On Bär's Conformal Lower Bound for the Spectrum of Generalized Dirac Operators

N. ANGHEL

Dedicated to the memory of Ruth Michler

Abstract

We use the orthogonal splitting of a certain Clifford module as a direct sum of generalised spinors and twistors to give a short and natural proof to Bär's conformal lower bound for the spectrum of a generalised Dirac operator on a compact Riemannian manifold.

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1 Introduction

Semmelmann [S] gave very short and elegant proofs for the by now classical Friedrich/Kirchberg lower bound on the spectrum of the Dirac operator on a Riemannian/Kählerian spin manifold with positive scalar curvature [F/K]. His proofs made full use of the orthogonal decomposition of the spinor bundle of the manifold twisted by the tangent bundle as a direct sum of spinor and twistor bundles. The purpose of this paper is to show that the same approach can be used to get Bär's conformal lower bound for the spectrum of a generalised Dirac operator [B, Theorem 1].

Let \( M \) be a compact connected \( n \)-dimensional Riemannian manifold and let \( Cl(M) \) be the Clifford bundle of algebras induced by the tangent bundle \( TM \) and the Riemannian metric \( \langle \cdot, \cdot \rangle \). \( Cl(M) \) comes naturally equipped with a metric and a connection, which extend the metric and the Levi-Civitá connection of \( M \). Let \( S \) be a bundle of Clifford modules over \( M \) equipped with a Hermitian metric \( \langle \cdot, \cdot \rangle \) and a metric connection \( \nabla \) such that unit vectors in \( TM \) act isometrically on \( S \) and such that the connection \( \nabla \) is compatible with that of \( Cl(M) \). In other words, \( S \) is a generalised Dirac bundle over \( M \) [GL]. In what follows we choose to view \( \nabla \) as a mapping \( \nabla : C^\infty(S) \to C^\infty(TM \otimes S) \).
Any generalized Dirac bundle generates a first-order elliptic differential operator $D : C^\infty(S) \to C^\infty(S)$, the generalized Dirac operator, defined as follows. Denote as usual by $\cdot$ the Clifford multiplication of $\text{Cl}(M)$ on $S$ and by $\mu : TM \otimes S \to S$ its restriction to $TM$. Then the generalized Dirac operator $D$ is the composite $\mu \circ \nabla$. Locally, $D$ admits the representation

$$D = \sum_{i=1}^{n} e_i \cdot \nabla e_i,$$

where $(e_1, e_2, \ldots, e_n)$ is a local orthonormal frame in $TM$.

Then (1.1) readily implies that for $f \in C^\infty(M)$ and $s \in C^\infty(S)$,

$$D(fs) = (\text{grad} f) \cdot s + fDs,$$

$$D^2(fs) = \Delta(f)s - 2\nabla_{\text{grad} f}s + fD^2s,$$

where $\Delta$ is the positive Laplacian on functions.

$D$ is selfadjoint with respect to the usual $L^2$-inner product $(s_1, s_2) = \int_M \langle s_1, s_2 \rangle$ in $C^\infty(S)$, and for $D^2$ the following Bochner-Weitzenböck formula holds [GL]:

$$D^2 = \nabla^* \nabla + \mathcal{R},$$

where $\mathcal{R}$ is a specific Hermitian curvature bundle morphism acting on $S$.

Under the assumption that $\mathcal{R}$ is pointwise bounded from below by a function $k \in C^\infty(M, \mathbb{R})$, i.e., $(\mathcal{R}s, s) \geq k(s, s)$, $s \in \cdot S$, Bär [B] proved the following lower estimate for the spectrum of the generalized Dirac operator $D$:

THEOREM 1.5. For every smooth function $f : M \to \mathbb{R}$ and every eigenvalue $\lambda$ of the generalized Dirac operator $D$ the following inequality holds:

$$\lambda^2 \geq \frac{n}{n-1} \min_M \left\{ k + \Delta f - \frac{n-2}{n-1} |\text{grad} f|^2 \right\}.$$ 

2 The Estimate

In this section we will give a short and natural proof to Theorem 1.5. Throughout, $\langle \cdot, \cdot \rangle$, respectively $| \cdot |$, will denote the pointwise inner product, respectively norm, in various bundles, such as $TM$, $T^*M$, $S$, $TM \otimes S$, and $(\cdot, \cdot)$, respectively $\| \cdot \|$, the associated global $L^2$-inner product, respectively $L^2$-norm.
Since $\mu : TM \otimes S \to S$ is onto, we have a canonical bundle isomorphism $TM \otimes S \simeq S \otimes \ker \mu$, implementable via the isomorphism $(\ker \mu)^\perp \overset{\mu}{\to} S$ induced by the Clifford multiplication $\mu$. By analogy with the spin manifold case we call $\ker \mu$ the bundle of generalized twistors and the differential operator $T : C^\infty(S) \to C^\infty(\ker \mu)$, $T = \text{proj}_{\ker \mu} \circ \nabla$, the generalized twistor operator.

Lemma 2.1. The inverse $I : S \to (\ker \mu)^\perp$ of the bundle isomorphism $(\ker \mu)^\perp \overset{\mu}{\to} S$ is given pointwise by the formula

$$I(s) = -\frac{1}{n} \sum_{i=1}^{n} e_i \otimes e_i \cdot s, \quad s \in S.$$  

Moreover,

$$|I(s)|^2 = \frac{1}{n} |s|^2, \quad s \in S,$$

$$\nabla s = I(Ds) + T(s), \quad s \in C^\infty(S),$$

$$|\nabla s|^2 = \frac{1}{n} |Ds|^2 + |T(s)|^2, \quad s \in C^\infty(S).$$

Proof. We have $I(s) = \sum_{i=1}^{n} e_i \otimes s_i$, for some $s_1, s_2, \ldots, s_n$ in $S$. Then $\sum_{i=1}^{n} e_i \cdot s_i = s$, since $\mu \circ I(s) = s$. Also, $e_j \cdot s_j = e_k \cdot s_k$, for any two indices $j$ and $k$, since $I(s)^\perp \ker \mu$ and $e_j \otimes e_j \cdot \sigma = e_k \otimes e_k \cdot \sigma \in \ker \mu$, for any $\sigma \in S$. Formula (2.2) follows. Using (2.2), (2.3) is now obvious.

It is clear that for $s \in C^\infty(S)$ there is $\sigma \in C^\infty(S)$ such that $\nabla s = I(\sigma) + T(s)$. Applying $\mu$ to this equation gives $Ds = \sigma$. Finally, (2.5) follows immediately from (2.3) and (2.4).

Fix now a function $f \in C^\infty(M, \mathbb{R})$ and assume that $s \in C^\infty(S)$ is an eigensection for $D$ with eigenvalue $\lambda$. Since $e^f \nabla(e^{-f}s) = -\text{grad} f \otimes s + \nabla s$ we have

$$|e^f \nabla(e^{-f}s)|^2 = |\text{grad} f|^2|s|^2 - \langle \nabla \text{grad} f, s \rangle - \langle s, \nabla \text{grad} f \rangle + \langle \nabla s, \nabla s \rangle.$$  

However, $\langle \nabla \text{grad} f, s \rangle + \langle s, \nabla \text{grad} f \rangle = \text{grad} f(|s|^2) = \langle df, d(|s|^2) \rangle$. Thus equations (2.5) and (2.6) imply

$$|\text{grad} f|^2|s|^2 - \langle df, d(|s|^2) \rangle + \langle \nabla s, \nabla s \rangle = \frac{1}{n} |e^f D(e^{-f}s)|^2 + |e^f T(e^{-f}s)|^2.$$  

Since $|e^f D(e^{-f}s)|^2 = |-\nabla f \cdot s + \lambda s|^2 = |\nabla f|^2 |s|^2 + \lambda^2 |s|^2$, (2.7) becomes

$$|e^f T(e^{-f}s)|^2 = \langle \nabla s, \nabla s \rangle - \lambda^2 \frac{n}{n-1} |s|^2 - \langle df, d(|s|^2) \rangle + \frac{n-1}{n} |\nabla f|^2 |s|^2.$$  (2.8)

Integrating now (2.8) over the manifold $M$ and using the facts that

$$||\nabla s||^2 = \langle \nabla s, \nabla s \rangle = \langle (D^2 - R) f, f \rangle \leq \langle (\lambda^2 - k) s, s \rangle,$$

one concludes that

$$\frac{n}{n-1} \min_M \left\{ k + \Delta f - \frac{n-1}{n} |\nabla f|^2 \right\}.$$  (2.9)

Equation (2.9) implies in particular that the real-valued function $\lambda^2 \frac{n}{n-1} - k - \Delta f + \frac{n-1}{n} |\nabla f|^2$ must be non-negative somewhere on $M$, i.e.,

$$\lambda^2 \geq \frac{n}{n-1} \min_M \left\{ k + \Delta f - \frac{n-1}{n} |\nabla f|^2 \right\}.$$  (2.10)

We proved therefore the following:

**Theorem 2.11.** For every smooth function $f : M \to \mathbb{R}$ and every eigenvalue $\lambda$ of the generalized Dirac operator $D$ the following inequality holds:

$$\lambda^2 \geq \frac{n}{n-1} \min_M \left\{ k + \Delta f - \frac{n-1}{n} |\nabla f|^2 \right\} \quad \square$$

Since $\frac{n-2}{n} < \frac{n}{n-1}$, Theorem 2.11 is weaker than Båå's result stated in Theorem 1.5. We included it here for its simple proof and because only a slight modification of it produces Båå's result. We proceed now to describe this modification.

The idea is to evaluate more generally, via (2.5), the norm of $e^{(\alpha+1) f} T(e^{-f} s)$ for some real constant $\alpha$. Working as in (2.6), (2.7), and (2.8), we get

$$|e^{(\alpha+1)f} \nabla (e^{-f}s)|^2 = e^{2\alpha f} |e^f \nabla (e^{-f}s)|^2 = (1 + 2\alpha) |\nabla f|^2 |e^{\alpha f}s|^2 -$$

$$\langle df, d(|e^{\alpha f}s|^2) \rangle + \langle \nabla s, \nabla (e^{2\alpha f}s) \rangle - \langle \nabla (e^{\alpha f}s), s \rangle.$$  (2.11)

$$|e^{(\alpha+1)f} (D(e^{-f}s))|^2 = e^{2\alpha f} \frac{1}{n} |e^f D(e^{-f}s)|^2 = \frac{1}{n} |\nabla f|^2 |e^{\alpha f}s|^2 + \lambda^2 |e^{\alpha f}s|^2.$$  (2.12)
\[(2.12) \quad |e^{\alpha f}T(e^{-f}s)|^2 = \langle \nabla s, \nabla (e^{2\alpha f}s) \rangle - \frac{\lambda^2}{n} |e^{\alpha f}s|^2 - \langle df, d(|e^{\alpha f}s|^2) \rangle + \left( \frac{n-1}{n} + 2\alpha \right) |\text{grad} f|^2 |e^{\alpha f}s|^2 - \langle \nabla \text{grad}(e^{\alpha f}s), s \rangle. \]

An inspection of (2.12) shows that in order to proceed as in (2.9) we need information about \( \langle \nabla \text{grad}(e^{\alpha f}s), s \rangle \).

**Lemma 2.13.**

\[ \langle \nabla \text{grad}(e^{\alpha f}s), s \rangle = \frac{1}{2} \left( \Delta (e^{\alpha f})s, s \right). \]

**Proof** By (1.3),

\[ D^2(e^{\alpha f}s) = \Delta (e^{\alpha f})s - 2\nabla \text{grad}(e^{\alpha f}s) + e^{2\alpha f}\lambda^2 s. \]

Since \( \langle D^2(e^{\alpha f}s), s \rangle = \left( e^{\alpha f}s, \lambda^2 s \right) \), the lemma follows. \( \square \)

Using the fact that \( \Delta (e^{\alpha f}) = e^{2\alpha f}(-4\alpha^2|\text{grad} f|^2 + 2\alpha \Delta f) \), (2.12) shows now that the equivalent of (2.9) is

\[ 0 \leq ||e^{(\alpha+1)f}T(e^{-f}s)||^2 \leq \int_M \left( \lambda^2 \frac{n-1}{n} - k - (1 + \alpha)\Delta f + \left( \frac{n-1}{n} + 2\alpha + 2\alpha^2 \right) |\text{grad} f|^2 \right) |e^{\alpha f}s|^2. \]

As before, we conclude that

\[ (2.14) \quad \lambda^2 \geq \frac{n}{n-1} \min_M \left\{ k + (1 + \alpha)\Delta f - \left( \frac{n-1}{n} + 2\alpha + 2\alpha^2 \right) |\text{grad} f|^2 \right\}. \]

Since (2.14) holds for arbitrary \( f \in C^\infty(M, \mathbb{R}) \) and \( \alpha \in \mathbb{R} \) we can replace \( f \) by \( \beta f, \beta = \frac{1}{1+\alpha}, \alpha \neq -1 \), to get, for every \( \beta \neq 0 \),

\[ (2.15) \quad \lambda^2 \geq \frac{n}{n-1} \min_M \left\{ k + \Delta f - \left( \frac{n-1}{n} \beta^2 - 2\beta + 2 \right) |\text{grad} f|^2 \right\}. \]

(2.15) is optimal when the quadratic polynomial in \( \beta, \frac{n-1}{n} \beta^2 - 2\beta + 2 \), reaches its minimum. This happens when \( \beta = \frac{n}{n-1} \), with minimum \( \frac{n-3}{n-1} \), in which case (2.15) becomes Bär's theorem.
References


N. ANGHEL
Department of Mathematics, University of North Texas
Denton, TX 76203, USA
E-mail: anghel@unt.edu