THE EFFECTIVENESS OF A GUIDED DISCOVERY METHOD OF TEACHING IN A COLLEGE MATHEMATICS COURSE FOR NON-MATHEMATICS AND NON-SCIENCE MAJORS

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THE EFFECTIVENESS OF A GUIDED DISCOVERY METHOD OF TEACHING IN A COLLEGE MATHEMATICS COURSE FOR NON-MATHMATICS AND NON-SCIENCE MAJORS

DISSERTATION

Presented to the Graduate Council of the North Texas State University in Partial Fulfillment of the Requirements

For the Degree of

Doctor of Education

By

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CHAPTER I

STATEMENT OF THE PROBLEM

A large proportion of the educational research of today is centered around the study of the teaching-learning process. A great deal is being learned about which teaching processes are and are not appropriate for effective learning under given conditions. This knowledge is being applied in the development of new curricula in many areas of study. At the present time, knowledge concerning the teaching-learning process is still limited. As a result, curriculum innovations are often based on hypotheses which have not been adequately tested. This necessitates the testing of new curricula and new teaching methods through carefully controlled research. Such research can be used to determine whether new teaching methods or new curricula actually accomplish the goals which they are designed to accomplish.

One of the teaching methods which has been developed as the result of research dealing with teaching and learning is the discovery method of teaching. This method of teaching has its early beginnings in attempts to formulate curricula and methods of teaching compatible
with the teachings of Gestalt psychology. Although research related to the discovery method of teaching has been conducted during the past thirty years, serious attempts to develop curricula emphasizing the discovery method of teaching were not made until after 1950. Curricula which emphasize the discovery method of teaching have been developed for elementary and secondary school mathematics and science. Programs for mathematics have been developed by the University of Illinois Committee on School Mathematics, the Greater Cleveland Mathematics Project, and the Madison Project of Syracuse University. Programs for science have been developed by the Science Curriculum Improvement Study and the American Association for the Advancement of Science. Attempts are presently being made to develop similar programs in the language arts and in the social studies. Since programs emphasizing discovery methods of teaching have been successfully developed for elementary and secondary school mathematics and science, it is of interest to consider the use of discovery methods of teaching at the college level.

Statement of the Problem

The problem under consideration was the effectiveness of a guided discovery method of teaching mathematics as compared to the effectiveness of an exposition method of
teaching mathematics in a college freshman mathematics course for non-mathematics and non-science majors.

**Purposes of the Study**

During the past few years several new programs for mathematics in the elementary and secondary school have been developed. Some of these programs not only emphasize new content and new organization of content, but new teaching methods as well. One of the most frequently advocated methods is the discovery method of teaching. 

In order to provide a reference for research related to the discovery method of teaching mathematics, a summary of research literature related to this method of teaching mathematics is presented. Since many mathematics educators advocate the use of the discovery method of teaching mathematics at the elementary and secondary levels, it was considered to have potential as a method of teaching at the college level as well. The purpose of this study was to ascertain the value, as determined by student achievement, of using a discovery method of teaching mathematics in a college freshman mathematics course for non-mathematics and non-science majors.

**Hypotheses**

The study was designed to test the following hypotheses:
1. Students taught by a guided discovery method will score significantly higher on the test, Cooperative Mathematics Tests, Structure of the Number System, than students taught by an exposition method.

2. Students taught by a guided discovery method will score significantly higher on the test, Cooperative Mathematics Tests, Algebra I, than students taught by an exposition method.

3. Students taught by a guided discovery method will score significantly higher on the Watson-Glaser Critical Thinking Appraisal than students taught by an exposition method.

4. Students taught by a guided discovery method will make significantly higher grades in the course than students taught by an exposition method.

5. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the test, Cooperative Mathematics Tests, Structure of the Number System, will not be dependent upon student ability.

6. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the test, Cooperative Mathematics Tests, Algebra I, will not be dependent upon student ability.
7. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the Watson-Glaser Critical Thinking Appraisal will not be dependent upon student ability.

8. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the student's grade in the course will not be dependent upon student ability.

Background and Significance of the Study

The discovery method of teaching has its foundations in the writings of the Gestalt psychologists. According to Gagne:

This idea, having its origin partly in the writings of members of the Gestalt school of psychology, has had a profound influence on mathematics and mathematics teachers. Basically the notion is that productive thinking, or inventive problem solving, is achieved by "insight," which represents an essential reorganization of the entire field in which the individual and the problem are located. What brings about this organization is a set of factors which tend to structure the insight process—like closure, pragnanz, symmetry, and others. 1

Wortheimer felt that when teaching it is preferable to proceed in a manner which favors discovery of the

essential nature of the problematic situation, of just what is needed to solve the problem, so that, even at the cost of elegance or brevity, the solution of the problem is sensible rather than blind and mechanical.

Writing in a similar vein, Ausubel states that in discovery learning "The learner must rearrange a given array of information, integrate it with existing cognitive structure, and reorganize or transform the integrated combination in such a way as to create a desired end product or discover a missing means-end relationship."

Many enthusiastic claims have been made concerning discovery teaching. For example, Kersh states that

... many would agree that when the student learns by discovery he (1) understands what he learns, and so is better able to remember and transfer it; (2) he learns something the psychologist calls a "learning set," or a strategy for discovering new principles, and (3) he develops an interest in what he learned.

Although the proponents of discovery teaching have made many claims favoring teaching by discovery methods, these claims have not been adequately tested through the use of

3Ibid., p. 91.
4Ibid., p. 117.
6Bert Y. Kersh.
carefully controlled research. According to Cronbach

In spite of the confident endorsements of teaching through discovery that we read in semi-
popular discourses on improving education, there is precious little substantiated knowledge about
what advantages accrue. We badly need research in which the right questions are asked and trustworthy
answers obtained. 7

Although many research studies comparing discovery
methods of teaching with other methods of teaching have
been conducted, many questions remain unanswered. Many
of the studies that have been conducted have yielded little
useful information because the experimental designs and
Experimental strategies employed have been inadequate.
After reviewing research literature related to discovery
methods of teaching, Worthen concluded that

Most "discovery" studies have been conducted in
laboratory settings and consequently have dealt
with small time samples, small numbers of subjects,
and very discrete and often manipulative learning
tasks. One might argue that such sampling of time,
subjects, and tasks is so restrictive and limited
in scope that any attempt to generalize the results
to classroom learning or instruction would be subject
to serious question. 8

Since evidence related to discovery methods of teaching
is limited, the educator interested in discovery methods

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7. Lee J. Cronbach, "The Logic of Experiments on
Discovery," Learning by Discovery: A Critical Appraisal,
edited by Lee J. Shulman and Eva R. Keisler (Chicago, 1966),
p. 77.
8. B. R. Worthen, "A Comparison of Discovery and Ex-
pository Sequencing in Elementary Mathematics Instruction,"
Research in Mathematics Education, The National Council of
of teaching must make his own decisions on the basis of the evidence available. In order to lessen the effort that such a task requires, the second chapter of this study contains a comprehensive review of research studies dealing with topics related to discovery methods of teaching mathematics.

Several programs emphasizing discovery methods of teaching have been developed for elementary and secondary level courses in mathematics. Examples of such programs are those developed by the University of Illinois Committee on School Mathematics and the Madison Project of Syracuse University. Some who have taught courses using teaching methods recommended by these groups and materials developed by these groups have been enthusiastic about the results. Therefore, it is of interest to consider the use of discovery methods of teaching mathematics at the college level. This study was designed to provide information concerning the value of using a guided

[References]

discovery method of teaching mathematics in a college freshman mathematics course for non-mathematics and non-science majors. The study utilized four sections of a regular college mathematics course and was conducted during an entire semester.

Definition of Terms

1. A guided discovery method of teaching. A guided discovery method of teaching is a method of teaching in which each concept or principle is taught through an instructional sequence characterized by a sequence of steps as described below.

In the first step of the sequence the students are asked to work exercises dealing with the concept or principle to be learned. These exercises are designed to guide the student to discovery of the concept or principle.

During the second step of the sequence the instructor tests for discovery. This occurs after the students have completed the introductory exercises. In order to determine whether discovery has occurred, the instructor asks carefully selected questions. These questions are such that if discovery has occurred they will be answered easily and quickly, but if discovery has not occurred they will be answered with difficulty or not at all.
If testing reveals that discovery has not occurred, the third step of the sequence is in the form of more exercises. These exercises are designed to give the students further guidance toward discovery of the concept or principle to be learned. After these exercises have been completed there should be another test for discovery. This sequence of exercises followed by tests for discovery should be repeated until a majority of the students have discovered the concept or principle to be learned.

After discovery has occurred the student is given exercises which provide him with an opportunity to use what he has discovered. Whenever possible these exercises should be designed to guide the student to the next discovery.

The final step in the sequence consists of naming the concept or principle discovered and in some cases this step also includes the development of a formal statement of the concept or principle discovered. If the students possess the verbal capacity to do so, the statement of the concept or principle should be formulated by the students. In order to provide adequate time for each student to discover for himself the concept to be learned, the formal statement of the concept should be delayed at least one class period. In some cases it may be desirable to delay the statement of the concept for a considerable length of
2. An expository method of teaching. An expository method of teaching is a method of teaching in which each concept or principle is taught through an instructional sequence characterized by a sequence of steps as described below.

The first step of the instructional sequence consists of a formal statement of the concept or principle to be learned. This statement is presented either by the instructor or in the instructional materials. If the concept or principle has a formal name, this name is presented along with the statement of the concept or principle.

The statement of the principle is followed by a discussion or by a short lecture by the instructor. The purpose of this discussion or lecture is to clarify the concept and to help the student understand the concept. This discussion or lecture should include illustrations of how the principle or concept can be used to solve specific problems.

After the first two steps of the instructional sequence have been completed, the students should be provided with opportunities to use the concept presented to solve specific problems. In many cases, the first two steps of the instructional sequence may be repeated several times before the final step is reached. That is, several concepts or principles may be presented during one
principles may be in the form of exercises to be completed between class periods.

3. A college mathematics course for non-mathematics and non-science majors. A college mathematics course for non-mathematics and non-science majors refers to the course College Mathematics at Southwestern State College, Weatherford, Oklahoma. This is a required course for all non-mathematics and non-science majors and is designed to serve as an introduction to mathematics for liberal arts majors. In the teacher education program it is a required course for all prospective elementary school teachers.

4. Ability. An ability is the actual power and tendency of an organism to respond in a predetermined way to a given set of stimuli. An ability is the actual power to perform an act and the tendency to perform that act in the appropriate situations. This act may be physical or mental. Since the power and tendency of an organism to respond may change with time, abilities are dependent upon time. In this study the term "ability" is used to refer to the class of abilities associated with mathematics. Since abilities are dependent upon time, the time referred to when the term "ability" is used is the beginning of the 1968 spring semester.

5. Achievement. In this study the term "achievement" is used to refer to the class of abilities associated with
that part of mathematics studied in College Mathematics, the time being the end of the 1968 spring semester. Among those abilities included under the term "achievement" are those related to the application of learning in new and different situations.

Limitations of the Study

1. The study was limited to students enrolled in four sections of College Mathematics at Southwestern State College, Weatherford, Oklahoma, during the spring term of the 1967-1968 school year.

2. The study investigated the effectiveness of only one of several discovery methods of teaching.

3. The study was limited to the effects of two teaching methods on student achievement.

Basic Assumptions

1. It was assumed that the four sections of College Mathematics used in the study were representative of all sections of College Mathematics taught at Southwestern State College.

2. It was assumed that activities outside the mathematics classroom and related to the learning of mathematics were balanced between the groups and therefore did not significantly affect the study.
CHAPTER II

SURVEY OF RELATED RESEARCH LITERATURE

Introduction

The purpose of this chapter is to present a comprehensive summary of research literature related to discovery methods of teaching. The studies reviewed in this chapter are limited to those which represent conclusions based on research and not mere personal opinion and which compare forms of instruction involving some degree of discovery by the learner with other methods of instruction. Since there are various forms of instruction through discovery this chapter will attempt to present some of the forms of discovery teaching and the results obtained through the use of these methods. In addition this chapter is an attempt to provide a reference for research related to discovery methods of teaching mathematics.

The studies reviewed in this chapter are organized according to the type of task or material learned by the subjects in the study reviewed. The review of the literature is presented according to the following subdivisions: the studies of George Katona and related studies, studies using coding problems as the criterion
task, studies in which the criterion task involved
discovery of word relationships, studies in which the
subjects learned sums of series, studies conducted
in the elementary school mathematics classroom, studies
conducted in the junior and senior high school math-
ematics classroom, and studies conducted in the college
mathematics classroom.

The Studies of George Katona
and Related Studies

George Katona, one of the more prominent members
of the gestalt school of psychology, was among the
earliest writers to report studies in which certain
forms of discovery methods of teaching were compared
with other methods of teaching. Katona¹ advanced the
hypothesis that there is more than one kind of learning.
This is in contrast to the hypothesis that all learning
can be thought of as the acquisition of associations
or connections. In order to obtain evidence to support
his hypothesis Katona selected sets of facts which
could be learned in more than one way. It was then
hypothesized that the way in which a set of facts is
learned will affect retention of these facts and the
way in which these facts will be used. The first set

¹George Katona, Organizing and Memorizing
(New York, 1940), pp. 331.
of facts used to test these hypotheses consisted of a class of card tricks. Several exploratory and one group experiment was conducted in which the learning task consisted of learning how to perform the card tricks.\(^2\) In each experiment one group of subjects was required to memorize the solution to one or more card tricks. The principle involved in the solution to the card tricks was explained to another group. The first group was called the memorization group and the second group was called the understanding group. In each of the experiments it was found that the two groups were similar in ability to perform the card tricks taught. After one or more weeks the subjects were asked to perform the card tricks taught. On these tests the performance of the understanding group was superior to the performance of the memorization group. Immediately after instruction the subjects were presented with card tricks different from, but similar to, the card tricks taught. When the subjects were asked to perform these tricks, the performance of the understanding group was superior to the performance of the memorization group.

Several additional experiments were conducted in which the subjects were shown card tricks and then asked to discover the solution to the trick without help. On the basis of these experiments it was concluded that:

\(^2\)Ibid., pp. 32-54.
... if the principle has been discovered without assistance, retention seems to endure even longer than after learning by understanding.
Tests made two months after the learning (or solving) periods with a few subjects showed that the best results were obtained by those who had solved the problem alone.

In order to obtain more information concerning the effects of methods of instruction on learning Katona\(^4\) conducted another series of experiments using a different set of learning tasks. In each learning task the subject was presented with a geometrical figure consisting of a set of squares. In each figure some of the squares had sides in common. The subject was asked to move some of the lines which formed sides of squares and produce a figure consisting of fewer, or in some cases more, squares than contained in the original figure. The new figure should include the same number of lines as the old figure and the squares should be the same size as in the original figure. Such figures can be constructed with match sticks and the task can then be completed by moving a limited number of match sticks. The final figure must include all the match sticks and it is not permissible to place one match stick on top of another. Although the figures were not always constructed using actual match sticks, the experiments of this series were called match-stick experiments.

\(^3\)Ibid., p. 50.

\(^4\)Ibid., pp. 81-107.
In each experiment one group of subjects was required to memorize the solution to one match-stick problem. Another group was shown the solution to three problems. The first group was called the memorization group and the second group was called the example group. During the learning period the subjects of the example group were presented with a match-stick problem and asked to attempt to solve it. After half a minute the subjects were shown a solution to the task. This procedure was repeated with three different match tasks. By studying the procedures used in the solution of these problems the subjects were expected to formulate for themselves (discover) the principle used to solve the problems. Immediately after the learning period the subjects of both groups were asked to exhibit solutions to the problems presented during the learning period. Then several new tasks were presented for solution. Four weeks later the subjects were again asked to solve the tasks presented during the learning session and they were also asked to solve some new tasks.

On the test administered immediately after the learning session both groups performed equally well on the tasks presented during the learning session, but four weeks later the example group exhibited superior performance on these tasks. On the test administered immediately after the learning session and on the test administered four weeks
later the example group exhibited superior performance on those tasks which were different from those presented during the learning session. In general, it was concluded that learning by examples is superior to learning by memorization.

During the learning session the subjects in the example group were expected to formulate for themselves the principle necessary for the solution of the problems. Investigation revealed that ability to verbalize such a principle is not correlated with ability to solve matchstick problems.

Hilgard, Irvine, and Whipple\(^5\) conducted an experiment similar to Katona's card-trick experiments. The experiment was designed to meet some of the criticism concerning Katona's experiment and to add additional dimensions to the investigation of problem-solving behavior. In particular, the following generalizations were tested

\[\ldots\ (e) \text{ The advantage of learning with understanding does not necessarily show up in original learning, for learning with understanding may take longer than learning by rote. (b) Retention after learning by understanding tends to be greater than retention after learning by rote. (c) Transfer to new related tasks is greater after learning by understanding than after learning by rote.}^6\]

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\(^6\) Ibid., p. 288.
The subjects for the experiment were sixty high school students. Each subject was assigned, by a random process, to one of two groups. One of the groups was called the understanding group and the other group was called the memorization group. On the first day of the experiment all subjects were taught two card tricks. The subjects in the memorization group learned the tricks by learning the order of the cards by rote memory. The understanding group was taught a scheme from which the proper order of the cards for any of a certain class of tricks could be derived. Both groups were given practice and help until the tricks could be performed without error. On the second day of the experiment the subjects were given two kinds of transfer tests. The tricks on the first test could be performed by simply transposing the order of the cards necessary to perform the original tricks. The tricks on the second test were based on entirely new principles.

It was found that the memorization group learned the original tricks much more quickly than the subjects in the understanding group. It was also found that there was no significant difference between the two groups in terms of retention after one day. The understanding group was significantly superior to the memorization group on both transfer tests with the superiority being the greatest on
the second transfer test. In general the results of the experiment agreed with the results obtained by Katona.

Gorman conducted an experiment similar to the matchstick studies conducted by Katona. It was predicted that only information appropriate to the criterion employed to evaluate success will be effective as guidance. It was further hypothesized that where the information is inappropriate to the criterion employed no significant effects will be discerned as a result of the guidance. It was hypothesized that, given an appropriate criterion, the effectiveness of guidance will increase directly as the amount of information supplied to the subject is increased. Finally, it was hypothesized that the effects of varying kinds and amounts of information will be similar for subjects of higher and lower intellectual ability. Two criteria for success were identified. One criterion was ability to find a method for solving the problem task and the other was ability to verbalize a principle involved in the solution of the task.

Two hundred thirty-three twelfth-year students were given varying amounts of instruction concerning the method of solution to be employed and the principle to be used in the solution of match-stick problems. Three levels of

information concerning the method of solution were identified. These were no information, some information, and an explicit statement of one method of solution to the problem. Three levels of information concerning the principle used in the solution of the problem were identified. These were no information, some information, and an explicit statement of the principle. Each subject was given one form of information concerning the method of solution and one form of information concerning the principle used in solving the problem. As a result, there were nine different instructional methods varying from no information concerning a method of solution and no information concerning the principle used in the solution to a statement of a method of solution and a statement of the principle used. Each of the nine instructional methods was used with a random subgroup of the total group of subjects.

After an eighteen-minute instructional period two tests were administered. The first test was designed to measure the ability of the subjects to solve problems only slightly different from those learned during the instructional period. This test was called the simple transfer test. The second test was designed to measure the ability of the subjects to solve problems based on the principle taught during the instructional period but dissimilar to the problems taught during the instructional period. This test was called the complex transfer test.
From the data obtained from these tests it was concluded that

1. Information given the student about the method of solving examples is more likely to be beneficial than information given about the principle—at least in the initial stages of problem solving.
2. Some appropriate guidance will prove more helpful to the student than no guidance. Leaving the student to discover for himself the solution of a problem will not prevent understanding, but will probably delay it.
3. The effectiveness of guidance does not depend solely on the amount of information imparted. More explicit forms of instruction will prove most helpful with students able to apply the information. For students of lesser ability, less explicit clues, designed to highlight structural relationships, may prove just as effective.

Scandura\(^9\) conducted three exploratory experiments which were somewhat related to the card-trick studies reported by Katona. In each experiment the task to be learned concerned certain types of arrangements of a set of sixty-four cards. The cards were specially designed for the experiment. Each card had two geometrical figures printed on the face of the card. The arrangements to be learned depended upon the types of figures printed on the cards. The purpose of the studies was to determine some of

\(^8\)Ibid., p. 18.

the variables and interrelationships which complicate experimental comparisons of expository and discovery methods of instruction and to provide a framework for future more precise experimentation. In each experiment the effectiveness of a discovery method of teaching was compared with the effectiveness of an expository method of teaching.

The first experiment utilized two classes of sixth-grade students as subjects. One class, the exposition class, was presented with all the information necessary to solve the problems. The information was presented as quickly as possible and all questions were answered as directly as possible. The other class, the discovery class, was given extensive experience with the prerequisite material before new material was presented. Meaning was stressed in that questions were asked and hints were given that were directed at the underlying principles involved in the problems. Instruction for both groups was given during two periods, but the instructional periods for the discovery group were longer than the instructional periods for the exposition group. Two types of tests were then given to each of the two groups. One test involved routine problems similar to those learned during the instructional period and the other test involved problems based on principles different from those learned during the instructional periods. There was no significant difference between the two groups on the first test but on the second test the
performance of the discovery group was superior to the performance of the exposition group.

In the second experiment the instructions for the discovery group were more direct and attempts were made to reduce instructional time. In the exposition class attempts were made to make the learning more meaningful. The subjects in this experiment were twenty-three fifth and sixth-grade students. Each class met five times. The first three periods were used for instruction and the last two periods were used for testing. On a test involving routine problems the performance of both groups was nearly the same. On a retention test and on a test involving problems based on different principles than the problems learned during the instructional periods the performance of the exposition group was superior to the performance of the discovery group.

In the second experiment the instructional time for the discovery group was longer than the instructional time for the exposition group. A third experiment was conducted in which the instructional time for both groups was the same. In order to accomplish this the subjects in the discovery group were given aid in finding systematic modes of attacking the problems. The subjects in the exposition group were taught an algorithm which could be used to solve a certain class of problems. In order to equate instructional time, practice was given in using the
algorithm until the instructional time matched that used for the discovery group. In this experiment there were only eight subjects in the discovery class and seven subjects in the exposition class. As a result no statistical analysis was performed. Examination of mean scores showed that the performance of the exposition class was superior to the performance of the discovery class on routine problems but the performance of the two groups was nearly the same on the problems based on new principles.

After an analysis of the results of the three experiments it was concluded that perhaps the most important variable in the instructional modes involved the "timing" of presentation of facts and ideas during instruction. According to Scandura,10 "A better understanding of the role of timing and its effects on learning, retention and transfer may be of practical import in helping make the teaching-learning process more efficient." Scandura stated further that

... generally it is assumed that the better the timing, the more conceptual the learning and the more transfer will obtain. When problem solving algorithms are presented directly and meaningfully at a point in time when S-feedback indicates good prerequisite comprehension, conceptual learning may be expected.11

10 Ibid., p. 155.
11 Ibid., p. 155.
The studies conducted by Katona have often been cited as providing evidence favoring discovery methods of teaching. His studies have also inspired several other studies. Most of these studies have also given some support to the hypothesis that discovery teaching is more effective than expository teaching. Closer examination of these studies reveals that the support is not as clear as often indicated. The results of the preceding studies should be viewed with caution because in each study there were at least two independent variables present and the interaction of these variables was not always considered or controlled. One of the variables present is the variable which lies along the continuum of all possible discovery methods of teaching. This variable varies from situations where all facts are learned without help to the situations where all information is given in such a way that there is no opportunity for search. The other variable lies along the continuum of all teaching methods which might be called methods of teaching by understanding. This variable varies from situations in which all information is presented in such a way that new information can be assimilated by the learner and correlated with past experience to situations in which information is presented for which the learner has a minimum of past experience and in which learning is by rote memorization.
Studies Using Coding Problems as the Criterion Task

Several investigations into the teaching-learning process have been conducted in which the learning task consisted of learning methods of deciphering encoded sentences. Such learning tasks have been considered similar to the learning tasks encountered in a typical classroom situation because coding problems are of an abstract nature and are based on clearly defined principles.

In one such study, conducted by Haslerud and Meyers, it was hypothesized that principles derived by the learner solely from concrete instances will be more readily used in a new situation than those given to him in the form of a statement of principle and instance. Seventy-six college students, ranging from freshmen to seniors, were used as subjects in an experimental group in a study designed to test this hypothesis. Twenty-four similar subjects were used as a control group. The experimental group was given two tests in which the task was to decipher twenty codes. The second test was given one week after the first test was given. The control group was given only the second test. The first test was designed to give the experimental
group two types of experience in problem solving. For some of the problems specific directions for deciphering the code were printed above the problem while no directions were given for the remaining problems. There were equal numbers of the two types of problems and the types of problems were alternated so that each subject would solve approximately an equal number of each type of problem. A different code was used in each problem. On the second test the subjects were required to identify the correct solution to each of the same twenty codes. For each subject in the experimental group there were four possible scores. The first score was the number of codes solved correctly on test one for which the rule was given. The second score was the number of codes solved correctly on test one for which no directions were given. The third score was the number of correct codes on test two for which rules had been given on test one and the fourth score was the number of correct codes on test two for which no direction had been given on test one.

It was found that on test two the performance of the experimental group was significantly superior to the performance of the control group. On test one significantly more codes were solved for which the rule was given than codes for which no direction was given. On test two the identification of correct solutions for codes for which no direction had been given on test one increased forty-six
per cent. The increase for codes for which the rule had been given was only ten per cent. This difference is highly significant. It was concluded that this experiment added strong support to the contention that independently derived principles are more transferable than those where the principle is given to the student.

Krebs13 conducted a study very similar to the study reported by Haslerud and Meyers. An initial test, very similar to the initial test used by Haslerud and Meyers, was administered to thirty-two ninth-grade students. After this test was completed a test containing coding items based on the same principles used in the initial test was administered. This test was administered again after six days and then again after forty-three days. Another group of fifty-eight subjects took the last three tests but not the initial learning test. This group was called the control group.

On the initial learning test more codes for which the coding principle was given (given principles) were deciphered than codes for which the coding principle was not given (derived principles) but illustrated by an example. On the first administration of the second test, the experimental subjects scored significantly higher than the control

subjects. After six days the difference was still significant. After forty-three days the difference had narrowed to borderline significance. The transfer indicated by these results was attributed to the teaching methods and not practice. An analysis of test data led to the rejection of the hypothesis that students will transfer derived principles better than given principles. Test results showed no difference. The hypothesis that the transfer effects of derived principles will be more permanent than those of given principles was also rejected.

Wittrock compared the effects of three variables upon the ability of subjects to decipher encoded sentences. The variables were rule, example, and order. There were two levels of the rule variable and the example variable. These levels were rule given or rule not given and example given or example not given. The levels of the order variable were rule given first or example given first. All possible combinations of these factors yield eight different instructional methods. Two hundred ninety-two upper level college students were assigned at random to the eight treatment methods. Instruction was provided through programmed booklets. Subjects were allowed to proceed through the booklets at their own pace. The first part of each

instruction booklet presented, in various sequences depending upon the experimental treatment, appropriate directions, a rule, a worked example of the rule, and an example of the rule to be worked. The remainder of the booklet was divided into ten series. In each series a rule was given or was to be derived by the subject and an enciphered sentence was presented followed either by the same sentence deciphered or by a space where the sentence was to be deciphered by the subject. The order of the stimuli was varied to fit the treatment.

A three-week retention test consisted of twenty-four enciphered sentences. The sentences were chosen to sample retention and transfer. Eight of the sentences were identical to those presented during the learning period. Another eight of the sentences were new examples of eight of the ten rules presented earlier. The remaining sentences were codes based on new rules. The sentences were arranged in random order.

On the original test the groups that were given the rule decoded more sentences than those who were not given the rule. No significant difference existed between the rule given group and the answer and rule given group although both groups were superior to the no rule and answer given group. There was a significant interaction between the rule and answer factors. When the rule was not given, giving the answer improved learning, but when
the rule was given, giving the answer did not enhance learning. When time to learn was considered, it was found that the rule not given and answer not given groups required significantly more time to learn than any of the other groups.

On the retention and transfer test the rule given and answer not given group was significantly better than the rule given and answer given group and the rule not given and answer not given group. The rule not given and answer not given group had a higher retention score than initial learning score. The other three groups’ retention scores were lower than their learning scores. On the second test a significant interaction was found between the rule and answer factors, just as on the initial learning test. However, the answer factor made no significant contribution to the retention and transfer scores. Giving specific answers improved initial scores whether or not rules were given, but improved retention and transfer scores only when rules were not given. It was concluded that when the criterion is initial learning of a few responses explicit and detailed direction seems to be most effective and efficient. When the criteria are retention and transfer, some intermediate amount of direction seems to produce the best results.

Three studies have been reviewed in which subjects deciphered codes. Two studies indicated that a certain
amount of discovery on the part of the student enhances retention and transfer whereas another study indicated that students will not transfer and retain derived principles better than given principles.

Studies in Which the Criterion Task Involved Discovery of Word Relationships

In the preceding section it was found that the task of learning methods of deciphering codes has been considered similar to the learning tasks encountered in typical classroom situations. Another learning task that has been considered similar to typical classroom learning is the task of learning a principle or relationship satisfied by four of a list of five words but not satisfied by the fifth word.

Stacey conducted a study in which the effects of varying amounts and kinds of information given during instruction on the learning of word relationships were investigated. The effects of reward and punishment upon learning were also investigated. In the learning situation used in the study the subjects were presented with a list of five English words. The task of the subject was to learn to respond to the stimulus consisting of the list of

five words by saying one of the words. In order to make the learning situation meaningful four of the words were related by some principle. The correct response consisted of the word which did not "belong" according to the principle. A further task of the subject was to identify the principle satisfied by four words of the list. For the total learning task a set of fifty lists was prepared. These lists involved ten different principles. During the experiment a list was presented to the subject and the subject was asked to make a response. If the response was correct the subject was rewarded by hearing the response "right" from the person who was presenting the list. If the response was incorrect the subject was punished by the response "wrong." The list of words was kept before the subject until a correct response was made. Then the next list was presented. The entire set of fifty lists was presented to each subject five times. During the first presentation of the set of fifty lists each subject was given one of five different kinds of instruction. In Method A the subject was given no intimation as to why one of the five words did not belong with the other four, nor was he told that there might be any particular reason underlying a correct choice. In Method B the subject was informed that there was a reason why one of the five words of each list did not belong with the other four words, but he was not told the reason. In Method C the subject was
given the correct response to each list during the first trial, but he was given no reason why the response was correct. In Method D, during the first trial, the subject was given the correct response to each list and told that there was a reason why the response was correct, but he was not told the reason. In Method E the subject was told the correct response to each list and he was told the principle or reason underlying the correct response.

The subjects used for the experiment were one hundred sixth-grade students. The subjects were assigned to the instructional methods by a random process in such a way that there were twenty subjects in each instructional group. Criterion scores were in the form of the number of incorrect responses made to each list during each of the five trials. Immediately after a subject had completed his fifth trial, the same set of fifty lists to which he had responded was once more placed before him. On being told the response he had made for each list, the subject was asked whether or not there had been any particular reason why that response had been made, and if so, what the reason was. A record was made of the reason given by each subject.

In addition a set of fifty lists of words based on principles similar to those used during the instructional period was used as a pretest and a posttest.

On the basis of the information obtained from the experiment the following conclusions were drawn:
1. It makes for better learning if the learner proceeds by a method of active participation involving self-discovery rather than by a method of passive participation involving only recognition or identification of information previously provided him.

2. The process of self-discovery on the part of the learner weakens errors or wrong habits and thereby eliminates them more quickly than does the process of authoritative identification.

3. The imperfect learning that occurs during a process of self-discovery is less detrimental to learning than that which occurs during a process in which the responses are identified for the learner.

4. The learner obtains as many or more facts, and discovers more correct reasons for them, by a process of self-discovery than by a process of authoritative identification.\(^\text{16}\)

Kittell\(^\text{17}\) studied the effects of minimum, intermediate, and maximum amounts of direction during discovery on transfer and retention. Subjects for the study were 132 sixth-grade pupils. The subjects were placed into three treatment groups through a process of stratified-random selection. The stratification was based on high, medium, and low reading achievement classifications. The learning material consisted of lists of five words in which four of the words satisfied some principle but the fifth word did not. The subjects of the minimum group were told that each group of three lists was based on a common underlying principle. No information was given this group as to the nature

\(^{16}\)Ibid., p. 100.

of the principles. The group receiving intermediate
direction was provided with all the information supplied
the minimum group plus a verbal statement of the principle.
The principles were printed immediately preceding the group
of lists to which they applied. The materials used by the
maximum treatment group included all the clues provided the
intermediate group plus oral statements of the three cor-
rect responses for each group of lists were given to the
subjects.

The training period was five weeks in length with
nine items based on three principles being presented and
repeated twice each week for a total presentation of
forty-five items based on fifteen principles. Analysis
of variance showed that the intermediate and maximum groups
were significantly superior to the minimum group when the
criterion was number of correct responses during the train-
ing period. On a transfer test each group was significantly
different from the other two. The superiority of the groups
was in this order: intermediate, maximum, and minimum. On
the ability to discover new principles the groups were in
this order: intermediate, maximum, and minimum. A retention
test was administered after two weeks and again after four
weeks. On both administrations of the retention test the
groups were in this order: intermediate, maximum, and mini-
um. It was concluded that subjects benefit from help
given them in their search for bases determining correct
responses but specification of answers in advance encourage reliance on rote memory rather than discovering and applying on the basis of underlying relations. It was also concluded that maximum and minimum amounts of direction during learning become less and less effective as the situations to which transfer is made become increasingly different and as the elapsed time between training and measurement of transfer become greater. The intermediate amount of direction increased in effectiveness between the first and second transfer situations and maintained the new level of effectiveness on the third transfer situation.

Using word relationships as the learning task Craig tested the hypothesis that increased direction of discovery activity increases learning without accompanying losses in retention or transfer. Fifty-three sophomore and junior education students were divided into two groups. Each was given a different amount of direction to help them discover the principle satisfied by four words of a list of five words. The subjects of one group were told that a principle existed but were not told what the principle was. This group was called the independent discovery group. The subjects of the other group were provided with a short statement of each principle. This group was called the

directed discovery group. The subjects of this group were required to discover how to apply the stated principle to the list of words.

In the analysis of the results it was found that when the criterion was the number of organizational relationships or principles learned the directed discovery group was superior to the independent discovery group. The retention of the two groups was compared over three different intervals of time, three days, seventeen days, and thirty-one days. On the first two tests there was no difference between the two groups but on the third test the directed discovery group was superior to the independent discovery group. On a test of ability to discover new relationships no difference was found between the two groups. It was concluded that the hypothesis made at the beginning of the study was supported by the results of the experiment.

Underwood and Richardson investigated the effects of level of response dominance and type of instructions on the learning of concepts. Only the results related to type of instruction are reviewed here. The learning task used in this study was similar to, but slightly different from, the learning tasks used in the preceding studies. In the learning task each subject was presented with the
names of four common objects. The task of the subject was to discover what single characteristic can be used to describe all four of the objects.

A total of 144 subjects was used in the experiment. These subjects were enrolled in an elementary psychology course at the time of the experiment. The subjects were divided into three equal groups. Each of the three groups was given a different set of instructions and then presented with the learning task. The subjects in one group were instructed to respond to each list of four words in a free-association fashion during the first presentation of the list. They were further told that it would be a good idea to vary their responses on subsequent presentations of the list until they started to get some responses correct. The subjects of a second group were led to discover the type of responses required before the learning task began. As a result, these subjects had more information than the first group concerning the nature of the learning task. The subjects in the third group were told the correct responses on the first presentation of the lists. They were allowed to study those responses until they could repeat them. Furthermore, the correct responses were printed on a card and the subjects were allowed to study the card during the process of the learning task. The learning task consisted of six lists of four words. Each list was based on a different principle.
After an analysis of test results it was concluded that the greater the amount of information given the subject concerning the nature of the concepts to be learned, the more rapid the acquisition of the concept.

Four studies have been reviewed in which subjects learned relationships satisfied by lists of words. In each study the number of relationships learned, when compared with the number of relationships learned in a typical classroom during an entire semester, was relatively small. In three of the studies all of the instruction was given in a single session. Stacey found that learning by a process of self-discovery is as effective or perhaps even more effective than learning by a process of authoritative identification. Kittell found that when learning by discovery an intermediate amount of direction is most effective, whereas Craig, as well as Underwood and Richardson, found that a maximum amount of direction is most effective.

Studies in Which the Subjects Learned Sums of Series

Several studies have been conducted in which subjects were required to learn formulas for sums of series. The formula given by the equation

\[ 1 + 3 + 5 + \ldots + (2n + 1) = n^2 \]

was used in several of these studies. This type of learning material was used because it is a type of learning material
actually used in the mathematics classroom. Therefore, the results of these studies, it has been argued, should be applicable to typical classroom learning.

Hendrix conducted several exploratory experiments in which the subjects were required to learn the formula given above. In Method I the subjects were given the statement, "The sum of the first n odd numbers is n-square." This rule was then verified for one particular instance. In Method II the subjects were asked to find the sum of the first two odd numbers, the sum of the first three odd numbers, the sum of the first four odd numbers, and so on until the speed of response of the subject indicated that the formula had been discovered. Any subject was excused from the room as soon as he had discovered the formula. In Method III the subjects were given the same directions as the subjects in Method II. In this group, however, when a subject gave evidence that he had discovered the formula he was immediately asked to state the rule he had discovered.

The experiment was repeated three times, once with eleventh-grade boys, once with twelfth-grade boys, and once with college girls. In all three studies only forty subjects were involved altogether. . . . and since some controls on all three runs of the experiment were admittedly

poor, all results must be examined with a view to further experimentation. On the basis of these studies the following hypotheses were offered:

1. For generation of transfer power, the unverbalized awareness method of learning a generalization is better than a method in which an authoritative statement of the generalization comes first.

2. Verbalizing a generalization immediately after discovery does not increase transfer power.

3. Verbalizing a generalization immediately after discovery may actually decrease transfer power.

Using forty-eight educational psychology students as subjects, Kerah compared the effectiveness of three teaching methods in teaching the same rule used in the study reported by Hendrix. One group of subjects, called the no help group, was required to discover the rule with no help. Another group, called the direct reference group, was given some direction in discovering the rule. A third group, called the rule given group, was given a verbal statement of the rule. An initial learning test was administered immediately after instruction. This test was readministered four weeks later. A questionnaire was administered during the same session in which the retest was administered. The responses on this questionnaire

21 Ibid., p. 198. 22 Ibid., p. 190.

reflected a definite difference in motivation among the three treatment groups. Motivation and interest was highest in the no help group. As a result it was concluded that the superiority of the discovery group may best be explained in terms of motivation.

In a second study, in which the same learning material was used, Kersh hypothesized that

... to the extent that the external direction provided to the learner is lessened during the attempts to discover the relationships which are considered essential to the understanding of a cognitive task: (a) the learner will tend to use the learned material more frequently after the learning period (i.e., to extend the practice period voluntarily) and as a result, (b) he will remember it longer and transfer his learning more effectively.  

In order to test this hypothesis ninety high school geometry students were divided into three equal groups. Each group was taught a set of rules for summing series by one of three different techniques. The Directed Learning group was taught the rules and their explanation entirely by a programmed learning technique. The Guided Discovery group was required to discover the explanation of the rules with guidance from the experimenter. They were taught tutorially using a form of Socratic questioning which required each subject to perform specific algebraic manipulations and to make inferences without help. The subjects of the Rote

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Learning group were simply told the rules and given no explanation. Tests of recall and transfer were given after three days, two weeks and six weeks. Each test was given to ten subjects of each group. The tests were administered in such a way that each subject took only one test.

The number of subjects who used the appropriate rule to solve a problem on a test was used as a measure of transfer. The number of subjects who wrote an acceptable statement of the rule was used as a measure of pure retention. It was found that the rate of forgetting did not differ significantly across the teaching treatment groups. The Rote Learning group was found to be consistently superior to each of the other treatment groups on each of the criterion tests. Upon analysis of the questionnaire it was found that the subjects in the Guided Discovery group used the rules significantly more often after the learning period than did the subjects in the other groups. From this it was concluded that learning by discovery may not produce superior learning but it does produce a greater interest in what is learned.

Another study using materials pertaining to the finding of sums of number series was conducted by Gagne’ and Brown.25

tenth-grade boys. Instruction was presented through the use of programed instructional materials. The effectiveness of the various instructional methods used was measured by a test which required the subjects to use concepts learned during instruction in novel situations.

Three methods of instruction with corresponding programed materials were compared. One method, called the rule and example method, presented each concept to be learned through a verbal statement of the concept followed by an example of the concept. This method was compared with two methods which encouraged discovery of verbal principles. One method, called the discovery method, utilized rather large steps in the instructional programs. The other method, called the guided discovery method, utilized approximately the same size steps as used with the rule and example method. The program for the guided discovery and the rule and example methods were constructed in accordance with the principles developed by B. F. Skinner. Both discovery methods require subjects to use previously learned concepts during the process of discovering new concepts. In the rule and example method each concept was presented separately and no special attempt was made to relate a concept with previously learned concepts.

On a transfer test the performance of the three groups was in the following order, from high to low: the guided discovery group, the discovery group, and the rule and
example group. It was concluded that

Discovery as a method appears to gain its effectiveness from the fact that it requires the individual learner to reinstate (and in this sense, to practice) the concepts he will later use in solving new problems. The extent that the G. D. (guided discovery) program was able to identify these concepts, it could then provide systematic practice in their use, and thus lead to a performance superior to that attained otherwise.26

In order to substantiate the findings of Gagne' and Brown, Eldredge27 conducted a replication of their study. The sample for the replication was composed of two ninth-grade algebra classes. One class received instruction through a guided discovery method and the other class received instruction through the rule and example method. In the Gagne' and Brown study all of the subjects were boys. In the replication each class contained thirteen boys and thirteen girls. On a criterion test no significant difference was found between the two experimental groups. This result does not correspond with the results obtained by Gagne' and Brown.

An examination of the programs used in the Gagne' and Brown study led to the conclusion that several variables not vital to the discovery method of learning had not been

26Ibid., p. 320.

27Gabriel M. Della-Piana, Garth M. Eldredge, and Blaine R. Worthen, Sequence Characteristics of Text Materials and Transfer of Learning, Part I: Experiments in Discovery Learning (Salt Lake City, 1965), pp. 4.00-4.42.
controlled. As a result, an improved set of programs was prepared and another study was conducted by Eldredge. In this study the subjects were again ninth-grade students. There were ninety-six subjects in all. The procedures used were the same as those used in the original study with the exception that the instructional programs had been revised. On a criterion test administered immediately after completion of instruction it was found that the performance of the guided discovery group was significantly superior to the performance of the rule and example group. This is in agreement with the results of the Gagne' and Brown study. After the first test was administered twenty-eight of the subjects were given an opportunity to explain orally the process by which they solved the criterion problems. Then a second criterion test was administered. On this test the performance of those who had an opportunity to verbalize the processes used in problem solving was superior to those who were not given this opportunity. This result is in contradiction to the hypothesis offered by Hendrix.28 After four weeks a retention test was administered. No treatment differences were found on this test.

Five studies have been reviewed in which subjects were required to learn formulas for finding sums of series. In each of these studies, with the exception of the second

study conducted by Kersh, it was concluded that some form of discovery method of teaching is more effective than a direct method of teaching.

Studies Conducted in the Elementary School Mathematics Classroom

The remaining studies which are reviewed are studies that have been conducted in typical classroom settings and that have utilized the learning materials that were regularly taught in these classrooms. In this section those studies that have been conducted in elementary mathematics classrooms are reviewed.

One of the very first studies in which a discovery method of teaching was compared with an exposition method of teaching was conducted by T. R. McConnell in 1934. McConnell investigated the effects of two teaching methods in teaching the one hundred basic addition and the one hundred basic subtraction facts. In Method A the number combinations were learned by sheer repetition. A studious effort was made to keep the child from discovering or verifying the answers to the number combinations. If a child made an error during practice the teacher immediately


corrected him by supplying the correct answer. In Method B all the number combinations were introduced through concrete situations. Each one of the two hundred number facts was presented meaningfully through the use of pictured situations. The child was expected to see through an active process of discovery and verification that the number combinations are true and reasonable. The child was expected to discover the answer to each of the number combinations.

Four hundred forty-one second-grade pupils served as subjects in an experiment designed to compare the effects of Teaching Method A and Teaching Method B. These subjects were selected from existing classes but the selection of subjects was made in such a way that matched pairs existed with one subject from each pair in each treatment group. Special materials were prepared for each group and extensive manuals were prepared for the teachers participating in the experiment. These manuals prescribed the conditions under which the Instructional materials were to be used. Three sets of interpolated tests were given during the experiment and seven tests were given at the close of the experiment. These tests were designed to measure accuracy, speed, transfer, ability to solve verbal problems, ability to detect errors, ability to learn new skill from a silent reading lesson, and maturity in manipulating number facts.
It was found that the group taught by Method A was definitely superior to the group taught by Method B on tests of speed but the group taught by Method B was definitely superior on tests of maturity in handling number facts and tests which put a premium on deliberate work and thoughtful manipulation. The group taught by Method B was superior on all transfer tests although only one of the differences was statistically significant. It was concluded that the results of the experiment did not support the extensive claims made for the superiority of teaching methods similar to Method B.

Thiele\textsuperscript{31} conducted a study which was an attempt to improve on the study conducted by McConnell. The drill method used in this study was similar to Method A as described by McConnell. In the generalization method the students were not only required to discover the number facts but they were required to discover relationships among the number facts as well. In this study, when the criterion was knowledge of the basic one hundred addition facts, it was found that the group taught by the generalization method was significantly superior to the group taught by the drill method. This superiority held for subjects of all levels of ability. When the criterion

test was a test of transfer of training it was again found that the generalization group was superior to the group taught by the drill method. It was concluded that the achievements of the pupils taught by the generalization method were greatly superior to those attained by the pupils taught by the drill method.

Another study in which second-grade pupils were taught the one hundred basic addition facts was conducted by Swenson. The purpose of this study was to compare the effects of three different methods of instruction on initial learning, transfer of training, and retroactive inhibition. The chief variable among the three methods of learning was degree of emphasis upon organization and generalization in the learning process. In one method of instruction, the generalization method, the addition facts were presented to the children in groups which were determined by a basic unifying idea or generalization. The facts which centered around a generalization were presented together in such a way that the teacher could, by skilful instruction, lead the pupils to their own formulation of the generalization. Children taught by this method were allowed to refer to concrete objects as often as they

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needed as aids in solving the abstract number combinations. In another method of instruction, the drill method, each addition fact was presented as a separate fact to be memorized. No attempt was made to present the facts in an organized fashion and the children were discouraged from reasoning out answers to number combinations. The children were told the answers to the number combinations and if at any point a child hesitated at giving an answer the teacher immediately supplied the correct answer. In a third method, called the drill-plus method, some attempt was made to make the addition facts meaningful. At the time of its introduction each number fact was presented by pictures of concrete objects. The children were also given the opportunity to verify the addition facts by manipulating concrete objects. In this method a limited amount of organization was used in determining the order of presentation of the addition facts.

A test on the one hundred addition facts was administered five times: once before the experiment began, again after the completion of an initial set of addition facts, again after the completion of a second set of facts, after the Christmas vacation, and again after the completion of the final set of facts. During the three close periods after the fifth administration of the addition test three transfer tests were administered. It was concluded that the evidence obtained from this study lends support to
the hypothesis that organization and generalization during the learning process are decisive factors in facilitating transfer. It was also concluded that when the learning results are viewed as a whole the generalization method is superior to the drill method and the drill-plus method. It was also found that the teaching method that was accompanied by the most transfer was also attended by the least retroactive inhibition.

In a study conducted in third-grade arithmetic classrooms, Fullerton compared the effects of a conventional teaching method and an experimental teaching method on the learning of the multiplication facts for whole numbers. In the experimental method students were led to discover the need for the particular phase of arithmetic to be taught. This was usually done through verbal problems. Once the problem was identified several solutions were suggested. The best method, based on the arithmetic background of the students, was then selected. Rules or statements of procedures were formulated by the pupils, either individually or as a class. These statements of rules or procedures were made after the pupils had gained experience with the procedure.

An experiment was conducted in order to compare the relative effectiveness of the two teaching methods. The subjects for the experiment were approximately 770 third-grade students in 28 classes. The regular classroom teacher taught the lessons and administered all of the tests. The material was taught in eight lessons. A pretest was given before the first lesson was taught and a posttest was given after the eighth lesson. Another test was given three and one-half weeks later. On the posttest there was a significant difference in favor of the experimental group in ability to recall the facts that were taught. There was also a significant difference in favor of the experimental group on a test of transfer of training. On the delayed test the experimental group maintained its superiority in terms of transfer of training. No attempt was made to compare the two groups with respect to ability to recall learned facts.

Anderson compared the effectiveness of two methods of teaching fourth-grade arithmetic. One method of teaching was based on connectionist theories of learning and the other teaching method was based on field theories of learning and emphasized understanding and generalization. The experiment was conducted in eighteen fourth-grade arithmetic classes. Each class was taught by a different teacher.

Ten classes were taught by the method based on connectionist theories. This method was called the drill method and the teacher spent approximately eleven minutes each day on instruction and twenty-four minutes each day on drill. Eight classes were taught by the method based on field theories. This method was called the meaning method. In these classes the teachers spent approximately twenty-seven minutes a day on instruction and eighteen minutes on drill.

Pretests for the study were made in November and the experimental teaching methods were initiated after the pretests had been administered. The instruction terminated the following May. At this time the posttests were administered. Unit tests were administered by the teachers throughout the instructional period. Tests of addition, subtraction, multiplication, division, and understanding of social concepts in arithmetic showed insignificant differences between the two treatment groups. The performance of the group taught by the meaning method was superior to the performance of the group taught by the drill method on a test of computational skill, although the significance of this difference was borderline. On a test of mathematical thinking ability the performance of the group taught by the meaning method was significantly superior to the performance of the group taught by the drill method. A questionnaire was given at the beginning and again at the end of the experimental period. This
questionnaire purported to measure the degree to which pupils liked arithmetic. An analysis of the responses to the questionnaire showed that the pupils taught by the two methods did not differ significantly in their rating of arithmetic as compared to other school subjects.

In sixteen sixth-grade mathematics classes, Worthen\(^\text{35}\) compared a guided discovery method of teaching with an exposition method of teaching. Eight of the classes were taught by the guided discovery method and eight of the classes were taught by an exposition method. Eight teachers participated in the study and each teacher taught one class by each of the teaching methods. Instruction was through "quasi-textual instructional programs." These programs presented the following mathematical concepts: (1) notation, addition, and multiplication of integers (positive, negative, and zero); (2) the distributive principle of multiplication over addition; and (3) exponential notation and multiplication and division of numbers expressed in exponential notation. These topics were presented during six weeks of instruction.

In order to determine the relative effectiveness of the two teaching methods four subtests were given during the instructional period. Each subtest was designed to

correspond with the instructional material completed just prior to the administration of the test. A test dealing with all the instructional material was administered twice, once five weeks after instruction and again eleven weeks after instruction. Transfer of heuristics was measured by two tests and pupil attitude toward mathematics was assessed by two attitude scales.

An analysis of data showed that the discovery method did not produce superior results in terms of initial learning but did produce superior results in retention five weeks after instruction. On the test of retention eleven weeks after instruction the two treatment groups were not significantly different. The results from a test on concept transfer showed no significant difference between the two groups. The results from the attitude scales showed no significant difference in attitude toward mathematics between the group taught by a discovery method and the group taught by an exposition method. It was found that the discovery group was superior to the exposition group in ability to transfer heuristics. It was concluded that, in general, the findings of this study supported many of the claims made by proponents of discovery methods of teaching.

In this section five studies conducted in elementary mathematics classrooms have been reviewed. In each of the studies, with the exception of the study conducted by
McConnell, some evidence was found which supported the hypothesis that discovery or generalization methods of teaching are superior to exposition or drill methods of teaching.

Studies Conducted in the Junior and Senior High School Mathematics Classroom

In this section eight studies that have been conducted in junior and senior high school classrooms are reviewed. Six of these studies deal with the learning of mathematics while the remaining two deal with content learned in technical education.

Michael compared two methods of teaching positive and negative numbers, the fundamental operations with them, and the solution of simple equations in ninth-grade algebra. In Method A the subjects were led to discover and understand the fundamental principles and relationships to be learned through exercises built around familiar situations such as those dealing with time, money, temperature, and others. No statement of rules of operation was made by the teacher or pupils in teaching or in pupil discussions. Method B emphasized the use of authoritative statements of the rules of operation combined with extensive practice or

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drill. No attempt was made, before practice began, to explain why the rules operated to give the correct results.

The students in fifteen ninth-grade algebra classes were used as subjects. Fifteen different teachers participated in the study. Tests were given to measure computational ability and ability to generalize. A questionnaire designed to measure attitude toward algebra was also administered. These tests were used as pretests and as posttests. The results were analyzed through the use of analysis of covariance. The analysis indicated that Method B produced significantly greater gains in computational skill and ability to make and use generalizations. Method A produced better results in attitude favorable to algebra, the subject of immediate interest.

In another study conducted with ninth-grade algebra students Sobel37 compared an abstract, verbalized, deductive method of teaching in which concepts were defined and presented by the teacher, followed by practice exercises with a concrete, nonverbalized, inductive teaching method in which students were guided through experiences involving applications, to discover and verbalize concepts. The experiment was conducted during the first four weeks of the school year.

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Special criterion tests were developed for the experiment during two pilot studies. After revisions were made two forms of the tests were completed. Each test contained two parts. One part dealt with concepts and the other part was designed to evaluate fundamental skills. On this test it was found that the performance of the subjects with high ability and taught by the inductive experimental method was superior to the performance of the subjects with high ability and taught by the deductive control method. For subjects with average ability there was no difference in performance between the subjects taught by the two different methods. Three months after the original test was given, a test was administered to the same subjects in order to determine the effect of method of presentation upon retention of concepts. On this test it was again found that the performance of the subjects of high ability and taught by the inductive experimental method was superior to the performance of those subjects of high ability and taught by the deductive control method. For subjects with average ability there was no difference due to teaching method used.

Wolfe investigated the problem of whether an abrupt change to expository methods will have any adverse effects.

on the performance of the student who has been accustomed
to learning by discovery methods. The sample for the study
was taken from a population of secondary school students
who were taught mathematics from the University of Illinois
Committee on School Mathematics materials during the pre-
ceding school year. Three hundred students were chosen
from classes in three junior high schools and two senior
high schools. One hundred sixty of the subjects were in
the ninth-grade and one hundred forty of the subjects were
in grade ten. Within a given class, one-half of the sub-
jects received expository instruction and one-half of the
class received instruction utilizing a discovery method.
Instruction was given through self-teaching programmed
materials. Each program contained approximately ten
pages of instructional materials. On the day after the
programs had been completed by the students an achievement
test was administered. On the next day a transfer test
was administered. No significant difference was found
between the group receiving expository instruction and
the group receiving instruction through a discovery method
with respect to achievement and with respect to ability to
transfer knowledge to new situations. It was concluded
that a change to expository instruction for students who
are accustomed to discovery methods will not seriously
interfere with the ability of students to learn mathe-
nematics.
Howitz conducted a study in which the relative effectiveness of a guided discovery method of teaching and a conventional method of teaching were compared. The subjects used in the study were 290 ninth-grade general mathematics students. Most of the subjects could be considered "slow learners." Seven teachers participated in the study. Each teacher taught one class using the guided discovery method of teaching and one class using the conventional method of teaching. A textbook was designed to be used in the classes taught by the guided discovery method. The content of this textbook was not similar to the content of traditional general mathematics textbooks. The textbook used in the classes taught by the conventional method basically presented a review of arithmetic. The study was conducted throughout an entire school year.

Pretests and posttests were given and the results were analyzed using the analysis of variance and analysis of covariance techniques. For each test a three-way analysis was used with teacher, treatment, and sex of subject as the three dimensions. On a standardized achievement test there was no significant difference in achievement as a result of the teaching methods. On a test specifically designed to test achievement of topics taught by the guided discovery method.

method the students taught by the guided discovery method showed superior achievement as compared to those taught by a conventional method. No difference was found between the two treatment groups in terms of attitude toward mathematics.

A second study utilizing general mathematics students as subjects was conducted by Price. The study was conducted in three tenth-grade classes. One class, called the control group, was taught by the traditional textbook-lecture-recitation method using the normal course of study for this class. The two other classes, called the experimental group, were taught by a technique designed to foster discovery. Before the study began the students were assigned to the three classes by a random process. The students in the experimental classes followed the same basic course plan as the control classes except for some additional topics that were treated in those classes. One of the experimental classes also made use of specially prepared transfer material. These exercises were used to provide extra experience with problems in critical thinking in non-mathematical areas.

Criterion tests were standardized tests designed to measure mathematical and thinking ability. These tests were first given to students during the first week of

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school. The tests were administered again after one semester of instruction. In addition a questionnaire concerning attitude toward the class was given. From the results of this testing program it was concluded that there was no difference in achievement among the three classes. The experimental groups showed a greater increase in mathematical reasoning than the control group, but the difference was not statistically significant. The experimental group receiving special transfer material showed a significant increase in critical thinking ability. Finally the groups taught by techniques designed to promote discovery had a positive change in attitude toward mathematics but the group taught by the traditional method showed a negative change.

Nichols attempted to assess the relative effectiveness of two teaching methods in teaching high school geometry. One method was similar to what is usually called a traditional method and the other method was a type of discovery method. In order to assess the relative effectiveness of these methods an experiment was conducted in which forty-two freshmen at the University of Illinois High School served as subjects. The criterion test was given as a pre-test two months before the study began. On the basis of the scores on this test and an intelligence test the

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Subjects were divided into two matched groups of equal size. Instruction was given during seventeen class sessions of sixty minutes each. The two groups were taught alternately by three teachers in order to randomize the teacher effect. At the end of the teaching period the criterion test was again administered. This test was designed to measure knowledge of vocabulary, critical thinking ability, ability to solve problems, and acquisition of fundamental skills. No significant difference was found between the group taught by the traditional method and the group taught by the discovery method with respect to the abilities measured by the criterion test.

A series of studies in which subjects were taught technical skills have been conducted. Although such studies may not be directly related to the learning of mathematics they do provide some evidence concerning the effectiveness of discovery methods of teaching. Since the extent to which the results of these studies generalize to the learning of mathematics is questionable only two of these studies are reviewed here.

In order to provide additional experimental and applied research evidence as to the relative effect of using a guided discovery method of teaching in situations providing numerous problem solving opportunities, Ray conducted a

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study in which the learning task consisted of learning micrometer measurement principles. A guided discovery method of teaching these principles was compared with a traditional direct and detailed method of instruction.

The subjects used in this study were 135 ninth-grade boys. Fifty-four subjects were taught by a directed discovery method, fifty-four were taught by a traditional direct and detailed method, and a third group of twenty-seven subjects, called the control group, received no instruction. The subjects in each group were classified according to three levels of mental ability. The two experimental groups received forty-seven minutes of instruction in one instructional session.

Three criterion tests were given. One test was given immediately after the instructional session. The second test was given one week later and the third test was given six weeks after instruction. The first test was designed to measure initial learning. The second and third tests were designed to measure retention and transfer. The results were analyzed using an analysis of variance with treatments by levels technique.

Analysis of data revealed no significant difference in initial learning due to the experimental teaching methods. There was no significant difference in retention after one week, but six weeks after instruction there was a significant difference in retention in favor of the group taught
by the directed discovery method. It was found that the group taught by the directed discovery method was significantly superior to the group taught by the traditional method when the criterion was performance on tests designed to measure ability to transfer learning to new situations. This was true for the transfer test given one week after instruction and for the test given six weeks after instruction. A hypothesis of no interaction between the two teaching methods and level of mental ability was tested and accepted in all cases. It was concluded that a directed discovery approach to teaching is superior to direct and detailed instruction with respect to retention of material initially learned and with respect to enabling pupils to make wide applications of material learned to new and related situations.

Moss attempted to formulate a teaching method based on the teachings of gestalt psychology. The teaching method developed was essentially a guided discovery method of teaching. In order to determine the effectiveness of this teaching method it was compared with a more traditional method of teaching. The subjects used in the comparison study were 131 junior and senior students in vocational-industrial curricula at a technical high school. The

43 Jerome Moss, Jr., "An Experimental Study of the Relative Effectiveness of the Direct-Detailed and the Directed Discovery Methods of Teaching."
learning task involved the learning of a highly organized non-manipulative learning task, letterpress imposition. The scope and complexity of this material does not differ greatly from many typical classroom lessons in which technical material is presented to a group of students.

The subjects were randomly assigned to three groups. One group was taught by the guided discovery method, one group was taught by the traditional method, and the third group received no instruction. Instruction was given to two of the groups during one instructional session. All instruction was presented with the use of a tape recorder in order to insure constancy of presentation. A test of initial learning was administered immediately after the learning period. Retention and transfer tests were given one and six weeks after instruction.

The results showed that the performance of the two groups receiving instruction was superior to the performance of the group receiving no instruction. The results showed further that there was no significant difference between the group receiving instruction by a guided discovery method and the group receiving instruction by a traditional method with respect to initial learning, retention, and transfer of training.

In this section six studies have been reviewed in which discovery methods of teaching junior or senior high-school mathematics were compared with traditional methods.
of teaching. Wolfe and Nichols found that students learned equally well under discovery methods of teaching and exposition methods of teaching. Howitz and Price reported studies in which students taught by discovery methods were superior to students taught by traditional methods, at least in certain facets of learning. That this superiority was actually a result of the different teaching methods is difficult to determine because the students taught by a discovery method actually learned a different set of facts than the students taught by an exposition method. As a result the findings of these studies should be viewed with caution. Sobel found that, with students of above average ability, teaching by a discovery method produced better results than those obtained when teaching by a traditional method. In contrast, Michael found that an exposition method of teaching produced superior results.

Two studies which were conducted with technical education students were reviewed. Ray found that a discovery method of teaching was better than a direct and detailed method of teaching, whereas Moss found that students learned equally well when either teaching method was used.

Studies Conducted in the College
Mathematics Classroom

Two studies have been conducted in which discovery and exposition methods of teaching mathematics to college
students have been compared. Both studies were conducted with college freshmen as subjects.

Weiner investigated the relative effectiveness of two teaching techniques in developing the functional competence of college students in a first semester course in mathematics. The subjects were students enrolled in a community college and who were in the same curriculum, electrical technology. Only male students were used as subjects. Two classes were used as a control group and two classes were used as an experimental group. Each class contained approximately twenty-two students. The teaching method used with the control group consisted, essentially, of starting with an explanation of the general concepts and then proceeding to illustrate and apply these concepts to various exercises. In the teaching method used with the experimental group attempts were made to lead up to the concepts to be learned through problematic situations. The instructional period consisted of four fifty-minute class sessions per week for approximately sixteen weeks. The same sequence and time schedule was used by all classes. There was no random assignment of subjects to classes, but an examination of scores

achieved by the subjects on several tests given before the experiment was begun indicated no significant difference between the treatment groups.

At the end of the instructional period the test, Davis Test of Functional Competence in Mathematics, was administered as a criterion test. No significant difference was found between subjects of high ability in the two treatment groups. It was found that the performance of the students of average ability in the experimental group was superior to the performance of the students of average ability in the control group. It was concluded that functional competence in mathematics can be improved by a teaching method based upon a problem-centered technique and students of weak to average mathematical background show the greatest improvement in functional competence when taught by such a technique.

Cummins developed a series of study-guide sheets which could be used in a beginning course in polynomial calculus. These materials were designed to develop understanding in the use of some of the fundamental ideas before these concepts were subjected to critical discussion by the class or before the results suggested as hypotheses were finally shown to be true by deductive methods. These

materials were designed to create an atmosphere in which the students could discover generalizations which could be verified deductively after the students had become familiar with them.

During each of two quarters one class was taught using these materials in addition to the standard textbook and another class was taught by the traditional method using only the textbook. Two teacher made tests were used as pretests and posttests. After a statistical analysis was made of the test scores it was concluded that the performance of the experimental group was significantly superior to the performance of the group taught by the traditional method. According to Cummins, "The results indicate that the method of teaching under examination was especially effective in promoting a deeper understanding of the calculus and that this gain was not at the sacrifice of proficiency in manipulations and applications."\(^{46}\)

In this section two studies conducted with college mathematics students have been reviewed. Weiner found that a problem-centered technique of teaching was effective in increasing the functional competence of students with average and below average ability. Cummins found that a discovery method of teaching polynomial calculus was effective in promoting understanding. The results of the study

\(^{46}\textit{Ibid.}, \text{ p.} \ 168.\)
by Cummins do not necessarily provide evidence supporting discovery methods of teaching because the students in the experimental group actually learned a different set of facts than the subjects in the control group.

Summary

The purpose of this chapter was to present a comprehensive summary of research literature related to discovery methods of teaching. The studies have been presented in groups with the organizing basis for the groups being the type of learning material used in the study.

In the first section the studies of George Katona and three related studies were reviewed. The studies conducted by Katona and the study reported by Hilgard, Irvine, and Whipple presented evidence indicating that teaching methods emphasizing understanding and pupil discovery are superior to teaching methods emphasizing rote memorization. Gorman found that appropriate guidance is better than no guidance and leaving the student to discover for himself the solution to a problem will not prevent understanding, but probably will delay it. Scandura found that expository methods of teaching are just as effective as and in some cases more effective than discovery methods of teaching.

In the second section three studies were reviewed in which the learning tasks involved learning methods of deciphering codes. Two of the studies indicated that learning
through discovery is no more effective than learning by memorization.

A group of four studies was reviewed in which subjects were required to learn word relationships. Two of the studies indicated that learning is most efficient when a maximum amount of direction is given during instruction. In the other two studies it was concluded that some form of learning by discovery is more effective than learning by a method in which all necessary information is identified by the instructor.

In the fourth group of studies five studies were reviewed in which sums of series were learned. In four of these studies it was concluded that a discovery method of teaching is more effective than a direct method of teaching.

Five studies were reviewed which reported the results of experiments conducted in elementary school mathematics classrooms. In four of the five studies some evidence was found which supported the hypothesis that learning by discovery is more effective than learning by a drill method.

In the sixth section eight studies conducted in junior and senior high school classrooms were reviewed. In three studies it was found that students learned equally well under discovery methods of teaching and exposition method of teaching. Four studies reported conclusions in favor of discovery methods of teaching. A sixth study found that
an exposition method of teaching produces results superior to those obtained when a discovery method is used.

In a final section two studies were reviewed which were conducted at the college level. In each study it was found that a form of discovery method of teaching is more effective than a traditional method.

It can be seen that, although agreement is not unanimous, the majority of the studies purport to provide evidence supporting the claim that learning by discovery is more effective than learning by memorization, drill, or authoritative identification. Further examination reveals that the superiority of learning by discovery is more likely to be revealed when the criterion is retention, transfer of training, or motivation than when the criterion is initial learning. Caution should be used when interpreting the results of these studies because the teaching methods used in the studies vary widely. What is called the discovery method of teaching in one study is not necessarily the same teaching method as the teaching method labelled with the same name in another study. In some instances what is called a discovery method of teaching in one study is called an exposition method of teaching in another study. After reviewing research literature related to discovery methods of teaching, Kersh concluded that...

... as is true in other areas of research, the evidence is somewhat equivocal, partly because it is difficult to equate studies in terms of the
amount and kind of direction that is provided. The experimental subjects rarely if ever are required to learn completely without help, and the kinds of help provided commonly differs.47

Similarly Worthen has stated that

To date these studies have failed to clarify many of the questions pertaining to discovery and expository instruction; rather, the findings of the various studies, when taken at face value, often seem to be contradictory. Perhaps the greatest factor which contributes to such equivocal research evidence is the differing specification among researchers as to what they mean by such terms as "discovery," "guided discovery," and "exposition." Since these terms have not yet been reduced to generally accepted operational definitions, it is highly probable that researchers working in what is nominally the same domain are not actually investigating the same phenomena at all.48

From such statements it can be seen that when attempting to interpret the results of studies on discovery methods of teaching it is imperative that the characteristics of each teaching method used be identified. Some of the characteristics of specific class procedures that should be considered are amount of information given as guidance, type of information given as guidance, "timing" of information given or sequence of presentation of information, and meaningfulness of the information given. More detailed analyses of some of the studies reviewed in this chapter


have been written by Wittrock,\textsuperscript{49} Cronbach,\textsuperscript{50} and Becker and McLeod.\textsuperscript{51}


\textsuperscript{50}Lee J. Cronbach, "The Logic of Experiments on Discovery," Learning by Discovery, pp. 76-92.

CHAPTER III

EXPERIMENTAL DESIGN AND EXPERIMENTAL PROCEDURES

Introduction

From the preceding chapter it can be seen that the results of experimental studies related to discovery methods of teaching are not wholly conclusive. As a result, generalizations concerning discovery methods of teaching must be accepted with caution. In order to provide further evidence concerning the effectiveness of discovery methods of teaching in an area where few investigations have been made, an experimental study was conducted. The purpose of this study was to ascertain the value, as determined by student achievement, of using a guided discovery method of teaching in a college freshman mathematics course for non-mathematics and non-science majors. The purpose of this chapter is to describe the study that was conducted. The setting of the study, the experimental design used in the study, the teaching methods compared, the testing program conducted in order to evaluate the relative effectiveness of the teaching methods, and the procedures used in analyzing the test data are described in this chapter.
The Setting of the Study

The purpose of this section is to describe the school at which the experiment was conducted, the subjects used in the experiment, and the course of study in which those subjects were enrolled. The experiment was conducted during the spring semester of the 1967-1968 school year at Southwestern State College, Weatherford, Oklahoma. Weatherford, Oklahoma has a population of approximately 6,000 and is located in western Oklahoma. Weatherford is a rural community, with most of its residents having an agricultural background.

Southwestern State College is a state-supported and fully accredited four-year college. The enrollment at Southwestern State College during the spring term of the 1967-1968 school year was approximately 4,300. Complete programs of study are available in many major areas. The college includes a reputable school of pharmacy and a graduate school which offers the Master of Teaching degree. The college was originally conceived as a teachers college; although liberal arts degrees are available in several fields of study, the majority of the students are still enrolled in the school of education. Many of the students who attend Southwestern State College live in rural communities in western Oklahoma and commute to school daily.

The subjects for the study were students enrolled in four of the fourteen sections of College Mathematics taught
at Southwestern State College during the spring semester of the 1967-1968 school year. College Mathematics is a required course for all students who do not complete College Algebra and Trigonometry as a part of their program of study. Students who do not normally enroll in College Mathematics are those majoring in mathematics, physics, chemistry, biological science, and students enrolled in the school of pharmacy. All other students are required to complete College Mathematics. College Mathematics has a twofold purpose. The course is designed to serve as an introduction to mathematics for liberal arts students and it is also designed to serve as one of two mathematics courses required of all prospective elementary-school teachers. A detailed outline of the material taught in the four sections of College Mathematics utilized in the study is presented in Appendix A.

The Experimental Design

In this section the nature of the experimental study conducted is presented and the procedures for executing the experiment are described. The experiment conducted was designed to conform to the model for a two by three factorial experiment. The two independent variables were method of teaching and level of ability. The dependent variable was achievement.
In the design of the experiment and in the analysis of data three levels of ability were considered. In order to determine the level of ability of each subject the subject's composite score on the American College Testing Program was utilized. The levels of ability were determined on the basis of the composite scores on the American College Testing Program of all students enrolled in College Mathematics during the spring semester of the 1966-1967 school year. Scores were available for 78.9 percent of these students. On the basis of these scores it was determined that students with a composite score of twenty or above ranked above the sixty-sixth percentile while students with a score of fifteen or below ranked below the thirty-third percentile. As a result, in the experiment Level I consisted of all subjects with a composite score on the American College Testing Program of twenty or above. Level II consisted of all subjects with a composite score higher than fifteen but lower than twenty and Level III consisted of all subjects with a composite score of fifteen or lower. In the analysis of data only the criterion scores of those subjects enrolled in the four sections of College Mathematics utilized in the study and for whom scores on the American College Testing Program were available were used. Composite scores were available for 79.3 percent of the students enrolled in the four sections of College Mathematics.
For the method of teaching variables there were two levels. One method of teaching was called the Guided Discovery Method and the other method of teaching was called the Exposition Method. The nature of each teaching method is described in a separate section. Of the four sections of College Mathematics utilized in the study two sections were taught by the Guided Discovery Method and two were taught by the Exposition Method. Two of the four sections of College Mathematics met from 10:00 a.m. to 10:50 a.m. on Tuesday, Thursday, and Friday during each week of the semester. One of these sections was taught by the Guided Discovery Method and the other section was taught by the Exposition Method. The remaining two sections of College Mathematics met from 2:00 p.m. to 2:50 p.m. on Monday, Tuesday, and Thursday of each week of the semester. One of these sections was taught by the Guided Discovery Method and the other section was taught by the Exposition Method. For both the morning classes and the afternoon classes the method to be used in a particular class was determined by the flip of a coin. The ten o'clock and two o'clock time periods were chosen in order to minimize the effects of time of day on learning. These time periods were chosen in order to avoid the depressing effects on learning of early time periods, late time periods, and time periods near noon. A time period for both morning and afternoon
was chosen in order to achieve a balance of effects of morning and afternoon time periods.

As a criterion for determining the relative effectiveness of the two teaching methods a single dependent variable was considered. This dependent variable was student achievement. Student achievement was measured by four tests. Three of these tests were standardized tests and the fourth test was a teacher-made test and was administered in several parts. A detailed description of the tests used and the methods and time of administration of these tests is given in a separate section.

Two instructors participated in the study. Both instructors were regular members of the staff of the mathematics department at Southwestern State College. Both instructors had several years experience as college teachers and both instructors were familiar with discovery methods of teaching. One of the instructors has participated in an institute for secondary mathematics teachers conducted at the University of Illinois. Max Beberman, one of the foremost advocates of discovery methods of teaching mathematics, was the director of this institute. In order to balance the effect of the instructor upon student achievement, each instructor taught one section of College Mathematics by the Guided Discovery Method and one section by the Exposition Method. In order to balance the effect of time of day each instructor taught by one method in the morning and by the
other method in the afternoon. The method of teaching to be used by each instructor at a given time period was determined by a random process. The restrictions on randomization were such that each teaching method was used during each time period and each instructor taught by each teaching method. Instructor A taught using the Guided Discovery Method in the morning and the Exposition Method in the afternoon. Instructor B taught using the Exposition Method in the morning and the Discovery Method in the afternoon.

In order to provide a means for a statistical comparison of the relative effectiveness of the two teaching methods used in the study, it was necessary to assign the subjects to the teaching methods through a random process. Special arrangements were made so that this would be possible. In the spring schedule of classes at Southwestern State College three sections of College Mathematics were listed under the 10 a.m. time period. These classes were listed as sections two, four, and five. All of these classes were listed as being taught at the same time and on the same days. No instructors were listed for sections four and five. During enrollment all students enrolling in either section four or section five were given a class card for section five and were instructed to meet their first class in the room listed for section five. The names of all of the students enrolling in these classes were obtained.
during enrollment. Three sections of College Mathematics were also listed under the 2 p.m. time period. They were sections seven, eight, and nine. No instructor was listed for sections eight and nine. During enrollment all students enrolling in either section eight or section nine were given a class card for section eight and were instructed to meet their first class in the room listed for section eight. The names of all students enrolling in sections eight and nine were obtained during enrollment.

Before the first day of regular classes at the beginning of the semester, the records of all students enrolling in sections four, five, eight, and nine of College Mathematics were checked. Of the 245 students enrolled in these sections, scores on the American College Testing Program, (A.C.T.), were available for 115 students. All the students were divided into four categories according to composite scores on the American College Testing Program. Group one consisted of all students with a composite score of twenty or above. Group two consisted of all students with a composite score less than twenty but more than fifteen. Group three consisted of all students with a composite score fifteen and less and group four consisted of all students for whom no scores were available. Each of these four categories were then divided into two categories, one category for those enrolled in a morning class and one category for those enrolled in an afternoon class. Each of the resulting eight
categories was then divided into two categories, one category for male students and one category for female students. Through this process each student was placed in one of sixteen categories. The number of students in each category is given in Table I.

**TABLE I**

**CLASSIFICATION OF SUBJECTS AT THE BEGINNING OF THE EXPERIMENT**

<table>
<thead>
<tr>
<th>A.C.T. Composite Score</th>
<th>Time of Day</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10 a.m.</td>
</tr>
<tr>
<td>20 and Above</td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>8</td>
</tr>
<tr>
<td>Female</td>
<td>5</td>
</tr>
<tr>
<td>16 through 19</td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>17</td>
</tr>
<tr>
<td>Female</td>
<td>9</td>
</tr>
<tr>
<td>15 and Below</td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>16</td>
</tr>
<tr>
<td>Female</td>
<td>7</td>
</tr>
<tr>
<td>No Score</td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>10</td>
</tr>
<tr>
<td>Female</td>
<td>8</td>
</tr>
<tr>
<td>Total</td>
<td>63</td>
</tr>
</tbody>
</table>

**Total Number of Subjects** 115

After all the students were classified, the students in each category were assigned to one of the two classes to be taught during the time period for which they had
process and the students in each category were assigned separately. The random assignment of subjects to classes was accomplished by assigning a number to each student. These numbers were placed on slips of paper and these slips of paper were placed in boxes labeled with the appropriate categories. Then, the two instructors took turns drawing slips out of the various categories. The categories were numbered and it was determined by the flip of a coin which instructor would draw a number from the first category. After the first draw, the instructors drew numbers alternately from the first category until all numbers had been drawn. The instructor who had not drawn the last number from the first category drew the first number from the second category. In general, numbers were drawn from a category until all numbers had been drawn. The instructor who had not drawn the last number from the category drew the first number from the next category. After all numbers had been drawn, class rolls were constructed for the two morning classes and the two afternoon classes. Then, by the flip of a coin it was determined which instructor should teach using the Guided Discovery Method in the morning and which instructor should teach using the Exposition Method in the morning.

Two very similar classrooms were used for the morning classes and two very similar classrooms were used for the
enrolled for one of the classes that met at 10 a.m. were all instructed to meet in the same classroom. The students who enrolled in one of the 2 p.m. classes were given similar instructions. During the first day of class the class rolls for each class were read and the students were told in which room their class was to meet. After the class rolls were read, the classes divided and from then on the classes met in separate rooms.

During the semester several students were excessively absent and it became necessary to drop them from the class in which they had been enrolled. Furthermore, composite scores on the American College Testing Program were not available for all of the students who enrolled in the four sections of College Mathematics used in the experiment. As a result, the number of students whose scores were used in the analysis of data was actually smaller than indicated by Table I. The distribution of the students whose scores were used in the final analysis of data is given in Table II.

At the beginning of the experiment, all subjects were assigned to classes through a random process. Therefore, except for chance differences, the classes should not have been significantly different from each other in terms of factors which could affect achievement. This similarity of classes could have been affected by the fact that eleven students for whom composite scores were available dropped from the classes during the semester. Of those that dropped
### TABLE II

**CLASSIFICATION OF SUBJECTS WHOSE SCORES WERE USED IN THE ANALYSIS OF DATA**

<table>
<thead>
<tr>
<th>A.C.T. Composite Score</th>
<th>Class</th>
<th>10 a.m. Instructor A Discovery Method</th>
<th>10 a.m. Instructor B Exposition Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 and Above</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>4</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>Female</td>
<td>3</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>16 through 19</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>7</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>Female</td>
<td>4</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>15 and Below</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>6</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>Female</td>
<td>4</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Total</td>
<td>28</td>
<td></td>
<td>30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2 p.m. Instructor A Exposition Method</th>
<th>2 p.m. Instructor B Discovery Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 and Above</td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>3</td>
</tr>
<tr>
<td>Female</td>
<td>3</td>
</tr>
<tr>
<td>16 through 19</td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>4</td>
</tr>
<tr>
<td>Female</td>
<td>6</td>
</tr>
<tr>
<td>15 and Below</td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>5</td>
</tr>
<tr>
<td>Female</td>
<td>3</td>
</tr>
<tr>
<td>Total</td>
<td>24</td>
</tr>
</tbody>
</table>

Total Number of Subjects: 104
nine were males and two were females. From this it can be seen that these students did not drop from class through a random process. In order to determine whether there is any reason to believe that the group consisting of all students for whom composite scores were available, who did not drop from class, and who were taught by the Guided Discovery Method was similar to the group consisting of all students for whom composite scores were available, who did not drop from class, and who were taught by the Exposition Method, at the beginning of the experiment, an analysis of variance was performed on the composite scores of these subjects. These scores were obtained before the experiment began, so they should give some indication of how the two groups compared before the beginning of the experiment. The analysis used to compare the two groups taught by the two teaching methods was an analysis of variance in which three levels of ability were considered as well as the two groups as determined by the two teaching methods. From Table III it can be seen that the F-ratio for the two groups determined by the two teaching methods did not reach significance and the F-ratio for interaction did not reach significance. The F-ratio for the two groups determined by the two teaching methods indicates that the means for these two groups are not significantly different. The F-ratio for interaction indicates further that at each level of ability the two groups determined by the two teaching methods were
### Table III

**Analysis of Variance Table for A.C.T. Composite Scores at the Beginning of the Experiment**

\( (N = 104) \)

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>0.013</td>
<td>1</td>
<td>0.013</td>
<td>0.003</td>
<td>( p &gt; .25 )</td>
</tr>
<tr>
<td>Ability</td>
<td>1491.936</td>
<td>2</td>
<td>745.968</td>
<td>166.797</td>
<td>( p &lt; .001 )</td>
</tr>
<tr>
<td>Interaction</td>
<td>17.108</td>
<td>2</td>
<td>8.505</td>
<td>1.902</td>
<td>( .1 &lt; p &lt; .25 )</td>
</tr>
<tr>
<td>Error</td>
<td>198.290</td>
<td>98</td>
<td>2.012</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>1947.346</strong></td>
<td><strong>103</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Similar. These statistical tests indicate that the two groups determined by teaching methods were not significantly different at the beginning of the experiment in terms of those abilities measured by the composite score of the American College Testing Program.

In summary, an experiment was conducted which was designed to conform to the model for a two by three factorial experiment. The two independent variables were method of teaching and level of ability. The dependent variable was achievement. The effects of two methods of teaching, the Guided Discovery Method and the Exposition Method, were compared in terms of their effect upon student achievement. The subjects for the experiment were 104 students enrolled in four sections of College Mathematics at Southwestern State College during the spring semester of the 1967-1968
school year. These students were divided into sixteen categories according to level of ability, the time of day during which the class in which they had enrolled met, and according to sex. Through a random process each student was assigned to a class taught by the Guided Discovery Method or to a class taught by the Exposition Method. An analysis of variance revealed that, at the beginning of the experiment, the group consisting of all students taught by the Guided Discovery Method was similar to the group consisting of all subjects taught by the Exposition Method.

Four sections of College Mathematics were utilized for the experiment. Two classes met at 10 a.m. and two classes met at 2 p.m. Two instructors participated in the study. Instructor A taught using the Guided Discovery Method in the morning and taught using the Exposition Method in the afternoon. Instructor B taught using the Exposition Method in the morning and taught using the Guided Discovery Method in the afternoon.

In order to determine the relative effectiveness of the two teaching methods the two teaching methods were compared in terms of their effect upon student achievement. Student achievement was measured by four different tests. On each of the four tests the scores of all the students taught by the Guided Discovery Method were considered as one group of scores and the scores of all the students
group of scores. By combining scores in this way, controls were provided for the differences between the two instructors and for the effects of time of day upon learning.

The Teaching Methods

Introduction

The purpose of this section is to describe the two teaching methods compared in the experiment, to show in which ways the two teaching methods differed, and to show in which ways the two teaching methods were similar. The two teaching methods differed in three basic aspects. In the Guided Discovery Method principles and concepts were presented inductively whereas in the Exposition Method concepts and principles were presented deductively. In the Guided Discovery Method concepts and principles were introduced and taught through examples whereas in the Exposition Method concepts and principles were introduced by the instructor giving an authoritative statement of the principle or concept and then attempts were made to justify the statement through deductive reasoning. The third aspect in which the two teaching methods differed was the order of presentation of the materials used to aid learning. In the Guided Discovery Method students were presented with examples of the concept or principle to be learned and then a statement of the concept or principle was given either by
instructor. In the Exposition Method the students were first presented with a statement of the principle and examples of the principle were given later.

There were several aspects with respect to which the two teaching methods were similar. The groups taught by the two methods studied the same concepts and with only a few exceptions these concepts were studied in the same order. In a few cases the order of presentation of concepts was varied in order to best utilize the different characteristics of the two teaching methods. In general, the same concepts were presented in each lesson to all classes. All classes met the same number of times and were given the same number of homework assignments. Attempts were made to make all homework assignments of approximately the same length.

In order to assure the effectiveness of the Guided Discovery Method, special instructional materials were prepared. These materials were in the form of duplicated lessons which were handed out to the students during each class period. In order to avoid any special effect that such materials might have, similar materials were prepared for the students who were taught by the Exposition Method. Care was taken to assure that the same concepts were presented in the two sets of materials and that the two sets of lessons contained approximately the same amount of written material.
The two teaching methods were similar in that both methods stressed understanding on the part of the student. In the Guided Discovery Method of teaching, attempts were made to help students understand what they learned by requiring the students to learn through examples and through active participation in the investigation of these examples. In the Exposition Method of teaching attempts were made to help students understand what they learned by justifying each concept by a deductive process and by illustrating the concept through examples after the concept had been presented. Thus the two teaching methods were similar in terms of the kinds and amounts of material presented but were different in terms of the order and the way in which these materials were presented for the individual concepts. A more detailed description of each of the two teaching methods is presented next.

The Guided Discovery Method of Teaching

The Guided Discovery Method of Teaching can be characterized as a method of teaching in which students learn through inductive processes, learn through examples, and a method of teaching in which formal statements of concepts or principles are not given to the students until the students have had some experience with the concept or principle and until the students are convinced that the concept or principle is valid. In particular, each concept or
principle is taught through an instructional sequence characterized by a sequence of steps as described below.

In the Guided Discovery Method of teaching each instructional sequence is begun by asking the students to work exercises which are examples of the concept or principle to be learned. As the exercises proceed they become more closely related to the concept to be learned. These exercises are selected and presented in such a way as to guide the student to discovery of the concept to be learned.

During the second step of the instructional sequence attempts are made to determine whether discovery has occurred. After the students have completed the initial exercises, the students are asked questions which are based on the concept to be learned. These questions are selected in such a way that they can be answered easily and quickly if the students understand the concept to be learned, but if the students have not discovered the concept they are very difficult to answer. If student responses indicate that the concept is not fully understood, the third step of the instructional sequence is in the form of more guidance. This guidance may be in the form of leading questions or in the form of additional exercises. As more guidance is given, the guidance becomes more specific. Additional guidance is given until student responses indicate that discovery has occurred. If the concept to be learned is
especially difficult, the guidance may eventually become an actual statement of the principle to be learned.

Usually the student's first introduction to a concept is in the form of exercises given at the end of the preceding lesson. Thus, the student first works with examples of the concept on his own. Then, during the next class session the students are asked to work more exercises dealing with the concept to be learned. Then, the instructor tests for discovery by asking a series of questions. Then, if necessary, the instructor gives additional guidance in the form of leading questions. These questions are discussed in class and answered by the class. Thus, the student is first given an opportunity to discover the principle on his own. If this is too difficult, the class is then given an opportunity to discover the concept as a group.

The final step in the instructional sequence consists of naming the concept or principle discovered and in some cases this step also includes the development of a formal statement of the concept or principle discovered. If the students possess the verbal capacity to do so, the statement of the concept or principle is formulated by the students. If the students do not possess sufficient verbal ability to formulate a precise statement of the principle, such a statement is given to them either by the instructor or in the instructional materials.
The steps of the instructional sequence need not be presented as a continuous unit. In some cases it is desirable to introduce a concept in one set of exercises, present more exercises dealing with the concept in the next exercise set, and then present a final exercise set in which the student is brought to a full understanding of the concept. Then, the naming of the principle and the presentation of a formal statement of the principle may be delayed for one or more lessons. A delay in the presentation of a formal statement of the principle is often desirable because this provides those students who have not yet discovered the principle with more time to reach a better understanding of the principle.

Since the Guided Discovery Method of teaching is based on the process of inductive reasoning and on students learning through examples, the students are often asked to state generalizations on the basis of an investigation of several exercises. Since these exercises have been carefully selected, the student can become overconfident and develop a tendency to accept generalizations after investigating only a minimum of cases of the generalization. In order to avoid this tendency, several special exercise sets were developed and presented during the semester. In these exercise sets, exercises were presented which seemed to indicate that a certain generalization was valid. After the exercise set had been completed, the students were questioned concerning
this generalization. If the students seemed to think that
the generalization was valid, additional exercises were
given which showed that the generalization was not valid.
This procedure was used to discourage students from accept-
ing generalizations after an inadequate amount of inves-
tigation.

One of the factors which has made the results of other
studies dealing with discovery methods of teaching question-
able is the fact that in many studies more instructional
time was devoted to the students who learned by the dis-
covery method than to the students who learned by the
exposition method. In order to equate instructional time
for the two teaching methods, it became necessary to pro-
vide the students taught by the Guided Discovery Method
with fairly explicit guidance. Providing direct forms of
guidance for the students taught by the Guided Discovery
Method was also considered appropriate because of the
results reported in the studies conducted by Craig1 and
Underwood and Richardson.2 By providing explicit forms of
guidance, it was possible for the students taught by the
Guided Discovery Method and the students taught by the

1Robert C. Craig, "Directed Versus Independent
Discovery of Established Relations," The Journal of
Educational Psychology, LXII (April, 1955), 225-234.

2Benton J. Underwood and Jack Richardson, "Verbal
Concept Learning as a Function of Instruction and
Dominance Level," Journal of Experimental Psychology,
VI (April, 1956), 227-230.
Exposition Method to study the same concepts and principles during equal amounts of instructional time.

Since no college textbooks for mathematics have been written in which discovery methods of teaching are emphasized, it was necessary to prepare special instructional materials for the students who were taught by the Guided Discovery Method. Before the beginning of the semester, a complete outline of topics to be presented during the semester was prepared. Then, thirty-three lessons were written in which these topics were presented. These lessons were duplicated and one lesson was given to the students during each class session. Several of these lessons are presented in Appendix B. For the sake of comparison corresponding sample lessons for the Exposition Method of teaching are presented in Appendix C.

The Exposition Method of Teaching

The Exposition Method of teaching can be characterized as a method of teaching in which students learn through deductive process, learn through studying authoritative statements of principles and concepts, and a method of teaching in which principles and concepts are stated in a formal manner before the students have had experience with the concepts and principles. In particular, each concept or principle is taught through an instructional sequence characterized by a sequence of steps as described below.
In the Exposition Method of teaching each instructional sequence is begun by presenting a formal statement of the concept or principle to be learned to the students. This statement is presented either by the instructor or in the instructional materials.

After the statement of the principle has been presented, materials designed to help the student understand the principle and designed to convince the student that the principle is valid are presented. These materials may include a deductive proof of the principle as well as several examples of applications of the principle or several examples of instances of the principle. The statement of the principle and the deductive proof may be presented either as a part of the written materials or as a lecture given by the instructor.

After the formal statement of the principle, the deductive proof of the principle, and examples of the principle have been presented, the student is provided with opportunities to use the concept or principle presented to solve specific problems. This is usually in the form of an exercise set given to the student and to be completed before the next class session. In some cases, several principles may be presented and discussed during a single class period. Then, the students are presented with an exercise set consisting of problems requiring the use and application of all the principles presented during the class session.
In contrast to the Guided Discovery Method of teaching, all the steps of the instructional sequence for the Exposition Method of teaching are usually presented in the same lesson. All the instruction related to a given concept or principle is given as a complete unit; although, inter-relationships among the concepts and principles are recognized and in many cases presented to the students.

Since the Exposition Method of teaching is a deductive method of teaching, if a student can not find the solution to an exercise, he is referred to the specific formula or principle that can be used to solve the problem. In contrast, if a student in a class taught by the Guided Discovery Method has difficulty working a problem, he is presented with an example problem that illustrates how the problem can be worked.

Special instructional lessons were prepared for the classes taught by the Exposition Method. These lessons were prepared in order to assure that the students taught by the Exposition Method and the students taught by the Guided Discovery Method would receive comparable instruction and study the same concepts in very nearly the same sequence. In this way there was no difference in the two groups taught by the two different instructional methods in terms of the type of instructional materials used. Neither group used the textbook usually used in College Mathematics. Thirty-three lessons were prepared for the classes taught by the
Exposition Method. One lesson was given to the students during each class session. Several of these lessons are presented in Appendix C. For the sake of comparison, corresponding sample lessons for the Guided Discovery Method of teaching are presented in Appendix B.

The Testing Program

In order to provide a means for comparing the relative effectiveness of the Guided Discovery Method of teaching and the Exposition Method of teaching, a testing program was conducted in which three standardized tests and a series of teacher-made tests were administered. Each of the standardized tests was chosen because it was designed to measure an aspect of achievement similar to the type of achievement emphasized in some portion of the course of study used in the experiment. The three standardized tests, combined, measured knowledge of content and ability to use facts learned for nearly all aspects of the course of study used in the experiment. The teacher-made tests were included because they were designed to specifically measure the achievement of the students in the subject matter taught during the semester in the course of study.

In the following sections a detailed description is given of the tests used in the testing program. The composite score of the American College Testing Program was used as a measure of ability in order to establish levels
of ability. These levels of ability were used in order to
determine whether the relative effectiveness of the Guided
Discovery Method of teaching and the Exposition Method of
teaching are dependent upon student ability. Since the
scores obtained from the American College Testing Program
were an important part of the experimental program, this
testing program is also described.

The American College Testing Program Examination

The American College Testing Program examination is
a three-hour test battery designed to test a broad area of
educational skills. It is designed to test knowledge of
facts as well as ability to use knowledge in the solution
of complex problems. The examination was developed in order
to provide a predictor for the success of college-bound
high school seniors and junior college students who intend
to transfer to a four-year college. During each year over
300,000 students complete the test and the results are sent
to over 700 colleges.

The test consists of four subtests. The items in each
subtest are multiple-choice items. Test I is an eighty-
item, fifty-minute test of English usage. Test II is a
forty-item, fifty-minute test of mathematics usage. Test
III is a fifty-two-item, forty-minute social studies reading

3Oscar Krisen Burce, editor, The Sixth Mental
test and Test IV is a fifty-two-item, forty-minute natural science reading test. Each edition of the examination is published as a single thirty-two-page booklet containing the four subtests.

In developing new forms of the test, specifications for test items are developed. Then writers are employed to select and write test items which meet these specifications. "Tryout" units are then administered to large representative samples of students. Then item analysis is conducted and the results on the new units are compared with the scores the students in the sample have achieved on the Iowa Tests of Educational Development. On the basis of this program of analysis national percentile norms are developed. In addition local norms and other data are provided for colleges that participate in the program.

The odd-even reliability coefficients of the four subtests are .90, .39, .86, and .83 for English, mathematics, social studies, and natural science respectively. The reliability of the composite score is .95. The intercorrelations between subtests are generally greater than .50. Such high intercorrelations cause the effectiveness of the test in differentiating among the various types of ability supposedly tested to be suspect. As a result, in the experimental study reported here, only the composite score was considered.
Test I: The Structure of the Number System

The Cooperative Mathematics Tests is a series of tests designed to measure achievement in the major content areas of mathematics from junior high school arithmetic through college calculus. In constructing the tests, forty-six mathematics teachers, junior high through college, were engaged to write items for the tests. These items were reviewed and edited and pretest forms were assembled. These pretest forms were administered to a national sample of students in May, 1960. The tests were then revised and re-pretested in a national program. The results indicated that the tests are now appropriate for the intended populations. Content validity is claimed on the basis that persons well-qualified to judge the relationship of test content to teaching objectives wrote the test items.

The test, the Structure of the Number System, is a forty-item, forty-minute test. All items are multiple-choice items with five alternatives. The test is designed to measure knowledge of number systems including bases other than ten, properties of operations with numbers through fractions, arithmetic judgment, modular arithmetic, and number lines.

The characteristics of the test were determined by administering the test to a national sample of over 1,000

seventh and eighth grade students. On the basis of the scores obtained from this sample reliabilities were calculated using the Kuder-Richardson Formula 20. The reliability of form A, the form used in this study, was found to be .86. Although the test was designed for seventh and eighth grade students, an examination of the test items lead to the conclusion that the test was appropriate for measurement of the content of the course used in the experimental study. The test was administered to two sections of College Mathematics during the fall semester of the 1967-1968 school year and an examination of those test scores indicated that the test was of appropriate difficulty and provided a satisfactory distribution of scores.

The test was administered during the last week of regular classes during the spring semester and was administered as the first of a series of three standardized tests. All tests were scored by hand. All tests were scored twice in order to insure accuracy.

Test II: Algebra I

The second test administered as a series of three standardized tests was the test Algebra I. This test is one of the tests in the Cooperative Mathematics Tests series. The test is a forty-item forty-minute test. All items are multiple-choice items with five alternatives.
The test is contained as a single unit in a seven page test booklet. The test was written and revised using the same procedure as used for Structure of the Number System. The characteristics of the test were determined by administering the test to a sample of 1,200 eighth and ninth grade students. The reliability of form A, the form used in this study, was found to be .65, as determined by the use of the Kuder-Richardson Formula 20.

The test is designed to measure ability to manipulate algebraic expressions, ability to solve algebraic equations, and ability to solve literal problems. In order to determine the appropriateness of using this test in College Mathematics, the test items were examined and compared with the topics discussed in the course College Mathematics. In addition the test was administered to two sections of College Mathematics during the semester preceding the experiment. On the basis of these investigations it was determined that the test is an appropriate measuring instrument for certain aspects of the course College Mathematics.

The test was administered during the last week of regular classes during the experiment. It was administered during the class period following the class period during which the Structure of the Number System was administered. Standard answer sheets were used and all answer
sheets were scored by hand. All scoring was checked twice in order to insure accuracy.

**Test III: The Watson-Glaser Critical Thinking Appraisal**

The Watson-Glaser Critical Thinking Appraisal is a test designed to measure critical thinking ability. The test exercises include problems, statements, arguments, and interpretations of data similar to those which an individual might encounter in daily life. The test consists of five subtests. Each subtest is designed to measure a specific aspect of critical thinking. Subtest I is a twenty-item, multiple-choice test. Each item has five alternatives. This subtest is designed to measure ability to discriminate among degrees of truth or falsity of inferences drawn from given data. Subtest II is a sixteen-item, multiple-choice test in which each item presents two choices. This subtest is designed to measure ability to recognize unstated assumptions or presuppositions which are taken for granted in given statements. Subtest III is a twenty-five-item, multiple-choice test in which each item presents two alternatives. This test is designed to measure ability to think deductively from given statements or premises. Subtest IV is a twenty-four-item, multiple-choice test in which there are two choices for each item. This test samples

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ability to weigh evidence and to distinguish between generalizations which, although not absolutely certain, do seem to be warranted. The final subtest is a fifteen-item test designed to measure ability to distinguish between arguments which are strong and relevant and those which are weak or irrelevant to a particular question. The administration times for each of the subtests are thirteen minutes, six minutes, eleven minutes, twelve minutes, and eight minutes respectively. For this test the total raw score was used as the criterion score.

The *Watson-Glaser Critical Thinking Appraisal* is a well known test and its construction and subsequent revision were based on over twenty-five years of study, research and experimentation of the measurement of critical thinking abilities. Two forms of the test are available. For this form consisted of 5,297 freshmen at fifteen four-year liberal arts colleges located in eleven different states. From this sample the split-half reliability coefficient was calculated and found to be .65.

The test, *Watson-Glaser Critical Thinking Appraisal*, was administered as the third of a series of standardized tests. The test was administered during the last week of regular classes during the semester and was administered during the class period following the class period during which the Algebra I test was administered. All answer
sheets were scored by hand. All scores were checked twice in order to insure accuracy.

**Test IV: Teacher-Made Tests**

In addition to the three standardized tests, a series of four teacher-made tests were administered during the course of the semester. These tests were designed to measure the specific course content covered during the time period between the administration of the previous test and the given test. None of the tests were comprehensive in nature. None of the tests were designed to measure content learned for the entire semester. In addition a series of homework assignments was collected and graded. The scores on all the homework assignments for each student were totaled, multiplied by one hundred and divided by the total maximum possible score. The raw scores on the four tests and the homework score were then totaled. This total score was used as the criterion score for Test IV.

The first teacher-made test was administered during the fifth week of school and was designed to test knowledge of the content of lessons one through twelve. The second test was administered during the eighth week of school and was designed to measure knowledge of the content of lessons thirteen through seventeen. The third teacher-made test was administered during the thirteenth week of school and covered lessons eighteen through twenty-six. The fourth
test was administered during the regular time for final examinations. This test was administered during the week after the three standardized tests had been administered. The fourth teacher-made test was not a comprehensive test and was designed to test knowledge of the content of lessons twenty-seven through thirty-three. Each of the teacher made tests was graded on the basis of one hundred possible points. By the way the homework score was calculated, the highest possible homework score was also one hundred points. Thus, each of the teacher-made tests and the homework score contributed equally to the final score used for Test IV.

The teacher-made tests were constructed in the way tests for College Mathematics are usually constructed. No statistics in terms of reliability, validity, and item discrimination were calculated. The tests were constructed, in cooperation, by the two instructors who participated in the study. The same tests were administered to all four sections of College Mathematics and each instructor scored the tests administered in the class he taught. Copies of the four tests are presented in Appendix I.

Summary

In order to provide a means for comparing the relative effectiveness of the Guided Discovery Method of teaching and the Exposition Method of teaching, four teacher-made tests and three standardized tests were administered. In
addition, homework papers were collected and graded during the semester. Three of the teacher-made tests were administered during the semester. Then, during the last week of classes three standardized tests were administered. The Structure of the Number System, Algebra I, and Watson-Clauer Critical Thinking Appraisal. Then, during the next week, which was the week of final examinations, the fourth teacher-made test was administered.

The raw scores for each of the standardized tests were used as criterion scores. Each of the teacher-made tests was based on one hundred possible points. In addition, the homework score was calculated in such a way as to be based on a possible score of one hundred points. The four test scores and the homework score were totaled and the total was used as the criterion score for Test IV.

Methods For Treating the Data

For purposes of statistical analysis the scores on each of the four criterion tests of the 104 subjects who participated in the study were divided into six categories. The criteria for determining the categories was teaching method and level of ability. There were two teaching methods, the Guided Discovery Method and the Exposition Method, and three levels of ability. The ability level of each student was determined by his composite score on the
American College Testing Program. Level I consisted of all students with a composite score of twenty or more. Level II consisted of all students with a composite score greater than fifteen and less than twenty. Level III consisted of all students with a composite score of fifteen or less. The six categories then were Level I and Guided Discovery Method, Level I and Exposition Method, Level II and Guided Discovery Method, Level II and Exposition Method, Level III and Guided Discovery Method, and Level III and Exposition Method.

Each category contained students from two different classes. In this way students enrolled in each time of day and taught by each of the two instructors were represented in each category. In this way controls were provided for the effects of time of day and for teacher effects.

The analysis of data used was the standard analysis of variance. The computational formulas used are those given by Lindquist. These formulas do not depend on an equal number of entries in each category. All calculations were done on a desk calculator and were checked twice in order to insure accuracy. The criterion used for accepting or rejecting hypotheses was the .05 level of significance.

An experimental study was conducted in which two teaching methods were compared. The teaching methods were the Guided Discovery Method of teaching and the Exposition Method of teaching. In addition, the relative effects of the two teaching methods as related to student ability were considered. The subjects for the experiment were 104 students enrolled in four sections of College Mathematics at Southwestern State College, Weatherford, Oklahoma, during the spring semester of the 1967-1968 school year. Two of the College Mathematics classes were taught at 10 a.m. and two of the classes were taught at 2 p.m. Two instructors participated in the study. Each instructor taught one class using the Guided Discovery Method and one class using the Exposition Method. At each time period the students were assigned at random to a class taught by the Guided Discovery Method or to a class taught by the Exposition Method.

Three standardized tests and a series of teacher-made tests were used as criterion tests. The standardized tests were the Structure of the Number System, Algebra I, and the Watson-Glass Critical Thinking Appraisal. These tests were administered during the last week of regular classes during the semester. During the semester three teacher-made tests were administered. In addition, a teacher-made final examination was administered. For each student the scores on
the four teacher-made tests were totaled along with a homework score. These total scores constituted the scores for a fourth criterion test in addition to the three standardized tests.

The raw scores for each of the four criterion tests were divided into six categories. The factors used in determining the categories were the two methods of teaching and three levels of ability. These scores were then analyzed, using a two by three factor analysis of variance. All hypotheses were accepted or rejected at the .05 level of significance.
CHAPTER IV

THE RESULTS OF THE EXPERIMENTAL STUDY

Introduction

An experimental study was conducted in which the relative effectiveness of a guided discovery method of teaching and an exposition method of teaching were compared. The subjects for the study were students enrolled in four sections of College Mathematics at Southwestern State College during the spring semester of the 1967-1968 school year. A total of 124 subjects participated in the study. Two of the sections of College Mathematics were taught at 10 a.m. One of these sections was taught by the Guided Discovery Method and the other section was taught by the Exposition Method. The remaining two sections of College Mathematics were taught at 2 p.m. One of these sections was taught by the Exposition Method and the other section was taught by the Guided Discovery Method. At each time period, the students were assigned to the classes through a random process.

In order to determine the relative effectiveness of the Guided Discovery Method of teaching and the Exposition Method of teaching, four criterion tests were administered. In order to balance the effects of time of day and the
differences between the two teachers who participated in the study, on each criterion test, the scores of the two classes taught by the Guided Discovery Method were considered as one group of scores and the scores of the two classes taught by the Exposition Method were considered as one group of scores. In addition, three levels of ability were considered. On each criterion test, the scores were divided into six categories. The factors determining the categories were the two teaching methods and the three levels of ability.

The data were analyzed using the standard three by two analysis of variance. The computational formulas used are those given by Lindquist. All calculations were performed on a Monroe desk calculator. All calculations were checked twice in order to insure accuracy. The statistics of interest were the F-ratio for the two groups determined by the two teaching methods and the F-ratio for the interaction between the two groups determined by the two teaching methods and the three groups determined by the three levels of ability. In order for the F-ratio to indicate that a given factor or set of factors have a significant effect in terms of the criterion measure, the F-ratio must be larger than one, and in most cases even larger. Therefore, an F-ratio of one or less would indicate that the combination of

1Dr. F. Lindquist, Design and Analysis of Experiments in Psychology and Education (1933, 1933, p. 12).
factors for which the ratio was calculated did not significantly affect the scores achieved by the subjects on the criterion test. This would be true regardless of what criteria for significance is used. In order for the F-ratio to indicate that a given combination of factors has a significant effect, the F-ratio must be larger than one and the exact size which the ratio must be in order to indicate a significant effect depends on the criteria for significance used and the degrees of freedom for the factors.

For each of the four criterion tests, three F-ratios were calculated. F-ratios were calculated for the teaching method factor, the ability level factor, and the interaction between the teaching method factor and the ability level factor. A significant F-ratio for the ability factor would indicate that the students in the three levels of ability achieve differently. That achievement is dependent upon ability is a well established fact, so the F-ratio for the ability factor is not of primary interest. A significant F-ratio for the interaction between the teaching method factor and the ability factor would indicate that the effects the two teaching methods on achievement are dependent upon ability and that the effects of the two teaching methods may be different at each level of ability. An insignificant F-ratio for the interaction effect would indicate that the two teaching methods have similar effects on achievement at each level of ability. If the F-ratio for
the interaction effect is insignificant, then a significant F-ratio for the teaching method factor would indicate that one teaching method is more effective than the other. An insignificant F-ratio for the teaching method factor would indicate that both teaching methods are equally effective. In all cases, the .05 level of significance was used to determine the significance of the effects of the various factors. The raw data used in the statistical analysis are given in Appendix E.

The Findings for Test I

The criterion scores for Test I were the raw scores achieved by the subjects on the test Structure of the Number System. This test is a forty-minute, forty-item multiple-choice test. The test is designed to measure knowledge of the properties of operations with numbers, number systems including bases other than ten, modular arithmetic, and arithmetic judgment.

From Table IV it can be seen that the F for the interaction effect of the ability factor and the factor determined by the teaching methods is not large enough to indicate a significant effect at any level of significance. On this basis, the following hypothesis is accepted: the relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the test, Cooperative Mathematics Test, Structure
of the Number System, will not be dependent upon student ability.

From Table IV it can be seen that the F for the factor determined by the teaching methods is not large enough to indicate a significant effect at any level of significance. This indicates that the Guided Discovery Method of teaching and the Exposition Method of teaching are equally effective when the criterion is the aspect of student achievement which is measured by the test, Structure of the Number System. Therefore, the following hypothesis is rejected: students taught by a guided discovery method will score significantly higher on the test, Cooperative Mathematics Tests, Structure of the Number System, than students taught by an exposition method.

### Table IV

**ANALYSIS OF VARIANCE TABLE FOR TEST I**

(N = 104)

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>5.009</td>
<td>1</td>
<td>5.009</td>
<td>.219</td>
<td>.14</td>
</tr>
<tr>
<td>Ability</td>
<td>933.016</td>
<td>2</td>
<td>466.509</td>
<td>20.38</td>
<td>&lt; .001</td>
</tr>
<tr>
<td>Interaction</td>
<td>33.717</td>
<td>2</td>
<td>16.858</td>
<td>.737</td>
<td>.69</td>
</tr>
<tr>
<td>Error</td>
<td>2211.534</td>
<td>98</td>
<td>22.688</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>2214.760</td>
<td>103</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Number of Mathematics Tests, Structure of the Number System, than students taught by an exposition method.
The Findings for Test II

The scores used in the analysis of data for Test II were the raw scores achieved on the test, Algebra I. This test is a standardized test and is one of the tests in the Cooperative Mathematics Tests series. Algebra I is a forty-minute, forty-item multiple-choice test. The test is designed to measure ability to manipulate algebraic expressions, ability to perform operations with algebraic expressions, ability to solve algebraic equations, and ability to solve literal problems.

From Table V it can be seen that the $F$ for the interaction effect of the ability factor and the factor determined by the teaching methods does not indicate a significant effect at any level of significance. As a result, the following hypothesis is accepted: the relative

TABLE V
ANALYSIS OF VARIANCE TABLE FOR TEST II

(N = 104)

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>$F$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>10.661</td>
<td>1</td>
<td>10.661</td>
<td>.111</td>
<td>.631</td>
</tr>
<tr>
<td>Ability</td>
<td>936.797</td>
<td>2</td>
<td>468.398</td>
<td>13.294</td>
<td>.001</td>
</tr>
<tr>
<td>Interaction</td>
<td>54.246</td>
<td>2</td>
<td>27.123</td>
<td>.911</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>358.420</td>
<td>98</td>
<td>3.636</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1137.624</td>
<td>103</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the test, Cooperative Mathematics Tests, Algebra I, will not be dependent upon student ability.

From Table V it can be seen that the F for the factor determined by the teaching method is not large enough to indicate a significant effect at any level of significance. From this it is concluded, that the Guided Discovery Method of teaching and the Exposition Method of teaching are equally effective when the criterion is that aspect of student achievement measured by the test, Algebra I. On the basis of this information, the following hypothesis is rejected: students taught by a guided discovery method will score significantly higher on the test, Cooperative Mathematics, Algebra I, than students taught by an exposition method.

The Findings for Test III

The criterion scores for Test III were the raw scores achieved by the subjects on the test, Wacker-Claxton Critical Thinking Appraisal. This test is designed to measure ability to discriminate among degrees of truth or falsity of inferences drawn from given data, ability to reason deductively from given statements or premises, ability to recognize unstated assumptions which are taken for granted in given statements or assertions, ability to weigh
TABLE VI
ANALYSIS OF VARIANCE TABLE FOR TEST III
(N = 101)

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>447.289</td>
<td>1</td>
<td>447.289</td>
<td>0.86</td>
<td>p &lt; .001</td>
</tr>
<tr>
<td>Ability</td>
<td>1599.687</td>
<td>2</td>
<td>799.844</td>
<td>14.628</td>
<td></td>
</tr>
<tr>
<td>Interaction</td>
<td>62.592</td>
<td>2</td>
<td>31.296</td>
<td>.572</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>5258.321</td>
<td>98</td>
<td>53.678</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>7067.662</td>
<td>102</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

evidence and to distinguish between generalizations from given data that are not warranted and generalizations which seem to be warranted, and ability to distinguish between arguments which are strong and relevant and those which are weak or irrelevant to a particular question at issue.

As revealed in Table VI, the F for interaction is less than one. This indicates that the Guided Discovery Method of teaching and the Exposition Method of teaching have the same relative effect on that aspect of achievement measured by the Nathan-Glaser Critical Thinking Appraisal at each level of ability. Accordingly, the following hypothesis is accepted: the relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the Nathan-Glaser Critical Thinking Appraisal will not be dependent upon student ability.
As revealed in Table VI, the F for the effects of the two teaching methods is less than one. Since the interaction effect is not significant this indicates that the difference in the effects of the two teaching methods is not significant when the criterion is achievement as measured by the Watson-Glaser Critical Thinking Appraisal. On the basis of this information, the following hypothesis is rejected: students taught by a guided discovery method will score significantly higher on the Watson-Glaser Critical Thinking Appraisal than students taught by an exposition method.

The Findings for Test IV

During the semester of the experimental study, four teacher-made tests were administered. These tests were designed to measure knowledge of the material presented during a given period of the semester. Each test tested knowledge of a particular unit of material. None of the tests were designed to test knowledge of the content of the entire course of study. The fourth teacher-made test was administered during the last week of the semester, but the test was not a comprehensive test. Each of the four teacher-made tests was graded on the basis of one hundred possible points. During the semester a series of homework assignments were collected and graded. At the end of the semester the points achieved by each student
TABLE VII
ANALYSIS OF VARIANCE TABLE FOR TEST IV
(N = 104)

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>6565.171</td>
<td>1</td>
<td>6565.171</td>
<td>1.857</td>
<td>.1 &lt; p &lt; .2</td>
</tr>
<tr>
<td>Ability</td>
<td>5530.646</td>
<td>2</td>
<td>2765.323</td>
<td>7.564</td>
<td>p &lt; .001</td>
</tr>
<tr>
<td>Interaction</td>
<td>5638.482</td>
<td>2</td>
<td>2819.231</td>
<td>.800</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>385031.111</td>
<td>98</td>
<td>3961.522</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>417669.185</td>
<td>103</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

On the homework assignments were totaled. This total was then divided by the total number of points possible and then multiplied by one hundred. Thus each student had a possible homework score of one hundred. At the end of the semester the scores achieved by each student on the four teacher made tests and the student's homework score were totaled. The resulting scores were used as the criterion scores for Test IV.

As indicated in Table VII, the F for interaction on Test IV is less than one. Therefore, the following hypothesis is accepted: the relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the student's grade in the course will not be dependent upon student ability.
As indicated in Table VII, the $F$ for the effects of the two teaching methods is greater than one but not great enough to indicate a significant effect at the .05 level of significance. This indicates that there was no significant difference between the Guided Discovery Method of teaching and the Exposition Method of teaching when the criterion is student achievement as measured by four teacher made tests and homework scores. Therefore, the following hypothesis is rejected: students taught by a guided discovery method will make significantly higher grades in the course than students taught by an exposition method.

Summary

Four criterion tests were used to determine the relative effectiveness of the Guided Discovery Method of teaching and the Exposition Method of teaching. The scores on each test were analyzed using the standard three by two factor analysis of variance. The $F$-ratio was calculated for the interaction effect of the two methods of teaching and three levels of ability. In addition, the $F$-ratio for the effects of the two teaching methods was calculated.

On each of the four criterion tests the $F$-ratio for the effects of the two teaching methods was not large enough to indicate a significant difference between the two teaching methods. As a result, of the hypotheses
identified in Chapter I, the following were rejected:

1. Students taught by a guided discovery method will score significantly higher on the test, Cooperative Mathematics Test, Structure of the Number System, than students taught by an exposition method.

2. Students taught by a guided discovery method will score significantly higher on the test, Cooperative Mathematics Test, Algebra I, than students taught by an exposition method.

3. Students taught by a guided discovery method will score significantly higher on the test, Watson-Glaser Critical Thinking Appraisal than students taught by an exposition method.

4. Students taught by a guided discovery method will make significantly higher grades in the course than students taught by an exposition method.

On each of the four criterion tests the F-ratio for the interaction of the ability factor and the teaching method factor was not large enough to indicate a significant interaction of these factors. This implies that the comparative effectiveness of the Guided Discovery Method of teaching and the Exposition Method of teaching was similar at each level of ability. As a result, of the hypotheses identified in Chapter I, the following were accepted:
5. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the test, Cooperative Mathematics Tests, Structure of the Number System, will not be dependent upon student ability.

6. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the test, Cooperative Mathematics Tests, Algebra I, will not be dependent upon student ability.

7. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the Ratson-Klasser Critical Thinking Manual will not be dependent upon student ability.

8. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the student's grade in the course will not be dependent upon student ability.

In summary, in terms of all of the factors considered, there was no significant difference between the Guided Discovery Method of teaching and the Exposition Method of teaching when the criterion is student achievement.
CHAPTER V

SUMMARY, FINDINGS AND CONCLUSIONS,
AND RECOMMENDATIONS

Summary

The purpose of this study was to provide a reference for research related to the discovery method of teaching mathematics and to ascertain the value, as determined by student achievement, of using a discovery method of teaching mathematics in a college freshman mathematics course for non-mathematics and non-science majors. In order to provide a reference for research related to the discovery method of teaching mathematics, a comprehensive summary of research literature related to this method of teaching mathematics was presented. In order to ascertain the value of using a discovery method of teaching mathematics in a college freshman mathematics course for non-mathematics and non-science majors an experimental study was conducted.

An experimental study was conducted in which the effects of two methods of teaching on student achievement were compared. The methods of teaching were the Guided Discovery Method of teaching and the Exposition Method of teaching. The subjects for the experiment were 101 students enrolled in four sections of College Mathematics at
Southwestern State College, Weatherford, Oklahoma. The experiment was conducted during the spring semester of the 1967-1968 school year. College Mathematics is a course designed for liberal arts students and is a required course for all students who do not complete college algebra and trigonometry as a part of their degree program. Two of the four sections of College Mathematics were taught at 10 a.m. and the remaining two sections were taught at 2 p.m. Two instructors participated in the study. The four classes were arranged in such a way that at each of the two time periods one class was taught by the Guided Discovery Method and one class was taught by the Exposition Method. Each instructor taught one class during each time period and each instructor taught one class using the Guided Discovery Method of teaching and one class using the Exposition Method of teaching.

Composite scores on the American College Testing Program were used to establish three levels of ability. Level I consisted of all students with a composite score of twenty or higher. Level II consisted of all students with a composite score higher than fifteen, but lower than twenty, and Level III consisted of all students with a composite score of fifteen or lower. The students who had enrolled for the 10 a.m. classes and the students who had enrolled for the 2 p.m. classes were divided into categories according to sex and level of ability. The students
in each category were then divided into groups through a random process. In this manner, of the students enrolled at 10 a.m., two groups were formed; and, of the students enrolled at 2 p.m., two groups were formed. During each time period, one of the two groups was taught by the Exposition Method and one of the groups was taught by the Guided Discovery Method. The teaching method to be used with each group was determined by a random process.

The purpose of the experimental study was to determine the relative effectiveness of the Guided Discovery Method of teaching and the Exposition Method of teaching. The Guided Discovery Method of teaching is a method of teaching in which students learn through inductive processes, learn through examples, and a method of teaching in which formal statements of concepts or principles are not given to the students until the students have had experience with the concept or principle and until the students are convinced that the concept or principle is valid. The Exposition Method of teaching is a method of teaching in which students learn through deductive processes, learn through studying authoritative statements of principles and concepts, and a method of teaching in which principles and concepts are stated in a formal manner at the beginning of the instructional sequence in which the principle or concept is introduced. Special instructional materials were prepared for both instructional methods in order to assure
that the instructional materials would be presented in a manner appropriate for each instructional method and to insure that the students taught by the two teaching methods actually learned the same concepts.

In order to determine the relative effectiveness of the two teaching methods, three standardized tests were administered and a series of teacher-made tests were administered. The standardized tests were Cooperative Mathematics Tests, Structure of the Number System, Cooperative Mathematics Tests, Algebra I, and the Watson-Glaser Critical Thinking Appraisal. These tests were administered during the last week of the semester. Three teacher-made tests were administered during the semester and a fourth teacher-made test was administered at the end of the semester. The raw scores on each of the standardized tests and the total scores obtained by summing each student's scores on the four teacher-made tests and a homework score were used as four criterion measures. For each of the four criterion measures a standard three by two factor analysis of variance was performed. This method of analysis was used to determine the relative effectiveness of the Guided Discovery Method of teaching and the Exposition Method of teaching.
Findings and Conclusions

For each of the four criterion measures the F-ratio for the effects of the two teaching methods was calculated. Of the four F-ratios calculated, none was large enough to indicate a significant difference between the two teaching methods at the .05 level of significance. As a result, of the eight hypotheses formulated at the beginning of the study, the following were rejected:

1. Students taught by a guided discovery method will score significantly higher on the test, Cooperative Mathematics Tests, Structure of the Number System, than students taught by an exposition method.

2. Students taught by a guided discovery method will score significantly higher on the test, Cooperative Mathematics Tests, Algebra I, than students taught by an exposition method.

3. Students taught by a guided discovery method will score significantly higher on the Nelson-Glaser Critical Thinking Appraisal than students taught by an exposition method.

4. Students taught by a guided discovery method will make significantly higher grades in the course than students taught by an exposition method.

For each of the four criterion measures the F-ratio for the interaction effects of the two teaching methods and the three levels of ability was calculated. Of the
four F-ratios calculated none was large enough to indicate a significant interaction between the teaching method factor and the ability factor. As a result, of the eight hypotheses formulated at the beginning of the study, the following were accepted:

5. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the test, Cooperative Mathematics Tests, Structure of the Number System, will not be dependent upon student ability.

6. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the test, Cooperative Mathematics Tests, Algebra I, will not be dependent upon student ability.

7. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the test, Watson-Glaser Critical Thinking Appraisal, will not be dependent upon student ability.

8. The relative effects of a guided discovery method of teaching and an exposition method of teaching on achievement as measured by the student's grade in the course will not be dependent upon student ability.

On the basis of the testing program and the statistical analysis, it is concluded that, in a college freshman
mathematics course for non-mathematics and non-science majors, the Guided Discovery Method of teaching and the Exposition Method of teaching are equally effective when the criterion is student achievement.

Recommendations

The results of the experiment reported in this study indicate that in a college mathematics course designed for liberal arts students the Guided Discovery Method of teaching and the Exposition Method of teaching are equally effective if the criteria are retention and ability to apply learning to situations similar to, but different from, the situations in which the learning occurred. If the objectives of instruction are to foster retention of learned facts and to foster the ability to use learning in new situations, then the Exposition Method of teaching is as effective as the Guided Discovery Method of teaching for obtaining the desired objectives.

Students taught by the Guided Discovery Method of teaching were required to discover concepts and principles. They were also required to formulate generalizations through inductive processes of reasoning and to discover methods of solutions to problems. It would appear that if the objectives of instruction are to develop the ability to discover concepts, principles, and methods of solutions to problems and the ability to formulate generalizations
through inductive processes, then the Guided Discovery Method of teaching should be more effective than the Exposition Method of teaching. The testing program conducted in relation to the experiment reported in this study was not designed to measure these aspects of achievement. A review of research literature reveals that very few experimental studies have been conducted in which these aspects of achievement have been considered. Therefore, further research is needed to determine whether guided discovery methods of teaching are more effective than exposition methods of teaching when these aspects of achievement are used as the criteria for determining the relative effectiveness of the compared teaching methods.

As has been indicated, many aspects of the relative effectiveness of discovery methods of teaching and exposition methods of teaching have not been adequately studied. In particular, research is needed which will help answer the following questions:

1. Does learning by discovery affect the relation the learner perceives between himself and the field of knowledge in which the discovery learning occurred? In particular, does learning by discovery foster a confidence in the learner that he has the ability to operate independently, learn for himself, and cope with the problems he faces through the use of his own intellectual abilities?
2. Does learning by discovery affect the attitude of the learner towards the field of knowledge in which the discovery learning occurred? Does learning by discovery produce an enduring interest in that field of knowledge and a desire to continue to learn from that field of knowledge?

3. Does learning by discovery create in the learner an appreciation of the value of the knowledge in the field in which the discovery learning occurred?

4. Does learning by discovery affect the way a learner behaves when he encounters an unprecedented problem? Does learning by discovery lead the learner to be less rigid in the way he attacks problems?

5. Does learning by discovery affect the way a learner behaves when he forgets how to solve a problem?
APPENDIX A

A COURSE OUTLINE FOR

COLLEGE MATHEMATICS

Introduction

The purpose of Appendix A is to present an outline of the topics considered in the course College Mathematics during the spring semester of the 1967-1968 school year. The subdivisions of the outline will be according to the lessons as they were given to the students.

The Course Outline

Lesson 1

A. Definition of terms
1. Set
2. Element
3. Natural number
4. Binary operation
5. Ordered pair

B. Exercises
1. The binary operation B defined by
   \[(a, b) B (c, d) = (a \times b, c)\]
2. The binary operation defined by
   \[(a, b) B (c, d) = (a \times d, b)\]
3. The binary operation B defined by
   \[(a, b) B (c, d) = (a \times c, b + d)\]

Lesson 2

A. The commutative principle
1. \(+\) is commutative
2. \(\times\) is commutative
3. The binary operation B defined by
   \[(a, b) B (c, d) = (a \times c, b + d)\] is commutative.
4. If \((a, b) B (c, d) = (2a + 2c, bd)\), then B is commutative.
B. The associative principle
1. \( + \) is associative
2. \( \times \) is associative

C. Exercises

Lesson 3
A. Identity elements
1. 0 is an identity for \( + \)
2. 1 is an identity for \( \times \)
3. \((0, 1)\) is an identity for \( B \) if
   \[(a, b) \# (c, d) = (a + c, bd)\]

B. Exercises

Lesson 4
A. The inverse of an element with respect to a binary operation and an identity
1. \(-a\) is the inverse of \( a \) with respect to \( + \)
2. If \((a, b) \# (c, d) = (a + c, b + d)\), then
   \((0, 0)\) is an identity for \( \# \) and \((-a, -b)\)
   is an inverse of \((a, b)\)

B. The solution of certain linear equations
C. Exercises

Lesson 5
A. If \( a \) \# \( b \) = \( x \) and \( a = b \) \# \( x \) have the same solution, then 0 is said to be the inverse
   of \( B \)
1. \(-a\) is the inverse of \( + \)
2. If \((a, b) \# (c, d) = (a + c, b + d)\) and
   \((a, b) \& (c, d) = (a - c, b - d)\), then
   \( \& \) is the inverse of \#.

B. Exercises

Lesson 6
A. Division is the inverse of multiplication
B. Exercises

Lesson 7
A. The distributive principle
1. Multiplication is distributive over addition
2. Addition is not distributive over multiplication
3. If \((a, b) \times (c, d) = (ac, bd)\) and
   \((a, b) \# (c, d) = (a + c, b + d)\), then
   \( \times \) is distributive over \( \# \).
4. If \((a, b) \# (c, d) = (a + c, bd)\) and
   \((a, b) \& (c, d) = (a \times c, bc)\), then
   \( \# \) is not distributive over \&.

B. Exercises
Lesson 8
A. Equivalence relations
1. \(=\) is an equivalence relation on the set of integers.
2. If \((a, b) = (c, d)\) means \(a = c\) and \(b = d\), then \(=\) is an equivalence relation on the set of all ordered pairs of integers.
3. If the relation \(R\) is defined by \((a, b) R (c, d)\) if and only if \(a + d = b - c\), then \(R\) is not an equivalence relation.
4. The relation \(R^1\) defined by \((a, b) R^1 (c, d)\) if and only if \(a + d = b + c\) is an equivalence relation.

B. Exercises

Lesson 9
A. Equivalence classes
1. The equivalence classes of modular arithmetic
2. Binary operations defined on equivalence classes

B. Exercises

Lesson 10
A. Clock arithmetic

B. Exercises

Lesson 11
A. The relationship between arithmetic on a clock and arithmetic with equivalence classes of natural numbers
B. The properties of clock arithmetic
C. Exercises

Lesson 12
A. The relation \(\equiv_n\) defined by \(a \equiv_n b\) if and only if the remainder obtained by dividing \(a\) by \(n\) is the same as the remainder obtained by dividing \(b\) by \(n\).
1. \(\equiv_n\) is an equivalence relation
2. The relationship between \(\equiv_n\) and arithmetic on the \(n\)-hour clock

B. Exercises

Lesson 13
A. The binary operation \(\odot\) defined by \((a, b) \odot (c, d) = (ad + bc, bd)\)
Lesson 11

A. Some properties of $\oplus$
1. $\oplus$ is commutative
2. $(0, 1)$ is an identity element for $\oplus$
3. Not every ordered pair $(a, b)$ has an inverse with respect to $\oplus$

Lesson 12

B. Some properties of $\times$
1. $\times$ is commutative
2. $(1, 1)$ is an identity element for $\times$
3. Not every ordered pair $(a, b)$ has an inverse with respect to $\times$

Lesson 13

C. Exercises

Lesson 14

A. Properties of elements of $R$
B. The binary operation $\oplus$ defined on $R$ as follows:
   If $X$ and $Y$ are elements of $R$, $(a, b)$ is an element of $X$ and $(c, d)$ is an element of $Y$, then $X \oplus Y$ is the element of $R$ containing $(a, b) \oplus (c, d)$.
C. The binary operation $\times$ defined on $R$ as follows:
   If $X$ and $Y$ are elements of $R$, $(a, b)$ is an element of $X$ and $(c, d)$ is an element of $Y$, then $X \times Y$ is the element of $R$ containing $(a, b) \times (c, d)$.
D. $\oplus$ and $\times$ are well defined
E. Exercises

Lesson 15

A. Properties of $+$ and $\times$
1. The element of $R$ containing $(0, 1)$ is an identity element for $+$
2. The element of $R$ containing $(1, 1)$ is an identity element for $\times$
3. The element of $R$ containing $(-a, b)$ is the inverse of the element of $R$ containing $(a, b)$ with respect to $+$
4. If \( a \) is not zero, then the element of \( \mathbb{R} \) containing \((b, a)\) is the inverse of the element of \( \mathbb{R} \) containing \((a, b)\) with respect to \( x \)

B. Exercises

Lesson 18
A. Solution of linear equations with elements of \( \mathbb{R} \)
B. Subtraction with elements of \( \mathbb{R} \)
C. Division with elements of \( \mathbb{R} \)
D. Exercises

Lesson 19
A. The element of \( \mathbb{R} \) containing \((a, b)\) may be considered to be equivalent to the fraction \(a/b\)
B. Operations with fractions
1. Addition
2. Multiplication
3. Subtraction
4. Division
5. Equivalent fractions
C. Exercises

Lesson 20
A. Definition of a mathematical system
B. The rigid translations of an equilateral triangle
C. The binary operation * defined on the set of rigid translations of a triangle
D. Exercises

Lesson 21
A. Properties of the binary operation *
B. The inverse of the binary operation *
C. Subsets of the set of translations of a triangle for which * is a complete binary operation
D. Exercises

Lesson 22
A. Binary operations defined by Cayley tables
B. Procedures for investigating the properties of binary operations defined by Cayley tables
C. Exercises

Lesson 23
A. Propositions
1. The negation of a proposition
2. Propositions of the form \( p \) and \( q \)
B. Exercises
Lesson 24
A. More on propositions
   1. Propositions of the form $p \lor q$
   2. The negation of a proposition of the form $p \land q$
   3. The negation of a proposition of the form $p \lor q$
B. Exercises

Lesson 25
A. Propositions of the form if $p$, then $q$
B. Exercises

Lesson 26
A. More on propositions of the form if $p$, then $q$
   1. The converse of an implication
   2. The contraposition of an implication
B. Exercises

Lesson 27
A. Methods of deductive reasoning
   1. The nature of axiom systems
   2. The law of detachment
B. Exercises

Lesson 28
A. Types of arguments that lead to valid conclusions
B. Types of arguments that lead to invalid conclusions
C. Exercises

Lesson 29
A. Drawing conclusions using Venn Diagrams
B. Exercises

Lesson 30
A. An example of a simple axiom system
B. Methods of proof
C. Exercises

Lesson 31
A. Another abstract axiom system
B. Exercises

Lesson 32
A. The definition of a group
B. Some simple group theorems
C. Exercises
Lesson 33

A. Examples of systems that are groups
B. Examples of systems that are not groups
C. Exercises
APPENDIX B

SAMPLE LESSONS FOR THE GUIDED DISCOVERY METHOD OF TEACHING
Throughout this course the term *set* will be used to refer to any well-defined collection, group, or class of objects or ideas. The phrase "well-defined" means that the set is described precisely enough so that one can tell whether or not any given object belongs to the set.

The objects contained in a set are called elements. In a set of dishes, each dish is an element of the set of dishes. In the case of a coin collection, each coin is an element of the set of coins.

One way of denoting a set is by listing all of its elements. If the elements of a set are listed, the list is placed within braces. Thus \( \{1, 2, 3, 4, 5, 6\} \) denotes the set consisting of all natural numbers from 1 through 6 inclusive. In some cases not all the elements of a set can be listed. One such set is the set of natural numbers. This set may be denoted as follows:

\[ \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots\} \]

Another concept which will be used extensively throughout this course is the concept of binary operation. A binary operation is a way of associating with two elements of a set a third element of that set. Addition is a binary operation on the set of natural numbers.
For example, the operation of addition associates with the natural numbers 3 and 5 the natural number 8. It associates with the numbers 598 and 683 the number 1281. Multiplication is also a binary operation on the set of natural numbers. This operation associates with the numbers 3 and 5 the number 15. It associates with the numbers 598 and 683 the number 408,434.

It is possible to define many binary operations on the set of natural numbers. Suppose that with two natural numbers we associate the first of the two numbers. This is an example of a binary operation on the set of natural numbers. This operation associates with 3 and 5 the number 3. It associates with 17 and 9 the number 17.

The concept of binary operation leads us to consider the concept of ordered pair. A binary operation associates with a pair of elements of a set an element of the set. The binary operation discussed in the preceding paragraph associates with the pair of numbers 3 and 5 the number 3. It associates with the pair of numbers 5 and 3 the number 5. In each case the same pair of numbers is used, but the results are different. This is true because the order in which the elements of the pair are listed is different. An ordered pair consists of a pair of elements listed in a specific
order. An ordered pair will be denoted by listing the elements of the pair with a comma between them and placing the list in parentheses. Thus (3, 5) denotes the ordered pair consisting of the numbers 3 and 5 with 3 coming before 5. (5, 3) denotes the same pair, but with 5 coming before 3. This shows that (3, 5) is not the same ordered pair as (5, 3). In the ordered pair (5, 3), 5 is called the first component of the ordered pair and 3 is called the second component of the ordered pair. Two ordered pairs are said to be equal if they have the same first component and the same second component.

Let us denote the set of all ordered pairs of natural numbers by the symbol $\mathcal{P}$. Some of the elements of $\mathcal{P}$ are (3, 9), (5, 1), (5, 5), (2871, 1), and (2, 3800972). Many interesting and unusual binary operations can be defined on the set $\mathcal{P}$. Remember that a binary operation is a way of associating with two elements of a set another element of the set. Thus a binary operation on $\mathcal{P}$ associates with a pair of ordered pairs an ordered pair. An example of a binary operation on $\mathcal{P}$ is the operation which associates with a pair of ordered pairs the second ordered pair. This binary operation associates with (4, 1) and (9, 5) the ordered pair (9, 5). It associates with (1, 17)
and $(39, 14)$ the ordered pair $(39, 14)$.

If we wish to consider the number which the binary operation of addition on the set of natural numbers associates with 7 and 4 we write $7 + 4$. Since the operation of addition associates with 7 and 4 the number 11, we write $7 + 4 = 11$. This is a very compact and convenient way of saying that the binary operation addition associates with 7 and 4 the natural number 11. When considering binary operations on $P$ we shall use the notation $(a, b) \circ (c, d) = (e, f)$ to denote that the binary operation $\circ$ associates with the pair of ordered pairs $(a, b)$ and $(c, d)$ the ordered pair $(e, f)$. Thus if $\circ$ denotes the binary operation on $P$ discussed in the preceding paragraph we have that

$(4, 1) \circ (9, 5) = (9, 5)$. Also

$(1, 17) \circ (39, 14) = (39, 14)$.

A binary operation on $P$ can often be defined using a general formula. $(a, b) \circ (c, d) = (e, f)$ is a formula which can be used to define the binary operation on $P$ discussed in the preceding paragraph. If $a$ is replaced by a natural number and if $b$ is replaced by a natural number, then $(a, b)$ becomes an element of $P$. Similarly if $c$ is replaced by a natural number and if $d$ is replaced by a natural number, then $(c, d)$ becomes an element of $P$. Then $(a, b) \circ (c, d) = (e, f)$ is a way of saying
that the binary operation $B$ associates with $(a, b)$ and $(c, d)$ the ordered pair $(c, d)$.

Consider the binary operation on $P$ defined by the formula

$$(a, b) B (c, d) = (a + c, b \times d).$$

This formula states that to find the first component of the ordered pair associated with $(a, b)$ and $(c, d)$ add the first components of $(a, b)$ and $(c, d)$. To find the second component multiply the second components of $(a, b)$ and $(c, d)$. According to the formula the ordered pair associated with $(2, 8)$ and $(9, 7)$ is $(2 + 9, 8 \times 7)$ or $(13, 56)$. Similarly,

$$(12, 3) B (4, 17) = (12 + 4, 3 \times 17)$$ or $$(12, 3) B (4, 17) = (16, 51).$$

**Exercise 1.**

In each of the following exercises a binary operation is defined. Use the definition to find the ordered pair associated with the given pair of ordered pairs.

**Example:** $(a, b) B (c, d) = (a \times b, c)$.

Complete the following.

$(3, 2) B (4, 7) =$

$(5, 1) B (9, 18) =$

**Solution:**

$(3, 2) B (4, 7) = (3 \times 2, 4) = (6, 4)$.

$(5, 1) B (9, 18) = (5 \times 1, 9) = (5, 9)$. 
1. \((a, b) \circ (c, d) = (a \times d, b)\).

**Example:** \((1, 9) \circ (7, 4) = (1 \times 4, 9) = (4, 9)\).

Complete the following.

\[(4, 3) \circ (18, 5) = \quad \]
\[(15, 2) \circ (1, 1) = \quad \]
\[(1, 1) \circ (15, 2) = \quad \]
\[(5, 7) \circ (9, 3) = \quad \]
\[(9, 3) \circ (5, 7) = \quad \]

2. \((a, b) \circ (c, d) = (a \times c, b + d)\).

**Example:** \((3, 2) \circ (9, 1) = (3 \times 9, 2 + 1) = (27, 3)\).

Complete the following.

\[(7, 1) \circ (9, 5) = \quad \]
\[(9, 5) \circ (7, 1) = \quad \]
\[(8, 2) \circ (13, 11) = \quad \]
\[(13, 11) \circ (8, 2) = \quad \]
\[(1, 1) \circ (2, 3) = \quad \]
\[(8, 3) \circ (1, 1) = \quad \]
\[(6, 9) \circ (14, 7) = \quad \]
\[(14, 7) \circ (6, 9) = \quad \]

3. Make up (invent) a formula of your own which defines a binary operation on \(P\). Use your formula to complete the following.
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(3, 1) B (2, 2) = 
(8, 2) B (7, 5) = 
(3, 11) B (8, 6) = 
(8, 6) B (3, 11) = 
(6, 8) B (11, 3) = 
(6, 8) B (3, 11) = 
(4, 7) B (1, 15) = 
(15, 1) B (4, 7) = 
(7, 4) B (1, 15) =
Lesson 2

Consider the binary operation on $P$ defined by the following formula:

$$(a, b) \triangleq (c, d) = (a \times c, b + d).$$

Using this formula, complete the following as quickly as possible. If you discover a short-cut, do not hesitate to use it. Be sure the short-cut works.

$$(1, 3) \triangleq (8, 4) =$$

$$(8, 4) \triangleq (1, 3) =$$

$$(2, 5) \triangleq (3, 7) =$$

$$(3, 7) \triangleq (2, 5) =$$

$$(3, 2) \triangleq (1, 6) =$$

$$(1, 6) \triangleq (8, 2) =$$

$$(12, 1) \triangleq (4, 9) =$$

$$(4, 9) \triangleq (12, 1) =$$

$$(5, 11) \triangleq (2, 2) =$$

$$(2, 2) \triangleq (5, 11) =$$

Note: If we write $2a$ we shall mean $2 \times a$. If we write $ab$, we shall mean $a \times b$.

Consider the binary operation of $P$ defined by the following:

$$(a, b) \triangleq (c, d) = (2a + 2c, bd)$$

Examples:

$$(3, 4) \triangleq (1, 2) = (2 \times 3 + 2 \times 1, 4 \times 2) = (8, 8)$$

$$(2, 5) \triangleq (1, 9) = (2 \times 2 + 2 \times 1, 5 \times 3) = (6, 45)$$
Using the above formula complete the following as quickly as possible.

\[(1, 3) \oplus (2, 6) = \_\_\_\_\_\_\_\]
\[(2, 8) \oplus (1, 3) = \_\_\_\_\_\_\_\]
\[(4, 6) \oplus (1, 1) = \_\_\_\_\_\_\_\]
\[(1, 1) \oplus (4, 6) = \_\_\_\_\_\_\_\]
\[(5, 4) \oplus (2, 11) = \_\_\_\_\_\_\_\]
\[(2, 11) \oplus (5, 4) = \_\_\_\_\_\_\_\]
\[(12, 15) \oplus (21, 42) = \_\_\_\_\_\_\_\]
\[(21, 42) \oplus (12, 15) = \_\_\_\_\_\_\_\]

Was it necessary to carry out all the implied calculations or could you find the answers to some of the exercises and from these conclude what the answers to the remaining exercises should be?

A binary operation on a set is a way of associating with two elements of a set an element of that set. Addition is a binary operation on the set of natural numbers. This binary operation associates with 9 and 12 the natural number 21. 21 is called the sum of 9 and 12. Suppose that it is necessary to find the sum of a list of natural numbers. Consider the list 7, 3, 9, 4, and 6. The sum can be found by finding the sum of 7 and 3. This sum is 10. Then find the sum of 10 and 9. This sum is 19. Then find \(19 + 4\). \(19 + 4 = 23\). Then find \(23 + 6\).

\(23 + 6 = 29\). This is the sum of 7, 3, 9, 4, and 6.
Using this procedure two numbers were added in each step of the procedure. This type of procedure was necessary because addition is a binary operation. Only two numbers can be added at a time. Consider the task of finding the sum of the list 4, 15, and 2.

$4 + 15 = 19$ and $19 + 2 = 21$. This procedure can be represented as follows:

$$(4 + 15) + 2 = 21.$$

The parentheses around $4 + 15$ indicates that the sum should be found first and that this result should be added to 2. Parentheses and brackets, $[ ]$, are often used to indicate which operations should be performed first.

Examples:

1. $[4 \times 2] + 6 = \underline{14}$

   The brackets indicate that $4 \times 2$ should be found first. The result should then be added to 6.

   $[4 \times 2] + 6 = 8 + 6 = 14.$

2. $3 + (2 + 7) = \underline{12}$

   The parentheses indicate that $2 + 7$ should be found first. Then 3 should be added to the result.

   $3 + (2 + 7) = 3 + 9 = 12.$

3. If the binary operation $\triangle$ is defined by

   $(a, b) \triangle (c, d) = (ad, c)$, then find

   $[(1, 3) \triangle (9, 2)] \triangle (8, 6).$
Solution: The brackets indicate that \((1, 3) \, B \, (1, 2)\) should be found first.

\((1, 3) \, B \, (9, 2) = (1 \times 2, 9) = (2, 9)\).

Then \((2, 9) \, B \, (8, 6)\) should be found.

\((2, 9) \, B \, (8, 6) = (2 \times 6, 8) = (12, 8)\).

Therefore:

\[
[(1, 3) \, B \, (9, 2)] \, B \, (8, 6) = (12, 8)
\]

Exercise 2.

Complete the following:

\[
\begin{align*}
[(17 + 3) + 7] + 7 &= \_\_\_\_\_\_\_\_\_\_ \\
18 + (3 \times 9) &= \_\_\_\_\_\_\_\_\_ \\
(18 + 3) + 9 &= \_\_\_\_\_\_\_\_\_ \\
(17 + 3) + 18 &= \_\_\_\_\_\_\_\_\_ \\
47 + (3 + 18) &= \_\_\_\_\_\_\_\_\_ \\
[14 + 6] + 27 &= \_\_\_\_\_\_\_\_\_ \\
14 + [6 + 27] &= \_\_\_\_\_\_\_\_\_ \\
\end{align*}
\]

When finding the sum of a list of natural numbers, does the way in which the numbers are grouped affect the final sum?

Using the binary operation on \(P\) defined by

\[(a, b) \, B \, (c, d) = (a + c, bd),\]

complete the following as quickly as possible. If you discover a short-cut, use it.

\[
\begin{align*}
[(1, 2) \, B \, (3, 4)] \, B \, (9, 1) &= \_\_\_\_\_\_\_\_\_\_ \\
(1, 2) \, B \, [(3, 4) \, B \, (9, 1)] &= \_\_\_\_\_\_\_\_\_\_ \\
\end{align*}
\]
\[(5, 2) B (9, 9) B (6, 5) = \]
\[(5, 2) B [(9, 9) B (6, 5)] = \]
\[(4, 7) B [(3, 8) B (1, 1)] = \]
\[[(4, 7) B (3, 8)] B (1, 1) = \]
\[(1, 6) B [(2, 11) B (11, 2)] = \]
\[[(1, 6) B (2, 11)] B (11, 2) = \]

Does the position of the brackets affect the final result?

Using the same binary operation as used above, complete the following:

\[(2, 1) B (8, 4) = \]
\[(8, 4) B (2, 1) = \]
\[(1, 6) B (2, 11) = \]
\[(2, 11) B (1, 6) = \]
\[(4, 7) B (3, 8) = \]
\[(3, 8) B (4, 7) = \]

In considering the final result does it matter which ordered pair comes first?

Can you write (discover, invent) a formula for a binary operation on \(\mathbb{P}\) so that the result will not depend on which ordered pair comes first? Try to find such a formula.

Try to write a formula for a binary operation on \(\mathbb{P}\) such that when the binary operation is used with three ordered pairs, the results will not depend on the
placement of the brackets. That is, write a formula so that
\[
[(a, b) B (c, d)] B (e, f) = (a, b) B [(c, d) B (e, f)].
\]
Complete the following.

\[
\begin{align*}
2 + 3 &= \underline{\quad} & 8 \times 7 &= \underline{\quad} \\
3 + 2 &= \underline{\quad} & 7 \times 6 &= \underline{\quad} \\
4 + 8 &= \underline{\quad} & 9 \times 6 &= \underline{\quad} \\
8 + 4 &= \underline{\quad} & 6 \times 9 &= \underline{\quad} \\
19 + 27 &= \underline{\quad} & 12 \times 28 &= \underline{\quad} \\
27 + 19 &= \underline{\quad} & 28 \times 12 &= \underline{\quad} \\
3 \times 4 &= \underline{\quad} & \frac{1}{2} &= \underline{\quad} \\
4 \times 3 &= \underline{\quad} & \frac{1}{2} &= \underline{\quad}
\end{align*}
\]

When finding the sum of two numbers, does the order in which the numbers occur affect the sum?

When finding the product of two numbers does the order in which the numbers occur affect the product?

Using the binary operation defined by

\[
(a, b) \ast (c, d) = (ab, c)
\]

complete the following.

\[
\begin{align*}
(1, 3) \ast (2, 4) &= \underline{\quad} \\
(2, 4) \ast (1, 3) &= \underline{\quad} \\
(4, 3) \ast (9, 2) &= \underline{\quad} \\
(9, 2) \ast (4, 3) &= \underline{\quad}
\end{align*}
\]

Does the order in which the ordered pairs occur affect the final result?
Using the binary operation defined by
\[(a, b) \ast (c, d) = (a + c, bc)\]
complete the following.
\[(1, 3) \ast (2, 4) = \quad \]
\[(2, 4) \ast (1, 3) = \quad \]
\[(6, 7) \ast (0, 1) = \quad \]
\[(0, 1) \ast (8, 7) = \quad \]

Does the order in which the ordered pairs occur affect the final result?

A binary operation associates with two elements of a set an element of the set. If the order in which the first two elements appear does not affect the result, the binary operation is said to be commutative. Is addition on the set of natural numbers a commutative binary operation? Is multiplication on the set of natural numbers a commutative binary operation?

Is the binary operation on \(P\) defined by
\[(a, b) \ast (c, d) = (ab, c)\]
a commutative binary operation? Is the binary operation on \(P\) defined by
\[(a, b) \ast (c, d) = (a + c, bd)\]
a commutative binary operation?

Consider the exercises on the use of grouping symbols (see page 160). When finding the sum of three natural numbers, does the way the numbers are grouped affect the sum?
Complete the following.

\[(2 \times 4) \times 7 = \] 
\[2 \times (4 \times 7) = \] 
\[(3 \times 9) \times 4 = \] 
\[3 \times (9 \times 4) = \] 
\[(4 \times 1) \times 18 = \] 
\[4 \times (1 \times 18) = \]

Does the way the numbers are grouped affect the product of three natural numbers?

Using the binary operation on \( P \) defined by
\[(a, b) B (c, d) = (ab, c)\]
complete the following.

\[((1, 2) B (4, 7)) B (9, 2) = \]
\[(1, 2) B [(4, 7) B (9, 2)] = \]
\[(3, 4) B [(8, 1) B (2, 7)] = \]
\[[(3, 4) B (8, 1)] B (2, 7) = \]

Does the way the ordered pairs are grouped affect the final result?

If \( B \) is defined by
\[(a, b) B (c, d) = (a + c, bd)\]
complete the following.

\[((1, 2) B (4, 7)) B (9, 2) = \]
\[(1, 2) B [(4, 7) B (9, 2)] = \]
\[(0, 1) B [(8, 1) B (2, 7)] = \]
\[[(0, 1) B (8, 1)] B (2, 7) = \]
Does the way the ordered pairs are grouped affect the final result?

Suppose that a binary operation is to be used on a list of elements of a set. If the way the elements of the list are grouped does not affect the final result, the binary operation is said to be associative. Which of the following are associative binary operations?

1. Addition on the set of natural numbers.
2. Multiplication on the set of natural numbers.
3. The binary operation on \( \mathbb{P} \) defined by
   \[(a, b) \circ (c, d) = (ac, c).\]
4. The binary operation on \( \mathbb{P} \) defined by
   \[(a, b) \circ (c, d) = (a + c, bd).\]

**Exercise 3.**

Using the binary operation defined by
\[(a, b) \circ (c, d) = (a + c, bd)\]
complete the following.

\[(8, 3) \circ (0, 1) = \_\_\_\_\_\_\_
\]
\[(4, 7) \circ [(2, 9) \circ (0, 1)] = \_\_\_\_\_\_
\]
\[\left( (5, 9) \circ (0, 1) \right) \circ (2, 6) = \_\_\_\_\_\_\_
\]
\[ (5, 9) \circ \left[ (0, 1) \circ (2, 6) \right] = \_\_\_\_\_\_\_
\]
\[(0, 1) \circ (8, 4) = \_\_\_\_\_\_
\]

Fill in the blanks.

\[(15, 3) \circ \_\_\_\_\_\_\_\_ = (15, 3)\]
\[(9, 5) \circ \_\_\_\_\_\_\_\_ = (9, 5)\]
Using the binary operation defined by 

\[(a, b) \ast (c, d) = (ac, bd)\]

complete the following.

\[(15, 3) \ast (1, 1) = \_\_\_
\]

\[(1, 1) \ast (9, 8) = \_\_\_
\]

\[(27, 42) \ast (1, 1) = \_\_\_
\]

\[\left[(293, 34) \ast (1, 1)\right] \ast (0, 1) = \_\_\_
\]

\[\left[(293, 34) \ast (1, 1)\right] \ast (0, 1) = \_\_\_
\]

Fill in the blanks.

\[(2, 4) \ast \_\_\_ = (2, 4)
\]

\[(3, 7) \ast \_\_\_ = (3, 7)
\]

\[\_\_\_ \ast (5, 1) = (5, 1)
\]

\[\_\_\_ \ast (8, 16) = (8, 16)
\]

Consider the binary operation defined by

\[(a, b) \ast (c, d) = (a + c, b + d)\]

Can you find an ordered pair \((x, y)\) such that

\[(a, b) \ast (x, y) = (a, b)\]

no matter which natural number is used in place of \(a\) and
no matter which natural number is used in place of \(b\)?

If so, which number should be used for \(x\)? Which number should be used for \(y\)?
Lesson 7

Using the binary operation \# defined by

\[(a, b) \# (c, d) = (a + c, b + d)\]

and the binary operation * defined by

\[(a, b) * (c, d) = (ac, bd),\]

fill in the blanks:

\[(4, -1) * [(1, 6) * (-8, 2)] = \]
\[
[(1, -1) * (1, 6)] \# [(4, -1) * (-8, 2)] = \]
\[(3, 9) * [(2, 7) \# (-3, -1)] = \]
\[
[(3, 9) * (2, 7)] \# [(3, 9) * (-3, -1)] = \]
\[
(-6, 3) * [(4, 9) \# (-5, 8)] = \]
\[
[(6, 3) * (4, 9)] \# [(6, 3) * (-5, 8)] = \]
\[
[-1, -3) * (4, 7)] \# [(-1, -3) * (5, 9)] = \]
\[
(-1, -3) * [(4, 7) \# (8, 9)] = \]
\[
(8, -7) * [(3, -5) \# (4, 9)] = \]
\[
[(8, -7) * (3, -5)] \# [(8, -7) \# \__ \__] = \]
\[
(1, 1) * [(3, -15) \# (2, 14)] = \]
\[
[\__ \# (3, -15)] \# [(1, 1) \# (2, 14)] = \]
\[
[(8, 3) * (8, 2)] \# [(8, 3) * (5, 7)] = \]
\[
\__ \# [(8, 2) \# (-3, 7)]

Using the binary operation @ defined by

\[(a, b) @ (c, d) = (a + c, ba)\]

and the binary operation & defined by

\[(a, b) \& (c, d) = (a + d, bc),\]

fill in the blanks.
\[
(3, 5) \circ [(9, 1) \& (8, 2)] = \\
[(3, 5) \circ (9, 1)] \& [(3, 5) \circ (8, 2)] = \\
(8, -1) \circ [(3, 2) \& (2, -3)] = \\
[(8, -1) \circ (3, 2)] \& [(8, -1) \circ (2, -3)] = \\
(-4, 7) \circ [(-1, 1) \& (8, -6)] = \\
[(-4, 7) \circ (-1, 1)] \& [(-4, 7) \circ (8, -6)] = \\
(-2, -5) \circ [(3, 6) \& (-4, 7)] = \\
[(-2, -5) \circ (3, 6)] \& [(-2, -5) \circ (-4, -7)] = \\
\]

Fill in the blanks.

7 \times (8 + 9) = \\
(7 \times 8) + (7 \times 9) = \\
14 \times [4 + (-9)] = \\
[14 \times 4] + [14 \times (-9)] = \\
37 \times [(-12) + 9] = \\
[37 \times (-12)] + [37 \times 9] = \\
[(-18) \times 7] + [(-18) \times 10] = \\
(-18) \times [7 + 10] = \\
[(-12) \times 41] + [(-12) \times (-18)] = -(-12) \times [41 + (-18)] \\
[16 \times 93] + [16 \times 92] = 16 \times [______ + 92] \\
[-47 \times 93] + [(-4) \times 106] = -4 \times [93 + ____] \\

A binary operation associates with two elements of a set an element of that set. Suppose the elements of a set are themselves sets. Consider the set \( S \) which has \( n \) elements the sets
A = \{1, 4, 7, 10, 13, 16, \ldots \}, \\
B = \{2, 5, 8, 11, 14, 17, \ldots \}, \text{ and} \\
C = \{3, 6, 9, 12, 15, 18, \ldots \}.

We shall define a binary operation \( \oplus \) on the set \( S \).

Let \( X \) and \( Y \) denote elements of \( S \). Then \( X \oplus Y \) denotes the element of \( S \) which \( \oplus \) associates with \( X \) and \( Y \).

To find \( X \oplus Y \), pick any element of \( X \), pick any element of \( Y \) and find their sum. Then \( X \oplus Y \) is the set containing this sum.

Examples:

Find \( A \oplus B \). 10 is an element of \( A \). 8 is an element of \( B \). \( 10 + 8 = 18 \). 18 is an element of \( C \).

Therefore \( A \oplus B = C \).

Find \( C \oplus A \). 6 is an element of \( C \) and 16 is an element of \( A \). 6 + 16 = 22 and 22 is an element of \( A \).

Therefore \( C \oplus A = A \).

Exercises 7.

Fill in the blanks.

\[
\begin{align*}
-12 \times [41 + (-18)] &= [(12) \times \underline{\phantom{0}}] + [(12) \times (-18)] \\
16 \times [58 + 92] &= [\underline{\phantom{0}} \times 58] + [16 \times 92] \\
-1 \times [93 + 106] &= [\underline{\phantom{0}} \times 93] + [\underline{\phantom{0}} \times 106] \\
37 \times [(-12) + 9] &= [37 \times \underline{\phantom{0}}] + [37 \times \underline{\phantom{0}}] \\
[6 \times 12] + [6 \times 84] &= \underline{\phantom{0}} \times [12 + 84] \\
[7 \times \underline{\phantom{0}}] + [7 \times 8] &= 7 \times [\underline{\phantom{0}} + \underline{\phantom{0}}] 
\end{align*}
\]
\[ [8 \times 6] + [19 \times 6] = [\quad + \quad] \times 6 \]
\[ [14 \times 9] + [21 \times 9] = [14 + \quad] \times \quad \]
\[ 2 \times [8 \times 7] = \quad \]
\[ [2 \times 8] \times [2 \times 7] = \quad \]
\[ -8 + [4 \times 9] = \quad \]
\[ [-8 + 4] \times [-8 + 9] = \quad \]
\[ 7 + [-6 \times 12] = \quad \]
\[ [7 + (-6)] \times [7 + 12] = \quad \]

Referring to the binary operation \( \circ \) defined just before exercise 7, fill in the blanks.

\[ C \circ B = \quad \]
\[ B \circ A = \quad \]
\[ C \circ C = \quad \]
\[ A \circ C = \quad \]
\[ B \circ \quad = B \]
\[ C \circ \quad = A \]

Is \( + \) a commutative binary operation? \( \quad \)

\[ A \circ (C \circ B) = \quad \]
\[ (A \circ C) \circ B = \quad \]
\[ B \circ (B \circ A) = \quad \]
\[ (B \circ B) \circ A = \quad \]

Is \( \circ \) an associative binary operation? \( \quad \)

Is there an identity element for \( \circ \)?
Lesson 3

Let $S$ be a set and let $*$ and $\circ$ be binary operations defined on $S$. If for any elements $A$, $B$, and $C$ of $S$

$$A \circ (B \circ C) = (A \circ B) \circ (A \circ C),$$

then $\circ$ is said to be distributive over $*$. 

1. Is multiplication distributive over addition? (See Lesson 7.)

2. Is addition distributive over multiplication? (See Lesson 7.)

3. Is $*$ distributive over $\circ$? (See Lesson 7.)

4. Is $\circ$ distributive over $*$? (See Lesson 7.)

In Lesson 1 it is stated that two ordered pairs are equal if they have the same first component and the same second component. We indicate the fact that two ordered pairs are equal by placing the symbol $=$ between them. If two ordered pairs are not equal, this can be indicated by placing the symbol $\neq$ between them. Place the appropriate symbol $=$ or $\neq$ between the following pairs of ordered pairs.

$(3, 7) \quad (8, 7)$

$(-4, 2) \quad (-4, 2)$

$(7 - 6, 9) \quad (1, 9)$

$(-8 \times 3, 6 - (-2)) \quad (5, 3)$

$(6 - 9, 4 + (-6)) \quad (-3, -2)$
If \( a + d = b - c \), then we shall place the symbol "\( R \)" between the pairs \((a, b)\) and \((c, d)\) and we shall say that \((a, b)\) is related to \((c, d)\). Consider the pairs \((3, 14)\) and \((7, 4)\). \(3 + 4 = 7\) and \(14 - 7 = 7\). Therefore \((3, 14) \ R \ (7, 4)\) and we say that \((3, 14)\) is related to \((7, 4)\). Is \((-6, 3)\) related to \((6, 1)\)?

Consider the pairs \((6, 4)\) and \((9, 3)\). \(6 + 3 = 9\) and \(4 - 9 = -5\). Since \(6 + 3\) and \(4 - 9\) are not equal, we say that \((6, 4)\) and \((9, 3)\) are not related. We indicate this by placing the symbol "\( R \)" between the ordered pairs. Thus \((6, 4) \ R \ (9, 3)\). Place the appropriate symbol "\( R \)" or "\( \not{R} \)" between the following pairs of ordered pairs:

\[
\begin{align*}
(6, 14) & \quad \_\_\_ \quad (3, 5) \\
(3, 5) & \quad \_\_\_ \quad (6, 14) \\
(7, 2) & \quad \_\_\_ \quad (9, 3) \\
(4, -2) & \quad \_\_\_ \quad (4, 6) \\
(4, 6) & \quad \_\_\_ \quad (4, -2) \\
(3, 7) & \quad \_\_\_ \quad (2, 2) \\
(2, 2) & \quad \_\_\_ \quad (-5, 5) \\
(3, 7) & \quad \_\_\_ \quad (-5, 5)
\end{align*}
\]

Many different relations can be defined on sets of ordered pairs. Consider the relation, \( R_1 \), defined as follows: \((a, b)\) is related to \((c, d)\) by \( R_1 \) if \( a + d = b + c\). If \( a + d = b + c\) then we write \((a, b) \ R_1 (c, d)\). If \( a + d \neq b + c\), then we write \((a, b) \not{R_1} (c, d)\).
(a, b) \not \in R_1 (c, d) \text{ and we say that } (a, b) \text{ is not related to } (c, d) \text{ by the relation } R_1. \text{ Consider the pairs } (4, 8) \text{ and } (2, 6). \ 4 + 6 = 10 \text{ and } 3 + 2 = 10. \text{ Therefore } (4, 8) \text{ is related to } (2, 6) \text{ by the relation } R_1 \text{ and we indicate this by writing } (4, 8) R_1 (2, 6).

Consider the pairs (9, 2) \text{ and } (8, 3). \ 9 + 3 = 12 \text{ and } 2 + 8 = 10. \text{ Since } 9 + 3 \text{ is not equal to } 2 + 8 \text{ we say that } (9, 2) \text{ is not related to } (8, 3) \text{ by the relation } R_1 \text{ and we indicate this by writing } (9, 2) \not R_1 (8, 3).

**Exercise 8.**

Determine which pairs of ordered pairs are related by 
\( R_1 \) and which pairs of ordered pairs are not related by \( R_1 \). 
That is, fill in the blanks using the appropriate symbol, 
\( R_1, \) or \( \not R_1. \)

\[
\begin{align*}
(3, 7) & \quad (3, 7) \\
(4, 8) & \quad (1, 5) \\
(1, 5) & \quad (4, 8) \\
(9, 6) & \quad (5, 1) \\
(5, 1) & \quad (9, 6) \\
(8, 16) & \quad (-3, 5) \\
(-3, 5) & \quad (8, 16) \\
(7, 2) & \quad (8, 3) \\
(8, 3) & \quad (7, 2) \\
(4, 11) & \quad (7, 14)
\end{align*}
\]
(−6, 9) _____ (−6, 9)
(15, 3) _____ (15, 3)
(18, 21) _____ (18, 21)
(7, 5) _____ (10, 8)

______ (10, 8) _____ (5, 3)
(7, 5) _____ (5, 3)
(4, 3) _____ (8, 7)

______ (8, 7) _____ (9, 8)
(4, 3) _____ (9, 8)

Fill in the blanks. (First see question 3 at the beginning of Lesson 3.)

(8, −7) * [(3, −5) # (4, 9)] =

[(____ * (3, −5)) # (____ * (4, 9))] =

(−1, 1) * [(3, 12) # (12, −3)] =

[(____ * ____)] # [(____ * ____)] =

(5, −2) * [(6, 4) # (6, 3)] =

[(____ * ____)] # [(____ * ____)] =

[(4, 6) # (−7, 2)] * (−12, 3) =

[(____ * ____)] # [(____ * ____)] =

[(−4, 3) * (8, 2)] # [(−4, 3) * (−3, 7)] =

[(____ * (8, 2) # (−3, 7))] =

[(3, 9) * (−2, 7)] # [(3, 9) * (−3, −1)] =

[(____ * (−2, 7))] # [(____ * (−3, −1))] =

[(−6, 3) * (4, 9)] # [(−6, 3) * (−5, 8)] =

[(____ * (−5, 8))] =

[(____) # (____)]
\[ [(-1, -3) \times (4, 1)] \setminus [(-1, -3) \times (8, 9)] = \]
\[ \hspace{1cm} \setminus \hspace{1cm} \setminus \]
\[ [(4, 9) \times (9, 3)] \setminus \]
\[ [(8, 7) \times (9, 9)] \setminus [(2, -1) \times (9, 9)] = \]
\[ \hspace{1cm} \setminus \hspace{1cm} \setminus \]

Let \( S \) be the set containing the elements

\( A = \{1, 5, 9, 13, 17, 21, 25, 29, \ldots \} \),

\( B = \{2, 6, 10, 14, 18, 22, 26, 30, \ldots \} \),

\( C = \{3, 7, 11, 15, 19, 23, 27, 31, \ldots \} \), and

\( D = \{4, 8, 12, 16, 20, 24, 28, 32, \ldots \} \).

\( \odot \) is the binary operation of \( S \) defined as follows: If \( X \) and \( Y \) are elements of \( S \), \( a \) is an element of \( X \) and \( b \) is an element of \( Y \), then \( X \odot Y \) is the element (set) of \( S \) containing \( a \times b \). For example, find \( A \odot C \). \( 9 \) is an element of \( A \) and \( 15 \) is an element of \( C \). \( 9 \times 15 = 135 \). \( 135 \) is an element of \( C \). Therefore \( A \odot C = C \).

Fill in the blanks.

\( B \odot C = \) __________

\( C \odot C = \) __________

\( A \odot D = \) __________

\( D \odot A = \) __________
Lesson 7

Let \( S \) be a set. A relation on \( S \) associates with two elements of \( S \) either the word, "yes," or the word, "no."

If \( R \) is a relation on \( S \), \( A \) and \( B \) are elements of \( S \), and \( R \) associates with \( A \) and \( B \) the word "yes," then we say that \( A \) is related to \( B \) and we indicate this by writing \( A R B \).

If \( R \) associates with \( A \) and \( B \) the word "no," then we say that \( A \) is not related to \( B \) and we indicate this by writing \( A \not{R} B \). If \( R \) is a relation on \( S \), then exactly one of the following is a true statement:

1. \( A \) is related to \( B \).
2. \( A \) is not related to \( B \).

\( A \) is related to \( B \) if and only if \( R \) associates with \( A \) and \( B \) the word "yes." \( A \) is not related to \( B \) if and only if \( R \) associates with \( A \) and \( B \) the word "no."

1. If \( a \) and \( b \) are integers, is exactly one of the following true?
   a. \( a = b \).
   b. \( a \neq b \).

   If the answer is yes, then "=" is a relation on the set of integers.

2. If \( (a, b) \) and \( (c, d) \) are ordered pairs of integers, is exactly one of the following true?
   a. \( (a, b) = (c, d) \).
   b. \( (a, b) \neq (c, d) \).
Is "=" a relation on the set of ordered pairs of integers?

3. Is R a relation on the set of ordered pairs of integers? (See Lesson 8.)

4. Is $R_1$ a relation on the set of ordered pairs of integers? (See Lesson 8.)

**Definition:** Let S be a set and let R be a relation on S. Then R is an equivalence relation if for any elements $a$, $b$, and $c$ of S,

1. $aRa$ (Reflexive Principle)
2. If $aRb$, then $bRa$. (Symmetric Principle)
3. If $aRb$ and $bRc$, then $aRc$. (Transitive Principle)

**Examples.**

1. Is "=" an equivalence relation on the set of integers?
   For "=" to be an equivalence relation, three things must be true.
   a. If $a$ is an integer, then "$a = a$" must be true. Is any integer equal to itself?
   b. If $a = b$, then $b = a$. Is this true?
   c. If $a = b$ and $b = c$, then $a = c$. If $a = b$ and $b = c$, then $a$ and $c$ are both equal to $b$. Does this imply that $a$ and $c$ are equal to each other?

2. Place the appropriate symbol, "=" or "$
eq$", between the following pairs of ordered pairs.
a. (3, 7) \sim (3, 7)
   (9, 2) \sim (9, 2)
   (-4, 6) \sim (-4, 6)
   (-18, -31) \sim (-18, -31)

b. (6, 9) \sim (3 + 3, 5 + 4)
   (3 + 3, 5 + 4) \sim (6, 9)
   (-8, 5) \sim (2 - 10, 15 - 10)
   (2 - 10, 15 - 10) \sim (-8, 5)
   (11, 6) \sim (5 + 6, 6)
   (5 + 6, 6) \sim (11, 6)

c. (3, 12) \sim (1 + 2, 6 + 6)
   (1 + 2, 6 + 6) \sim (4 - 1, 13 - 1)
   (3, 12) \sim (4 - 1, 13 - 1)
   (-1, 7) \sim (5 - 9, 9 - 2)
   (5 - 9, 9 - 2) \sim (8 - 12, 12 - 5)
   (-1, 7) \sim (8 - 12, 12 - 5)

Is "\sim" an equivalence relation on the set of ordered pairs of integers?

3. Is R an equivalence relation on the set of ordered pairs of integers? (See Lesson 8.)

4. Is R_1 an equivalence relation on the set of ordered pairs of integers? (See Lesson 8.)

Let S be the set containing the elements
\[ A = \{1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, \ldots \} \]
\[ B = \{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, \ldots \} \]
Let $\circ$ be a binary operation on $S$ defined as follows: If $X$ and $Y$ are elements of $S$, $a$ is an element of $X$, and $b$ is an element of $Y$, then $X \circ Y$ is the element (set) of $S$ containing $a + b$. For example, find $A \circ C$. 9 is an element of $A$ and 15 is an element of $C$. $9 + 15 = 24$. 24 is an element of $D$. Therefore, $A \circ C = D$.

Exercise 9.

Fill in the blanks.

\[
\begin{align*}
A \circ B &= \quad \\
B \circ A &= \\
B \circ D &= \\
D \circ B &= \\
C \circ D &= \\
D \circ C &= \\
A \circ D &= \\
D \circ A &= \\
(A \circ C) \circ D &= \\
A \circ (C \circ D) &= \\
(B \circ A) \circ B &= \\
B \circ (A \circ B) &= \\
C \circ (D \circ A) &= \\
(C \circ D) \circ A &= \\
\end{align*}
\]
Is the binary operation \(\circ\) commutative?

Is the binary operation \(\circ\) associative?

Is there an identity element for the binary operation \(\circ\)?

Using the binary operation \(\circ\) and the binary operation \(\oplus\) as defined in Lesson 8, fill in the blanks.

\[
\begin{align*}
B \circ (C \circ A) &= \\
(B \circ C) \circ (B \circ A) &= \\
D \circ (B \circ C) &= \\
(D \circ B) \circ (D \circ C) &= \\
A \circ (B \circ A) &= \\
(A \circ B) \circ (A \circ A) &= \\
\end{align*}
\]

Does \(\circ\) distribute over \(\oplus\)?

Is \(\circ\) a commutative binary operation?

Is \(\circ\) an associative binary operation?

Is there an identity element for \(\circ\)?

Fill in the blanks.

\[
\begin{align*}
A \circ & = D \\
B \circ & = D \\
C \circ & = D \\
D \circ & = D \\
A \circ & = A \\
B \circ & = A \\
C \circ & = A \\
D \circ & = A
\end{align*}
\]
APPENDIX C

SAMPLE LESSONS FOR THE EXPOSITION

METHOD OF TEACHING
Throughout this course the term set will be used to refer to any well-defined collection, group, or class of objects or ideas. The phrase "well-defined" means that the set is described precisely enough so that one can tell whether or not any given object belongs to the set. The objects contained in a set are called elements. In a set of dishes, each dish is an element of the set of dishes. In the case of a coin collection, each coin is an element of the set of coins.

One way of denoting a set is by listing all of its elements. If the elements of a set are listed, the list is placed within braces. Thus \( \{1, 2, 3, 4, 5, 6\} \), denotes the set consisting of all natural numbers from 1 through 6 inclusive. In some cases not all the elements of a set can be listed. One such set is the set of natural numbers. This set may be denoted as follows: \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots \} \).

Another concept which will be used extensively throughout this course is the concept of binary operation. A binary operation is a way of associating with two elements of a set a third element of that set. Addition is a binary operation on the set of natural numbers. For example, the operation of addition associated with the
natural numbers 3 and 5 the natural number 8. It associates with the numbers 598 and 683 the number 1281. Multiplication is also a binary operation on the set of natural numbers. This operation associates with the numbers 3 and 5 the number 15. It associates with the numbers 598 and 683 the number 408,434.

It is possible to define many binary operations on the set of natural numbers. Suppose that with two natural numbers we associate the first of the two numbers. This is an example of a binary operation on the set of natural numbers. This operation associates with 3 and 5 the number 3. It associates with 17 and 9 the number 17.

The concept of binary operation leads us to consider the concept of ordered pair. A binary operation associates with a pair of elements of a set an element of the set. The binary operation discussed in the preceding paragraph associates with the pair of numbers 3 and 5 the number 3. It associates with the pair of numbers 5 and 3 the number 5. In each case the same pair of numbers is used, but the results are different. This is true because the order in which the elements of the pair are listed is different.

An *ordered pair* consists of a pair of elements listed in a specific order. An ordered pair will be denoted by listing the elements of the pair with a comma between them and placing the list in parentheses. Thus (3, 5) denotes
the ordered pair consisting of the numbers 3 and 5 with 3 coming before 5. $(5, 3)$ denotes the same pair, but with 5 coming before 3. This shows that $(3, 5)$ is not the same ordered pair as $(5, 3)$. In the ordered pair $(5, 3)$, 5 is called the first component of the ordered pair and 3 is called the second component of the ordered pair. Two ordered pairs are said to be equal if they have the same first component and the same second component.

Let us denote the set of all ordered pairs of natural numbers by the symbol $P$. Some of the elements of $P$ are $(3, 9)$, $(5, 1)$, $(5, 5)$, $(3911, 1)$, and $(2, 3800972)$. Many interesting and unusual binary operations can be defined on the set $P$. Remember that a binary operation is a way of associating with two elements of a set another element of the set. Thus a binary operation on $P$ associates with a pair of ordered pairs an ordered pair. An example of a binary operation on $P$ is the operation which associates with a pair of ordered pairs the second ordered pair. This binary operation associates with $(1, 1)$ and $(9, 5)$ the ordered pair $(9, 5)$. It associates with $(1, 17)$ and $(39, 14)$ the ordered pair $(39, 14)$.

If we wish to consider the number which the binary operation of addition on the set of natural numbers associates with 7 and 4 we write $7 + 4$. Since the operation of addition associates with 7 and 4 the number 11, we
write \(7 + 4 = 11\). This is a very compact and convenient way of saying that the binary operation addition associates with 7 and 4 the natural number 11. When considering binary operations on \(P\) we shall use the notation 
\[
(a, b) \ast (c, d) = (e, f)
\]
to denote that the binary operation \(\ast\) associates with the pair of ordered pairs 
\((a, b)\) and \((c, d)\) the ordered pair \((e, f)\). Thus if \(\ast\) denotes the binary operation on \(P\) discussed in the preceding paragraph we have that 
\[
(1, 3) \ast (9, 5) = (9, 5).
\]
Also 
\[
(1, 17) \ast (39, 14) = (39, 14).
\]
A binary operation on \(P\) can often be defined using a general formula. 
\[
(a, b) \ast (c, d) = (e, f)
\]
is a formula which can be used to define the binary operation on \(P\) discussed in the preceding paragraph. If \(a\) is replaced by a natural number and if \(b\) is replaced by a natural number, then \((a, b)\) becomes an element of \(P\). Similarly if \(e\) is replaced by a natural number and if \(f\) is replaced by a natural number, then \((e, f)\) becomes an element of \(P\). Then 
\[
(a, b) \ast (c, d) = (e, f)
\]
is a way of saying that the binary operation \(\ast\) associates with \((a, b)\) and \((c, d)\) the ordered pair \((e, f)\).

Consider the binary operation on \(P\) defined by the formula
\[
(a, b) \ast (c, d) = (a + c, b + d).
\]
This formula states that to find the first component of the ordered pair associated with \((a, b)\) and \((c, d)\) add the
first components of \((a, b)\) and \((c, d)\). To find the second component multiply the second components of \((a, b)\) and \((c, d)\). According to the formula the ordered pair associated with \((4, 3)\) and \((9, 7)\) is \((4 + 9, 3 \times 7)\) or \((13, 21)\). Similarly \((12, 3)\) \(\times (4, 17) = (12 + 4, 3 \times 17)\) or \((12, 3) \times (4, 17) = (16, 51)\).

**EXERCISE I.**

In each of the following exercises a binary operation is defined. Use the definition to find the ordered pair associated with the given pair of ordered pairs.

Example: \((a, b) \times (c, d) = (a \times b, c)\).

Complete the following.

\((3, 2) \times (4, 7) = \_\_\_\_

\((5, 1) \times (9, 18) = \_\_\_\_

**Solution:**

\((3, 2) \times (4, 7) = (3 \times 2, 4) = (6, 4)\)

\((5, 1) \times (9, 18) = (5 \times 1, 9) = (5, 9)\)

1. \((a, b) \times (c, d) = (a \times d, b)\)

Example: \((1, 9) \times (7, 19) = (1 \times 7, 9) = (7, 9)\).

Complete the following.

\((4, 3) \times (13, 5) = \_\_\_\_

\((1, 1) \times (15, 2) = \_\_\_\_

\((5, 7) \times (9, 3) = \_\_\_\_

\((15, 21) \times (1, 1) = \_\_\_\_

\((9, 3) \times (5, 7) = \_\_\_\_


2. \((a, b) \ast (c, d) = (a \times c, b + d)\).

Example: \((3, 2) \ast (9, 1) = (3 \times 9, 2 + 1) = (27, 3)\).

Complete the following.

(7, 1) \ast (9, 5) =

(1, 1) \ast (8, 3) =

(6, 9) \ast (14, 7) =

(9, 5) \ast (7, 1) =

(14, 7) \ast (6, 9) =

(13, 11) \ast (8, 2) =

(6, 3) \ast (1, 1) =

(8, 2) \ast (13, 11) =

3. Make up (invent) a formula of your own which defines a binary operation on \(P\). Use your formula to complete the following.

(3, 1) \ast (2, 2) =

(6, 8) \ast (3, 11) =

(15, 1) \ast (4, 7) =

(8, 2) \ast (8, 6) =

(8, 6) \ast (3, 11) =

(7, 4) \ast (1, 15) =

(8, 2) \ast (7, 5) =

(6, 8) \ast (11, 3) =

(4, 7) \ast (1, 15) =
A binary operation associates with two elements of a set an element of the set. If the order in which the first two elements appear does not affect the final result, the binary operation is said to be commutative.

Examples:

Addition on the set of natural numbers is a commutative binary operation. If a list of numbers is to be added, the result does not depend on the order in which the numbers are listed.

\[ 7 + 3 = 10 \]
\[ 3 + 7 = 10 \]
\[ 4 + 11 = 15 \]
\[ 11 + 4 = 15 \]

Multiplication on the set of natural numbers is a commutative binary operation. For example:

\[ 7 \times 12 = 84 \]
\[ 12 \times 7 = 84 \]
\[ 4 \times 26 = 104 \]
\[ 26 \times 4 = 104 \]

The binary operation on \( P \) defined by

\[ (a, b) B (c, d) = (a \times c, b + d) \]

is a commutative binary operation. Verify this by completing the following.

\[ (1, 3) B (8, 4) = \]
Consider the binary operation \( \circ \) defined by the following:
\[
(a, b) \circ (c, d) = (2a + 2c, bd)
\]
This is a commutative binary operation.

Examples:
\[
\begin{align*}
(3, 4) \circ (1, 2) &= (2\cdot3 + 2\cdot1, 4\cdot2) = (8, 8) \\
(1, 2) \circ (3, 4) &= (2\cdot1 + 2\cdot3, 2\cdot4) = (8, 8)
\end{align*}
\]
Verify that this binary operation is commutative by completing the following.
\[
\begin{align*}
(1, 3) \circ (2, 0) &= \\
(2, 8) \circ (1, 3) &= \\
(4, 6) \circ (1, 1) &= \\
(1, 1) \circ (4, 6) &= \\
(5, 4) \circ (2, 11) &= \\
(2, 11) \circ (5, 4) &= \\
(12, 15) \circ (21, 42) &=
\end{align*}
\]
The binary operation of addition on the set of natural numbers associates with 9 and 12 the natural number 21. 21 is called the sum of 9 and 12. Suppose that it is necessary to find the sum of a list of natural numbers. Consider the list 7, 3, 9, 4, and 6. The sum can be found by finding the sum of 7 and 3. This sum is 10. Then find the sum of 10 and 9. This sum is 19. Then find 19 + 4. 19 + 4 = 23. Then find 23 + 6. 23 + 6 = 29. This is the sum of 7, 3, 9, 4, and 6.

Using this procedure two numbers were added in each step of the procedure. This type of procedure was necessary because addition is a binary operation. Only two numbers can be added at a time. Consider the task of finding the sum of the list 4, 15, and 2. 4 + 15 = 19. 19 + 2 = 21. Therefore the desired sum is 21. This procedure can be represented as follows:

\[(4 + 15) + 2 = 21.\]

The parentheses around 4 + 15 indicates that this sum should be found first and that the result should be added to 2. Parentheses and brackets, \([ \quad ]\), are often used to indicate which operations should be performed first.

Examples:

1. \([4 \times 2] + 6 = \)

(21, 12) \& (12, 15) = 21
The brackets indicate that \(4 \times 2\) should be found first. The result should then be added to 6.

\[
[4 \times 2] + 6 = 8 + 6 = 14.
\]

2. \(3 + (2 + 7) = \ldots\)

The parentheses indicate that \(2 + 7\) should be found first. Then 3 should be added to the result.

\[3 + (2 + 7) = 3 + 9 = 12.\]

3. If the binary operation \(B\) is defined by

\[(a, b) \, B \, (c, d) = (ad, c),\]

then find \([ (1, 3) \, B \, (9, 2) ] \, B \, (8, 6) \).

Solution: The brackets indicate that \((1, 3) \, B \, (9, 2)\) should be found first.

\[(1, 3) \, B \, (9, 2) = (1 \times 2, 9) = (2, 9)\]

Then \((2, 9) \, B \, (8, 6)\) should be found.

\[(2, 9) \, B \, (8, 6) = (2 \times 6, 8) = (12, 8).\]

Therefore,

\[\[(1, 3) \, B \, (9, 2)] \, B \, (8, 6) = (12, 8).\]

Suppose that a binary operation is to be used on a list of elements of a set. If the way the elements of the list are grouped does not affect the final result, the binary operation is said to be associative.

Exercise 2.

Addition on the set of natural numbers is an associative binary operation. Verify this by completing the following.
Consider the binary operation on $F$ defined by 

$$(a, b) \circ (c, d) = (a + c, 3b).$$

This binary operation is an associative binary operation. Verify this by completing the following.

$$(1, 2) \circ (3, 4) \circ (9, 1) = \_ \_ \_ \_ \_ \_ \_$$

$$(1, 2) \circ [(3, 4) \circ (9, 1)] = \_ \_ \_ \_ \_ \_ \_$$

$$(5, 2) \circ (9, 9) \circ (6, 5) = \_ \_ \_ \_ \_ \_ \_$$

$$(5, 2) \circ [(9, 9) \circ (6, 5)] = \_ \_ \_ \_ \_ \_ \_$$

$$(1, 6) \circ [(2, 11) \circ (11, 2)] = \_ \_ \_ \_ \_ \_ \_$$

$$(1, 6) \circ (2, 11) \circ (11, 2) = \_ \_ \_ \_ \_ \_ \_$$

The binary operation used above is a commutative binary operation. Verify this by completing the following.

$$(2, 1) \circ (8, 4) = \_ \_ \_ \_ \_ \_ \_$$

$$(8, 4) \circ (2, 1) = \_ \_ \_ \_ \_ \_ \_$$

$$(1, 6) \circ (2, 11) = \_ \_ \_ \_ \_ \_ \_$$

$$(2, 11) \circ (1, 6) = \_ \_ \_ \_ \_ \_ \_$$

$$(4, 7) \circ (3, 8) = \_ \_ \_ \_ \_ \_ \_$$

$$(3, 8) \circ (4, 7) = \_ \_ \_ \_ \_ \_ \_$$
Lesson 3

Review.

Which of the following binary operations are commutative?

1. Addition on the set of natural numbers.
2. Multiplication on the set of natural numbers.
3. The binary operation on \( \mathbb{F} \) defined by:
   \((a, b) \circ (c, d) = (ab, c)\).
4. The binary operation on \( \mathbb{F} \) defined by:
   \((a, b) \circ (c, d) = (a + c, bd)\).

Which of the following binary operations are associative?

1. Addition on the set of natural numbers.
2. Multiplication on the set of natural numbers.
3. The binary operation on \( \mathbb{F} \) defined by:
   \((a, b) \circ (c, d) = (ab, c)\).
4. The binary operation on \( \mathbb{F} \) defined by:
   \((a, b) \circ (c, d) = (a + c, bd)\).

A binary operation associates with two elements of a set an element of the set. Let \( S \) denote a set and let "a" and "b" denote elements of the set. Let "\( \circ \)" denote a binary operation on \( S \). Then we shall denote the element of \( S \) which \( \circ \) associates with \( a \) and \( b \) by \( a \circ b \). Suppose there is an element of \( S \), denoted by the symbol "e", such that for each element \( a \) of \( S \), \( e \circ a = a \) and \( a \circ e = a \). If such an
element, e, exists it is called a neutral element or an identity element.

Examples.

1. Given the set
   \[ \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots \} \]
   0 is an identity element for addition on this set.

   For example:
   
   \[
   \begin{align*}
   7 + 0 &= 7 \\
   0 + 7 &= 7 \\
   18 + 0 &= 18 \\
   0 + 18 &= 18 \\
   921 + 0 &= 921 \\
   0 + 921 &= 921
   \end{align*}
   \]

2. 1 is an identity element for multiplication on the set of natural numbers. For example:

   \[
   \begin{align*}
   3 \times 1 &= 3 \\
   1 \times 3 &= 3 \\
   17 \times 1 &= 17 \\
   1 \times 17 &= 17 \\
   42 \times 1 &= 42 \\
   1 \times 42 &= 42 \\
   385 \times 1 &= 385 \\
   1 \times 385 &= 385
   \end{align*}
   \]

3. \((0, 1)\) is an identity element for the binary operation on \(P\) defined by \((a, b) \circ (c, d) = (a + c, bd)\).
Verify this by completing the following.

(8, 3) B (0, 1) = 
(0, 1) B (8, 3) = 
(2, 9) B (0, 1) = 
(0, 1) B (2, 9) = 
(0, 1) B (5, 9) = 
(5, 9) B (0, 1) = 

Fill in the blanks.

(15, 3) B ______ = (15, 3)
(9, 5) B ______ = (9, 5)
(7, 4) B ______ = (7, 4)
______ B (8, 14) = (8, 14)
______ B (9, 17) = (9, 17)

In each of the last five exercises the blank should have been filled with (0, 1). The answer should be (0, 1) in each case because (0, 1) is an identity element for the binary operation B.

Exercise 2.

(1, 1) is an identity element for the binary operation on P defined by

\[(a, b) B (c, d) = (ac, bd).\]

Verify this by completing the following.

(15, 3) B (1, 1) = 
(1, 1) B (9, 8) = 
(9, 8) B (1, 1) = 

(27, 34) B (1, 1) =
(1, 1) B (27, 34) =
(293, 47) B (1, 1) =
(1, 1) B (293, 47) =

Fill in the blanks.
(2, 4) B ____ = (2, 4)
(3, 7) B ____ = (3, 7)
____ B (5, 1) = (5, 1)
____ B (8, 16) = (8, 16)

(0, 0) is an identity element for the binary operation defined by

\[(a, b) \ B (c, d) = (a + c, b \cdot d).\]

Verify this by completing the following.

(2, 4) B (0, 0) =
(0, 0) B (2, 4) =
(3, 19) B (0, 0) =
(0, 0) B (3, 19) =

\[((2, 3) B (9, 2)) B (0, 0) =
(0, 0) B [(8, 7) B (4, 13)] =

Fill in the blanks.
____ + 7 = 7
4 + ____ = 4
18 \times ____ = 18
924 \times ____ = 924
138 + ____ = 138
296 \times \underline{\phantom{0}} = 296

\underline{\phantom{0}} \times 48321 = 48321

\underline{\phantom{0}} + 48321 = 48321
Lesson 7

Let S be a set and let \( \circ \) and \( \cdot \) be binary operations defined on S. If for any elements A, B, and C of S

\[ A \circ (B \cdot C) = (A \circ B) \cdot (A \circ C), \]

then \( \circ \) is said to be distributive over \( \cdot \).

Examples.

1. Multiplication is distributive over addition.
   That is, \( a \cdot (b + c) = (a \cdot b) + (a \cdot c) \).
   For example,
   \[ 7 \cdot (8 + 9) = 7 \cdot 17 = 119 \text{ and } \]
   \[ (7 \cdot 8) + (7 \cdot 9) = 56 + 63 = 119. \]
   Therefore
   \[ 7 \cdot (8 + 9) = (7 \cdot 8) + (7 \cdot 9), \]
   \[ [-13] \cdot 7 + [-13] \cdot 10 = -126 + -180 = -306, \]
   \[ -13 \cdot [7 + 10] = -13 \cdot 17 = -306. \]
   Therefore
   \[ [-13] \cdot 7 + [-13] \cdot 10 = -13 \cdot [7 + 10]. \]

2. Addition is not distributive over multiplication.
   For example,
   \[ 2 + (8 \cdot 7) = 2 + 56 = 58, \]
   \[ (2 + 8) \cdot (2 + 7) = 10 \cdot 9 = 90. \]
   Therefore
   \[ 2 + (8 \cdot 7) \neq (2 + 8) \cdot (2 + 7). \]

3. The binary operation \( \ast \) defined by
   \[ (a, b) \ast (c, d) = (ac, bd) \]
is distributive over the binary operation $\#$ defined by
\[(a, b) \# (c, d) = (a + c, b + d).\]

For example,
\[
(1, -1) \# [(1, 6) \# (-8, 2)] = (1, -1) \# (-7, 8)
\]
\[
= (-23, -6)
\]
\[
[(1, -1) \# (1, 6)] \# [(1, -1) \# (-8, 2)] = (1, -6) \# (-32, -2)
\]
\[
= (-28, -8)
\]
Therefore
\[
(1, -1) \# [(1, 6) \# (-8, 2)] \neq [(1, -1) \# (1, 6)] \# [(1, -1) \# (-8, 2)]
\]

4. The binary operation $\circ$ defined by
\[(a, b) \circ (c, d) = (a + c, bd)\]
is not distributive over the binary operation $\&$ defined by
\[(a, b) \& (c, d) = (a \circ d, bd).\]

This can be shown as follows:
\[
(3, 5) \circ [(9, 1) \& (8, 2)] = (3, 5) \circ (11, 8) = (14, 40)
\]
\[
[(3, 5) \circ (9, 1)] \& [(3, 5) \circ (8, 2)] = (12, 5) \& (11, 10) = (22, 55)
\]
Therefore
\[
(3, 5) \circ [(9, 1) \& (8, 2)] \neq [(3, 5) \circ (9, 1)] \& [(3, 5) \circ (8, 2)]
\]
One counterexample is sufficient to show that $\circ$ is not distributive over $\&$.

Use the fact that multiplication is distributive over addition to fill in the blanks.
\[
(16 \times 50) + (16 \times 92) = 16 \times \phantom{50} + 92
\]
\[
[-12 \times 4] + [(-12) \times (-13)] = \phantom{-12} \times [4 + (-18)]
\]
Exercise 7.

Using the fact that ∗ is distributive over §, fill in the blanks.

\[ (4, -1) \times \un{[1, 6] \times (-8, 2)} = \un{[1, 6] \times (2, 9)} \times \un{[1, 6] \times (-8, 2)} \]
\[ (3, 9) \times \un{[(-2, 7) \times (-3, -1)]} = \un{[3, 9] \times [(-2, 7) \times (-3, -1)]} \]
\[ (-6, 3) \times \un{[2, 9] \times (-5, 3)]} = \un{(-6, 3) \times [2, 9] \times (-5, 3)]} \]
\[ (1, 1) \times \un{\{1, 1\} \times (-1, -1)} = \un{(1, 1) \times \{1, 1\} \times (-1, -1)} \]
\[ (3, 9) \times \un{\{1, 9\} \times (-1, -1)} = \un{(3, 9) \times \{1, 9\} \times (-1, -1)} \]
\[ (3, -5) \times \un{(4, 9) \times (-1, -1)} = \un{(3, -5) \times (4, 9) \times (-1, -1)} \]
\[ (3, 9) \times \un{\{1, 9\} \times (-1, -1)} = \un{(3, 9) \times \{1, 9\} \times (-1, -1)} \]
\[ (-6, 3) \times \un{\{1, 9\} \times (-1, -1)} = \un{(-6, 3) \times \{1, 9\} \times (-1, -1)} \]
\[ (-1, -1) \times \un{(4, 9) \times (1, 2)} = \un{(-1, -1) \times (4, 9) \times (1, 2)} \]
\[ (h, 9) \times \un{\{9, 3\} \times (-6, 1)]} = \un{(h, 9) \times \{9, 3\} \times (-6, 1)]} \]
\[ (1, 2) \times \un{\{3, 12\} \times (1, 2)} = \un{(1, 2) \times \{3, 12\} \times (1, 2)} \]
\[ (2, 7) \times \un{(9, 9) \times (2, -1)} = \un{(2, 7) \times \{9, 9\} \times (2, -1)} \]
\[ (8, 7) \times \un{(6, 3) \times (5, -2)} = \un{(8, 7) \times \{6, 3\} \times (5, -2)} \]
Solve the following equations.

\[4 = 7 + x\]
\[12 = x - 7\]
\[-3 + x = -10\]
\[x - 4 = -6\]
\[(4, -6) = (9, 4) \# (x, y)\]
\[(1, -1) = (x, y) \# (7, -3)\]
\[(-2, 7) \# (x, y) = (4, 5)\]
\[(5, -6) = (3, 7) \# (x, y)\]
\[(x, y) \# (6, -9) = (-4, 12)\]
\[2x = 16\]
\[32 = 16x\]
\[17x = 68\]
Lesson 8

In Lesson 1 it is stated that two ordered pairs are equal if they have the same first component and the same second component. We indicate the fact that two ordered pairs are equal by placing the symbol \(=\) between them. If two ordered pairs are not equal, this can be indicated by placing the symbol \(\neq\) between them. For example
\((-4, 2) = (-4, 2)\) and \((3, 7) \neq (3, 7)\).

For integers the following are true:
1. If \(a\) is an integer, then \(a = a\).
2. If \(a = b\), then \(b = a\).
3. If \(a = b\) and \(b = c\), then \(a = c\).

For pairs of integers, it can be verified that the following are true:
1. \((a, b) = (a, b)\).
2. If \((a, b) = (c, d)\), then \((c, d) = (a, b)\).
3. If \((a, b) = (c, d)\) and \((c, d) = (e, f)\), then \((a, b) = (e, f)\).

Let \(S\) be a set. A relation on \(S\) associates with two elements of \(S\) either the word, "yes," or the word, "no."
If \(R\) is a relation on \(S\), \(A\) and \(B\) are elements of \(S\), and \(R\) associates with \(A\) and \(B\) the word "yes" then we say that \(A\) is related to \(B\) and we indicate this by writing \(A \mathbin{R} B\). If \(R\) associates with \(A\) and \(B\) the word "no" then we say that \(A\) is not related to \(B\) and we indicate this by writing \(A \nmid B\).
If $R$ is a relation on $S$ and $A$ and $B$ are elements of $S$, then exactly one of the following is a true statement:

1. $A$ is related to $B$.
2. $A$ is not related to $B$.

$A$ is related to $B$ if and only if $R$ associates with $A$ and $B$ the word "yes." $A$ is not related to $B$ if and only if $R$ associates with $A$ and $B$ the word "no."

Examples of Relations.

1. "=" is a relation on the set of integers. If $a$ and $b$ are integers, then exactly one of the following is a true statement:
   a. $a = b$.
   b. $a \neq b$.

2. "<" is a relation on the set of ordered pairs of integers. If $(a, b)$ and $(c, d)$ are ordered pairs of integers, then exactly one of the following is a true statement:
   a. $(a, b) < (c, d)$.
   b. $(a, b) \geq (c, d)$.

3. The statements
   a. $(a, b)R(c, d)$, ($(a, b)$ is related to $(c, d)$) if $a + d = b - c$, and
   b. $(a, b) \not{R}(c, d)$, ($(a, b)$ is not related to $(c, d)$), if $a + d \neq b - c$,

define a relation on the set of ordered pairs of integers.
For example, \((3, 9) \in R(4, 2)\) because \(3 + 2 = 5\) and \(9 - 4 = 5\). \((7, 6) \notin (2, 6)\) because \(7 + 8 = 15\), \(7 - 2 = 4\) and \(15 \neq 4\).

4. Another example of a relation on the set of ordered pairs of integers is the relation \(R_1\) defined by,
   a. \((a, b) \in R_1(c, d)\) if \(a + d = b + c\).
   b. \((a, b) \notin R_1(c, d)\) if \(a + d \neq b + c\).
Another way of defining \(R_1\) is by the statement,
\((a, b) \in R_1(c, d)\) if and only if \(a + d = b + c\).
\((4, 8) \in R_1(2, 6)\) because \(4 + 6 = 10\) and \(8 + 2 = 10\).
\((9, 2) \notin R_1(6, 3)\) because \(9 + 3 = 12, 2 + 3 = 10\) and \(12 \neq 10\).

Definition: Let \(S\) be a set and let \(R\) be a relation on \(S\). Then \(R\) is an equivalence relation if for any elements \(A\), \(B\), and \(C\) of \(S\),
1. \(A R A\). (Reflexive Principle)
2. If \(A R B\), then \(B R A\). (Symmetric Principle)
3. If \(A R B\) and \(B R C\), then \(A R C\). (Transitive Principle)

Examples of Equivalence Relations.
1. \(\equiv\) is an equivalence relation on the set of integers.
2. \(\equiv\) is an equivalence relation on the set of ordered pairs of integers. This is true because,
   a. Any ordered pair is equal to itself. That is, \((a, b) = (a, b)\).
   b. If \((a, b) = (c, d)\), then \(a = c\) and \(b = d\).
If \( a = c \), then \( c = a \). If \( b = d \), then \( d = b \).

Therefore \((c, d) = (a, b)\).

c. Suppose \((a, b) = (c, d)\) and \((a, d) = (e, f)\).

Since \((a, b) = (c, d)\), \(a = c\) and \(b = d\).

Since \((c, d) = (e, f)\), \(c = e\) and \(d = f\).

Since \(a = c\) and \(c = a\), then \(a = e\).

Since \(b = d\) and \(d = f\), then \(b = f\).

Therefore, \((a, b) = (e, f)\).

3. The relation \( R \) defined by \((a, b)R(c, d)\) if and only if \( a + d = b - c \) is not an equivalence relation.

a. \( R \) does not satisfy the reflexive principle.

For example, \((3, 7)\not\equiv(3, 7)\) because \(3 + 7 \neq 7 - 3\).

b. \( R \) does not satisfy the symmetric principle

because \((3, 6)R(1, 2)\), but \((1, 2)\not\equiv(3, 6)\).

c. \( R \) does not satisfy the transitive principle

because \((3, 6)R(1, 2)\) and \((1, 2)R(1, 0)\), but \((3, 6)\not\equiv(1, 0)\).

4. The relation \( R_1 \) defined by \((a, b)R_1(c, d)\) if and only if \( a + d = b + c \) is an equivalence relation.

**Exercises 8**

Place the appropriate symbol, "\(=\)" or "\(\not\equiv\)" between the following pairs of ordered pairs.

\((3, 7)\, \equiv \, (8, 7)\)

\((-4, 2)\, \equiv \, (-4, 2)\)

\((7 - 6, 9)\, \equiv \, (1, 9)\)
(-8 + 3, 6 - (-2)) ___ (5, 6)
(6 - 9, 4 + (-6)) ___ (-3, -2)
(12, -3) ___ (6 + 6, 9 - 6)

Place the appropriate symbol, "R" or "L" between the following pairs of ordered pairs.

(6, 14) ___ (3, 5)
(7, 2) ___ (9, 3)
(4, 6) ___ (4, -2)
(3, 7) ___ (2, 2)
(3, 5) ___ (6, 14)
(4, -2) ___ (4, 6)
(3, 7) ___ (-5, 5)

Place the appropriate symbol, "R_1" or "L_1" between the following pairs of ordered pairs.

(3, 7) ___ (3, 7)
(4, 8) ___ (1, 5)
(5, 1) ___ (9, 6)
(-6, 9) ___ (-6, 9)
(4, 11) ___ (7, 13)
(7, 5) ___ (9, 8)
(8, 3) ___ (7, 2)

Fill in the blanks in such a way that each of the following is true.

(3, ___) = (3, -17)
(___, ___) = (-9, 47)
(6, 9)R(____, 1)
(4, 3)R(2, ____)
(____, 7)R(5, 1)
(4, 3)R(____, 3)
(6, 9)R(____, 6)
(____, 4)R(5, 7)
(____, 15)R(3, 21)
(____, 7)R(20, 9)
(-3, 8)R(____, 5)
(-4, -9)R(____, 4)

In each of the following state the name of the principle illustrated.

(3, 2)R(3, 2).

(3, 2)R(6, 5) and (6, 5)R(4, 3). Therefore

(4, 7)R(3, 6). Therefore, (3, 6)R(4, 7).

(9, 8)R(6, 5) and (6, 5)R(9, 6). Therefore,

(9, 8)R(9, 8).
Lesson 9

A binary operation associates with two elements of a set an element of that set. Suppose the elements of a set are themselves sets. Consider the set $S$ which has as elements the sets:

- $A = \{1, 5, 9, 13, 17, 21, 25, 19, \ldots\}$,
- $B = \{2, 6, 10, 14, 18, 22, 26, 30, \ldots\}$,
- $C = \{3, 7, 11, 15, 19, 23, 27, 31, \ldots\}$, and
- $D = \{4, 8, 12, 16, 20, 24, 28, 32, \ldots\}$.

We shall define a binary operation $\oplus$ on the set $S$. Let $X$ and $Y$ denote elements of $S$. Then $X \oplus Y$ denotes the element of $S$ which associates with $X$ and $Y$. To find $X \oplus Y$, pick any element of $X$, pick any element of $Y$ and find their sum. Then $X \oplus Y$ is the set containing this sum.

Examples.

Find $A \oplus B$. 9 is an element of $A$. 18 is an element of $B$. $9 + 18 = 27$. 27 is an element of $C$. Therefore $A \oplus B = C$.

Find $C \oplus A$. 7 is an element of $C$ and 17 is an element of $A$. $7 + 17 = 24$ and 24 is an element of $D$. Therefore $C \oplus A = D$.

1. The binary operation $\oplus$ is a commutative binary operation. This can be shown as follows: Let $X$ and $Y$ be elements of $S$. To find $X \oplus Y$ choose an element $a$ of $X$ and an element $b$ of $Y$. Then $X \oplus Y$ is the element (set) of $S$ containing $a + b$. To find $Y \oplus X$ we
we must choose an element of Y and an element of X.
Then the sum of these two numbers must be found. \( Y \oplus X \)
is the set containing this sum. \( b \) is an element of Y, so choose \( b \). \( a \) is an element of X, so choose \( a \). Then
\( Y \oplus X \) is the set containing \( b + a \). \( b + a = a + b \) and
\( X \ominus Y \) contains \( a + b \). This shows that \( Y \oplus X = X \ominus Y \).
Therefore \( \ominus \) is a commutative binary operation.

2. \( \ominus \) is an associative binary operation. This can be shown using a procedure similar to that used above.

3. \( D \) is an identity element for the binary operation \( \ominus \).

4. Each element has an inverse with respect to the
identity \( D \). The inverse of \( A \) is \( C \) because \( A \ominus C = D \).
The inverse of \( B \) is \( B \) because \( B \ominus B = D \). The inverse
of \( C \) is \( A \) because \( C \ominus A = D \). The inverse of \( D \) is \( D \)
because \( D \ominus D = D \).

We now define another binary operation on the set \( S \),
the binary operation \( \otimes \). If \( X \) and \( Y \) are elements of the
set \( S \), \( a \) is an element of \( X \) and \( b \) is an element of \( Y \), then
\( X \otimes Y \) is the element (set) of \( S \) containing \( a \times b \). For
example, find \( A \otimes C \). \( 9 \) is an element of \( A \) and \( 15 \) is an
element of \( C \). \( 9 \times 15 = 135 \). \( 135 \) is an element of \( C \).
Therefore \( A \otimes C = 135 \).

1. \( \otimes \) is a commutative binary operation.

2. \( \otimes \) is an associative binary operation. This can be shown as follows: Let \( X, Y, \) and \( Z \) be elements of \( S \).
Let a be an element of X, b be an element of Y, and c be an element of Z. Then \((X @ Y) @ Z\) is the set containing \((a \times b) \times c\). \(X @ (Y @ Z)\) is the set containing \(a \times (b \times c)\). Since \((a \times b) \times c = a \times (b \times c)\), \((X @ Y) @ Z\) and \(X @ (Y @ Z)\) must be the same set. Therefore \((X @ Y) @ Z = X @ (Y @ Z)\).

3. A is an identity element for the binary operation \@\.

This can be verified by showing that each of the following is true:

\[
\begin{align*}
A @ A &= A \\
B @ A &= B \\
A @ B &= B \\
A @ D &= D
\end{align*}
\]

4. Each of the elements A and C has an inverse with respect to the binary operation \@ and the identity element A.

The inverse of each element can be determined by filling the blanks.

\[
\begin{align*}
A @ \_ &= A \\
C @ \_ &= A
\end{align*}
\]

Since certain elements have inverses under the binary operations \(\@\) and \(\odot\), certain types of equations are solvable.

1. Solve the equation \(B @ X = C\) for X.

Solution:

The inverse of B with respect to \(\odot\) is B since \(B @ B = D\).

Since \(B @ X = C\), \(B @ (B @ X) = B @ C\). Since \(\odot\) is
associative: \((3 \ bullets \ 2) \ bullets \ X = 3 \ bullets \ (2 \ bullets \ X)\). \(6 \ bullets \ X = 5 \ bullets \ C\).

Since \(D\) is an identity element for \(\circ\), \(X = 3 \ bullets \ C\) and \(X = A\).

2. Solve the equation \(C \ bullets \ X = B\).

Solution:
\(C\) is the inverse of \(G\) since \(C \circ G = A\). Then
\(C \ bullets (C \ bullets X) = C \ bullets B\)
\((C \ bullets C) \ bullets X = B\)
\(A \ bullets X = B\)

\(X = B\) since \(A\) is an identity element for \(\circ\).

Exercise 9.

Fill in the blanks.

\[
\begin{align*}
A \ bullets B &= \quad & C \ bullets C &= \\
B \ bullets D &= \quad & D \ bullets B &= \\
C \ bullets D &= \quad & B \ bullets C &= \\
D \ bullets A &= \quad & C \ bullets C &= \\
B \ bullets C &= \quad & B \ bullets D &= \\
A \ bullets A &= \quad & A \ bullets B &= \\
\end{align*}
\]

Solve the following equations for \(X\).

\[
\begin{align*}
A \ bullets X &= D \\
X \ bullets C &= B \\
D \ bullets X &= A \\
C \ bullets X &= D \\
A \ bullets X &= D \\
C \ bullets X &= B \\
\end{align*}
\]
A \times X = D
D \times X = D
B \times X = A
X \times C = A

Consider the set S which has as elements the sets

A = \{1, 4, 7, 10, 13, 16, \ldots \},
B = \{2, 5, 8, 11, 14, 17, \ldots \},
C = \{3, 6, 9, 12, 15, 18, \ldots \}.

Let \triangle be the binary operation on S defined as follows:
If X and Y are elements of S, a is an element of X and b is an element of Y, then \(X \triangle Y\) is the element (set) containing \(a + b\). Fill in the blanks.

\[G \triangle 3 = \ldots\]
\[B \triangle A = \ldots\]
\[C \triangle C = \ldots\]
\[A \triangle C = \ldots\]
\[G \triangle G = \ldots\]
\[S \triangle C = \ldots\]
\[G \triangle A = \ldots\]
APPENDIX D

TRACKER MADE TESTS
I. Using the binary operation $\text{B}$ defined by 
$(a, b) \text{B} (c, d) = (a + c, b - d)$ complete the following.

1. $(3, 4) \text{B} (2, 6) = \underline{\hspace{2cm}}$
2. $(7, 3) \text{B} (1, -4) = \underline{\hspace{2cm}}$
3. $(-3, 9) \text{B} (4, -3) = \underline{\hspace{2cm}}$
4. $(6, 7) \text{B} (11, 2) = \underline{\hspace{2cm}}$
5. $(-3, 5) \text{B} (-7, \frac{1}{3}) = \underline{\hspace{2cm}}$
6. $(1, -6) \text{B} (3, -9) = \underline{\hspace{2cm}}$
7. $(2, -1) \text{B} (-6, -11) = \underline{\hspace{2cm}}$

II. Using the binary operation $\text{O}$ defined by 
$(a, b) \text{O} (c, d) = (a + 2b, c \times d)$ complete the following.

1. $(3, 4) \text{O} (2, 6) = \underline{\hspace{2cm}}$
2. $(1, 3) \text{O} (9, 2) = \underline{\hspace{2cm}}$
3. $(9, 2) \text{O} (1, 3) = \underline{\hspace{2cm}}$
4. $(1, 5) \text{O} (-1, 6) = \underline{\hspace{2cm}}$
5. $(-1, 6) \text{O} (4, 5) = \underline{\hspace{2cm}}$
6. $(3, -7) \text{O} (-2, -9) = \underline{\hspace{2cm}}$
7. $(-2, -9) \text{O} (3, -7) = \underline{\hspace{2cm}}$
8. Is the binary operation $\text{O}$ commutative?
III. Place the appropriate symbol, $\leq$ or $\neq$, in each of the following blanks.

1. $3 \underline{} = 3$
2. $4 \underline{} = 10$
3. $10 \underline{} = 4$
4. $7 \underline{} = 18$
5. $19 \underline{} = 27$
6. $8 \underline{} = 20$
7. $20 \underline{} = 44$
8. $8 \underline{} = 44$

IV. 1. Does the relation $\leq$ satisfy the reflexive principle? __________

2. Does the relation $\leq$ satisfy the symmetric principle? __________

3. Does the relation $\leq$ satisfy the transitive principle? __________

4. Is the relation $\leq$ an equivalence relation? __________

V. The binary operation $\leq$ is defined as follows:

$a \leq b$ is the whole number less than $b$ related to $a$ by the relation $\leq$. Using the binary operation $\leq$ complete the following table.
Is the binary operation $x_6$ commutative?

Is there an identity element for the binary operation $x_6$?

Which elements have inverses with respect to the binary operation $x_6$?

VI. If there are solutions, find all possible solutions to each of the following equations.

1. $5 \cdot x_6 x = 3$
   \[ x = \]

2. $1 \cdot x_6 x = 4$
   \[ x = \]

3. $5 \cdot x_6 x = 1$
   \[ x = \]

4. $1 \cdot x_6 x = 3$
   \[ x = \]

5. $2 \cdot x_6 x = 5$
   \[ x = \]

6. $0 \cdot x_6 x = 0$
   \[ x = \]

7. $2 \cdot x_6 x = 4$
   \[ x = \]

8. $3 \cdot x_6 x = 0$
   \[ x = \]

VII. Is the binary operation $x_6$ associative?

Give examples to support your answer.
MATH 163
Test II

I. Perform the indicated operations.

1. \((3,5) + (7,8) = \)

2. \((1,7) + (3,3) = \)

3. \((18,10) + (21,7) = \)

4. \((8,13) \times (1,1) = \)

5. \((11,9) \times (2,3) = \)

6. \((5,4) \times (16,8) = \)

7. \((12,57) \times (234,531) = \)

8. \((2,5) \div (0,7) = \)

9. \((6,7) \div (4,5) = \)

10. \((-12,6) \div (9,3) = \)

11. \((-4,3) \div (12,10) = \)

12. \((2,10) \div (-6,3) = \)

13. \((-254761,321587) \div (78,326213) = \)

14. \((0,379212,2560) \div (-9,336217,351267) = \)

15. \((6,7) \times (-8,9) = \)

16. \((-3,11) \times (6,5) = \)

17. \((6,-12) \times (-8,13) = \)

18. \((-13,4) \times (-7,3) = \)

19. \((978273,578215) \times (705398,877298) = \)

20. \((-459751,77235) \div (3776916,3776916) = \)

II. Place the appropriate symbol, \(\ast\), \(\div\), \(\times\), or \(\equiv\)
in each of the following blanks.
III. Solve the following equations.

1. \((7,9) + x = (3,4)\)
   \(x = \underline{\quad}\)

2. \((2,1) + x = (-6,-2)\)
   \(x = \underline{\quad}\)

3. \((6,-2) + x = (-3,-1)\)
   \(x = \underline{\quad}\)

4. \((18,27) + x = (2,3)\)
   \(x = \underline{\quad}\)

5. \((4,8) + x = (-5,-10)\)
   \(x = \underline{\quad}\)

6. \((3,7) + x = (6,14)\)
   \(x = \underline{\quad}\)

7. \((3,6) + x = (-9,2)\)
   \(x = \underline{\quad}\)

8. \((-62,61,39,25) + x = (-62,61,39,25)\)
   \(x = \underline{\quad}\)

9. \((553,301,366,361) + x = (20873,20875)\)
   \(x = \underline{\quad}\)

10. \((0,77355) + x = (0,-112736)\)
    \(x = \underline{\quad}\)

IV. Below are listed some of the principles we have studied.

1. The commutative principle for \(*\).
2. The commutative principle for \(\times\).
3. The associative principle for \(*\).
4. The associative principle for $x$.
5. $x$ is distributive over $\cdot$.
6. Identity element for $\cdot$.
7. Identity element for $x$.
8. Inverse of an element with respect to $\cdot$.
9. Inverse of an element with respect to $x$.
10. Reflexive principle of $\leq$.
11. Symmetric principle of $\leq$.
12. Transitive principle of $\leq$.

Below are instances of some of the above principles.
Place the number of the principle in the blank.

A. $(8,17) \times (3,7) = (4,7) \times (3,17)$
B. $(3,7) \times (9,3) = (11,7)$
C. $[(3,17) + (9,3)] \times (2,5) = [(3,17) \times (2,5)] + [(9,3) \times (2,5)]$
D. $(8,7) + [(7,3) \times (9,2)] = [(7,3) \times (9,2)] \times (8,7)$
E. $(4,7) + [(2,2)] = (5,7) + (1,7)$
F. $(18,12) \geq (18,12)$
G. $(8,5) \times [(3,3) \times (4,1)] = [(8,5) \times (8,3)] \times (4,1)$
H. $(4,19) + (0,1) = (4,19)$
I. If $(8,3) \equiv (2,5)$, then $(2,5) \equiv (0,3)$
J. $(4,2) \times (3,3) = (4,2)$
II. Solve the following equations.
1. \((3/3) + x = (1/2)\) \quad x = \\
2. \(x + (3/7) = (-7/5)\) \quad x = \\
3. \(y \times (1/3) = (2/3)\) \quad y = \\
4. \((-2/7) \times (1/3) \times y\) \quad y = \\
5. \((2/3) = (1/2) + (1/5) \times y\) \quad y = \\

III. Consider the binary operation \(\circ\) defined by the
following table.

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>N</th>
<th>P</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>P</td>
<td>M</td>
<td>Q</td>
<td>M</td>
</tr>
<tr>
<td>N</td>
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<tr>
<td>Q</td>
<td>N</td>
<td>N</td>
<td>P</td>
<td>Q</td>
</tr>
</tbody>
</table>

1. Is there an identity element for the binary operation o?  

2. Is the binary operation o commutative?  

3. Give an example to support your answer to question 2.  

4. Which elements of the set M, N, P, Q have inverses with respect to o?  

5. Is the binary operation o associative?  

6. Give an example to support your answer to question 5.  

IV. Consider the binary operations t' and x' defined by the following tables.

<table>
<thead>
<tr>
<th>t'</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tr>
<td>5</td>
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<td>2</td>
<td>3</td>
<td>4</td>
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</table>

<table>
<thead>
<tr>
<th>x'</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
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<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
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<td>4</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>
1. Is the binary operation \( t \) commutative? 

2. Give an example to support your answer to question 1.

3. Is there an identity element for \( x^2 \)? If so, what is it?

4, 5, 6, and 7. Write the inverse, if any, of each of the following with respect to \( t \).

4

5

8, 9, 10, and 11. Write the inverse, if any, of each of the following with respect to \( x \).

4

5

12. Is \( x \) distributive over \( t \)? Give an example to support your answer.

13. Is \( t \) distributive over \( x \)? Give an example to support your answer.

V. Write the negation of each of the following.

1. It is clear.

2. It is false that it is windy.
3. He is hurt or he is pretending.

4. It is cloudy and windy.

VI. Given the proposition, "If a toom is a boom, then a zoom is a room,"
1. Write the converse of the proposition.

2. Write the contrapositive of the proposition.

VII. Answer the following true or false.
1. $3 + 4 = 7$ and $2 + 9 = 11$
2. If $3 + 7 = 10$ then $9 + 5 = 12$.
3. $4 + 8 = 9$ or $7 = 6 + 1$
4. $5 + 7 = 12$ and $12 - 4 = 6$
5. If $3 = 9 - 7$, then $6 + 2 = 8$. 
1. In each of the following state whether the argument is valid or invalid.

1. Given: If a person is a college student, then he is intelligent.
   Given: John is a college student.
   Conclusion: John is intelligent.

2. Given: If a person is blind, then he is not an airplane pilot.
   Given: Wesley is not blind.
   Conclusion: Wesley is an airplane pilot.

3. Given: If a person is a singer, then he is temperamental.
   Given: Sally is not temperamental.
   Conclusion: Sally is not a singer.

4. Given: If you are a Democrat, then you are not a Republican.
   Given: You are not a Democrat.
   Conclusion: You are a Republican.

5. Given: If a polygon is a square, then its
Given: The diagonals of this polygon are equal.
Conclusion: This polygon is a square.

6. Given: If a natural number ends in 5 or 0, then it is divisible by 5.

Given: If a natural number \( n \) is divisible by 5, then there exists a natural number \( m \) such that \( n = 5 \times m \).

Conclusion: If a natural number \( n \) ends in 5 or 0, then there exists a natural number \( m \) such that \( n = 5 \times m \).

7. Given: If a quadrilateral is a parallelogram then its opposite sides are parallel.

Given: The opposite sides of this quadrilateral are not parallel.

Conclusion: This quadrilateral is not a parallelogram.

8. Given: If two angles are right angles, then they are equal.

Given: Angles \( A \) and \( B \) are right angles.

Conclusion: Angle \( A \) is equal to angle \( B \).
9. Given: If a triangle is equilateral, then it is equiangular.
   Given: If a polygon is equiangular, then it is a regular polygon.
   Conclusion: If a triangle is equilateral, then it is a regular polygon.

10. Given: If two angles are right angles, then they are equal.
    Given: These two angles are equal.
    Conclusion: These two angles are right angles.

II. In each of the following draw a Venn diagram and state whether the argument is valid or invalid.

1. Given: Some politicians are dishonest people.
   Given: No banker is a dishonest person.
   Conclusion: No banker is a politician.

2. Given: No horses are dogs.
   Given: No dogs are cats.
   Conclusion: No horses are cats.
3. Given: Sally is an dependable person.
Given: Sally has brown skin.
Conclusion: Some people with brown skin are dependable.

4. Given: All wise people drive carefully on icy streets.
Given: Mr. Brown drives carefully on icy streets.
Conclusion: Mr. Brown is wise.

5. Given: All squares are rectangles.
Given: All rectangles are parallelograms.
Conclusion: All squares are parallelograms.

III. In each of the following two propositions are given. If these are accepted as true, what conclusion, if any, follows from them. If no conclusion follows, write "no conclusion".
1. Given: All college students are clever.
Given: John is a college student.
Conclusion:

3. Given: Some coeds are beautiful.
   Given: Some blondes are beautiful.
   Conclusion:

4. Given: No traffic policeman has a sense of humor.
   Given: All fat people have a sense of humor.
   Conclusion:

5. Given: All rectangles are parallelograms.
   Given: All parallelograms are quadrilaterals.
   Conclusion:

IV. Consider the following axiom system.

Undefined terms:
1. Divisible by 3
2. Odd
3. Not divisible by 4
4. Multiple of 3

Axioms
P1. 8 is divisible by 4.
P2. If a number is divisible by 3, then it is odd.
P3. If a number is odd, then it is not divisible by 4.
P4. A number is divisible by 3 if and only if it is a multiple of 3.

"Prove" the following theorems.
T1. If a number is divisible by 3, then it is not divisible by 4.

T2. 8 is not divisible by 3.

T3. If a number is a multiple of 3, then it is divisible by 3.

T4. 8 is not a multiple of 3.

T5. If a number is divisible by 4, then it is not odd.

V. Complete the following truth tables.

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<th>not p</th>
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APPENDIX E

THE RAW DATA
The Coding Procedure

In order to facilitate the handling of the raw data each student was assigned a student number. The number consisted of a capital letter followed by a four digit number. This number identified the student in terms of sex, ability level, and class. From the class number it could be determined at what time the class met, who the instructor for the class was, and what teaching method was used in the class.

The first letter in the student number indicates the sex of the student. An "M" indicates male and an "F" indicates female.

Each student number consisted of a letter followed by a four digit number. The first digit in the number indicates the class in which the student was enrolled. The digit "1" denotes the class taught by Instructor A at 10 a.m. and taught by the Guided Discovery Method. The digit "2" denotes the class taught by Instructor B at 10 a.m. and taught by the Exposition Method. The digit "3" denotes the class taught by Instructor A at 2 p.m. and taught by the Exposition Method. The digit "4" denotes the class taught by Instructor B at 2 p.m. and taught by the Guided Discovery Method.

The second digit of the four digit number indicates the student's ability level as determined by his composite score on the American College Testing Program. The numeral
"1" indicates a composite score of twenty or higher. The numeral "2" indicates a composite score of more than fifteen but less than twenty. The numeral "3" indicates that the student's composite score is fifteen or less.

The last two digits of the four digit number were used to differentiate among the students of the same sex and in the same class. The last two digits range from "01" and up.

Interpreting the Data

In the column labeled "ACT Score" the initial composite scores on the American College Testing Program of all the subjects are listed. In the column labeled "Test I" the raw scores of all the subjects on the tests, Cooperative Mathematics Tests, Structure of the Number System, are listed. The scores for all the subjects on the test, Cooperative Mathematics Tests, Algebra I, are listed in the column labeled "Test II." The raw scores of all the subjects on the Watson-Glaser Critical Thinking Appraisal are listed in the column labeled "Test III." The criterion score for each subject on Test IV consisted of the total of the subjects scores on four teacher made tests and the student's homework score. Those total scores are listed in the column labeled "Test IV."
### The Raw Data

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BIBLIOGRAPHY

Books


Articles


Michael, R. C., "The Relative Effectiveness of Two Methods of Teaching Certain Topics in Ninth Grade Algebra," The Mathematics Teacher, XLIX (February, 1949), 23-87.


Publications of Learned Organizations


Public Documents


Unpublished Materials


