COMPLETING THE SPACE OF STEP FUNCTIONS

APPROVED:

Russell O. Saladz
Major Professor

H.C. Varnish
Minor Professor

E.E. Gooe
Chairman of the Department of Mathematics

Robert B. Touloum
Dean of the Graduate School
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In this thesis a study is made of the space $X$ of all step functions on $[0,1]$. This investigation includes determining a completion space, $X^*$, for the incomplete space $X$, defining integration for $X^*$, and proving some theorems about integration in $X^*$.

The thesis is divided into three chapters. Chapter I is an introduction to the thesis. Chapter II defines a step function and the space of all step functions $X$, includes proof that $X$ is an incomplete metric space, and concludes by showing $X$ has a completion metric space, $X^*$. Chapter III investigates $X^*$, especially with respect to integration. The integral of a member of $X^*$ is defined and then it is proven that this integral actually exists and is unique. Some of the properties of integrals are shown to be true for integration in $X^*$. The last theorem of Chapter II shows a relationship between the Riemann integral and the integral of a member of $X^*$. 
COMPLETING THE SPACE OF STEP FUNCTIONS

THESIS

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By

Linda K. Massey, B. S.

Denton, Texas
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CHAPTER I
INTRODUCTION

In this thesis a study is made of the space $X$ of all step functions on $[0,1]$. In Chapter II step functions are defined and the space $X$ is proven to be a metric space. $X$ is an incomplete metric space since there is at least one Cauchy sequence of elements of $X$ which converges to an element not in $X$. The incomplete metric space $X$ is shown to have a completion metric space, $X^*$. 

Chapter III investigates $X^*$, especially with respect to integration. The integral of a member of $X^*$ is defined and then it is proven that this integral actually exists and is unique. Some of the properties of integrals are shown to be true for integration in $X^*$. The last few theorems of Chapter III investigate specific members of $X^*$. 
A step function is appropriately named since its graph appears to be a series of steps, without the riser of the step, or one long, continuous step. Some definitions allow the very end of the step, a point, to be separated from the step and thus lie above or below the step; this will not be allowed in the definition of a step function in this thesis. Thus a step function is either a constant function over the domain or it is a discontinuous function which is constant on subintervals of the domain.

In this thesis a step function will be defined entirely in terms of subintervals of \([0,1]\) which are left half-closed. There are infinitely many step functions on \([0,1]\); all of these will make up the space of all step functions on \([0,1]\). It is important to have an intuitive idea of a step function, but to clarify matters, a formal definition is in order.

A function \(S\), whose domain is \([0,1]\) is called a step function if there is a partition \(P = \{X_0, X_1, \cdots, X_n\}\) of \([0,1]\) such that \(S\) is constant on each left half-closed subinterval of \(P\). That is to say, for each \(k = 1, 2, \cdots, n\) there is a real number \(S_k\) such that

\[
S(x) = S_k \text{ if } x_{k-1} \leq x < x_k \text{ except for } x_n = 1
\]
then,

$$S(x) = S_k \text{ if } x_{k-1} \leq x \leq x_k$$

The integral of a step function $S$ from 0 to 1, denoted by the symbol $\int_0^1 S(x) \, dx$ is defined by the following formula:

$$\int_0^1 S(x) \, dx = \sum_{k=1}^{n} S_k \cdot (x_k - x_{k-1}).$$

A metric space is a mathematical system $(X,d)$ consisting of a set $X$ of elements and a real single-valued function $d(x,y)$ defined on all ordered pairs $(x,y)$ of elements of $X$, having the following properties:

1. $d(x,y) \geq 0$.
2. $d(x,y) = 0$ if and only if $x = y$.
3. $d(x,y) = d(y,x)$.
4. $d(x,z) \leq d(x,y) + d(y,z)$.

If $X$ is a metric space, and $x_1, x_2, x_3, \ldots$ is a sequence of points in $X$, the sequence is a Cauchy sequence if for every positive real number $\epsilon$, there exists a positive integer $N$ such that, if $m$ and $n$ are integers each greater than $N$, then $d(x_m, x_n) < \epsilon$.

If $x_1, x_2, x_3, \ldots$ is a sequence of points in the metric space $X$, the sequence $x_1, x_2, x_3, \ldots$ is said to be convergent if there is a point $x$ in $X$ such that, for every positive real number $\epsilon$, there is a positive integer $N$ such that if $n$ is an integer larger than $N$, then $d(x_n, x) < \epsilon$. The point $x$ is then called a limit of the sequence $x_1, x_2, x_3, \ldots$. 
A metric space \((X,d)\) is complete if and only if every Cauchy sequence in \(X\) converges to a point \(p \in X\).

Define \(X\) to be the set of all step functions on \([0,1]\).
If \(f, g \in X\) define \(d(f,g) = \int_0^1 |f(x) - g(x)| \, dx\).

**Theorem 2.1.** \(\int_0^1 |f(x) - g(x)| \, dx \geq 0\).

Since \(|f(x) - g(x)| \geq 0\), then \(\int_0^1 |f(x) - g(x)| \, dx \geq 0\).

**Theorem 2.2.** \(\int_0^1 |f(x) - g(x)| \, dx = 0\) if and only if \(f(x) = g(x)\),

(a) Suppose \(\int_0^1 |f(x) - g(x)| \, dx = 0\). Since \(|f(x) - g(x)| \geq 0\), suppose \(|f(x) - g(x)| > 0\) for \(x' \in [0,1]\). Let \(h(x) = |f(x) - g(x)| \geq 0\). \(h(x)\) is a step function on \([0,1]\); let \(k = 1, \ldots, n\) be the partition of \([0,1]\) so that \(h(x) = h_k\) if \(x_{k-1} \leq x < x_k\), except for \(x_n = 1\), then 

\[ h(x) = h_k \text{ if } x_{k-1} \leq x < x_k. \]

By definition of an integral of a step function, \(\int_0^1 h(x) \, dx = \sum_{k=1}^{n} h_k (x_k - x_{k-1})\). There is at least one value, \(x'\), so that \(h_k (x' - x_k) > 0\) and since \(h(x)\) is non-negative, then \(\sum_{k=1}^{n} h_k (x_k - x_{k-1}) > 0\). Thus \(\int_0^1 h(x) \, dx > 0\), a contradiction. Suppose \(|f(x) - g(x)| = 0\).

By definition of a step function, since no isolated points are allowed, \(f(x) = g(x)\).

(b) Suppose \(f(x) = g(x)\), then \(f(x) - g(x) = 0\) and \(|f(x) - g(x)| = 0\); thus \(\int_0^1 |f(x) - g(x)| \, dx = 0\).
Theorem 2.3.

\[
\int_0^1 |f(x) - g(x)| \, dx = \int_0^1 |g(x) - f(x)| \, dx
\]

Thus,

\[
\int_0^1 |f(x) - g(x)| \, dx = \int_0^1 |g(x) - f(x)| \, dx.
\]

Theorem 2.4.

\[
\int_0^1 |f(x) - h(x)| \, dx \leq \int_0^1 |f(x) - g(x)| \, dx + \int_0^1 |g(x) - h(x)| \, dx.
\]

\[
\int_0^1 |f(x) - h(x)| \, dx = \int_0^1 |f(x) - g(x) + g(x) - h(x)| \, dx
\]

Thus

\[
\int_0^1 |f(x) - h(x)| \, dx \leq \int_0^1 |f(x) - g(x)| \, dx + \int_0^1 |g(x) - h(x)| \, dx.
\]

By Theorems 2.1, 2.2, 2.3 and 2.4, \((X, d)\) is a metric space. \((X, d)\) is not a complete metric space because the following is an example of a Cauchy sequence of step functions which converges to a non-constant continuous function.

Consider \(f(x) = x\). Let \(P\) be a sequence of partitions of \([0,1]\) such that \(P = \left\{ \left[0, \frac{1}{n} \right], \ldots, \left[\frac{n-1}{n}, 1 \right] \right\}\), for \(n \in \mathbb{N}\). Let \(M_f\) equal the least upper bound of \(x\) on each subinterval of \(P\). Let this sequence of step functions be denoted by \(f_n\).

Note \(f_n \to f\) in \(C([0,1])\) if \(\left\| f_n - f \right\|_{\infty} \to 0\) or \(\left\| f_n - f \right\|_{\infty} \to 0\). Let \(\epsilon > 0\), choose \(n > \frac{1}{2\epsilon}\) and a particular subinterval \(\left(\frac{k-1}{n}, \frac{k}{n}\right) = \{a,b\}\).
\[ \int_a^b f_n(x) - f(x) \, dx = \int_a^b f_n - f \, dx = \int_a^b \frac{b}{2n^2} - \frac{k}{n} \, dx = \frac{1}{2n^2}. \]

Since \( \int_a^b |f_n - f| < \frac{1}{2n^2} \), then \( \int_0^1 |f_n - f| < n \left( \frac{1}{2n^2} \right) = \frac{1}{2n} < \epsilon \)

\([a,b]\) is subinterval in \([0,1]\), \(f_n\) converges to \(f\).

\((X,d)\) is not complete because this sequence of step functions did not converge to a step function.

A metric space \(X^*\) is called a completion of a metric space \(X\) if \(X^*\) is complete and \(X\) is isometric to a dense subset of \(X^*\).

Consider \((X,d)\) as defined previously. Let \(C[X]\) denote the collection of all Cauchy sequences in \(X\) and let \(\nu\) be the relation in \(C[X]\) defined by \(\{a^i\} \sim \{b^i\}\) if and only if
\[
\lim_{n \to \infty} d(a_n, b_n) = 0; \quad \text{that is, if } \lim_{n \to \infty} \int_0^1 |a_n - b_n| = 0.
\]

Theorem 2.5. The relation \(\nu\) is an equivalence relation.

(a) \(\{a^i\} \sim \{a^i\}\).
\[
\lim_{n \to \infty} \int_0^1 |a_n - a_n| = 0.
\]
Thus \(\{a^i\} \sim \{a^i\}\)

(b) If \(\{a^i\} \sim \{b^i\}\), then \(\{b^i\} \sim \{a^i\}\). Since
\[
\{a^i\} \sim \{b^i\}, \quad \lim_{n \to \infty} \int_0^1 |a_n - b_n| = 0, \quad \text{and since } |a_n - b_n| = |b_n - a_n|,
\]
then the \(\lim_{n \to \infty} \int_0^1 |a_n - b_n| = 0 = \lim_{n \to \infty} \int_0^1 |b_n - a_n|\). Thus \(\{b^i\} \sim \{a^i\}\).

(c) If \(\{a^i\} \sim \{b^i\}\) and \(\{b^i\} \sim \{c^i\}\), then \(\{a^i\} \sim \{c^i\}\).
Since \([a_n] \ni \{b_n\} \text{ and } \{b_n\} \ni \{c_n\}\), then
\[
\lim_{n \to \infty} \frac{1}{n} |a_n - b_n| = 0 \text{ and }
\lim_{n \to \infty} \frac{1}{n} |b_n - c_n| = 0.
\]
\[
\lim_{n \to \infty} \frac{1}{n} |a_n - b_n| + \lim_{n \to \infty} \frac{1}{n} |b_n - c_n| > 0.
\]
\[
\lim_{n \to \infty} \frac{1}{n} |a_n - b_n + b_n - c_n| = \lim_{n \to \infty} \frac{1}{n} |a_n - c_n|.
\]
So,
\[
\lim_{n \to \infty} \frac{1}{n} |a_n - b_n| + \lim_{n \to \infty} \frac{1}{n} |b_n - c_n| > 0.
\]
\[
\lim_{n \to \infty} |a_n - c_n|.
\]
Since \(\lim_{n \to \infty} \frac{1}{n} |a_n - b_n| + \lim_{n \to \infty} \frac{1}{n} |b_n - c_n| = 0\), then
\[
0 > \lim_{n \to \infty} \frac{1}{n} |a_n - c_n|, \text{ so }
\]
\[
0 = \lim_{n \to \infty} \frac{1}{n} |a_n - c_n|.
\]

By (a), (b) and (c), \(\nu\) is an equivalence relation.

Now let \(X^*\) denote the quotient set \(C[X]/\nu\); that is, \(X^*\) consists of equivalence classes \([\{a_n\}]\) of Cauchy sequences \([a_n] \in C[X]\).
Let $e$ be the function defined by $e([a_n], [b_n]) = \lim_{n \to \infty} \int_0^1 |a_n - b_n|$ where $[a_n], [b_n] \in X^*$.

Theorem 2.6. The function $e$ is well-defined, that is, if $\{a_n\} \vee \{A_n\}$ and $\{b_n\} \vee \{B_n\}$, then $\lim_{n \to \infty} \int_0^1 |a_n - b_n| = \lim_{n \to \infty} \int_0^1 |A_n - B_n|$.

Set $r = \lim_{n \to \infty} \int_0^1 |a_n - b_n|$ and

$$R = \lim_{n \to \infty} \int_0^1 |A_n - B_n|$$

and let $\epsilon > 0$.

Note: $\int_0^1 |a_n - b_n| \leq \int_0^1 |a_n - A_n| + \int_0^1 |A_n - B_n| + \int_0^1 |B_n - b_n|$.

Now, there exists $n_1 \in \mathbb{N}$ so that if $n > n_1$, then $\int_0^1 |a_n - A_n| < \frac{\epsilon}{3}$.

And there exists $n_2 \in \mathbb{N}$ so that if $n > n_2$, then $\int_0^1 |b_n - B_n| < \frac{\epsilon}{3}$.

And there exists $n_3 \in \mathbb{N}$ so that if $n > n_3$, then $\int_0^1 |A_n - B_n| < \frac{\epsilon}{3}$.

If $n > \max(n_1, n_2, n_3)$ then $\int_0^1 |a_n - b_n| < R + \epsilon$ and

$$\lim_{n \to \infty} \int_0^1 |a_n - b_n| = r,$$

so

$$r \leq R + \epsilon.$$

But this inequality holds for every $\epsilon > 0$; hence, $r \leq R$.

In the same manner it may be shown that $R \leq r$; thus, $r = R$.

In other words, $e$ does not depend upon the particular Cauchy sequence chosen to represent any equivalence class.

Theorem 2.7. The function $e$ is a metric on $X^*$.

(P1) If $f, g \in X^*$, then $e(f, g) \geq 0$; that is, if $[a_n] \vee f$ and $[b_n] \vee g$, then $\int |a_n - b_n| \geq 0$. Since $|a_n - b_n| \geq 0$, then $\int |a_n - b_n| \geq 0$. 


(P₂) If \( f, g \in X^* \), then \( e(f,g) = 0 \) if and only if \( f = -g \).

Assume \( \{a_n\} \in f \) and \( \{b_n\} \in g \).

(a) If \( \lim |a_n - b_n| = 0 \), then \( \{a_n\} \cup \{b_n\} \), by definition.

(b) If \( \{a_n\} \cup \{b_n\} \), then \( \int |a_n - b_n| = 0 \).

(P₃) If \( f, g \in X^* \), then \( e(f,g) = e(g,f) \). Assume \( \{a_n\} \in f \) and \( \{b_n\} \in g \), then \( e(f,g) = \lim |a_n - b_n| = \lim |b_n - a_n| = e(g,f) \).

(P₄) If \( f,g,h \in X^* \), then \( e(f,h) \leq e(f,g) + e(g,h) \). If \( \{a_n\} \in f \), \( \{b_n\} \in g \), \( \{c_n\} \in h \), then \( e(f,h) = \lim \int |a_n - c_n| = \lim \int |a_n - b_n + b_n - c_n| \leq \lim \int |a_n - b_n| + \lim \int |b_n - c_n| = e(f,g) + e(g,h) \).

Thus, \( e \) is a metric on \( X^* \).

Now for each \( p \in X \), the sequence \( \{p, p, p, \ldots\} \) is Cauchy. Set \( \hat{p} = [(p, p, p, \ldots)] \) and \( \hat{X} = \{\hat{p}; p \in X\} \). Then \( \hat{X} \subset X^* \).

Theorem 2.3. \( X \) is isometric to \( \hat{X} \) and \( \hat{X} \) is dense in \( X^* \).

For every \( p,q \in X \), \( e(\hat{p}, \hat{q}) = \lim d(p,q) = d(p,q) \) so \( X \) is isometric to \( \hat{X} \). To show \( \hat{X} \) is dense in \( X^* \), show every point in \( X^* \) is the limit of a sequence in \( \hat{X} \). Let \( \alpha = [(a_1, a_2, \ldots)] \) be any point in \( X^* \). Then \( \{a_n\} \) is a Cauchy sequence in \( X \). Hence, \( \lim e(\hat{a}_m, \alpha) = \lim [\lim d(a_m, a_n)] = \lim d(a_m, a_n) = 0 \).

Accordingly, \( \{\hat{a}_n\} \to \hat{\alpha} \), and thus, \( \alpha \) is the limit of the sequence \( \{\hat{a}_1, \hat{a}_2, \ldots\} \) in \( \hat{X} \). Thus \( \hat{X} \) is dense in \( X^* \).
Theorem 2.9. Let \( \{b_1, b_2, \ldots\} \) be a Cauchy sequence in \( X \) and let \( \{a_1, a_2, \ldots\} \) be a sequence in \( X \) such that \( d(a_n, b_n) < \frac{1}{n} \) for every \( n \in \mathbb{N} \), then

(i) \( \{a_n\} \) is also a Cauchy sequence in \( X \).

(ii) \( \{a_n\} \) converges to \( p \) if and only if \( \{b_n\} \) converges to \( p \).

(i) By the triangle inequality \( d(a_m, a_n) \leq d(a_m, b_m) + d(b_m, b_n) + d(b_n, a_n) \). Let \( \epsilon > 0 \). Then there exists \( n_1 \in \mathbb{N} \) such that \( \frac{1}{n} < \frac{\epsilon}{3} \). Hence, for \( n, m > n_1 \),

\[
d(a_m, a_n) < \frac{\epsilon}{3} + d(b_m, b_n) + \frac{\epsilon}{3}.
\]

By hypothesis \( \{b_1, b_2, \ldots\} \) is a Cauchy sequence so there is \( n_2 \in \mathbb{N} \) such that for \( n, m > n_2 \),

\[
d(b_m, b_n) < \frac{\epsilon}{3}.
\]

Set \( n_0 = \max(n_1, n_2) \). Then, for \( n, m > n_0 \),

\[
d(a_m, a_n) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

Thus, \( \{a_n\} \) is a Cauchy sequence.

(ii) By the Triangle inequality, \( d(b_n, p) \leq d(b_n, a_n) + d(a_n, p) \). Hence \( \lim d(b_n, p) \leq \lim d(b_n, a_n) + \lim (a_n, p) \). But the \( \lim d(b_n, a_n) \leq \lim \left( \frac{1}{n} \right) = 0 \). Thus, if \( a_n \) converges to \( p \), then \( \lim d(b_n, p) \leq \lim (a_n, p) = 0 \), so \( b_n \) converges to \( p \).

Similarly, if \( \{b_n\} \) converges to \( p \), then \( \{a_n\} \) converges to \( p \).
Theorem 2.10. Every Cauchy sequence in \((X^*, e)\) converges and so \((X^*, e)\) is a completion of \(X\).

Let \(\{A_1, A_2, \cdots\}\) be a Cauchy sequence in \(X^*\). Since \(\hat{X}\) is dense in \(X^*\) for every \(n \in \mathbb{N}\), there is \(\hat{a}_n \in \hat{X}\) such that \(e(\hat{a}_n, A_n) < \frac{1}{n}\). Then, by Theorem 2.9, \(\{\hat{a}_1, \hat{a}_2, \cdots\}\) is also a Cauchy sequence and by Theorem 2.8, \(\{\hat{a}_1, \hat{a}_2, \cdots\}\) converges to \(B = [\{a_1, a_2, \cdots\}] \in X^*\). Hence, by Theorem 2.9, \(\{A_n\}\) also converges to \(B\) and, therefore, \((X^*, e)\) is complete.
CHAPTER III

INTEGRATION IN $X^*$

Do the integrals of members of $X^*$ exist? How is an integral of a member of $X^*$ to be defined? What are some of the properties of integrals which hold true for integration in $X^*$? What are some specific elements of $X^*$? These are some of the questions that Chapter III will answer.

Definition. If $A \in X^*$ and $\{a_n\}$ is a Cauchy sequence which is an element of $A$, then the integral of $A$, denoted by the symbol $\int A$, is defined by the following,

$$\int A = \lim_{n \to \infty} a_n.$$

**Theorem 3.1.** If $A \in X^*$ and $\{a_n\} \in A$, then $\lim \int a_n$ exists.

Since $\{a_n\}$ is a Cauchy sequence, $\int |a_n - a_m|$ converges to zero as $n$ and $m$ go to infinity. Since $|\int a_n - \int a_m|$ converges to zero as $n$, $m$ go to infinity, the limit $\lim \int a_n$ exists.

Now, to make sure the definition of the integral of a member of $X^*$ is a valid definition, it must be proved that all Cauchy sequences from $A \in X^*$ give the same limit.

**Theorem 3.2.** All sequences from $A \in X^*$ give the same limit.

Consider $\{a_n\} \in A$ and $\{b_n\} \in A$ where $\{a_n\}$ and $\{b_n\}$ are not the same sequence. By Theorem 3.1, $\lim \int a_n$ and $\lim \int b_n$ exist. Since $d(a_n, b_n) = 0$, then $\int |a_n - b_n|$ converges to zero as $n$ goes to infinity. Now, $|\int a_n - \int b_n| \leq \int |a_n - b_n|$.
Thus, \( \int a_n - \int b_n \) converges to zero as \( n \) goes to infinity.

\[
\lim_{n \to \infty} \int a_n = \lim_{n \to \infty} \int b_n \text{ as } n \text{ goes to infinity.}
\]

Since the integral of a member of \( X^* \) exists and is unique, then the definition is a valid one.

**Definition.** If \( A, B \in X^* \), \( \{a_n\} \in A \) and \( \{b_n\} \in B \), then define \( A + B \) by the following:

\[
A + B = [\{a_n + b_n\}].
\]

**Theorem 3.3.** \( A + B \) is well-defined, that is if \( \{a_n\} \) and \( \{x_n\} \in A \) and if \( \{b_n\} \) and \( \{y_n\} \in B \) then \( \{a_n + b_n\} = [\{x_n + y_n\}] \).

Let \( \varepsilon > 0 \). Since \( \{a_n\} \cup \{x_n\} \) there exists \( n_1 \in \mathbb{N} \) so that if \( n > n_1 \), \( \int_0^1 |a_n - x_n| < \frac{\varepsilon}{2} \)

and since \( \{b_n\} \cup \{y_n\} \) there exists \( n_2 \in \mathbb{N} \) so that if \( n > n_2 \), \( \int_0^1 |b_n - y_n| < \frac{\varepsilon}{2} \). Now,

\[
\lim_{n \to \infty} \int_0^1 |a_n + b_n| - \lim_{n \to \infty} \int_0^1 |x_n + y_n| \leq \lim_{n \to \infty} \int_0^1 |a_n - x_n + b_n - y_n| \leq \lim_{n \to \infty} \int_0^1 |a_n - x_n| + |b_n - y_n| < \varepsilon. \]

Since \( d(a_n + b_n, x_n + y_n) < \varepsilon \), then by definition \( \{a_n + b_n\} = [\{x_n + y_n\}] \).

**Definition.** The characteristic function of \( [a,b) \), denoted \( C[a,b) \), is defined as \( C[a,b)(x) = 1 \) if \( x \in [a,b) \)

and \( C[a,b)(x) = 0 \) if \( x \notin [a,b) \). If \( A \in X^* \) and \( A = [\{a_n\}] \)

define \( C[a,b) \cdot A \) by the following: \( C[a,b) \cdot A = [\{C[a,b) \cdot a_n\}] \).
If \( A \in X^* \) and \( A = \{a_n\} \) then define \( \int_a^b A \) by the following formula: 

\[
\int_a^b A = \int_0^1 C(a,b) \cdot A \text{ where } 0 \leq a < b \leq 1.
\]

**Theorem 3.4.** If \( A \in X^* \) then \( \int_a^b A \) exists and is unique. 

\[
\int_a^b A = \int_0^1 C(a,b) \cdot A = \int_0^1 \{C(a,b) \cdot a_n\}.
\]

Choose a particular \( n \). Now consider \( \int_0^1 C(a,b) \cdot a_n \). Let \( C(a,b) \cdot a_n = b_n \) where \( b_n = a_n \) if \( x \in [a,b) \) and \( b_n = 0 \) if \( x \notin [a,b) \). Now, \( b_n \) is a step function since \( a_n \) is a step function and \( b_n = 0 \) is a constant function. This is true for every \( n \). Also, since \( \{a_n\} \) is a Cauchy sequence of step functions then \( \{b_n\} = \{a_n\} \) for \( x \in [a,b) \) is Cauchy and \( \{b_n\} = 0 \) is Cauchy so \( \{b_n\} \) is a Cauchy sequence of step functions thus \( \{b_n\} \in B \) where \( B \in X^* \) and thus, by Theorem 3.1, \( \int_0^1 \{b_n\} \) exists and is unique.

**Theorem 3.5.** If \( F \in X^* \) then \( \int_a^b F + \int_b^c F = \int_a^c F \).

Let \( C_x = C[a,b) \) be the characteristic function on \( [a,b) \), \( C_y = C[b,c) \) be the characteristic function on \( [b,c) \) and \( C_z = C[a,c) \) be the characteristic function on \( [a,c) \) and let \( \{a_n\} \in F \). 

\[
\int_a^b F + \int_b^c F = \lim_{n \to \infty} \int_0^1 C_x \cdot a_n + \lim_{n \to \infty} \int_0^1 C_y \cdot a_n
\]

\[
= \lim_{n \to \infty} \int_0^1 C_x \cdot a_n + C_y \cdot a_n = \lim_{n \to \infty} \int_0^1 (C_x + C_y) \cdot a_n
\]

\[
= \lim_{n \to \infty} \int_0^1 C_z \cdot a_n = \lim_{n \to \infty} \int_0^1 a_n \cdot C_z. \text{ Thus } \int_a^b F + \int_b^c F = \int_a^c F.
\]

**Definition.** Let \( F \) be a function defined and bounded on \([0,1]\) and assume that \( m, M \) are such that \( m \leq F(x) \leq M \) for \( x \in [0,1] \). Let \( p \) be a partition of \([0,1]\) where 

\[
p = \{0 = x_0, x_1, \ldots, x_n = 1\}. \text{ } p \text{ can also be described by} \]
Theorem 3.6. If $F$ is Riemann integrable then
$$R\int_0^1 F = A$$
where $A \in X^*$. 

Let $p = [0=x_1, x_2, \cdots, x_n=1]$. 

I. The Riemann $(R)$ upper integral is defined as
$$(R)\int_{[0,1]} F = \text{g.l.b. } \bar{\Sigma}_p F \text{ for every } p \text{ of } [0,1].$$

The Riemann $(R)$ lower integral is defined as
$$(R)\int_{[0,1]} F = \text{l.u.b. } \underline{\Sigma}_p F \text{ for every } p \text{ of } [0,1].$$

If $\int_{[0,1]} F = \int_{[0,1]} F$ then $F$ is said to be Riemann $(R)$ integrable on $[0,1]$ and $R\int_0^1 F$ is equal to the common value of $\int$ and $\int$, that is the value when $\int = \int$. 

II. $F$ is Riemann integrable on $[0,1]$ if and only if for $\epsilon > 0$, there exists a particular $p$ of $[0,1]$ so that
$$\bar{\Sigma}_p - \underline{\Sigma}_p < \epsilon$$
which implies $M_p - m_p < \epsilon$. 

Let $p = [0=x_1, x_2, \cdots, x_n=1]$. 

If $F$ is Riemann integrable then
$$R\int_0^1 F = \int A$$
where $A \in X^*$. 

$I_1, I_2, \cdots, I_n$, where $I_i = [x_{i-1}, x_i]$. Let

$\delta_i = \text{length of } I_i$. Let $M_i = \text{least upper bound (l.u.b.)}$

of $F$ on $I_i$ and $m_i = \text{greatest lower bound (g.l.b.)}$ of $F$

on $I_i$. \(S_i = M_i - m_i = \text{l.u.b.} \{f(x_2) - f(x_1) | x_1, x_2 \in I_i\},\)

this is said to be the saltus of $F$ on $I_i$. 

\[
\sum_{p} F = M_1(x_1-x_0) + M_2(x_2-x_1) + \cdots + M_n(x_n-x_{n-1})
\]

\[
= \sum_{i=1}^{n} M_i(x_i-x_{i-1}) = \sum_{i=1}^{n} M_i \delta_i
\]

\[
\sum_{p} F = \sum_{i=1}^{n} m_i(x_i-x_{i-1}) = \sum_{i=1}^{n} m_i \delta_i
\]

\[
\sum_{p} F - \sum_{p} F = \sum_{i=1}^{n} S_i(x_i-x_{i-1}) = \sum_{i=1}^{n} S_i(\delta_i)
\]
Choose $\varepsilon_1 = 1$. Let a sequence of positive numbers be defined by the following: $\varepsilon_1 = 1, \varepsilon_2 = \frac{1}{2}, \ldots, \varepsilon_n = \frac{1}{n}, \ldots$.

Let $p_n$ be a sequence of partitions so that each $p_n$ satisfies II with $\varepsilon = \varepsilon_n$. Let a sequence of step functions $F_n$ be defined on subintervals of $p_n$ by $F_n = \text{l.u.b. of } F$ on $[x_{k-1}, x_k) = m_k$ where $x_1 \leq x_{k-1} < x_k \leq x_n$. Let $f_n$ be a sequence of step functions defined on subintervals of $p_n$ by $f_n = \text{g.l.b. of } F$ on $[x_{k-1}, x_k) = m_k$ where $x_1 \leq x_{k-1} < x_k \leq x_n$.

(a) $f_n$ converges to $F_n$, that is $\lim_{n \to \infty} \int (F_n - f_n) = 0$.

By II, $\int F_n - \int f_n < \varepsilon_n$. So $\int F_n - f_n < \varepsilon_n$. Since $\lim \varepsilon_n = 0$, then $\lim_{n \to \infty} \int (F_n - f_n) = 0$. Thus $f_n$ converges to $F_n$.

(b) $f_n$ converges to $F$ and $F_n$ converges to $F$. Since $f_n$ converges to $F_n$, then there exists $n_2 \in \mathbb{N}$ so that if $n > n_2$, then $\int (F_n - f_n) < \varepsilon_n$. Since $f_n \leq F \leq F_n$ then $\int f_n \leq \int F \leq \int F_n$ and for $n > n_2$, $\int (F_n - f_n) \leq \int (F - f_n) \leq \int (p_n - f_n) < \varepsilon_n$.

Since $\lim \varepsilon_n = 0$, then $\lim_{n \to \infty} \int (F_n - f_n) = 0$. Also, for $n > n_2$, $\int (F_n - F) \leq \int (F_n - f_n) < \varepsilon_n$. Thus $\lim_{n \to \infty} \int (F_n - F) = 0$. Thus $f_n$ converges to $F$ and $F_n$ converges to $F$.

(c) $f_n$ and $F_n$ are Cauchy sequences of step functions.

Let $\varepsilon > 0$. Since by part b, $f_n$ converges to $F$, there exists $n_3 \in \mathbb{N}$ so that if $n > n_3$ this implies $\int_0^1 |f_n - F| < \varepsilon/2$ and $m > n_3$ implies $\int_0^1 |f_m - F| < \varepsilon/2$.

$\int |f_n - f_m| = \int |f_n - F + F - f_m| \leq \int |f_n - F| + |F - f_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So $\int |f_n - f_m| < \varepsilon$. Thus $f_n$ is Cauchy.
Let $\varepsilon > 0$. Since by part b, $F_n$ converges to $F$, there exists $n_0 \in \mathbb{N}$ so that if $n > n_0$ this implies
\[ \int_0^1 |F_n - F| < \varepsilon/2 \text{ and } m > n_0 \text{ implies } \int_0^1 |F_m - F| < \varepsilon/2. \]

\[ \int |F_n - F_m| = \int |F_n - F + F - F_m| \leq \int |F_n - F| + |F - F_m| \]
\[ = \int |F_n - F| + \int |F_m - F| \]
\[ < \varepsilon/2 + \varepsilon/2 = \varepsilon. \]

So $\int |F_n - F_m| < \varepsilon$. Thus $F_n$ is Cauchy.

Now, $F_n$ and $f_n$ are Cauchy sequences of step functions and $\{F_n\} \cup \{f_n\}$ since by part b, $\lim_{n \to \infty} \int (F_n - f_n) = 0$. So there is some $A \in X^*$ so that $\{F_n\}$ and $\{f_n\} \in A$. By theorem 3.2, $\bigcap A$ is unique, and since $F_n$ and $f_n$ converges to $F$, then $\mathcal{R} \int F = A$, where $A \in X^*$. 
BIBLIOGRAPHY
