed a framework for additional model weighting has emerged. However, a differential approach may also readily suffice; therefore, emphasis will center squarely on this conception.

The differential approach.--Rather than weight each customer and supplier by its own weighted per ton mile rate, an average composite ton mile rate for all outbound and inbound warehouse shipments may be developed in which supply tonnages are additionally weighted by the percentage relationship of inbound rates to outbound rates. This was basically the approach delineated in Chapter III when ton miles were minimized within the constraints of the existing freight rate structure. Thus, it will be recalled that inbound tonnages from suppliers to the warehouse were weighted by 0.5. While this approach is characterized by a lesser degree of sophistication its simplicity and speed may well make it a worthwhile orientation.

Attention will now center on the actual determination of a least cost location through the utilization of freight rates as an added information input.

## Hypothetical Least Cost Determination

To facilitate the determination of a least cost warehouse location with average per ton mile freight rates serving as information inputs, the development of a hypothetical situation will be generic to the task. Such development involves the delineation of a series of customers and suppliers, the products to be handled through the warehousing operation, tonnage estimates per product, and, of course, the identification of average per ton mile freight rates for the products in question.

It will now be assumed that two major suppliers (manufacturers) have elected to service three large product users through a warehousing operation. In this regard products A and B will encompass the entire spectrum of demand for these three customers, and the average per ton mile freight rate for these products is now presented in Table XI.

TABLE XI

AVERAGE PER TON MILE FREIGHT RATES

| Product | Less than Truckload Class Rate | Truckload Commodity Rate |
|---|---|---|
| A | .10¢ | .06 |
| B | .07 | .04 |

These rates in Table XI when multiplied times the tonnage weights assigned each customer and supplier produce per ton mile cost weightings for each point which provides the basis for cost minimization. Table XII will now present these total cost weightings based on the assumed estimates of tonnage demand for a stipulated period; say three years.

Notice in Table XII that supply and demand tonnages for products A and B are in equilibrium with the exception of the 10,000 ton stock contingency reserve that supplier $S_4$ will provide the warehousing operation. Moreover, the tonnage estimates presented in Table XII apply only to the

TABLE XII

TOTAL PER TON MILE COST WEIGHTINGS

| Points | Product | Tonnages | Rate | Cost | Cost |
|---|---|---|---|---|---|
| Customer $P_1$ | A | 110,000 | 0.10¢ | $11,000 | $11,000 |
| Customer $P_2$ | A | 50,000 | 0.10 | 5,000 | |
| Customer $P_2$ | B | 80,000 | 0.07 | 5,600 | 10,600 |
| Customer $P_3$ | B | 100,000 | 0.07 | 7,000 | 7,000 |
| Supplier $S_4$ | A | 110,000 | 0.06 | 6,600 | |
| Supplier $S_4$ | B | 80,000 | 0.04 | 3,200 | |
| Supplier $S_4$ | B* | 10,000 | 0.04 | 400 | 10,200 |
| Supplier $S_5$ | A | 50,000 | 0.06 | 3,000 | |
| Supplier $S_5$ | B | 100,000 | 0.04 | 4,000 | 7,000 |

*Stockout Contingency Reserve.

demand for product A and B that will be supplied from the warehouse. Also, the spatial arrangement of points may be such that it is more economical for a supplier to ship direct to a given customer rather than through a warehouse. Therefore, those products which may be shipped more economically direct are excluded from the projections of tonnage demand.

With the development of the cost weightings for each customer and supplier point now complete, the objective now shifts to the balance point and confirmation models as a means to minimize the distances associated with these weightings, hence total transportation costs. Figure 30 presents this new orientation.

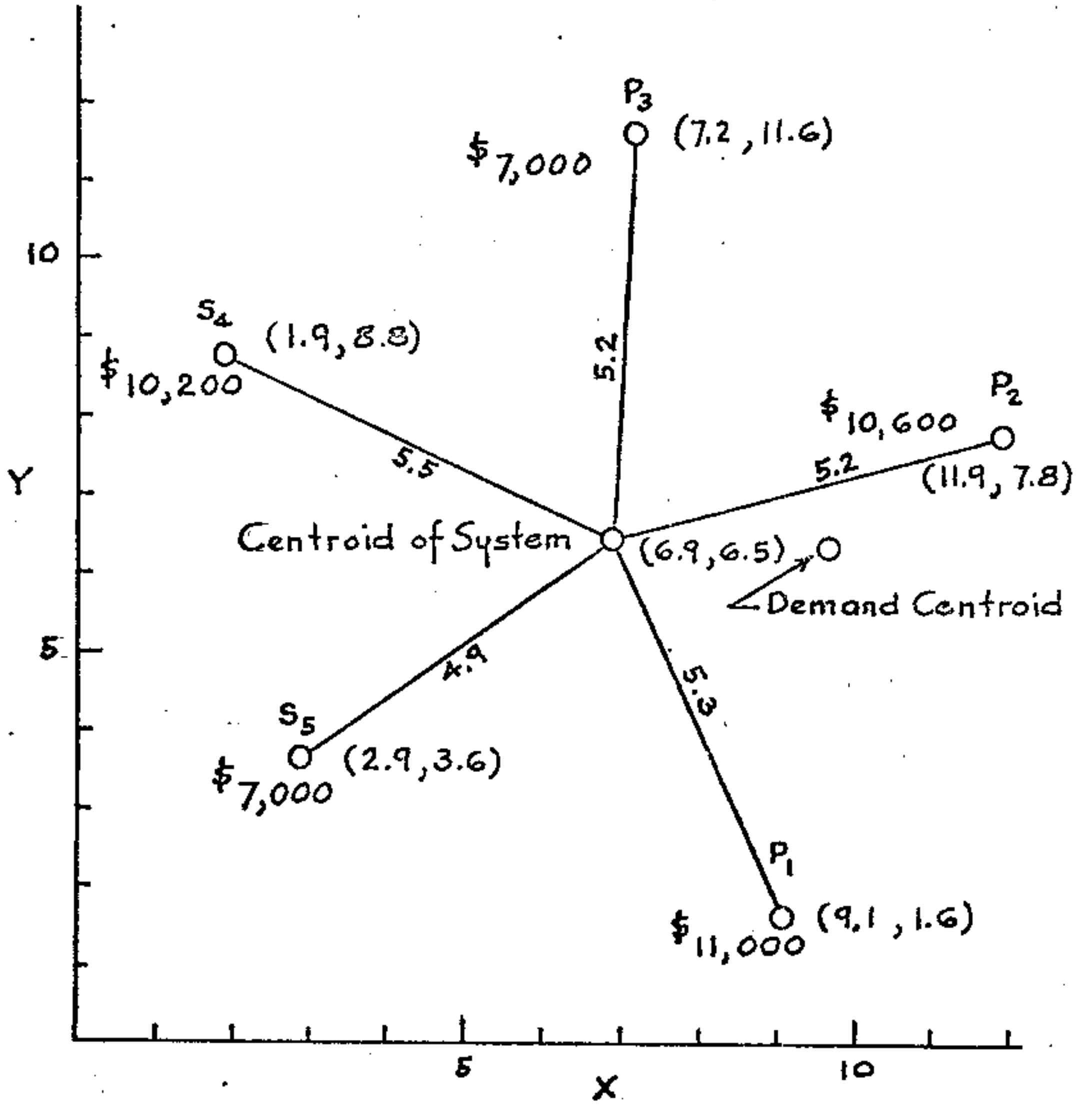


Fig. 30--Potential minimization of transportation expense through a centroid location

In Figure 30, the balance point formulation has produced an overall centroid location at coordinates 6.9 for $\bar{X}$ and 6.5 for $\bar{Y}$. As denoted, this point may serve as the optimal or near optimal warehouse location site, unless offset by the existence of extreme variations in weightings or extreme asymmetry in the spatial array of points. Therefore, the ton mile rate confirmation model must be implemented to confirm the coordinates of the centroid. This formulation, utilizing the weightings, coordinates and vector radii delineated in Figure 30, is presented as follows:

$$\bar{X} = \frac{\frac{9.1 \cdot 11}{5.3} + \frac{11.9 \cdot 10.6}{5.2} + \frac{7.2 \cdot 7}{5.2} + \frac{1.9 \cdot 10.2}{5.5} + \frac{2.9 \cdot 7}{4.9}}{\frac{11}{5.3} + \frac{10.6}{5.2} + \frac{7}{5.2} + \frac{10.2}{5.5} + \frac{7}{4.9}}$$

$$\bar{X} = 6.9,$$

$$\bar{Y} = \frac{\frac{1.6 \cdot 11}{5.3} + \frac{7.8 \cdot 10.6}{5.2} + \frac{11.6 \cdot 7}{5.2} + \frac{8.8 \cdot 10.2}{5.5} + \frac{3.6 \cdot 7}{4.9}}{\frac{11}{5.3} + \frac{10.6}{5.2} + \frac{7}{5.2} + \frac{10.2}{5.5} + \frac{7}{4.9}}$$

$$\bar{Y} = 6.5.$$

The confirmation model has produced coordinates of 6.9 and 6.5 which represents a perfect check on the centroid coordinates of 6.9 and 6.5 for $\bar{X}$ and $\bar{Y}$, respectively. Therefore, for the hypothetically developed situation the ton mile rate centroid is the true point of cost minimization. In essence, the minimal cost associated with a warehouse location at coordinates 6.9 and 6.5 is depicted in Table XIII.

TABLE XIII

MINIMAL TRANSPORTATION EXPENSE FOR THE OPTIMAL CENTROID LOCATION DEPICTED IN FIGURE 30

| Points | Ton Mile Costs | Mileage | Expense |
|---|---|---|---|
| Customer $P_1$ | \$11,000 | 530 | \$ 5,830,000 |
| Customer $P_2$ | 10,600 | 520 | 5,512,000 |
| Customer $P_3$ | 7,000 | 520 | 3,640,000 |
| Supplier $S_4$ | 10,200 | 550 | 5,610,000 |
| Supplier $S_5$ | 7,000 | 490 | 3,430,000 |
| Total | \$45,800 | 2,610 | \$24,022,000 |

With the presentation of the total transportation expense of shipping products A and B over an assumed three year period now in view, no other warehouse location based upon the same ton mile costs can result in a lower total of transportation expense.

## Nonlinear Freight Rates as Information Inputs

By utilizing the assumption of linearity, per ton mile freight rates remain constant over distance and all that needs be determined is an average per ton mile rate for both outbound and inbound shipments. However, since freight rates are generally nonlinear with respect to distance the utilization of the linear assumption produces freight rates as model inputs which generally understate and overstate the rates associated with given points. It will be recalled that due to the tapering principle, rates per ton mile decline with expansions in distance.

However, by employing nonlinear rates as information inputs, the rates which most closely approximate shipping to a given customer or from a given point of supply may be identified. Generic to such an identification is the delineation of a potential warehouse location and the distances emanating from this location which may be utilized as a guide to a rate per ton mile determination.

To arrive at the prescribed guide the ton mile centroid for the system in question is utilized in which supply tonnages have been weighted to reflect the general transport cost difference between the firm's inbound and outbound warehouse shipments. Of course if the centroid is non-optimal the location of ton mile minimization within these same constraints should be utilized.

It is now evident that by substituting the rates so obtained into the model, the market centroid will shift to depict the influence of nonlinear freight rates as an added information input. Such a shift will be in the direction of the points characterized by the heavier tonnages, as they will also be characterized by the higher rates per ton mile. Moreover, if the check on the initial centroid indicated optimality, the additional inclusion of nonlinear rates as information inputs should not serve to offset the optimality of the new location. On the other hand, if the initial centroid was not confirmed and a least cost location was delineated for the purpose of identifying freight rates, the shift in this location should be undertaken with the potential least cost location subject to confirmation.

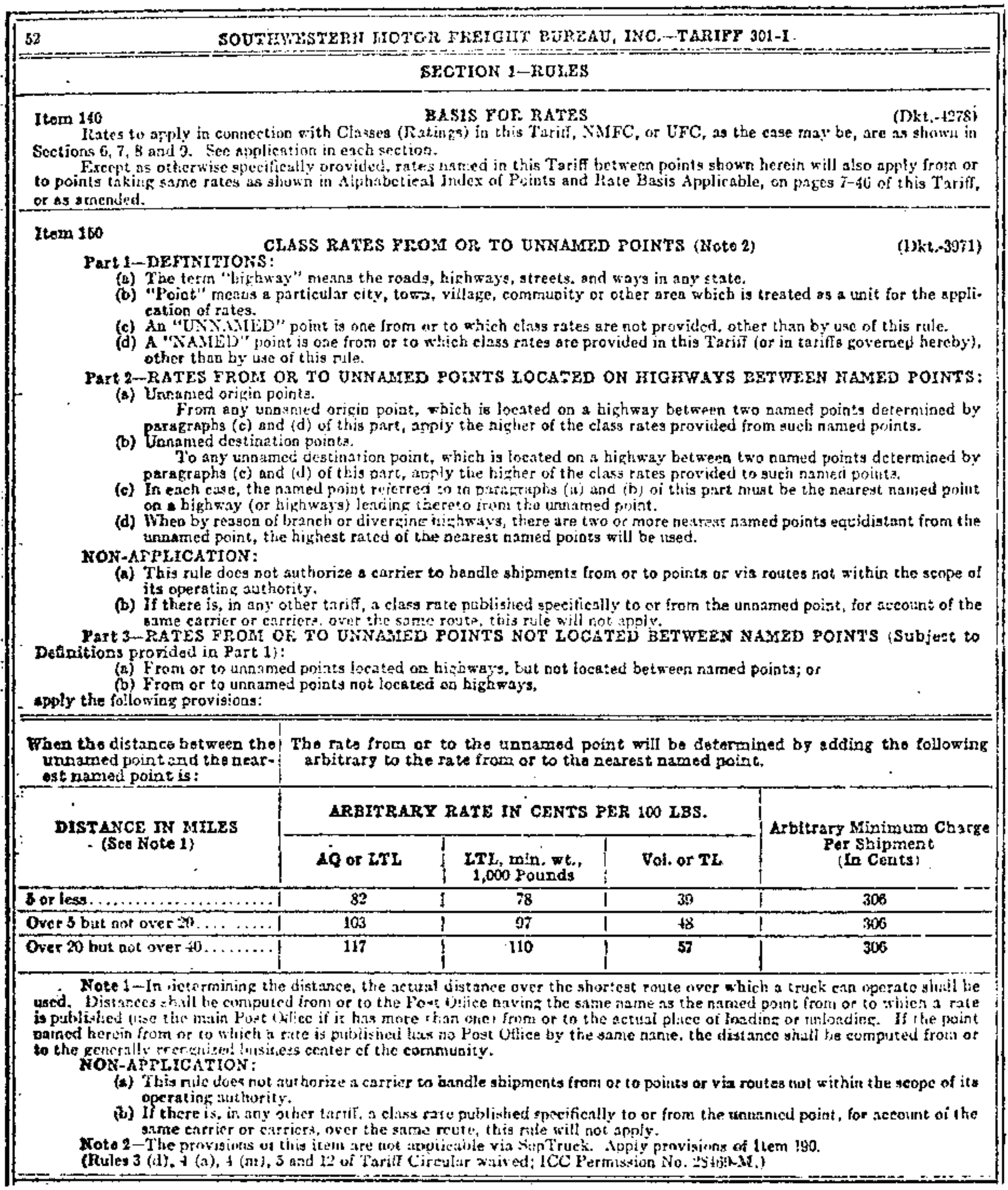
Moreover, by substituting nonlinear freight rates into the model and by assuming the optimality of the new centroid, any other location based on the prevailing information inputs must produce a higher total transportation expense. Therefore, by holding tonnages and ton mile rates constant and by varying air mile distances, any new location must be nonoptimal. Similarly, if the ton mile rate centroid is nonoptimal and a least cost location is confirmed any other location will likewise be nonoptimal.

## Rate Per Ton Mile Delineation

To facilitate the determination of rates as model inputs it will be initially assumed for discussion purposes that the initial point of ton mile minimization and the entire structure of points are located commensurate with named points. These points in turn serve as the final basis for rate determination. Therefore, since rates are quoted in terms of cents per hundred weight between these points all that has to be determined is the distance between the confirmed least ton mile point and each customer and each supplier. Once this distance is determined the rate per hundred weight is then converted into a per ton mile rate and is substituted into the model.

In the event that the initial confirmed centroid or point of minimum ton miles or any of the points encompassing the system are not located commensurate with the alphabetical index of points appearing in the front of each tariff, rules are generally promulgated in each tariff to provide a basis for rate identification. Figure 31 on the following page is an example of such formalization for class rates (3, p. 52). Moreover, commodity rates also provide for such rulings, but generally they are depicted in a tighter framework. However, if no rulings on commodity rates are forthcoming contact with the tariff publishing house or rate bureau in the territory in question may provide an approximate guideline.

With the determination of rates per hundred weight now having been generated, concern will center on air miles or quoted distances between points as a basis for converting rates into per ton mile information inputs.



52 SOUTHWESTERN MOTOR FREIGHT BUREAU, INC.—TARIFF 301-I

**SECTION 1—RULES**

**Item 140** **BASIS FOR RATES** (Dkt.-4278)

Rates to apply in connection with Classes (Ratings) in this Tariff, NMFC, or UFC, as the case may be, are as shown in Sections 6, 7, 8 and 9. See application in each section.

Except as otherwise specifically provided, rates named in this Tariff between points shown herein will also apply from or to points taking same rates as shown in Alphabetical Index of Points and Rate Basis Applicable, on pages 7-40 of this Tariff, or as amended.

**Item 150**

**CLASS RATES FROM OR TO UNNAMED POINTS (Note 2)** (Dkt.-3971)

**Part 1—DEFINITIONS:**

(a) The term "highway" means the roads, highways, streets, and ways in any state.
(b) "Point" means a particular city, town, village, community or other area which is treated as a unit for the application of rates.
(c) An "UNNAMED" point is one from or to which class rates are not provided, other than by use of this rule.
(d) A "NAMED" point is one from or to which class rates are provided in this Tariff (or in tariffs governed hereby), other than by use of this rule.

**Part 2—RATES FROM OR TO UNNAMED POINTS LOCATED ON HIGHWAYS BETWEEN NAMED POINTS:**

(a) Unnamed origin points.

From any unnamed origin point, which is located on a highway between two named points determined by paragraphs (c) and (d) of this part, apply the higher of the class rates provided from such named points.

(b) Unnamed destination points.

To any unnamed destination point, which is located on a highway between two named points determined by paragraphs (c) and (d) of this part, apply the higher of the class rates provided to such named points.

(c) In each case, the named point referred to in paragraphs (a) and (b) of this part must be the nearest named point on a highway (or highways) leading thereto from the unnamed point.
(d) When by reason of branch or diverging highways, there are two or more nearest named points equidistant from the unnamed point, the highest rated of the nearest named points will be used.

**NON-APPLICATION:**

(a) This rule does not authorize a carrier to handle shipments from or to points or via routes not within the scope of its operating authority.
(b) If there is, in any other tariff, a class rate published specifically to or from the unnamed point, for account of the same carrier or carriers, over the same route, this rule will not apply.

**Part 3—RATES FROM OR TO UNNAMED POINTS NOT LOCATED BETWEEN NAMED POINTS** (Subject to Definitions provided in Part 1):

(a) From or to unnamed points located on highways, but not located between named points; or
(b) From or to unnamed points not located on highways,

apply the following provisions:

| When the distance between the unnamed point and the nearest named point is: | The rate from or to the unnamed point will be determined by adding the following arbitrary to the rate from or to the nearest named point. | | | |
|---|---|---|---|---|
| **DISTANCE IN MILES** (See Note 1) | **ARBITRARY RATE IN CENTS PER 100 LBS.** | | | **Arbitrary Minimum Charge Per Shipment (In Cents)** |
| | **AQ or LTL** | **LTL, min. wt., 1,000 Pounds** | **Vol. or TL** | |
| 5 or less | 82 | 78 | 39 | 306 |
| Over 5 but not over 20 | 103 | 97 | 48 | 306 |
| Over 20 but not over 40 | 117 | 110 | 57 | 306 |

**Note 1**—In determining the distance, the actual distance over the shortest route over which a truck can operate shall be used. Distances shall be computed from or to the Post Office having the same name as the named point from or to which a rate is published (use the main Post Office if it has more than one) from or to the actual place of loading or unloading. If the point named herein from or to which a rate is published has no Post Office by the same name, the distance shall be computed from or to the generally recognized business center of the community.

**NON-APPLICATION:**

(a) This rule does not authorize a carrier to handle shipments from or to points or via routes not within the scope of its operating authority.
(b) If there is, in any other tariff, a class rate published specifically to or from the unnamed point, for account of the same carrier or carriers, over the same route, this rule will not apply.

**Note 2**—The provisions of this item are not applicable via SupTruck. Apply provisions of Item 190.

(Rules 3 (d), 4 (a), 4 (m), 5 and 12 of Tariff Circular waived; ICC Permission No. 25469-M.)

Fig. 31--Class rate rulings for locations at other than named points

Scaling distances as a basis for rate per ton mile determination.--Once rates in terms of cents per hundred weight have been determined for each customer and supplier, scaled air miles from a confirmed centroid or point of least ton miles to each of these points may be utilized as a basis for facilitating the conversion of these rates into per ton mile inputs. Obviously, by utilizing air miles actual ground distances will be slightly understated. This occurs because class rates, and for those commodity rates which are expressed as a percentage of class 100, are based on the shortest applicable ground distance via rail or truck, depending on the mode of transportation, between named points. Therefore, depending on the degree of sophistication desired by the individual researcher air miles may be converted into ground miles.

Basic to the conversion task is the development of an average distance factor between aerial and ground mileages derived from samples. For example, by sampling ground mileages aerial distances may be found to differ from ground mileages by a factor of approximately 0.4. If this is the situation for the sample in question aerial mileages may be converted to ground mileages by multiplying aerial mileages by 1.4. However, it must be indicated that such conversion is by no means mandatory.

The utilization of quoted distances as a basis for rate per ton mile determination.--In the event that the initial point of least ton miles and the points encompassing the system are located commensurate with named points, the actual distances between these points and the point of minimum ton miles are found in each class tariff under the heading Applicable Rates Bases. Moreover, if the final least cost location is located within the boundary limits of a named point any other location within this boundary will produce identical transportation costs. This would occur because the utilization of quoted distances as a basis for determining transport costs (tonnage x mileage x rate per ton mile) would result in identical mileage quotes. Also, by utilizing quoted distances or air mile distances a final optimal location slightly external to the boundary limits of a named point may produce a lower freight rate by moving into its limits. Such a factor may be viewed as a secondary consideration which may compromise an optimal location. Such considerations will be covered in Chapter V.

It should now be evident that by holding tonnages and ton mile rates constant and by varying air distances a confirmed centroid location will be superior to any other location. However, by varying locations within the boundary limits of a named point the utilization of quoted distances produces greater flexibility. Yet the utilization of quoted distances may be viewed as a highly accurate approximate approach, approximate only because lower quoted distances could conceivably accrue from another location. Naturally, by holding only tonnages constant and by varying both rates and distances from alternative locations, the model may be placed in a highly accurate approximate classification. This occurs only because the peculiarities of the rate structure could also conceivably produce relatively lower rates from another location.

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# CHAPTER V

# PROCEDURAL STEPS AND SECONDARY LOCATIONAL CONSIDERATIONS

Now that the complete conceptual base for freight rate application to the model has been generated, attention will proceed to a view of the overall required procedural steps to facilitate the application of the model by the interested practitioner. The presentation of these steps is essentially laid upon the theoretical and proven bases generated in Chapters III and IV. Once these steps have been depicted possible secondary locational considerations which may compromise a scientifically determined least cost location will be considered. As a result, such a presentation should serve to lend practicality and scope to the overall locational construct.

## Major Preliminary Procedural Steps

For the firm faced with the problem of determining an optimal or near optimal location over time a series of key preliminary steps or stages must be performed prior to delving into actual model application. These preliminary steps are highly significant in that they reveal the information inputs that serve as the basis for model implementation. Basically, these preliminary planning steps involve

(1) the determination of the time period over which the warehouse location is to be optimized; (2) the identification of all relevant purchaser and supplier points which are to be tentatively supplied from the to-be-determined warehouse; (3) the projection of demand requirements over time; (4) the determination of which customers will actually be supplied by the warehouse; and (5) depending upon the degree of sophistication desired by the practitioner, the determination of the constraints of freight rate application.

With the accomplishment of the preliminary stages of analysis the locations researcher is then in a position to engage in direct model implementation; and with this framework now having been presented, emphasis will proceed to time period determination.

## Time Optimization Period

The first key question that the researcher (who wishes to determine a least cost warehouse location over time) must entertain, involves over what period of time is the warehouse to be optimized? The determination of this time consideration is in turn dictated by the length of time in the future that reasonable demand forecasts may be obtained. According to one source, a warehouse should be built to handle the anticipated level of demand to be reached in a five year period (1, p. 53). Beyond this time, consideration tends to center on relocation. However, in the final analysis, the ability to generate reasonable forecasts or estimates of tonnage usage should govern the time period over which the location is to be optimized. By so doing, dynamism will be introduced, thus producing a warehouse location which will be optimized or nearly optimized over the designated time period.

Those models which rely solely on last year's volume as a guide are essentially arriving at a static location, which may shortly be divorced from actual market conditions. Also, the usage of prior volume per se neglects the fact that financing, land purhcase, and construction may often require a year or more between the planning and implementation stage. Thus, by relying on static analysis the resulting warehouse location may never even provide a semblance of first year optimization.

## Point Determination

Once the time period over which a least cost warehouse location will reside has been determined, attention centers on the identification of all relevant customer and supplier points which are to be tentatively involved in the to-be-determined warehouse network. This point identification also involves the delineation of all current purchasers to be tentatively supplied, and if the locations researcher so desires, all potential customers. Likewise, the same conception is equally applicable to supplier points.

If both potential consumers and suppliers over the designated time period are to be included, consideration should be given to the assignment of probability weights which, for example, reflect the degree of entrenchment of potential customers with competitors and which also denote the possibility of market encroachments. Marginal current purchasers may likewise be assessed such probability weights.

## Demand Determination

After such point identification over the designated time period has been accomplished, the volume or demand requirements of the customers and suppliers to be tentatively involved in the warehouse complex are determined. This analysis refers to only those products or product classifications which will actually be involved in the warehouse operation.

With the above in view, the demand requirements of the individual purchasers and suppliers may be assessed either through a build-up or breakdown approach. The build-up method essentially ascertains consumer needs for each designated consumer point for each year in question and arrives at an overall tonnage requirement for the allotted period. The breakdown approach involves assessing the overall demand or market potential for the products in question and assigns volume or tonnage requirements to each consumer point based on percentage purchases of the products in question. The resulting volume requirements for each consumer point are then expressed in tonnages for each product classification. Of course, to facilitate the analysis of demand, the researcher should make use of economic base studies for the area in question and regional business forecasts.

Once the analysis of consumer demand has been accomplished it is then a simple matter to assign tonnage weightings to the respective supply point or points, thus depicting a desired balance between supply and demand for the shipments included in the warehouse network.

The final result of this demand analysis is now seen in the assignment of tonnage weights to the identified points as they appear on a map or on some representation depicting the scaled geographical relationships between the points involved. The points are also identified on this map as being either a customer or a supplier. Of course, if potential consumers, for example, were included in the analysis, the received probability weights are then multiplied times their tonnage weighting assignments to reflect their potential for warehouse service.

With the assignment of tonnage weights to each tentative purchaser and supplier point now having been discussed, attention will center on the identification of those customers and suppliers who will actually comprise the warehouse network over time and will, therefore, be included in the final model framework.

## Absolute Supplier and Purchaser Delineation

So far the procedural analysis has involved the identification of tentative supplier and purchaser points in the general area to be encompassed in the warehouse network and their associated tonnage weightings. These points are essentially viewed as tentative, because depending on the geographical relationship between a supplier and a customer that is serviced by this supplier, it may be more economical for the supplier to ship direct to the consuming point than from a warehouse location. If this were the case the consuming and supplier point would be deleted from the analysis, or assuming that the supplier served other consuming points (in the analysis) the tonnage requirements of the deleted consumer would be subtracted from the overall tonnage assignment of the supplier. Similarly, the consumer may receive tonnage requirements from other suppliers included in the analysis, hence the tonnage requirements would be deleted only for the amount of tonnage that could be more economically shipped direct.

The determination of the final purchaser and supplier points along with the final tonnage requirements is based on a survey of the consumer and supplier points as they appear on a map or some other similar representation. This survey involves the identification of the general area of warehouse location, and based upon this identification it should be relatively easy to discern which customers or products should be shipped direct. An illustration of the conception appears in Figure 32.

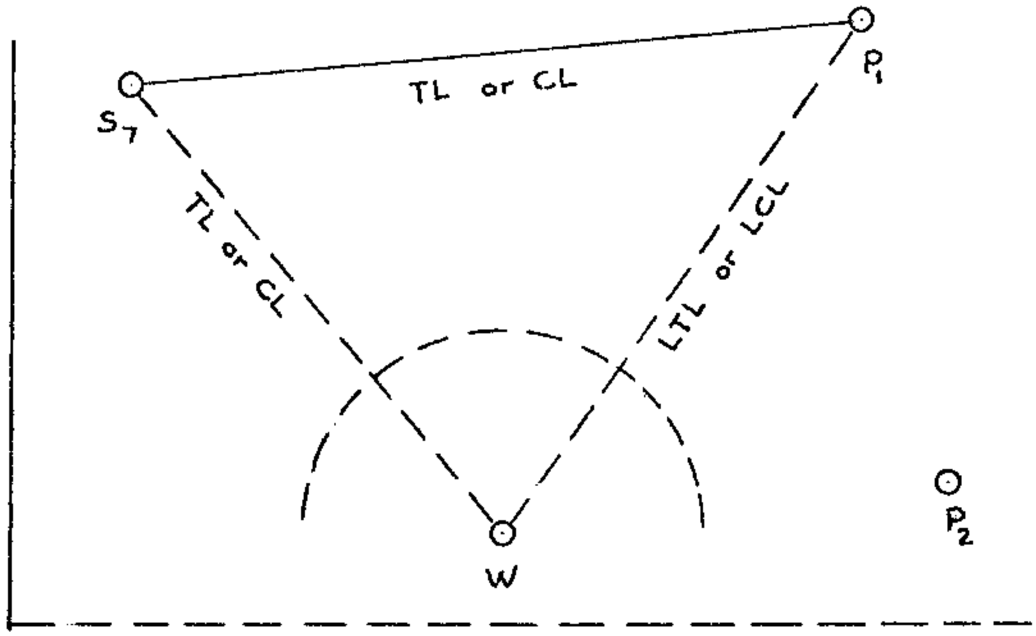

Fig. 32--Partial presentation of tentative points and potential warehouse constraints

Note in Figure 32, which depicts a general area in the form of an arc in which an optimum warehouse location may be included, it may be more economical to ship directly from supplier $S_7$ to purchaser $P_1$ than to a warehouse location potentially located at point W. As a result, the tonnages assigned point $P_1$ may be reduced by the amount of the direct shipment from $S_7$ to $P_1$. Likewise, the tonnages associated with point $S_7$ will be reduced by a like amount.

This is assuming that point $P_1$ is supplied by other supplier points and that also $S_7$ serves other consuming points. By engaging in such analysis the true points and tonnages to be involved in the warehouse network may be identified.

However, in the event that there is no "close" geographical relationship between purchasers and suppliers, then the depicted analysis is not needed. On the other hand, the locations researcher may be faced with a situation in which it is more difficult to ascertain whether it is more economical to ship direct or from a warehouse location. When such a situation may become evident the model should be implemented both with and without the involved points and/or tonnages, and cost comparisons between the two should indicate the best system.

## General Freight Rate Procedural Review

After the identification of the consumer and supplier points, along with the appropriate tonnage requirements for the products handled by these points, has been accomplished, only freight rate determination remains prior to actual model implementation. Depending upon the degree of sophistication desired by the locations researcher, such rate determination may involve one of several distinct approaches.

Basically, there is a linear approach, which assumes that freight rates are directly proportional to distance, and a nonlinear approach which recognizes that freight rates per ton mile taper off over distance. The more sophisticated approach is the nonlinear inclusion.

The linear approach.--By utilizing the assumption that freight rates are directly proportional to distance, freight rates may be easily included in the model framework. All that has to be accomplished is the determination of a per ton mile freight rate for each designated customer and supplier point. As a result, the respective per ton mile rates may then be multiplied times the projected tonnage weights associated with each point. The resulting ton mile cost weightings then serve as the necessary inputs for model implementation.

A simpler conception involves the "differential approach" which weights the tonnage associated with each supplier by the general transport cost differential between car or truckload and less than car or truckload rates. This essentially recognizes the presence of transport economies on inbound warehouse shipments. In this type of application all supply point tonnages would be weighted by 0.5. Thus the weightings associated with each point would remain in the form of tonnages.

However, to implement ton mile cost weightings for each point, the locations researcher needs to determine appropriate per ton mile rates for each point. By so doing more of the nuances which pervade the locational decision are considered. To facilitate this rate delineation, average rates on both outbound and inbound shipments for each involved product classification may be determined from a prevailing warehouse operation. Weighted per ton mile rates may then be assigned to each point which reflect the per ton mile freight rate importance of each product.

In the event that a warehousing system is not in current operation average rates may be developed from tariff distance scales. For a complete discussion of this particular conception see the Chapter IV heading, _Average Per Ton Mile Rates as Information Inputs_.

_The nonlinear approach_.--The approach most oriented in the realities of the existing freight rate structure involves the determination of nonlinear freight rates. By employing nonlinear rates as information inputs, the rates which most closely approximate shipping to a given customer or from a given point of supply may be identified. To accomplish this goal, a point of minimum ton mileage in which all supply points have been weighted by 0.5 needs to be determined. As a result, model implementation occurs before final freight rate determination. Once this point has been ascertained, the tentative location serves as a point of departure for freight rate delineation. For a complete discussion of such rate determination see the

Chapter IV heading, Nonlinear Freight Rates as Information Inputs.

Once these rates are ascertained nonlinear rates are assigned to the appropriate tonnages at each point and the model is reimplemented to depict the shift in the initial location due to the impact of nonlinear rates.

## Major Procedural Model Steps

In keeping with the unity of the procedural presentation the focus of orientation should aptly reflect a brief view of the model steps to be performed. Thus a major step review of both the balance point and coordinate model will be presented along with the required confirmation model.

### Balance Point Procedural Steps

Assuming that the required per ton mile cost weightings (tonnage x rate per ton mile) have been assigned to the designated customer and supplier points, the basics for model implementation are present. First, all consumer points are assigned numerical values in descending order of weight importance. Symbolically, the presentation appears as follows:

$$P_1,\ W_1R_1 \geqq P_2,\ W_2R_3 \geqq P_n,\ W_nR_3 \quad ,$$

where

P = point designation as identified on a map or other spatial representation

W = tonnage weight assigned to each point
R = per ton mile freight rates.

In the event the differential approach is being utilized, all consumer points are assigned numerical values in descending order on the basis of tonnage weights only. However, supply tonnages are weighted by 0.5.

Next, continuing numerical values are assigned to supplier points. For example, if the last purchaser point was number 6 the next value to be assigned a supplier point would be number 7. Uniquely enough, the assignment of numerical symbols does not have to reflect descending weight values in the case of supplier points.

By following this approach to numerical point assignments added perspective may be added to the locational problem through the determination of a centroid of demand, as well as the overall system centroid. On the other hand, all points may be ranked in a descending order based upon weights per se without creating a distinction between customers and suppliers.

After the necessary numbering of points has been accomplished, the overall system centroid is determined through the systematic determination of balance points. To facilitate this objective the following formulation is utilized for determining the point of balance between the first two points designated $P_1$ and $P_2$:

$$b_1 = \frac{d_1 \ (W_2R_2)}{W_1R_1 + W_2R_2} \quad ,$$

where

$\bar{B}_1$ = balance point between points $P_1$ and $P_2$

$W_1, W_2$ = tonnages associated with $P_1$ and $P_2$

$R_1, R_2$ = freight rates per ton mile associated with points

$d_1$ = scaled or computed distances between $P_1$ and $P_2$

$b_1 = P_1 \smile \bar{B}_1$ or the distance from $P_1$ to $\bar{B}_1$ on line $d_1$.

After determining the point of balance $\bar{B}_1$ between points $P_1$ and $P_2$ a line is drawn from $\bar{B}_1$ to $P_3$ and the following general formula is implemented to determine $\bar{B}_2$ or the centroid of the first three indicated points:

$$b_n = \frac{d_n (W_{n+1}R_{n+1})}{W_1R_1 + \ldots + W_{n+1}R_{n+1}} \quad ,$$

where

$\bar{B}_n$ = balance point of 2 node system

$W_n$ = tonnages associated with respective points

$R_n$ = freight rates per ton mile associated with respective points

$d_n = \bar{B}_{n-1} \smile P_{n+1}$ or the scaled or computed distance between $\bar{B}_{n-1}$ and $P_{n+1}$

$b_n = \bar{B}_{n-1} \smile \bar{B}_n$ or the distance from $\bar{B}_{n-1}$ to $\bar{B}_n$ on line $d_n$.

The procedure is then successively continued until the overall centroid of the system is identified. In other words, the last point of balance determined is indicative of the ultimate centroid which depicts complete system equilibrium.

## Coordinate Approach Procedural Steps

An alternative approach to warehouse location through centroid determination is seen in the Cartesian coordinate formulation. This method involves encompassing all designated supplier and customer points in the positive quadrant of a Cartesian coordinate system. This, of course, assumes that all points have received appropriate per ton mile cost weights. The vertical axis of this coordinate system denotes the ordinate or the Y axis and the horizontal axis denotes the abscissa or X axis. Values are then assigned to both axes, and these values are used to indicate the distances of each customer and supplier from both the vertical and horizontal axis. As a result an X and Y coordinate value will be assigned each point in the analysis.

When the required coordinates have been assigned to the respective points the following formulations are then utilized to determine the appropriate coordinates of a centroid location:

$$\overline{X} = \frac{X_1W_1R_1 + X_2W_2R_2 + \ldots + X_nW_nR_n}{W_1R_1 + W_2R_2 + \ldots + W_nR_n} ,$$

$$\overline{Y} = \frac{Y_1W_1R_1 + Y_2W_2R_2 + \ldots + Y_nW_nR_n}{W_1R_1 + W_2R_2 + \ldots + W_nR_n},$$

where

$\overline{X}$ = coordinate of centroid location on X axis

$\overline{Y}$ = coordinate of centroid location on Y axis

X = coordinate for customer and supplier points on X axis

Y = coordinate for customer and supplier points on Y axis

W = tonnage weights associated with each customer and supplier

R = rate per ton mile associated with each customer and supplier.

## The Establishment of Tolerance Limits

As this study has shown, the determination of a centroid location does not insure optimal warehouse location. Basically, a centroid location may be near optimal rather than optimal per se or it may be purely nonoptimal. Therefore, a confirmation model is applied to the centroid location to confirm or deny its optimality or near optimality. For example, if the usage of the confirmation model produces X and Y coordinates which are synonymous with the coordinates of the centroid, such a location is viewed as being optimal. However, if the coordinates produced by the confirmation model do not perfectly check the centroid's coordinates, the centroid location may be viewed as being near optimal. In this regard, if the confirmation model produces coordinates which fall very close to the centroid's coordinates, the centroid location is near optimal and for all practical purposes may be viewed as being optimal. The determination of how close these confirmation coordinates have to be to the centroid location, to depict what may be viewed as an optimal location, depends upon the establishment of tolerance limits.

If, for example, a tolerance limit of ± .5 miles were established, this would indicate that the centroid location would be viewed as being near optimal if the coordinates produced by the confirmation model fall within one square mile of the coordinates of a centroid location. Naturally, the size of the tolerance limits depend upon the scope of the problem and the amount of error that may be tolerated. Of course, if the confirmation model produces coordinates which fall external to these tolerance limits the centroid is viewed as being nonoptimal per se. When this occurs the actual point of least cost is established through cost comparisons with the lowest tentative point so discovered subject to the confirmation of its coordinates.

Attention will now proceed to the procedural steps for confirmation and then some useful generalizations will be presented to help the researcher determine the general area where cost comparisons should be made. The latter is based upon a lack of centroid optimality or near optimality within the designated tolerance limits.

## Procedural Steps for Confirmation

The methodology for generating confirmation of centroid optimality or near optimality involves scaling or computing the distances from the centroid location to each customer and supplier point. These inputs along with the appropriate coordinates for each point are then substituted into the following model:

$$\overline{X} = \frac{\frac{X_1(W_1R_1)}{V_1} + \frac{X_2(W_2R_2)}{V_2} + \ldots + \frac{X_n(W_nR_n)}{V_n}}{\frac{W_1R_1}{V_1} + \frac{W_2R_2}{V_2} + \ldots + \frac{W_nR_n}{V_n}} ,$$

$$\overline{Y} = \frac{\frac{Y_1(W_1R_1)}{V_1} + \frac{Y_2(W_2R_2)}{V_2} + \ldots + \frac{Y_n(W_nR_n)}{V_n}}{\frac{W_1R_1}{V_1} + \frac{W_2R_2}{V_2} + \ldots + \frac{W_nR_n}{V_n}} ,$$

where

$\overline{X}$ = coordinate of cost centroid or potential least cost site on X axis

$\overline{Y}$ = coordinate of cost centroid or potential least cost site on Y axis

$X_n$ = coordinate for customer or supplier point on X axis

$Y_n$ = coordinate for customer or supplier point on Y axis

$W_n$ = tonnage weight associated with a given point

$V_n$ = vector or distance of customer or supplier from warehouse at centroid or potential least cost location

$R_n$ = per ton mile freight rate associated with a designated point.

To facilitate the usage of the confirmation model, Table XIV presents a tabular presentation of the methodology to be followed in calculating the radius vectors which are to be included in the confirmation model. Note the vectors or distances may be measured rather than computed.

Table XV is also presented which demonstrates the application of the confirmation model in tabular form.

## General Guidelines to Optimum Site Determination

Basically, a centroid location may be offset as an optimal or near optimal point due to the existence of two conditions. The first refers to the existence of a disproportionately heavy point or cluster of weights, as it relates to the entire system of weights. The second refers to the existence of extreme asymmetry in the spatial array of points.

With these conditions now in view some general guides will be presented to help the researcher ascertain the

# TABLE XIV

## METHOD OF COMPUTING RADIUS VECTORS ($V_j$) FROM COORDINATES OF CENTROID OR OTHER POTENTIAL LEAST COST LOCATION

| $P_j$ (1) | $X_j$ (2) | $\overline{X}$ (3) | $X_j - \overline{X}$ (4) | $(X_j - \overline{X})^2$ (5) | $Y_j$ (6) | $\overline{Y}$ (7) | $(Y_j - \overline{Y})$ (8) | $(Y_j - \overline{Y})^2$ (9) | (5)+(9) (10) | $\sqrt{(5)+(9)}$ ($V_j$) |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | |

$P_j = P_1, P_2 \ldots P_n$ = Customer and supplier points.
$X_j = X_1, X_2 \ldots X_n$ = X - Coordinate of $P_j$.
$Y_j = Y_1, Y_2 \ldots Y_n$ = Y - Coordinate of $P_j$.
$V_j = V_1, V_2 \ldots V_n$ = Distance from centroid or other potential least cost site to customer and supplier points.
$\overline{X}$ = Coordinate of centroid location or other potential least cost site on the X axis.
$\overline{Y}$ = Coordinate of centroid location or other potential least cost site on the Y axis.

TABLE XV

TABULAR PRESENTATION OF THE CONFIRMATION MODEL

| $P_j$ (1) | $X_j$ (2) | $V_j$ (3) | $W_j$ (4) | $R_j$ (5) | $W_j \cdot R_j$ (6) | (6) $\cdot X_j$ (7) | (6) ÷ (3) $\Sigma$(8) | (7) ÷ (3) $\Sigma$(9) | (9) ÷ (8) (10) |
|---|---|---|---|---|---|---|---|---|---|
| For X-Axis | | | | | | | | | $\bar{X}$ |
| | | | | | | | | | |
| | | | | | | | | | |

| $P_j$ (1) | $Y_j$ (2) | $V_j$ (3) | $W_j$ (4) | $R_j$ (5) | $W_j \cdot R_j$ (6) | (6) $\cdot X_j$ (7) | (6) ÷ (3) $\Sigma$(8) | (7) ÷ (3) $\Sigma$(9) | (9) ÷ (8) (10) |
|---|---|---|---|---|---|---|---|---|---|
| For Y-Axis | | | | | | | | | $\bar{Y}$ |
| | | | | | | | | | |
| | | | | | | | | | |

$P_j$ = $P_1$, $P_2$ . . . $P_n$ = Customer and supplier points.
$\bar{X}$ = Coordinate for centroid or other potential least cost location on the X axis.
$\bar{Y}$ = Coordinate for centroid or other potential least cost location on the Y axis.
$X_j$ = $X_1$, $X_2$ . . . $X_n$ = Coordinate of customer and supplier points on X axis.
$Y_j$ = $Y_1$, $Y_2$ . . . $Y_n$ = Coordinate of customer and supplier points on Y axis.
V = $V_1$, $V_2$ . . . $V_n$ = Distance from centroid or other potential least cost site to customer and supplier points.
$W_j$ = $W_1$, $W_2$ . . . $W_n$ = Tonnage weights associated with $P_j$'s.
$R_j$ = $R_1$, $R_2$ . . . $R_n$ = Per ton mile freight rate associated with $P_j$'s.

general area in which cost comparisons are to be generated, with the tentative lowest point so obtained subject to the confirmation of its coordinates. In the event the centroid has been offset by a single dominant point or a dominant cluster of weights the actual point of least cost will lie between the centroid and the single dominant point or cluster of weights. Also the heavier the weighting in relation to the entire system of weights the closer the proximity of the true point of least cost to the heavier weighting. Naturally, the converse is also apparent. Moreover, if the centroid is offset by a single dominant point the possibility exists for the establishment of a "line of force" or a line upon which the actual point of least cost will readily lie. This conception is completely presented in Chapter III.

Turning to extreme asymmetry in the spatial array of points, this possibility exists when there is a distant relationship between a single supply point (or cluster of supply points) and the demand points which it serves. When such a situation occurs the true point of least cost will lie between the centroid of the system and the series of demand points. Thus, cost comparisons should be undertaken only within these constraints. Also, if the centroid of the system is offset by a single distant supply point the possibility again exists for the establishment of a "line of force." Likewise, this conception is completely presented in Chapter III.

### Secondary Site Considerations

Once a confirmed least transportation cost warehouse location has been identified the obvious emphasis centers on site evaluation. Hopefully, the site will possess the necessary secondary considerations to permit an optimal or near optimal location. There are, however, many secondary factors or considerations which may comprise such an optimal location. Most noticeable of these secondary factors would be locations in lakes, rivers, mountain ranges or other physical impractibilities. Yet, even if the site meets the basic physical constraints of location still other secondary forces may compromise the optimality of the location. These other potential compromising forces may be classified as general and specific secondary considerations for purposes of elaboration. For example, general secondary forces would refer to those locational considerations which are common to virtually any type of warehouse location. On the other hand specific secondary forces would tend to involve those locational considerations which are peculiar to a given firm.

With the above in view the logical starting point for site evaluation once an optimal or near optimal site has been mathematically determined centers on basics--i.e., the determination of whether the site possesses the physical requirements to support a warehouse location. In the event that a least cost location does not meet basic physical criteria, other alternative sites are then evaluated as to transport costs and physical criteria. As a result of this evaluation the most optimal site is evaluated as to general secondary considerations and needed specifics. Noteworthy is the fact that the scientifically determined least cost site sets the framework or the starting point for the analysis of alternative sites in regard to the desired criteria.

Since there are so many general and specific secondary factors, other than basic physical constraints, which may actually compromise a least cost warehouse location, a generalized and specific checklist of factors is now presented.

## Macro Secondary Checklist

Assuming that the site meets the basic physical constraints of site location, the following key macro or general secondary forces must be considered:

I. Macro or General Secondary Forces
    A. Availability or ultimate availability of acreage and adjoining land
    B. Zoning requirements of land and adjacent land

C. Availability of easements
    1. Present and future roads
    2. Rail lines
    3. Power lines
    4. Pipelines
    5. Sewers

D. Reasonableness of purchase price

E. Availability of manpower and skills associated with warehouse operation.

These secondary factors are, of course, generally applicable to any desired warehouse location, and in the event that a site meets basics and fails to meet the dictated general criteria, other sites are evaluated as to costs, basics, and their ability to meet general secondary considerations. Once a low transport cost site which meets these characteristics is found the site may then be evaluated as to specifics. The discussion will now proceed to the specific checklist.

## Micro Secondary Checklist

The presentation of a micro or specific secondary checklist is not meant to be all inclusive. However, the list is suggestive of many key criteria which may or may not compromise a least cost site. Hence, the list may be considered as a guide to significant issues. The list appears as follows:

II. Micro or Specific Secondary Forces
  A. Transportation
    1. Rail
      a. Lines servicing site
      b. Switch service available
      c. Rail cars servicing site
      d. Outbound routes, service and transit time to customers
      e. Inbound service routes and transit times from producing plants
      f. Damage experience by other warehouse in close geographical proximity--particularly as related to local yards
      g. Car supply
        1) Assigned
        2) Free runners
        3) Seasonal fluctuations of car supply in area
        4) History of embargoes such as results of dock strikes
      h. Transit applications
      i. Average demurrage agreements
      j. Local rail management's attitude toward service and customers
      k. Participation of railroad in building trackage to the site

    l. General financial and physical condition of the railroad
    m. Security problems
    n. Willingness of railroad to support rate requests
    o. Alternate rail service in case of storms or strikes
    p. Car size and weight restrictions
    q. Piggyback and container service
2. Motor carriers
    a. Common carriers available
    b. Opportunity for contract and private carriage
    c. Consolidated delivery opportunities
    d. Carriers' claims and financial history
    e. Availability of equipment and type
    f. Location of site within terminal area or commercial zone
    g. Security or labor problems
    h. Distance of site to good transportation arteries and interstate highways
    i. Service to highways in event of frozen roads
    j. State highway limitation on size and weight of vehicles

        - k. Physical height and weight restrictions related to access highways
- B. Labor force
    - 1. Turnover rate at other warehouses in area
    - 2. Prevailing labor rates
    - 3. Unions
    - 4. Satisfactory disposition of labor disputes
    - 5. Labor problems of vendors and carriers in relation to warehouse
    - 6. Service skills available
- C. Taxes--city, county, state
    - 1. Real estate
        - a. Basis for assessed evaluation
        - b. Rate per one hundred dollars of assessed evaluation and per square foot
    - 2. Personal property tax
    - 3. Inventory tax
        - a. Restrictions
        - b. Exemptions
        - c. Rate basis
        - d. Rate assessment date
    - 4. Payroll taxes
    - 5. Fuel taxes
    - 6. Projections on future tax increases
    - 7. Tax relief granted to attract new industry

D. Site data
  1. Cost
    a. Survey fees
    b. Unpaid assessments
    c. Fees for clearing old buildings and trees
    d. Grading and field costs
    e. Below ground cost, piling, expanded, footings
    f. Road building
    g. Rail lines
  2. Utilities
    a. Water
      1) Rates
      2) Size of mains
      3) Pressure
      4) Cost of extending to site
      5) Connection costs
      6) Planned lines and assessments
    b. Gas
      1) Present and proposed or planned rates
      2) Size of main
      3) Pressure
      4) Cost of extending to site including connecting cost
    c. Electrical
      1) Rates

        2) Capacity of lines
        3) Cost of transformers and other electrical equipment
        4) Cost of extending to site
    d. Sewer
        1) Rates
        2) Size of main
            a) Storm
            b) Sanitation
        3) Cost of extending to site including connection
        4) Planned lines and assessments
3. Building restrictions
4. Civil unrest potentiality

E. Local factors
1. Attitude of community and state toward new industry
2. Willingness to support changes in zoning
3. Possibility of utility concessions to new industry
4. Commitments on adequate police and fire service
5. Effectiveness of local business organizations such as the chamber of commerce and industrial associations

F. Legal factors

  1. Review of state and local ordinances
  2. Review of abstract and title.

The presentation of these specific secondary considerations may provide insight into factors which may or may not compromise an optimum or a near optimum location. It is, therefore, suggested that these criteria be evaluated in the light of the structure or needs of the individual firm conducting least cost locational analysis. Additionally, while the above checklist was partially derived from an article published by the Manager of Distributive Services, Hunt-Wesson Foods (1, pp. 53-55), the individual firm may wish to add to the checklist in the light of other discernible requirements.

# CHAPTER BIBLIOGRAPHY

1. Ray, Ron, "Checklist for Planning a Distribution Center," Distribution Manager (June, 1969), pp. 50-55.

# CHAPTER VI

# CONCLUSION

The conclusions generated in this research study will be developed around the initial purported hypotheses. Additionally, a section will be included to reveal the research contributions generated by the study.

## Hypothesis Number One

The first hypothesis, which was concerned with the creation and proof of a noncoordinate centroid determining model for warehouse location, was developed and proven. Such development and proof was based upon the reiterative *balancing of moments* between two weighted points. This indicates that given a point of balance between two weighted points, the *first moment*, which consists of the product of the first weight multiplied times its distance from the point of balance, *equals* the product of the *second moment*, which in turn involves a similar product involving both weight and distance of a second point.

Such a balancing of moments between weighted points proceeds by progression. This involves concentrating the weights of a first two point system at its point of balance and scaling a line to a third weighted point. Again a point of balance is achieved through a balancing of moments between these two points, the result being viewed as the centroid of a three point system. The procedure is continued until the last point of balance is obtained for a series of weighted points, with this last point of balance being viewed as the centroid of the system. The obtainment of this point indicates perfect system equilibrium and, therefore, potential cost minimization.

Since the approach requires a successive balancing of moments between weighted points, the last point of balance must be the centroid of the system. Therefore, conceptually the model is self-proving. However, corollary proof was seen when the balance point methodology absolutely confirmed the centroid of a known configuration. In this case, an isosceles triangle.

In terms of application the balance point approach to centroid determination required ordering the points of the system so that

$$P_1, W_1 \geqq P_2, W_2 \geqq P_3, W_3$$

where

P's = customer and supplier points

W's = per ton mile cost weightings associated with respective customer and supplier points.

The next step involves applying the following model formulation for determining the point of balance ($\overline{B}_1$) between $P_1$ and $P_2$:

$$b_1 = \frac{d_1 \; W_2}{W_1 + W_2} \quad ,$$

where

$\bar{B}_1$ = balance point of 2 point system

$d_1$ = scaled distance between $P_1$ and $P_2$

$b_1 = P_1 \sim \bar{B}_1$ or the distance from $P_1$ to $\bar{B}_1$ on line $d_1$.

Next the following general formula is systematically utilized to arrive at a final centroid for points $P_3$ through $P_n$ for the system in question.

$$b_n = \frac{d_n \; W_{n+1}}{W_1 + \ . \ . \ . + W_{n+1}} \quad ,$$

where

$\bar{B}_n$ = centroid of any 2 node system

$d_n = \bar{B}_{n-1} \sim P_{n+1}$ or the scaled or computed distance between $\bar{B}_{n-1}$ and $P_{n+1}$

$b_n = \bar{B}_{n-1} \sim \bar{B}_n$ or the distance from $\bar{B}_{n-1}$ to $\bar{B}_n$ on line $d_n$.

## Hypothesis Number Two

The second hypothesis of this study, which was concerned with proving the consistent optimality or near optimality of a centroid location, was not proven. However, the study results did not disprove the potential optimality of a centroid location, for a centroid location may or may not be optimal or near optimal depending upon the degree of asymmetry and the degree of weight variation in a designated array of weighted consumer and supplier points.

Proof of this was generated through the usage of the following mathematically proven confirmation model:

$$\bar{X} = \frac{\frac{W_1X_1}{V_1} + \frac{W_2X_2}{V_2} + \frac{W_3X_3}{V_3} + \frac{W_nX_n}{V_n}}{\frac{W_1}{V_1} + \frac{W_2}{V_2} + \frac{W_3}{V_3} + \frac{W_n}{V_n}} ,$$

$$\bar{Y} = \frac{\frac{W_1Y_1}{V_1} + \frac{W_2Y_2}{V_2} + \frac{W_3Y_3}{V_3} + \frac{W_nY_n}{V_n}}{\frac{W_1}{V_1} + \frac{W_2}{V_2} + \frac{W_3}{V_3} + \frac{W_n}{V_n}} ,$$

where

$\bar{X}$, $\bar{Y}$ = coordinates of cost centroid or potential least cost site on X and Y axes

$X_nY_n$ = coordinates for customer and supplier points on X and Y axes

$V_n$ = vector or distance of customer or supplier from warehouse at centroid or potential least cost site

$W_n$ = weights assigned to each point.

The mathematical validity of this model was obtained by determining the minimum value of the following ton mile equation:

$$T\ M = V_1W_1 + \ .\ .\ .\ + V_nW_n\ . \tag{1}$$

This was accomplished by substituting the following formula for determining the distance between two points into equation (1):

$$V = [(X_1 - X_2)^2 + (Y_1 - Y_2)^2]^{1/2} , \qquad (2)$$

and solving for the resulting equation

$$T\ M = [(X_1-X_2)^2 + (Y_1-Y_2)^2]^{1/2}\ W_1 + \ldots + [(X_1-X_2)^2 + (Y_1-Y_2)^2]^{1/2}\ W_n$$

by partially differentiating with respect to X and Y and equating each of the expressions to zero. The result was seen in the form of the presented confirmation model which would confirm the minimum value of equation (1).

The usage of this confirmation model was seen in the form of tests applied to asymmetrical arrays of points coupled with varying weights. These tests indicated that a centroid location may serve as the point of ton mile minimization. In effect, absolute confirmation was generated for a centroid location in several tests, while other testing depicted near optimality.

However, further tests indicated that a centroid location may be offset as the point of ton mile minimization or as a near optimal point if a dominant weighted point in relationship to the entire system of weights was introduced into the array of points. Also, tests indicated that the centroid may be offset due to the impact of a dominant cluster or a disproportionately heavy cluster of weighted points as they relate to the entire system.

Finally, tests indicated that the impact of extreme asymmetry in the spatial arrangement of points may serve to offset the centroid as the minimal or near minimal point. In this regard a series of points with equal weightings were offset by such extreme asymmetry. In fact, it was shown that the only consideration which may offset the optimality or near optimality of a centroid location for a series of equally weighted points was extreme asymmetry. It was also indicated that extreme asymmetrical point arrays, perhaps, would appear most commonly when a supply point or cluster of supply points bears a distant relationship to the demand markets of points which it is to serve.

Based upon these tests, it was therefore concluded that hypothesis number two cannot be completely proven. Yet, in the absence of the designated limiting factors in the forms of extreme asymmetry and extreme weighting variations the centroid location would depict general optimality or near optimality. As a result, hypothesis number two may not be completely disproven.

These conclusions also indicate that the centroid determining literature does have a basis of mathematical validity. Indeed, this study has shown that it is feasible to determine an optimal or near optimal location through a centroid location. Therefore, the depicted literature which attests to the lack of optimality of a centroid location per se is not completely correct. Basically, this attack is primarily centered on examples of a centroid location which does not indicate optimality or near optimality. As a result, it has been erroneously concluded that the centroid is never optimal.

On the other hand, the literature has depicted arrays of points and weightings which denote absolute centroid optimality. Thus, the literature has also presented the conclusion that the centroid is the optimal point of least cost. As has been shown, this conclusion is likewise incorrect.

The above conclusions now suggest that hypothesis number two should have indicated that a centroid location may or may not produce an optimal or near optimal location, depending upon the presence of the designated limitations. Such an hypothesis would have then been proven generally correct.

## Ramifications of Hypothesis Number Two

The fact that a centroid location may not always be optimal or near optimal (within designated limits of tolerance) should not be construed as minimizing the significance of a centroid location, for such a scientifically determined location still has value. For example, when a centroid location is indicated by the application of the confirmation model as being nonoptimal per se, the centroid may be used as a guide to the general area where cost comparisons should be made. Cost comparisons are then made in this area with the site possessing the most obvious potentiality being subject to the confirmation of its coordinates.

The usage of extensive tests, however, has resulted in the uncovering of some useful generalizations which will prove to be of value to the locations researcher. For example, in the event that a centroid location has been offset by a dominant weight or cluster of weights, the actual point of least cost will lie in the general area between the dominant point or cluster and the centroid.

In the event that the centroid has been offset by a single dominant point, a line may be drawn from the centroid to the dominant point. This is unique because the point of least cost may lie at a location along this line. If the researcher suspects that a dominant point has offset a centroid location the coordinates produced by the confirmation check on the centroid location should fall on this line. If not, this tends to indicate that some factor other than a single dominant point was also compromising the centroid.

Also, if the centroid has been offset by extreme asymmetry such as in the case of a distant relationship between clustered supply points and demand points, the point of least cost will fall between the system centroid and the series of demand points. In the event that a single distant supply point has served to offset the centroid, a line may be drawn from this point to the centroid of the demand subset. Again, the point of least cost should lay along this line. If the confirmation check on the centroid location produces coordinates which do not fall on this line, this indicates that some factor other than a single distant supply point was likewise compromising the centroid.

Therefore, while a centroid location is unfortunately nonoptimal under certain conditions, its usage may serve to substantially reduce the search dilemma of the locations researcher. And when the centroid is confirmed as being optimal or near optimal (within designated tolerance limits) the procedural approach advocated in this dissertation becomes far more professional than a pure cost comparison or trial and error approach.

## Hypothesis Number Three

The third hypothesis of this study, which was concerned with the creation of a methodology which generated nonlinear freight rates as a model input, was developed. This methodology required the determination of the point of ton mile minimization in which supply tonnages were weighted to reflect the general transport cost differential between the firm's inbound and outbound warehouse shipments. By determining this location the distances emanating from the location are used as the guide to a rate per ton mile determination.

The actual mechanics of this freight rate delineation were developed in Chapter IV. By following this methodology the obtained nonlinear rates are then substituted into the model in the form of new weights, and the actual point of cost minimization may then be obtained.

## Research Contributions

As a result of undertaking this particular subject area for study, the following research contributions are presented:

1. The development of a noncoordinate centroid determination model which is calculated through a reiterative procedure for balancing moments.

2. The presentation of the mathematical proof behind the trial and error or confirmation model.

3. The presentation of the optimizing limitations on a centroid location through the existence of extreme asymmetry in the spatial array of points or through the existence of a dominant weight or cluster of weights as they relate to the entire system.

4. The development of a synthesis between the existing states of the literature. This refers to using a centroid location as the starting point of analysis and

confirming its coordinates within designated tolerance limits through the application of the confirmation model. If the confirming model does not indicate optimality or near optimality (within acceptable tolerance limits) the centroid may be used to depict the general area for cost comparisons with the tentatively optimum site subject to the confirmation of its coordinates.

5. The presentation of generalizations which may aid the researcher in determining the more specific area where cost comparisons are to be engendered.

6. The development of the first analysis of nonlinear freight rate inclusion in the depicted models.

7. The development of the first overall systematic procedural presentation of a manually applied warehouse location program, which may insure an optimal warehouse location. In no manner is a final mathematically confirmed site viewed as an approximation. Moreover, the procedural presentation advocates warehouse locational site optimization over time.

# APPENDIX

# APPENDIX A

To further lend credence to the conception of ton mile minimization or near minimization through a centroid location, the following points and associated weightings are presented to provide a basis for the confirmation of such minimization.

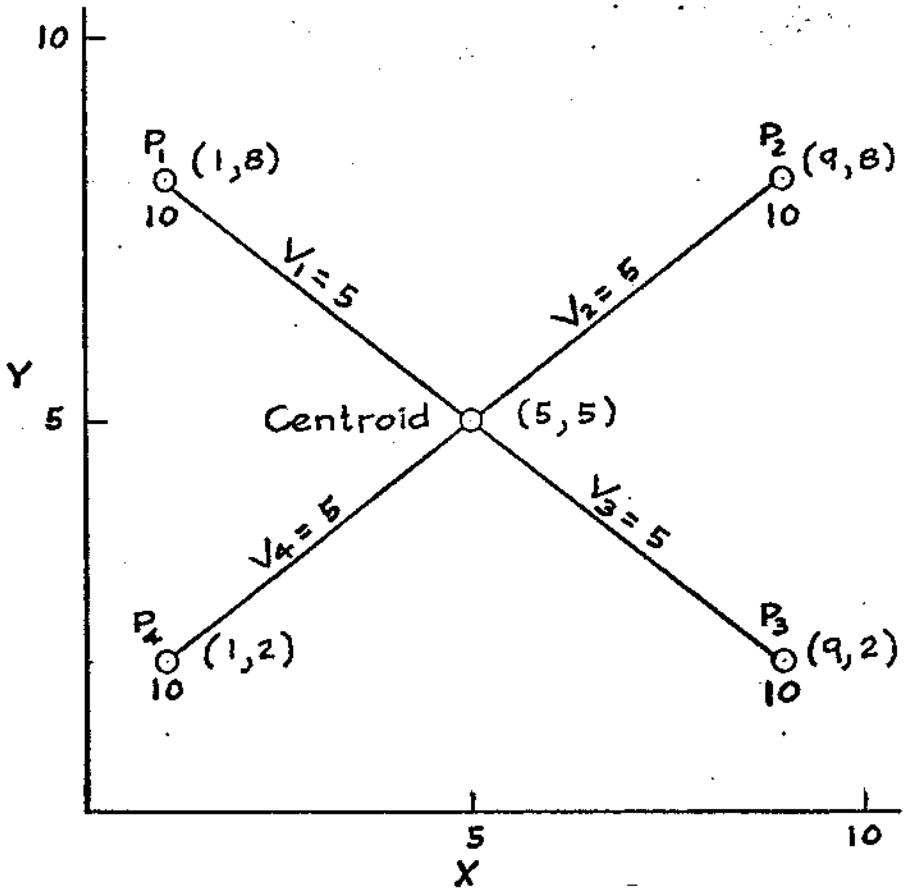

Fig. 33--Spatial array demonstrating potential ton mile minimization

In Figure 33,

P's = customers

V's = distances from centroid to customer points.

In the presented figure ton mileage from a central location to the designated customer points should be at a minimum from a centroid location. In symbolic terms this ton mile minimization is expressed as follows:

$$T\ M = V_1W_1 + V_2W_2 + V_3W_3 + V_4W_4\ . \qquad (1)$$

To prove that equation (1) may represent a minimum solution as applied to a centroid location use is made of the following concepts and mathematical operations.

From analytical geometry (2, p. 8) the distance between any two points can be computed through the following formula:

$$V = [(X_1-X_2)^2 + (Y_1-Y_2)^2]^{1/2} \qquad (2)$$

where

V = the distance between the two points

$X_1$, $Y_1$ = the X and Y coordinates of the first point

$X_2$, $Y_2$ = the X and Y coordinates of the second point.

Therefore, the distances between the centroid and $P_1$, $P_2$, $P_3$, and $P_4$ as depicted in Figure 33 are determined as follows:

$$V_1 = [(X_1-X)^2 + (Y_1-Y)^2]^{1/2}\ ,$$

$$V_2 = [(X_2-X)^2 + (Y_2-Y)^2]^{1/2} ,$$

$$V_3 = [(X_3-X)^2 + (Y_3-Y)^2]^{1/2} ,$$

$$V_4 = [(X_4-X)^2 + (Y_4-Y)^2]^{1/2} ,$$

where

$X, Y$ = coordinates of the centroid

$X_1, Y_1$ = coordinates of $P_1$

$X_2, Y_2$ = coordinates pf $P_2$

$X_3, Y_3$ = coordinates of $P_3$

$X_4, Y_4$ = coordinates of $P_4$.

By substituting equation (2) as applied to the centroid and points $P_1$, $P_2$, $P_3$, and $P_4$, in equation (1) the following is produced:

$$TM = W_1[(X_1-X)^2 + (Y_1-Y)^2]^{1/2} + W_2[(X_2-X)^2 + (Y_2-Y)^2]^{1/2} + W_3[(X_3-X)^2 + (Y_3-Y)^2]^{1/2} + W_4[(X_4-X)^2 + (Y_4-Y)^2]^{1/2}. \tag{3}$$

Using calculus (1, p. 446) the minimum value of equation (3) or the location in Figure 33 which will produce the minimum value of equation (3) can be determined by partially differentiating with respect to X and Y and equating each of the expressions obtained to zero. The steps are as follows:

$$\frac{\partial TM}{\partial X} = \left\{[(X_1-X)^2 + (Y_1-Y)^2]^{1/2}\right\} W_1$$
$$+ \left\{[(X_2-X)^2 + (Y_2-Y)^2]^{1/2}\right\} W_2$$
$$+ \left\{[(X_3-X)^2 + (Y_3-Y)^2]^{1/2}\right\} W_3$$
$$+ \left\{[(X_4-X)^2 + (Y_4-Y)^2]^{1/2}\right\} W_4 .$$

Proceeding with the partial differentiation of equation (3) with respect to X produces the following:

$$1/2 \left\{[(X_1-X)^2 + (Y_1-Y)^2]^{-1/2} \frac{\partial}{\partial X} (X_1-X)^2\right\} W_1$$
$$+ 1/2 \left\{[(X_2-X)^2 + (Y_2-Y)^2]^{-1/2} \frac{\partial}{\partial X} (X_2-X)^2\right\} W_2$$
$$+ 1/2 \left\{[(X_3-X)^2 + (Y_3-Y)^2]^{-1/2} \frac{\partial}{\partial X} (X_3-X)^2\right\} W_3$$
$$+ 1/2 \left\{[(X_4-X)^2 + (Y_4-Y)^2]^{-1/2} \frac{\partial}{\partial X} (X_4-X)^2\right\} W_4 = 0.$$

Continuing the differentiation,

$$1/2 \left\{[(X_1-X)^2 + (Y_1-Y)^2]^{-1/2} 2 (X_1-X)\right\} W_1$$
$$+ 1/2 \left\{[(X_2-X)^2 + (Y_2-Y)^2]^{-1/2} 2 (X_2-X)\right\} W_2$$
$$+ 1/2 \left\{[(X_3-X)^2 + (Y_3-Y)^2]^{-1/2} 2 (X_3-X)\right\} W_3$$
$$+ 1/2 \left\{[(X_4-X)^2 + (Y_4-Y)^2]^{-1/2} 2 (X_4-X)\right\} W_4 = 0.$$

Substituting $V_1$, $V_2$, $V_3$, $V_4$ from equation (2) and clearing produces:

$$\frac{W_1(X_1-X)}{V_1} + \frac{W_2(X_2-X)}{V_2} + \frac{W_3(X_3-X)}{V_3} + \frac{W_4(X_4-X)}{V_4} = 0 \ ,$$

and multiplying by respective weights or W's there results

$$\frac{W_1X_1}{V_1} + \frac{W_1X}{V_1} + \frac{W_2X_2}{V_2} + \frac{W_2X}{V_2} + \frac{W_3X_3}{V_3} + \frac{W_3X}{V_3} + \frac{W_4X_4}{V_4} + \frac{W_4X}{V_4} = 0 \ ,$$

and clearing,

$$\frac{W_1X_1}{V_1} + \frac{W_2X_2}{V_2} + \frac{W_3X_3}{V_3} + \frac{W_4X_4}{V_4} = \left[ \frac{W_1}{V_1} + \frac{W_2}{V_2} + \frac{W_3}{V_3} + \frac{W_4}{V_4} \right] X \ .$$

Solving for the X coordinate of the centroid now produces

$$X = \frac{\frac{W_1X_1}{V_1} + \frac{W_2X_2}{V_2} + \frac{W_3X_3}{V_3} + \frac{W_4X_4}{V_4}}{\frac{W_1}{V_1} + \frac{W_2}{V_2} + \frac{W_3}{V_3} + \frac{W_4}{V_4}} \ . \tag{4}$$

Similarly, the value of the Y coordinate is found to be

$$Y = \frac{\frac{W_1Y_1}{V_1} + \frac{W_2Y_2}{V_2} + \frac{W_3Y_3}{V_3} + \frac{W_4Y_4}{V_4}}{\frac{W_1}{V_1} + \frac{W_2}{V_2} + \frac{W_3}{V_3} + \frac{W_4}{V_4}} \ . \tag{5}$$

Equations (4) and (5) are now used to confirm the location in Figure 33 which will produce the minimum value of equation (3), hence equation (1). Substituting the information inputs provided in Figure 33 into equations (4) and (5) produces the following:

$$X = \frac{\frac{10 \times 1}{5} + \frac{10 \times 9}{5} + \frac{10 \times 1}{5} + \frac{10 \times 9}{5}}{\frac{10}{5} + \frac{10}{5} + \frac{10}{5} + \frac{10}{5}} = \frac{40}{8} = 5,$$

$$Y = \frac{\frac{10 \times 8}{5} + \frac{10 \times 8}{5} + \frac{10 \times 2}{5} + \frac{10 \times 2}{5}}{\frac{10}{5} + \frac{10}{5} + \frac{10}{5} + \frac{10}{5}} = \frac{40}{8} = 5.$$

Accordingly, the location in Figure 33, which produces the minimum value of equation (1), occurs at coordinates 5 and 5 for X and Y respectively, the exact location of the centroid in Figure 33 as determined by the balance point method to be subsequently developed.

Obviously, Figure 33 was characterized by balanced tonnages and symmetry of the array itself. However, the conception of ton mile minimization through a centroid location is likewise applicable to an asymmetrical array coupled with varying weights or tonnages. Figure 34 denotes such a presentation with the centroid of the system located at the intersection of the $\overline{X}$ coordinate equal to 7.9 and the $\overline{Y}$ coordinate equal to 5.7. Confirmation of ton mile minimization through a centroid location may therefore be generated by substituting the information inputs from this figure into equations (4) and (5).

To lend a higher degree of accuracy to the conception, the vectors or distances from the centroid to each point were computed by again utilizing the following formula for determining the distance between two points:

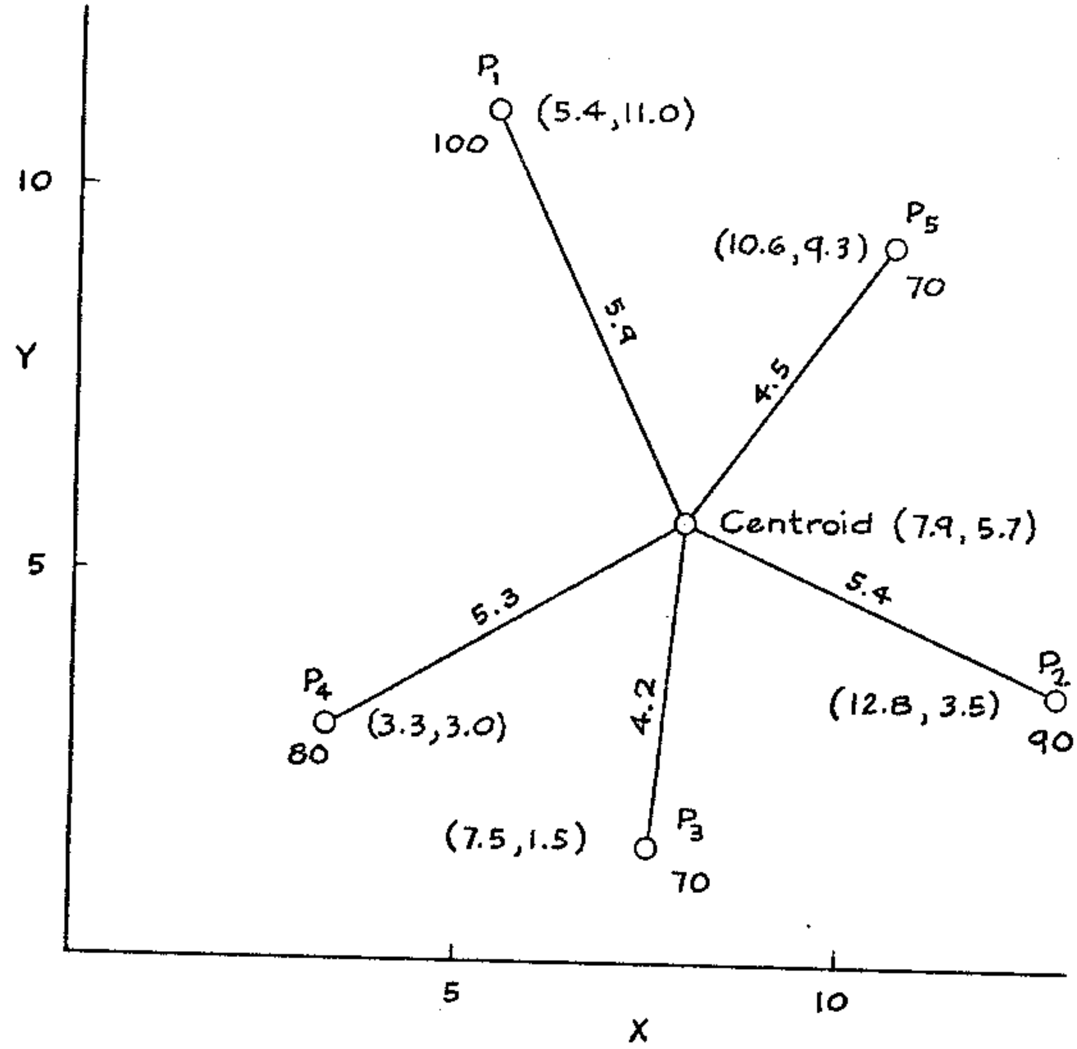

Fig. 34--Potential ton mile minimization at the centroid location for an asymmetrical array characterized by varying weights

$$V_n = [(X_n - \overline{X})^2 + (Y_n - \overline{Y})^2]^{1/2} ,$$

although scaled distances readily suffice. The results of these calculations are as follows:

$$V_1 = [(2.5)^2 + (5.3)^2]^{1/2} = 5.9$$

$$V_2 = [(4.9)^2 + (2.2)^2]^{1/2} = 5.4$$

$$V_3 = [(0.4)^2 + (4.2)^2]^{1/2} = 4.2$$

$$V_4 = [(4.6)^2 + (2.7)^2]^{1/2} = 5.3$$

$$V_5 = [(2.7)^2 + (3.6)^2]^{1/2} = 4.5.$$

Substituting these vectors and the weights and coordinates generated from Figure 34 now produces:

$$\overline{X} = \frac{\frac{5.4 \cdot 100}{5.9} + \frac{12.8 \cdot 90}{5.4} + \frac{7.5 \cdot 70}{4.2} + \frac{3.3 \cdot 80}{5.3} + \frac{10.6 \cdot 70}{4.5}}{\frac{100}{5.9} + \frac{90}{5.4} + \frac{70}{4.2} + \frac{80}{5.3} + \frac{70}{4.5}}$$

$$\overline{X} = \frac{645.4}{81.3} = 7.9,$$

$$\overline{Y} = \frac{\frac{11.0 \cdot 100}{5.9} \quad \frac{3.5 \cdot 90}{5.4} \quad \frac{1.5 \cdot 70}{4.2} \quad \frac{3.0 \cdot 80}{5.3} \quad \frac{9.3 \cdot 70}{4.5}}{\frac{100}{5.9} \quad \frac{90}{5.4} \quad \frac{70}{4.2} \quad \frac{80}{5.3} \quad \frac{70}{4.5}}$$

$$\overline{Y} = \frac{460.2}{81.3} = 5.7.$$

Confirmation of ton mile minimization through a centroid location is now forthcoming as the centroid's coordinates of 7.9 and 5.7 for $\overline{X}$ and $\overline{Y}$, respectively, check with the coordinates 7.9 and 5.7 as determined by the formulation for confirming the point of least ton miles. As a result, proof is generated as to the possible optimality of a centroid location for an asymmetrical array of points characterized

However, it will also be shown in Figure 35 that an asymmetrical array of points characterized by varying weights may also be indicative of near optimality.

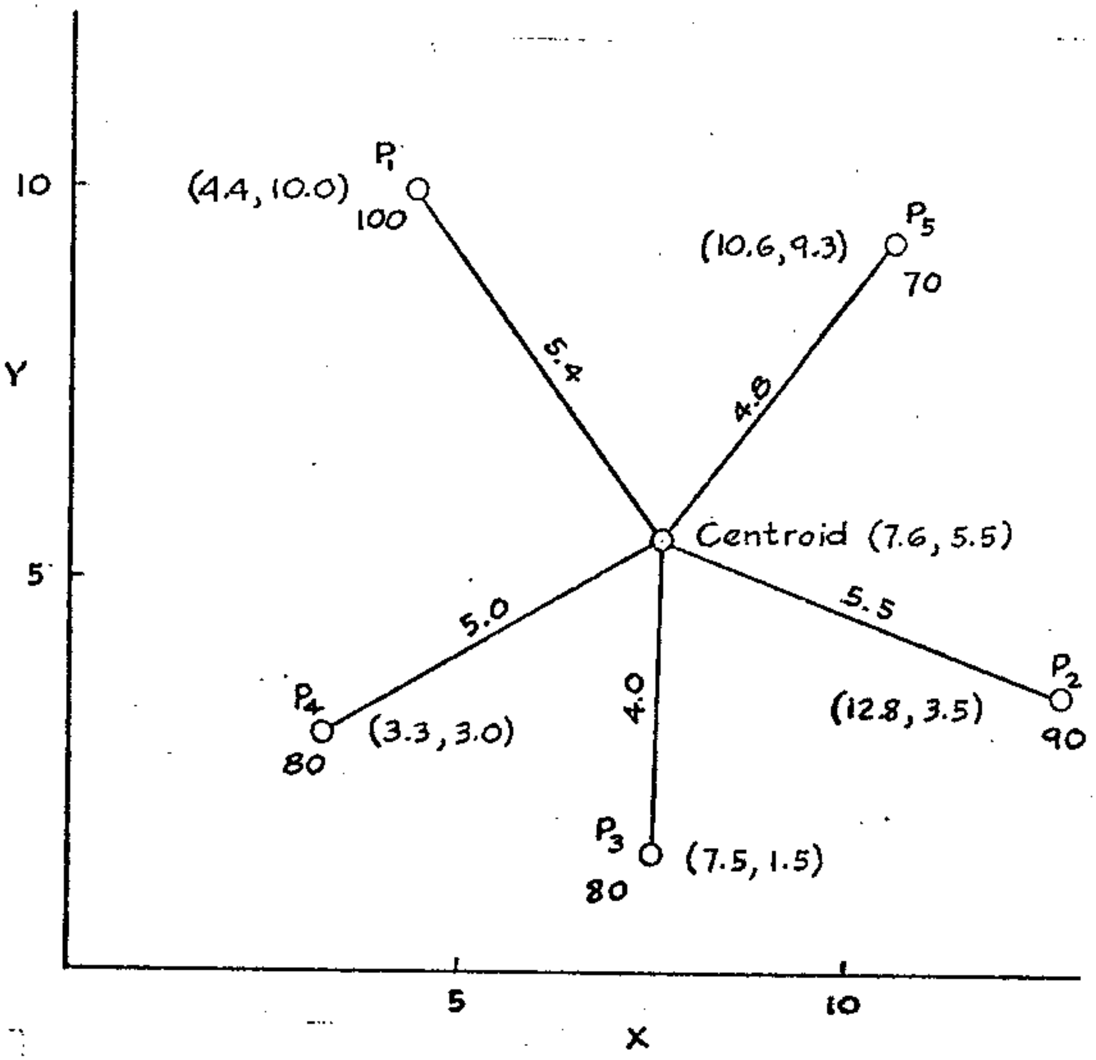

Fig. 35--Near optimality at the centroid location for an asymmetrical array characterized by varying weights

In Figure 35 the coordinates of the centroid are located at 7.6 and 5.5 for $\overline{X}$ and $\overline{Y}$ respectively as determined by the balance point formulation. Confirmation of near centroid optimality may be generated by again substituting the information inputs from this figure into equations (4) and (5). In this situation the vectors or distances from the centroid to each point have been scaled.

The substitution of the vectors, weights, and coordinates from Figure 35 now produces:

$$\bar{X} = \frac{\frac{4.4 \cdot 100}{5.4} + \frac{12.8 \cdot 90}{5.5} + \frac{7.5 \cdot 80}{4.0} + \frac{3.3 \cdot 80}{5.0} + \frac{10.6 \cdot 70}{4.8}}{\frac{100}{5.4} + \frac{90}{5.5} + \frac{80}{4.0} + \frac{80}{5.0} + \frac{70}{4.8}}$$

$$\bar{X} = \frac{648.3}{85.5} = 7.6,$$

$$\bar{Y} = \frac{\frac{10.0 \cdot 100}{100} + \frac{3.5 \cdot 90}{90} + \frac{1.5 \cdot 80}{80} + \frac{3.0 \cdot 80}{80} + \frac{9.3 \cdot 70}{70}}{\frac{100}{5.4} + \frac{90}{5.5} + \frac{80}{4.0} + \frac{80}{5.0} + \frac{70}{4.8}}$$

$$\bar{Y} = \frac{455.8}{85.8} = 5.4.$$

The coordinates generated by the confirmation model are equal to 7.6 and 5.4 for $\bar{X}$ and $\bar{Y}$ respectively, hence it can be seen that the coordinates of the centroid of 7.6 and 5.5 do not represent an absolute check. As a result, the centroid location indicates near optimality. Yet, if a range of tolerance were established the centroid could easily be viewed as being optimal. For example, given centroid coordinates of 7.6 and 5.8 for both $\bar{X}$ and $\bar{Y}$, respectively, confirmation model coordinates of 7.7 for $\bar{X}$ and 5.6 for $\bar{Y}$ could fall within a selected range of tolerance and the centroid could be viewed as being optimal; when in actuality it is near optimal.

However, it must be indicated that the introduction of a higher degree of asymmetry into the spatial array of points and a higher degree of weight variation than that depicted in Figures 34 and 35 may serve to offset the optimality or near optimality of a centroid location. Such developments are presented in the discussion on centroid limitations. Still, the centroid conception is significant because the researcher may seize upon this tool when the centroid is optimal or near optimal, and even when the centroid is clearly nonoptimal the centroid may be used as a guide to facilitate cost comparisons and ultimate confirmation.

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# APPENDIX B

## Determination Models

As was evidenced in Chapters II, III and IV two corollary types of centroid determination models are now in effect for the warehouse location problem. In Chapter II the development of the Cartesian coordinate approach was presented, while Chapters III and IV presented the development of a new type of centroid determination through a balance point formulation which may be implemented without the need of coordinates. Both approaches may be applied to three unique types of orientation. These refer to a mile centroid, a ton mile centroid, and a ton mile rate centroid. Therefore, for the purpose of emphasis a symbolic presentation will be in order for both the balance point formulation and the coordinate formulation as applied to these three respective types of orientation.

### Balance Point Determination Models

Basically, the same general theory of the balance point formulation is applicable to any centroid determination orientation. This refers to the systematic determination of balance points or centers of gravity between two designated points. With this in mind the respective types of balance point applications will now be explored.

*Mile centroid*.--For theoretical purposes the development of a mile centroid balance point model is useful. For example, if it is assumed that transportation expense is a direct function of mileage per se the determination of a mile centroid provides a potentially useful point of departure for determining a point of least cost.

Moreover, when the orientation centers on a mile centroid no consideration need be given the respective points as to an ordering of points according to magnitudes. In essence, the following balance point formulation is implemented for the first two points designated $P_1$ and $P_2$.

$$b_1 = \frac{d_1}{2} \quad ,$$

where

$\bar{B}_1$ = balance point or centroid between points $P_1$ and $P_2$

$d_1$ = scaled or computed distances between $P_1$ and $P_2$

$b_1 = P_1 \sim \bar{B}_1$ or the distance from $P_1$ to $\bar{B}_1$ on line $d_1$.

With the determination of the point of balance $\bar{B}_1$ between $P_1$ and $P_2$ now delineated, the following general formula may be implemented for points $P_3$ through $P_n$:

$$b_n = \frac{d_n}{n+1} \quad ,$$

where

$\bar{B}_n$ = balance point of any two node system

$d_n = \bar{B}_{n-1} \sim P_{n+1}$ or the scaled or computed distance between $\bar{B}_{n-1}$ and $P_{n+1}$

$b_n = \overline{B}_{n-1} \sim \overline{B}_n$ or the distance from $\overline{B}_{n-1}$ to $\overline{B}_n$ on line $d_n$.

Notice that the general equation requires that all preceding centroids for each pair of nodes must be computed prior to computing the nth balance point and that n in the general equation will always be one less than the highest numbered point in a particular balance point calculation. Therefore, by assuming that $\overline{B}_6$ has been determined for an eight point system, the final mile centroid or balance point $\overline{B}_7$ may be determined by substituting in the general equation as follows:

$$b_7 = \frac{d_7}{8} ,$$

where

$$d_7 = \overline{B}_6 \sim P_8$$

$$b_7 = \overline{B}_6 \sim \overline{B}_7 .$$

*Ton mile centroid*.--Rather than engage in the determination of a mile centroid which ignores the impact of varying weights which may be associated with the points in question, the balance point formulation may additionally be characterized by the inclusion of weights in the form of tonnages as information inputs. As a result the mile centroid is converted to a ton mile centroid. Moreover, when an array of points is characterized by synonymous weightings or tonnages for each point the mile centroid and ton mile centroid will be identical, yet with the addition of varying tonnages to these same points, the centroid clearly shifts in the direction of the heavier weightings.

Naturally, if freight rates per ton mile were constant over distance, or linear, and if no distinctions were made between volume and less than volume shipments, the ton mile centroid, if optimal, would be the actual point of least transportation cost. Based upon this assumption the ton mile centroid is ascertained by initially ordering the points encompassing the system so that:

$$P_1, W_1 \geqq P_2, W_2 \geqq P_3, W_3 \quad ,$$

where

P's = points referring to both customers and suppliers

W's = tonnage weightings associated with the respective points.

And by applying the following formulation for determining the point of balance $\overline{B}_1$ between points $P_1$ and $P_2$:

$$b_1 = \frac{d_1 W_1}{W_1 + W_2}$$

where

$\overline{B}_1$ = point of balance between points $P_1$ and $P_2$

$d_1$ = scaled or computed distance between $P_1$ and $P_2$

$b_1 = P_1 \sim \overline{B}_1$ or the distance from $P_1$ to $\overline{B}_1$ on line $d_1$.

Next the following general formula is systematically utilized to arrive at the final centroid for points $P_3$ through $P_n$ for the system in question:

$$b_n = \frac{d_n W_{n+1}}{W_1 + \ . \ . \ . \ + W_{n+1}} \ ,$$

where

$\overline{B}_n$ = centroid of any 2 node system

$d_n = \overline{B}_{n-1} \sim P_{n+1}$ or the scaled or computed distance between $\overline{B}_{n-1}$ and $P_{n+1}$

$b_n = \overline{B}_{n-1} \sim \overline{B}_n$ or the distance from $\overline{B}_{n-1}$ to $\overline{B}_n$ on line $d_n$.

The general equation also requires that all preceding centroids for each pair of nodes must be computed prior to computing the nth balance point, and for any balance point formulation n in the general equation will always be one less than the highest numbered point in a particular balance point calculation. Additionally, greater realism may be imparted to the goal of ton mile minimization through such minimization within the constraints of the existing freight rate structure. This requires weighting supply tonnages by 0.5 or some other factor to reflect the general transport cost difference between volume and less than volume shipments.

Ton mile rate centroid.--To give consideration to the pure cost ramifications of warehouse location, per ton mile freight rates may be implemented in the ton mile centroid formulation. As a result, the tonnage weights assigned each point are converted to per ton mile costs due to the additional assignment of freight rates to each of these points. Thus, the orientation of the model centers on minimizing the distances associated with these per ton mile costs, hence total transportation expense. Naturally, the application of these rates readily considers the volume and less than volume distinctions between inbound and outbound shipments, and nonlinear rates may be substituted into the model to give effect to the tapering principle of declining per ton mile rates over expanding distances.

As with the ton mile centroid model, the addition of rates as an information input requires an initial ordering of the points encompassing the system so that:

$$P_1, W_1 \cdot R_1 \geqq P_2, W_2 \cdot R_2 \geqq P_3, W_3 \cdot R_3 \text{ ,}$$

where

P's = points referring to both customers and suppliers

W's = tonnage weightings associated with the respective points

R's = average per ton mile freight rates assigned the respective points.

Next the following formulation is implemented to ascertain the point of balance between the first two points designated $P_1$ and $P_2$:

$$b_1 = \frac{d_1 \ (W_2R_2)}{W_1R_1 + W_2R_2} \quad ,$$

where

$\overline{B}_1$ = balance point between points $P_1$ and $P_2$

$W_1,W_2$ = tonnages associated with $P_1$ and $P_2$

$R_1,R_2$ = freight rates per ton mile associated with points $P_1$ and $P_2$

$d_1$ = scaled or computed distances between $P_1$ and $P_2$

$b_1 = P_1 \sim \overline{B}_1$ or the distance from $P_1$ to $\overline{B}_1$ on line $d_1$.

With the determination of the balance point formulation per points $P_1$ and $P_2$ now complete, the systematic determination of balance points per points $P_3$ through $P_n$ is accomplished through the application of the following general formula:

$$b_n = \frac{d_n \ (W_{n+1} \ R_{n+1})}{W_1R_1 + \ . \ . \ . + W_{n+1} \ R_{n+1}} \ ,$$

where

$\overline{B}_n$ = balance point of 2 node system

$W_n$ = tonnages associated with respective points

$R_n$ = freight rates per ton mile associated with respective points

$d_n = \overline{B}_{n-1} \sim P_{n+1}$ or the scaled or computed distance between $\overline{B}_{n-1}$ and $P_{n+1}$

$b_n = \overline{B}_{n-1} \sim \overline{B}_n$ or the distance from $\overline{B}_{n-1}$ to $\overline{B}_n$ on line $d_n$.

Lastly, the determination of the last centroid for the system in question represents the ultimate centroid for this respective system.

## Coordinate Determination Models

As a corollary application to the problem of centroid determination, coordinate models may also be implemented. This basically involves placing all customer and supplier points in the postive quadrant of a Cartesian coordinate system with the vertical Y axis denoting the ordinate and the horizontal X axis denoting the abscissa. As a result, all X and Y values for each customer or supplier will be expressed in positive terms.

Once this construct has been developed the various types of centroid determination models may be implemented. Attention will now center on the mile centroid as the initial focus of this orientation.

Mile centroid.--The coordinate approach like the balance point approach, as applied to the determination of a mile centroid, is useful when it is assumed that transportation expense is a direct function of mileage. To accomplish such an orientation the following model is utilized to identify the coordinates of a mile centroid:

$$\bar{X} = \frac{X_1 + X_2 + \ . \ . \ . + X_n}{n} \quad ,$$

$$\bar{Y} = \frac{Y_1 + Y_2 + \ . \ . \ . + Y_n}{n} \quad ,$$

where

$\overline{X}$ = coordinate of centroid on X axis

$\overline{Y}$ = coordinate of centroid on Y axis

X = coordinate for customer or supplier point on X axis

Y = coordinate for customer or supplier point on Y axis.

Ton mile centroid.--Rather than just concentrate on the delineation of the $\overline{X}$ and $\overline{Y}$ coordinates of a mile centroid the impact of varying tonnages may also be readily instituted in the coordinate mile centroid formulation. Such an implementation converts the mile centroid to a definite ton mile orientation, and produces the following formulation:

$$\overline{X} = \frac{X_1W_1 + X_2W_2 + \ldots + X_nW_n}{W_1 + W_2 + \ldots + W_n},$$

$$\overline{Y} = \frac{Y_1W_1 + Y_2W_2 + \ldots + Y_nW_n}{W_1 + W_2 + \ldots + W_n},$$

where

$\overline{X}$ = coordinate of ton mile centroid on X axis

$\overline{Y}$ = coordinate of ton mile centroid on Y axis

X = coordinate for customer or supplier point on X axis

Y = coordinate for customer or supplier point on Y axis

W = tonnage weights associated with each customer and supplier.

Ton mile rate centroid.--Since the ton mile centroid, if optimal, would serve as the point of least cost only if freight rates were linear with respect to distance and only if all differentials between volume and less than volume shipments were abolished, the inclusion of freight rates in the coordinate ton mile centroid formulation is also necessary. Such a rate orientation is produced as follows:

$$\overline{X} = \frac{X_1W_1R_1 + X_2W_2R_2 + \ . \ . \ . + X_nW_nR_n}{W_1R_1 + W_2R_2 + \ . \ . \ . + W_nR_n} ,$$

$$\overline{Y} = \frac{Y_1W_1R_1 + Y_1W_1R_1 + \ . \ . \ . + Y_nW_nR_n}{W_1R_1 + W_2R_2 + \ . \ . \ . + W_nR_n} ,$$

where

$\overline{X}$ = coordinate of ton mile rate centroid on X axis

$\overline{Y}$ = coordinate of ton mile rate centroid on Y axis

X = coordinate for customer or supplier on X axis

Y = coordinate for customer or supplier on Y axis

W = tonnage weights associated with each customer and supplier

R = average per ton mile freight rates associated with each customer and supplier.

## Confirmation Models

As was readily indicated in Chapter III, there are no limitations as applied to the determination of a centroid location. However, it will be recalled that there are limitations as to the optimality or near optimality of a centroid location. Therefore, for the purpose of emphasis, a symbolic presentation will also be in order for the confirmation models as applied to the respective centroid orientations and to a potential least cost point if a given centroid has proven nonoptimal. The basis for such confirmation centers on checking the coordinates of a centroid location or potential least cost site through the substitution of distances or vector radii emanating from the proposed site into the model per se. If absolute optimality is desired the coordinates produced by the confirmation should depict an absolute check. However, a range of tolerance may be developed for the location to be checked, and if the coordinates produced by the confirmation model fall within this range the location may be viewed as being near optimal.

Moreover, the confirmation model is readily oriented in terms of coordinates and likewise involves encompassing the systems within the positive quadrant of a Cartesian coordinate system, where the vertical Y axis denotes the ordinate and where the horizontal X axis portrays the abscissa. With these conceptions again in view, the study will focus on the mile confirmation symbol.

## The Mile Confirmation Model

When confronted with a mile centroid orientation, the only factor which may serve to offset the optimality or near optimality of such a centroid centers on extreme asymmetry, as tonnage weightings are not associated with the points encompassing the system. Yet to facilitate confirming the coordinates of a mile centroid, or the confirmation of the coordinates of a potential least mileage point if the centroid has proven nonoptimal, the following confirmation model is utilized:

$$\bar{X} = \frac{\frac{X_1}{V_1} + \frac{X_2}{V_2} + \ldots + \frac{X_n}{V_n}}{\frac{1}{V_1} + \frac{1}{V_2} + \ldots + \frac{1}{V_n}} ,$$

$$\bar{Y} = \frac{\frac{Y_1}{V_1} + \frac{Y_2}{V_2} + \ldots + \frac{Y_n}{V_n}}{\frac{1}{V_1} + \frac{1}{V_2} + \ldots + \frac{1}{V_n}} ,$$

where

$\bar{X}$ = coordinate of centroid or potential least mileage site on X axis

$\bar{Y}$ = coordinate of centroid or potential least mileage site on Y axis

$X_n$ = coordinate for customer or supplier point on X axis

$Y_n$ = coordinate for customer or supplier point on Y axis

$V_n$ = vector or distance of customer or supplier from warehouse at centroid or potential least cost location.

## The Ton Mile Confirmation Model

With the addition of tonnages to the location problem the centroid may additionally be offset as an optimal or near optimal site by the impact of a dominant point or cluster, hence tonnage weightings are included in the confirmation model as indicated:

$$\bar{X} = \frac{\frac{X_1W_1}{V_1} + \frac{X_2W_2}{V_2} + \ . \ . \ . + \frac{X_nW_n}{V_n}}{\frac{W_1}{V_1} + \frac{W_2}{V_2} + \ . \ . \ . + \frac{W_n}{V_n}},$$

$$\bar{Y} = \frac{\frac{Y_1W_1}{V_1} + \frac{Y_2W_2}{V_2} + \ . \ . \ . + \frac{Y_nW_n}{V_n}}{\frac{W_1}{V_1} + \frac{W_2}{V_2} + \ . \ . \ . + \frac{W_n}{V_n}},$$

where

$\bar{X}$ = coordinate of ton mile centroid or potential least ton mile site on X axis

$\bar{Y}$ = coordinate of ton mile centroid or potential least ton mile site on Y axis

$X_n$ = coordinate for customer or supplier point on X axis

$Y_n$ = coordinate for customer or supplier point on Y axis

$V_n$ = vector or distance of customer or supplier from warehouse at centroid or potential least ton mile location

$W_n$ = tonnage weight associated with a given point.

## The Ton Mile Rate Confirmation Model

While the ton mile centroid may be compromised by the impact of extreme tonnage weightings associated with a point or cluster of points, the ton mile rate centroid may likewise be compromised, except here the weightings refer to extreme cost weights associated with a given point or cluster. Therefore, to ascertain if the centroid has been offset as an optimal or near optimal location by either extreme asymmetry and/or extreme cost weightings, or to confirm a potential point of least cost if the centroid has been compromised, the following confirmation model prevails:

$$\overline{X} = \frac{\dfrac{X_1(W_1R_1)}{V_1} + \dfrac{X_2(W_2R_2)}{V_2} + \ldots + \dfrac{X_n(W_nR_n)}{V_n}}{\dfrac{W_1R_1}{V_1} + \dfrac{W_2R_2}{V_2} + \ldots + \dfrac{W_nR_n}{V_n}},$$

$$\overline{Y} = \frac{\dfrac{Y_1(W_1R_1)}{V_1} + \dfrac{Y_2(W_2R_2)}{V_2} + \ldots + \dfrac{Y_n(W_nR_n)}{V_n}}{\dfrac{W_1R_1}{V_1} + \dfrac{W_2R_2}{V_2} + \ldots + \dfrac{W_nR_n}{V_n}},$$

where

$\overline{X}$ = coordinate of cost centroid or potential least cost site on X axis

$\overline{Y}$ = coordinate of cost centroid or potential least cost site on Y axis

$X_n$ = coordinate for customer or supplier point on X axis

$Y_n$ = coordinate for customer or supplier point on Y axis

$V_n$ = vector or distance of customer or supplier from warehouse at centroid or potential least cost location

$W_n$ = tonnage weight associated with a given point

$R_n$ = per ton mile freight rate associated with a designated point.

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