AN ABSTRACT INDEX THEOREM
ON NON-COMPACT RIEMANNIAN MANIFOLDS

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ABSTRACT. We prove an abstract index theorem for essentially self-adjoint Fredholm supersymmetric first-order elliptic differential operators on Hermitian vector bundles over complete oriented Riemannian manifolds. According to our main result the supersymmetric $L^2$-index of such an operator can be expressed as the sum of a "local contribution" (the familiar Atiyah-Singer index form, suitably restricted to and integrated over a finite region) and a "boundary contribution" (which depends only on the restriction of the operator at large distances). This is done by splicing together local parametrices and Green's operators defined "at infinity". The result yields (in fact is equivalent to) a generalisation of the relative index theorem of Gromov and Lawson.

1. Preliminaries

Let $S$ be a Hermitian vector bundle over a complete (non-compact) oriented Riemannian manifold $M$. The inner product and the norm on $L^2(S)$, the Hilbert space of $L^2$-integrable sections of $S$, will be denoted by $(\cdot, \cdot)$, respectively $\| \cdot \|$; they are induced as usual by the pointwise inner product $(\cdot, \cdot)$ on $S$ and the canonical volume form $d\text{vol}$ on $M$.

Let $D$ be a first-order formally self-adjoint elliptic differential operator on $C^\infty_0(S)$, the space of compactly supported smooth sections of $S$. We assume that $D$ is also essentially self-adjoint, i.e., its minimal (graph) and

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maximal (distributional) closures in the \( L^2 \)-space coincide. The unique closed extension of \( D \) to the \( L^2 \)-space will be also denoted by \( D \). Its domain, the first Sobolev space \( W^1(S) \), consists in sections \( s \in L^2(S) \) such that the distributional image \( Ds \) is in \( L^2(S) \). \( W^1(S) \) is a Hilbert space equipped with the inner product \( (\cdot, \cdot)_1 = (\cdot, \cdot) + (D^\cdot, D^\cdot) \). It is also the completion of \( C_0^\infty(S) \) with respect to the norm \( \| \cdot \|_1 \). Similarly the second Sobolev space \( W^2(S) \) will be the completion of \( C_0^\infty(S) \) with respect to the norm \( \| \cdot \|_2 \) induced by the inner product \( (\cdot, \cdot)_2 = (\cdot, \cdot) + (D^\cdot, D^\cdot) + (D^2^\cdot, D^2^\cdot) \). \( W^2(S) \) is also the minimal \( L^2 \)-closure of \( D^2 \) with domain \( C_0^\infty(S) \). We ignore whether \( D^2 \) is essentially self-adjoint, although this happens very often. The following symbol formula holds for \( D \):

\[
(1.1) \quad D(fs) = \sigma(df)s + fDs, \quad f \in C^\infty(M), \ s \in C^\infty(S),
\]

where \( \sigma(\xi) \in End(S_x), x \in M, \xi \in T^*_x M \) is the principal symbol map.

Such operators appear naturally in geometry and mathematical physics. For instance the generalized Dirac operators (and in particular all geometric operators) introduced by Gromov and Lawson in [8] are of this type. So, more generally, are the operators with finite propagation speed defined by Chernoff [6].

Finally, we assume that \( D \) is supersymmetric in the sense that there is a Hermitian involution \( \epsilon \in End(S) \) such that \( D\epsilon + \epsilon D = 0 \). Thus \( S = S^+ \oplus S^- \), \( S^\pm = \{ \xi = \pm 1 \} \), and so \( D \) is an odd operator with respect to this \( \mathbb{Z}_2 \)-grading of \( S \).

If \( D \), viewed as an unbounded operator in the \( L^2 \)-space, is a Fredholm operator then (just as in the compact case) it is of interest to characterize the \( L^2 \)-index of the restriction \( D^+ \) of \( D \) to \( S^+ \):

\[
L^2 \text{-} \text{index}(D^+) := \dim \{ s \in L^2(S^+) : Ds = 0 \} - \dim \{ s \in L^2(S^-) : Ds = 0 \},
\]

in purely local terms, involving the operator and the manifold. This is possible only to a certain extent, and to assess it we will make use of the existence of a good splicing of local parametrices for \( D \) onto Green's operators manufactured by restricting \( D \) to suitable ends of the manifold. Before embarking to this task we make some useful observations about Fredholm-ness.
2. Fredholmness

In this section we give equivalent characterizations for Fredholmness for the class of operators defined in the previous section. The descriptions (iii) and (iv) in the theorem below are new, as far as we know.

Theorem 2.1. The following statements about essentially self-adjoint first-order elliptic differential operators are equivalent:

(i) The bounded operator $D : W^1(S) \to L^2(S)$ is a Fredholm operator, i.e., it has closed range and finite dimensional kernel and cokernel.

(ii) If $H = \ker(D)$ and $H^\perp$ is the orthogonal complement of $H$ in $L^2(S)$, then $H$ is finite dimensional and there is a constant $c > 0$ such that

$$ ||Ds|| \geq c||s||, \quad s \in H^\perp \cap W^1(S). $$

(iii) There is a bounded positive (in the $L^2$-sense) operator $P : W^2(S) \to L^2(S)$ and a bundle morphism $R \in \text{End}(S)$, $R$ positive at infinity (i.e., there is a compact subset $K \subset\subset M$ and $k > 0$ such that pointwise on $S\setminus M \setminus K$, $R \geq k$), such that on $W^2(S)$,

$$ D^2 = P + R. $$

(iv) There is a constant $c > 0$ and a compact subset $K \subset\subset M$ such that

$$ ||Ds|| \geq c||s||, \quad s \in W^1(S), \quad \text{Supp}(s) \cap K = \emptyset. $$

Remark 2.2. It is clear that in (iii) and (iv) the respective Sobolev spaces can be replaced by $C_0^{\infty}(S)$.

Proof of Theorem 2.1.

(i) $\implies$ (ii) This implication is true for more general operators [10] and involves a standard use of the closed graph theorem.

(ii) $\implies$ (iii) Let $\{K_m\}_m$ be an exhaustion of $M$ by compact subsets. If $c$ is as in (ii), let $f_m \in C^{\infty}(M)$ be a real valued function such that $-c^2/2 \leq f_m \leq c^2/2$ and

$$ f_m = \begin{cases} c^2/2 & \text{on } K_m \\ -c^2/2 & \text{on } M \setminus K_{m+1}. \end{cases} $$
Obviously multiplication by \(-f_m\) provides a bundle morphism positive at infinity and we claim that there is an \(m\) such that \(P := D^2 + f_m\) is a bounded positive operator on \(W^2(S)\). Assume not. Then for any \(m\) there is a section \(s_m \in C_0^\infty(S)\), \(||s_m|| = 1\), with

\[
(2.3) \quad ((D^2 + f_m)s_m, s_m) < 0.
\]

Using the decomposition \(L^2(S) = H \oplus H^\perp\) we can find \(h_m \in H\), \(\sigma_m \in H^\perp\) such that \(s_m = h_m + \sigma_m\); \(\sigma_m \in W^1(S)\), since \(H \subset W^1(S)\). Also \(||h_m|| \leq 1\), \(||\sigma_m|| \leq 1\), for all \(m\). Now \(\{h_m\}_m\) is a bounded sequence in a finite dimensional Hilbert space; therefore it has a convergent subsequence, assumed to be the sequence itself. We have

\[
((D^2 + f_m)s_m, s_m) = ||D\sigma_m||^2 + (f_mh_m, h_m) + (f_m\sigma_m, h_m) + (f_mh_m, \sigma_m) + (f_m\sigma_m, \sigma_m),
\]

and since \(||D\sigma_m||^2 \geq c^2||\sigma_m||^2\) by (ii), it is fairly straightforward to check that if \(h_m \xrightarrow{L^2} h \in H\) then,

\[
\lim_m (f_mh_m, h_m) = \frac{c^2}{2}||h||^2, \quad \lim_m (f_m\sigma_m, h_m) = 0,
\]

and \((f_m\sigma_m, \sigma_m) \leq \frac{c^2}{2}\), for any \(m\).

As a result,

\[
\liminf_m ((D^2 + f_m)s_m, s_m) \geq \liminf_m c^2||\sigma_m||^2 + c^2/2 ||h||^2 - \limsup_m c^2/2 ||\sigma_m||^2.
\]

However, \(\lim_m ||\sigma_m||^2 = 1 - ||h||^2\), and so

\[
\liminf_m ((D^2 + f_m)s_m, s_m) \geq c^2/2 > 0,
\]

which contradicts 2.3.

(iii) \implies (iv) By hypothesis there is a decomposition \(D^2 = P + R\) on \(C_0^\infty(S)\), where \(P\) is a bounded positive operator and \(R \in \text{End}(S)\) is such that \(R \geq k\) pointwise, outside some compact subset \(K \subset M\), for some \(k > 0\). Let \(s \in C_0^\infty(S)\), \(\text{Supp} \ s \cap K = \emptyset\). Then

\[
||Ds||^2 = (D^2s, s) = (Ps, s) + (Rs, s) \geq (Rs, s) \geq k||s||^2.
\]
This proves (iv), taking $c = k^{1/2}$.

(iii) Assume (iv). We need to show only that $H := \ker(D)$ in finite dimensional and $D(W^1(S))$ is closed in $L^2(S)$, since $\coker(D) \cong \ker(D)$, as $D$ is self-adjoint.

Both these statements will follow at once from the following claim: if $s_m \in W^1(S)$ is $L^2$-bounded and $\{Ds_m\}_m$ is convergent in $L^2(S)$, then $\{s_m\}_m$ has a $L^2$-convergent subsequence.

To prove the claim fix a compact subset $L \subset M$ such that $K \subset \text{Int}(L)$ ($K$ being as in (iv)). Since $\{||s_m||_1\}$ is bounded, by Rellich's lemma [12] the sequence $\{s_m|_L\}_m$ has a convergent subsequence in $L^2(S|_L)$ (assumed as before to be the sequence itself). Choose now $f \in C^\infty(M)$, $0 \leq f \leq 1$, $f = 0$ on $K$, and $f = 1$ on $M \setminus L$. Then from 1.1 and (iv) for $s = fs_m$, we get

$$||\sigma(df)s_m + fD(s_m)|| = ||D(fs_m)|| \geq c||fs_m|| \geq c||s_m||_{M \setminus L},$$

where $||s_m||_{M \setminus L} = (\int_{M \setminus L} (s_m, s_m))^{1/2}$. Thus

$$c||s_m||_{M \setminus L} \leq |\sigma(df)|_{\infty} ||s_m||_{L \setminus L} + ||Ds_m||.$$ 

Clearly the above inequality holds for any difference $s_m - s_n$, which shows that the sequence $\{s_m|_{M \setminus L}\}_m$ is a Cauchy sequence in $L^2(S|_{M \setminus L})$. We conclude that $\{s_m\}_m$ converges in the $L^2$-norm, since its restrictions to $L$ and $M \setminus L$ converge. This proves the claim.

Suppose that $H := \ker(D)$ is not finite dimensional. Let then $\{h_m\}$ be an orthonormal basis for $H$. Since $||h_m|| = 1$ and $Dh_m = 0$ the claim applies and therefore a subsequence of $\{h_m\}_m$ converges. This is impossible, however, since $||h_m - h_n|| = 1$, $m \neq n$. Thus $\ker(D)$ is finite dimensional.

Let now $s_m \in W^1(S)$ be such that $\{Ds_m\}_m$ converges in $L^2(S)$. Without loss of generality we can assume that $s_m \in H^\perp$, $s_m \neq 0$. $\{s_m\}$ must be $L^2$-bounded. If not, $\{s_m/||s_m||\}$ is bounded and $D(s_m/||s_m||) \xrightarrow{L^2} 0$. Using the claim again we can conclude that (a subsequence of) $\{s_m/||s_m||\}_m$ is $L^2$-convergent to, say $s \in H^\perp$. As $D$ is closed, $Ds = 0$ or $s \in H$. Thus $s = 0$, a contradiction since $||s_m/||s_m|| = 1$. Therefore $\{s_m\}$ is bounded and one more time the claim shows that $\{s_m\}_m$ has $L^2$-limit points. If $s$ is such a limit point then $Ds_m \to Ds$, and so $D(W^1(S))$ is a closed subspace of $L^2(S)$. □
3. THE ABSTRACT INDEX THEOREM

Throughout this section $D$ will be an essentially self-adjoint Fredholm supersymmetric first-order elliptic differential operator defined on (sections of) a graded Hermitian vector bundle $(S, \varepsilon)$ over a complete oriented Riemannian manifold $M$.

The $L^2$-index of $D$, $L^2$-index$(D) := \dim L^2-\ker(D) - \dim L^2-\ker(D^*)$ certainly vanishes, since $D$ is essentially self-adjoint. However certain restrictions of $D$ may have interesting indices, in particular $D^+$, the restriction of $D$ to $S^+$. This section is devoted to exploring $L^2$-index$(D^+)$. The following lemma will get us started.

Lemma 3.1. Let $Q$ be a bounded operator from $L^2(S)$ into itself such that $DQ = 1 - R$ and $QD = 1 - R'$, with $R, R'$ (extendible to) bounded operators on $L^2(S)$. Assume that $R, R'$, and $DR'$ can be extended to trace class operators on $L^2(S)$. Then

$$L^2$-index$(D^+) = \frac{1}{2}(\text{trace}(\varepsilon R) + \text{trace}(\varepsilon R')).$$

Proof. The proof is an adjustment to the non-compact case of an argument given in [5] for the compact situation. We sketch it for the sake of completeness.

Let $G$ be the Green's operator associated to $D$ [5, 8], i.e., the bounded operator $G$ from $L^2(S)$ into itself, such that $DG = 1 - P_H$ and $GD = 1 - P_H$, where $P_H$ is the orthogonal projection onto $H$, the $L^2$-kernel of $D$. Strictly speaking the above equations define $G$ uniquely only on $H^\perp$. We take it to be 0 on $H$. If $Q, R, R'$ are as stated, then clearly $eG = -Ge$, $eP_H = P_H e$, $P_H = P_H R = R' P_H$, and $DR' = RD$. Thus,

$$\text{trace}(\varepsilon DR'G) = -\text{trace}(R'GD\varepsilon) = \text{trace}(\varepsilon P_H) - \text{trace}(\varepsilon R'),$$

$$\text{trace}(\varepsilon RDG) = \text{trace}(DG\varepsilon R) = \text{trace}(\varepsilon R) - \text{trace}(\varepsilon P_H).$$

Since $L^2$-index$(D^+) = \text{trace}(\varepsilon P_H)$, the lemma follows. □

The Green's operator of $D$ is a global example of an operator $Q$ as in the above lemma. What we seek, however, are examples where $Q, R, R'$, are (as much as possible) local, i.e., they are operators represented by
Schwartzian kernels depending only on the local expression of $D$. In such cases Lemma 3.1 gives a nontrivial realization of $L^2 - \text{index}(D^+)$. To construct such a $Q$ we start first with a local parametrix $Q_0$ for $D$ [2, 13, 8], i.e., a bounded pseudo-differential operator $Q_0 : L^2(S) \to L^2(S)$ such that

\begin{equation}
DQ_0 = 1 - R_0 \text{ and } Q_0 D = 1 - R_0',
\end{equation}

where $R_0$ and $R_0'$ are smoothing operators and where $Q_0$, $R_0$, and $R_0'$ all have Schwartzian kernels depending only on the local expression of $D$ and supported near the diagonal of $M \times M$. Of course $R_0$, $R_0'$ will not be trace class in general, as in the compact case. As a result, Lemma 3.1 does not apply directly to $Q_0$.

Next we are going to modify $Q_0$ outside a compact set by replacing it with a Green's operator associated to a suitable restriction of $D$. To this end let $K$ be a compact subset of $M$ and $c > 0$ such that Condition (iv) of Theorem 2.1 holds, i.e.,

\begin{equation}
||Ds|| \geq c||s||, \quad s \in C_0^\infty(S), \operatorname{Supp} s \cap K = \emptyset.
\end{equation}

For $\Omega := M \setminus K$ and $C_0^\infty(\Omega, S) := \{s \in C_0^\infty(S) : \operatorname{Supp} s \cap K = \emptyset\}$, let $D_\Omega$ denote the graph closure of the restriction of $D$ to $C_0^\infty(\Omega, S)$ in $L^2(\Omega, S)$, the $L^2$-closure of $C_0^\infty(\Omega, S)$ in $L^2(S)$. Clearly $D_\Omega$ is the restriction of $D$ to its domain, $\operatorname{dom}(D_\Omega)$. Also let $P_\Omega$ denote the orthogonal projection from $L^2(\Omega, S)$ onto $H_\Omega := \{s \in L^2(\Omega, S) : D|_\Omega(s|_\Omega) = 0 \text{ distributionally}\}$.

Lemma 3.4. The following two equations define uniquely a bounded operator $G_\Omega : L^2(\Omega, S) \to L^2(\Omega, S)$, the Green's operator of $D_\Omega$:

\begin{align*}
D_\Omega G_\Omega &= 1 - P_\Omega, \text{ on } L^2(\Omega, S) \\
G_\Omega D_\Omega &= 1, \text{ on } \operatorname{dom}(D_\Omega)
\end{align*}

Moreover, if $f \in C^\infty(M)$ is a bounded real valued function such that $f = 0$ on an open neighborhood of $K$, then $P_\Omega f$ defines a Hilbert-Schmidt operator on $L^2(S)$.

Proof. Equation 3.3 shows that $D_\Omega$ is one-to-one and has closed range, $\operatorname{ran}(D_\Omega)$, in $L^2(\Omega, S)$. We have the following direct orthogonal decomposition: $H_\Omega \oplus \operatorname{ran}(D_\Omega) = L^2(\Omega, S)$. Thus $G_\Omega$, defined to be $0$ on $H_\Omega$ and $D_\Omega^{-1}$
on \( \text{ran}(D_{\Omega}) \), has the properties mentioned above. The boundedness of \( G_{\Omega} \)
 is also a consequence of Equation 3.3. \( G_{\Omega} \) is an operator given by a locally
\( L^1 \)-Schwartzian kernel supported in \( \bar{\Omega} \times \bar{\Omega} \) which is smooth off the diagonal
of \( \Omega \times \Omega \) [8]. Likewise, \( P_{\Omega} \) is an integral operator with \( C^\infty \)-Schwartzian
kernel on \( \bar{\Omega} \times \Omega \) given by the Bergman kernel

\[
(3.5) \quad P_{\Omega}(x, y) = \sum_i s_i(x) \otimes s_i^t(y), \quad (x, y) \in \Omega \times \Omega,
\]

where \( \{s_i\}_i \) is any orthonormal basis of \( H_{\Omega} \) [5, 8]. The local trace function
of \( P_{\Omega}, \ p_{\Omega}(x):=\text{tr} P_{\Omega}(x, x) = \sum_i \langle s_i(x), s_i(x) \rangle, \ x \in \Omega, \)
has the following remarkable property [8]:

\[
(3.6) \quad \text{For any closed subset } \bar{\Omega}\text{ of } M \text{ such that } K \subset M \setminus \bar{\Omega}, \quad \int_{\bar{\Omega}} p_{\Omega}(x) \, d\nu(x) < \infty.
\]

Property 3.6 is a direct consequence of Equation 3.3, the symbol formula
1.1 for first-order differential operators, and the fact that the convergence
in Equation 3.5 is uniform (for any \( C^k \)-norm, \( k \geq 0 \), in fact) on the compact
subsets of \( \Omega \times \Omega \) [5].

For the second part of Lemma 3.4 it is enough to show that \( fP_{\Omega} : \ L^2(\Omega, S) \rightarrow L^2(\Omega, S) \) is a Hilbert-Schmidt operator. Clearly the Hilbert-Schmidt norm of \( fP_{\Omega} \) is given by \( \|fP_{\Omega}\|_{HS}^2 = \sum_i \|f s_i\|^2 \). However,

\[
\sum_i \|f s_i\|^2 \leq \sup_M |f|^2 \int_{\text{Suppf}} p_{\Omega}(x) \, d\nu(x) < \infty,
\]

by Property 3.6. \( \square \)

Assume now that the smooth function \( f \) of Lemma 3.4 has the additional
property of being \( 1 \) outside some compact set, and define on \( L^2(S) \)
the bounded operator \( \tilde{Q} \) by

\[
(3.7) \quad \tilde{Q} := Q_0(1 - f) + G_{\Omega} f.
\]

Theorem 3.8. There are Hilbert-Schmidt operators \( \tilde{R} \) and \( \tilde{R}' \) on \( L^2(S) \)
such that \( D \tilde{Q} = 1 - \tilde{R} \) and \( \tilde{Q} D = 1 - \tilde{R}' \). Moreover, \( D \tilde{R}' \) is also a Hilbert-
Schmidt operator, and in fact if \( P_{\Omega} \chi_{\Omega}, \tilde{\Omega} \) as in Equation 3.6, is a trace class
operator on \( L^2(S) \), then \( \tilde{R}, \tilde{R}', \) and \( D \tilde{R}' \) are all trace class operators.
Proof. The definition of $\tilde{Q}$ in Equation 3.7, coupled with Equation 3.2 and Lemma 3.4, shows that $D\tilde{Q} = 1 - \tilde{R}$, and $\tilde{Q}D = 1 - \tilde{R}'$, if $\tilde{R}$ and $\tilde{R}'$ are chosen to be

$$\tilde{R} := R_0(1 - f) + P_{\Omega}f$$
$$\tilde{R}' := R_0'(1 - f) + G_{\Omega} \sigma(df) - Q_0 \sigma(df).$$

Now $R_0(1 - f)$ and $R_0'(1 - f)$ are trace class operators on $L^2(S)$, just as in the compact case, since they are given by smooth compactly supported Schwartzian kernels. Thus, invoking Lemma 3.4, $\tilde{R}$ is a Hilbert-Schmidt operator. Moreover, if $Q_0$ is chosen to have narrow enough support, $Q_0 \sigma(df)$ has range contained in $L^2(\Omega, S)$ and so Equation 3.3 implies that

$$||D(G_{\Omega} \sigma(df) - Q_0 \sigma(df))s|| \geq ||(G_{\Omega} \sigma(df) - Q_0 \sigma(df))s||, \quad s \in L^2(S),$$

or equivalently,

$$(3.9) \quad ||(R_0 \sigma(df) - P_{\Omega} \sigma(df))s|| \geq ||(G_{\Omega} \sigma(df) - Q_0 \sigma(df))s||, \quad s \in L^2(S).$$

Since $R_0 \sigma(df) - P_{\Omega} \sigma(df)$ is a Hilbert-Schmidt operator, so is $G_{\Omega} \sigma(df) - Q_0 \sigma(df)$, and with it $\tilde{R}'$. Clearly

$$(3.10) \quad D\tilde{R}' = DR_0'(1 - f) + R_0 \sigma(df) - P_{\Omega} \sigma(df)$$

is a Hilbert-Schmidt operator too.

Now, if $P_{\Omega} \chi_{\tilde{S}}$ is of trace class, then so are $P_{\Omega}f$ and $P_{\Omega} \sigma(df)$. Using Equations 3.9 and 3.10, we conclude that $\tilde{R}$, $\tilde{R}'$, and $D\tilde{R}'$ are all trace class operators. \qed

Remark 3.11. According to the above theorem, Lemma 3.1 applies to $\tilde{Q}$, $\tilde{R}$, $\tilde{R}'$, if $P_{\Omega} \chi_{\tilde{S}}$ is a trace class operator. We are not aware of any situation where $P_{\Omega} \chi_{\tilde{S}}$ fails to be so.

Next we come to our main result in this section.

Theorem 3.12. Let $D$ be an essentially self-adjoint Fredholm supersymmetric first-order elliptic differential operator defined on a $\mathbb{Z}_2$-graded Hermitian vector bundle $(S, \epsilon)$ over a complete oriented Riemannian manifold
Let $K$ and $L$ be two compact subsets of $M$ such that the Fredholmness Condition (iv) of Theorem 2.1 above holds for $K$, and $K \subset \text{Int}(L)$. Let $f \in C^\infty(M)$ be a real valued function $= 0$ on an open neighborhood of $K$ and $= 1$ on an open neighborhood of $M \setminus \text{Int}(L)$. Then

$$L^2 - \text{index}(D^+) = \int_M \omega(x)(1 - f(x)) \, d\text{vol}(x) + I_\Omega,$$

where $\omega \, d\text{vol}$ is a density manufactured out of the local expression of $D$ and $I_\Omega$ is a "boundary" contribution which depends only on the restriction of $D$ (and $f$) to $\Omega = M \setminus K$ (both $\omega$ and $I_\Omega$ to be accurately described in the body of the proof).

**Proof.** If $P_\hbar \chi_R$ is a trace class operator on $L^2(S)$ then the result follows from Theorem 3.8 and Lemma 3.1. Indeed we have

$$L^2 - \text{index}(D^+) = \text{trace} \frac{1}{2}(\epsilon R_0 + \epsilon R_0')(1 - f) + \frac{1}{2} \text{trace} \epsilon P_\hbar f + \frac{1}{2} \text{trace} \epsilon W,$$

where $W := G_\Omega \sigma(df) - Q_0 \sigma(df)$. However,

$$\text{trace} \frac{1}{2}(\epsilon R_0 + \epsilon R_0')(1 - f) = \int_M \omega(x)(1 - f(x)) \, d\text{vol}(x),$$

where $\omega(x)$ stands for the local quantity

$$\omega(x) := \frac{1}{2} \text{tr} (\epsilon(x) R_0(x,x) + \epsilon(x) R_0'(x,x)),$$

and $\text{tr}$ denotes the pointwise trace. Also, cutting further the support of $Q_0$ if necessary, it is easily seen that $W$ depends only on the restriction of $D$ to $\Omega$. Therefore,

$$I_\Omega := \frac{1}{2} \text{trace} \epsilon P_\hbar f + \frac{1}{2} \text{trace}(\epsilon G_\Omega \sigma(df) - \epsilon Q_0 \sigma(df))$$

depends only on the restriction of $D$ to $\Omega$. Particular examples where $M$ have well understood ends (e.g., cylindrical or conical ends) show that $\frac{1}{2} \text{trace} P_\hbar f$ represents a regularized eta invariant [3] and $\frac{1}{2} \text{trace}(\epsilon G_\Omega \sigma(df) - \epsilon Q_0 \sigma(df))$ is a "small" contribution due to the presence of the cut-off function $f$. 
If \( P_n \chi_\Omega \) is not a trace class operator, then we use a construction inspired by [9]. Set

\[
Q := (1 + W)\tilde{Q} = (1 + \tilde{R} - R'_0(1 - f))\tilde{Q}.
\]

On one hand, using \( Q = (1 + W)\tilde{Q} \) and Theorem 3.8, we get \( DQ = 1 - R \) and \( QD = 1 - R' \), where

\[
R = R_0(1 - f) + P_n f - (R_0 \sigma(df) - P_n \sigma(df))(Q_0(1 - f) + G_n f)
\]

\[
R' = R'_0(1 - f) + (G_n \sigma(df) - Q_0 \sigma(df'))^2 +
(G_n \sigma(df) - Q_0 \sigma(df'))R'_0(1 - f).
\]

On the other hand, using \( Q = (1 + \tilde{R} - R'_0(1 - f))\tilde{Q} \), we obtain for \( R \) and \( R' \), and \( DR' \), the equivalent expressions,

\[
R = \tilde{R}^2 + DR'_0(1 - f)\tilde{Q}
\]

\[
R' = \tilde{R}'^2 - R'_0(1 - f)(1 - \tilde{R}')
\]

\[
DR' = D\tilde{R}' \tilde{R}' - DR'_0(1 - f)(1 - \tilde{R}').
\]

Now Equations 3.14 show that \( R, R' \), and \( DR' \) are all trace class operators, since \( R'_0(1 - f) \) and \( DR'_0(1 - f) \) are trace class operators, \( \tilde{R}, \tilde{R}' \), and \( D\tilde{R}' \) are Hilbert-Schmidt operators, and \( \tilde{Q} \) and \( 1 - \tilde{R}' \) are bounded operators. Equations 3.13 show that \( R - R_0(1 - f) \) and \( R' - R'_0(1 - f) \), depend only on the restriction of \( D \) (and \( f \)) to \( \Omega \); furthermore, invoking the equivalence of Equations 3.13 and 3.14 one more time, they are trace class operators too. If we set

\[
I_\Omega := \frac{1}{2} \text{trace} (\epsilon P_n f - \epsilon(R_0 \sigma(df) - P_n \sigma(df))\tilde{Q}) +
\]

\[
\frac{1}{2} \text{trace} (\epsilon W^2 + \epsilon W R'_0(1 - f)),
\]

the theorem follows. \( \Box \)

In the above theorem the local density \( \omega d\text{vol} \) has little practical value. It is desirable, of course, to be able to replace it by better understood local densities, such as the top degree component of the Atiyah-Singer index form \( ch \sigma(D^+) \wedge \tau(M) \) [4]. We are going to do just that via the following generalization of the relative index theorem of Gromov and Lawson [8,7].
Corollary 3.15. (The relative index theorem) Let $D_j$, $j = 1, 2$, be two essentially self-adjoint Fredholm supersymmetric first-order elliptic differential operators defined on $\mathbb{Z}_2$-graded Hermitian vector bundles $(S_j, e_j)$, over complete oriented Riemannian manifolds $M_j$. Assume also that the two operators agree at infinity i.e., there are compact sets $K_j \subset M_j$, $j = 1, 2$, and an orientation preserving isometry $F : M_1 \setminus K_1 \to M_2 \setminus K_2$ which is covered by a bundle isometry $\bar{F} : S_1|_{M_1 \setminus K_1} \to S_2|_{M_2 \setminus K_2}$ so that on $M_1 \setminus K_1$, $\bar{F}e_1 = e_2\bar{F}$ and $D_1 = \bar{F}^{-1} \circ D_2 \circ \bar{F}$. Then

$$L^2 - \text{index}(D_j^+) - L^2 - \text{index}(D_j^+) =$$

$$\int_{M_1} \text{ch} \sigma(D_j^+) \wedge \tau(M_1) - \int_{M_2} \text{ch} \sigma(D_j^+) \wedge \tau(M_2),$$

where the right hand side of the above equation is to be interpreted appropriately.

**Proof.** We can choose $K_j$, $j = 1, 2$, as stated in the corollary and such that Condition 3.10 is satisfied for $D_j$. Next we identify $M_1 \setminus K_1$ and $M_2 \setminus K_2$; let $\Omega := M_1 \setminus K_1 \setminus K_2$. By Theorem 3.12, for $j = 1, 2$,

$$L^2 - \text{index}(D_j^+) = \int_{M_j} \omega_j(1 - f_j(\omega)) d \text{vol}(\omega) + I^j_\Omega.$$

Since $D_1|\Omega = D_2|\Omega$ and $f_1|\Omega = f_2|\Omega$, we conclude that $\omega_1|\Omega = \omega_2|\Omega$ and $I^j_\Omega = I^2$. Hence

$$L^2 - \text{index}(D_j^+) - L^2 - \text{index}(D_j^+) =$$

$$\int_{M_1} \omega_1(1 - f_1) d \text{vol} - \int_{M_2} \omega_2(1 - f_2) d \text{vol} = \int_{K_1} \omega_1 d \text{vol} - \int_{K_2} \omega_2 d \text{vol}.$$

Now the passage from $\omega$'s to the Atiyah-Singer index forms will be achieved as in [8] by means of the Atiyah-Singer index theorem, applied to a suitable compact manifold.

Let $N \subset \Omega$ be a compact hypersurface which separates off the infinite part of $\Omega$, i.e., there is a constant $k > 0$ so that every point in $\Omega \setminus N$ is at distance $\leq k$ from $K_j$ in $M_j$, or cannot be connected to $K_j$ by a path in $M_j \setminus N$, for $j = 1, 2$. Deform now the (identical) Riemannian metrics,
bundles and operators in a small tubular neighborhood of \( N \) in \( \Omega \) so that the metric becomes a Riemannian product on \((-\epsilon, \epsilon) \times N \), so that the bundle \( S_1 = S_2 \) becomes \((-\epsilon, \epsilon) \times S_N, S_N \) the restriction of \( S \) to \( N \), and so that the operator \( D_1 = D_2 \) becomes \( \sigma(dt)(0, x) \partial/\partial t + D_N(x), t \in (-\epsilon, \epsilon), x \in N, D_N \) the restriction of \( D_0 = D_1 \) to \( N \). (This is always possible!). After these alterations the operators continue to be Fredholm and to agree on \( \Omega \). Let \( M'_j, j = 1, 2 \), be the compact manifold with boundary obtained by deleting the connected component(s) of \( M_j \setminus N \) contained in \( \Omega \). Next glue \( M'_1 \) and \( M'_2 \) along \( N \) to get a compact oriented Riemannian manifold

\[ \widetilde{M} := M'_1 \cup_N (-M'_2). \]

(\( \widetilde{M} \) has the orientation of \( M_1 \) and the opposite orientation of \( M_2 \).) The altered operators \( D_1 \) and \( D_2 \) join naturally to give a supersymmetric elliptic differential operator \( \widetilde{D} \) on the compact manifold \( \widetilde{M} \). By the Atiyah-Singer index theorem [4],

\[
\text{index}(\widetilde{D}^+) = \int_{M'_1} \text{ch}(D_1^+) \wedge \tau(M_1) - \int_{M'_2} \text{ch}(D_2^+) \wedge \tau(M_2)
\]

\[
= \int_{K_1} \text{ch}(D_1^+) \wedge \tau(M_1) - \int_{K_2} \text{ch}(D_2^+) \wedge \tau(M_2),
\]

since \( M'_1 \setminus K_1 = M'_2 \setminus K_2 \), and the altered operators coincide there. However, using local parametrices we also see that

\[
\text{index}(\widetilde{D}^+) = \int_{M'_1} \omega_1 \ d\text{vol} - \int_{M'_2} \omega_2 \ d\text{vol} = \int_{K_1} \omega_1 \ d\text{vol} - \int_{K_2} \omega_2 \ d\text{vol}.
\]

Thus,

\[
\int_{K_1} \omega_1 d\text{vol} - \int_{K_2} \omega_2 d\text{vol} = \int_{K_1} \text{ch}(D_1^+) \wedge \tau(M_1) - \int_{K_2} \text{ch}(D_2^+) \wedge \tau(M_2).
\]

Defining

\[
\int_{M'_1} \text{ch}(D_1^+) \wedge \tau(M_1) - \int_{M'_2} \text{ch}(D_2^+) \wedge \tau(M_2) := \int_{K_1} \text{ch}(D_1^+) \wedge \tau(M_1) - \int_{K_2} \text{ch}(D_2^+) \wedge \tau(M_2),
\]
justifiably so since $D_1|_{\Omega} = D_2|_{\Omega}$, the corollary follows. □

Remark 3.16. If the two operators in Corollary 3.15 agree only on some ends it is possible to get a partial relative index theorem, just as in [8, Theorem 4.35]. The proper statement and the details of the proof are left to the reader.

Remark 3.17. The relative index theorem stated above can sometimes be used to derive very concrete index theorems. For an example involving Callias-type operators, see [1].

Corollary 3.18. (The abstract index theorem) Let $D$ be an essentially self-adjoint Fredholm supersymmetric first-order elliptic differential operator defined on a $\mathbb{Z}_2$-graded Hermitian vector bundle $(S, \varepsilon)$ over a complete oriented Riemannian manifold $M$. Let $K$ be a compact subset of $M$ such that the Fredholmness Condition (iv) of Theorem 2.1 holds for $K$. Then

$$L^2 - \text{index}(D^+)_\Omega = \int_K \text{ch} \sigma(D^+) \wedge \tau(M) + J_\Omega,$$

where $\text{ch} \sigma(D^+) \wedge \tau(M)$ is the Atiyah-Singer index form, manufactured out of the local expression of $D$ and the local geometry of $M$ and $J_\Omega$ is a "boundary" contribution which depends only on the restriction of $D$ to $\Omega = M \setminus K$.

Proof. Define $J_\Omega := L^2 - \text{index}(D^+)_\Omega - \int_K \text{ch} \sigma(D^+) \wedge \tau(M)$. Then Corollary 3.15 (The relative index theorem) shows that $J_\Omega$ is independent of the local expression of $D$ on $K$, i.e., it depends only on the restriction of $D$ to $\Omega$. □

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