EQUIVALENCE CLASSES OF SUBQUOTIENTS OF PSEUDODIFFERENTIAL OPERATOR MODULES ON THE LINE

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Certain subquotients of Vec($\mathbf{R}$)-modules of pseudodifferential operators from one tensor density module to another are categorized, giving necessary and sufficient conditions under which two such subquotients are equivalent as Vec($\mathbf{R}$)-representations. These subquotients split under the projective subalgebra, a copy of $\mathfrak{sl}_2$, when the members of their composition series have distinct Casimir eigenvalues. Results were obtained using the explicit description of the action of Vec($\mathbf{R}$) with respect to this splitting.

In the length five case, the equivalence classes of the subquotients are determined by two invariants. In an appropriate coordinate system, the level curves of one of these invariants are a pencil of conics, and those of the other are a pencil of cubics.

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CHAPTER 1

INTRODUCTION

Let $\mathrm{Diff}^k_{\lambda,\lambda+p}$ be the $\mathrm{Vec}\,\mathbb{R}$-module of differential operators of order $\leq k$ from $F(\lambda)$ to $F(\lambda+p)$, where $F(\lambda)$ and $F(\lambda+p)$ are the tensor density modules of $\mathrm{Vec}\,\mathbb{R}$ with lowest weights λ and $\lambda+p$ respectively. Its quotient by $\mathrm{Diff}^{k-l}_{\lambda,\lambda+p}$ is of length l and has composition series

$$F(p-k), F(p-k+1), \ldots, F(p-k+l-1).$$

We refer to this quotient as $\mathrm{SQ}^{k,l}_{\lambda,p}$. It splits under the projective subalgebra $\mathfrak{sl}_2$ if its composition series has distinct Casimir eigenvalues. For $l \leq 5$, we will state necessary and sufficient conditions under which two such quotients $\mathrm{SQ}^{k,l}_{\lambda,p}$ and $\mathrm{SQ}^{k',l}_{\lambda',p'}$ are equivalent as $\mathrm{Vec}\,\mathbb{R}$-modules. Our results follow from the action of $\mathrm{Vec}\,\mathbb{R}$ with respect to the projective splitting. They extend to pseudodifferential operators: k can be taken to be an element of $\mathbb{C}$.

This problem was first considered by Lecomte and Ovsienko [9]. They treated only the case $p = 0$, where equivalence requires $k = k'$. They found that $\mathrm{SQ}^{k,l}_{\lambda,0}$ and $\mathrm{SQ}^{k,l}_{\lambda',0}$ are generically equivalent for $l \leq 4$, and equivalent only in the conjugate case $\lambda' = l - \lambda$ for $l \geq 5$.

When p is allowed to vary, equivalence requires $p - k = p' - k'$. Here the length 5 case exhibits a new phenomenon: the equivalence class of $\mathrm{SQ}^{k,5}_{\lambda,p}$ depends only on two invariants in (p, λ)-space, called I_q and I_c. After a suitable change of coordinates, the level of curves of I_q are a pencil of conics determined by four points, and those of I_c are a pencil of cubics determined by nine points. We also investigate a certain type of "lacunary" subquotient of $\mathrm{SQ}^{k,5}_{\lambda,p}$ with composition series $F(p-k)$, $F(p-k+2)$, $F(p-k+4)$. The equivalence classes of these subquotients are determined solely by I_q.

CHAPTER 2

TENSOR DENSITY MODULES OF THE LIE ALGEBRA $\mathrm{Vec}\,\mathbb{R}$

2.1. Definitions

$\mathrm{Vec}\,\mathbb{R}$ is the infinite-dimensional Lie algebra of polynomial vector fields on the line. In this section we give the basic vocabulary necessary to discuss representations of $\mathrm{Vec}\,\mathbb{R}$, define the projective subalgebra of $\mathrm{Vec}\,\mathbb{R}$, a copy of $\mathfrak{sl}_2$, and define the Casimir operator of a representation of the projective subalgebra.

Definition. $\mathrm{Vec}\,\mathbb{R} := \big\{ fD : f \in \mathbb{C}[x] \big\}$.

Here D is the usual derivative, $\frac{d}{dx}$. The algebra operation is the usual Lie bracket:

$$[fD, gD] = fD \circ gD - gD \circ fD = (fg' - gf')D.$$

A basis of $\mathrm{Vec}\,\mathbb{R}$ consists of the elements $e_n := x^{n+1}D$, $n \geq -1$. We have

$$[e_n, e_m] = (m-n)e_{m+n}.$$

We now define the projective subalgebra $\mathfrak{a}$ of $\mathrm{Vec}\,\mathbb{R}$, which will prove useful for our computations. The following lemma is clear.

Lemma 2.1. *The space* $\mathfrak{a} := \mathrm{Span}_{\mathbb{C}}\{e_{-1}, e_0, e_1\}$ *is a subalgebra of* $\mathrm{Vec}\,\mathbb{R}$ *isomorphic to* $\mathfrak{sl}_2$.

Definition. Let π be a representation of $\mathrm{Vec}\,\mathbb{R}$ on a space V. Eigenvectors of $\pi(e_0)$ of eigenvalue λ are called **λ-weight vectors**. If they belong to the kernel of $\pi(e_{-1})$, then they are called **lowest weight vectors** of weight λ.

Definition. Let (π, V) be a representation of $\mathfrak{a}$. The Casimir operator of π is defined to be

$$\pi(Q) := \pi(xD)^2 + \pi(xD) - \pi(D)\pi(x^2D).$$

Proposition 2.2. *Let* (π, V) *be a representation of* $\mathrm{Vec}\,\mathbb{R}$. *The Casimir operator commutes with the action of* $\mathfrak{sl}_2$ *on* V.

Proof. A straightforward computation shows that for any $k \geq -1$,

$$[Q, e_k] = (2e_0 - k - 1)ke_k - (k-1)e_{k+1}e_{-1} - (k+1)e_1 e_{k-1}.$$

Taking $k = 0$, 1, and -1 gives the result. □

2.2. Tensor Density Modules

The tensor density modules of $\operatorname{Vec}\mathbb{R}$ are deformations of the basic module $\mathbb{C}[x]$. It was conjectured by Kostrikin [8] and proven by Mathieu [11] that these modules, along with their localizations, comprise all the irreducible representations of $\operatorname{Vec}\mathbb{R}$.

Definition. For λ in $\mathbb{C}$, define $\pi_\lambda : \operatorname{Vec}\mathbb{R} \to \operatorname{End}\mathbb{C}[x]$ by

$$\pi_\lambda(fD)g = fg' + \lambda f'g, \quad \text{that is,} \quad \pi_\lambda(fD) = fD + \lambda f'.$$

To denote $\mathbb{C}[x]$ under the action π_λ, we use the notation $F(\lambda) := dx^\lambda \mathbb{C}[x]$.

Lemma 2.3. *For all λ in $\mathbb{C}$, $\big(\pi_\lambda, F(\lambda)\big)$ is a $\operatorname{Vec}\mathbb{R}$-representation with*

$$\pi_\lambda(e_k)(dx^\lambda x^n) = \big(n + \lambda(k+1)\big)dx^\lambda x^{k+n}.$$

Proof. Simply check that π_λ preserves the Lie bracket. □

Lemma 2.4. *$F(\lambda)$ has weights $\lambda, \lambda+1, \lambda+2, \ldots,$ corresponding to the weight vectors dx^λ, $dx^\lambda x$, $dx^\lambda x^2$, …, respectively. Up to a scalar, its unique lowest weight vector is dx^λ.*

Proof. The first sentence is clear. The second follows from $\pi_\lambda(e_{-1})dx^\lambda f = dx^\lambda f'$. □

Lemma 2.5. *The Casimir operator of $F(\lambda)$ is the scalar $\lambda^2 - \lambda$.*

Proof. Apply the definition of the Casimir operator to obtain the result. □

Proposition 2.6. *For all $\lambda \neq 0$, $F(\lambda)$ is an irreducible representation of $\operatorname{Vec}\mathbb{R}$. The only proper non-trivial $\operatorname{Vec}\mathbb{R}$-subrepresentation of $F(0)$ is $\mathbb{C}$.*

Proof. If W is a non-trivial subrepresentation of $F(\lambda)$ with $\lambda \neq 0$, then applying $\pi_\lambda(e_{-1})$ repeatedly to a non-zero element of W yields dx^λ, and applying $\pi_\lambda(e_1)$ and $\pi_\lambda(e_2)$ repeatedly to dx^λ yields all basis elements $dx^\lambda x^k$. Thus W is equal to $F(\lambda)$.

It is clear that $\mathbb{C}$ is a subrepresentation of $F(0)$. If W is a non-trivial subrepresentation of $F(0)$ containing a non-constant element f, then f generates both 1 and x under $\pi_0(e_{-1})$, and x generates the rest of $\mathbb{C}[x]$ under $\pi_0(e_1)$. □

LEMMA 2.7. *$\big(\pi_{-1}, F(-1)\big)$ is equivalent to $(\mathrm{ad}, \mathrm{Vec}\,\mathbb{R})$ as a $\mathrm{Vec}\,\mathbb{R}$-representation.*

PROOF. The reader may check that $gD \mapsto dx^{-1}g$ is an equivalence. □

LEMMA 2.8. *For $\lambda \neq \mu$, $\big(\pi_\lambda, F(\lambda)\big)$ and $\big(\pi_\mu, F(\mu)\big)$ are not $\mathrm{Vec}\,\mathbb{R}$-equivalent.*

PROOF. This is immediate from the fact that $F(\lambda)$ and $F(\mu)$ have different weights. □

2.3. The Tensor Density Modules Under the Projective Subalgebra

PROPOSITION 2.9. *For λ not in $-\frac{1}{2}\mathbb{N}$, $\big(\pi_\lambda, F(\lambda)\big)$ is irreducible under the action of $\mathfrak{a}$. For λ in $-\frac{1}{2}\mathbb{N}$, the only non-trivial $\mathfrak{a}$-subrepresentation of $\big(\pi_\lambda, F(\lambda)\big)$ is the space*

$$L(\lambda) := \mathrm{Span}_{\mathbb{C}} \left\{ dx^\lambda x^k : 0 \leq k \leq -2\lambda \right\}.$$

PROOF. Recall that the kernel of $\pi_\lambda(e_{-1})$ is $\mathbb{C}dx^\lambda$. Since $\pi_\lambda(e_1)dx^\lambda x^k = (k+2\lambda)dx^\lambda x^{k+1}$, the kernel of $\pi_\lambda(e_1)$ is $\mathbb{C}dx^\lambda x^{-2\lambda}$ for $\lambda \in -\frac{1}{2}\mathbb{N}$, and zero otherwise. The result follows. □

2.4. Intertwining Maps

In this section we classify the $\mathrm{Vec}\,\mathbb{R}$- and $\mathfrak{a}$-intertwining maps between tensor density modules.

LEMMA 2.10. *Under $\mathrm{Vec}\,\mathbb{R}$, $F(0)/\mathbb{C}$ is equivalent to $F(1)$.*

PROOF. The reader may check that $f \mapsto dxf'$ is a surjective $\mathrm{Vec}\,\mathbb{R}$-map from $F(0)$ to $F(1)$ with kernel $\mathbb{C}$. The result follows. □

PROPOSITION 2.11. *If $\epsilon : F(\lambda) \to F(\mu)$ is a $\mathrm{Vec}\,\mathbb{R}$-intertwining map, then either $\lambda = \mu$ and ϵ is a scalar, or $\lambda = 0$, $\mu = 1$, and $\epsilon \in \mathbb{C}dxD$.*

PROOF. Assume that $\epsilon : F(\lambda) \to F(\mu)$ is a non-trivial $\mathrm{Vec}\,\mathbb{R}$-intertwining map. Let k_0 be minimal such that $\epsilon(dx^\lambda x^{k_0}) \neq 0$. Since $\epsilon(dx^\lambda x^{k_0})$ is an element of the kernel of $\pi_\mu(e_{-1})$, it is $c dx^\mu$ for some $c \neq 0$. Since ϵ preserves weights, we have $\mu = \lambda + k_0$, thus $\mu - \lambda \in \mathbb{N}$.

If $\mu = \lambda \neq 0$, then $F(\lambda)$ is irreducible by Proposition 2.6, so Schur's lemma implies that ϵ is scalar. If $\mu = \lambda = 0$, then k_0 is zero and $\epsilon - c$ annihilates 1, so we find that $\epsilon = c$. If $\mu \neq \lambda$, then $k_0 \geq 1$, so the kernel of ϵ is nontrivial. Therefore by Proposition 2.6, $\lambda = 0$, $\operatorname{Ker} \epsilon = \mathbb{C}$, and $\mu = 1$. In this case ϵ and $f \mapsto cdxf'$ agree on x, so an easy argument shows that they agree everywhere. $\square$

Proposition 2.12. *If $\epsilon : F(\lambda) \to F(\mu)$ is an $\mathfrak{a}$-intertwining map, then either $\lambda = \mu$ and ϵ is a scalar, or $\lambda \in -\frac{1}{2}\mathbb{N}$, $\mu = 1 - \lambda$, and ϵ is a multiple of $\mathbb{C}dx^{1-2\lambda}D^{1-2\lambda}$.*

Proof. Assume that $\epsilon : F(\lambda) \to F(\mu)$ is a non-trivial $\mathfrak{a}$-intertwining map. Let k_0 be minimal such that $\epsilon(dx^\lambda x^{k_0})$ is not zero. Since $\epsilon(dx^\lambda x^{k_0})$ is an element of the kernel of $\pi_\mu(e_{-1})$, it is cdx^μ for some nonzero scalar c in $\mathbb{C}$. Since ϵ preserves weights, we have $\mu = \lambda + k_0$, so $\mu - \lambda \in \mathbb{N}$.

If $\mu = \lambda \notin -\frac{1}{2}\mathbb{N}$, then $F(\lambda)$ is irreducible by Proposition 2.9, so Schur's lemma implies that ϵ is scalar. If $\mu = \lambda \in -\frac{1}{2}\mathbb{N}$, then k_0 is 0 and $\epsilon - c$ annihilates 1. In this case also we obtain $\epsilon = c$. If $\mu \neq \lambda$, then $k_0 \geq 1$ and the kernel of ϵ is nontrivial. Therefore by Proposition 2.9, $\lambda \in -\frac{1}{2}\mathbb{N}$ and $\operatorname{Ker} \epsilon = L(\lambda)$, so $k_0 = -2\lambda + 1$, and $\mu = 1 - \lambda$. The reader may check that $dx^{1-2\lambda}D^{1-2\lambda} : F(\lambda) \to F(1 - \lambda)$ is an $\mathfrak{a}$-map. It follows easily that ϵ is a multiple thereof.

$\square$

Corollary 2.13. *For $\lambda \in -\frac{1}{2}\mathbb{N}$, $F(\lambda)/L(\lambda)$ is $\mathfrak{a}$-equivalent to $F(1 - \lambda)$.*

By Lemma 2.7, we have in particular the following corollary.

Corollary 2.14. *$\big(\mathrm{ad}, \operatorname{Vec}\mathbb{R}/\mathfrak{a}\big)$ is $\mathfrak{a}$-equivalent to $\big(\pi_2, F(2)\big)$ as a $\operatorname{Vec}\mathbb{R}$ representation.*

CHAPTER 3

EXTENSIONS OF THE TENSOR DENSITY MODULES

3.1. $\mathrm{Hom}_{\mathbb{C}}\big(F(\lambda), F(\mu)\big)$

Let us write $\mathrm{Hom}_{\lambda,\mu}$ for the space $\mathrm{Hom}_{\mathbb{C}}\big(F(\lambda), F(\mu)\big)$, and $h_{\lambda,\mu}$ for the natural action of $\mathrm{Vec}\,\mathbb{R}$ on it:

$$h_{\lambda,\mu}(X)T := \pi_\mu(X) \circ T - T \circ \pi_\lambda(X).$$

The following well-known lemma is convenient for calculations. We leave its proof to the reader.

LEMMA 3.1. $\mathrm{Hom}_{\lambda,\mu} = dx^{\mu-\lambda}\mathbb{C}[x][[D]]$. *In particular,* $h_{\lambda,\mu}(e_{-1})$ *acts surjectively on* $\mathrm{Hom}_{\lambda,\mu}$ *with kernel* $dx^{\mu-\lambda}\mathbb{C}[[D]]$.

The following proposition is also well-known. It may be deduced from the fact that, regarding multiplication by g as an order 0 differential operator, we have

$$D^j \circ g = \sum_{i=0}^{j} \binom{j}{i} g^{(i)} D^{j-i}.$$

PROPOSITION 3.2. *Let* λ *and* p *be in* $\mathbb{C}$, *and let* f *and* g *be in* $\mathbb{C}[x]$. *For any* j *in* $\mathbb{N}$,

$$h_{\lambda,\lambda+p}(gD)(dx^p f D^j) = dx^p\Bigg(\big(gf' + (p-j)g'f\big)D^j - f\sum_{i=1}^{j}\binom{j}{i}\left(\lambda + \frac{j-i}{i+1}\right) g^{(i+1)} D^{j-i}\Bigg).$$

3.2. 1-cohomology

Here we recall some standard facts from Lie algebra cohomology; see for example the book by Guichardet [7]. Let us fix a representation π of a Lie algebra $\mathfrak{g}$ on a space V, and a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

DEFINITION. (1) The space of **$\mathfrak{h}$-relative n-cochains of $\mathfrak{g}$ with values in V** is

$$C^n(\mathfrak{g}, \mathfrak{h}, V) := \mathrm{Hom}_{\mathfrak{h}}\big(\Lambda^n(\mathfrak{g}/\mathfrak{h}), V\big).$$

(2) The **coboundary operator** $\partial : C^n(\mathfrak{g}, \mathfrak{h}, V) \to C^{n+1}(\mathfrak{g}, \mathfrak{h}, V)$ is defined by

$$\partial\alpha(X_0 \wedge \ldots \wedge X_n) := \sum_{i=0}^{n} (-1)^i \pi(X_i)\alpha\Big(\bigwedge_{j\neq i} X_j\Big) + \sum_{i<j} (-1)^{i+j} \alpha\Big([X_i, X_j] \wedge \bigwedge_{k\neq i,j} X_k\Big).$$

(3) The space $Z^n(\mathfrak{g}, \mathfrak{h}, V)$ of **$\mathfrak{h}$-relative n-cocycles** is the kernel of the restriction of ∂ to $C^n(\mathfrak{g}, \mathfrak{h}, V)$.

(4) The space $B^n(\mathfrak{g}, \mathfrak{h}, V)$ of **$\mathfrak{h}$-relative n-coboundaries** is the image of the restriction of ∂ to $C^{n-1}(\mathfrak{g}, \mathfrak{h}, V)$.

It is well-known that the square $\partial^2 : C^n(\mathfrak{g}, \mathfrak{h}, V) \to C^{n+2}(\mathfrak{g}, \mathfrak{h}, V)$ of the coboundary operator is zero, so $B^n(\mathfrak{g}, \mathfrak{h}, V) \subset Z^n(\mathfrak{g}, \mathfrak{h}, V)$. The space of **$\mathfrak{h}$-relative n-cohomology classes** of V is

$$H^n(\mathfrak{g}, \mathfrak{h}, V) := Z^n(\mathfrak{g}, \mathfrak{h}, V)/B^n(\mathfrak{g}, \mathfrak{h}, V).$$

If we write simply $C^n(\mathfrak{g}, V)$, $Z^n(\mathfrak{g}, V)$, $B^n(\mathfrak{g}, V)$, or $H^n(\mathfrak{g}, V)$, it is understood that $\mathfrak{h} = 0$.

In fact, we will only consider 1-cohomology. Observe that for v in $C^0(\mathfrak{g}, V) = V$ and α in $C^1(\mathfrak{g}, V) = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, V)$, we have $\partial v(X) = \pi(X)v$ and

$$\partial\alpha(X \wedge Y) = \pi(X)\alpha(Y) - \pi(Y)\alpha(X) - \alpha[X, Y].$$

3.3. Extensions

Again, all of the statements in this section are elementary and may be found in Guichardet [7]. Maintain $\mathfrak{g}$ and $\mathfrak{h}$ as in the previous section, and fix two representations (ϕ_1, V_1) and (ϕ_2, V_2) of $\mathfrak{g}$. Write h for the two-sided action of $\mathfrak{g}$ on $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$:

$$h(X)T := \phi_2(X) \circ T - T \circ \phi_1(X).$$

A representation π of $\mathfrak{g}$ on the space $V_1 \oplus V_2$ is called an **extension of V_1 by V_2** if it is of the form

$$\pi = \begin{pmatrix} \phi_1 & 0 \\ \alpha & \phi_2 \end{pmatrix},$$

where $\alpha : \mathfrak{g} \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$. The condition that π is a representation is easily seen to be simply the condition that α be a 1-cocycle. If α is $\mathfrak{h}$-relative, then $\pi|_{\mathfrak{h}}$ is the direct sum $\phi_1|_{\mathfrak{h}} \oplus \phi_2|_{\mathfrak{h}}$.

Two extensions π and π' of V_1 by V_2 are equivalent if the corresponding cohomology classes $[\alpha]$ and $[\alpha']$ are proportional: $[\alpha'] = c[\alpha]$ for some non-zero scalar c. This is not the only situation in which they are equivalent, but it is the only situation in which there is an equivalence of the form

$$\epsilon = \begin{pmatrix} c_1 & 0 \\ \gamma & c_2 \end{pmatrix},$$

where c_1 and c_2 are non-zero scalars and γ is an element of $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$. Indeed, we have the following lemma.

Lemma 3.3. *$\epsilon = \left(\begin{smallmatrix} c_1 & 0 \\ \gamma & c_2 \end{smallmatrix}\right)$ is an equivalence from $\pi = \left(\begin{smallmatrix} \phi_1 & 0 \\ \alpha & \phi_2 \end{smallmatrix}\right)$ to $\pi' = \left(\begin{smallmatrix} \phi_1 & 0 \\ \alpha' & \phi_2 \end{smallmatrix}\right)$ if and only if $\alpha' = \frac{c_2}{c_1}\alpha - \frac{1}{c_1}\partial\gamma$.*

3.4. $\mathfrak{a}$-split extensions of $F(\lambda)$ by $F(\lambda + p)$

We now consider the case where $F(\lambda)$ and $F(\lambda + p)$ are tensor density modules of $\mathrm{Vec}\,\mathbb{R}$ whose Casimir operators, $\lambda^2 - \lambda$ and $(\lambda+p)^2 - (\lambda+p)$, are distinct, that is, p is neither 0 nor $1 - 2\lambda$. Proofs of the following two elementary results may be found in [4].

Proposition 3.4. *If p is neither 0 nor $1 - 2\lambda$, then every element of $Z^1(\mathrm{Vec}\,\mathbb{R}, \mathrm{Hom}_{\lambda,\lambda+p})$ is cohomologous to an element of $Z^1(\mathrm{Vec}\,\mathbb{R}, \mathfrak{a}, \mathrm{Hom}_{\lambda,\lambda+p})$.*

Definition. For every λ in $\mathbb{C}$ and p in $2 + \mathbb{N}$, define a 1-cochain $\beta_p^\lambda : \mathrm{Vec}\,\mathbb{R} \to \mathrm{Hom}_{\lambda,\lambda+p}$ by $\beta_p^\lambda(\mathfrak{a}) = 0$, and for $k \geq 2$

$$\beta_p^\lambda(e_k) := \tfrac{6}{(k-2)!} h_{\lambda,\lambda+p}(e_1)^{k-2}(dx^p D^{p-2}).$$

Lemma 3.5. *The 1-cochains β_p^λ are $\mathfrak{a}$-relative. Moreover, $C^1(\mathrm{Vec}\,\mathbb{R}, \mathfrak{a}, \mathrm{Hom}_{\lambda,\lambda+p})$ is $\mathbb{C}\beta_p^\lambda$ for $p \in 2 + \mathbb{N}$, and zero otherwise.*

We remark that this lemma may be proven quickly using the facts that $\mathrm{Vec}\,\mathbb{R}/\mathfrak{a}$ is $\mathfrak{a}$-equivalent to $F(2)$, as stated in Corollary 2.14, and that, up to a scalar, $dx^p D^{p-2}$ is the unique lowest weight vector in $\mathrm{Hom}_{\lambda,\lambda+p}$ of weight 2.

The following theorem is due to Feigin and Fuchs [6]. Similar results were obtained in different settings by Martin and Piard [10] over $\mathrm{Vec}\,S^1$, by Bouarroudj and Ovsienko [2] for smooth vector fields rather than polynomials, and by Boe, Nakano, and Weisner [1] in positive characteristic.

A proof hinging on $\mathfrak{a}$-relativity was given in [4]. It can be shown that $\Lambda^2(\operatorname{Vec}\mathbb{R}/\mathfrak{a})$ is $\mathfrak{a}$-equivalent to $\bigoplus_{r=0}^{\infty} F(2r+5)$. Therefore, writing w_{2r+5} for the lowest weight vectors of weight $2r+5$ in $\Lambda^2(\operatorname{Vec}\mathbb{R}/\mathfrak{a})$, we see that β_p^λ is a cocycle if and only if $\partial\beta_p^\lambda(w_{2r+5}) = 0$ for all r in $\mathbb{N}$. Moreover, since $\partial\beta_p^\lambda(w_{2r+5})$ is itself a lowest weight vector of weight $2r+5$, it must be a multiple of $dx^p D^{p-2r-5}$ if p is in $2r+5+\mathbb{N}$, and zero otherwise. The multiples are difficult to compute, but they can be deduced from results of Cohen, Manin, and Zagier [3].

THEOREM 3.6. *For $p \neq 0$ or $1-2\lambda$, $H^1(\operatorname{Vec}\mathbb{R}, \operatorname{Hom}_{\lambda,\lambda+p}) = H^1(\operatorname{Vec}\mathbb{R}, \mathfrak{a}, \operatorname{Hom}_{\lambda,\lambda+p})$. Moreover,*

(1) *$H^1(\operatorname{Vec}\mathbb{R}, \operatorname{Hom}_{\lambda,\lambda+2})$ is $\mathbb{C}[\beta_2^\lambda]$ for $\lambda \neq -\frac{1}{2}$.*

(2) *$H^1(\operatorname{Vec}\mathbb{R}, \operatorname{Hom}_{\lambda,\lambda+3})$ is $\mathbb{C}[\beta_3^\lambda]$ for $\lambda \neq -1$.*

(3) *$H^1(\operatorname{Vec}\mathbb{R}, \operatorname{Hom}_{\lambda,\lambda+4})$ is $\mathbb{C}[\beta_4^\lambda]$ for $\lambda \neq -\frac{3}{2}$.*

(4) *$H^1(\operatorname{Vec}\mathbb{R}, \operatorname{Hom}_{\lambda,\lambda+5})$ is $\mathbb{C}[\beta_5^\lambda]$ for $\lambda = -4$ or 0, and 0 otherwise.*

(5) *$H^1(\operatorname{Vec}\mathbb{R}, \operatorname{Hom}_{\lambda,\lambda+6})$ is $\mathbb{C}[\beta_6^\lambda]$ for $\lambda = \frac{1}{2}(-5 \pm \sqrt{19})$, and 0 otherwise.*

(6) *$H^1(\operatorname{Vec}\mathbb{R}, \operatorname{Hom}_{\lambda,\lambda+p})$ is 0 in all other cases where $p \neq 0$ or $1-2\lambda$.*

CHAPTER 4

EQUIVALENCES

4.1. Differential Operators

Recall from Lemma 3.1 that $\mathrm{Hom}_{\lambda,\mu} = dx^{\mu-\lambda}\mathbb{C}[x][[D]]$. The space of differential operators from $F(\lambda)$ to $F(\mu)$ is $\mathrm{Diff}_{\lambda,\mu} := dx^{\mu-\lambda}\mathbb{C}[x][D]$. Clearly $\mathrm{Diff}_{\lambda,\mu}$ is a $\mathrm{Vec}\,\mathbb{R}$-submodule of $\mathrm{Hom}_{\lambda,\mu}$, and its **order filtration**

$$\mathrm{Diff}^k_{\lambda,\mu} := \mathrm{Span}_{\mathbb{C}[x]}\left\{D^s : 0 \leq s \leq k\right\}$$

is also $\mathrm{Vec}\,\mathbb{R}$-invariant. The following result is fundamental.

Lemma 4.1. $\mathrm{Diff}^k_{\lambda,\lambda+p} / \mathrm{Diff}^{k-1}_{\lambda,\lambda+p}$ *is* $\mathrm{Vec}\,\mathbb{R}$-*equivalent to* $F(p-k)$.

Proof. The reader may check that $dx^p f D^k + \mathrm{Diff}^{k-1}_{\lambda,\lambda+p} \mapsto dx^{p-k} f$ is an equivalence. □

The topic of this thesis is the set of equivalence classes of the reducible subquotients of the collection of $\mathrm{Vec}\,\mathbb{R}$-modules $\mathrm{Diff}^k_{\lambda,\lambda+p}$ as λ and p vary. As we will explain, we will also consider subquotients of modules of pseudodifferential operators.

4.2. Subquotients

Definition. For λ, p in $\mathbb{C}$ and k, l in $\mathbb{N}$, set

$$\mathrm{SQ}^{k,l}_{\lambda,p} := \mathrm{Diff}^k_{\lambda,\lambda+p} / \mathrm{Diff}^{k-l}_{\lambda,\lambda+p}\,.$$

The subquotient $\mathrm{SQ}^{k,l}_{\lambda,p}$ of $\mathrm{Diff}_{\lambda,\lambda+p}$ is of length l and has Jordan-Hölder composition series

$$F(p-k), F(p-k+1), \ldots, F(p-k+l-1).$$

In general, it does not split under $\mathrm{Vec}\,\mathbb{R}$ as the direct sum of these modules. However, as we will now prove, it does split under the projective subalgebra $\mathfrak{a}$ if the Casimir operators of the composition series elements are distinct. The following lemma, which characterizes this situation, is left to the reader.

LEMMA 4.2. *The Casimir operators of the composition series of* $\mathrm{SQ}_{\lambda,p}^{k,l}$ *are not distinct if and only if*

$$p-k\in-\tfrac{1}{2}\mathbb{N}\qquad\text{and}\qquad p-k+l-1\in 1+\tfrac{1}{2}\mathbb{N}. \tag{1}$$

PROPOSITION 4.3. *If* (1) *is false, then* $\mathrm{SQ}_{\lambda,p}^{k,l}$ *is* $\mathfrak{a}$*-equivalent to* $\bigoplus_{i=1}^{l}F(p-k+i-1)$*.*

PROOF. Since $F(p-k), F(p-k+1),\ldots,F(p-k+l-1)$ have distinct Casimir operators, each is $\mathfrak{a}$-equivalent to the corresponding eigenspace of the Casimir operator of $\mathrm{SQ}_{\lambda,p}^{k,l}$ $\square$

In light of the preceding discussion, $\mathrm{SQ}_{\lambda,p}^{k,l}$ is equivalent to a representation ϕ of $\mathrm{Vec}\,\mathbb{R}$ on $\bigoplus_{i=1}^{l}F(p-k+i-1)$ which, when regarded as a block matrix with entries $\phi_{mn}:\mathrm{Vec}\,\mathbb{R}\to\mathrm{Hom}_{n,m}$, takes the form

$$\phi=\begin{pmatrix}\pi_{p-k} & 0 & 0 & 0 & \cdots & 0\\ & \pi_{p-k+1} & 0 & 0 & \cdots & 0\\ & & \pi_{p-k+2} & 0 & \cdots & 0\\ & & & \pi_{p-k+3} & \cdots & 0\\ \vdots & \phi_{mn} & \vdots & \vdots & \ddots & 0\\ & & \cdots & & & \pi_{p-k+l-1}\end{pmatrix}.$$

That is, ϕ is lower triangular and its diagonal entries are the tensor density actions. Moreover, it follows from the previous proposition that the lower triangular entries ϕ_{mn} are $\mathfrak{a}$-relative, and thus by Lemma 3.5, $\phi_{n+1,n}=0$ and ϕ_{mn} is a multiple of β_{m-n}^{n} for $p-k\le n<m\le p-k+l-1$. As in the proof of Theorem 3.6, these multiples are difficult to compute, but may be deduced from [3]; see also [4] and [5]. The result is as follows.

THEOREM 4.4. *If* (1) *is false, then* $\mathrm{SQ}_{\lambda,p}^{k,l}$ *is equivalent to a representation* $\phi_{\lambda,p}^{k,l}$ *of* $\mathrm{Vec}\,\mathbb{R}$ *on*

$\bigoplus_{i=1}^{l} F(p-k+i-1)$ *of the form*

$$\begin{pmatrix} \pi_{p-k} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \pi_{p-k+1} & 0 & 0 & \cdots & 0 \\ \phi_{p-k+2,p-k} & 0 & \pi_{p-k+2} & 0 & \cdots & 0 \\ \phi_{p-k+3,p-k} & \phi_{p-k+3,p-k+1} & 0 & \pi_{p-k+3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \phi_{p-k+l-1,p-k} & \phi_{p-k+l-1,p-k+1} & \cdots & & 0 & \pi_{p-k+l-1} \end{pmatrix},$$

where for $p-k \leq n \leq m-2$ *and* $m \leq p-k+l-1$,

$$\phi_{mn} = b_{mn}(\lambda, p)\beta^{n}_{m-n},$$

where $b_{mn}(\lambda, p)$ *is the scalar defined in (3) and (4) of* [5].

By definition, the parameter k in the representation $\phi^{k,l}_{\lambda,p}$ is in $\mathbb{N}$, being the maximal order of the differential operators in the subquotient. However, by an obvious Zariski density argument based on the formula for $b_{mn}(\lambda, p)$, the formula for $\phi^{k,l}_{\lambda,p}$ defines a representation of $\mathrm{Vec}\,\mathbb{R}$ for all λ, p, and k in $\mathbb{C}$ and all l in $\mathbb{N}$. These representations correspond to subquotients of **pseudodifferential operator modules**: $\phi^{k,l}_{\lambda,p}$ is equivalent to $\Psi^{k}_{\lambda,\lambda+p}/\Psi^{k-l}_{\lambda,\lambda+p}$.

We conclude this section by reproducing the formulas for $b_{n+2,n}(\lambda, p)$, $b_{n+3,n}(\lambda, p)$, and $b_{n+4,n}(\lambda, p)$, the only $b_{mn}(\lambda, p)$ we will need here; see (3) and (6)-(8) of [5]. It will be convenient to establish some preliminary notation. Set

$$c := 2\lambda + p - 1, \qquad N := n + \tfrac{3}{2}, \qquad \tilde{c} := 3c^2 - 2Np. \tag{2}$$

We will define $b_{n+2,n}$, $b_{n+3,n}$, and $b_{n+4,n}$ in terms of intermediate scalars $B_{n+2,n}$, $B_{n+3,n}$, and $B_{n+4,n}$:

$$B_{n+2,n}(\lambda, p) := \tilde{c} + 2p - (N-1)^2 - \tfrac{3}{4},$$

$$B_{n+3,n}(\lambda, p) := \sqrt{3}c\big(\tilde{c} - (N - \tfrac{3}{2})p - 3\big),$$

$$B_{n+4,n}(\lambda, p) := \big(\tilde{c} + N^2 - \tfrac{15}{4}\big)^2 - 4\big(Np - \tfrac{1}{5}(N^2-6)\big)^2 - \tfrac{3}{5}\big(N^2 - \tfrac{9}{4}\big)\big(N^2 + \tfrac{9}{4}\big).$$

The formulas for $b_{n+2,n}$, $b_{n+3,n}$, and $b_{n+4,n}$ are

$$\begin{aligned}
b_{n+2,n}(\lambda,p) &:= -\tfrac{1}{12}\binom{p-n}{2}\frac{B_{n+2,n}(\lambda,p)}{n+\frac{1}{2}},\\
b_{n+3,n}(\lambda,p) &:= \tfrac{\sqrt{3}}{18}\binom{p-n}{3}\frac{B_{n+3,n}(\lambda,p)}{n(n+1)(n+2)},\\
b_{n+4,n}(\lambda,p) &:= -\tfrac{5}{48}\binom{p-n}{4}\frac{B_{n+4,n}(\lambda,p)}{n\big(n+\frac{1}{2}\big)\big(n+\frac{3}{2}\big)\big(n+\frac{5}{2}\big)(n+3)}.
\end{aligned}$$

4.3. Equivalence Classes of $\mathrm{SQ}^{k,l}_{\lambda,p}$

We wish to know the conditions under which $\mathrm{SQ}^{k,l}_{\lambda,p}$ and $\mathrm{SQ}^{k',l}_{\lambda',p'}$ are $\mathrm{Vec}\,\mathbb{R}$-equivalent. Henceforth we admit subquotients of pseudodifferential operators, so k may be in $\mathbb{C}$.

Lecomte and Ovsienko [9] examined the case that $p = p' = 0$, where $k = k'$ is necessary for equivalence. They found that $\mathrm{SQ}^{k,l}_{\lambda,0}$ and $\mathrm{SQ}^{k,l}_{\lambda',0}$ are generically equivalent for $l \leq 4$, and equivalent for $l \geq 5$ only when λ' is λ or $1-\lambda$, the conjugate case. We will give necessary and sufficient conditions under which $\mathrm{SQ}^{k,l}_{\lambda,p}$ and $\mathrm{SQ}^{k',l}_{\lambda',p'}$ are $\mathrm{Vec}\,\mathbb{R}$-equivalent for $l \leq 5$.

Lemma 4.5. *If $\mathrm{SQ}^{k,l}_{\lambda,p}$ is $\mathrm{Vec}\,\mathbb{R}$-equivalent to $\mathrm{SQ}^{k',l'}_{\lambda',p'}$, then $l = l'$ and $p-k = p'-k'$.*

Proof. The two modules have the same composition series only if $l = l'$ and $p - k = p' - k'$. □

Proposition 4.6. *Assume that $p-k = p'-k'$ and that (1) is false. Suppose that ϵ is an endomorphism of $\bigoplus_{i=1}^{l} F(p-k+i-1)$. Then it is an $\mathfrak{a}$-equivalence from $\phi^{k,l}_{\lambda,p}$ to $\phi^{k',l}_{\lambda',p'}$ if and only if, regarded as a block matrix with entries $\epsilon_{mn} : F(m) \to F(n)$, it is block diagonal with non-zero scalars on the diagonal.*

Proof. Under our assumptions, the Casimir operators of $\phi^{k,l}_{\lambda,p}$ and $\phi^{k',l}_{\lambda',p'}$ are equal, and are block diagonal with scalars on the diagonal. Since ϵ commutes with this operator, it is also diagonal. To see that the diagonal entries ϵ_{nn} are scalars, note that $\epsilon_{nn} : F(n) \to F(n)$ is an $\mathfrak{a}$-equivalence and apply Proposition 2.12. □

Proposition 4.7. *Assume that* $p-k=p'-k'$ *and that* (1) *is false. Then* $\mathrm{SQ}^{k,l}_{\lambda,p}$ *and* $\mathrm{SQ}^{k',l}_{\lambda',p'}$ *are* $\mathrm{Vec}\,\mathbb{R}$*-equivalent if and only if there are non-zero scalars* ϵ_{nn}*, where* $n=p-k,p-k+1,\ldots,p-k+l-1$*, such that for all* (m,n) *with* $p-k\le n$*,* $m\le p-k+l-1$*, and* $n+2\le m$*, we have*

$$\epsilon_{mm}b_{mn}(\lambda,p)=b_{mn}(\lambda',p')\epsilon_{nn}. \tag{3}$$

Proof. If $\mathrm{SQ}^{k,l}_{\lambda,p}$ and $\mathrm{SQ}^{k',l}_{\lambda',p'}$ are $\mathrm{Vec}\,\mathbb{R}$-equivalent, then there is an equivalence ϵ from $\phi^{k,l}_{\lambda,p}$ to $\phi^{k',l}_{\lambda',p'}$. By Proposition 4.6, ϵ is block diagonal, and (3) is the (m,n) entry of the equation $\epsilon\circ\phi^{k,l}_{\lambda,p}=\phi^{k',l}_{\lambda',p'}\circ\epsilon$. Conversely, if (3) can be solved for the ϵ_{nn}, then the resulting block diagonal matrix ϵ is an equivalence from $\phi^{k,l}_{\lambda,p}$ to $\phi^{k',l}_{\lambda',p'}$. □

Definition. The subquotients $\mathrm{SQ}^{k,l}_{\lambda,p}$ and $\mathrm{SQ}^{k',l}_{\lambda',p'}$ are said to satisfy the **same vanishing condition** if $b_{mn}(\lambda,p)$ and $b_{mn}(\lambda',p')$ are either both zero or both non-zero for all $p-k\le n$, $m\le p-k+l-1$, and $n+2\le m$.

Corollary 4.8. *If* $\mathrm{SQ}^{k,l}_{\lambda,p}$ *and* $\mathrm{SQ}^{k',l}_{\lambda',p'}$ *are* $\mathrm{Vec}\,\mathbb{R}$*-equivalent, then they satisfy the same vanishing condition.*

Henceforth, let us write b_{mn}, b'_{mn}, B_{mn}, and B'_{mn} for $b_{mn}(\lambda,p)$, $b_{mn}(\lambda',p')$, $B_{mn}(\lambda,p)$, and $B_{mn}(\lambda',p')$ respectively. Similarly, by analogy with (2) we will write c' for $2\lambda'+p'-1$ and $\tilde{c}'$ for $3(c')^2-2Np'$.

Observe that if $p-k=p'-k'$ and $(p',c')=(p,-c)$, then the formulas for the b_{ij} show that $\mathrm{SQ}^{k,l}_{\lambda,p}$ and $\mathrm{SQ}^{k',l}_{\lambda',p'}$ are equivalent for $l\le 5$. In fact, this is true for all l, as in this case $\Psi^k_{\lambda,p}$ and $\Psi^k_{1-p-\lambda,p}$ are conjugate. This explains why the equivalence class of $\mathrm{SQ}^{k,l}_{\lambda,p}$ depends only on $(k,p,\tilde{c})$.

The following result is proven in [9] for $p=0$.

Theorem 4.9. *Assume that* $p-k=p'-k'$*, and define* $n:=p-k$*.*

(1) $\mathrm{SQ}^{k,1}_{\lambda,p}$ *and* $\mathrm{SQ}^{k',1}_{\lambda',p'}$ *are* $\mathrm{Vec}\,\mathbb{R}$*-equivalent.*
(2) *For* $n\neq 0$*,* $\mathrm{SQ}^{k,2}_{\lambda,p}$ *and* $\mathrm{SQ}^{k',2}_{\lambda',p'}$ *are* $\mathrm{Vec}\,\mathbb{R}$*-equivalent.*

(3) *For* $n \neq 0,\ -\frac{1}{2},$ *or* -1, $\mathrm{SQ}^{k,3}_{\lambda,p}$ *and* $\mathrm{SQ}^{k',3}_{\lambda',p'}$ *are* $\mathrm{Vec}\,\mathbb{R}$*-equivalent if and only if they satisfy the same vanishing condition, that is,* $b_{n+2,n}$ *and* $b'_{n+2,n}$ *are both zero or both non-zero.*

(4) *For* $n \neq 0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2},$ *or -2,* $\mathrm{SQ}^{k,4}_{\lambda,p}$ *and* $\mathrm{SQ}^{k',4}_{\lambda',p'}$ *are* $\mathrm{Vec}\,\mathbb{R}$*-equivalent if and only if they satisfy the same vanishing condition, that is, the elements of each of the pairs* $\left(b_{n+2,n}, b'_{n+2,n}\right)$, $\left(b_{n+3,n+1}, b'_{n+3,n+1}\right)$, *and* $\left(b_{n+3,n}, b'_{n+3,n}\right)$ *are either both zero or both non-zero.*

Proof. In all cases, the assumption on n implies that (1) is false. In Case (1), both modules are equivalent to $F(n)$, and in Case (2), Theorem 4.4 implies that both modules are equivalent to $F(n) \oplus F(n+1)$. In Case (3), if both $b_{n+2,n}$ and $b'_{n+2,n}$ are zero, then both modules are equivalent to $\bigoplus_{i=0}^{2} F(n+i)$. If both $b_{n+2,n}$ and $b'_{n+2,n}$ are non-zero, apply Proposition 4.7 with $\epsilon_{nn} = \epsilon_{n+1,n+1} = 1$ and $\epsilon_{n+2,n+2} = b_{n+2,n}\big/b'_{n+2,n}$.

In Case (4), it is again always possible to solve (3) for ϵ. We will only describe the case that none of the b_{mn} is zero. We may take $\epsilon_{nn} = 1$,

$$\epsilon_{n+2,n+2} = b'_{n+2,n}\big/b_{n+2,n}, \quad \epsilon_{n+3,n+3} = b'_{n+3,n}\big/b_{n+3,n}, \quad \epsilon_{n+1,n+1} = \epsilon_{n+3,n+3} b_{n+3,n+1}\big/b'_{n+3,n+1}.$$

□

4.4. The Length Five Case

In this section we study the length five case, which exhibits a new phenomenon: the equivalence classes have continuous invariants. As in Theorem 4.9, we assume throughout that $n := p - k = p' - k'$, and we maintain the notation (2). Define

$$I_q(\lambda, p) := B_{n+4,n}\big/B_{n+4,n+2}B_{n+2,n},$$

$$I_c(\lambda, p) := B_{n+4,n}B_{n+3,n+1}\big/B_{n+4,n+1}B_{n+3,n},$$

$$I_r(\lambda, p) := B_{n+3,n+1}B_{n+4,n+2}B_{n+2,n}\big/B_{n+4,n+1}B_{n+3,n}.$$

As usual we shall abbreviate $I_\bullet(\lambda, p)$ and $I_\bullet(\lambda', p')$ by $I_\bullet$ and $I'_\bullet$. Note that $I_c = I_q I_r$. The following theorem is our main result.

THEOREM 4.10. *Assume that* $k \neq 0$, 1, 2, *or* 3, *and that* $n \neq 0$, $-\frac{1}{2}$, -1, $-\frac{3}{2}$, -2, $-\frac{5}{2}$, *or* -3. *Then* $\mathrm{SQ}^{k,5}_{\lambda,p}$ *and* $\mathrm{SQ}^{k',5}_{\lambda',p'}$ *are* $\mathrm{Vec}\,\mathbb{R}$*-equivalent if and only if they satisfy the same vanishing condition and one of the following mutually exclusive conditions hold:*

(1) *Two or more of* $B_{n+4,n}$, $B_{n+4,n+2}B_{n+2,n}$, *and* $B_{n+4,n+1}B_{n+3,n+1}B_{n+3,n}$ *are 0.*

(2) $B_{n+4,n}B_{n+4,n+2}B_{n+2,n} \neq 0$, $B_{n+4,n+1}B_{n+3,n+1}B_{n+3,n} = 0$, *and* $I_q = I'_q$.

(3) $B_{n+4,n}B_{n+4,n+1}B_{n+3,n+1}B_{n+3,n} \neq 0$, $B_{n+4,n+2}B_{n+2,n} = 0$, *and* $I_c = I'_c$.

(4) *Of the six* B_{ij}*'s, only* $B_{n+4,n} = 0$, *and* $I_r = I'_r$.

(5) *None of the* B_{ij}*'s is zero,* $I_q = I'_q$, *and* $I_c = I'_c$.

PROOF. By the assumption on n, (1) is false, so Proposition 4.7 applies: $\mathrm{SQ}^{k,5}_{\lambda,p}$ and $\mathrm{SQ}^{k',5}_{\lambda',p'}$ are $\mathrm{Vec}\,\mathbb{R}$-equivalent if and only if (3) can be solved for ϵ.

The condition on k rules out cases in which $\mathrm{SQ}^{k,5}_{\lambda,p}$ is split by the image of the splitting $\Psi^3_{\lambda,p} = \mathrm{Diff}^3_{\lambda,p} \oplus \Psi^{-1}_{\lambda,p}$. In these cases one can use Theorem 4.9 to determine whether $\mathrm{SQ}^{k,5}_{\lambda,p}$ and $\mathrm{SQ}^{k',5}_{\lambda',p'}$ are equivalent. Observe that under this condition, each of the six b_{ij}'s is zero if and only if the corresponding B_{ij} is zero.

Using the formulas for b_{ij} in terms of B_{ij}, verify that

$$I_q = -\tfrac{2}{5}N(N^2 - \tfrac{9}{4})b_{n+4,n}\big/b_{n+4,n+2}b_{n+2,n},$$

$$I_c = -\tfrac{64}{45}\frac{N(N^2-1)}{(N^2-\frac{1}{4})^2}b_{n+4,n}b_{n+3,n+1}\big/b_{n+4,n+1}b_{n+3,n},$$

$$I_r = -\tfrac{9}{32}\frac{(N^2-\frac{1}{4})^2(N^2-\frac{9}{4})}{N(N^2-1)}b_{n+4,n+1}b_{n+3,n}\big/b_{n+4,n+2}b_{n+3,n+1}b_{n+2,n}.$$

Thus if none of $B_{n+4,n}$, $B_{n+4,n+2}$, $B_{n+2,n}$ is zero, then (3) gives

$$I'_q/I_q = \frac{(b'_{n+4,n}/b_{n+4,n})}{(b'_{n+4,n+2}/b_{n+4,n+2})(b'_{n+2,n}/b_{n+2,n})} = \frac{(\epsilon_{n+4,n+4}/\epsilon_{n,n})}{(\epsilon_{n+4,n+4}/\epsilon_{n+2,n+2})(\epsilon_{n+2,n+2}/\epsilon_{n,n})} = 1.$$

Similarly, if none of $B_{n+4,n}$, $B_{n+4,n+1}$, $B_{n+3,n+1}$, $B_{n+3,n}$ is zero, then

$$I'_c/I_c = \frac{(b'_{n+4,n}/b_{n+4,n})(b'_{n+3,n+1}/b_{n+3,n+1})}{(b'_{n+4,n+1}/b_{n+4,n+1})(b'_{n+3,n}/b_{n+3,n})} = \frac{(\epsilon_{n+4,n+4}/\epsilon_{n,n})(\epsilon_{n+3,n+3}/\epsilon_{n+1,n+1})}{(\epsilon_{n+4,n+4}/\epsilon_{n+1,n+1})(\epsilon_{n+3,n+3}/\epsilon_{n,n})} = 1.$$

The same kind of argument shows that if none of B_{41}, B_{30}, B_{42}, B_{31}, and B_{20} is zero, then $I'_r/I_r = 1$. It is easy to see that there are no other obstructions to equivalence beyond the same vanishing condition. The result follows. $\square$

CHAPTER 5

THE GENERIC LENGTH FIVE CASE

In this section we study the equivalence class of $\mathrm{SQ}^{k,5}_{\lambda,p}$ in the generic case that none of the six relevant B_{ij} is zero. We maintain the same assumptions on $n = p - k$ and on k as in Theorem 4.10. By Case (5) of that theorem, I_q and I_c are complete invariants for the equivalence class of $\mathrm{SQ}^{k,5}_{\lambda,p}$, so we are reduced to studying the level curves of I_q and I_c.

5.1. The Level Curves of I_q

Consider the A-level curve $I_q = A$ of I_q, which may be written

$$B_{n+4,n} = A B_{n+4,n+2} B_{n+2,n}.$$

In $(p, \tilde{c})$-coordinates, $B_{n+4,n} = 0$ is a conic and $B_{n+4,n+2} B_{n+2,n} = 0$ is two lines. The intersection of the conic with the two lines is four points, and these four points are on all of the level curves of I_q. Thus, the family of level curves $I_q = A$ as A varies is the pencil of conics through four fixed points.

The center of the quadrilateral formed by these four points turns out to be $\big(\frac{1}{5}N, N^2 + \frac{15}{4}\big)$ in $(p, \tilde{c})$-coordinates. Therefore we will study the level curves in the coordinate system $(\hat{p}, \hat{\Lambda})$ in which the quadrilateral is centered at the origin:

$$\hat{p} := p - \tfrac{1}{5}N, \qquad \hat{\Lambda} := \tilde{c} - N^2 - \tfrac{15}{4}.$$

In this new coordinate system,

$$\begin{aligned} B_{n+2,n} &= \hat{\Lambda} + 2\hat{p} + \tfrac{12}{5}N + 2, \\ B_{n+4,n+2} &= \hat{\Lambda} - 2\hat{p} - \tfrac{12}{5}N + 2, \\ B_{n+4,n} &= \big(\hat{\Lambda} + 2N^2\big)^2 - 4\big(N\hat{p} + \tfrac{6}{5}\big)^2 - \tfrac{36}{25}\big(N^2 - 1\big)\big(N^2 - \tfrac{9}{4}\big). \end{aligned}$$

Proposition 5.1. *In $(\hat{p}, \hat{\Lambda})$-coordinates, the level curves of I_q comprise the pencil of conics through the following four points, which are inscribed in a circle:*

$$P_{20} := \big(\tfrac{4}{5}N - \tfrac{5}{2}, -4N + 3\big), \quad P_{42} := \big(\tfrac{4}{5}N + \tfrac{5}{2}, 4N + 3\big),$$

$$Q_{20} := \left(-\tfrac{4}{5}N + \tfrac{1}{2}, -\tfrac{4}{5}N - 3\right), \quad Q_{42} := \left(-\tfrac{4}{5}N - \tfrac{1}{2}, -\tfrac{4}{5}N - 3\right).$$

PROOF. Elementary algebra shows that P_{20} and Q_{20} are the points of intersection of the line $B_{n+2,n} = 0$ and the conic $B_{n+4,n} = 0$. Since the transformation $N \mapsto -N$, $\hat{p} \mapsto -\hat{p}$, $\hat{\Lambda} \mapsto \hat{\Lambda}$ leaves $B_{n+4,n}$ fixed and exchanges $B_{n+2,n}$ and $B_{n+4,n+2}$, we see that P_{42} and Q_{42} are the points of intersection of $B_{n+4,n+2}$ and $B_{n+4,n}$.

As is made clear by Proposition 5.3, the inscribing circle is

$$\left(\hat{p} - \tfrac{24}{5}N\right)^2 + \left(\hat{\Lambda} - \tfrac{1}{2}\right)^2 = 32N^2 + \tfrac{25}{2}.$$

□

COROLLARY 5.2. *At $N = 0$, the quadrilateral $P_{20}P_{42}Q_{20}Q_{42}$ is a trapezoid with two sides parallel to the $\hat{p}$-axis. As $N \to \infty$, it approaches a trapezoid with two sides parallel to the $\hat{\Lambda}$-axis. Its diagonals have the following slopes.*

(1) *$\overline{P_{20}Q_{20}}$ and $\overline{P_{42}Q_{42}}$ have slopes -2 and 2, respectively.*

(2) *$\overline{P_{20}P_{42}}$ and $\overline{Q_{20}Q_{42}}$ have slopes $\frac{8}{5}N$ and $-\frac{8}{5}N$, respectively.*

(3) *$\overline{P_{20}Q_{42}}$ and $\overline{P_{42}Q_{20}}$ have slopes -3 and 3, respectively.*

REMARK. Note that when $N = \frac{15}{8}$, P_{42} is equal to Q_{42}; when $N = -\frac{15}{8}$, P_{20} is equal to Q_{20}; when $N = \frac{5}{4}$, P_{20} is equal to Q_{42}; and when $N = -\frac{5}{4}$, P_{42} is equal to Q_{20}. These are the only values of N at which two of the four points are equal.

PROPOSITION 5.3. *After multiplication by $(A-1)(A-N^2)B_{n+4,n+2}B_{n+2,n}$, the equation of the level curve $I_q(\lambda, p) = A$ becomes*

$$(A-N^2)\Big((A-1)\hat{\Lambda} + 2(A-N^2)\Big)^2 - 4(A-1)\Big((A-N^2)\hat{p} + \tfrac{6}{5}N(A-1)\Big)^2$$
$$= -\tfrac{1}{25}(N^2-1)\Big(5(A-1) + 4(N^2-1)\Big)\Big(16N^2(A-1) - 25(A-N^2)\Big).$$

PROOF. The proof is a long but straightforward algebra exercise. □

PROPOSITION 5.4. *I_q has the following types of level curves.*

(1) *When $A = 1$ or N^2, the level curves are parabolas.*

(2) *When* $(A-1)(A-N^2) > 0$*, the level curves are hyperbolas. In particular, when* A *is* $-9N^2/(16N^2-25)$, $-\frac{4}{5}\left(N^2-\frac{9}{4}\right)$*, or* ∞*, the level curves are degenerate hyperbolas: the pairs of lines given in Corollary 5.2.*

(3) *When* $(A-1)(A-N^2) < 0$*, the level curves are ellipses. In particular, when* $A = \frac{4}{5}N^2 + \frac{1}{5}$*, the level curve is a circle.*

PROOF. This is immediate from Proposition 5.3. □

We now give plots of several level curves at $N = 5$. The points P_{20}, P_{42}, Q_{20}, and Q_{42} are labeled in red. The final figure in this section is an overlay of the three pairs of lines, the two parabolas, and the circle, as well as two hyperbolas and two ellipses.

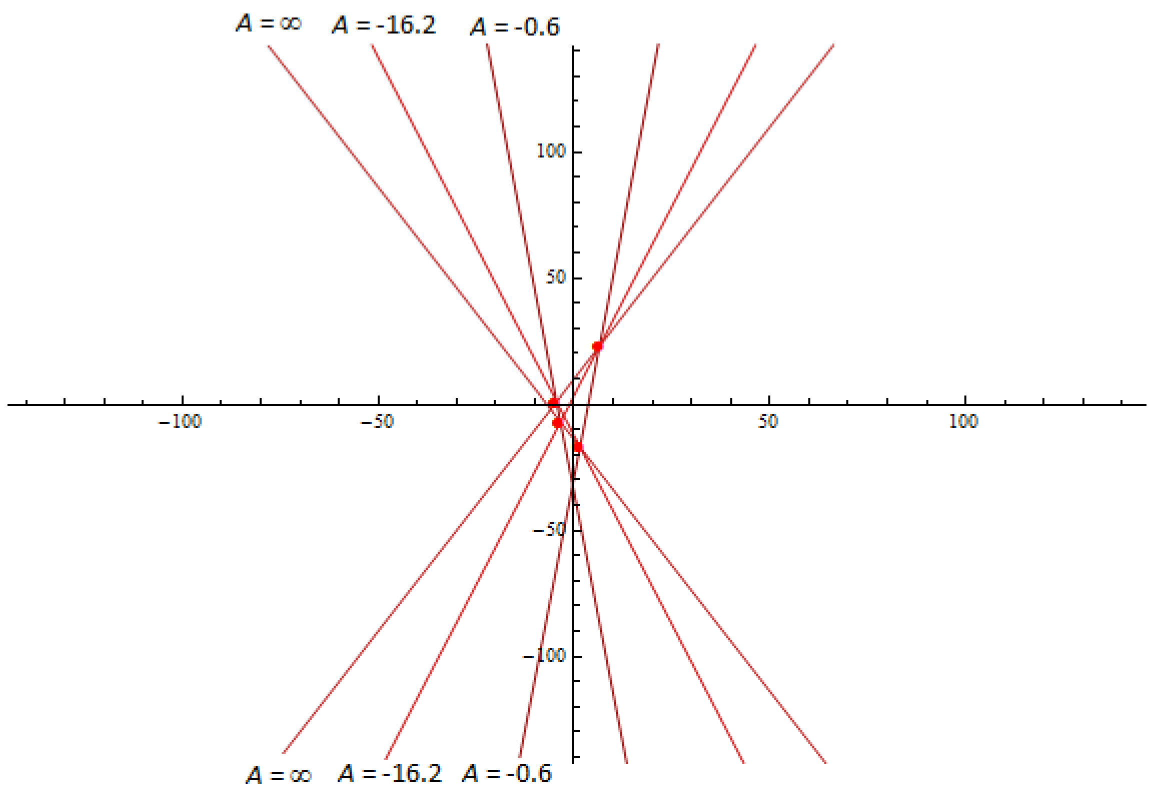

FIGURE 5.1. When $N = 5$, the three pairs of lines among the level curves of I_q occur at $A = \infty$, -0.6, and -16.2.

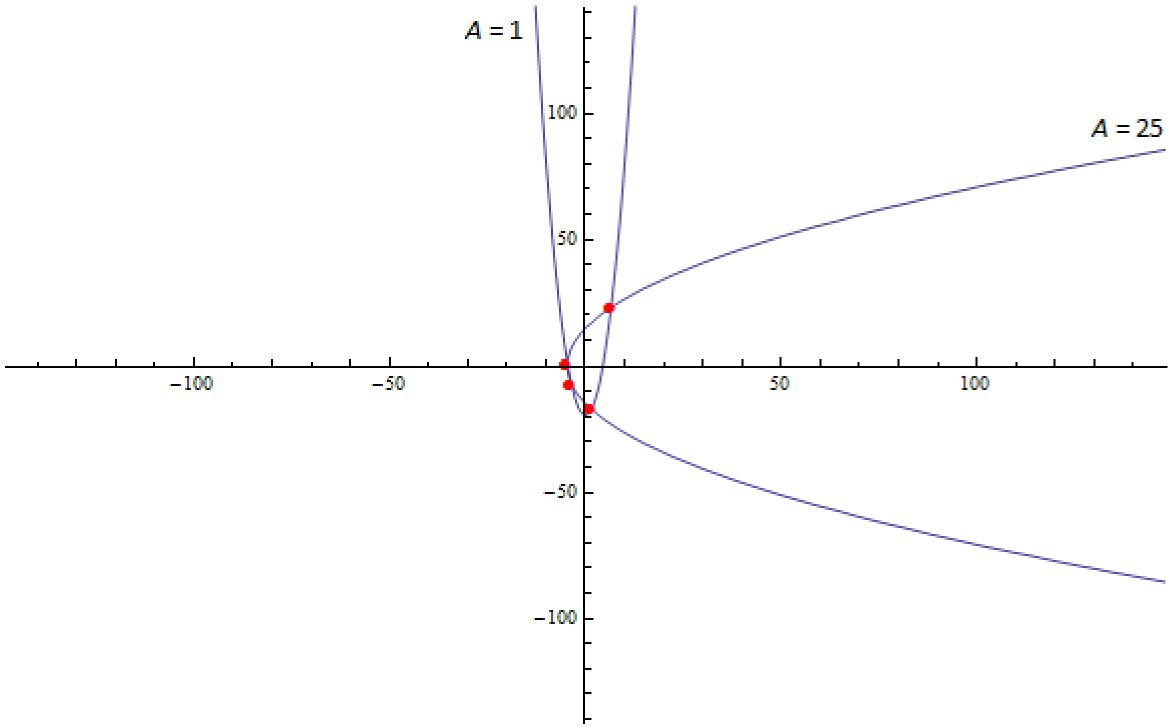

FIGURE 5.2. As stated in Proposition 5.4, the two parabolas occur when $A = N^2$ and when $A = 1$. When $N = 5$, they occur at $A = 1$ and 25.

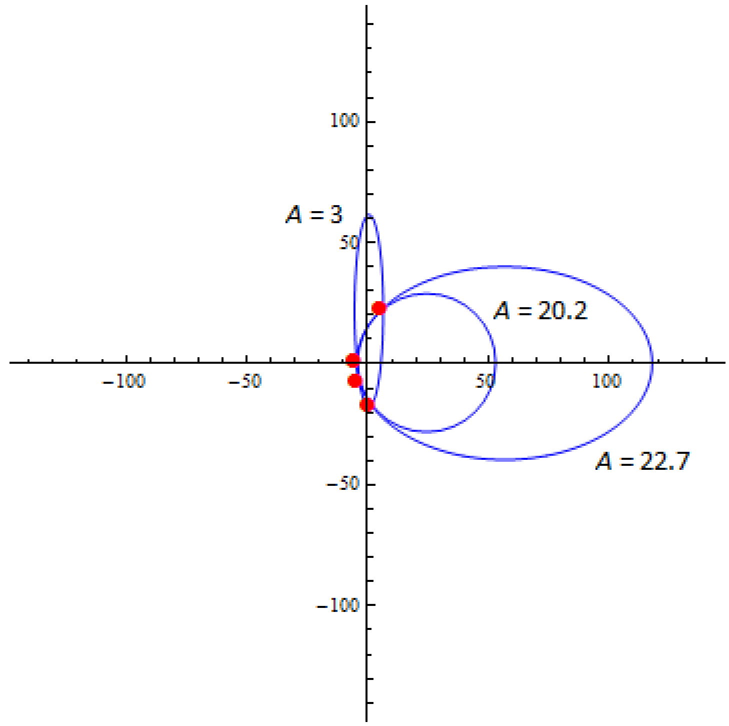

FIGURE 5.3. Here are three ellipses, one of which is a circle. Ellipses occur in the level curves of I_q when $(A-1)(A-N^2) < 0$, and we get a circle when $A = \frac{4}{5}N^2 + \frac{1}{5}$. For $N = 5$, this is when $A = 20.2$.

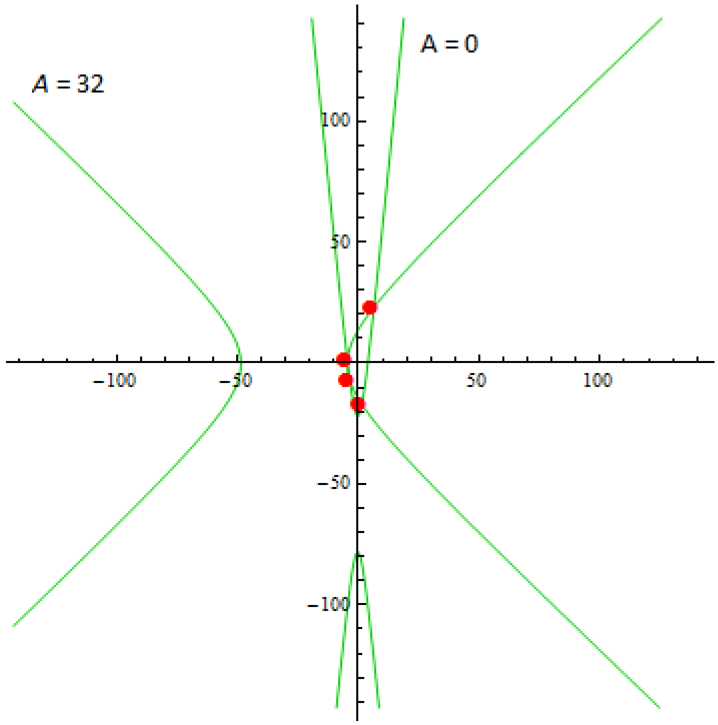

FIGURE 5.4. Here are two hyperbolas, one North-South opening and the other East-West opening. Hyperbolas occur in the level curves of I_q when $(A-1)(A-N^2) > 0$.

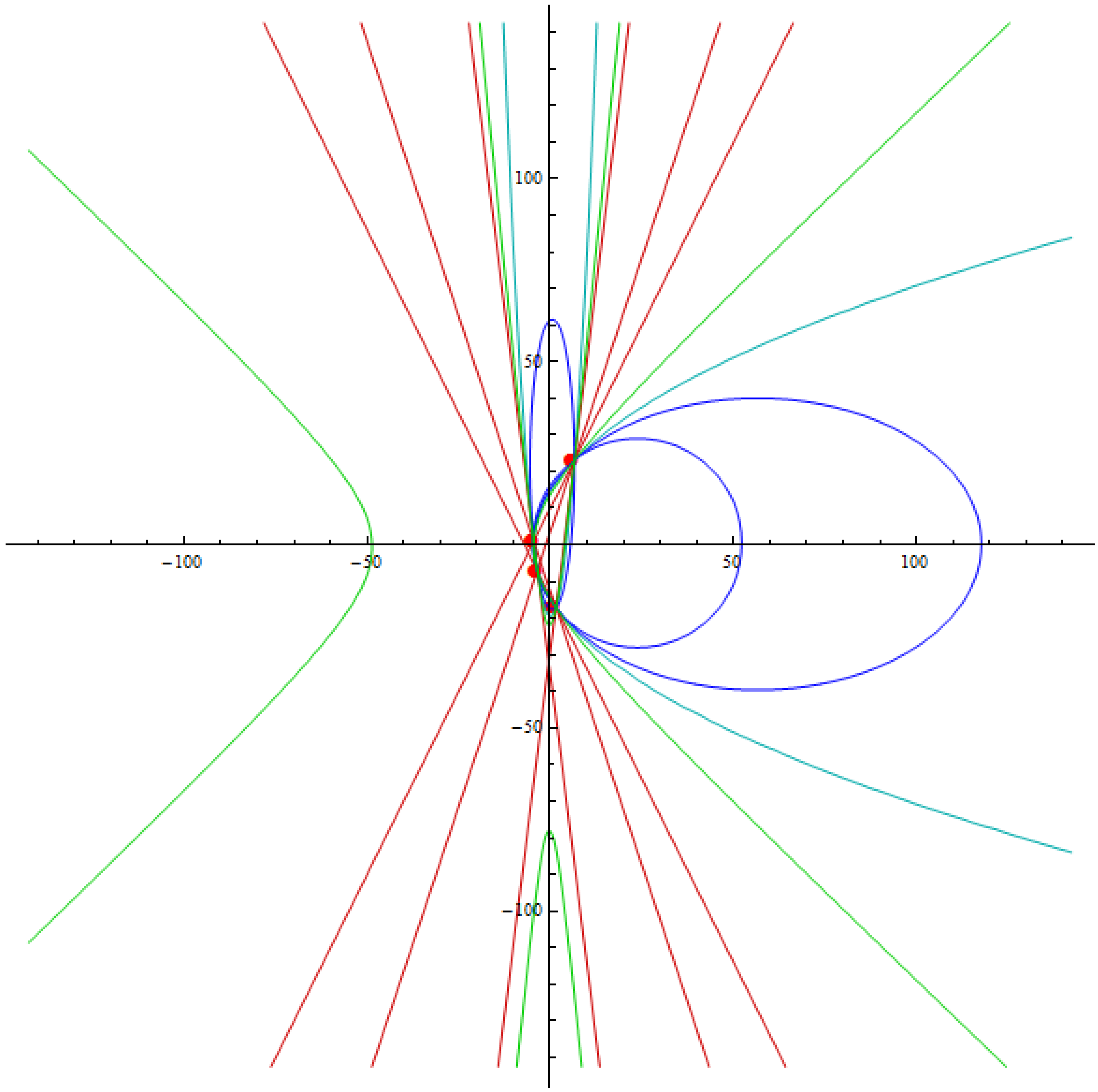

FIGURE 5.5. Various level curves of $I_q(\lambda, p)$ when $N = 5$

5.2. The Level Curves of I_c

Here we briefly discuss the A-level curve $I_c = A$ of I_c, which may be written

$$B_{n+4,n}B_{n+3,n+1} = AB_{n+4,n+1}B_{n+3,n}.$$

In $(\hat{p}, \hat{\Lambda})$-coordinates,

$$B_{n+4,n}B_{n+3,n+1} = \Big(\big(\hat{\Lambda} + 2N^2\big)^2 - 4\big(N\hat{p} + \tfrac{6}{5}\big)^2 - \tfrac{36}{25}\big(N^2 - 1\big)\big(N^2 - \tfrac{9}{4}\big)\Big)(\hat{\Lambda} + 3),$$

$$B_{n+3,n}B_{n+4,n+1} = \big(\hat{\Lambda} + 2N\hat{p} + \tfrac{7}{5}N^2 + \tfrac{15}{4}\big)\big(\hat{\Lambda} - (N - \tfrac{3}{2})\hat{p} + \tfrac{4}{5}N^2 + \tfrac{3}{10}N + \tfrac{3}{4}\big)$$
$$\times \big(\hat{\Lambda} - (N + \tfrac{3}{2})\hat{p} + \tfrac{4}{5}N^2 - \tfrac{3}{10}N + \tfrac{3}{4}\big).$$

Thus $B_{n+4,n}B_{n+3,n+1} = 0$ is a conic multiplied by a line, and $B_{n+3,n}B_{n+4,n+1} = 0$ is three lines. The intersection of $B_{n+4,n}B_{n+3,n+1} = 0$ with the three lines is nine points, and these nine points are on all of the level curves of I_c. Thus the family of level curves $I_c = A$ as A varies is the pencil of cubics through nine fixed points. (By Chasle's theorem, it would suffice to take any eight of the nine.)

THEOREM 5.5. *Assume that $k \neq 0$, 1, 2, or 3, and that $n \neq 0$, $-\frac{1}{2}$, -1, $-\frac{3}{2}$, -2, $-\frac{5}{2}$, or -3. The generic equivalence class of $\mathrm{SQ}^{k,5}_{\lambda,p}$ is six pairs of conjugate subquotients.*

PROOF. Let us fix a subquotient $\mathrm{SQ}^{k,5}_{\lambda,p}$. Assume the six b_{ij}'s are non-zero. By Theorem 4.10, a subquotient $\mathrm{SQ}^{k',5}_{\lambda',p'}$ is equivalent to $\mathrm{SQ}^{k,5}_{\lambda,p}$ if and only $p-k = p'-k'$, $\mathrm{SQ}^{k,5}_{\lambda,p}$ and $\mathrm{SQ}^{k',5}_{\lambda',p'}$ satisfy the same vanishing condition, and (λ, p) and (λ', p') lie on the same level curves of both I_q and I_c. By Bézout's theorem, the number of points of intersection of a conic and a cubic is six. □

We will now plot various level curves of $I_c(\lambda, p)$ for $N = 5$. The value of A increases as the figures progress. Note that for $A = 0$ we obtain the line $B_{n+3,n+1} = 0$ and the hyperbola $B_{n+4,n} = 0$, while as $A \to \pm\infty$ we obtain the three lines $B_{n+4,n+1}B_{n+3,n} = 0$ in the limit.

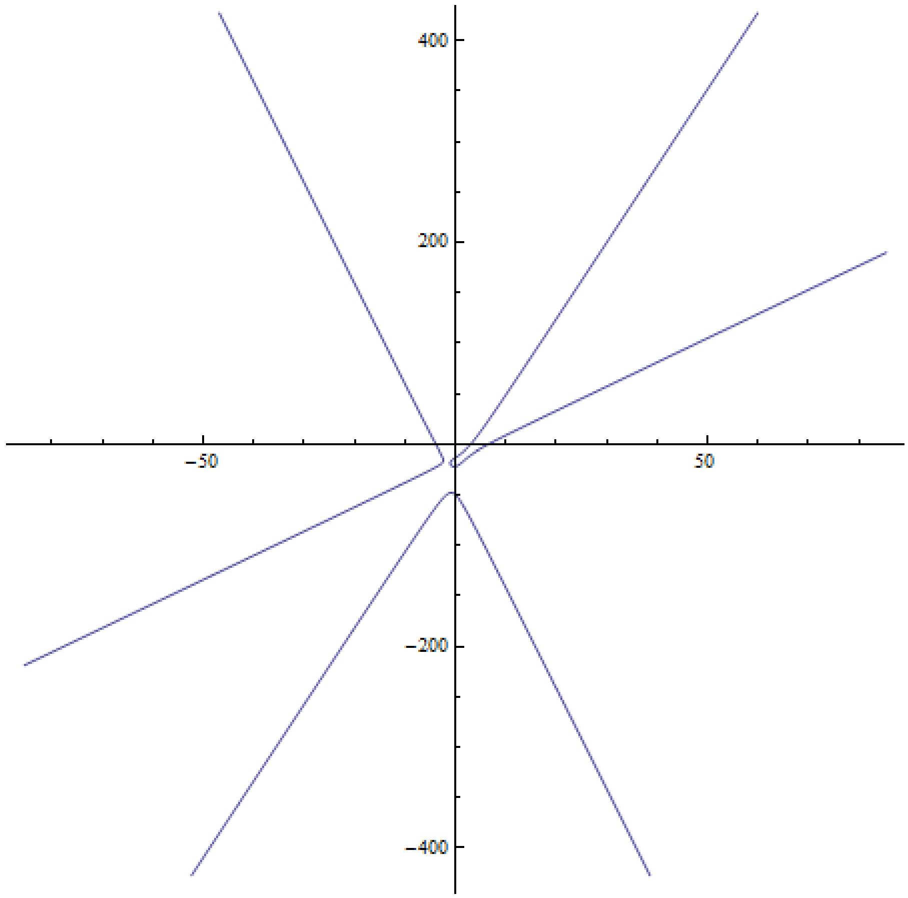

FIGURE 5.6. $I_c = -4$ when $N = 5$

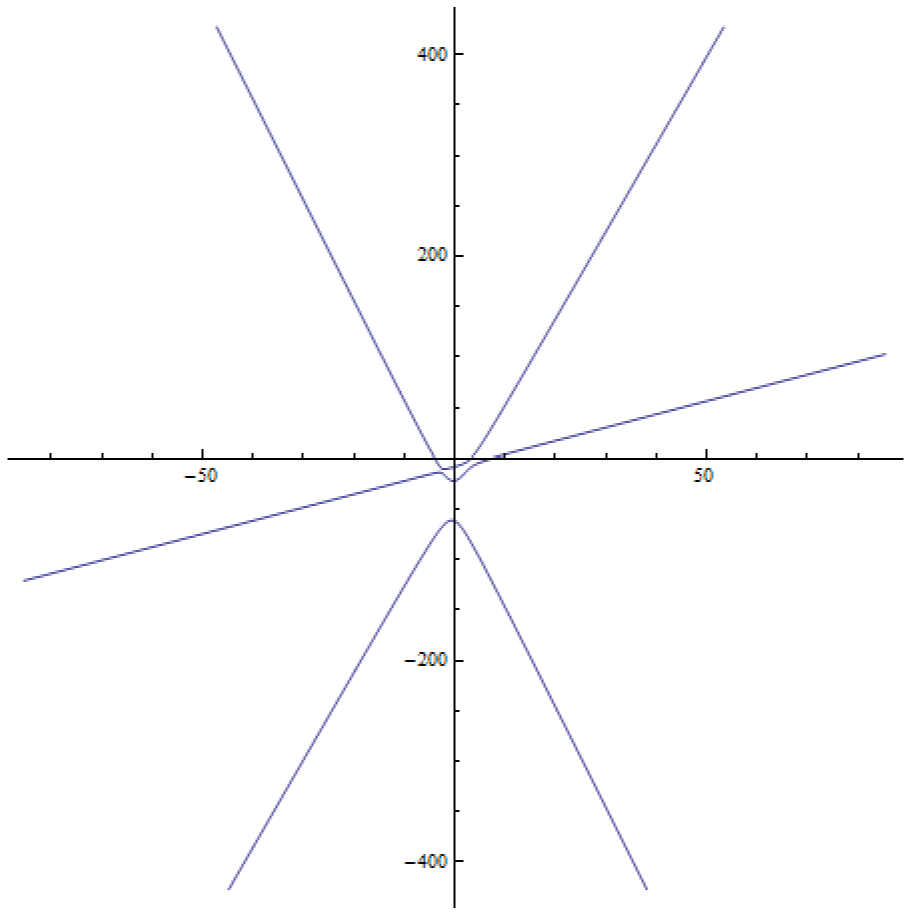

FIGURE 5.7. $I_c = -1$ when $N = 5$

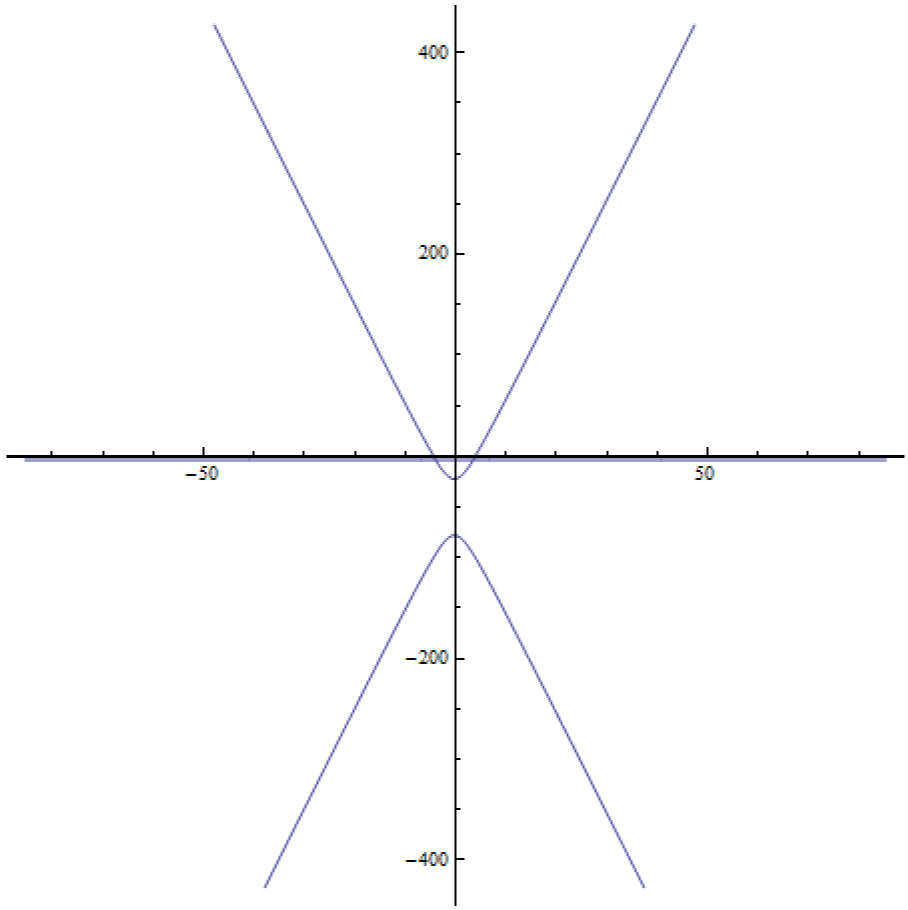

FIGURE 5.8. $I_c = 0$ when $N = 5$

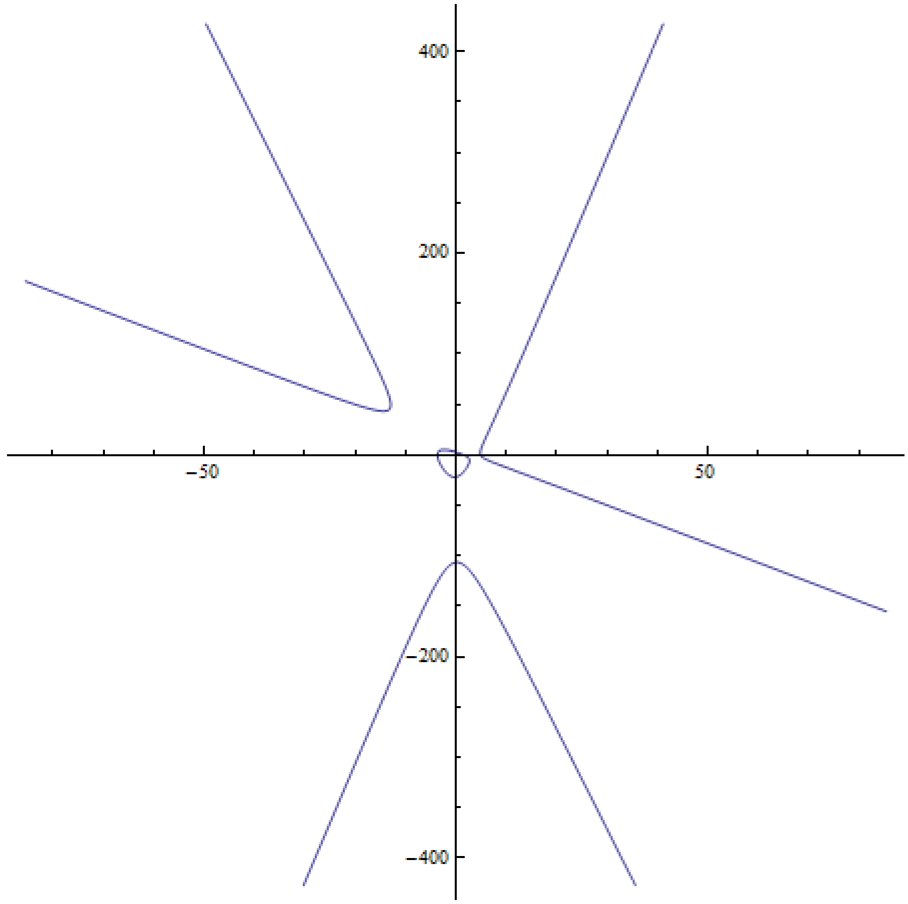

FIGURE 5.9. $I_c = 0.5$ when $N = 5$

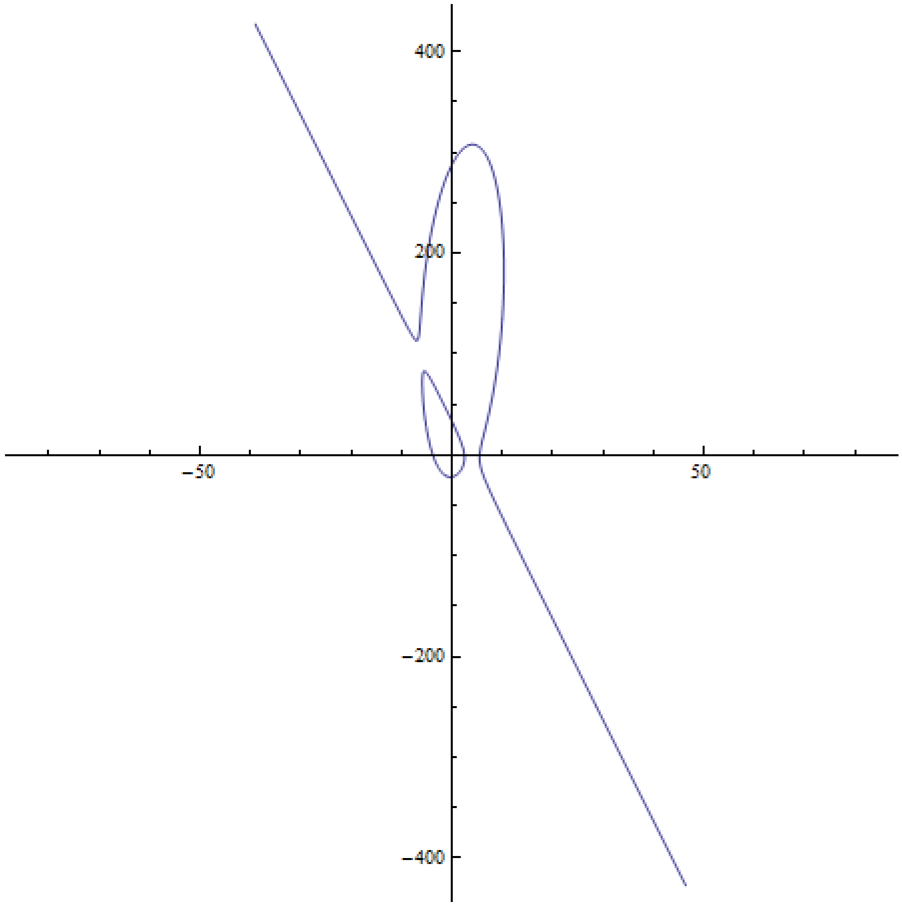

FIGURE 5.10. $I_c = 1.06$ when $N = 5$

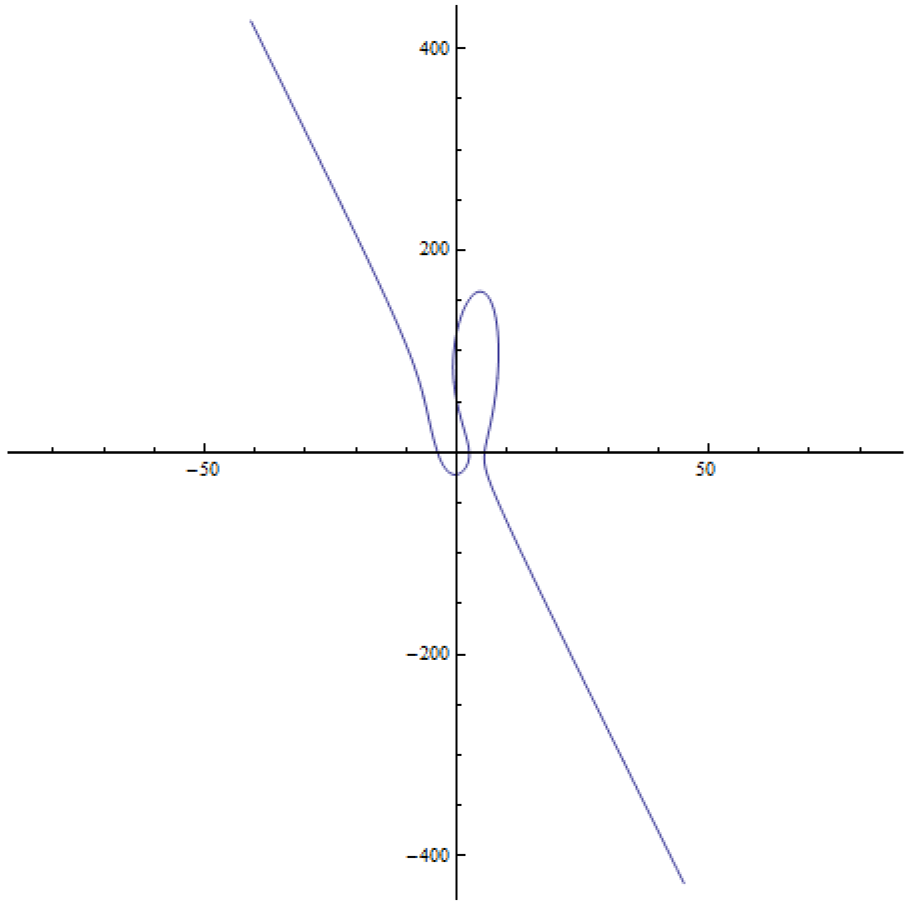

FIGURE 5.11. $I_c = 1.1$ when $N = 5$

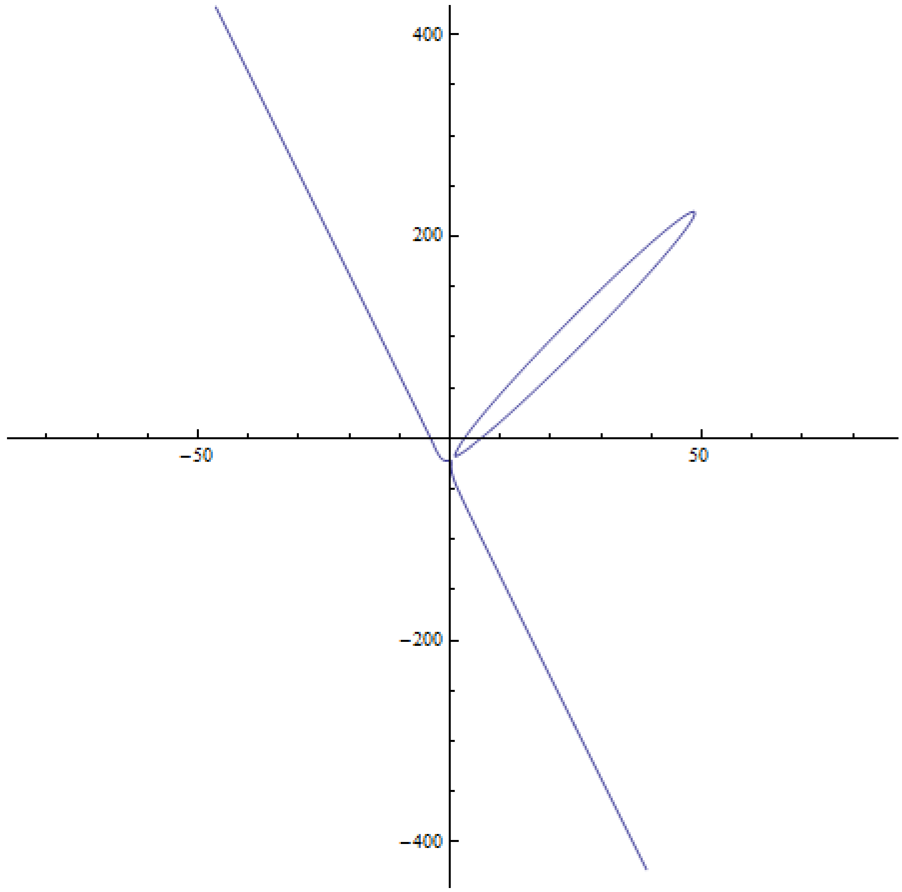

FIGURE 5.12. $I_c = 9$ when $N = 5$

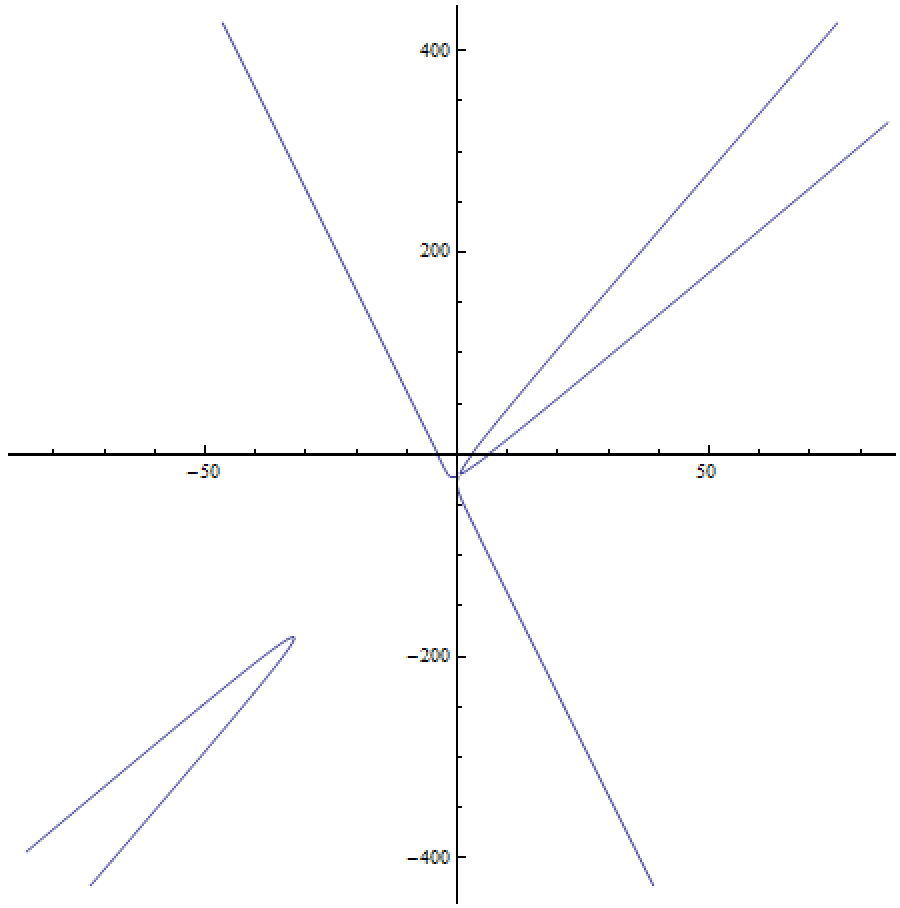

FIGURE 5.13. $I_c = 15$ when $N = 5$

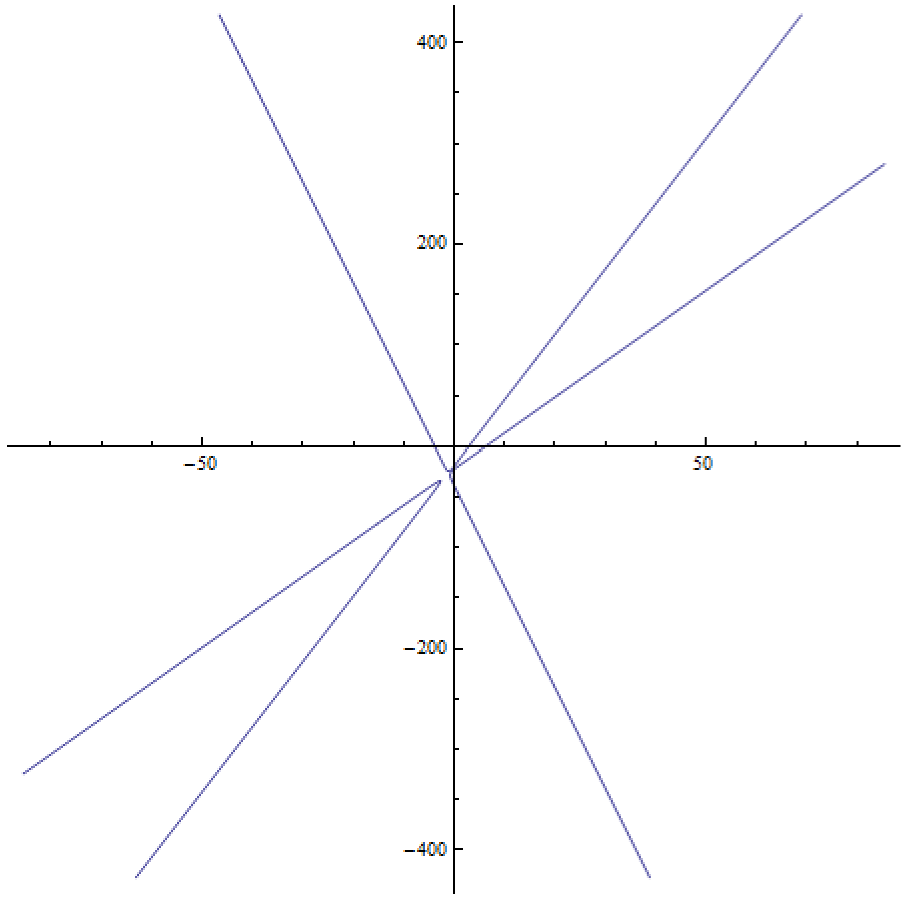

FIGURE 5.14. $I_c = 250$ when $N = 5$

5.3. The Lacunary Case

We will now define certain $\mathrm{Vec}\,\mathbb{R}$-subquotients of $\mathrm{SQ}^{k,5}_{\lambda,p}$ whose equivalence classes depend generically only on the invariant $I_q(\lambda,p)$. As before, let us write b_{mn}, b'_{mn}, B_{mn}, and B'_{mn} for $b_{mn}(\lambda,p)$, $b_{mn}(\lambda',p')$, $B_{mn}(\lambda,p)$, and $B_{mn}(\lambda',p')$ respectively. We assume throughout this section that $n = p' - k' = p - k$, and that it is not 0, $-\frac{1}{2}$, -1, $-\frac{3}{2}$, -2, $-\frac{5}{2}$, or -3. With this condition on n, Theorem 4.4 applies.

PROPOSITION 5.6. *The following block matrix defines a representation of* $\mathrm{Vec}\,\mathbb{R}$ *on the space* $F(n) \oplus F(n+2) \oplus F(n+4)$*:*

$$\begin{pmatrix} \pi_n & 0 & 0 \\ b_{n+2,n}\beta_2^n & \pi_{n+2} & 0 \\ b_{n+4,n}\beta_4^n & b_{n+4,n+2}\beta_2^{n+2} & \pi_{n+4} \end{pmatrix}.$$

This representation is a subquotient of $\mathrm{SQ}^{k,5}_{\lambda,p}$*, called the* lacunary subquotient $\mathrm{Lac}^k_{\lambda,p}$.

PROOF. As is noted in [5], for $n \neq 0$ there is a $\mathrm{Vec}\,\mathbb{R}$-submodule $\Psi^{k,\hat{1}}_{\lambda,p}$ of $\Psi^k_{\lambda,p}$ with composition series

$$F(n),\ F(n+2),\ F(n+3),\ F(n+4),\ldots.$$

This submodule contains $\Psi^{k-2}_{\lambda,p}$ but not $\Psi^{k-1}_{\lambda,p}$. It exists because of the zeroes on the first subdiagonal of the block matrix shown in Theorem 4.4. The lacunary subquotient is defined to be

$$\mathrm{Lac}^k_{\lambda,p} := \Psi^{k,\hat{1}}_{\lambda,p}/\Psi^{k-3,\hat{1}}_{\lambda,p}.$$

□

THEOREM 5.7. *Assume* $k \neq 0$, 1, 2, *or* 3, *and* $n = p - k = p' - k'$ *is not* 0, $-\frac{1}{2}$, -1, $-\frac{3}{2}$, -2, $-\frac{5}{2}$, *or* -3. *Then* $\mathrm{Lac}^k_{\lambda,p}$ *and* $\mathrm{Lac}^{k'}_{\lambda',p'}$ *are* $\mathrm{Vec}\,\mathbb{R}$*-equivalent if and only if* $B_{n+4,n}$, $B_{n+4,n+2}$, *and* $B_{n+2,n}$ *satisfy the same vanishing condition and one of the following mutually exclusive conditions holds:*

(1) $B_{n+4,n}B_{n+4,n+2}B_{n+2,n} = 0$.

(2) $B_{n+4,n}B_{n+4,n+2}B_{n+2,n} \neq 0$, *and* $I_q = I'_q$.

PROOF. Let $\mathrm{Lac}^{k}_{\lambda,p}$ and $\mathrm{Lac}^{k'}_{\lambda',p'}$ be two lacunary subquotients. Their composition series are the same if and only if $p-k=p'-k'$. As in Propositions 4.6 and 4.7, one proves that

$$\epsilon = \begin{pmatrix} \epsilon_{nn} & \epsilon_{n,n+2} & \epsilon_{n,n+4} \\ \epsilon_{n+2,n} & \epsilon_{n+2,n+4} & \epsilon_{n+2,n+4} \\ \epsilon_{n+4,n} & \epsilon_{n+4,n+2} & \epsilon_{n+4,n+4} \end{pmatrix}$$

is an equivalence from $\mathrm{Lac}^{k}_{\lambda,p}$ to $\mathrm{Lac}^{k'}_{\lambda',p'}$ if and only if it is diagonal, invertible, and $\epsilon_{mm}b_{mn} = b'_{mn}\epsilon_{nn}$. If none of $B_{n+4,n}$, $B_{n+4,n+2}$, and $B_{n+2,n}$ is zero and $I_q = I'_q$, then the reader may check that

$$\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b'_{n+2,n}/b_{n+2,n} & 0 \\ 0 & 0 & b'_{n+4,n}/b_{n+4,n} \end{pmatrix}$$

satisfies these properties. □

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