COMPLETING THE SPACE OF STEP FUNCTIONS

APPROVED:

Russell G. Bilyeu
Major Professor

H. C. Parrish
Minor Professor

F. E. Connor
Chairman of the Department of Mathematics

Robert B. Toulouse
Dean of the Graduate School

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In this thesis a study is made of the space X of all step functions on [0,1]. This investigation includes determining a completion space, X*, for the incomplete space X, defining integration for X*, and proving some theorems about integration in X*.

The thesis is divided into three chapters. Chapter I is an introduction to the thesis. Chapter II defines a step function and the space of all step functions X, includes proof that X is an incomplete metric space, and concludes by showing X has a completion metric space, X*. Chapter III investigates X*, especially with respect to integration. The integral of a member of X* is defined and then it is proven that this integral actually exists and is unique. Some of the properties of integrals are shown to be true for integration in X*. The last theorem of Chapter III shows a relationship between the Riemann integral and the integral of a member of X*.

COMPLETING THE SPACE OF STEP FUNCTIONS

THESIS

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Linda K. Massey, B. S.

Denton, Texas

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CHAPTER I

INTRODUCTION

In this thesis a study is made of the space X of all step functions on [0.1]. In Chapter II step functions are defined and the space X is proven to be a metric space. X is an incomplete metric space since there is at least one Cauchy sequence of elements of X which converges to an element not in X. The incomplete metric space X is shown to have a completion metric space, X*.

Chapter III investigates X*, especially with respect to integration. The integral of a member of X* is defined and then it is proven that this integral actually exists and is unique. Some of the properties of integrals are shown to be true for integration in X*. The last few theorems of Chapter III investigate specific members of X*.

CHAPTER II

A COMPLETE METRIC SPACE

A step function is appropriately named since its graph appears to be a series of steps, without the riser of the step, or one long, continuous step. Some definitions allow the very end of the step, a point, to be separated from the step and thus lie above or below the step; this will not be allowed in the definition of a step function in this thesis. Thus a step function is either a constant function over the domain or it is a discontinuous function which is constant on subintervals of the domain.

In this thesis a step function will be defined entirely in terms of subintervals of [0,1] which are left half-closed. There are infinitely many step functions on [0,1]; all of these will make up the space of all step functions on [0,1]. It is important to have an intuitive idea of a step function, but to clarify matters, a formal definition is in order.

A function S, whose domain is [0,1] is called a step function if there is a partition $P = \{X_0, X_1, \cdots X_n\}$ of [0,1] such that S is constant on each left half-closed subinterval of P. That is to say, for each $k = 1, 2, \cdots, n$ there is a real number S_k such that

$$S(x) = S_k \text{ if } x_{k-1} \leq x < x_k \text{ except for } x_k = 1$$

then,

$$S(x) = S_k \text{ if } x_{k-1} \leqq x \leqq x_k$$

The integral of a step function S from 0 to 1, denoted by the symbol $\int_0^1 S(x)dx$ is defined by the following formula:

$$\int_0^1 S(x)dx = \sum_{k=1}^{n} S_k \cdot (x_k - x_{k-1}).$$

A metric space is a mathematical system (X,d) consisting of a set X of elements and a real single-valued function d(x,y) defined on all ordered pairs (x,y) of elements of X, having the following properties:

P1. $d(x,y) \geqq 0$.

P2. $d(x,y) = 0$ if and only if $x = y$.

P3. $d(x,y) = d(y,x)$.

P4. $d(x,z) \leqq d(x,y) + d(y,z)$.

If X is a metric space, and $x_1, x_2, x_3, \cdots$ is a sequence of points in X, the sequence is a Cauchy sequence if for every positive real number ϵ, there exists a positive integer N such that, if m and n are integers each greater than N, then $d(x_m, x_n) < \epsilon$.

If $x_1, x_2, x_3, \cdots$ is a sequence of points in the metric space X, the sequence $x_1, x_2, x_3, \cdots$ is said to be convergent if there is a point x in X such that, for every positive real number ϵ, there is a positive integer N such that if n is an integer larger than N, then $d(x_n, x) < \epsilon$. The point x is then called a limit of the sequence $x_1, x_2, x_3, \cdots$.

A metric space (X,d) is complete if and only if every Cauchy sequence in X converges to a point $p \in X$.

Define X to be the set of all step functions on $[0,1]$. If $f,g \in X$ define $d(f,g) = \int_0^1 |f(x)-g(x)|dx$.

Theorem 2.1. $\int_0^1 |f(x) - g(x)|dx \geqq 0$.

Since $|f(x) - g(x)| \geqq 0$, then $\int_0^1 |f(x) - g(x)|dx \geqq 0$.

Theorem 2.2. $\int_0^1 |f(x) - g(x)|dx = 0$ if and only if $f(x) = g(x)$.

(a) Suppose $\int_0^1 |f(x) - g(x)|dx = 0$. Since $|f(x) - g(x)| \geqq 0$, suppose $|f(x) - g(x)| > 0$ for $x' \in [0,1]$. Let $h(x) = |f(x) - g(x)| \geqq 0$. $h(x)$ is a step function on $[0,1]$; let $k = 1, \cdots, n$ be the partition of $[0,1]$ so that $h(x) = h_k$ if $x_{k-1} \leqq x < x_k$ except for $x_k = 1$, then $h(x) = h_k$ if $x_{k-1} \leqq x \leqq x_k$. By definition of an integral of a step function, $\int_0^1 h(x)dx = \sum_{k=1}^{n} h_k(x_k - x_{k-1})$. There is at least one value, x', so that $h_k(x_k' - x_{k-1}') > 0$ and since $h(x)$ is non-negative, then $\sum_{k=1}^{n} h_k(x_k - x_{k-1}) > 0$. Thus $\int_0^1 h(x)dx > 0$, a contradiction. Suppose $|f(x) - g(x)| = 0$. By definition of a step function, since no isolated points are allowed, $f(x) = g(x)$.

(b) Suppose $f(x) = g(x)$, then

$$f(x) - g(x) = 0 \text{ and}$$

$$|f(x) - g(x)| = 0; \text{ thus}$$

$$\int_0^1 |f(x) - g(x)| = 0.$$

Theorem 2.3.

$$\int_0^1 |f(x) - g(x)|dx = \int_0^1 |g(x) - f(x)|dx$$

$$|f(x) - g(x)| = |g(x) - f(x)|.$$

Thus,

$$\int_0^1 |f(x) - g(x)|dx = \int_0^1 |g(x) - f(x)|dx.$$

Theorem 2.4.

$$\int_0^1 |f(x) - h(x)|dx \leqq \int_0^1 |f(x) - g(x)|dx + \int_0^1 |g(x) - h(x)|dx.$$

$$\int_0^1 |f(x) - h(x)|dx = \int_0^1 |f(x) - g(x) + g(x) - h(x)|dx$$

$$\int_0^1 |f(x) - g(x) + g(x) - h(x)|dx \leqq \int_0^1 \left(|f(x) - g(x)|dx + |g(x) - h(x)|dx\right)$$

$$\int_0^1 \left(|f(x) - g(x)|dx + |g(x) - h(x)|dx\right) = \int_0^1 |f(x) - g(x)|dx + \int_0^1 |g(x) - h(x)|dx$$

Thus

$$\int_0^1 |f(x) - h(x)|dx \leqq \int_0^1 |f(x) - g(x)|dx + \int_0^1 |g(x) - h(x)|dx.$$

By Theorems 2.1, 2.2, 2.3 and 2.4, (X,d) is a metric space. (X,d) is not a complete metric space because the following is an example of a Cauchy sequence of step functions which converges to a non-constant continuous function.

Consider $f(x) = x$. Let P be a sequence of partitions of $[0,1]$ such that $P = [\frac{0}{n}, \frac{1}{n}, \cdots \frac{n}{n}]$, for $n \epsilon$ positive integers. Let M_f equal the least upper bound of x on each subinterval of P. Let this sequence of step functions be denoted by f_n. Note $f_n \geqq f$ and M_f of $[\frac{k-1}{n}, \frac{k}{n}) = \frac{k}{n}$. Let $\epsilon > 0$, choose $n > \frac{1}{2\epsilon}$ and a particular subinterval $[\frac{k-1}{n}, \frac{k}{n}) = [a,b)$.

So $\int_a^b |f_n - f| = \int_a^b f_n - f = \int_a^b f_n - \int_a^b f =$

$$\int_a^b M_f - \int_a^b x = \int_a^b b - \int_a^b x = \frac{1}{n} \cdot \frac{k}{n} - \left(\frac{k^2}{2n^2} - \frac{k^2-2k+1}{2n^2}\right) = \frac{1}{2n^2}.$$

Since $\int_a^b |f_n - f| < \frac{1}{2n^2}$, then $\int_0^1 |f_n - f| < n(\frac{1}{2n^2}) = \frac{1}{2n} < \varepsilon$

$[a,b)$ is subinterval in $[0,1]$, f_n converges to f.

(X,d) is not complete because this sequence of step functions did not converge to a step function.

A metric space X* is called a completion of a metric space X if X* is complete and X is isometric to a dense subset of X*.

Consider (X,d) as defined previously. Let C[X] denote the collection of all Cauchy sequences in X and let ν be the relation in C[X] defined by $\{a_n\}\ \nu\ \{b_n\}$ if and only if $\lim_{n\to\infty} d(a_n, b_n) = 0$; that is, if $\lim_{n\to\infty}\int_0^1 | a_n - b_n| = 0$.

Theorem 2.5. The relation ν is an equivalence relation.

(a) $\{a_n\}\ \nu\ \{a_n\}$.

$$\lim_{n\to\infty} \int_0^1 |a_n - a_n| = 0.$$

Thus $\{a_n\}\ \nu\ \{a_n\}$

(b) If $\{a_n\}\ \nu\ \{b_n\}$, then $\{b_n\}\ \nu\ \{a_n\}$. Since $\{a_n\}\ \nu\ \{b_n\}$, $\lim_{n\to\infty}\int_0^1 |a_n - b_n| = 0$, and since $|a_n - b_n| = |b_n - a_n|$, then the $\lim_{n\to\infty}\int_0^1 |a_n - b_n| = 0 = \lim_{n\to\infty}\int_0^1 |b_n - a_n|$. Thus $\{b_n\}\ \nu\ \{a_n\}$.

(c) If $\{a_n\}\ \nu\ \{b_n\}$ and $\{b_n\}\ \nu\ \{c_n\}$, then $\{a_n\}\ \nu\ \{c_n\}$.

Since $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\}$, then

$$\lim_{n\to\infty} \int_0^1 |a_n - b_n| = 0 \text{ and}$$

$$\lim_{n\to\infty} \int_0^1 |b_n - c_n| = 0$$

$$\lim_{n\to\infty} \int_0^1 |a_n - b_n| + \lim_{n\to\infty} \int_0^1 |b_n - c_n| \geqq$$

$$\lim_{n\to\infty} \int_0^1 |a_n - b_n + b_n - c_n|$$

$$\lim_{n\to\infty} \int_0^1 |a_n - b_n + b_n - c_n| = \lim_{n\to\infty} \int_0^1 |a_n - c_n|.$$

So,

$$\lim_{n\to\infty} \int_0^1 |a_n - b_n| + \lim_{n\to\infty} \int_0^1 |b_n - c_n| \geqq$$

$$\lim_{n\to\infty} |a_n - c_n|.$$

Since $\lim_{n\to\infty} \int_0^1 |a_n - b_n| + \lim_{n\to\infty} \int_0^1 |b_n - c_n| = 0$, then

$$0 \geqq \lim_{n\to\infty} \int_0^1 | a_n - c_n|, \text{ so}$$

$$0 = \lim_{n\to\infty} \int_0^1 |a_n - c_n|.$$

By (a), (b) and (c), $\sim$ is an equivalence relation.

Now let X^* denote the quotient set $C[X]|\sim$; that is, X^* consists of equivalence classes $[\{a_n\}]$ of Cauchy sequences $\{a_n\} \in C[X]$.

Let e be the function defined by $e([\{a_n\}], [\{b_n\}]) = \lim_{n\to\infty} \int_0^1 |a_n - b_n|$ where $[\{a_n\}], [\{b_n\}] \in X^*$.

<u>Theorem 2.6.</u> The function e is well-defined, that is, if $\{a_n\} \vee \{A_n\}$ and $\{b_n\} \vee \{B_n\}$, then $\lim_{n\to\infty} \int_0^1 |a_n - b_n| = \lim_{n\to\infty} \int_0^1 |A_n - B_n|$.

Set $r = \lim_{n\to\infty} \int_0^1 |a_n - b_n|$ and

$R = \lim \int_0^1 |A_n - B_n|$ and let $\epsilon > 0$.

Note: $\int_0^1 |a_n - b_n| \leqq \int_0^1 |a_n - A_n| + \int_0^1 |A_n - B_n| + \int_0^1 |B_n - b_n|$.

Now, there exists $n_1 \in N$ so that if $n > n_1$, $\int_0^1 |a_n - A_n| < \frac{\epsilon}{3}$, there exists $n_2 \in N$ so that if $n > n_2$, then $\int_0^1 |b_n - B_n| < \frac{\epsilon}{3}$, and there exists $n_3 \in N$ so that if $n > n_3$, then $\left|\int_0^1 |A_n - B_n| - R\right| < \frac{\epsilon}{3}$.

If $n > \max(n_1, n_2, n_3)$ then $\int_0^1 |a_n - b_n| < R + \epsilon$ and $\lim_{n\to\infty} \int_0^1 |a_n - b_n| = r$, so

$$r \leqq R + \epsilon.$$

But this inequality holds for every $\epsilon > 0$; hence, $r \leqq R$. In the same manner it may be shown that $R \leqq r$; thus, $r = R$.

In other words, e does not depend upon the particular Cauchy sequence chosen to represent any equivalence class.

<u>Theorem 2.7.</u> The function e is a metric on X^*.

(P_1) If $f, g \in X^*$, then $e(f,g) \geqq 0$; that is, if $\{a_n\} \in f$ and $\{b_n\} \in g$, then $\int |a_n - b_n| \geqq 0$. Since $|a_n - b_n| \geqq 0$, then $\int |a_n - b_n| \geqq 0$.

(P_2) If f, g ϵ X*, then e(f,g) = 0 if and only if f = g.

Assume $\{a_n\} \epsilon$ f and $\{b_n\} \epsilon$ g.

(a) If $\lim \int |a_n - b_n| = 0$, then $\{a_n\} \nu \{b_n\}$, by definition.

(b) If $\{a_n\} \nu \{b_n\}$, then $\int |a_n - b_n| = 0$.

(P_3) If f, g ϵ X*, then e(f,g) = e(g,f). Assume $\{a_n\} \epsilon$ f and $\{b_n\} \epsilon$ g, then $e(f,g) = \lim \int |a_n - b_n| = \lim \int |b_n - a_n| = e(g,f)$.

(P_4) If f,g,h ϵ X*, then $e(f,h) \leq e(f,g) + e(g,h)$. If $\{a_n\} \epsilon$ f, $\{b_n\} \epsilon$ g, $\{c_n\} \epsilon$ h, then $e(f,h) = \lim \int |a_n - c_n| =$

$$\lim \int |a_n - b_n + b_n - c_n| \leq$$

$$\lim \int |a_n - b_n| + \lim \int |b_n - c_n| =$$

$$e(f,g) + e(g,h).$$

Thus, e is a metric on X*

Now for each p ϵ X, the sequence {p, p, p, $\cdots$} is Cauchy. Set $\hat{p}$ = [{p, p, p, $\cdots$}] and $\hat{X} = \{\hat{p};\ p \epsilon X\}$. Then $\hat{X} \subset X^*$.

Theorem 2.8. X is isometric to $\hat{X}$ and $\hat{X}$ is dense in X*.

For every p,q ϵ X, $e(\hat{p},\hat{q}) = \lim d(p,q) = d(p,q)$ so X is isometric to $\hat{X}$. To show $\hat{X}$ is dense in X*, show every point in X* is the limit of a sequence in $\hat{X}$. Let $\alpha = [\{a_1, a_2, \cdots\}]$ be any point in X*. Then $\{a_n\}$ is a Cauchy sequence in X. Hence, $\lim e(\hat{a}_m,\alpha) = \lim [\lim d(a_m, a_n)] = \lim d(a_m, a_n) = 0$.

Accordingly, $\{\hat{a}_n\} \rightarrow \alpha$, and thus, α is the limit of the sequence $\{\hat{a}_1, \hat{a}_2, \cdots\}$ in $\hat{X}$. Thus $\hat{X}$ is dense in X*.

<u>Theorem 2.9.</u> Let $\{b_1, b_2, \cdots\}$ be a Cauchy sequence in X and let $\{a_1, a_2, \cdots\}$ be a sequence in X such that $d(a_n, b_n) < \frac{1}{n}$ for every $n \in N$, then

(i) $\{a_n\}$ is also a Cauchy sequence in X.

(ii) $\{a_n\}$ converges to p if and only if $\{b_n\}$ converges to p.

(i) By the triangle inequality $d(a_m, a_n) \leqq d(a_m, b_m) + d(b_m, b_n) + d(b_n, a_n)$. Let $\epsilon > 0$. Then there exists $n_1 \in N$ such that $\frac{1}{n} < \frac{\epsilon}{3}$. Hence, for $n, m > n_1$, $d(a_m, a_n) < \epsilon/3 + d(b_m, b_n) + \frac{\epsilon}{3}$. By hypothesis $\{b_1, b_2, \cdots\}$ is a Cauchy sequence so there is $n_2 \in N$ such that for $n, m > n_2$, $d(b_m, b_n) < \epsilon/3$.

Set $n_0 = \max(n_1, n_2)$. Then, for $n, m > n_0$, $d(a_m, a_n) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. Thus, $\{a_n\}$ is a Cauchy sequence.

(ii) By the Triangle inequality, $d(b_n, p) \leqq d(b_n, a_n) + d(a_n, p)$. Hence $\lim d(b_n, p) \leqq \lim d(b_n, a_n) + \lim(a_n, p)$. But the $\lim d(b_n, a_n) \leqq \lim (\frac{1}{n}) = 0$. Thus, if a_n converges to p, then $\lim d(b_n, p) \leqq \lim (a_n, p) = 0$, so b_n converges to p.

Similarly, if $\{b_n\}$ converges to p, then $\{a_n\}$ converges to p.

<u>Theorem 2.10.</u> Every Cauchy sequence in (X^*,e) converges and so (X^*,e) is a completion of X.

Let $\{A_1,A_2, \cdots\}$ be a Cauchy sequence in X^*. Since $\hat{X}$ is dense in X^* for every $n \in N$, there is $\hat{a}_n \in \hat{X}$ such that $e(\hat{a}_n,A_n) < \frac{1}{n}$. Then, by Theorem 2.9, $\{\hat{a}_1, \hat{a}_2, \cdots\}$ is also a Cauchy sequence and by Theorem 2.8, $\{\hat{a}_1, \hat{a}_2, \cdots\}$ converges to $B = [\{a_1,a_2, \cdots\}] \in X^*$. Hence, by Theorem 2.9, $\{A_n\}$ also converges to B and, therefore, (X^*, e) is complete.

CHAPTER III

INTEGRATION IN X*

Do the integrals of members of X* exist? How is an integral of a member of X* to be defined? What are some of the properties of integrals which hold true for integration in X*? What are some specific elements of X*? These are some of the questions that Chapter III will answer.

Definition. If $A \in X^*$ and $\{a_n\}$ is a Cauchy sequence which is an element of A, then the integral of A, denoted by the symbol $\int A$, is defined by the following,

$$\int A = \lim \int a_n.$$

Theorem 3.1. If $A \in X^*$ and $\{a_n\} \in A$, then $\lim \int a_n$ exists.

Since $\{a_n\}$ is a Cauchy sequence, $\int |a_n - a_m|$ converges to zero as n and m go to infinity. Since $|\int a_n - \int a_m|$ converges to zero as n, m go to infinity, the limit $\lim \int a_n$ exists.

Now, to make sure the definition of the integral of a member of X* is a valid definition, it must be proved that all Cauchy sequences from $A \in X^*$ give the same limit.

Theorem 3.2. All sequences from $A \in X^*$ give the same limit.

Consider $\{a_n\} \in A$ and $\{b_n\} \in A$ where $\{a_n\}$ and $\{b_n\}$ are not the same sequence. By Theorem 3.1, $\lim \int a_n$ and $\lim \int b_n$ exist. Since $d(a_n, b_n) = 0$, then $\int |a_n - b_n|$ converges to zero as n goes to infinity. Now, $|\int a_n - \int b_n| \leq \int |a_n - b_n|$.

Thus, $|\int a_n - \int b_n|$ converges to zero as n goes to infinity $\lim \int a_n = \lim \int b_n$ as n goes to infinity.

Since the integral of a member of X* exists and is unique, then the definition is a valid one.

Definition. If $A, B \in X^*$, $\{a_n\} \in A$ and $\{b_n\} \in B$, then define A + B by the following:

$$A + B = [\{a_n + b_n\}].$$

Theorem 3.3. A + B is well-defined, that is if $\{a_n\}$ and $\{x_n\} \in A$ and if $\{b_n\}$ and $\{y_n\} \in B$ then $[\{a_n+b_n\}]=[\{x_n+y_n\}]$.

Let $\varepsilon > 0$. Since $\{a_n\} \nu \{x_n\}$ there exists $n_1 \in N$ so that if $n > n_1$, $\int_0^1 |a_n - x_n| < \frac{\epsilon}{2}$

and since $\{b_n\} \nu \{y_n\}$ there exists $n_2 \in N$ so that if $n > n_2$ $\int_0^1 |b_n - y_n| < \frac{\epsilon}{2}$. Now,

$\lim\limits_{n\to\infty} \int_0^1 |a_n + b_n| - \lim\limits_{n\to\infty} \int_0^1 |x_n + y_n| \leq \lim\limits_{n\to\infty} \int_0^1 |a_n - x_n + b_b - y_n| \leq \lim\limits_{n\to\infty} \int_0^1 |a_n - x_n| + |b_n - y_n| < \epsilon$. Since $d(a_n + b_n, x_n + y_n) < \epsilon$, then by definition $[\{a_n + b_n\}] = [\{x_n + y_n\}]$.

Definition. The characteristic function of $[a,b)$, denoted $C[a,b)$, is defined as $C[a,b)(x) = 1$ if $x \in [a,b)$ and $C[a,b)(x) = 0$ if $x \notin [a,b)$. If $A \in X^*$ and $A = [\{a_n\}]$ define $C[a,b)\cdot A$ by the following: $C[a,b)\cdot A = [\{C[a,b)\cdot a_n\}]$.

If $A \in X^*$ and $A = [\{a_n\}]$ then define $\int_a^b A$ by the following formula: $\int_a^b A = \int_0^1 C[a,b)\cdot A$ where $0 \le a < b \le 1$.

<u>Theorem 3.4.</u> If $A \in X^*$ then $\int_a^b A$ exists and is unique.

$\int_a^b A = \int_0^1 C[a,b)\cdot A = \int_0^1 \{C[a,b)\cdot a_n\}$. Choose a particular n. Now consider $\int_0^1 C[a,b)\cdot a_n$. Let $C[a,b)\cdot a_n = b_n$ where $b_n = a_n$ if $x \in [a,b)$ and $b_n = 0$ if $x \notin [a,b)$. Now, b_n is a step function since a_n is a step function and $b_n = 0$ is a constant function. This is true for every n. Also, since $\{a_n\}$ is a Cauchy sequence of step functions then $\{b_n\} = \{a_n\}$ for $x \in [a,b)$ is Cauchy and $\{b_n\} = 0$ is Cauchy so $\{b_n\}$ is a Cauchy sequence of step functions thus $\{b_n\} \in B$ where $B \in X^*$ and thus, by Theorem 3.1, $\int_0^1 [\{b_n\}]$ exists and is unique.

<u>Theorem 3.5.</u> If $F \in X^*$ then $\int_a^b F + \int_b^c F = \int_a^c F$.

Let $C_x = C[a,b)$ be the characteristic function on $[a,b)$, $C_y = C[b,c)$ be the characteristic function on $[b,c)$ and $C_z = C[a,c)$ be the characteristic function on $[a,c)$ and let $\{a_n\} \in F$. $\int_a^b F + \int_b^c F = \lim_{n\to\infty} \int_0^1 C_x\cdot a_n + \lim_{n\to\infty} \int_0^1 C_y\cdot a_n$

$= \lim_{n\to\infty} \int_0^1 C_x\cdot a_n + C_y\cdot a_n = \lim_{n\to\infty} \int_0^1 (C_x + C_y)\cdot a_n$

$= \lim_{n\to\infty} \int_0^1 C_z\cdot a_n = \lim_{n\to\infty} \int_a^c a_n$. Thus $\int_a^b F + \int_b^c F = \int_a^c F$.

<u>Definition.</u> Let F be a function defined and bounded on $[0,1]$ and assume that m,M are such that $m \le F(x) \le M$ for $x \in [0,1]$. Let p be a partition of $[0,1]$ where $p = \{0 = x_0, x_1, \cdots, x_n = 1\}$. p can also be described by

$I_1, I_2, \cdots, I_i \cdots, I_n$, where $I_i = [x_{i-1}, x_i]$. Let δ_i = length of I_i. Let M_i = least upper bound (l.u.b.) of F on I_i and m_i = greatest lower bound (g.l.b.) of F on I_i. $S_i = M_i - m_i = \text{l.u.b.}\{f(x_2) - f(x_1) | x_1, x_2 \in I_i\}$, this is said to be the saltus of F on I_i.

$$\overline{\Sigma}_p F = M_i(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$$
$$= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} M_i \delta_i$$
$$\underline{\Sigma}_p F = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} m_i \delta_i$$
$$\overline{\Sigma}_p F - \underline{\Sigma}_p F = \sum_{i=1}^{n} S_i(x_i - x_{i-1}) = \sum_{i=1}^{n} S_i(\delta_i)$$

I. The Riemann (R) upper integral is defined as $(R)\overline{\int}_{[0,1]} F = \text{g.l.b. } \overline{\Sigma}_p F$ for every p of [0,1].

The Riemann (R) lower integral is defined as $(R)\underline{\int}_{[0,1]} F = \text{l.u.b.} \underline{\Sigma}_p F$ for every p of [0,1].

If $\overline{\int}_{[0,1]} F = \underline{\int}_{[0,1]} F$ then F is said to be Riemann (R) integrable on [0,1] and $R\int F$ is equal to the common value of $\overline{\int}$ and $\underline{\int}$, that is the value when $\overline{\int} = \underline{\int}$.

II. F is Riemann integrable on [0,1] if and only if for $\epsilon > 0$, there exists a particular p of [0,1] so that $\overline{\Sigma}_p - \underline{\Sigma}_p < \epsilon$ which implies $M_p - m_p < \epsilon$.

Let $p = [0 = x_1, x_2, \cdots, x_n = 1]$.

<u>Theorem 3.6.</u> If F is Riemann integrable then $R\int F = \int A$ where $A \in X^*$.

Choose $\varepsilon_1 = 1$. Let a sequence of positive numbers be defined by the following: $\varepsilon_1 = 1$, $\varepsilon_2 = \frac{1}{2}$ ..., $\varepsilon_n = \frac{1}{n}$ .... Let p_n be a sequence of partitions so that each p_n satisfies II with $\varepsilon = \varepsilon_n$. Let a sequence of step functions F_n be defined on subintervals of p_n by F_n = l.u.b. of F on $[x_{k-1}, x_k) = M_k$ where $x_1 \leq x_{k-1} < x_k \leq x_n$. Let f_n be a sequence of step functions defined on subintervals of p_n by f_n = g.l.b. of F on $[x_{k-1}, x_k) = m_k$ where $x_1 \leq x_{k-1} < x_k \leq x_n$.

(a) f_n converges to F_n, that is $\lim_{n\to\infty} \int(F_n - f_n) = 0$. By II, $\int F_n - \int f_n < \varepsilon_n$. So $\int F_n - f_n < \varepsilon_n$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, then $\lim_{n\to\infty} \int(F_n - f_n) = 0$. Thus f_n converges to F_n.

(b) f_n converges to F and F_n converges to F. Since f_n converges to F_n, then there exists $n_2 \varepsilon N$ so that if $n > n_2$, then $\int(F_n - f_n) < \varepsilon_n$. Since $f_n \leq F \leq F_n$ then $\int f_n \leq \int F \leq \int F_n$ and for $n > n_2$ $\int(F - f_n) \leq \int(F_n - f_n) < \varepsilon_n$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, then $\lim_{n\to\infty} \int(F - f_n) = 0$. Also, for $n > n_2$, $\int(F_n - F) \leq \int(F_n - f_n) < \varepsilon_n$. Thus $\lim_{n\to\infty} \int (F_n - F) = 0$. Thus f_n converges to F and F_n converges to F.

(c) f_n and F_n are Cauchy sequences of step functions. Let $\varepsilon > 0$. Since by part b, f_n converges to F, there exists $n_3 \varepsilon N$ so that if $n > n_3$ this implies $\int_0^1 |f_n - F| < \varepsilon/2$ and $m > n_3$ implies $\int_0^1 |f_m - F| < \varepsilon/2$. $\int |f_n - f_m| = \int |f_n - F + F - f_m| \leq \int |f_n - F| + |F - f_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So $\int |f_n - f_m| < \varepsilon$. Thus f_n is Cauchy.

Let $\varepsilon > 0$. Since by part b, F_n converges to F, there exists $n_0 \varepsilon N$ so that if $n > n_0$ this implies $\int_0^1 |F_n - F| < \varepsilon/2$ and $m > n_0$ implies $\int_0^1 |F_m - F| < \varepsilon/2$.

$$\int |F_n - F_m| = \int |F_n - F + F - F_m| \leq \int |F_n - F| + |F - F_m|$$
$$= \int |F_n - F| + \int |F_m - F|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So $\int |F_n - F_m| < \varepsilon$. Thus F_n is Cauchy.

Now, F_n and f_n are Cauchy sequences of step functions and $\{F_n\} \nu \{f_n\}$ since by part b, $\lim_{n\to\infty} \int (F_n - f_n) = 0$. So there is some $A \varepsilon X^*$ so that $\{F_n\}$ and $\{f_n\} \varepsilon A$. By theorem 3.2 $\int A$ is unique, and since F_n and f_n converges to F, then $R \int F = A$, where $A \varepsilon X^*$.

BIBLIOGRAPHY

Apostol, Tom M., _Calculus_, Waltham, Massachusetts, Blaisdell Publishing Company, 1967.

Hartnett, William E., _An Introduction to the Fundamental Concepts of Analysis_, New York, John Wiley and Sons, Inc., 1964.