ON THE STIETJES INTEGRAL

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This paper is a study of the Stieltjes integral, a generalization of the Riemann integral normally studied in introductory calculus courses. The purpose of the paper is to investigate many of the basic manipulative properties of the integral. A basic knowledge of the real number system is assumed. All functions considered are real-valued functions.

The first chapter is a general introduction that contains assumed definitions and theorems. Included in this chapter is material concerning sequences, continuity, differentiation, and the Riemann integral.

The second chapter introduces a possible upper and lower sum approach to the Stieltjes integral. A rigorous definition of the Stieltjes integral is given, and the Stieltjes integral of a function with respect to a nondecreasing function is investigated. The second chapter concludes with a proof that a continuous function is Stieltjes integrable with respect to a nondecreasing function.

The third chapter is concerned with the development of the basic manipulative properties of the integral. An introduction to the concept of a function of bounded variation is
included, and a proof that a continuous function is Stieltjes integrable with respect to a function of bounded variation is included. The third chapter concludes with a proof of a formula concerning integration by parts.

The fourth and final chapter includes a proof of the Belt Theorem.
ON THE STIELTJES INTEGRAL

THESIS

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CHAPTER I

INTRODUCTION

This paper is a study of the Stieltjes integral, a generalization of the Riemann integral normally studied in introductory calculus courses. A basic knowledge of the real number system will be assumed. All functions will be real valued functions.

The first chapter includes assumed definitions and theorems.

The second chapter introduces a possible upper and lower sum approach to the Stieltjes integral. A rigorous definition of the Stieltjes integral is given, and the Stieltjes integral of a function with respect to a nondecreasing function is investigated.

The third chapter includes an introduction to the concept of a function of bounded variation. A more general criterion for the existence of the integral is developed, and many of the basic properties of the integral are proved.

The fourth and final chapter returns to the Stieltjes integral of a function with respect to a nondecreasing function and culminates in a proof of the Belt Theorem.

The following notation conventions will hold throughout this paper:
1. $x \in M$ means that $x$ is an element of the set $M$
2. $N \subset M$ means that $N$ is a proper subset of $M$
3. $N \subseteq M$ means that $N$ is a subset of $M$
4. $[a, b] = \{x : a \leq x \leq b\}$
5. $(a, b) = \{x : a < x < b\}$
6. $[a, b) = \{x : a \leq x < b\}$
7. $(a, b] = \{x : a < x \leq b\}$
8. $\text{Lub } M$ denotes the least upper bound of the set $M$
9. $\text{Glb } M$ denotes the greatest lower bound of the set $M$.

**Definition 1.1.** An infinite sequence is a function whose domain is the set of positive integers. The symbol $\{a_n\}_{n=1}^{\infty}$ will be used to represent an infinite sequence of real numbers. [1, p. 9]

**Definition 1.2.** If $f$ is a function from $A$ onto $B' \subseteq B$ and $g$ is a function from $B'$ into $C$, then the function $h(x) = g(f(x))$ is called the composition of $g$ with $f$ and is denoted by $g \circ f$. [1, p. 9]

**Definition 1.3.** Let $f$ be a function with domain $D$; then
1. $f$ is said to be nondecreasing if given $x, y \in D$ such that $x < y$, then $f(x) \leq f(y)$;
2. $f$ is said to be increasing if given $x, y \in D$ such that $x < y$, then $f(x) < f(y)$;
3. $f$ is said to be nonincreasing if given $x, y \in D$ such that $x < y$, then $f(x) \geq f(y)$;
4. $f$ is said to be decreasing if given $x, y \in D$ such that $x < y$, then $f(x) > f(y)$.
Definition 1.4. Let \( f \) be an infinite sequence. Then \( h \) is said to be an infinite subsequence of \( f \) if there is a non-decreasing function \( g \) from the positive integers into the positive integers such that \( h = f \circ g \). [1, p. 10]

Definition 1.5. The statement \( \{a_n\}_{n=1}^{\infty} \) converges to \( a \) means that if \( \varepsilon > 0 \), then there exists \( N \) such that if \( n \geq N \), then \( |a_n - a| < \varepsilon \).

Definition 1.6. The statement \( \{a_n\}_{n=1}^{\infty} \) converges means that there exists a number \( a \) such that \( \{a_n\}_{n=1}^{\infty} \) converges to \( a \).

Definition 1.7. The statement \( \{a_n\}_{n=1}^{\infty} \) diverges means that \( \{a_n\}_{n=1}^{\infty} \) does not converge.

Theorem 1.1. If the sequence \( \{a_n\}_{n=1}^{\infty} \) converges to \( a \), then every subsequence of \( \{a_n\}_{n=1}^{\infty} \) converges to \( a \).

Definition 1.8. Let \( f \) be a function with domain \( D \); then
1. \( f \) is said to be bounded above if there exists a real number \( M \) such that \( f(x) \leq M \) for all \( x \in D \);
2. \( f \) is said to be bounded below if there exists a real number \( M \) such that \( f(x) \geq M \) for all \( x \in D \);
3. \( f \) is said to be bounded if \( f \) is bounded above and bounded below.

Theorem 1.2. If \( \{a_n\}_{n=1}^{\infty} \) is a nondecreasing sequence bounded above, then \( \{a_n\}_{n=1}^{\infty} \) converges.

Definition 1.9. The statement \( \alpha \) is a limit superior of \( \{a_n\}_{n=1}^{\infty} \) means that if \( \varepsilon > 0 \), then
1. \( \{n: a_n > \alpha + \varepsilon\} \) is either empty or else finite;
2. \( \{n: a_n > \alpha - \varepsilon\} \) is infinite.
Definition 1.10. The statement that \( a \) is a limit inferior of \( \{a_n\}_{n=1}^{\infty} \) means that if \( \varepsilon > 0 \), then

1. \( \{n: a_n < a - \varepsilon\} \) is either empty or else finite;
2. \( \{n: a_n < a + \varepsilon\} \) is infinite.

Theorem 1.3. If the sequence \( \{a_n\}_{n=1}^{\infty} \) is bounded, then it has a limit superior and a limit inferior.

Theorem 1.4. The sequence \( \{a_n\}_{n=1}^{\infty} \) converges if, and only if, it has both a limit superior and a limit inferior and they are equal.

Theorem 1.5. The sequence \( \{a_n\}_{n=1}^{\infty} \) converges if, and only if, given \( \varepsilon > 0 \) there exists \( N \) such that if \( n \geq N \) and \( m \geq N \), then \( |a_n - a_m| < \varepsilon \).

Definition 1.11. The statement that \( p \) is a limit point of a set \( M \) of real numbers means that if \( \varepsilon > 0 \), then there exists \( q \in M \) such that \( q \neq p \) and \( |q - p| < \varepsilon \).

Definition 1.12. The statement that a set \( M \) of real numbers is closed means that if \( p \) is a limit point of \( M \), then \( p \in M \).

Definition 1.13. The statement that \( p \) is an interior point of a set \( M \) of real numbers means that there exists \( \varepsilon > 0 \) such that \( (p - \varepsilon, p + \varepsilon) \subseteq M \).

Definition 1.14. The statement that a set \( M \) of real numbers is open means that every point of \( M \) is an interior point of \( M \).

Theorem 1.6. If \( M \) is a closed and bounded set, and \( Q \) is a covering of open sets for \( M \), then there exists a finite subcollection of \( Q \) that covers \( M \).
Definition 1.15. The statement that \( f \) is continuous at \( x_0 \) means that \( x_0 \) is an element of the domain of \( f \) and if \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( |x - x_0| < \delta \) and \( x \) is an element of the domain of \( f \), then \( |f(x) - f(x_0)| < \varepsilon \).

Definition 1.16. The statement that \( f \) is continuous over \( M \) means that \( f \) is continuous at every point of \( M \).

Theorem 1.7. Let \( c \) be some fixed real number and let \( a < b \). If \( f(x) = c \) for every \( x \in [a, b] \), then \( f \) is continuous over \( [a, b] \).

Theorem 1.8. If \( f \) is continuous at \( x_0 \) and \( g \) is continuous at \( x_0 \), then \( f + g \) and \( fg \) are continuous at \( x_0 \).

Theorem 1.9. If \( M \) is a closed and bounded set, and \( f \) is continuous over \( M \), then \( f \) is bounded over \( M \).

Definition 1.17. The statement that \( f \) is differentiable at \( x_0 \) means that \( x_0 \) is an element of the domain of \( f \), \( x_0 \) is a limit point of the domain of \( f \), and there exists a real number \( L \) such that if \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( 0 < |x - x_0| < \delta \) and \( x \) is an element of the domain of \( f \), then

\[
\left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \varepsilon.
\]

The number \( L \) will be denoted by \( f'(x_0) \).

Theorem 1.10. If \( f \) is differentiable at \( x_0 \), then \( f \) is continuous at \( x_0 \).

Definition 1.18. The statement that \( f \) is uniformly continuous over \( M \) means that if \( \varepsilon > 0 \), there exists \( \delta > 0 \)
such that if \(|x_1 - x_2| < \delta\) and \(x_1, x_2 \in M\), then
\[|f(x_1) - f(x_2)| < \varepsilon.\]

**Theorem 1.11.** The function \(f\) is continuous over \([a, b]\) if, and only if, \(f\) is uniformly continuous over \([a, b]\).

**Theorem 1.12.** If \(f\) is continuous over a bounded closed set \(M\), then there exist \(x_1 \in M\) and \(x_2 \in M\) such that
\[f(x_1) = \operatorname{lub} \{f(x) : x \in M\}\] and \(f(x_2) = \operatorname{glb} \{f(x) : x \in M\}.\)

**Theorem 1.13.** If \(f\) is continuous over \([a, b]\), \(f(a) < 0\), and \(f(b) > 0\), then there exists \(c \in (a, b)\) such that \(f(c) = 0.\)

**Theorem 1.14.** Suppose \(f\) is continuous over \([a, b]\), \(f(c) = \operatorname{glb} \{f(x) : x \in [a, b]\}\), and \(f(d) = \operatorname{lub} \{f(x) : x \in [a, b]\}\). If \(f(c) < m < f(d)\), then there exists \(t \in [a, b]\) such that \(f(t) = m.\)

**Theorem 1.15.** If \(f\) is continuous over \([a, b]\), differentiable over \((a, b)\), and \(f(a) = f(b)\), then there exists \(c \in [a, b]\) such that \(f'(c) = 0.\)

**Theorem 1.16.** If \(f\) is continuous over \([a, b]\) and differentiable over \((a, b)\), then there exists \(t \in (a, b)\) such that
\[f'(t) = \frac{f(b) - f(a)}{b - a}.\]

**Theorem 1.17.** If \(f\) is a nondecreasing function, then \(f\) has at most a countable number of points of discontinuity.

**Definition 1.19.** If \(a < b\), then the statement that \(\sigma\) is a subdivision of \([a, b]\) means that \(\sigma = \{x_\rho\}_{\rho=0}^n\) is a finite increasing sequence with \(x_0 = a\) and \(x_n = b\). It will be
convenient to use the notation $\sigma: a = x_0 < \ldots < x_n = b$

to denote such a subdivision.

**Definition 1.20.** The statement that $f$ is a step function over an interval $[a, b]$ means that there exists a subdivision $\sigma: a = x_0 < \ldots < x_n = b$ of $[a, b]$ such that for $1 \leq i \leq n$, $f$ assumes only one value in the interval $(x_{i-1}, x_i)$. [1, p. 49]

**Theorem 1.18.** Let $f$ be a continuous function over $[a, b]$. Then given $\varepsilon > 0$ there exists a step function $\phi(x)$ such that for all $x \in [a, b]$

$$|f(x) - \phi(x)| < \varepsilon .$$

**Definition 1.21.** The statement that $\{c_i\}_{i=1}^n$ is a marking of $\sigma$: $a = x_0 < \ldots < x_n = b$ means that $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$.

**Definition 1.22.** The statement that $|\sigma|$ is the norm of $\sigma$: $a = x_0 < \ldots < x_n = b$ is a subdivision of $[a, b]$ and $|\sigma| = \text{lub} \left\{ |x_i - x_{i-1}| : i = 1, 2, \ldots, n \right\}$.

**Definition 1.23.** The statement that $f$ is R-integrable over $[a, b]$ means that $[a, b]$ is a subset of the domain of $f$ and there exists a number $I$ such that if $\varepsilon > 0$, there exists $\delta > 0$ such that if $\sigma: a = x_0 < \ldots < x_n = b$ is a subdivision of $[a, b]$ with $|\sigma| < \delta$ and $\{c_i\}_{i=1}^n$ is any marking of $\sigma$, then

$$\left| \sum_{i=1}^{n} f(c_i) (x_i - x_{i-1}) - I \right| < \varepsilon .$$
CHAPTER BIBLIOGRAPHY

CHAPTER II

THE INTEGRAL DEFINED

In this chapter a theory similar to Darboux integration theory is developed. A rigorous definition of the Stieltjes integral is then given, and the two concepts are contrasted. The chapter concludes with an existence theorem for the Stieltjes integral.

Definition 2.1. Let f be a bounded function defined over [a,b], and let g be a nondecreasing function defined over [a,b]. Let \( \sigma : a = x_0 < \ldots < x_n = b \) be a subdivision of [a,b]. Then define:

1. \( \overline{\sum}_{\sigma}^* = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x) [g(x_i) - g(x_{i-1})] \)
2. \( \underline{\sum}_{\sigma}^* = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x) [g(x_i) - g(x_{i-1})] \)
3. \( \int_a^b f \, dg = \inf \left\{ \overline{\sum}_{\sigma}^* : \sigma \text{ is a subdivision of } [a,b] \right\} \)
4. \( \int_a^b f \, dg = \sup \left\{ \underline{\sum}_{\sigma}^* : \sigma \text{ is a subdivision of } [a,b] \right\} \)

Theorem 2.1. If each of \( \sigma \) and \( \sigma' \) is a subdivision of [a,b] and \( \sigma \subseteq \sigma' \), then \( \overline{\sum}_{\sigma}^* \leq \overline{\sum}_{\sigma'}^* \) and \( \underline{\sum}_{\sigma}^* \leq \underline{\sum}_{\sigma'}^* \).

Proof. Let f be a bounded function defined over [a,b], and let g be a nondecreasing function defined over [a,b]. Let \( \sigma : a = x_0 < \ldots < x_n = b \) be a subdivision of [a,b]. If
\[ g(a) = g(b), \text{ then } \sum_{\sigma}^* - \sum_{\sigma'}^* = \text{lub } f(x) [g(x_j) - g(x_{j,-})] \]
\[ \text{for } x \in [x_{j,-}, x_j] \]
\[ - \text{lub } f(x) [g(x_{j,-}) - g(x_j)] = \text{lub } f(x) [g(x_{j,1}) - g(y)] \]
\[ x \in [x_{j,-}, y] \]
\[ \geq \text{lub } f(x) [g(x_{j,1}) - g(x_j)] \]
\[ x \in [x_{j,-}, x_j] \]
\[ - \text{lub } f(x) [g(x_j) - g(y)] = 0. \]
\[ x \in [x_{j,-}, x_j] \]

Therefore \( \sum_{\sigma}^* - \sum_{\sigma'}^* \leq 0 \), similarly \( \sum_{\sigma}^* \leq \sum_{\sigma'}^* \). Now assume that if \( \sigma'' \) is any subdivision of \([a, b]\) containing \( \sigma \) and exactly \( n - 1 \) distinct points not in \( \sigma \), then \( \sum_{\sigma''}^* \leq \sum_{\sigma}^* \) and \( \sum_{\sigma}^* \leq \sum_{\sigma''}^* \). Let \( \sigma' = \sigma \cup \{y_1, y_2, \ldots, y_n\} \), and \( \sigma'' = \sigma \cup \{y_1, \ldots, y_{n-1}\} \). Then \( \sum_{\sigma''}^* \leq \sum_{\sigma}^* \) and \( \sum_{\sigma}^* \leq \sum_{\sigma''}^* \) by the induction hypothesis. But also \( \sum_{\sigma''}^* \leq \sum_{\sigma'''}^* \) and \( \sum_{\sigma}^* \leq \sum_{\sigma'''}^* \) by the case \( n = 1 \) above. Therefore \( \sum_{\sigma'}^* \leq \sum_{\sigma}^* \) and \( \sum_{\sigma}^* \leq \sum_{\sigma'}^* \). Therefore, by mathematical induction, if \( \sigma \subset \sigma' \), then \( \sum_{\sigma}^* \leq \sum_{\sigma'}^* \) and \( \sum_{\sigma'}^* \leq \sum_{\sigma}^* \).

**Theorem 2.2.** Each lower sum \( \sum_{\sigma}^* \) is less than or equal to every upper sum \( \sum_{\sigma'}^* \).

**Proof.** Let \( f \) be a bounded function defined over \([a, b]\), and let \( g \) be defined and nondecreasing over \([a, b]\). Let \( \sigma : a = x_1 < \ldots < x_n = b \) be any subdivision of \([a, b]\). Let \( \sigma' : a = y_1 < \ldots < y_m = b \) be a subdivision of \([a, b]\). Suppose \( \sigma = \sigma' \), then
\[
\sum_{\sigma}^{*} - \sum_{\tau}^{*} = \sum_{i=1}^{n} \left[ \text{lub} f(x_i) - \text{glb} f(x_i) \left[ g(x_i) - g(x_{i-1}) \right] \right] \geq 0,
\]
therefore \( \sum_{\sigma}^{*} \geq \sum_{\tau}^{*} \). Now suppose that \( \sigma \neq \tau \). Then since \( \sigma \subseteq \sigma \cup \tau \) and \( \sigma' \subseteq \sigma \cup \tau' \), it follows from
Theorem 2.1 that \( \sum_{\sigma'}^{*} \geq \sum_{\tau}^{*} \), and \( \sum_{\sigma \cup \tau'}^{*} \geq \sum_{\sigma}^{*} \).
Hence \( \sum_{\sigma'}^{*} \geq \sum_{\tau}^{*} \) since \( \sum_{\sigma \cup \tau'}^{*} \geq \sum_{\sigma}^{*} \).

Theorem 2.3. If both \( \int_{a}^{b} f \, dg \) and \( \int_{a}^{b} f \, dg \) exist, then if
\( \sigma : a = x_0 < ... < x_n = b \) is any subdivision of \( [a,b] \),
\[ \sum_{\sigma}^{*} \leq \int_{a}^{b} f \, dg \] and \( \int_{a}^{b} f \, dg \leq \sum_{\sigma}^{*} \).

Proof. Let both \( \int_{a}^{b} f \, dg \) and \( \int_{a}^{b} f \, dg \) exist, and let
\( \sigma' : a = x_0 < x_1 < ... < x_n = b \) be a subdivision of \( [a,b] \).
Then \( f \) is defined over \( [a,b] \) and \( g \) is defined and non-decreasing over \( [a,b] \). Therefore \( \sum_{\sigma}^{*} \) and \( \sum_{\sigma}^{*} \) are defined, and by the definition of a least upper bound and a greatest lower bound, \( \int_{a}^{b} f \, dg \leq \sum_{\sigma}^{*} \) and \( \int_{a}^{b} f \, dg \geq \sum_{\sigma}^{*} \).

Theorem 2.4. If both \( \int_{a}^{b} f \, dg \) and \( \int_{a}^{b} f \, dg \) exist, then
\( \int_{a}^{b} f \, dg \leq \int_{a}^{b} f \, dg \).
Proof. Let both \( \int_a^b f \, d g \) and \( \int_a^b g \, d f \) exist. Suppose
\[
\int_a^b f \, d g < \int_a^b g \, d f.
\]
Then there exists \( \sigma : a = x_0 < \ldots < x_n = b \) such that
\[
\int_a^b f \, d g \leq \sum_{i=1}^n \left| f(x_i) (g(x_i) - g(x_{i-1})) \right| < \int_a^b g \, d f
\]
and there exists \( \sigma' : a = y_0 < \ldots < y_m = b \) such that
\[
\int_a^b g \, d f \leq \sum_{i=1}^m \left| g(y_i) (f(y_i) - f(y_{i-1})) \right| < \int_a^b f \, d g
\]
which contradicts Theorem 2.2.

Therefore
\[
\int_a^b f \, d g \leq \int_a^b g \, d f.
\]

**Definition 2.2.** The statement that \( f \) is \( S \)-integrable over \([a,b]\) with respect to \( g \) means that both \( f \) and \( g \) are defined over \([a,b]\) and there exists a number \( I \) such that if \( \varepsilon > 0 \), then there exists \( \delta > 0 \) such that if
\[
\sigma : a = x_0 < \ldots < x_n = b \text{ is any subdivision of } [a,b] \text{ with } |\sigma| < \delta \text{ and } \{c_i\}_{i=1}^n \text{ is any marking of } \sigma,
\]
then
\[
\left| \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] - I \right| < \varepsilon.
\]
If \( f \) is \( S \)-integrable over \([a,b]\) with respect to \( g \), then the number \( I \) will be denoted by \( \int_a^b f \, d g \).

**Definition 2.3.** \( \int_a^b f \, d g = 0 \), and \( \int_a^b g \, d f = - \int_a^b f \, d g. \)
Theorem 2.5. If $f$ is $S$-integrable over $[a,b]$ with respect to an increasing function $g$, then $f$ is bounded over $[a,b]$.

Proof. Let $f$ be $S$-integrable over $[a,b]$ with respect to an increasing function $g$. Assume that $f$ is not bounded over $[a,b]$, and let $\varepsilon > 0$. There exists $\delta > 0$ such that if $\sigma : a = x_0 < \ldots < x_n = b$ is any subdivision of $[a,b]$ with $|\sigma| < \delta$ and $\{c_i\}_{i=1}^n$ is any marking of $\sigma$, then

$$\left| \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f \, dg \right| < \varepsilon .$$

Let $\sigma : a = x_0 < \ldots < x_n = b$ be a subdivision of $[a,b]$ with $|\sigma| < \delta$, and let $\{c_i\}_{i=1}^n$ be a marking of $\sigma$. Let $[x_{j-1}, x_j]$ be a subinterval of $[a,b]$ such that $f$ is not bounded over $[x_{j-1}, x_j]$. Suppose $f$ is not bounded above over $[x_{j-1}, x_j]$. Let $c_j \in [x_{j-1}, x_j]$ such that $f(c_j) > \sup_{c \in [x_{j-1}, x_j]} f(c) + 2\varepsilon .

Let $d_i = c_i$ if $i \neq j$ and $d_j = c_j$. Then $\{d_i\}_{i=1}^n$ is also a marking of $\sigma$, and

$$\varepsilon > \left| \sum_{i=1}^n f(d_i)[g(x_i) - g(x_{i-1})] - \int_a^b f \, dg \right|$$

$$\geq \left| \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] - \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] \right|$$

$$- \left| \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f \, dg \right|$$
which is a contradiction. If \( f \) is not bounded below over \([x_{i-1}, x_i]\), then a similar argument yields a contradiction. Therefore it must be true that \( f \) is bounded over each \([x_{i-1}, x_i]\) and therefore over \([a, b]\).

**Theorem 2.6.** If \( f \) is \( S \)-integrable over \([a, b]\) with respect to an increasing function \( g \), then

\[
\int_a^b f \, dg = \int_a^b f \, dg = \int_a^b f \, dg.
\]

**Proof.** Let \( f \) be \( S \)-integrable over \([a, b]\) with respect to an increasing function \( g \). By Theorem 2.5 \( f \) is bounded, and therefore has a greatest lower bound \( G \). But since \( G \leq f(x) \) for all \( x \in [a, b] \), \( G [g(b) - g(a)] \leq \sum^\sigma_r \) for every subdivision \( \sigma \) of \([a, b]\). Therefore

\[
\int_a^b f \, dg = \operatorname{glb} \left\{ \sum^\sigma_r : \sigma \text{ is a subdivision of } [a, b] \right\}
\]

is a real number. Similarly, \( \int_a^b f \, dg \) is a real number.

Let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that if \( \sigma : a = x_0 < \ldots < x_n = b \) is a subdivision of \([a, b]\) with \( |\sigma| < \delta \), and \( \{c_i\}_{i=1}^n \) is any marking of \( \sigma \), then

\[
\left| \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f \, dg \right| < \frac{\varepsilon}{4}.
\]

Let \( \sigma : a = x_0 < \ldots < x_n = b \) be a subdivision of \([a, b]\)
with $|\sigma| < \delta$ and let $\{\bar{\sigma}_i\}_{i=1}^n$ be a marking of $\sigma$ such that

$$\frac{\epsilon}{4[g(b) - g(a)]} < \inf_{x \in [x_{i-1}, x_i]} f(x)$$

for $i = 1, 2, \ldots, n$.

Also let $\{\bar{\alpha}_i\}_{i=1}^n$ be a marking of $\sigma$ such that

$$\frac{\epsilon}{4[g(b) - g(a)]} < \sup_{x \in [x_{i-1}, x_i]} f(x)$$

for $i = 1, 2, \ldots, n$.

Then

$$\int_a^b f d\alpha - \int_a^b f d\beta < \sum_{\sigma}^* - \sum_{\sigma}^*$$

$$< \frac{\epsilon}{4[g(b) - g(a)]} \int_a^b f d\alpha + f(\bar{\alpha}_i) [g(x_i) - g(x_{i-1})]$$

$$= \frac{\epsilon}{4} \left[ f(\bar{\alpha}_i) [g(x_i) - g(x_{i-1})] + \frac{\epsilon}{4} \right]$$

$$- \sum_{\sigma}^* f(\bar{\alpha}_i) [g(x_i) - g(x_{i-1})] + \frac{\epsilon}{4}$$

$$= \frac{\epsilon}{4} \left[ f(\bar{\alpha}_i) [g(x_i) - g(x_{i-1})] + \frac{\epsilon}{4} \right]$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$ 

Therefore $\int_a^b f d\alpha = \int_a^b f d\beta$.

Suppose that $\int_a^b f d\alpha < \int_a^b f d\beta$. Let $\sigma : a = y_1 < \ldots < y_m = b$ be a subdivision of $[a, b]$ such that $\int_a^b f d\alpha < \int_a^b f d\beta$, and let $\epsilon' = \sum_{\sigma}^* - \int_a^b f d\alpha$. Let $\delta > 0$.
such that if \( \sigma : a = x_0 < ... < x_n = b \) is any subdivision of \([a,b]\) with \(|\sigma| < \delta \) and \( \{c_i\}_{i=1}^{n} \) is any marking of \( \sigma \), then

\[
\left| \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f\,dg \right| < \varepsilon.
\]

Let \( \sigma : a = x_0 < ... < x_n = b \) be a subdivision of \([a,b]\) with \(|\sigma| < \delta \) and let \( \mathcal{G}_1 \cup \mathcal{U}_2 = \sigma : a = x_0 < ... < x_n = b \). Let \( \{c_i\}_{i=1}^{n} \) be a marking of \( \sigma \). Since \( \mathcal{G}_1 \subseteq \sigma \), \(|\sigma| < \delta \), and since \( \mathcal{G}_1 \subseteq \sigma \), by Theorem 2.1 \( \sum_{c_i}^* \leq \sum_{\sigma}^* \). Also

\[
\sum_{c_i}^* \leq \sum_{\sigma}^* - \int_a^b f\,dg.
\]

Therefore

\[
\left| \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f\,dg \right| < \varepsilon.
\]

Let \( \sigma : a = z_0 < ... < z_k = b \) be a subdivision of \([a,b]\) with \(|\sigma| < \delta \) and let \( \mathcal{G}_1 \cup \mathcal{U}_2 = \sigma : a = x_0 < ... < x_n = b \). Let \( \{c_i\}_{i=1}^{n} \) be a marking of \( \sigma \). Since \( \mathcal{G}_1 \subseteq \sigma \), \(|\sigma| < \delta \), and since \( \mathcal{G}_1 \subseteq \sigma \), by Theorem 2.1 \( \sum_{c_i}^* \leq \sum_{\sigma}^* \). Also

\[
\sum_{c_i}^* \leq \sum_{\sigma}^* - \int_a^b f\,dg.
\]

Therefore

\[
\left| \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f\,dg \right| < \varepsilon.
\]

which is a contradiction. Therefore \( \int_a^b f\,dg \leq \int_a^b f\,dg \),

and similarly \( \int_a^b f\,dg \leq \int_a^b f\,dg \). Therefore

\[
\int_a^b f\,dg = \int_a^b f\,dg = \int_a^b f\,dg.
\]

Example. There exist functions \( f \) and \( g \) such that

\[
\int_a^b f\,dg = \int_a^b f\,dg,
\]

but \( f \) is not \( S \)-integrable over \([a,b]\) with respect to \( g \). Consider the following functions:
\[
\begin{align*}
f(x) &= \begin{cases} 
1 & \text{if } x \in [0, \frac{1}{2}) \\
2 & \text{if } x \in [\frac{1}{2}, 1] .
\end{cases} \\
g(x) &= \begin{cases} 
x - 1 & \text{if } x \in [0, \frac{1}{2}] \\
x & \text{if } x \in (\frac{1}{2}, 1] .
\end{cases}
\end{align*}
\]

Clearly both \( f \) and \( g \) are defined and bounded over \([0,1]\), and \( g \) is increasing over \([0,1]\). Therefore by Definition 2.1

\[
\int_a^b f \, dg \text{ and } \int_a^b f \, dg \text{ exist. Let } \epsilon > 0 \text{ and } t = \min \left\{ \frac{\epsilon}{2}, \frac{1}{4} \right\} .
\]

Let \( \sigma : 0 < (\frac{1}{2} - t) < \frac{1}{2} < 1 \). Then

\[
\int_a^b f \, dg - \int_a^b f \, dg \leq \sum_{\sigma}^* - \sum_{\sigma}^* = t < \epsilon .\]

Hence \( \int_a^b f \, dg = \int_a^b f \, dg = \frac{3}{2} \),

and by Theorem 2.6 if \( f \) is to be \( S \)-integrable over \([0,1]\) with respect to \( g \), then \( \int_a^b f \, dg = \frac{3}{2} . \)

But let \( \epsilon = \frac{1}{8} \), and let \( \delta > 0 \). Let \( \sigma : a = x_0 < \ldots < x_n = b \) be any subdivision of \([0,1]\) with \( |\sigma| < \min \left\{ \delta, \frac{1}{4} \right\} \)

and \( \frac{1}{2} \notin \sigma \). Let \( \frac{1}{2} \in [x_{j-1}, x_j] \) and let \( \{ c_i \}_{i=1}^n \) be a marking of \( \sigma \) such that \( c_j = \frac{2x_{j-1} + 1}{4} \). Then \( f(c_j) = 1 \), and

\[
1 < g(x_j) - g(x_{j-1}) < \frac{1}{4} .\]

Then
Therefore, by Definition 2.2 f is not S-integrable over [0,1] with respect to g.

**Remark.** It is interesting to note that a slight adjustment in Definition 2.2 would lead to an integral, normally called the Riemann-Stieltjes integral, for which \( \int_a^b f \, dg \) would imply that f is integrable over \([a,b]\) with respect to g. A definition and discussion of the Riemann-Stieltjes integral may be found in [2, pp. 83-86].

**Theorem 2.7.** If f is defined over \([a,b]\), and g is defined and nondecreasing over \([a,b]\), and each of f and g is discontinuous at the point \(p \in [a,b]\), then f is not S-integrable over \([a,b]\) with respect to g.

**Proof.** Let f be defined over \([a,b]\), and let g be defined and nondecreasing over \([a,b]\). Let each of f and g be discontinuous at \(p \in [a,b]\). Then there exists \(k_1 > 0\) such that for every \(\delta > 0\) there exists \(x \in [a,b]\) such that \(|x - p| < \delta\) and \(|f(x) - f(p)| < k_1\). And there exists \(k_2 > 0\) such that for every \(\delta > 0\) there exists \(y \in [a,b]\) such that \(|y - p| < \delta\) and \(|g(y) - g(p)| < k_2\). Now, suppose that f is S-integrable...
over \([a,b]\) with respect to \(g\). Then there exists \(\delta > 0\) such that if \(\sigma: a = x_0 < \ldots < x_n = b\) is a subdivision of \([a,b]\) with \(|\sigma| < \delta\), and \(\{c_i\}_{i=1}^n\) is any marking of \(\sigma\), then
\[
\left| \sum_{i=1}^n f(c_i)\left[g(x_i) - g(x_{i-1})\right] - \int_a^b f dg \right| < \frac{k_1 k_2}{2}.
\]

Let \(\sigma: a = x_0 < \ldots < x_n = b\) be any subdivision of \([a,b]\) with \(|\sigma| < \delta\) such that \(p \notin \sigma\) unless \(p = a\) or \(p = b\). Let \(p \in [x_{j-1}, x_j]\). Then there exists \(w \in [x_{j-1}, x_j]\) such that
\[
|g(w) - g(p)| > k_2.\]
Also there exists \(z \in [x_{j-1}, x_j]\) such that \(|f(z) - f(p)| > k_1\). Let \(\{c_i\}_{i=1}^n\) be a marking of \(\sigma\) such that \(c_j = p\), and let \(\{c'_i\}_{i=1}^n\) be a marking of \(\sigma\) such that \(c_i = c'_i\) for \(i \neq j\) and \(c'_j = z\). Then
\[
\left| \sum_{i=1}^n f(c_i)\left[g(x_i) - g(x_{i-1})\right] - \int_a^b f dg \right| < \frac{k_1 k_2}{2},
\]
and
\[
\left| \sum_{i=1}^n f(c'_i)\left[g(x_i) - g(x_{i-1})\right] - \int_a^b f dg \right| < \frac{k_1 k_2}{2}.
\]

Therefore
\[
k_1 k_2 > \left| \sum_{i=1}^n f(c_i)\left[g(x_i) - g(x_{i-1})\right] - \int_a^b f dg \right| +
\]
\[
\left| \sum_{i=1}^n f(c'_i)\left[g(x_i) - g(x_{i-1})\right] - \int_a^b f dg \right| >
\]
\[
\left| \sum_{i=1}^n f(c_i)\left[g(x_i) - g(x_{i-1})\right] - \sum_{i=1}^n f(c'_i)\left[g(x_i) - g(x_{i-1})\right] \right| =
\left| f(p) - f(z) \right| \left| g(x_j) - g(x_{j-1}) \right| \geq k_1 k_2
\]
which is a contradiction. Therefore the assumption that \(f\) is \(S\)-integrable over \([a,b]\) with respect to \(g\) must be false.
Theorem 2.8. If \( f \) is \( S \)-integrable over \([a,b]\) with respect to \( g \), and \( fg' \) is \( R \)-integrable over \([a,b]\), then

\[
\int_a^b f dg = \int_a^b fg'.
\]

Proof. Let \( f \) be \( S \)-integrable over \([a,b]\) with respect to \( g \), and let \( fg' \) be \( R \)-integrable over \([a,b]\). Let \( \varepsilon > 0 \). Then there exists \( \delta_1 > 0 \) such that if \( \sigma_1: a = x_0 < \ldots < x_n = b \) is a subdivision of \([a,b]\) with \( |\sigma_1| < \delta_1 \), and \( \{c_i\}_{i=1}^n \) is any marking of \( \sigma_1 \), then

\[
\left| \sum_{i=1}^n f(c_i) [g(x_i') - g(x_i'')] - \int_a^b f dg \right| < \frac{\varepsilon}{3}.\]

And there exists \( \delta_2 > 0 \) such that if \( \sigma_2: a = x_0 < \ldots < x_m = b \) is any subdivision of \([a,b]\) with \( |\sigma_2| < \delta_2 \), and \( \{c_i\}_{i=1}^m \) is any marking of \( \sigma_2 \), then

\[
\left| \sum_{i=1}^m f(c_i') g'(c_i') [x_i'' - x_i'] - \int_a^b fg' \right| < \frac{\varepsilon}{3}.\]

Now let \( \delta = \min \{\delta_1, \delta_2\} \), and let \( \sigma: a = x_0 < \ldots < x_n = b \) be a subdivision of \([a,b]\) with \( |\sigma| < \delta \). Let \( \{c_i\}_{i=1}^n \) be a marking of \( \sigma \). Since \( g \) is differentiable over \([a,b]\), then \( g \) is differentiable on each \([x_{i-1}, x_i]\), and by Theorem 1.10 \( g \) is continuous over each \([x_{i-1}, x_i]\). Therefore by Theorem 1.16 there exists \( k_i \in [x_{i-1}, x_i] \) such that

\[
g'(k_i) = \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}}.\]

Thus \( \{k_i\}_{i=1}^n \) is a marking of \( \sigma \). Hence

\[
\left| \sum_{i=1}^n f(c_i) [g(x_i') - g(x_i'')] - \int_a^b fg' \right| < \varepsilon.
\]
Therefore by Definition 2.2 \( \int_a^b f dg = \int_a^b fg' \).

**Theorem 2.9.** If \( f \) is continuous over \([a, b]\) and \( g \) is nondecreasing over \([a, b]\), then \( f \) is \( S \)-integrable over \([a, b]\) with respect to \( g \).

**Proof.** Let \( f \) be continuous over \([a, b]\) and \( g \) be non-decreasing over \([a, b]\). Then \( f \) is defined over \([a, b]\), and

\[
\int_a^b f dg \text{ exists. If } g(b) = g(a), \text{ then let } \sigma: a = x_0 < \ldots < x_n = b \text{ be any subdivision of } [a, b] \text{ and } \{c_i\}_{i=1}^n \text{ any marking of } \sigma. \text{ Then }
\]

\[
\left| \sum_{i=1}^n f(c_i) [g(x_i) - g(x_{i-1})] - 0 \right| = 0 < \varepsilon \text{ for every } \varepsilon > 0.
\]

Now assume that \( g(b) > g(a) \). Let \( \varepsilon > 0 \). By Theorem 1.11 \( f \) is uniformly continuous over \([a, b]\). Hence there exists \( \delta > 0 \) such that if \( x, y \in [a, b] \) and \( |x - y| < \delta \), then

\[
|f(x) - f(y)| < \frac{\varepsilon}{2[g(b) - g(a)]}.
\]
Let \( \sigma : a = x_0 < \ldots < x_n = b \) be any subdivision of \([a,b]\) with \( |\sigma| < \delta \), and let \( \{c_i\}_{i=1}^n \) be any marking of \( \sigma \). By Theorem 1.12, there exist \( s_i, t_i \in [x_{i-1}, x_i] \) such that

\[
s_i = \text{lub } f(x) \quad \text{and} \quad t_i = \text{glb } f(x).
\]

Therefore

\[
x \in [x_{i-1}, x_i] \quad x \in [x_{i-1}, x_i]
\]

\[
\left| \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f \, dg \right| \leq \left| \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] - \sum_{\sigma}^* \right| + \left| \overline{\sum}_{\sigma}^* - \int_a^b f \, dg \right|
\]

\[
\leq \sum_{i=1}^n \left| f(c_i) - f(s_i) \right| \left| g(x_i) - g(x_{i-1}) \right| + \sum_{i=1}^n \left| f(s_i) - f(t_i) \right| \left| g(x_i) - g(x_{i-1}) \right|
\]

\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \text{Therefore by Definition 2.2 } f \text{ is } S\text{-integrable over } [a,b] \text{ with respect to } g.
\]
CHAPTER BIBLIOGRAPHY


CHAPTER III

GENERAL PROPERTIES OF THE INTEGRAL

In this chapter the concept of a function of bounded variation is introduced. A more general existence theorem is proved, and many of the basic properties of the integral are proved.

**Definition 3.1.** The statement that $f$ is of bounded variation over $[a,b]$ means that there exists a real number $k$ such that if $\sigma: a = x_0 < \ldots < x_n = b$ is any subdivision of $[a,b]$, then
\[
\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < k.
\]

**Definition 3.2.** Let $f$ be a function defined over $[a,b]$.

1. $T_{[a,b]} = \{k: k > \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \text{ for any subdivision } \sigma: a = x_0 < \ldots < x_n = b \text{ of } [a,b]\}$;

2. $k_{[a,b]} = \text{glb } T_{[a,b]} \text{ if } T_{[a,b]} \neq \emptyset.$

**Lemma 3.1.** If $f$ is a function of bounded variation over $[a,b]$, then $f$ is of bounded variation over $[x,y] \subseteq [a,b]$.

**Proof.** Let $f$ be of bounded variation over $[a,b]$, and let $[x,y] \subseteq [a,b]$. Let $k \in T_{[a,b]}$. Let $\sigma: a = p_0 < \ldots < p_m = y$ be any subdivision of $[x,y]$. Then $\sigma': a \leq x = p_0 < \ldots < p_m = y \leq b$ is a subdivision of $[a,b]$, and
\[ \sum_{i=1}^{n} |f(p_i) - f(p_{i-1})| \]

\[ \leq \sum_{i=1}^{n} |f(p_i) - f(p_{i-1})| + |f(a) - f(x)| + |f(b) - f(y)| < k. \]

Therefore \( T \neq \emptyset \) and \( f \) is of bounded variation over \([x,y]\).

**Lemma 3.2.** If \( f \) is of bounded variation over \([a,c]\) and \( b \in (a,c) \), then \( k_{[a,b]} = k_{[a,c]} + k_{[b,c]} \).

**Proof.** Let \( f \) be of bounded variation over \([a,c]\), and let \( b \in (a,c) \). Then by Lemma 3.1 \( k_{[a,b]} \) and \( k_{[b,c]} \) both exist. Let \( \sigma : a = x_0 < \ldots < x_n = b \) be a subdivision of \([a,c]\) such that \( x_i < b < x_{i+1} \). Let \( b \in [x_{i-1}, x_i] \). Then

\[ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \]

\[ = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + |f(x_i) - f(x_{i-1})| \]

\[ + \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \]

\[ \leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + |f(b) - f(x_{i-1})| \]

\[ + \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + |f(x_i) - f(b)| \]

\[ \leq k_{[a,b]} + k_{[b,c]} \]

Consequently, \( k_{[a,b]} \leq k_{[a,c]} + k_{[b,c]} \).

Now assume that \( k_{[a,c]} < [k_{[a,b]} + k_{[b,c]}] \). Then since \( k_{[a,c]} \) is the greatest lower bound of \( T_{[a,c]} \), there exists \( \lambda \in T_{[a,c]} \) such that \( k_{[a,c]} \leq \lambda < [k_{[a,b]} + k_{[b,c]}] \). Let

\[ \varepsilon = \frac{k_{[a,b]} + k_{[b,c]} - \lambda}{2} \]

and let \( \sigma' : a = x_0 < \ldots < x_n = b \)
be a subdivision of \([a,b]\) such that

\[
k_{[a,b]} < \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \epsilon
\]

and \(\sigma_a: a = y_0 < \ldots < y_m = b\) be a subdivision of \([b,c]\) such that

\[
k_{[b,c]} < \sum_{i=1}^{n} |f(y_i) - f(y_{i-1})| + \epsilon.
\]

Let \(\sigma_a \cup \sigma_b = \sigma_c: a = z_0 < \ldots < z_{m+n} = c\). Then

\[
k_{[a,b]} + k_{[b,c]} - 2 \epsilon < \sum_{i=1}^{m+n} |f(z_i) - f(z_{i-1})|,
\]

and

\[
k_{[a,b]} + k_{[b,c]} - 2 \left[ \frac{k_{[a,b]} + k_{[b,c]} - \lambda}{2} \right] < \sum_{i=1}^{m+n} |f(z_i) - f(z_{i-1})|.
\]

Thus \(\lambda < \sum_{i=1}^{m+n} |f(z_i) - f(z_{i-1})|\), but this contradicts the fact that \(\lambda \in T_{[a,c]}\). Therefore \(k_{[a,c]} \geq k_{[a,b]} + k_{[b,c]}\), and it must be true that \(k_{[a,b]} + k_{[b,c]} = k_{[a,c]}\).

**Theorem 3.1.** The function \(f\) is of bounded variation over \([a,b]\) if, and only if, \(f\) is the difference of two non-decreasing functions defined over \([a,b]\).

**Proof.** Let \(f\) and \(g\) be two nondecreasing functions defined over \([a,b]\), and let \(h = f - g\). Let \(\sigma: a = x_0 < \ldots < x_n = b\) be any subdivision of \([a,b]\). Then

\[
\sum_{i=1}^{n} |h(x_i) - h(x_{i-1})| = \sum_{i=1}^{n} \left| [f(x_i) - f(x_{i-1})] + [g(x_{i-1}) - g(x_i)] \right|
\]
\[
\left\lvert \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] + \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})] \right\rvert \\
= [f(b) - f(a)] + [g(b) - g(a)].
\]

Therefore, by Definition 3.1 \( h \) is of bounded variation.

Let \( f \) be a function of bounded variation over \([a, b]\).

Then by Lemma 3.1, for each \( x \in (a, b) \), \( T_{[a, x]} \neq \emptyset \), and therefore \( k_{[a, x]} \) exists for each \( x \in (a, b) \). Define:

\[
g(x) = \begin{cases} 
0 & \text{if } x = a \\
k_{[a, x]} & \text{if } x \in (a, b],
\end{cases}
\]

and let \( h(x) = g(x) - f(x) \). Then \( g(x) - h(x) = f(x) \). Let \( x_1, x_2 \in [a, b] \) such that \( x_1 < x_2 \). If \( x_1 = a \), then \( g(x_1) = 0 \), and for every \( x \in (a, b] \), \( g(x) = k_{[a, x]} \geq 0 \), which means that \( g(x_1) \leq g(x_2) \). If \( x_1 \neq a \), then

\[
g(x_1) = k_{[a, x_1]} \leq k_{[a, x_1]} + k_{[x_1, x_2]} = k_{[a, x_2]} = g(x_2).
\]

Therefore \( g \) is nondecreasing over \([a, b]\). Assume that \( f(x_2) - f(x_1) > k_{[x_1, x_2]} \). Then \( f(x_2) - f(x_1) > 0 \) since \( k_{[x_1, x_2]} \geq 0 \). Let \( G = \frac{|f(x_2) - f(x_1)|}{2} + k_{[x_1, x_2]} \). Then

\[
k_{[x_1, x_2]} < G < |f(x_2) - f(x_1)|.
\]

Let \( k' \in T_{[x_1, x_2]} \) such that \( k_{[x_1, x_2]} < k' < G \). By Definition 3.2 \( |f(x_2) - f(x_1)| < k' \), which means that \( k' < k' \), a contradiction. Therefore, it must be true that \( f(x_2) - f(x_1) \leq k_{[x_1, x_2]} \). Then

\[
h(x_1) = g(x_1) - f(x_1)
\leq g(x_1) - [f(x_2) - k_{[x_1, x_2]}] = [g(x_1) + k_{[x_1, x_2]}] - f(x_2)
= k_{[a, x_2]} - f(x_2) = g(x_2) - f(x_2) = h(x_2).
\]
Therefore $h$ is also nondecreasing over $[a, b]$, and $f$ is the difference of two nondecreasing functions defined over $[a, b]$.

**Corollary 3.1.** If $f$ is nondecreasing, then $f$ is of bounded variation.

**Proof.** Let $f$ be a nondecreasing function defined over $[a, b]$. Let $g(x) = 0$ for all $x \in [a, b]$. Then $g$ is nondecreasing over $[a, b]$ and $f = f - g$. Therefore by Theorem 3.1, $f$ is of bounded variation.

**Theorem 3.2.** If $f$ is continuous over $[a, b]$ and $g$ is of bounded variation over $[a, b]$, then $f$ is $S$-integrable over $[a, b]$ with respect to $g$.

**Proof.** Let $f$ be continuous over $[a, b]$ and $g$ be of bounded variation over $[a, b]$. Then by Theorem 3.1 there exist two nondecreasing functions $g_i$ and $g_\perp$ defined on $[a, b]$ such that $g = g_i - g_\perp$. By Theorem 2.9 $f$ is $S$-integrable over $[a, b]$ with respect to both $g_i$ and $g_\perp$. Let $\frac{\varepsilon}{2} > 0$. There exists $\delta_i > 0$ such that if $\sigma_i : a = x_0 < \ldots < x_n = b$ is any subdivision of $[a, b]$ with $|\sigma_i| < \delta_i$ and $\{c_i\}_{i=1}^n$ is any marking of $\sigma_i$, then

$$\left| \sum_{i=1}^n f(c_i)[g_i(x_i) - g_i(x_{i-1})] - \int_a^b f dg_i \right| < \frac{\varepsilon}{2}.$$  

Also there exists $\delta_\perp > 0$ such that if $\sigma_\perp : a = y_0 < \ldots < y_m = b$ is any subdivision of $[a, b]$ with $|\sigma_\perp| < \delta_\perp$ and $\{c_\perp\}_{i=1}^m$ is any marking of $\sigma_\perp$, then

$$\left| \sum_{i=1}^m f(c_\perp)[g_\perp(y_i) - g_\perp(y_{i-1})] - \int_a^b f dg_\perp \right| < \frac{\varepsilon}{2}.$$
Let \( \delta = \min \{ \delta_1, \delta_2 \} \), and let \( \sigma : a = z_0 < \ldots < z_k = b \) be a subdivision of \([a, b]\) with \( |\sigma| < \delta \). Let \( \{ z_i \}_{i=1}^k \) be any marking of \( \sigma \). Then

\[
\left| \sum_{i=1}^k f(z_i)[g(z_i) - g(z_{i-1})] - \left[ \int_a^b f \, dg - \int_a^b f \, dg \right] \right|
\]

\[
= \left| \sum_{i=1}^k f(z_i)[g(z_i) - g(z_{i-1})] - \sum_{i=1}^k f(z_i)[g(z_i) - g(z_{i-1})] - \int_a^b f \, dg + \int_a^b f \, dg \right|
\]

\leq \left| \sum_{i=1}^k f(z_i)[g(z_i) - g(z_{i-1})] - \int_a^b f \, dg \right| + \left| \sum_{i=1}^k f(z_i)[g(z_i) - g(z_{i-1})] - \int_a^b f \, dg \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Therefore, by Definition 2.2 \( f \) is \( S \)-integrable over \([a, b]\) with respect to \( g \).

**Theorem 3.3.** If \( f \) is a function such that \( f(x) = c \) for some real number \( c \) whenever \( x \in [a, b] \), and \( g \) is a function defined over \([a, b]\), then

\[
\int_a^b f \, dg = c[g(b) - g(a)].
\]

**Proof.** Let \( f(x) = c \) for every \( x \in [a, b] \), where \( c \) is some fixed real number. Let \( g \) be a function defined over \([a, b]\). Let \( \epsilon > 0 \) and let \( \delta = 1 \). Also let \( \sigma : a = x_0 < \ldots < x_k = b \) be a subdivision of \([a, b]\) with \( |\sigma| < \delta \), and let \( \{ z_i \}_{i=1}^k \) be a marking of \( \sigma \). Then

\[
\left| \sum_{i=1}^k f(z_i)[g(x_i) - g(x_{i-1})] - c[g(b) - g(a)] \right|
\]
Therefore, by Definition 2.2 \( f \) is \( S \)-integrable over \([a,b]\) with respect to \( g \) and
\[
\int_a^b f \, dg = c[g(b) - g(a)].
\]

**Corollary 3.2.** If \( g \) is a function defined over \([a,b]\), then \( g(x) = g(a) + \int_a^x dg \).

**Proof.** Let \( g \) be a function defined over \([a,b]\). If \( x = a \), then by Definition 2.3 \( \int_a^x dg = 0 \), and
\[
g(x) = g(a) + 0 = g(a) + \int_a^x dg.
\]
If \( x \neq a \), then by Theorem 3.3 \( \int_a^x dg = g(x) - g(a) \), and
\[
g(x) = g(a) + (g(x) - g(a)) = g(a) + \int_a^x dg.
\]

**Theorem 3.4.** If \( f \) is \( S \)-integrable over \([a,b]\) with respect to \( g \) and \( k \) is any real number, then
\[
\int_a^b kf \, dg = \int_a^b f \, (kg) = k \int_a^b f \, dg.
\]

**Proof.** Let \( f \) be \( S \)-integrable over \([a,b]\) with respect to \( g \). Let \( k \) be any real number, and let \( \varepsilon > 0 \). Suppose \( k \neq 0 \). Then there exists \( \delta > 0 \) such that if
If $k = 0$, then for any subdivision $\sigma : a = x_0 < \ldots < x_n = b$ of $[a,b]$ and any marking $\{c_i\}_{i=0}^n$ of $\sigma$, it follows that

$$\left| \sum_{i=0}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - k \int_a^b f dg \right| = 0 < \varepsilon \quad \text{and}$$

$$\left| \sum_{i=0}^{n} kg(c_i)[g(x_i) - g(x_{i-1})] - k \int_a^b f dg \right| = 0 < \varepsilon .$$
Hence, by Definition 2.2 $k f$ is $S$-integrable over $[a, b]$ with respect to $g$, and $f$ is $S$-integrable over $[a, b]$ with respect to $kg$, and

$$k \int_a^b f \, dg = \int_a^b k f \, dg = \int_a^b f \, kg \, dg.$$ 

**Theorem 3.5.** If each of $f_1$ and $f_2$ is $S$-integrable over $[a, b]$ with respect to $g$, then

$$\int_a^b (f_1 + f_2) \, dg = \int_a^b f_1 \, dg + \int_a^b f_2 \, dg.$$

**Proof.** Let each of $f_1$ and $f_2$ be $S$-integrable over $[a, b]$ with respect to $g$. Let $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that if $\mathcal{C}_i : a = x_0 < \ldots < x_n = b$ is a subdivision of $[a, b]$ with $|\mathcal{C}_i| < \delta_1$ and $\{c_i\}_{i=1}^n$ is any marking of $\mathcal{C}_i$, then

$$\left| \sum_{i=1}^n f_1(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f_1 \, dg \right| < \frac{\varepsilon}{2}.$$

Also there exists $\delta_2 > 0$ such that if $\mathcal{C}_2 : a = x_0 < \ldots < x_n = b$ is any subdivision of $[a, b]$ with $|\mathcal{C}_2| < \delta_2$ and $\{c_i\}_{i=1}^n$ is any marking of $\mathcal{C}_2$, then

$$\left| \sum_{i=1}^n f_2(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f_2 \, dg \right| < \frac{\varepsilon}{2}.$$

Now let $\delta = \min \{\delta_1, \delta_2\}$. Let $\mathcal{C} : a = x_0 < \ldots < x_n = b$ be a subdivision of $[a, b]$ with $|\mathcal{C}| < \delta$, and let $\{c_i\}_{i=1}^n$ be any marking of $\mathcal{C}$. Then

$$\left| \sum_{i=1}^n [f_1(c_i) + f_2(c_i)][g(x_i) - g(x_{i-1})] - \left[ \int_a^b f_1 \, dg + \int_a^b f_2 \, dg \right] \right|$$
Therefore, by Definition 2.2 \( f_1 + f_2 \) is \( S \)-integrable over \([a,b]\) with respect to \( g \), and

\[
\int_a^b (f_1 + f_2)dg = \int_a^b f_1 dg + \int_a^b f_2 dg.
\]

**Theorem 3.6.** If \( f \) is \( S \)-integrable over \([a,b]\) with respect to both \( g_1 \) and \( g_2 \), then

\[
\int_a^b f \cdot (g_1 + g_2) = \int_a^b f \cdot g_1 + \int_a^b f \cdot g_2.
\]

**Proof.** Let \( f \) be \( S \)-integrable over \([a,b]\) with respect to both \( g_1 \) and \( g_2 \). Let \( \varepsilon > 0 \). Then there exists \( \delta_1 > 0 \) such that if \( \sigma_1 : a = x_0 < \ldots < x_n = b \) is any subdivision of \([a,b]\) with \( |\sigma_1| < \delta_1 \) and \( \{c_i\}_{i=1}^n \) is any marking of \( \sigma_1 \), then

\[
\left| \frac{1}{n} \sum_{i=1}^n f(c_i)[g_1(x_i) - g_1(x_{i-1})] - \int_a^b f \cdot g_1 \right| < \frac{\varepsilon}{2}.
\]

Also there exists \( \delta_2 > 0 \) such that if \( \sigma_2 : a = x_0 < \ldots < x_n = b \) is any subdivision of \([a,b]\) with \( |\sigma_2| < \delta_2 \) and \( \{c_i\}_{i=1}^n \) is any marking of \( \sigma_2 \), then

\[
\left| \frac{1}{n} \sum_{i=1}^n f(c_i)[g_2(x_i) - g_2(x_{i-1})] - \int_a^b f \cdot g_2 \right| < \frac{\varepsilon}{2}.
\]

Now let \( \delta = \min \{\delta_1, \delta_2\} \). Let \( \sigma : a = x_0 < \ldots < x_n = b \) be any subdivision of \([a,b]\) with \( |\sigma| < \delta \), and let \( \{c_i\}_{i=1}^n \) be any marking of \( \sigma \). Then
\[
\left| \sum_{i=1}^{n} f(c_i)[g_1(x_i) + g_2(x_i) - g_1(x_{i-1}) - g_2(x_{i-1})] \right|
- \left| \int_{a}^{b} f \, dg + \int_{a}^{b} f \, dg \right|
\leq \left| \sum_{i=1}^{n} f(c_i)[g_1(x_i) - g_1(x_{i-1})] \right| - \int_{a}^{b} f \, dg
+ \left| \sum_{i=1}^{n} f(c_i)[g_2(x_i) - g_2(x_{i-1})] \right| - \int_{a}^{b} f \, dg
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Therefore, by Definition 2.2 \( f \) is \( S \)-integrable over \([a,b]\) with respect to \( g_1 + g_2 \), and
\[
\int_{a}^{b} f \, dg = \int_{a}^{c} f \, dg + \int_{c}^{b} f \, dg.
\]

**Theorem 3.7.** If \( f \) is \( S \)-integrable over \([a,b]\) with respect to \( g \) and \( a < c < b \), then
\[
\int_{a}^{b} f \, dg = \int_{a}^{c} f \, dg + \int_{c}^{b} f \, dg.
\]

**Proof.** Let \( f \) be \( S \)-integrable over \([a,b]\) with respect to \( g \), and let \( a < c < b \). Then for every subdivision \( \mathcal{V} : a = x_0 < \ldots < x_\alpha = b \) of \([a,b]\) and for every marking \( \{c_i\}_{i=1}^{\alpha} \) of \( \mathcal{V} \), the sum
\[
\sum_{i=1}^{\alpha} f(c_i)[g(x_i) - g(x_{i-1})]
\]exists.

Therefore, for every subdivision \( \mathcal{V}_1 : a = x_0 < \ldots < x_\alpha = c \) of \([a,c]\) and every marking \( \{c_i\}_{i=1}^{\alpha} \) of \( \mathcal{V}_1 \), the sum
\[
\sum_{i=1}^{\alpha} f(c_i)[g(x_i) - g(x_{i-1})]
\]exists. Similarly, for every subdivision \( \mathcal{V}_2 : c = x_0 < \ldots < x_\alpha = b \) of \([c,b]\) and every
marking \( \{c_i\}_{i=1}^{\infty} \) of \( \sigma_a \), the sum \( \sum_{i=1}^{\infty} f(c_i)[g(x_i) - g(x_{i-1})] \) exists. Let \( \{\sigma^i\}_{i=1}^{\infty} \) be a sequence of subdivisions of \([a,c]\) such that \( |\sigma^i| < 2^i \) for \( i = 1, 2, \ldots \). Let \( \{c_i\}_{i=1}^{\infty} \) be any marking of \( \sigma^i : a = x_0 < \ldots < x_n = c \) and let \( S^i_1 = \sum_{i=1}^{\infty} f(c_i)[g(x_i) - g(x_{i-1})] \). Let \( S^1 = \left\{ \sum_{i=1}^{\infty} \sigma^i_1 \right\}_{n=1}^{\infty} \). In a similar manner define \( \{\sigma^i\}_{i=1}^{\infty} \) to be a sequence of subdivisions of \([c,b]\) with \( |\sigma^i| < 2^i \) for \( i = 1, 2, \ldots \). Let \( S^1_1 = \left\{ \sum_{i=1}^{\infty} \sigma^i_1 \right\}_{n=1}^{\infty} \) be the related sequence of sums. Assume that both \( S^1 \) and \( S^1_1 \) diverge. Then by Theorem 1.5 there exists \( \varepsilon_1 > 0 \) such that given any \( N \), there exist integers \( i \) and \( j \), both greater than or equal to \( N \), such that

\[
| \sum_{\sigma^i_1} - \sum_{\sigma^j_1} | > \varepsilon_1 .
\]

Similarly, there exists \( \varepsilon_2 > 0 \) such that given any \( N \), there exist integers \( u \) and \( v \), both greater than or equal to \( N \) such that

\[
| \sum_{\sigma^u} - \sum_{\sigma^v} | > \varepsilon_2 .
\]

But \( \varepsilon_1 + \varepsilon_2 = \varepsilon > 0 \), and there exists \( \delta > 0 \) such that if \( \sigma : a = x_0 < \ldots < x_n = b \) is any subdivision of \([a,b]\) with \( |\sigma| < \delta \) and \( \{c_i\}_{i=1}^{\infty} \) is any marking of \( \sigma \), then

\[
\left| \sum_{i=1}^{\infty} f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f dg \right| < \frac{\varepsilon}{2} .
\]

Let \( N \) be the least positive integer such that \( 2^{-N} < \delta \). Let
1 \geq N \text{ and } j \geq N \text{ such that } \left| \sum_{\sigma_i} - \sum_{\sigma_j} \right| \geq \varepsilon_1.

Similarly, let \( u \geq N \) and \( v \geq N \) such that \( \left| \sum_{\sigma_u} - \sum_{\sigma_v} \right| \geq \varepsilon_2 \).

Since \( i \) and \( j \) are arbitrary labels, assume that \( \sum_{\sigma_i} \geq \sum_{\sigma_j} \).

Similarly assume \( \sum_{\sigma_u} \geq \sum_{\sigma_v} \). But \( \sigma_i \cup \sigma_u = \sigma^u \): a sequence of numbers \( x_0 < \ldots < x_n = b \), a subdivision of \([a,b]\), and if \( \{c_i\}_{i=1}^n \) is a marking of \( \sigma_i \), and \( \{c'_i\}_{i=1}^m \) is a marking of \( \sigma_u \), then \( \{c_i\}_{i=1}^n \cup \{c'_i\}_{i=1}^m \) is a marking of \( \sigma^u \). Similarly, \( \sigma_j \cup \sigma_v = \sigma^v \): a sequence of numbers \( y_0 < \ldots < y_m = b \) is a subdivision of \([a,b]\) and has a marking \( \{d_i\}_{i=1}^m \) associated with it. Then

\[
\left| \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - \int_{a}^{b} f dg \right| < \frac{\varepsilon}{2},
\]

and

\[
\left| \sum_{i=1}^{m} f(d_i)[g(y_i) - g(y_{i-1})] - \int_{a}^{b} f dg \right| < \frac{\varepsilon}{2}.
\]

Therefore \( \varepsilon = \varepsilon_1 + \varepsilon_2 \).

\[
\leq \left[ \sum_{\sigma_i} - \sum_{\sigma_j} \right] + \left[ \sum_{\sigma_u} - \sum_{\sigma_v} \right]
\]

\[
= \left[ \sum_{\sigma_i} + \sum_{\sigma_u} - \int_{a}^{b} f dg \right] - \left[ \sum_{\sigma_j} + \sum_{\sigma_v} - \int_{a}^{b} f dg \right]
\]

\[
\leq \left| \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - \int_{a}^{b} f dg \right| + \left| \sum_{i=1}^{m} f(d_i)[g(y_i) - g(y_{i-1})] - \int_{a}^{b} f dg \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
which is a contradiction. Therefore it is not true that both
S' and S diverge. Assume that S' converges to A. Let
\( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that if \( \sigma: a = x_0 < \ldots < x_n = b \) is a subdivision of \([a,b]\) with \( |\sigma| < \delta \) and
\( \{c_i\}_{i=1}^n \) is any marking of \( \sigma \), then
\[
\left| \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - \int_{a}^{b} f\,dg \right| < \frac{\varepsilon}{2}.
\]
Let \( N_0 \) be the least positive integer such that \( 2^{-N_0} < \delta \).
There exists \( N_1 \) such that if \( n > N_1 \) then
\[
\left| \sum_{i=1}^{n} f(c_i) - A \right| < \frac{\varepsilon}{2}.
\]
Let \( N = \max\{N_0, N_1, 2^{-N_1}\} \), and let \( \varepsilon > N \). Then \( \sum_{i=1}^{n} f(c_i) + k = A \) for
some real number \( k, |k| < \frac{\varepsilon}{2} \). Let \( \sigma*: c = x_0 < \ldots < x_n = b \) be a subdivision of \([c,b]\) with \( |\sigma*| < \delta \) and
\( \{c_i^*\}_{i=1}^n \) be any marking of \( \sigma* \). Then \( \sigma_1^* \cup \sigma* = \sigma: a = y_0 < \ldots < y_n = b \) is a subdivision of \([a,b]\) with \( |\sigma| < \delta \), and if
\( \{c_i^*\}_{i=1}^p \) is the marking associated with \( \sigma_1^* \), then
\[
\{c_i^*\}_{i=1}^p \cup \{c_i^*\}_{i=1}^n = \{c_i\}_{i=1}^n
\]
is a marking of \( \sigma \), and
\[
\left| \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - \left[ \int_{a}^{b} f\,dg - A \right] \right| = \left| \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})] + \sum_{i=1}^{p} f(c_i^*) + k - \int_{a}^{b} f\,dg \right| \leq \left| \sum_{i=1}^{n} f(c_i)[g(y_i) - g(y_{i-1})] - \int_{a}^{b} f\,dg \right| + |k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
Therefore, by Definition 2.2 \( f \) is \( S \)-integrable over \([c,b]\) with respect to \( g \).

Again let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that if \( \sigma_i : c = x_0 < \ldots < x_n = b \) is any subdivision of \([c,b]\) with \( |\sigma_i| < \delta \), and \( \{c_i\}_{i=1}^n \) is any marking of \( \sigma_i \), then

\[
\left| \frac{1}{n} \sum_{i=1}^{n} f(c_i) [g(x_i) - g(x_{i-1})] - \int_c^b f \, dg \right| < \frac{\varepsilon}{2}.
\]

Also there exists \( \delta_2 > 0 \) such that if \( \sigma_a : a = x_0 < \ldots < x_n = b \) is any subdivision of \([a,b]\) with \( |\sigma_a| < \delta_2 \), and \( \{c_i\}_{i=1}^n \) is any marking of \( \sigma_a \), then

\[
\left| \frac{1}{n} \sum_{i=1}^{n} f(c_i) [g(x_i) - g(x_{i-1})] - \int_a^b f \, dg \right| < \frac{\varepsilon}{2}.
\]

Let \( \delta = \min \{\delta_1, \delta_2\} \), and let \( \sigma : a = x_0 < \ldots < x_n = c \) be a subdivision of \([a,c]\) with \( |\sigma| < \delta \), and let \( \{c_i\}_{i=1}^n \) be a marking of \( \sigma \). Let \( N \) be the least positive integer such that \( 2^{-N} < \delta \). Let \( \{c_i^*\}_{i=1}^t \) be the marking associated with \( \sigma^*_N \). Then \( \sigma^*_N \cup \sigma = \sigma^* : a = y_0 < \ldots < y_m = b \) is a subdivision of \([a,b]\) with \( |\sigma^*| < \delta \), and

\[
\{c_i\}_{i=1}^n \cup \{c_i^*\}_{i=1}^t = \{c_i^*\}_{i=1}^m
\]

is a marking of \( \sigma^* \). Therefore

\[
\left| \frac{1}{n} \sum_{i=1}^{n} f(c_i) [g(x_i) - g(x_{i-1})] - \left[ \int_a^b f \, dg - \int_c^b f \, dg \right] \right| \\
\leq \left| \frac{1}{n} \sum_{i=1}^{n} f(c_i) [g(x_i) - g(x_{i-1})] - \int_a^b f \, dg \right| \\
+ \left| \sum_{i=1}^{n} - \int_c^b f \, dg \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Therefore, by Definition 2.2 f is $S$-integrable over $[a,c]$ with respect to $g$, and

$$\int_a^b fdg = \int_a^c fdg + \int_c^b fdg.$$ 

If in the above it had been assumed that $S^\infty$ converges to some limit $B$, then a similar argument could be used to show that the same relation holds.

Therefore $f$ is $S$-integrable over $[a,c]$ and over $[c,b]$, and

$$\int_a^b fdg = \int_a^c fdg + \int_c^b fdg.$$ 

Theorem 3.8. Let $a < c < b$. If $f$ is $S$-integrable over $[a,c]$ and over $[c,b]$ with respect to $g$, and each of $f$ and $g$ is continuous at $c$, then $f$ is $S$-integrable over $[a,b]$ with respect to $g$ and

$$\int_a^b fdg = \int_a^c fdg + \int_c^b fdg.$$ 

Proof. Let $a < c < b$. Let $f$ be $S$-integrable over $[a,c]$ and over $[c,b]$ with respect to $g$, and let each of $f$ and $g$ be continuous at $c$. Let $\varepsilon > 0$. Let $\delta_1 > 0$ such that if $x \in [a,b]$ and $|x - c| < \delta_1$, then $|f(x) - f(c)| < \varepsilon$. Let $\delta_2 > 0$ such that if $x \in [a,b]$ and $|x - c| < \delta_2$, then $|g(x) - g(c)| < \frac{\varepsilon}{4}$. Let $\delta_3 > 0$ such that if $\sigma = a = x_0 < \ldots < x_n = c$ is any subdivision of $[a,c]$ with $|\sigma| < \delta_3$, and $\{c_i\}_{i=0}^{n}$ is any marking of $\sigma$, then
Let $\delta_4 > 0$ such that if $\sigma : c = x_0 < \ldots < x_\alpha = b$ is any subdivision of $[c,b]$ with $|\sigma| < \delta_4$, and $\{c_i\}_{i=1}^n$ is any marking of $\sigma$, then

$$\left| \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^c f dg \right| < \frac{\epsilon}{4}.$$ 

Let $\delta = \min \{\delta_1, \delta_2, \delta_3, \delta_4\}$. Let $\sigma : a = x_0 < \ldots < x_\alpha = b$ be any subdivision of $[a,b]$ with $|\sigma| < \delta$, and let $\{c_i\}_{i=1}^n$ be a marking of $\sigma$. Let $c \in [x_{j-1}, x_j]$. Assume $c = x_j$. Then

$$\left| \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] - \left[ \int_a^c f dg + \int_c^b f dg \right] \right|$$

$$\leq \left| \sum_{i=1}^{j-1} f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^c f dg \right|$$

$$+ \left| \sum_{i=j+1}^n f(c_i)[g(x_i) - g(x_{i-1})] - \int_c^b f dg \right| < \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon.$$ 

A similar proof follows in the case that $c = x_{j-1}$. Now assume that $c \in (x_{j-1}, x_j)$. Then

$$\left| \sum_{i=1}^n f(c_i)[g(x_i) - g(x_{i-1})] - \left[ \int_a^c f dg + \int_c^b f dg \right] \right|$$

$$\leq \left| \sum_{i=1}^{j-1} f(c_i)[g(x_i) - g(x_{i-1})] + f(c)[g(c) - g(x_{j-1})] - \int_a^c f dg \right|$$

$$+ \left| [f(c_j) - f(c)][g(c) - g(x_{j-1})] \right|$$

$$+ \left| [f(c_j) - f(c)][g(x_j) - g(c)] \right|$$
\[
+ \sum_{i=1}^{n} f(c_i) \left[ g(x_i) - g(x_{i-1}) \right] + f(c) \left[ g(x_j) - g(c) \right] - \int_{c}^{b} f \, dg \n\]
\[
< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon .
\]
Therefore \( f \) is \( S \)-integrable over \([a,b]\) with respect to \( g \), and
\[
\int_{a}^{b} f \, dg = \int_{a}^{c} f \, dg + \int_{c}^{b} f \, dg.
\]

Lemma 3.3. If the functions \( f \) and \( g \) are both defined on \([a,b]\), \( \sigma : a = x_0 < \ldots < x_n = b \) is any subdivision of \([a,b]\), and \( \{c_i\}_{i=1}^{n} \) is any marking of \( \sigma \), then
\[
\sum_{i=1}^{n} f(c_i) \left[ g(x_i) - g(x_{i-1}) \right]
\]
\[
= - \sum_{i=0}^{n} g(x_i) \left[ f(c_{i+1}) - f(c_i) \right] + f(b)g(b) - f(a)g(a)
\]
where \( c_0 = a \) and \( c_{n+1} = b \).

Proof. Let the functions \( f \) and \( g \) be defined over \([a,b]\). Let \( \sigma : a = x_0 < \ldots < x_n = b \) be a subdivision of \([a,b]\), and let \( \{c_i\}_{i=1}^{n} \) be a marking of \( \sigma \). Also let \( c_0 = a \) and \( c_{n+1} = b \). Then
\[
\sum_{i=1}^{n} f(c_i) \left[ g(x_i) - g(x_{i-1}) \right] = \sum_{i=1}^{n} \left[ f(c_i)g(x_i) - f(c_i)g(x_{i-1}) \right]
\]
\[
= \sum_{i=1}^{n} \left[ f(c_i)g(x_i) - f(c_{i+1})g(x_i) \right] - f(c_i)g(x_0) + f(c_n)g(x_n)
\]
\[
= \sum_{i=1}^{n} g(x_i) \left[ f(c_i) - f(c_{i+1}) \right] - f(c_i)g(a) + f(c_n)g(a)
\]
\[
- f(c_0)g(a) + f(c_n)g(b) + f(c_{n+1})g(b) - f(c_{n+1})g(b)
\]
\[
= \sum_{i=1}^{n} g(x_i) \left[ f(c_i) - f(c_{i+1}) \right] + g(a) \left[ f(c_0) - f(c_{n+1}) \right]
\]
Theorem 3.9. If \( f \) is \( S \)-integrable over \([a,b]\) with respect to \( g \), then \( g \) is \( S \)-integrable over \([a,b]\) with respect to \( f \), and

\[
\int_a^b f \, dg = f(b)g(b) - f(a)g(a) - \int_a^b g \, df.
\]

Proof. Let \( f \) be \( S \)-integrable over \([a,b]\) with respect to \( g \). Let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that if \( \sigma : a = x_0 < \ldots < x_n = b \) is any subdivision of \([a,b]\) with \( |\sigma| < \delta \), and \( \{c_i\}_{i=0}^n \) is any marking of \( \sigma \), then

\[
\left| \sum_{i=0}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f \, dg \right| < \varepsilon.
\]

Let \( \sigma : a = x_0 < \ldots < x_n = b \) be a subdivision of \([a,b]\) with \( |\sigma| < \frac{\delta}{2} \), and let \( \{c_i\}_{i=0}^n \) be a marking of \( \sigma \). Then \( c_i - a < \delta \), \( b - c_n < \delta \), and \( c_i - c_{i-1} < \delta \) for \( i = 2, 3, \ldots, n \). Define \( c_0 = a \) and \( c_{n+1} = b \). If for any \( 1 \leq i \leq n + 1 \), \( c_{i-1} = c_i \), then let \( i \in U \). Then

\[
\{c_i : 0 \leq i \leq n + 1 \text{ and } i \in U\} = \sigma' : a = y_0 < \ldots < y_m = b \text{ is a subdivision of } [a,b] \text{ with } |\sigma'| < \delta.
\]

Note that either \( c_i = c_{i+1} = x_i \) or \( x_i \in [c_i, c_{i+1}] \). Let \( x_i \in U' \) if \( i \in U \). Then \( \{x_i\}_{i=0}^n - U' = \{d_i\}_{i=0}^n \) is a marking of \( \sigma' \).
But notice that
\[
\sum_{i=0}^{n} f(x_i)[g(c_{i+1}) - g(c_i)] = \sum_{i=1}^{m} f(d_i)[g(y_i) - g(y_{i-1})],
\]
and so by Lemma 3.3
\[
\left| \sum_{i=0}^{n} g(c_i)[f(x_i) - f(x_{i-1})] - \left[ - \int_{a}^{b} f dg + f(b)g(b) - f(a)g(a) \right] \right|
\]
\[
= \left| - \sum_{i=0}^{n} f(x_i)[g(c_{i+1}) - g(c_i)] + f(b)g(b) - f(a)g(a) \right|
\]
\[
- \left[ - \int_{a}^{b} f dg + f(b)g(b) - f(a)g(a) \right] \right|
\]
\[
= \left| \sum_{i=1}^{m} f(d_i)[g(y_i) - g(y_{i-1})] - \int_{a}^{b} f dg \right| < \varepsilon.
\]
Therefore, by Definition 2.2 \( g \) is \( S \)-integrable over \([a,b]\) with respect to \( g \), and
\[
\int_{a}^{b} f dg = - \int_{a}^{b} g df + f(b)g(b) - f(a)g(a).
\]
CHAPTER BIBLIOGRAPHY


CHAPTER IV

THE BELT THEOREM

This chapter contains a further study of the integral of a function with respect to a nondecreasing function, and culminates in a proof of the Belt Theorem.

Theorem 4.1. If $g$ is continuous and nondecreasing over $[a,b]$, and $f$ is $S$-integrable over $[g(a),g(b)]$ with respect to $h$, then $f \circ g$ is $S$-integrable over $[a,b]$ with respect to $h \circ g$, and

$$\int_{g(a)}^{g(b)} f dh = \int_{h \circ g(a)}^{h \circ g(b)} f \circ g dh \circ g.$$  \hfill (1)

Proof. Let $g$ be continuous and nondecreasing over $[a,b]$. Let $f$ be $S$-integrable over $[g(a),g(b)]$ with respect to $h$. If $g(a) = g(b) = c$, then by Definition 2.3

$$\int_{g(a)}^{g(b)} f dh = 0.$$  \hfill (2)

But since $g$ is nondecreasing over $[a,b]$, then $g(x) = c$ for every $x \in [a,b]$. Therefore $f \circ g(x) = f(c)$, a constant, for every $x \in [a,b]$. Therefore, by Theorem 3.3

$$\int_{a}^{b} f \circ g dh \circ g = f(c)[h \circ g(b) - h \circ g(a)]$$

$$= f(c)[h(c) - h(c)] = 0.$$  \hfill (3)

Therefore

$$\int_{g(a)}^{g(b)} f dh = \int_{a}^{b} f \circ g dh \circ g.$$  \hfill (4)
Now suppose that \( g(a) < g(b) \). Let \( \epsilon > 0 \). Then there exists \( \delta > 0 \) such that if \( \sigma : g(a) = x_0 < \ldots < x_n = g(b) \) is any subdivision of \([g(a), g(b)]\) with \( |\sigma| < \delta \), and \( \{c_i\}_{i=1}^n \) is any marking of \( \sigma \), then

\[
\left| \sum_{i=1}^{n} f(c_i)[h(x_i) - h(x_{i-1})] - \int_{g(a)}^{g(b)} f dh \right| < \epsilon.
\]

Since \( g \) is continuous over \([a, b]\), by Theorem 1.11 there exists \( \delta_a > 0 \) such that if \( p, q \in [a, b] \) and \( |p - q| < \delta_a \), then \( |g(p) - g(q)| < \delta \). Let \( \sigma : a = x_0 < \ldots < x_n = b \) be any subdivision of \([a, b]\) with \( |\sigma| < \delta \), and let \( \{c_i\}_{i=1}^n \) be any marking of \( \sigma \). Then for \( 1 \leq i \leq n \), \( |x_i - x_{i-1}| < \delta_a \), and so \( |g(x_i) - g(x_{i-1})| < \delta \). Let \( U = \{i : 1 \leq i \leq n \text{ and } g(x_{i-1}) = g(x_i)\} \). Let \( \{y'_i, y''_i, \ldots, y'_m\} = \{x_i \in \sigma : i \in U\} \) where it is understood that \( a = y'_0 < y'_1 < \ldots < y'_m \). Let \( y'_i = y_i \) if \( i = 0, 1, \ldots, m - 1 \) and \( y'_m = b \). Let \( d_i = c_i \) if \( y'_i = x_i \) and \( i \in U \). Then \( \sigma_a : g(a) = g(y'_0) < \ldots < g(y'_m) = g(b) \) is a subdivision of \([g(a), g(b)]\) with \( |\sigma_a| < \delta \), and \( \{g(d_i)\}_{i=1}^m \) is a marking of \( \sigma_a \). Therefore

\[
\left| \sum_{i=1}^{n} f \circ g(c_i)[h \circ g(x_i) - h \circ g(x_{i-1})] - \int_{g(a)}^{g(b)} f dh \right|
= \left| \sum_{j=1}^{m} f(g(d_j))[h(g(y_j)) - h(g(y_{j-1}))] - \int_{g(a)}^{g(b)} f dh \right| < \epsilon.
\]

Therefore, by Definition 2.2 \( f \circ g \) is \( S \)-integrable over \([a, b]\) with respect to \( h \circ g \), and

\[
\int_{g(a)}^{g(b)} f dh = \int_{a}^{b} f \circ gdh \circ g.
\]
Theorem 4.2. If each of $f_i$ and $f_\alpha$ is S-integrable over $[a,b]$ with respect to a nondecreasing function $g$, and $f_i(x) \leq f_\alpha(x)$ for every $x \in [a,b]$, then
\[ \int_a^b f_i \, dg \leq \int_a^b f_\alpha \, dg. \]

Proof. Let each of $f_i$ and $f_\alpha$ be S-integrable over $[a,b]$ with respect to a nondecreasing function $g$, and let $f_i(x) \leq f_\alpha(x)$ for every $x \in [a,b]$. Assume that
\[ \int_a^b f_i \, dg - \int_a^b f_\alpha \, dg = \varepsilon > 0. \]
Then there exists $\delta_i > 0$ such that if $\sigma : a = x_0 < \ldots < x_n = b$ is any subdivision of $[a,b]$ with $|\sigma| < \delta_i$, and $\{c_i\}_{i=1}^n$ is any marking of $\sigma$, then
\[ \left| \sum_{i=1}^n f_i(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f_i \, dg \right| < \frac{\varepsilon}{2}. \]
Also there exists $\delta_\alpha > 0$ such that if $\sigma : a = x_0 < \ldots < x_n = b$ is any subdivision of $[a,b]$ with $|\sigma| < \delta_\alpha$, and $\{c_i\}_{i=1}^n$ is any marking of $\sigma$, then
\[ \left| \sum_{i=1}^n f_\alpha(c_i)[g(x_i) - g(x_{i-1})] - \int_a^b f_\alpha \, dg \right| < \frac{\varepsilon}{2}. \]
Let $\delta = \min \{\delta_i, \delta_\alpha\}$. Let $\sigma : a = x_0 < \ldots < x_n = b$ be any subdivision of $[a,b]$ with $|\sigma| < \delta$, and let $\{c_i\}_{i=1}^n$ be any marking of $\sigma$. Then since $f_i(x) \leq f_\alpha(x)$ for every $x \in [a,b]$, and $g$ is nondecreasing over $[a,b]$, it follows that
\[ \sum_{i=1}^n f_\alpha(c_i)[g(x_i) - g(x_{i-1})] \]
Therefore $\epsilon = \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$$\geq \left| \sum_{i=1}^{n} f_i(c_i)(g(x_i) - g(x_{i-1})) - \int_{a}^{b} f_i \, dg \right|$$

$$+ \left| \sum_{i=1}^{n} f_i(c_i)(g(x_i) - g(x_{i-1})) - \int_{a}^{b} f_i \, dg \right|$$

$$\geq \left| \sum_{i=1}^{n} f_i(c_i)(g(x_i) - g(x_{i-1})) - \sum_{i=1}^{n} f_i(c_i)(g(x_i) - g(x_{i-1})) \right|$$

$$+ \int_{a}^{b} f_i \, dg - \int_{a}^{b} f_i \, dg = \eta + \epsilon \geq \epsilon$$, which is a contradiction. Therefore it must be true that

$$\int_{a}^{b} f_i \, dg \leq \int_{a}^{b} f_i \, dg.$$

**Corollary 4.1.** If $f$ is a bounded function which is $S$-integrable over $[a,b]$ with respect to a nondecreasing function $g$, then

$$G[g(b) - g(a)] \leq \int_{a}^{b} f \, dg \leq L[g(b) - g(a)],$$

where $G = \text{glb}\{f(x): x \in [a,b]\}$ and $L = \text{lub}\{f(x): x \in [a,b]\}$.

**Proof.** Let $f$ be a bounded function which is $S$-integrable over $[a,b]$ with respect to a nondecreasing function $g$. Let $G = \text{glb}\{f(x): x \in [a,b]\}$ and $L = \text{lub}\{f(x): x \in [a,b]\}$. By Theorem 3.3

$$\int_{a}^{b} L \, dg = L[g(b) - g(a)],$$

and

$$\int_{a}^{b} G \, dg = G[g(b) - g(a)].$$
And by Theorem 4.2, since $G \leq f(x) \leq L$ for every $x \in [a,b]$,

$$G[g(b) - g(a)] \leq \int_a^b f(x) \, dx \leq L[g(b) - g(a)].$$

Corollary 4.2. If $f$ and $|f|$ are both $S$-integrable over $[a,b]$ with respect to a nondecreasing function $g$, then

$$\left| \int_a^b f(x) \, dg \right| \leq \int_a^b |f(x)| \, dg.$$

Proof. Let $f$ and $|f|$ both be $S$-integrable over $[a,b]$ with respect to a nondecreasing function $g$. Clearly $|f(x)| \geq f(x)$ for every $x \in [a,b]$, and $-|f(x)| \leq f(x)$ for every $x \in [a,b]$. By Theorem 3.4

$$\int_a^b -|f| \, dg = -\int_a^b |f| \, dg.$$

Therefore by Theorem 4.2

$$-\int_a^b |f| \, dg \leq \int_a^b f(x) \, dg \leq \int_a^b |f| \, dg,$$

and

$$\left| \int_a^b f(x) \, dg \right| \leq \int_a^b |f| \, dg.$$

Theorem 4.3. If $f$ is continuous over $[a,b]$, and $g$ is nondecreasing over $[a,b]$, then there exists $t \in [a,b]$ such that

$$f(t)[g(b) - g(a)] = \int_a^b f(x) \, dg.$$

Proof. Let $f$ be continuous over $[a,b]$, and let $g$ be nondecreasing over $[a,b]$. By Theorem 2.9 $f$ is $S$-integrable over $[a,b]$ with respect to $g$. By Theorem 1.9 $f$ is bounded
over \([a,b]\), and both \(L = \text{lub} \{f(x): \ x \in [a,b]\}\) and \(G = \text{glb} \{f(x): \ x \in [a,b]\}\) exist. By Corollary 4.1 \(G[g(b) - g(a)]\) \(
\leq \int_a^b f \, dg \leq L[g(b) - g(a)].\) Therefore if \(g(a) = g(b)\), then \(\int_a^b f \, dg = 0\), and given \(t \in [a,b]\), then \(f(t)[g(b) - g(a)] = 0\).

If \(g(b) > g(a)\), then

\[
G \leq \frac{\int_a^b f \, dg}{g(b) - g(a)} \leq L,
\]

and by Theorem 1.14 there exists \(t \in [a,b]\) such that

\[
f(t) = \frac{\int_a^b f \, dg}{g(b) - g(a)}.
\]

Therefore \(f(t)[g(b) - g(a)] = \int_a^b f \, dg\).

**Lemma 4.1.** If \(f\) is a bounded function and is continuous over \([a,b]\) except at the points of a finite set \(A\), and \(g\) is nondecreasing and \(\{x: \ g\ \text{is discontinuous at} \ x\}\) contains no point of \(A\), then \(f\) is \(S\)-integrable over \([a,b]\) with respect to \(g\).

**Proof.** Let \(f\) be a bounded function that is continuous over \([a,b]\) except at the points of a finite set \(A = \{x_1, x_2, \ldots, x_n\}\). Let \(|f(x)| \leq L\) for all \(x \in [a,b]\). Let \(g\) be a nondecreasing function defined on \([a,b]\) with
\{ x : g \text{ is discontinuous at } x \} \cap A = \emptyset. \text{ Let } \varepsilon > 0. \text{ Let } n = 1, \text{ and suppose that } x_n = b. \text{ Let } k_n = (b - a)2^{-n} \text{ for } n = 1, 2, 3, \ldots. \text{ Let } \{y_n\}_{n=1}^\infty \text{ be a sequence of elements of } [a, b] \text{ with } y_n = b - k_n. \text{ Since } \lim_{n \to \infty} k_n = 0, \text{ then } \lim_{n \to \infty} y_n = b.

Note that } f \text{ is continuous over } [a, y_n] \text{ for } n = 1, 2, 3, \ldots. \text{ Therefore } f \text{ is } S\text{-integrable over } [a, y_n] \text{ with respect to } g \text{ for } n = 1, 2, 3, \ldots \text{ by Theorem 2.9. Since } g \text{ is continuous at } b, \text{ there exists } \delta > 0 \text{ such that if } x \in [a, b] \text{ such that } |x - b| < \delta, \text{ then }

\left| g(x) - g(b) \right| < \frac{\varepsilon}{L}.

Let } N \text{ be the least positive integer such that } k_N < \delta. \text{ Let } m > n > N. \text{ Then } y_1 = b - k_1 \in [a, b] \text{ such that } |y_1 - b| < \delta \text{ and } y_2 = b - k_2 \in [a, b] \text{ such that } |y_2 - b| < \delta. \text{ Therefore }

\left| g(y_1) - g(y_2) \right| < \frac{\varepsilon}{L}

and

\left| \int_a^{y_1} f dg - \int_a^{y_2} f dg \right| = \left| \int_{y_1}^{y_2} f dg \right| \\
\leq L \left[ g(y_2) - g(y_1) \right] < \varepsilon.

Thus by Theorem 1.5 \( \left\{ \int_a^{b - k_n} f dg \right\} \to \infty \text{ converges to some limit } L. \)

Let } \delta_1 > 0 \text{ such that if } k_n < \delta_1, \text{ then }

\left| \int_a^{b - k_n} f dg - I \right| < \frac{\varepsilon}{2}.
Let $\delta_2 > 0$ such that if $x \in [a,b]$ such that $b - x < \delta_2$, then

$$|g(b) - g(x)| < \frac{\epsilon}{2L}.$$ 

Let $\delta = \min \{\delta_1, \delta_2\}$. Let $x \in [a,b]$ such that $|x - b| < \delta$, and let $k_{\delta} < \delta$. Then

$$\left| \int_a^x f \, dg - I \right| \leq \left| \int_a^b f \, dg - \int_a^{b - k_{\delta}} f \, dg \right| + \left| \int_a^{b - k_{\delta}} f \, dg - I \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ 

Therefore $\lim_{x \to b} \int_a^x f \, dg = I$.

Let $\delta_1 > 0$ such that if $x \in [a,b]$ and $|x - b| < \delta_1$, then

$$\left| \int_a^x f \, dg - I \right| < \frac{\epsilon}{4}.$$ 

Let $\delta_3 > 0$ such that if $x \in [a,b]$ and $|x - b| < \delta_3$, then

$$|g(x) - g(b)| < \frac{\epsilon}{4L}.$$ 

Let $p = \min \left\{-\frac{\delta_1}{2}, \frac{1}{4}(b - a), \frac{\delta_3}{2} \right\}$. Let $t = b - p$. Let $\delta_3 > 0$ such that if $\sigma: a = z_0 < \ldots < z_{n_\sigma} = t$ is any subdivision of $[a,t]$ with $|\sigma| < \delta_3$ and $\{c_i\}_{i=1}^{n_\sigma}$ is any marking of $\sigma$, then

$$\left| \sum_{i=1}^{n_\sigma} f(c_i)[g(z_i) - g(z_{i-1})] - \int_a^x f \, dg \right| < \frac{\epsilon}{4}.$$ 

Let $\delta = \min \left\{-\frac{\delta_3}{2}, \delta_3 \right\}$, and let $\sigma: a = z_0 < \ldots < z_{n_\sigma} = b$ be any subdivision of $[a,b]$ with $|\sigma| < \delta$, and let $\{c_i\}_{i=1}^{n_\sigma}$
be any marking of $\sigma$. Suppose that $t \in [z_{j-1}, z_j]$. Note that $t - z_{j-1} < \frac{\xi}{2}$ and $b - t < \frac{\xi}{2}$. Therefore $b - z_{j-1} < \xi$ and $|g(t) - g(z_{j-1})| < \frac{\xi}{4L}$. Similarly $|g(b) - g(z)| < \frac{\xi}{4L}$.

Thus

$$\left| \sum_{i=1}^{j-1} f(c_i)[g(z_i) - g(z_{i-1})] - I \right|$$

$$\leq \left| \sum_{i=1}^{j-1} f(c_i)[g(z_i) - g(z_{i-1})] + f(t)[g(t) - g(z_{j-1})] - f(t)[g(t) - g(z_{j-1})] + \int_a^t f \, dg - \int_a^t f \, dg - I \right|$$

$$\leq \left| \sum_{i=1}^{j-1} f(c_i)[g(z_i) - g(z_{i-1})] + f(t)[g(t) - g(z_{j-1})] - \int_a^t f \, dg \right|$$

$$+ \left| \int_a^t f \, dg - I \right| + \left| f(t)[g(t) - g(z_{j-1})] \right|$$

$$+ \left| \sum_{i=1}^{j-1} f(c_i)[g(z_i) - g(z_{i-1})] \right|$$

$$\leq \left| \sum_{i=1}^{j-1} f(c_i)[g(z_i) - g(z_{i-1})] + f(t)[g(t) - g(z_{j-1})] - \int_a^t f \, dg \right|$$

$$+ \left| \int_a^t f \, dg - I \right| + L|g(t) - g(z_{j-1})| + L|g(b) - g(z_j)|$$

$$< \frac{\xi}{4} + \frac{\xi}{4} + \frac{\xi}{4} + \frac{\xi}{4} = \xi$$

Therefore $f$ is $S$-integrable over $[a,b]$ with respect to $g$ and

$$\int_a^b f \, dg = I.$$

A similar argument may be presented if it is assumed that $x_i = a$. If $x_i \in (a,b)$, then by an argument similar to that above $f$ can be shown to be $S$-integrable over $[a,x_i]$ with
respect to \( g \) and over \([x_1, b]\) with respect to \( g \), consequently by Theorem 3.8 \( f \) would be \( S \)-integrable over \([a, b]\) with respect to \( g \). Therefore for the case when \( n = 1 \), \( f \) is \( S \)-integrable over \([a, b]\) with respect to \( g \).

Now assume that if \( n \leq N \), then \( f \) is \( S \)-integrable over \([a, b]\) with respect to \( g \). Suppose that \( x_1, x_2, \ldots, x_{N+1} \) are the points of discontinuity for \( f \) arranged so that \( x_1 < x_2 < \ldots < x_{N+1} \). By Theorem 1.17 there exists \( y \in (x_{N+1}, x_{N+1}) \) such that \( g \) is continuous at \( y \). Then by the induction hypothesis and by Theorem 3.7 \( f \) is \( S \)-integrable over \([a, y]\) with respect to \( g \), and by the case \( n = 1 \) above \( f \) is \( S \)-integrable over \([y, b]\) with respect to \( g \). Thus by Theorem 3.8 \( f \) is \( S \)-integrable over \([a, b]\) with respect to \( g \).

**Theorem 4.5.** Suppose \( \mathcal{F} \) is an infinite set of non-decreasing functions defined for all real numbers, and there exists a real number \( M \) such that if \( g \in \mathcal{F} \), then \( |g(x)| \leq M \) for every \( x \). Then there exist a nondecreasing function \( \phi \) and an infinite sequence \( \{\phi_n\}_{n=1}^{\infty} \) of distinct functions in \( \mathcal{F} \) such that for each \( x \)

\[
\lim_{n \to \infty} \phi_n(x) = \phi(x),
\]

and

\[ |\phi(x)| \leq M. \]

Furthermore if \( a < b \) and \( f \) is continuous over \([a, b]\), then
Proof. Let $\Phi$ be an infinite set of nondecreasing functions defined for all real numbers with the property that if $g \in \Phi$ then $|g(x)| \leq M$ for all $x$, where $M$ is a real number. Let $r_1, r_2, \ldots, r_i, \ldots$ be a listing of the rational numbers with the property that every rational number is listed one, and only one, time. Let $\{\phi^{(n)}_n\}_{n=1}^{\infty}$ be an infinite sequence of distinct elements of $\Phi$. Then the sequence \[ \{\phi^{(n)}_n(r_i)\}_{n=1}^{\infty} \] of real numbers is bounded by $M$, and by Theorem 1.3 has a limit superior. Let $\phi(r_i)$ be this limit superior, and let $\{\phi^{(n)}_n\}_{n=1}^{\infty}$ be a subsequence of $\{\phi^{(n)}_n\}_{n=1}^{\infty}$ that converges to $\phi(r_i)$ at $x = r_i$. Let $\phi^* = \phi^{(n)}_n$. Then $\{\phi^{\ast}_n(r_i)\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, and by Theorem 1.3 has a limit superior. Let $\phi(r_j)$ be this limit superior, and let $\{\phi^{(n)}_n\}_{n=1}^{\infty}$ be a subsequence of $\{\phi^{(n)}_n\}_{n=1}^{\infty}$ that converges to $\phi(r_j)$ at $x = r_j$. Let $\phi^* = \phi^{(n)}_n$. In this manner the sequence $\{\phi^*_n\}_{n=1}^{\infty}$ may be defined so that $\{\phi^*_n(r_i)\}_{n=1}^{\infty}$ converges to a real number $\phi(r_i)$ for every rational number $r_i$. Suppose that $T = \{x: \{\phi^*_n(x)\}_{n=1}^{\infty} \text{ diverges}\}$ is uncountable. For every $x$ $\{\phi^*_n(x)\}_{n=1}^{\infty}$ is bounded by $M$, and by Theorem 1.3 $\lim \{\phi^*_n(x)\}_{n=1}^{\infty} = \overline{\alpha}_x$ and $\lim \{\phi^*_n(x)\}_{n=1}^{\infty} = \underline{\alpha}_x$ exist. Also, by Theorem 1.4 $x \in T$ if, and only if, $\overline{\alpha}_x - \underline{\alpha}_x > 0$. Since $T$ is uncountable, one of
must be uncountable since \( T = \bigcup_{i=1}^{\infty} T_i \). Suppose \( T_p \) is uncountable. Let \( N \) be the least positive integer such that \( 2M + 1 < \frac{N}{p} \). Let \( x_1, x_2, \ldots, x_N \in T_p \) such that \( x_1 < x_2 < \ldots < x_N \).

Let \( y_1, y_2, \ldots, y_M \) be rational numbers such that \( x_1 < y_1 < x_2 < y_2 < \ldots < y_{M-1} < x_N \). Suppose that \( \bar{a}_{n_i} - \phi(y_i) = \infty > 0 \).

By Definition 1.9 \( \{ n : \phi^*_n(x_i) > \bar{a}_{n_i} - \frac{\alpha}{3} \} \) is infinite, but \( \{ n : \phi^*_n(y_i) > \phi(y_i) + \frac{\alpha}{3} \} \) is empty or finite. This means that an infinite number of \( \phi^*_n \) are not nondecreasing.

But \( \phi^*_n \in \Phi \) for each \( n \), and every element of \( \Phi \) is nondecreasing. Therefore \( \bar{a}_{n_i} < \phi(y_i) \). By using a similar argument it follows that

\[
\bar{a}_{n_i} < \bar{a}_{n_i} < \phi(y_1) < \bar{a}_{n_2} < \bar{a}_{n_2} < \ldots < \bar{a}_{n_M}.
\]

Therefore \( \bar{a}_{n_i} - a_{n_i} > 2M + 1 \). But there exist \( \phi^*_n(x_M) > \bar{a}_{n_i} - \frac{1}{4} \) and \( \phi^*_n(x_1) < a_{n_1} + \frac{1}{4} \). Therefore \( \phi^*_n(x_M) - \phi^*_n(x_1) > 2M \), which contradicts the fact that \( |g(x)| < M \) for every \( g \in \Phi \).

Therefore \( T \) must be countable. Let \( d_1, d_2, \ldots, d_i, \ldots \) be a listing of the elements of \( T \) in such a way that each element of \( T \) is listed one and only one time. Then \( \{ \phi^*_n(d_i) \}_{n=1}^{\infty} \) is bounded by \( M \), and by Theorem 1.3 has a limit superior.
Let \( \phi (d_i) \) be this limit superior, and let \( \{ \phi^{(n)}_n \}_{n=1}^{\infty} \) be a subsequence of \( \{ \phi^{(n)}_n \}_{n=1}^{\infty} \) that converges to \( \phi (d_i) \) at \( x = d_i \).

Let \( \phi_i = \phi^{(n)}_n \). Now, in a manner similar to that used in forming the sequence \( \{ \phi^{(n)}_n \}_{n=1}^{\infty} \) above, the sequence \( \{ \phi_n \}_{n=1}^{\infty} \) may be formed. Notice that \( \{ \phi_n(x) \}_{n=1}^{\infty} \) converges for every \( x \). Therefore let \( \phi (x) = \lim \phi_n(x) \) for all \( x \). Let \( x < y \), and suppose that \( \phi (x) - \phi (y) = \alpha > 0 \). Since, by Theorem 1.4, both \( \phi (x) \) and \( \phi (y) \) are limit superiors, then

\[
\{ n : \phi_n(x) > \phi (x) - \frac{\alpha}{3} \} \text{ is infinite, yet } \{ n : \phi_n(y) > \phi (y) + \frac{\alpha}{3} \} \text{ is finite or empty. This means that an infnite number of } \phi_n \text{ are not nondecreasing, which is a contradiction since each } \phi_n \text{ is in } \Phi . \text{ Therefore } \phi \text{ is a nondecreasing function.}
\]

Suppose that there exists an \( x \) such that \( |\phi (x)| = M + \beta > M \). Since \( \lim_{n \to \infty} \phi_n(x) = \phi (x) \) then there exists \( N \) such that if \( n > N \) then \( |\phi_n(x) - \phi (x)| < \frac{\beta}{2} \).

Therefore \( |\phi_n(x)| > |\phi (x)| - \frac{\beta}{2} = M + \beta - \frac{\beta}{2} = M + \frac{\beta}{2} > M \), which contradicts the fact that for every \( g \in \Phi \), \( |g(x)| \leq M \) for all \( x \). Therefore \( |\phi (x)| \leq M \) for all \( x \).

Let \( a < b \), and let \( f \) be a continuous function over \([a, b]\).

Let \( \varepsilon > 0 \). Let \( k > 0 \) such that \( k[\phi (b) - \phi (a) + 1] < \frac{\varepsilon}{4} \).

Let \( N_1 \) be a positive integer such that if \( n > N_1 \), then
\[ |\phi_n(a) - \phi(a)| < \frac{1}{2} \]. Let \( N_2 \) be a positive integer such that if \( n > N_2 \), then \( |\phi_n(b) - \phi(b)| < \frac{1}{2} \). Then given \( n \geq \max \{N_1, N_2\} \)

\[
k[\phi_n(b) - \phi_n(a)] < k \left[ \phi(b) + \frac{1}{2} - \phi(a) + \frac{1}{2} \right]
\]

\[= k[\phi(b) - \phi(a) + 1] < \frac{e}{4}.\]

By Theorem 1.18 a step function \( g \) approximating \( f \) may be chosen so that \(|f(x) - g(x)| < k\) for all \( x \in [a,b] \). By Theorem 1.17 there are at most a countable number of points of \([a,b]\) where \( \phi \) or any \( \phi_i \), \( i = 1, 2, \ldots \), is discontinuous. From Theorem 1.18 it is clear that \( g \) may be chosen in such a way that if \( G : a = x_0 < \ldots < x_m = b \) is the subdivision of \([a,b]\) associated with \( g \), then none of \( \phi, \phi_1, \phi_2, \ldots \) are discontinuous at any of \( x_1, x_2, \ldots, x_m \). Also, \( g \) may be defined by requiring \( g(x) = c_i \) if \( x \in [a,x_i] \), and \( g(x) = c_i \) if \( x \in (x_{i-1},x_i] \) where \( i = 2, 3, \ldots, m \). Assume \( g \) has the above mentioned properties. Notice that \( f(x) - g(x) = t(x) \) is a continuous function at \( a,b \), and for every \( x \in (x_{i-1},x_i) \) where \( i = 1, 2, \ldots, n \). Therefore by Lemma 4.1 \( t \) is \( S \)-integrable over \([a,b]\) with respect to each of \( \phi, \phi_1, \phi_2, \ldots \). But \(|t(x)| < k\) for all \( x \). Therefore by Corollary 4.1 given \( n > \max \{N_1, N_2\} \),

\[
\left| \int_a^b t d\phi - \int_a^b t d\phi_n \right| \leq \left| \int_a^b t d\phi \right| + \left| \int_a^b t d\phi_n \right|
\]
Let \( \delta, > 0 \) such that if \( x \in [a, b] \) and \( |x - x_i| < \delta \), then

\[
|\phi(x) - \phi(x_i)| < \frac{\varepsilon}{64 [l_{c_1} + |c_2| + 1]}
\]

Let

\[
y_i = \max \left\{ x_i - \frac{\delta}{3}, \frac{a + x_i}{2} \right\},
\]

and

\[
y_\ast = \min \left\{ x_i + \frac{\delta}{3}, x_i + \frac{1}{3}(x_\ast - x_i) \right\}.
\]

Then by Lemma 4.1 \( g \) is \( S \)-integrable over \([y_i, y_\ast]\) with respect to each of \( \phi, \phi_i, \phi_\ast, \ldots \), and by Theorem 4.2

\[
\left| \int_{y_i}^{y_\ast} g d\phi \right| \leq [l_{c_1} + |c_2|] [\phi(y_\ast) - \phi(y_i)]
\]

\[
= [l_{c_1} + |c_2|] [\phi(y_\ast) - \phi(x_i) + \phi(x_i) - \phi(y_i)]
\]

\[
< \frac{\varepsilon}{64} + \frac{\varepsilon}{64} = \frac{\varepsilon}{32}.
\]

Let \( N_3 \) be a positive integer such that if \( n \geq N_3 \), then

\[
|\phi_n(y_i) - \phi(y_i)| < \frac{\varepsilon}{64 [l_{c_1} + |c_2| + 1]}
\]

Let \( N_4 \) be a positive integer such that if \( n > N_4 \), then

\[
|\phi_n(y_\ast) - \phi(y_\ast)| < \frac{\varepsilon}{64 [l_{c_1} + |c_2| + 1]}
\]

Then for \( n \geq \max \{N_3, N_4\} \),

\[
\left| \int_{y_i}^{y_\ast} g d\phi_n \right| \leq [l_{c_1} + |c_2|] [\phi_n(y_\ast) - \phi_n(y_i)]
\]

\[
< [l_{c_1} + |c_2|] \left[ \phi(y_\ast) + \frac{\varepsilon}{64 [l_{c_1} + |c_2| + 1]} \right]
\]
Therefore if \( n \geq \max \{ N_3, N_4 \} \),
\[
\left| \int_{y_1}^{y_2} g \, d\phi - \int_{y_1}^{y_2} g \, d\phi_n \right| \leq \left| \int_{y_1}^{y_2} g \, d\phi \right| + \left| \int_{y_1}^{y_2} g \, d\phi_n \right|
\]
\[
< \frac{\varepsilon}{32} + \frac{\varepsilon}{16} < \frac{\varepsilon}{8}
\]
Let \( N_5 \) be a positive integer such that if \( n \geq N_5 \), then
\[
|\phi_n(a) - \phi(a)| < \frac{\varepsilon}{16 [ |c_i| + 1 ]}
\]
Then notice that since \( g(x) = c_i \) for every \( x \in [a, y_1] \), by
Theorem 3.3, \( g \) is \( S \)-integrable over \([a, y_1]\) with respect to
each of \( \phi \), \( \phi_1 \), \( \ldots \). Let \( n \geq \max \{ N_3, N_4, N_5 \} \), then
\[
\left| \int_{a}^{y_1} g \, d\phi - \int_{a}^{y_1} g \, d\phi_n \right|
\]
\[
\leq |c_1| \left[ |\phi(y_1) - \phi_n(y_1)| + |\phi(a) - \phi_n(a)| \right]
\]
\[
< \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = \frac{\varepsilon}{8}
\]
In a similar manner \( y_3, y_4, \ldots, y_{m-1} \) and \( N_b, N_1, \ldots, N_{m+1} \)
may be defined in such a way that if \( y_a = a, y_{m-1} = b \), and
\( n \geq \max \{ N_1, N_2, \ldots, N_{m+1} \} \), then
\[
\left| \int_{y_{ap}}^{y_{ap+1}} g \, d\phi - \int_{y_{ap}}^{y_{ap+1}} g \, d\phi_n \right| < \frac{\varepsilon}{2^{p+2}} \text{ where } p = 1, 2, \ldots, m - 1,
\]
and
\[
\left| \int_{y_{ap}}^{y_{ap+1}} g \, d\phi - \int_{y_{ap}}^{y_{ap+1}} g \, d\phi_n \right| < \frac{\varepsilon}{2^{p+3}} \text{ where } p = 0, 1, \ldots, m - 1.
\]
Let \( N = \max \{ N_1, N_2, \ldots, N_{m+1} \} \), and let \( n \geq N \). Then
\[
\left| \int_a^b f \, d\phi - \int_a^b f \, d\phi_n \right| = \left| \int_a^b (g + t) \, d\phi - \int_a^b (g + t) \, d\phi_n \right| \\
\leq \left| \int_a^b g \, d\phi - \int_a^b g \, d\phi_n \right| + \left| \int_a^b t \, d\phi - \int_a^b t \, d\phi_n \right| \\
\leq \left| \int_a^b g \, d\phi - \int_a^b g \, d\phi_n \right| + \ldots + \left| \int_{y_{n-1}}^{y_n} g \, d\phi - \int_{y_{n-1}}^{y_{n-1}} g \, d\phi_n \right| \\
+ \left| \int_a^b t \, d\phi - \int_a^b t \, d\phi_n \right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.
\]

Therefore \( \lim_{n \to \infty} \int_a^b f \, d\phi_n = \int_a^b f \, d\phi \).
CHAPTER BIBLIOGRAPHY


BIBLIOGRAPHY

Books


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