SCTM 63-60(14)

DETERMINATION OF REQUIRED COMPONENT RELIABILITY FROM SYSTEM RELIABILITY REQUIREMENTS

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ABSTRACT

The primary purpose of this study was to find a method for establishing component reliability requirements from system reliability specifications, because component reliability requirements are presently arbitrarily and vaguely defined. These requirements are considered as two separate problems--determination of allowable component catastrophic failure probabilities and determination of allowable distributions of component performance variables.

The results of the study consisted of:

1. A method for determining a unique allocation of component catastrophic failure probability, resulting in a minimum cost, for an assumed cost versus failure probability model.

2. A method for determining a unique allocation of the standard deviation of the component performance variables, resulting in a minimum cost, for an assumed cost versus standard deviation model.

3. Approximate methods for each of the above allocations, without assuming specific cost models.

March 3, 1960

TABLE OF CONTENTS

TABLE OF CONTENTS (Contd)

CHAPTER I -- INTRODUCTION

A. Problem Description

In many engineering fields, notably missile and weapon development, the design characteristics include numerical reliability requirements, as well as size, weight, performance, etc., requirements. Assuming for the moment that a definition of reliability has been agreed upon, part of the initial design task is to decide how reliable each component must be in order to meet the stated system reliability requirements. This is the problem discussed in this paper.

The method for expressing system failure probability in terms of probabilities of the "failure events" of the system will be described in Section A of Chapter II and in Section A of Chapter III. In trying to determine allowable "event" failure probabilities from the stated system requirements, the major problem is that there is no unique solution. In fact, the solution space has n - 1 degrees of freedom, where n is the number of failure events. For example, if allowable system failure probability, Q, is known and if Q is related to the probabilities of ten failure events, $A_1, A_2, \ldots, A_{10}$, by the relationship:

$$Q = \sum_{i=1}^{10} P(A_i) ;$$

where $P(A_i)$ is the probability of A_i--there are nine degrees of freedom in choosing the allowable failure probabilities of the ten failure events.

To solve this problem uniquely, it is necessary to impose some further constraints. Reliability is certainly some function of design man-hours, allowable size, allowable weights, production quality control, type of testing, and many other factors. However, the allowable size and weight as well as the allowable time until production units are available are specified in the design characteristics. But the cost of the system is not exactly specified. By putting more money into a certain component, more people and more facilities can be devoted to development, better quality control can be put into the production line, more extensive testing may be employed. Hence, the more money placed in the development program, the higher one might expect the reliability of the device to be.

For these reasons it was decided to find the solution that yields the minimum cost.[1] A procedure for doing this is described in this report.

Another major problem is determining exactly what constitutes a failure event. That is, if a resistor changes from a design value of 1 K ohm to a value of 3 K ohms, is this a failure event? For instance, if the distribution of the resistance values of resistors were considered, something of the form of Figure 1 might be expected.

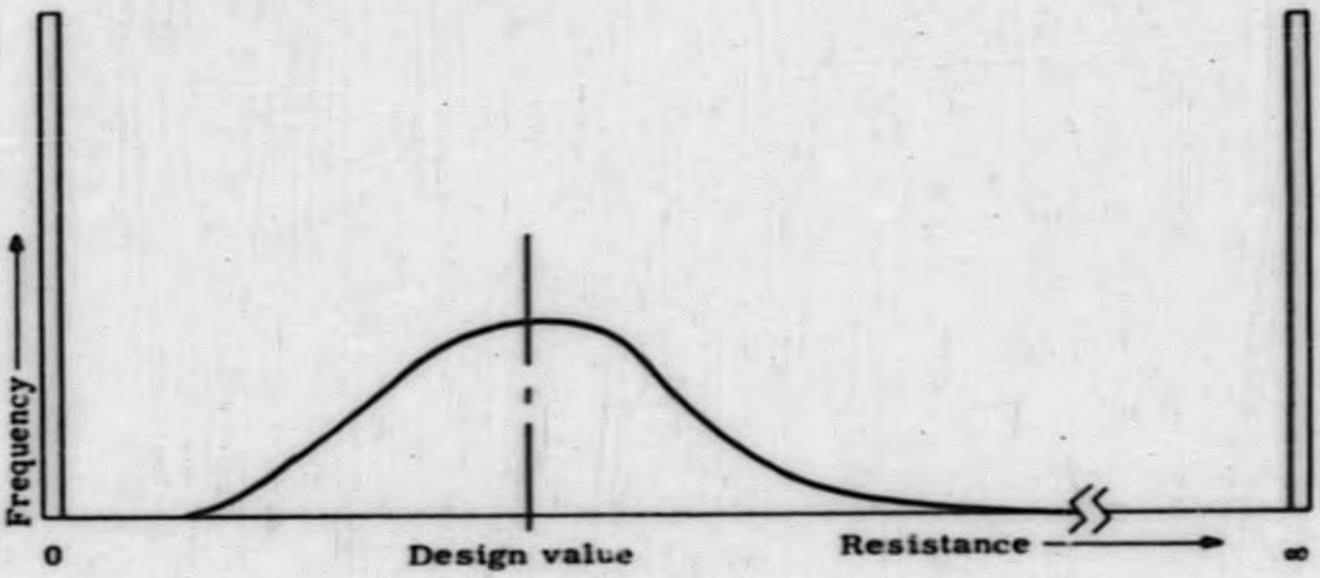

Figure 1

[1]If cost is not the paramount consideration in a particular situation, the value concept of game theory may be substituted, and the described procedure may be used.

Generally either 0 or ∞ is a failure. Whether a drift from the design value constitutes a failure is quite dependent upon the values of the other parameters in the circuit. For this reason, the predicted failure events will be discussed in two general categories:

1. Complete or catastrophic failures (e.g., an open resistor, a broken tube). Chapter II.
2. Out-of-tolerance failures (e.g., a resistor whose resistance is higher than the design value but not open, a tube with g_m less than design value but positive). Chapter III.

B. Basic Definitions

Before discussing the problem, a specific definition of reliability must be adopted since reliability has been defined in a number of ways. For the purposes of this problem, reliability is defined as a measure. Specifically, reliability is the probability of successful operation. This definition is briefly amplified below.

The definition of probability is that given by Cramer[2] as follows:

> "Whenever we say that the probability of an event E with respect to an experiment ξ is equal to P, the concrete meaning of this assertion will thus simply be the following: In a long series of repetitions of ξ, it is practically certain that the relative frequency of E will be approximately equal to "P"."

"Successful operation" is quite vague and perhaps needs some clarification. In the problem being discussed, it is assumed that successful system operation is clearly delineated in the design characteristics received by the design organization. The definition of successful component operation is discussed at some length in Chapter IIIA.

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[2] Cramer, H., see Reference 1, p. 148.

CHAPTER II -- COMPLETE FAILURES

A. Relationship Between Component and System Failure Probabilities

In a system, the manner in which a component fails is very important. For example, consider a simple component consisting of two cables in parallel where either cable will carry the required current, and where each cable has the following probabilities:

P(open cable) = q;

P(short to ground in cable) = r;

P(good cable) = s;

$q + r + s = 1.$

If one tried to reason by success probability (s) alone, it would be impossible to decide whether success of one cable (s) or success of two cables(s^2) were necessary for successful component operation. This is true because success of one cable may or may not be sufficient to ensure component success, dependent on whether the other cable failed by shorting to ground or by opening.

For this reason, equations relating the various modes of failure are derived rather than equations relating the success probabilities or reliabilities.

If the various failure probabilities of a system or a component are computed, then reliability, if desired, can be computed by recognizing:

$$\text{Reliability} = 1 - \sum \text{failure probabilities},$$

where the failure probabilities are mutually exclusive.

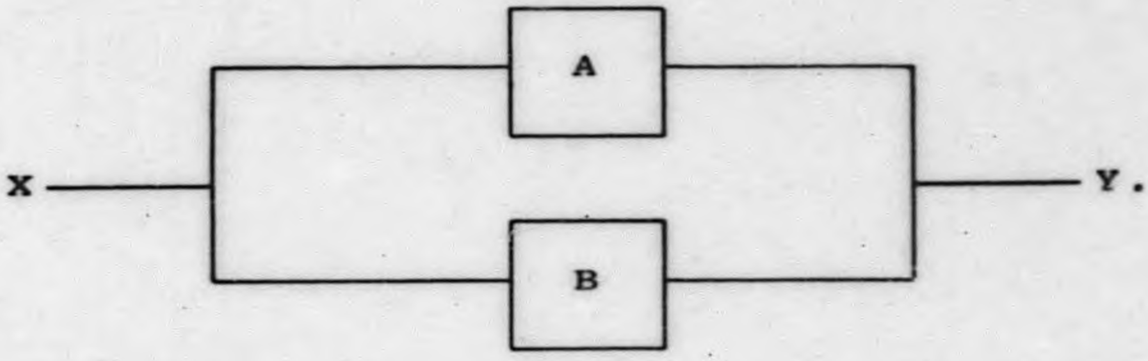
It should be noted that probabilities of failure events, rather than component failure probabilities, are used. However, component failure probabilities can be synthesized from the event failure probabilities quite easily. For example, the failure probability of a relay might be stated in terms of the probabilities of the following events:

1. Open coil
2. Open contact
3. Prematurely closed contact.

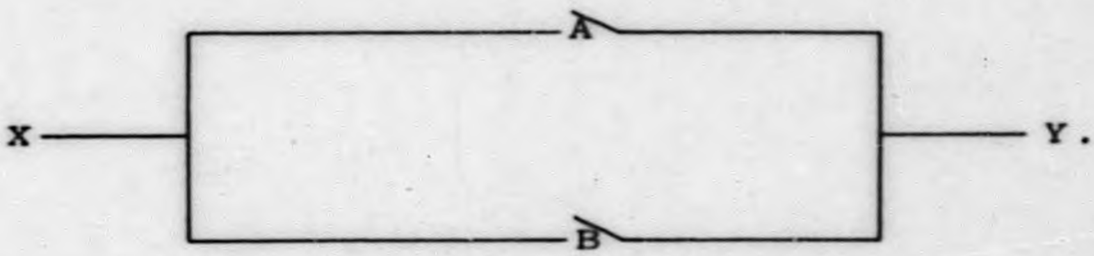
To find a relationship between probabilities of failure events and system failure probability, the failure events themselves must be anticipated. These events are occasionally not obvious (e.g., circulating ground currents or electromagnetic waves prematuring explosive switches, supposedly inert potting either breaking down electrical insulation or filtering into a switch and precluding switch operation); however, it is assumed in this paper that a careful engineering examination of the proposed system will yield the significant failure events.

1. Derivation of Failure Model

After the failure events are defined, the next step is to synthesize these events so that they properly describe system operation. This synthesis is described in Reference 2. However, for the purposes of this problem, an approximate synthesis, which will be called a "failure model," will be used. The derivation of the "failure model" will be approached from an extension of elementary set theory as follows: one group of failure events (such as the group of events leading to system failure to operate) is considered. A block diagram is then constructed from the definitions of the events and from a knowledge of the system. To illustrate, consider a simple system where two failure events (A, B) are defined. From the knowledge of the system, it is known that both events must occur to cause a certain system failure. Thus the block diagram may look like the following.



This block diagram may be thought of as a switch diagram with switches replacing the blocks as follows:



Success is now a complete circuit from X to Y. Then the failure event, A, is an open switch at A, and the failure event, B, is an open switch at B. From these models, it is obvious that both failure events A <u>and</u> B must occur to effect a failure of the model. Thus, Q, system failure probability, is represented by the equation:

$$Q = P(A \cap B) \; ; \qquad \text{(IIA1)}$$

where the symbol, $\cap$, means "and," and the equation represents the probability of the intersection of set A and set B.

This relationship is described in Figure 2. Generally,

$$P(A \cap B) = P(A)P(B|A)$$

$$= P(B)P(A|B) \; ,$$

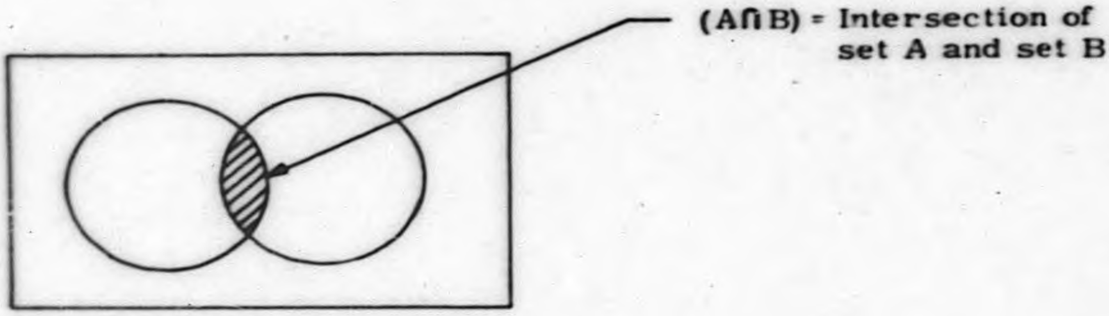


Figure 2

where the vertical line indicates "given." If the events A and B are assumed to be statistically independent:

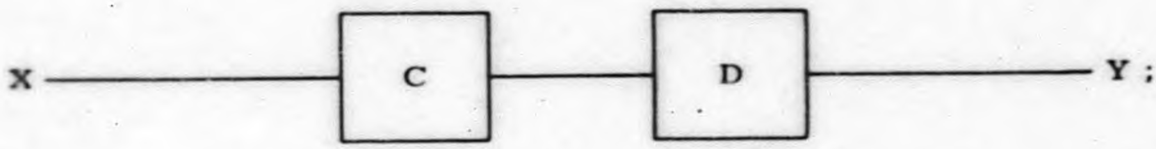
$$P(A|B) = P(A)$$

$$P(B|A) = P(B) .$$

Therefore:

$$Q = P(A \cap B) = P(A)P(B) . \qquad \text{(IIA2)}$$

Now consider another case where two failure events (C, D) are defined. This time consider that either one or both of these events will cause a failure to operate. Thus, the block diagram appears:



and the circuit diagram is:

Obviously in this circuit C or D or both will cause a circuit failure. (At this point, a standard notation will be adopted so that or implies and/or.) Thus the block diagram and the circuit diagram are simply models of the equipment.

The failure probability, Q, of the equipment will now be considered. If either C or D occurs, a dud will result, and the following equation may be written:

$$Q = P(C \cup D) \; . \tag{IIA3}$$

where the symbol "$\cup$" means "or" and the equation represents the union of set P(C) and set P(D).

The above equation may be written as follows:

$$Q = P(C) + P(D) - P(C \cap D) \; . \tag{IIA4}$$

The logic of this equation may be seen from Figure 3.

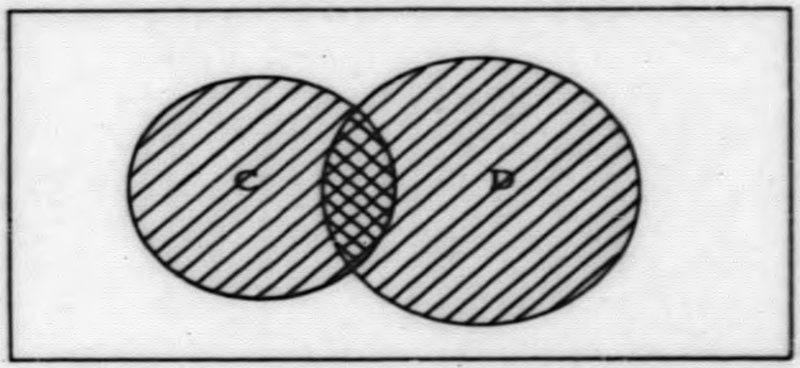


Figure 3

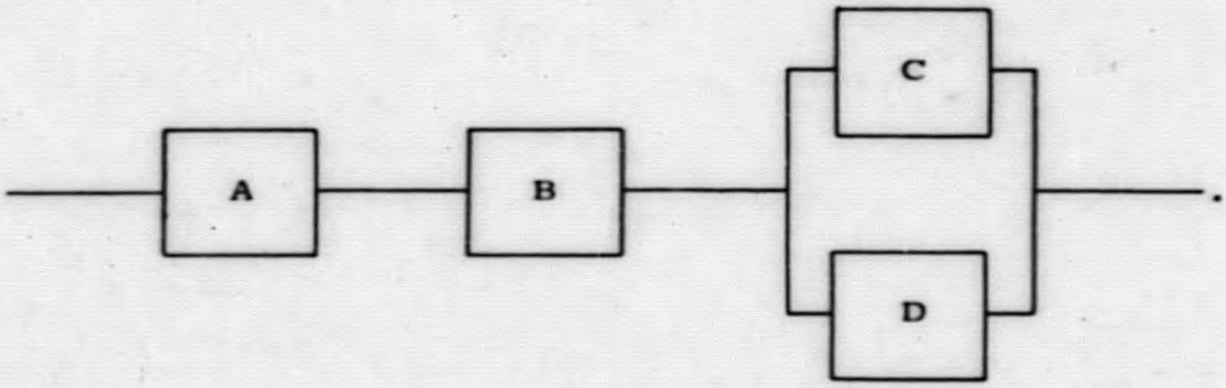
The shaded area is $(C \cup D)$. To obtain this union, the area (C) is added to the area (D). But the cross-hatched area has been added twice. Therefore, it must be subtracted once. This leads to the expression:

$$\text{Area}\,(C \cup D) = \text{Area}\,(C) + \text{Area}\,(D) - \text{Area}\,(C \cap D) \; .$$

Representing the areas as probabilities results in Equation IIA4. Equation IIA4 will be written:[3]

$$P(C \cup D) = P(C) + P(D) . \qquad \text{(IIA5)}$$

Now that two simple cases have been discussed, consider the following block diagram:



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[3] This approximation is justified because the term $P(C \cap D)$--the overlapped area of Figure 3--is assumed to be numerically small, as may be seen from the following reasoning:

$$P(C \cap D) = P(C)P(D), \text{ as explained before.}$$

It is assumed that system reliability requirements are such that the probabilities of failure events, such as C and D, must be on the order of 10^{-2} or smaller in order to have a system that is near the reliability requirements. Moreover, the estimate of events such as P(C) and P(D) is only made to one significant figure. For an example consider P(C) and P(D) are both 0.01. Then

$$\begin{aligned} Q &= P(C) + P(D) - P(C \cap D) \\ &= 0.02 - 0.0001 \\ &= 0.0199. \end{aligned}$$

But, by the rules of significant digits, this number should be rounded off to 0.02. This 0.02 is the same result that would be obtained if we wrote the equation $Q = P(C) + P(D)$. If the numbers used for failure event

The diagram, which is just a representation of the equipment, says that if A or B or (C and D) occurs, an equipment failure results.

The following equation results from a consideration of the above diagram:

$$Q = P(A \cup B \cup C \cap D)$$

$$= P(A) + P(B) + P(C \cap D) - P(A \cap B) - P(A \cap C \cap D)$$

$$- P(B \cap C \cap D) + P(A \cap B \cap C \cap D) .$$

Again the overlapped area is neglected;[4] so the approximate equation is:

$$Q = P(A) + P(B) + P(C \cap D) ;$$

and, again assuming that C and D are statistically independent, the equation becomes:

$$Q = P(A) + P(B) + P(C)P(D) .$$

Now, assuming statistical independence, a probability model for either a series or a parallel circuit may be generated. Therefore, all

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estimations are smaller, the simplification of the equation becomes even more obvious.

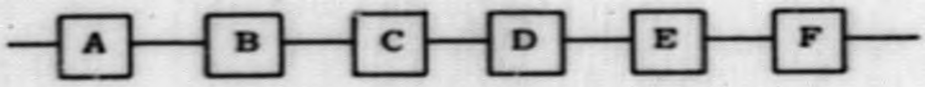
Moreover, deleting the terms that account for the overlap, such as $P(C \cap D)$, tends, if anything, to make the numerical result of equation more conservative. This is probably desirable in that a conservative estimate made early in the development program tends to offset the effect of unpredicted failure events.

[4]Notice that, in general, $P(A \cap B)$ is neglected when A and B are in series, but $P(C \cap D)$ is not neglected when C and D are in parallel. This is done because the probabilities of failure events that are in parallel are generally significantly greater than those in series (i.e., a high failure rate is the reason for paralleling components).

models which are a combination of series and parallel events may be derived by the extension of the logic given above.

2. Examples

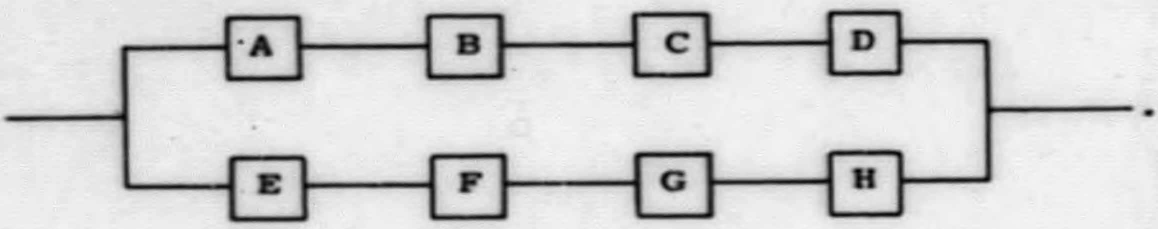
A series chain of events represented by the model



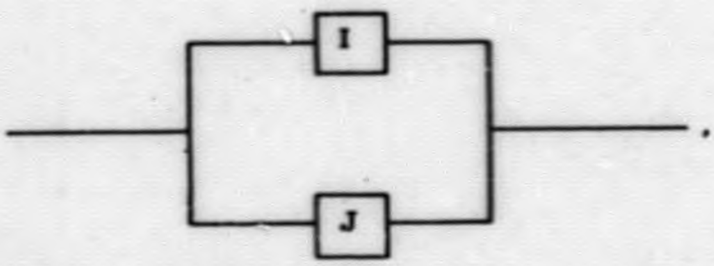
is simply described, under the assumptions and approximations given above by:

$$Q = P(A) + P(B) + P(C) + P(D) + P(E) + P(F) .$$

For another example, consider the following block diagram:



This can be reduced to the diagram:



where

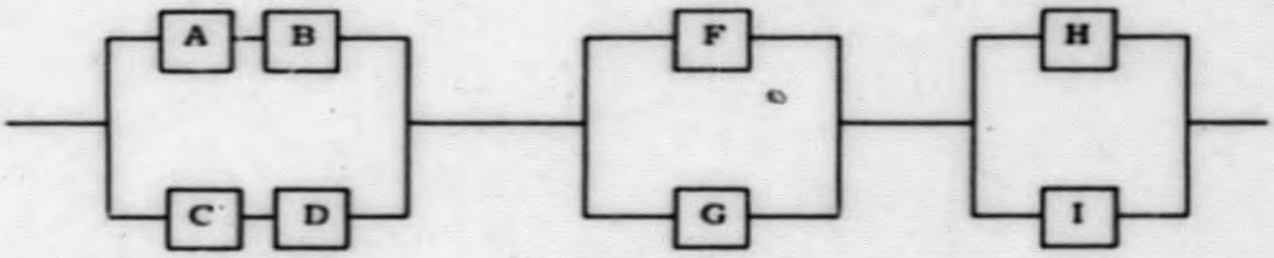
$$P(I) = P(A) + P(B) + P(C) + P(D) .$$

and

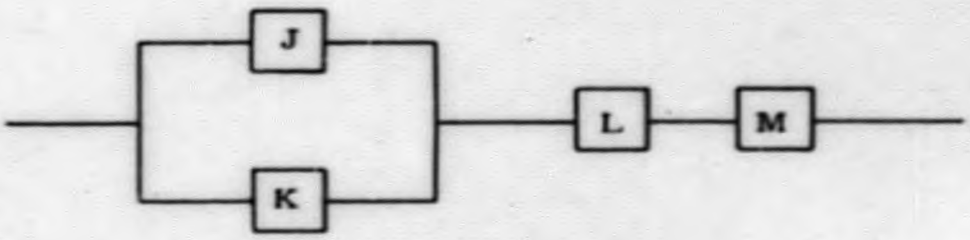
$$P(J) = P(E) + P(F) + P(G) + P(H) ,$$

and this circuit is an elementary parallel circuit evaluated previously.

As a last example, consider the block diagram:



This may be reduced to:



where

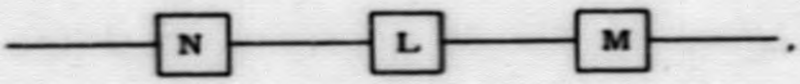
$$P(J) = P(A) + P(B) ,$$

$$P(K) = P(C) + P(D) .$$

$$P(L) = P(F)P(G) .$$

$$P(M) = P(H)P(I) .$$

This diagram may be further reduced to:



where

$$P(N) = P(J)P(K) .$$

Now the diagram has been reduced to a form evaluated previously. In a similar fashion any complex block diagram that is a series-parallel combination may be reduced in steps to a very elementary diagram.

By the method outlined above, system failure probability can be represented by a polynomial of degree equal to the highest number of modes of failure in parallel. For example, a system consisting of series and dual parallel modes of failure may be represented by an equation of the form:

$$Q = X_1 + X_2X_3 + (X_4 + X_5 + \ldots + X_n)(X_{n+1} + \ldots X_{n+m})$$

$$+ \text{ other terms of the same form ;} \qquad \text{(IIA6)}$$

where X_i is the probability of a failure event, and Q is the probability of system failure.

B. Determination of Allowable Catastrophic Failure Probabilities to Obtain Minimum Cost

In Section A of this chapter, a method for representing system failure probability in terms of the probabilities of the failure events was explained. The equation that represents this method is of the general form of Equation IIA6.

The problem to be discussed in this section is the determination of the X_i's, given Q, that will result in a minimum cost. To do this, the relationship between cost and probability of failure events must be known or assumed.

1. Discussion of Cost Versus Failure Probability Function

In Reference 3, a cost versus reliability function is assumed such that the cost for a given reliability is the same for all primitive components. In Reference 4 this is refined to the form:

$$\text{Cost} = K(-\ln r)^{-a} ;$$

where r is reliability and K and a are constants that vary from one device to another. This function seems reasonable and is practical for the problem to which it is applied. However, in allocating allowable failure probabilities in a system that is not a simple series circuit, failure probabilities rather than reliabilities are necessary if a component has more than one mode of failure. Nevertheless, the function given above might reasonably be used with 1 - X, where X is a certain failure probability, substituted for r.

In Reference 5, the cost of a certain component is assumed to be:

$$\text{Cost} = mc ;$$

where c is "basic" cost of one element and m is the number of elements used in parallel in the component. This function was used to determine the number of elements that should be placed in parallel in each component to yield minimum system cost, and seems quite reasonable for that problem. However, it does not seem appropriate for the present study because here it is assumed that reliability can be varied without changing the number of elements in parallel.

For the purposes of this study, a cost versus reliability function will be assumed that has two arbitrary parameters which will be determined for each failure event.

More explicitly, a function relating probability of failure to cost will be assumed such that there is one parameter that is determined by the cost of producing a certain predicted probability of failure and another parameter that is a measure of the rate of change of failure probability with respect to cost.

In hypothesizing a function relating cost and failure probability, there is a practical consideration that simplifies the task. That is, in a practical problem, the range of failure probability of interest is only from about 0 to 0.1. This conclusion results from over-all reliability requirements that are quite close to 1.

As a first approximation to the function, an exponential, $Y = \exp(-X)$ (where Y is cost and $X (0 \leq X \leq 1)$ is failure probability), was proposed. This exponential is partially satisfying in that the cost increases as failure probability decreases and that the cost always remains positive. But this function does not satisfy the logical requirement that cost should be infinite when the failure probability is zero. To satisfy this requirement the function was modified to be:

$$Y = \frac{\exp(-X)}{X}. \qquad \text{(IIB1)}$$

The next step was to introduce constants that could be chosen from the cost and failure probability estimates for a particular event. It seems reasonable to assume that, from knowledge of similar devices, a prediction of the probability of each failure event can be made. Similarly, an estimate of the cost of producing a device just this reliable can be made from previous cost records. This information will determine one constant.

Next, the manner in which failure probability varies with cost needs to be determined. Certainly the more money put into a device, the lower the failure probability should be. In some devices, a degree of reliability improvement can be achieved by simply putting more money into the production processes. That is, better and more expensive materials can be used, and more thorough quality control and production tests can be inaugurated. In other devices, reliability improvement involves putting more men and equipment into the development program. In still other devices, reliability improvement must await advancements in basic research, and additional money invested in the development program will produce little reliability improvement in these devices within the time scales of the development program.

This fact that curves passing through the same point on a cost versus failure probability plot may have different slopes calls for an additional constant to be chosen for each failure event. The two constants that must be determined for each failure event modify Equation IIB1 to

$$Y = \frac{K_1}{X}\exp(-K_2X) . \tag{IIB2}$$

This expression will be used in the following development.

As an example of choosing the constants K_1 and K_2, assume that with a cost of \$10,000, a failure event's predicted probability is 0.01 and that to reduce the failure probability to 0.008, the additional cost would be \$5000.

$$\begin{aligned} 10000 &= \frac{K_1}{0.01}\exp[-K_2(0.01)] ; \\ 100 &= K_1\exp(-0.01K_2) . \end{aligned} \tag{IIB3}$$

$$15000 = \frac{K_1}{0.008}\exp[-K_2(0.008)] \; ; \qquad \text{(IIB4)}$$

$$120 = K_1\exp[-0.008K_2] \; .$$

Dividing Equation IIB4 by Equation IIB3:

$$1.2 = \frac{\exp(-0.008K_2)}{\exp(-0.01K_2)} = \exp(0.002K_2) \; ;$$

$$K_2 = \frac{1}{0.002}\ln(1.2) \cong (500)(0.182) = 91 \; .$$

Using this in Equation IIB3:

$$100 = K_1\exp(-0.91) \cong K_1 0.403 \; ;$$

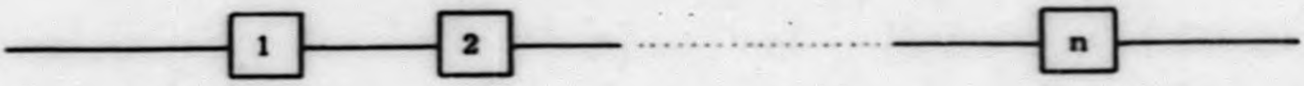
$$K_1 = 248 \; .$$

Some question should be raised about how good a model is the function that has been assumed. Of course, no model can be better than the available reliability and cost estimates. In this instance, the data for the estimates are sparse. Moreover, within the limitations of the data, many curves could be chosen that would fit the two data points and the point $(0, \infty)$.

The justification for assuming any curve is that, lacking a relationship between cost and failure probability, the allocation of failure probabilities consists of arbitrarily choosing one of many solutions. The only justification for choosing the form $K_1\exp(-K_2X)/X$ rather than some other function is that it seems reasonable and is relatively simple mathematically.

2. Application to Series Events

Consider a system consisting of n modes of failure in series.



With each mode of failure there is an assumed cost relationship. That is, for the i^{th} mode of failure:

$$Y_i = \frac{K_{1i}\exp(-K_{2i}X_i)}{X_i}, \quad i = 1, 2, \ldots, n, \tag{IIB5}$$

where:

Y_i = cost associated with i^{th} mode of failure

X_i = failure probability of i^{th} mode of failure

K_{1i} and K_{2i} = constants relating to the i^{th} mode of failure. These constants are assumed to be known at this point.

The allowable system failure probability, Q, is given by the equation:

$$Q = \sum_{i=1}^{n} X_i . \tag{IIB6}$$

The total cost (Z) is:

$$Z = \sum_{i=1}^{n} Y_i . \tag{IIB7}$$

The objective is to minimize Z, subject to the constraint Equation IIB6. To minimize Z, the method of Lagrange multipliers[5] is used:

$$Z = \sum_i Y_i = \sum_i \frac{K_{1i} \exp(-K_{2i} X_i)}{X_i} .$$

$$f(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} \frac{K_{1i} \exp(-K_{2i} X_i)}{X_i} + \lambda \left[\left(\sum_{i=1}^{n} X_i \right) - Q \right] .$$

$$\frac{\partial f}{\partial X_1} = \frac{-K_{11}[\exp(-K_{21} X_1)](1 + K_{21} X_1)}{X_1^2} + \lambda = 0 ,$$

. . . .

$$\frac{\partial f}{\partial X_i} = \frac{-K_{1i}[\exp(-K_{2i} X_i)](1 + K_{2i} X_i)}{X_i^2} + \lambda = 0 .$$

These n equations contain n + 1 unknowns (n x's and λ).

An additional equation,

$$Q = \sum_{i=1}^{n} X_i . \qquad \text{(IIB6)}$$

provides the necessary number of equations for a unique[6] solution.

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[5]See Reference 6, pp. 249-57.

[6]For further discussion, see Appendix.

To find this solution, each of the first n equations is written in the form:

$$\lambda = \frac{K_{1i}[\exp(-K_{2i}X_i)](1 + K_{2i}X_i)}{X_i^2}.$$

Equating two of these relationships (say 1 and 2):

$$\frac{K_{11}[\exp(-K_{21}X_1)](1 + K_{21}X_1)}{X_1^2} = \frac{K_{12}[\exp(-K_{22}X_2)](1 + K_{22}X_2)}{X_2^2}.$$

Rearranging:

$$\frac{\exp(-K_{21}X_1)}{\exp(-K_{22}X_2)} = \frac{K_{12}}{K_{11}}\left(\frac{X_1}{X_2}\right)^2\left(\frac{1 + K_{22}X_2}{1 + K_{21}X_1}\right).$$

or:

$$\exp(-K_{21}X_1 + K_{22}X_2) = \frac{K_{12}}{K_{11}}\left(\frac{X_1^2}{1 + K_{21}X_1}\right)\left(\frac{1}{X_2^2} + \frac{K_{22}}{X_2}\right).$$

In general, if the 1^{st} equation is equated to the j^{th}:

$$\exp(-K_{21}X_1 + K_{2j}X_j) = \frac{K_{1j}}{K_{11}}\left(\frac{X_1^2}{1 + K_{21}X_1}\right)\left(\frac{1}{X_j^2} + \frac{K_{2j}}{X_j}\right). \quad \text{(IIB8)}$$

The system of equations has now been reduced to n equations, with n - 1 in the form of Equation IIB8 and the equation:

$$Q = \sum_{i=1}^{n} X_i \, . \tag{IIB6}$$

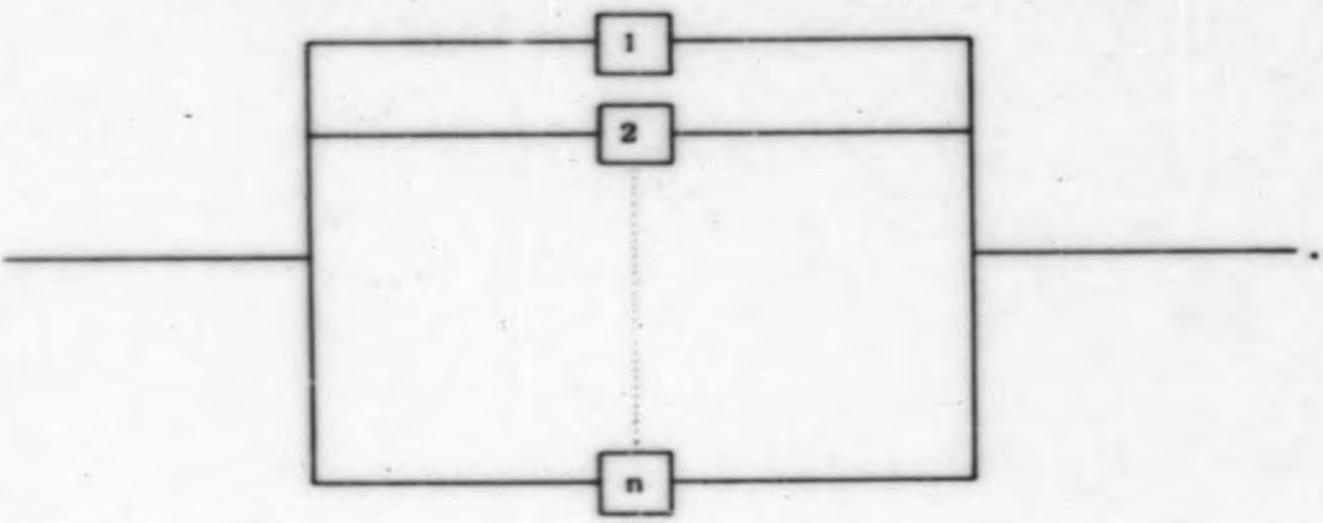
To solve this system, guess X_1 (a good guess would be Q/n); calculate the remaining X's from Equation IIB8. A method for solving these equations is to plot the left side of Equation IIB8 versus X_j and to plot the right side of Equation IIB8 versus X_j. The point of intersection is the solution. Then calculate a $\hat{Q}$ from Equation IIB6. If $\hat{Q}$ does not agree with the given Q, estimate X_1 again, calculate a new $\hat{Q}$, etc. After the desired Q has been bracketed by two estimates, say $\hat{Q}_1$ and $\hat{Q}_2$, a new guess for X_1 might be formed by a linear interpolation between the trial values of X_1 that led to $\hat{Q}_1$ and $\hat{Q}_2$. For instance, if X_{11} led to $\hat{Q}_1$ and X_{12} led to $\hat{Q}_2$, the new trial for $X_1(X_{1T})$ would be:

$$X_{1T} = X_{11} + \frac{Q - \hat{Q}_1}{\hat{Q}_2 - \hat{Q}_1}(X_{12} - X_{11}) \, .$$

This process can be repeated until Q and $\hat{Q}$ agree. An example solution is given in Section IIB5.

3. Application to Parallel Events

Now a simple system of n elements in parallel will be considered.



The cost is again represented by:

$$Z = \sum_{i=1}^{n} Y_i . \tag{IIB7}$$

The same equation relating cost and unreliability is assumed.

$$Y_i = \frac{K_{1i}\exp(-K_{2i}X_i)}{X_i} . \tag{IIB5}$$

The equation relating failure probabilities is:

$$Q = \prod_i X_i . \tag{IIB9}$$

Using the same general approach as before:

$$f(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} \frac{K_{1i}\exp(-K_{2i}X_i)}{X_i} + \lambda\left[\left(\prod_i X_i\right) - Q\right] .$$

$$\frac{\partial f}{\partial X_1} = \frac{-K_{11}[\exp(-K_{21}X_1)](1 + K_{21}X_1)}{X_1^2} + \lambda \prod_{j \neq 1} X_j = 0 ,$$

. . . .

$$\frac{\partial f}{\partial X_i} = \frac{[-K_{1i}\exp(-K_{2i}X_i)](1 + K_{2i}X_i)}{X_i^2} + \lambda \prod_{j \neq i} X_j = 0 .$$

Rearranging:

$$\lambda \prod_{j \neq i} X_j = \frac{K_{1i}[\exp(-K_{2i}X_i)](1 + K_{2i}X_i)}{X_i^2} .$$

or:

$$\lambda \prod_j X_j = \frac{K_{1i}[\exp(-K_{2i}X_i)](1 + K_{2i}X_i)}{X_i} .$$

Equating the 1^{st} such expression to the i^{th}:

$$\frac{K_{11}[\exp(-K_{21}X_1)](1 + K_{21}X_1)}{X_1} = \frac{K_{1i}[\exp(-K_{2i}X_i)](1 + K_{2i}X_i)}{X_i} .$$

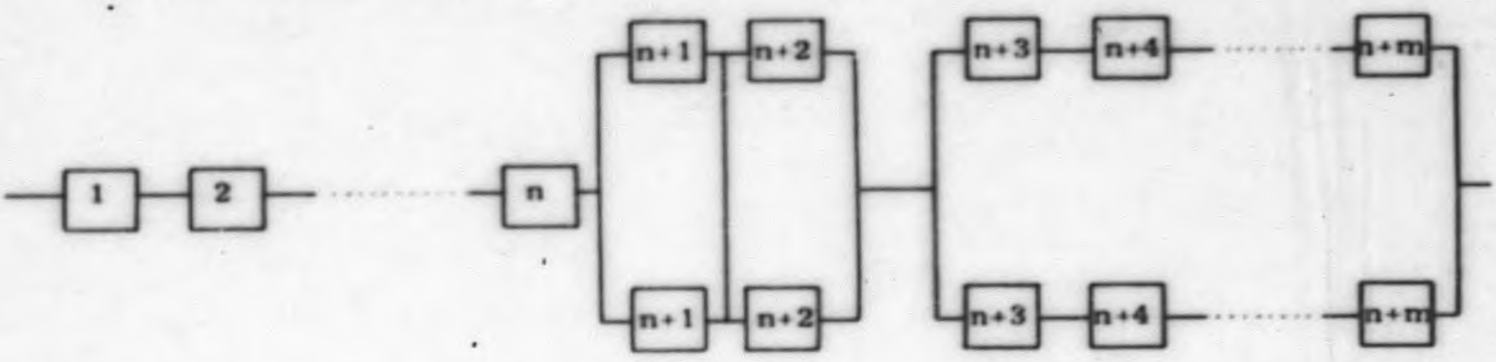
The expression is nearly the same as that derived in the series case. Hence, the solution is achieved in the same manner. That is, rearranging in the same manner as before:

$$\exp(K_{2i}X_i - K_{21}X_1) = \frac{K_{1i}}{K_{11}}\left(\frac{X_1}{1 + K_{21}X_1}\right)\left(\frac{1}{X_i} + K_{2i}\right) . \qquad \text{(IIB10)}$$

Again this may be solved by a graph similar to that used in the series case and illustrated in Section IIB5.

4. <u>Application to Series Parallel Events</u>

Series and parallel arrangements have been discussed. Now a combination of series and parallel events will be considered. In this consideration, the assumed circuit will be made more practical in that only two identical events will be in parallel. Consider the diagram:



with n events in series, two sets of parallel events in series, and m - 2 sets of identical series events in parallel.

Then, using the same definitions as before:

$$Q = \sum_{i=1}^{n} X_i + \sum_{i=n+1}^{n+2} X_i^2 + \left(\sum_{i=n+3}^{n+m} X_i\right)^2 . \qquad \text{(IIB11)}$$

$$Z = \sum_{i=1}^{n} Y_i + 2 \sum_{i=n+1}^{m} Y_i . \qquad \text{(IIB12)}$$

$$Y_i = \frac{K_{1i}\exp(-K_{2i}X_i)}{X_i} . \qquad \text{(IIB5)}$$

$$f = \sum_{i=1}^{n} \frac{K_{1i}\exp(-K_{2i}X_i)}{X_i} + 2 \sum_{i=n+1}^{m} \frac{K_{1i}\exp(-K_{2i}X_i)}{X_i}$$

$$+ \lambda\left[\sum_{i=1}^{n} X_i + \sum_{i=n+1}^{n+2} X_i^2 + \left(\sum_{i=n+3}^{n+m} X_i\right)^2 - Q\right] .$$

$$\frac{\partial f}{\partial X_1} = -\frac{K_{11}[\exp(-K_{21}X_1)](1 + K_{21}X_1)}{X_1^2} + \lambda = 0 ,$$

$$\frac{\partial f}{\partial X_{n+1}} = -\frac{2K_{1,n+1}[\exp(-K_{2,n+1}X_{n+1})](1 + K_{2,n+1}X_{n+1})}{X_{n+1}^2}$$

$$+ 2\lambda X_{n+1} = 0 ,$$

$$\frac{\partial f}{\partial X_{n+3}} = -\frac{2K_{1,n+3}[\exp(-K_{2,n+3}X_{n+3})](1 + K_{2,n+3}X_{n+3})}{X_{n+3}^2}$$

$$+ 2\lambda \sum_{i=n+3}^{n+m} X_i = 0 .$$

$$\lambda = \frac{K_{11}[\exp(-K_{21}X_1)](1 + K_{21}X_1)}{X_1^2}, \tag{IIB13}$$

$$\lambda = \frac{K_{1,n+1}[\exp(-K_{2,n+1}X_{n+1})](1 + K_{2,n+1}X_{n+1})}{X_{n+1}^3}, \tag{IIB14}$$

$$\lambda = \frac{K_{1,n+3}[\exp(-K_{2,n+3}X_{n+3})](1 + K_{2,n+3}X_{n+3})}{X_{n+3}^2 \sum_{i=n+3}^{n+m} X_i}. \tag{IIB15}$$

Using the same method as in the series case, X_2 through X_n may be found in terms of X_1.

Equating expressions IIB13 and IIB14:

$$\frac{K_{11}[\exp(-K_{21}X_1)](1 + K_{21}X_1)}{X_1^2}$$

$$= \frac{K_{1,n+1}[\exp(-K_{2,n+1}X_{n+1})](1 + K_{2,n+1}X_{n+1})}{X_{n+1}^3}.$$

Rearranging:

$$\exp(-K_{21}X_1 + K_{2,n+1}X_{n+1}) = \frac{K_{1,n+1}}{K_{11}}\left(\frac{X_1^2}{1 + K_{21}X_1}\right)\left(\frac{1}{X_{n+1}^2}\right)\left(\frac{1}{X_{n+1}} + K_{2,n+1}\right).$$

By a graph of the type described previously, X_{n+1} and X_{n+2} (which is related to X_1 by an exactly similar equation) may be found in terms of X_1.

Attention is now directed toward finding X_{n+3} through X_{n+m} in terms of X_1. Equating Equations IIB13 and IIB15:

$$\frac{K_{11}[\exp(-K_{21}X_1)](1 + K_{21}X_1)}{X_1^2}$$

$$= \frac{K_{1,n+3}[\exp(-K_{2,n+3}X_{n+3})](1 + K_{2,n+3}X_{n+3})}{X_{n+3}^2 \sum_{i=n+3}^{n+m} X_i}. \qquad \text{(IIB16)}$$

The only additional difficulty is in evaluating

$$\sum_{i=n+3}^{m} X_i .$$

To approximate this expression, it is convenient to get the best approximation of X_i, $i = n + 3, \ldots, n + m$, in the form:

$$X_i = A_i X_{n+3} .$$

To accomplish this, two equations of the form Equation IIB15 are equated--specifically the $n + 3^{rd}$ and the $n + i^{th}$ ($i = 4, 5, \ldots, m$):

$$\frac{K_{1,n+3}[\exp(-K_{2,n+3}X_{n+3})](1 + K_{2,n+3}X_{n+3})}{X_{n+3}^2 \sum_{i=n+3}^{n+m} X_i}$$

$$= \frac{K_{1,n+i}[\exp(-K_{2,n+i}X_{n+i})](1 + K_{2,n+i}X_{n+i})}{X_{n+i}^2 \sum_{i=n+3}^{n+m} X_i}.$$

The

$$\sum_{i=n+3}^{n+m} X_i$$

cancel, resulting in the identical equation that was solved for the series case. Thus, $X_{n+i}(i = 4, 5, \ldots, m)$ may be found in terms of X_{n+3}.

Then by guessing, X_{n+3}, $X_{n+i}(i = 4, 5, \ldots, m)$ may be found. Therefore,

$$\sum_{i=n+3}^{n+m} X_i$$

may be found. Returning to Equation IIB16 and rearranging:

$$\exp[-K_{2,n+3}X_{n+3} + K_{21}X_1] = \frac{X_{n+3}^2 \sum_{i=n+3}^{n+m} X_i}{1 + K_{2,n+3}X_{n+3}}$$

$$\left(\frac{K_{11}}{K_{1,n+3}}\right)\left(\frac{1}{X_1}\right)\left(\frac{1}{X_1} + K_{21}\right).$$

From this expression, X_1 may be found and then all of the remaining X_i's may be found. Again a $\hat{Q}$ can be computed, a new X_{n+3} can be tried, and an iterative procedure can be carried on until Q and $\hat{Q}$ agree.

The solution to this problem in an actual system will be quite involved numerically, as is illustrated by the following example.

5. Example

A simple system consists of three failure events, A, B, and C in series.

The failure probabilities of A, B, and C are X_1, X_2, and X_3, respectively.

The following information is assumed:

Maximum allowable system failure probability, $Q = 0.01$.

Cost relationships:

Failure event	Probability	Cost
X_1	0.01	$ 10,000
X_1	0.005	100,000
X_2	0.01	20,000
X_2	0.005	100,000
X_3	0.01	10,000
X_3	0.005	200,000

Finding the cost versus X_1 relationship:

$$10,000 = \frac{K_1 \exp[-K_2(0.01)]}{0.01},$$

$$100 = K_1 \exp(-0.01 K_2).$$

$$100,000 = \frac{K_1 \exp[-K_2(0.005)]}{0.005} ,$$

$$500 = K_1 \exp(-0.005K_2) .$$

$$5 = \exp(-0.005K_2 + 0.01K_2) ,$$

$$5 = \exp(0.005K_2) .$$

$$K_2 \cong \frac{1.61}{0.005} = 322 .$$

$$100 = K_1 \exp(-3.22) ,$$

$$K_1 \cong \frac{100}{0.04} = 2500 .$$

Therefore:

$$\mathrm{Cost}_A = Y_1 = \frac{2500\exp(-322X_1)}{X_1} .$$

Similarly:

$$\mathrm{Cost}_B = Y_2 = \frac{1243\exp(-183X_2)}{X_2} ,$$

$$\mathrm{Cost}_C = Y_3 = \frac{10,000\exp(-460X_3)}{X_3} .$$

$$\text{Total Cost} = Z = \frac{2500\exp(-322X_1)}{X_1} + \frac{1243\exp(-183X_2)}{X_2}$$

$$+ \frac{10{,}000\exp(-460X_3)}{X_3} .$$

$$Q = X_1 + X_2 + X_3 .$$

Using Equation IIB8:

$$\exp(-322X_1 + 183X_2) = \frac{1243}{2500}\left(\frac{X_1^2}{1 + 322X_1}\right)\left(\frac{1}{X_2^2} + \frac{183}{X_2}\right) . \quad \text{(IIB17)}$$

$$\exp(-322X_1 + 460X_3) = \frac{10{,}000}{2500}\left(\frac{X_1^2}{1 + 322X_1}\right)\left(\frac{1}{X_3^2} + \frac{460}{X_3}\right) . \quad \text{(IIB18)}$$

Guess $X_1 = 0.003$. A graphical solution yields:

from Figure 4 $X_2 = 0.0024$;

from Figure 5 $X_3 = 0.0044$.

$\hat{Q} = 0.003 + 0.0024 + 0.0044 = 0.0098.$

This number is close enough to the required Q(0.01) for a practical case. However, to demonstrate the iterative procedure, the new estimate for X_1 is:

$$0.003 + \frac{0.01 - 0.0098}{3} = 0.0031 .$$

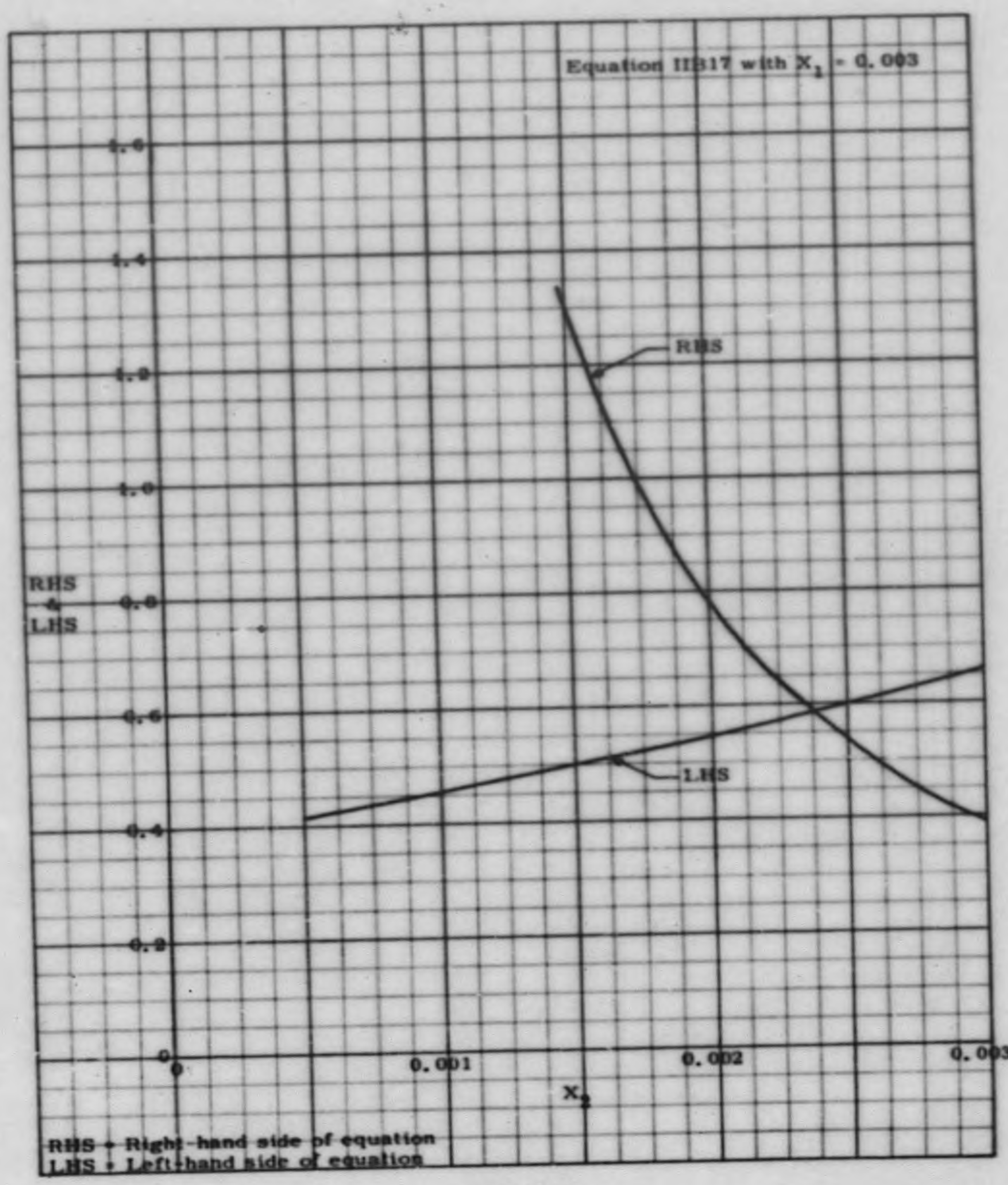

Equation III17 with X_1 = 0.003
1.6
1.4
1.2
1.0
RHS & LHS
0.8
0.6
0.4
0.2
0
0
0.001
0.002
0.003
X_2
RHS
LHS
RHS = Right-hand side of equation
LHS = Left-hand side of equation

Figure 4

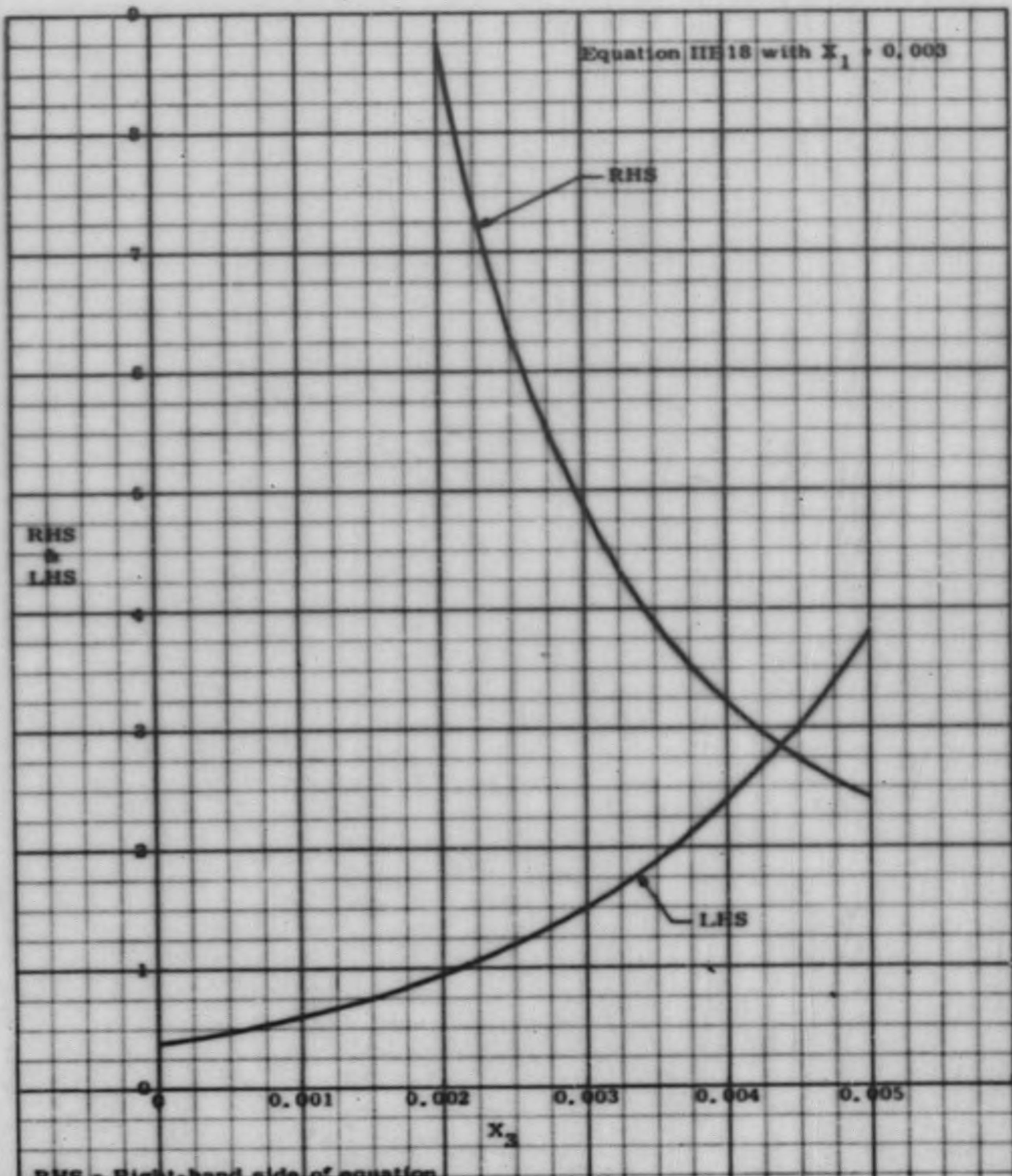

Equation III 18 with X_1 = 0.003
RHS
LHS
RHS & LHS
0
1
2
3
4
5
6
7
8
9
0
0.001
0.002
0.003
0.004
0.005
X_3
RHS - Right-hand side of equation
LHS - Left-hand side of equation

Figure 5

A graphical solution yields:

from Figure 6 $X_2 = 0.00244$;

from Figure 7 $X_3 = 0.0045$.

$\hat{Q} = 0.0031 + 0.00244 + 0.0045 = 0.01004$.

This agrees with Q to three significant figures; therefore, the requirements are:

$X_1 = 0.0031$;

$X_2 = 0.0024$;

$X_3 = 0.0045$.

6. Approximations

Lacking computer facilities and/or the inclination to perform extensive numerical calculations, some approximations are the next best approach. To arrive at some of these rules consider the basic equations:

$$Q = f(X_1, X_2, \ldots, X_n) ;$$

$$Z = g(X_1, X_2, \ldots, X_n) ;$$

where Q is system failure probability, the X_i's, $i = 1, 2, \ldots, n$, are probabilities of failure events, and Z is system cost.

To find how system reliability varies with component reliability, the total differential of Q is taken, resulting in:

$$dQ = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} dX_i , \quad \frac{\partial f}{\partial X_i} \geq 0 .$$

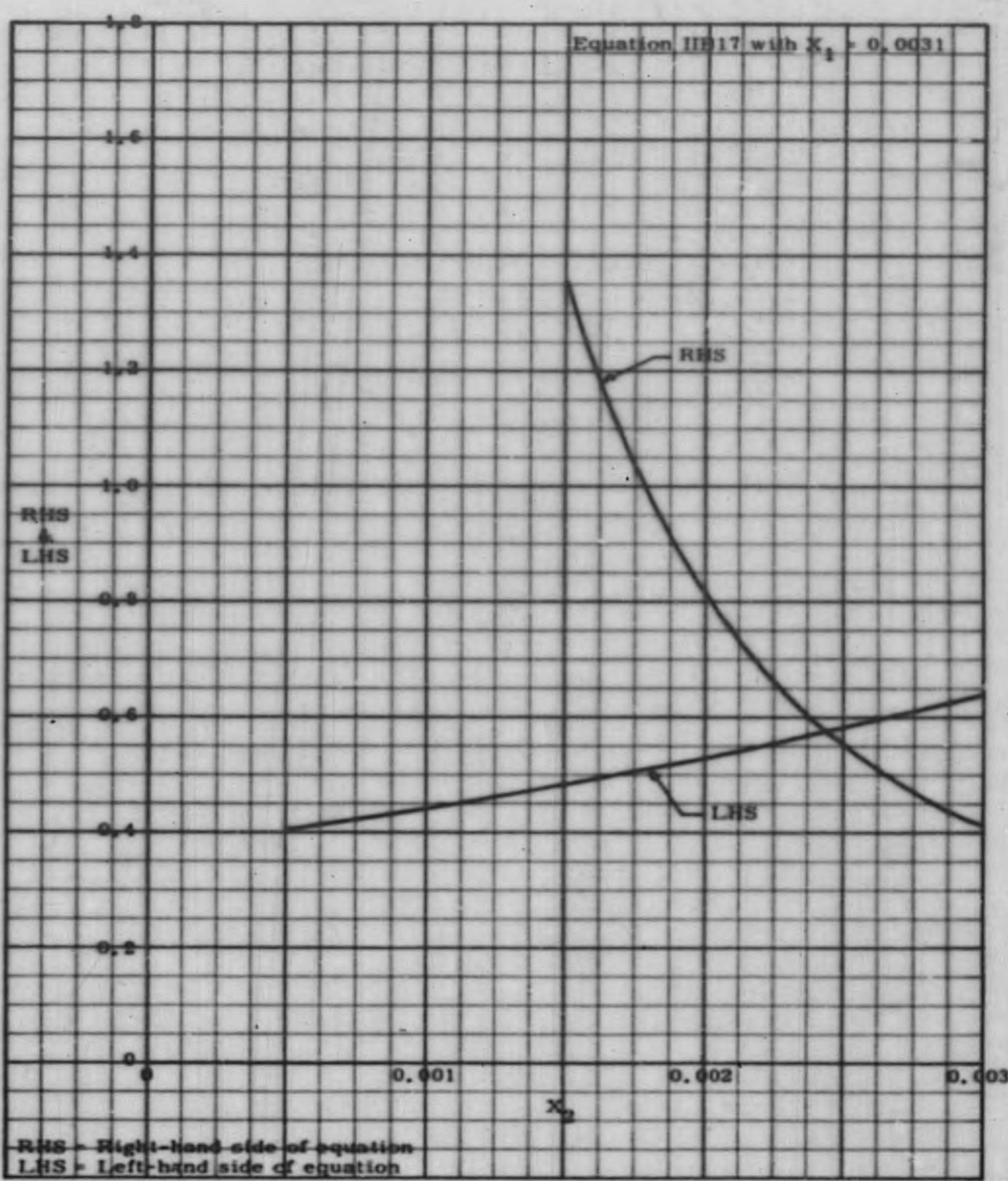
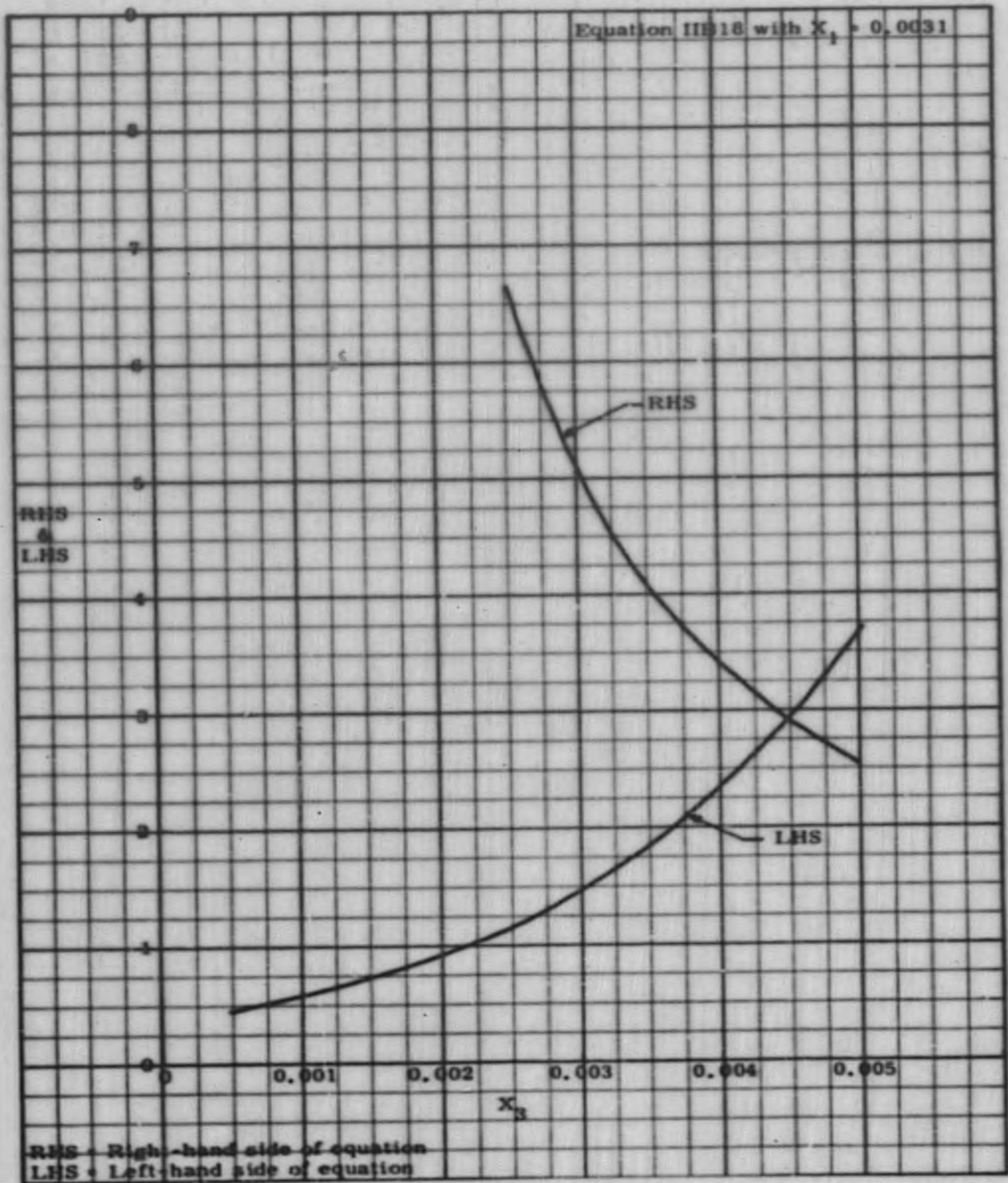

Equation IIB17 with X_1 = 0.0031
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
RHS & LHS
RHS
LHS
0
0.001
0.002
0.003
X_2
RHS = Right-hand side of equation
LHS = Left-hand side of equation

Figure 6



Equation IIB18 with X_1 = 0.0031
RHS
LHS
RHS & LHS
0
1
2
3
4
5
6
7
8
9
0
0.001
0.002
0.003
0.004
0.005
X_3
RHS = Right-hand side of equation
LHS = Left-hand side of equation

Figure 7

Or approximately:

$$\Delta Q = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} \Delta X_i \ .$$

For this reason $\partial f/\partial X_i$ will be called the failure sensitivity of the event X_i. That is, $\partial f/\partial X_i$ is a measure of how the failure probability of the system varies with the failure probability, X_i. Similarly $\partial g/\partial X_i$ is a measure of how system cost varies with the failure probability X_i and will be called the cost sensitivity of X_i.

Now one is actually interested in comparing the rate of change of system failure probability with respect to cost as one of the event failure probabilities changes to this same rate of change as one of the other event failure probabilities changes. The rate of change of failure probability with respect to cost as X_i is varied is $(\partial f/\partial X_i)/(\partial g/\partial X_i)$ and will be called the system sensitivity of X_i. Then, in order to lower system failure probability, one would change the event failure probabilities that result in a maximum change in the absolute value of system failure probability for a given cost.

The foregoing argument expressed mathematically is that, if all X's except X_K are constants,

$$Q = f(a_1, a_2, \ldots, X_K, \ldots, a_n) \ ;$$

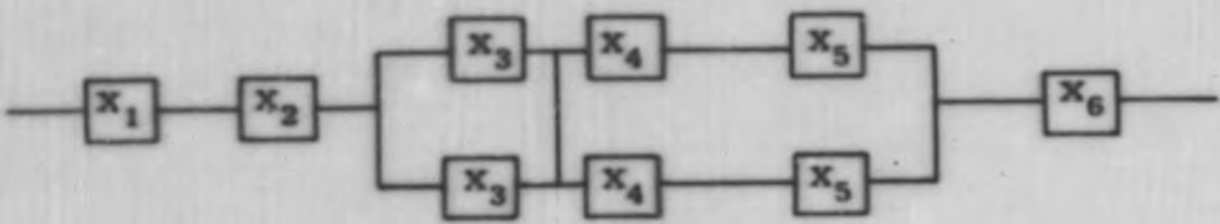
$$Z = g(a_1, a_2, \ldots, X_K, \ldots, a_n) \ .$$

Then:

$$\frac{dQ}{dZ} = \frac{df}{dX_K} / \frac{dg}{dX_K} \ .$$

This relationship gives the rate of change of reliability with cost as only X_K varies, and will be called the system sensitivity of X_K.

Now let us apply this to a sample system.



where the X_i's are the probabilities of failure events. Let us assume that the cost (Y_i) is related to the failure probabilities by:

$$Y_i = \frac{K_{1i}\exp(-K_{2i}X_i)}{X_i};$$

where the K's are known constants and Z (total cost) is:

$$Z = Y_1 + Y_2 + 2(Y_3 + Y_4 + Y_5) + Y_6 . \qquad \text{(IIB19)}$$

The equation for system failure probability (Q) is:

$$Q = X_1 + X_2 + X_6 + X_3^2 + (X_4 + X_5)^2 . \qquad \text{(IIB20)}$$

Finding failure sensitivities:

$$\frac{\partial Q}{\partial X_1} = \frac{\partial Q}{\partial X_2} = \frac{\partial Q}{\partial X_6} = 1 ;$$

$$\frac{\partial Q}{\partial X_3} = 2X_3 ;$$

$$\frac{\partial Q}{\partial X_4} = \frac{\partial Q}{\partial X_5} = 2(X_4 + X_5)\,.$$

Similarly the cost sensitivities are all of the form:

$$\frac{\partial Z}{\partial X_i} = \frac{CK_{1i}[\exp(-K_{2i}X_i)](1 + K_{2i}X_i)}{X_i^2}\,.$$

where C is the coefficient of Y_i in Equation IIB19.

Now the system sensitivity of X_i (the rate of change of system failure probability with cost as only X_i varies) will be represented by the symbol $dQ/dZ|\Delta X_i$. These values are:

$$\frac{dQ}{dZ}\Big|\Delta X_1 = -\frac{X_1^2\exp(K_{21}X_1)}{K_{11}(1 + K_{21}X_1)}\,.$$

$$\frac{dQ}{dZ}\Big|\Delta X_2 = -\frac{X_2^2\exp(K_{22}X_2)}{K_{12}(1 + K_{22}X_2)}\,.$$

$$\frac{dQ}{dZ}\Big|\Delta X_3 = -\frac{X_3^3\exp(K_{23}X_3)}{K_{13}(1 + K_{23}X_3)}\,.$$

$$\frac{dQ}{dZ}\Big|\Delta X_4 = -\frac{X_4^2(X_4 + X_5)\exp(K_{24}X_4)}{K_{14}(1 + K_{24}X_4)}\,.$$

$$\frac{dQ}{dZ}\Big|\Delta X_5 = -\frac{X_5^2(X_4 + X_5)\exp(K_{25}X_5)}{K_{15}(1 + K_{25}X_5)}\,.$$

$$\frac{dQ}{dZ}\bigg|\Delta X_6 = -\frac{X_6^2 \exp(K_{26}X_6)}{K_{16}(1 + K_{26}X_6)}.$$

The optimum manner of change would be to look at the expressions 1 through 6 and change the X_i in direct relationship to the magnitude of the expression. This is analogous to the vector concept of moving along the gradient to move in the direction of the greatest space rate of increase. A more practical approach is to slightly change the X_i that has the maximum absolute value (the value will be negative because failure rate varies inversely with cost) of system sensitivity, recompute the system sensitivities, and repeat the process until the required system Q is obtained.

To get one step more practical, if the system failure model has been derived and the X_i's have been estimated, resulting in a failure probability that is above requirements, which X_i or X_i's should be changed? The failure sensitivities can be computed from the system failure model. The other requirement for an approximate answer is the cost sensitivities. These depend on the cost versus X_i functions which are difficult to estimate. But if relative cost sensitivity estimates can be made (e.g., it will cost twice as much to change X_1 from 0.001 to 0.0008 as to change X_2 the same amount), relative system sensitivities can be computed to decide which component reliability requirement changes would be most inexpensive.

CHAPTER III -- OUT-OF-TOLERANCE FAILURES

In this chapter out-of-tolerance failures will be considered. Certainly, catastrophic failures are also out of tolerance, but it is not convenient to consider them as such because they do not usually fall within the continuous distributions that will be used for considering out-of-tolerance failures.

A. Relationships Between Component and System Tolerances

Before turning to the problem of allocating tolerances to individual parts it is necessary to discuss how tolerances of parts are combined to produce system tolerances.

1. Statement of Tolerance Problems

Many tolerance problems can be expressed in terms of an equation of the form $Y = f(X_1, X_2, \ldots, X_n)$. For example, if physical dimensions are considered, the total dimension is a function of the dimensions of the parts (e.g., the total length of a bar (L) made up of three shorter bars placed end-to-end is the sum of the lengths). Thus, this is in the form $Y = f(X_1, X_2, \ldots, X_n)$.

Another problem often encountered is usually described by the stress-strength curves. To give an example of this type of problem, consider a device that is put into the service for which it is designed. The total environment (including the electrical and/or mechanical load and environmental variables of temperature, humidity, shock, vibration, etc.) is bound to be variable.

Standard design procedure is to assume or calculate the "average maximum" environment and design the average device to have a strength that is some "K" factor above the strength necessary to withstand this environment.

For a simple example, let us consider a structural member which is designed to support a load, and for simplification let us assume that all the total environmental variables are constant except the load. Then the standard design procedure is to find the design limit load and design the ultimate strength of the rod "K" times the design limit load.[7] This may be pictured:

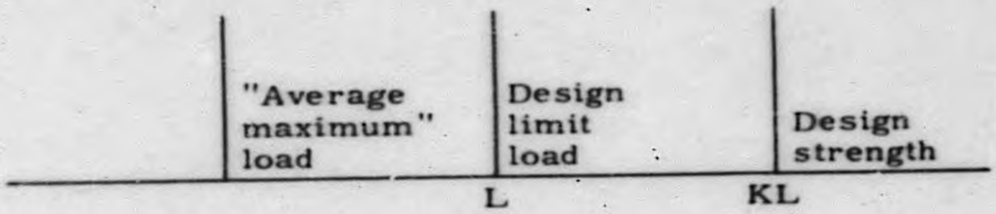


Figure 8

Now the actual loads and strength of the rods will have some distribution about the means. These will be pictorially represented as continuous distributions.

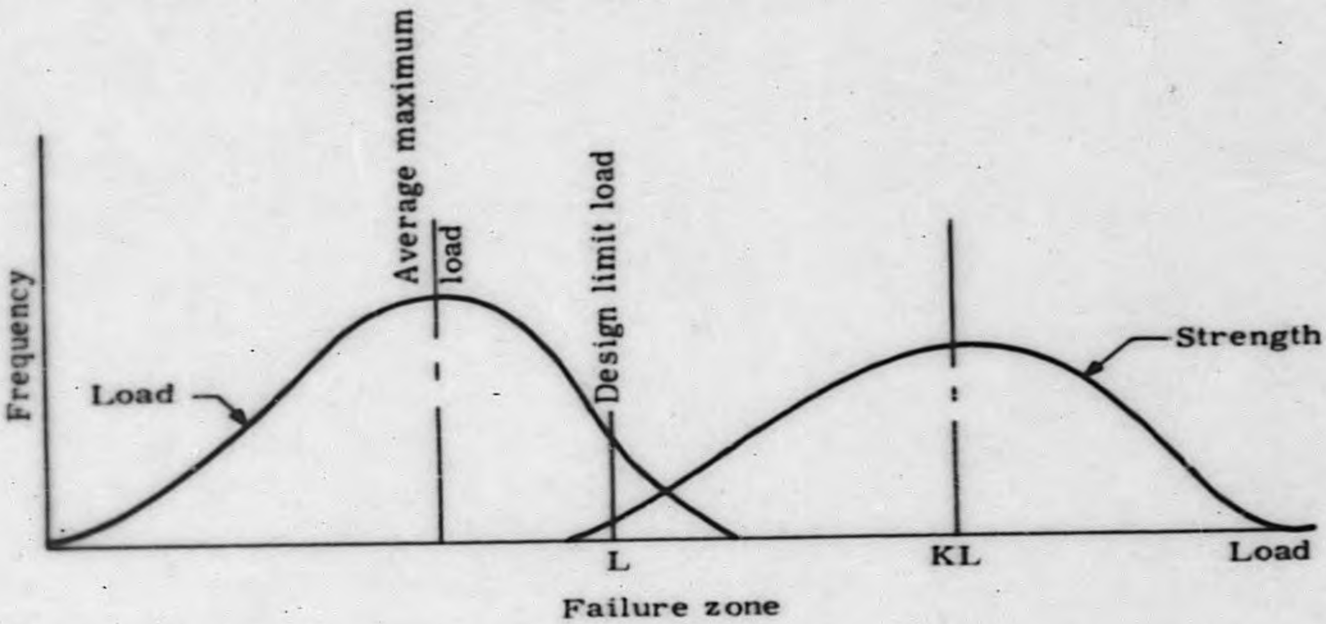


Figure 9

[7]For simplicity only ultimate strengths are considered. Yield strengths would also be considered in an actual analysis.

Notice in Figure 9 that it is possible for the variations in the load and strength of the member to be such that there is an overlap of the two distributions. Any overlap means that there is some chance (a given probability) for the environmental stress to exceed the strength of the product. As the overlap of the distributions in Figure 9 increases, a larger percentage of the units will fail, and the reliability of the structural member will decrease.

In a more practical situation, the actual stress is a combination of all the things that tend to cause failure, and strength is the resistance to these stresses; the actual distributions of stress and strength in the general case, however, are difficult to estimate. Nevertheless, the thinking that is pictorially represented by these two distributions in Figure 9 with the overlap representing failure is a useful guide when considering the reliability of a device.

This stress-strength type of problem can also be put in the form of $Y = f(X_1, X_2, \ldots, X_n)$. To demonstrate this fact, consider that X_1 is strength and X_2 is stress; then, if $Y = f(X_1, X_2) = X_1 - X_2 \leq 0$, there is a failure. Thus, if the distribution of Y where Y is a function of $(X_1, X_2, \ldots, X_n)$ can be found, the portion of the distribution of $Y = X_1 - X_2$ that is less than zero can also be found.

2. General Statistical Treatment

Many tolerance problems have been reduced to that of finding the distribution of $Y = f(X_1, X_2, \ldots, X_n)$ given the distribution of the X_i's. The exact method for finding the distribution of Y where $Y = f(X_1, X_2, \ldots, X_n)$ is discussed in Reference 7. Various approximations are discussed in References 8, 9, and 10. Naturally the degree of approximation and the amount of labor are directly related.

For the purpose of this problem, a practical approximation will be discussed in some detail and then better approximations obtained at the expense of considerable labor will be mentioned. Consider:

$$Y = f(X_1, X_2, \ldots, X_n) ,$$

where the X_i's are independent. If Y is represented by its first-order Taylor series expansion about (a_i), $i = 1, 2, 3, \ldots, n$:

$$Y \cong f(a_i) + \sum_{j=1}^{n} f_{x_j}(a_i)(x_j - a_j) . \qquad \text{(IIIA1)}$$

Now the items of interest are the expected value or mean of $Y[E(y) = \mu_y]$ and some measure of the dispersion, specifically the variance or the expected value of $(y - \mu_y)^2[E(y - \mu_y)^2 = \sigma_y^2]$. To find these expected values it is necessary to recall that the expected value of a sum is equal to the sum of the expected values of the summands and that the expected value of a product is equal to the product of the expected values of the factors, given the factors are independent. When evaluating expected values, it should be recalled that the expected value of $(X_i - \mu_i) = 0$.

Now if a_i is set equal to μ_i in Equation IIIA1, there results:

$$y \cong f(\mu_i) + \sum_{j=1}^{n} f_{x_j}(\mu_i)(x_j - \mu_j) . \qquad \text{(IIIA2)}$$

Then:

$$\mu_y = E(y) \cong E[f(\mu_i)] + E\left[\sum_{j=1}^{n} f_{x_j}(\mu_i)(x_j - \mu_j)\right] . \qquad \text{(IIIA3)}$$

Realizing that the expected value of a constant is the constant, and that the expected value of a constant times a function is the constant times the expected value of the function, Equation IIIA3 becomes:

$$\mu_y \cong f(\mu_i) + \sum_{j=1}^{n} f_{x_j}(\mu_i)E(x_j - \mu_j) ; \quad \mu_y \cong f(\mu_i) . \qquad \text{(IIIA4)}$$

Now $\sigma_y^2 = E(y - \mu_y)^2$ will be found. Setting $\mu_y = f(\mu_i)$ in Equation IIIA2:

$$y - \mu_y \cong \sum_{j=1}^{n} f_{x_j}(\mu_i)(x_j - \mu_j) .$$

$$E(y - \mu_y)^2 \cong E\left[\sum_{j=1}^{n} f_{x_j}(\mu_i)(x_j - \mu_j)\right]^2 ;$$

$$\cong E\left[\sum_{j=1}^{n}\sum_{k=1}^{n} f_{x_j}(\mu_i) f_{x_k}(\mu_i)(x_j - \mu_j)(x_k - \mu_k)\right] ;$$

$$\cong \sum_{j=1}^{n}\sum_{k=1}^{n} f_{x_j}(\mu_i) f_{x_k}(\mu_i) E[(x_j - \mu_j)(x_k - \mu_k)] .$$

If $j \neq k$ and x_j and x_k are independent, then:

$$E[(x_j - \mu_j)(x_k - \mu_k)] = E(x_j - \mu_j)E(x_k - \mu_k) = 0 .$$

Thus:

$$E(y - \mu_y)^2 \cong \sum_{j=1}^{n} f_{x_j}^2(\mu_i) E[x_j - \mu_j]^2 .$$

$$\sigma_y^2 = E(y - \mu_y)^2 \cong \sum_{j=1}^{n} f_{x_j}^2(\mu_i)\sigma_{x_j}^2 . \qquad \text{(IIIA5)}$$

Approximate equations for the mean and standard deviation of Y are derived above. These equations are good approximations so long as the stated assumption of independence is valid and the first-order Taylor series is a good approximation. The Taylor series will be a good approximation if the standard deviation of the X_i's is small compared to the means of the X_i's, or if the functional relationship between Y and the X_i's is of a certain form.[8] Only the mean and standard deviation of Y are given by the above formula. These do not describe the distribution of Y. However, if Y is assumed to be normally distributed, the mean and standard deviation completely describe the distribution. The assumption of a normal distribution is often quite reasonable as demonstrated by the central limit theorem. If this assumption does not seem reasonable, then more moments of the distribution may be found by using a higher order Taylor series, and these moments may be used to better approximate the distribution by various methods.[9]

Thus, for any circuit that can be described by a mathematical model, an approximate expression for the mean and standard deviation of the output can be found. It is quite important to recognize that failure rate in use is the parameter of interest. Therefore, the mean and standard deviation of interest are those present at time of use. Now the "use" value of a parameter, X, is actually:

$$X = X_o + \Delta X ;$$

where X_o is the initial value and ΔX is the change in value that occurs before use. This equation is governed by the previously derived equations, so:

$$\mu_X = \mu_{X_o} + \mu_{\Delta X} ; \qquad \text{(IIIA6)}$$

or the "use" mean is the sum of the initial mean and the mean change. Similarly, the "use" standard deviation is related to the initial standard

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[8] A discussion of sums and products is contained in Section IIIA3.

[9] Pearson method, Gram-Charlier or Edgeworth's series - see References 9 and 10.

deviation and the standard deviation of the change by:

$$\sigma_X^2 = \sigma_{X_o}^2 + \sigma_{\Delta X}^2 . \qquad \text{(IIIA7)}$$

Thus a ±5-percent resistor does <u>not</u> mean that all of the resistors will be within ±5 percent of the mean value when the equipment is to be used. Rather the ±5 percent refers to the resistance measurement made by the manufacturer.

Frequency, temperature, loading, and aging all affect the value of the resistor. That is, the manufacturer's data cannot be used directly, but information to estimate the distribution of the parameters of interest at the time of intended use must be obtained. This information may be available from the manufacturer, from experiments, or from available information on similar components. It may be noted that the mean as well as dispersion, skewness, and higher moments of a distribution may shift with time. Quite often this information will be far from complete, but it seems reasonable to assume that the mean and standard deviation of a certain parameter associated with a particular kind of component can be predicted.

3. Sums and Products of Independent Random Variables

First, the sum of two independent variables will be considered.

$$Y = X_1 + X_2 .$$

The exact expressions for the mean and standard deviations of Y in terms of the means and standard deviation of X_1 and X_2 are:[10]

$$\mu_y = \mu_{x_1} + \mu_{x_2} .$$

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[10] See Reference 11, p. 98.

$$\sigma_y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 .$$

It may be noted that the approximate formulas given in Equation IIIA2 are exactly true in the case of sums. This should be expected because all partial derivatives other than the first are equal to zero, and in this case the approximations become equalities.

Next $Y = X_1X_2$ where X_1 and X_2 are independent will be considered. Then:

$$\mu_y = E(Y) = E(X_1X_2) = E(X_1)E(X_2) ;$$

$$\mu_y = \mu_{x_1}\mu_{x_2} .$$

This is the same as the approximate formula. To find σ_y:

$$\frac{\partial Y}{\partial X_1} = X_2 ;$$

$$\frac{\partial Y}{\partial X_2} = X_1 ;$$

$$\frac{\partial^2 Y}{\partial X_1^2} = 0 ;$$

$$\frac{\partial^2 Y}{\partial X_2^2} = 0 ;$$

$$\frac{\partial^2 Y}{\partial X_1 \partial X_2} = \frac{\partial^2 Y}{\partial X_2 \partial X_1} = 1 .$$

Therefore:

$$y - \mu_y = \mu_2(x_1 - \mu_1) + \mu_1(x_2 - \mu_2) + (x_1 - \mu_1)(x_2 - \mu_2) .$$

Recalling that $E(x_1 - \mu_1)$ is zero,

$$E(y - \mu_y)^2 = E\left[\mu_2^2(x_1 - \mu_1)^2\right] + E\left[\mu_1^2(x_2 - \mu_2)^2\right]$$

$$+ E\left[(x_1 - \mu_1)^2(x_2 - \mu_2)^2\right] ;$$

$$\sigma_y^2 = \mu_2^2\sigma_{x_1}^2 + \mu_1^2\sigma_{x_2}^2 + \sigma_{x_1}^2\sigma_{x_2}^2 .$$

That is, the approximate formula neglects the term $\sigma_{x_1}^2\sigma_{x_2}^2$. Recalling that the assumption made in the original analysis was $\sigma_{x_1} << \mu_{x_1}$, the exact case is rewritten:

$$\sigma_y^2 = \mu_2^2\sigma_{x_1}^2 + \sigma_{x_2}^2\left(\mu_1^2 + \sigma_{x_1}^2\right) ;$$

$$= \mu_2^2\sigma_{x_1}^2 + \sigma_{x_2}^2\mu_1^2 ,$$

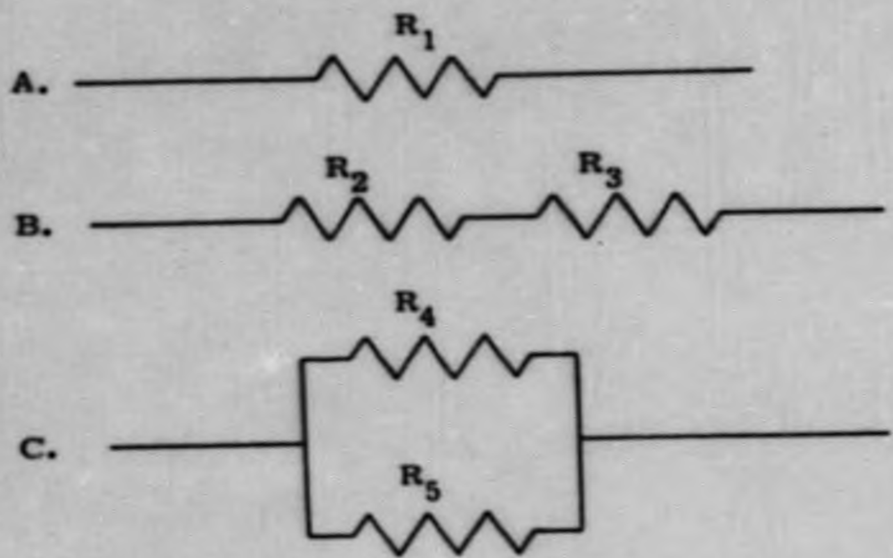
if:

$$\sigma_{x_1}^2 << \mu_1^2 .$$

So with the same assumption made earlier, the same approximate formula is derived.

4. Example of Tolerance of Resistors

Throughout this discussion, it will be assumed that, "in use", all resistors are ±10-percent resistors and that the in-use distributions are such that the mean is equal to the nominal value and that the standard deviation is 3 percent of the mean. The tolerances (via the standard deviations) of three circuits will be compared.



In order for these circuits to have the same mean resistance, the mean values of the individual resistors are related as follows:

$$\mu_{R1} = 2\mu_{R2} = 2\mu_{R3} = \frac{\mu_{R4}}{2} = \frac{\mu_{R5}}{2} .$$

The standard deviation (σ) of the resistance of Circuit A is simply:

$$\sigma_A = 0.03\mu_{R1} .$$

The standard deviation of the resistance of Circuit B is computed as follows:

$$R_B = R_2 + R_3 ;$$

$$\frac{\partial R_B}{\partial R_2} = 1 = \frac{\partial R_B}{\partial R_3} ;$$

$$\sigma_B^2 = \sigma_2^2 + \sigma_3^2 = (0.03\mu_{R2})^2 + (0.03\mu_{R3})^2$$

$$\sigma_B = 0.03\frac{\mu_{R1}}{\sqrt{2}} \cong 0.021\mu_{R1} \cong 0.707\sigma_A .$$

The standard deviation of the resistance of Circuit C is computed as follows:

$$R_C = \frac{R_4 R_5}{R_4 + R_5} ;$$

$$\frac{\partial R_C}{\partial R_4} = \frac{R_4 R_5 + R_5^2 - R_4 R_5}{(R_4 + R_5)^2} = \left[\frac{R_5}{R_4 + R_5}\right]^2 ;$$

$$\frac{\partial R_C}{\partial R_5} = \left[\frac{R_4}{R_4 + R_5}\right]^2 ;$$

$$\sigma_C^2 = \left[\frac{\mu_{R5}}{\mu_{R4} + \mu_{R5}}\right]^4 \sigma_{R4}^2 + \left[\frac{\mu_{R4}}{\mu_{R4} + \mu_{R5}}\right]^4 \sigma_{R5}^2 ;$$

$$\sigma_C^2 = \left[\frac{2\mu_{R1}}{4\mu_{R1}}\right]^4 (0.03\mu_{R4})^2 + \left[\frac{2\mu_{R1}}{4\mu_{R1}}\right]^4 (0.03\mu_{R5})^2 ;$$

$$\sigma_C^2 = (0.03)^2\left[(1/2)^4(2\mu_{R1})^2 + (1/2)^4(2\mu_{R1})^2\right] ;$$

$$\sigma_C = 0.03\mu_{R1}\sqrt{(1/2)^2 + (1/2)^2} ;$$

$$\sigma_C = 0.03\frac{\mu_{R1}}{\sqrt{2}} = 0.021\mu_{R1} = 0.707\sigma_A .$$

Thus, by combining resistors in either series or parallel the standard deviation of resistance, and hence the tolerance, is smaller than that of the single resistor.

B. Allocation of Allowable Component Standard Deviation to Obtain Minimum Costs

Now assume that the system can be represented by the equation:

$$Y = f(X_1, X_2, \ldots, X_n) ;$$

where Y is the output parameter of interest and where the X's are component parameters. Let us further assume a relationship of the form:

$$C_{X_i} = f\left(\sigma_{X_i}\right) i = (1, 2, \ldots, n) ,$$

where σ_{X_i} is the standard deviation of the X_i^{th} component parameter, f is a general function, and C_{X_i} is the cost associated with σ_{X_i} .

Assuming that the variables (X_1, X_2, ..., X_n) are independent and that the final output is normally distributed,[11] the equation relating the means and sigmas of the component parameters to the mean and sigma of the output parameter of interest (Equations IIIA4 and IIIA5) can be used to describe the output distribution. Then this distribution must have a mean and a sigma such that no more than a given portion of the output distribution falls outside of the specification limits. Given this mean and standard deviation, it is no problem to find a combination of means and standard deviations of the component parameters that will give this result. Rather the problem is to choose the best of many possible solutions. The solution for the means of the component parameters is not extremely difficult because there is usually little difference in cost, time, etc., whether the mean of a component parameter is chosen as A or B (e.g., the load resistor is chosen with a mean of 1 or 1.2 K). In choosing the standard deviations of the various component parameters, there may be considerable differences in cost, time, size, weight, etc., that accompany the different solutions that will result in the proper output standard deviation. Again, because of specifications on everything but cost, the solution that yields the minimum cost will be chosen.

1. Discussion of Cost Versus Standard Deviation Functions

The relationship between cost and standard deviation of a component parameter is, in general, quite complex. However, there are some considerations that can guide the choice of relationships.

The price of a component is determined both by the initial dispersion and by the amount of stability (resistance to changes with time and environment). As a general discussion of cost versus initial dispersion, it will be assumed that the manufacturer produces a normal distribution and that the cost of purchased units is based on the production costs of

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[11] If the X_i's are known not to be independent, new variables which are independent can usually be defined, and if the output distribution is thought not to be normal, the distribution may be better approximated by taking more moments. See Reference 9.

the units that must be screened to give the buyer the units desired. Thus, if the units produced have a mean of μ_p and a sigma of σ_p, and if the cost per unit of these units is C_p, the unit cost of those units bought is dependent on these factors, and also upon the part of the distribution bought. Thus, the unit cost of purchased units, C_o, under the above assumptions, is:

$$C_o = C_p \frac{\int_{-\infty}^{+\infty} \frac{1}{\sigma_p\sqrt{2\pi}} \exp\left[\frac{(\mu_p - X)^2}{2\sigma_p^2}\right] dX}{\int_{A}^{B} \frac{1}{\sigma_p\sqrt{2\pi}} \exp\left[\frac{(\mu_p - X)^2}{2\sigma_p^2}\right] dX}; \qquad \text{(IIIB1)}$$

where the limits A and B describe the limits of purchased product. Thus, the cost of the units bought is a function of the production cost and the ratio of the allowable tolerance to the standard deviation of the produced product. Of course, there must be some relationship between C_p and σ_p. Therefore, it might be said that the cost is a function of production cost and allowable spread. As an example of this relationship, C_o divided by C_p is plotted against the ratio of $(A-B)/\sigma_p$, assuming A - B is centered at μ_p, in Figure 10.

In a more practical situation the manufacturer will probably be able to supply points of costs versus spread or cost versus standard deviation, and these points can be used to approximate the function of cost versus the original standard deviation of purchased parts (σ_o). Then the relationship between the standard deviation of the change that occurs with time (σ_Δ) and other environments must be investigated. This relationship is difficult to hypothesize, since it seems to be quite dependent on the type of device. Hence, the relationship must be obtained through experimentation.

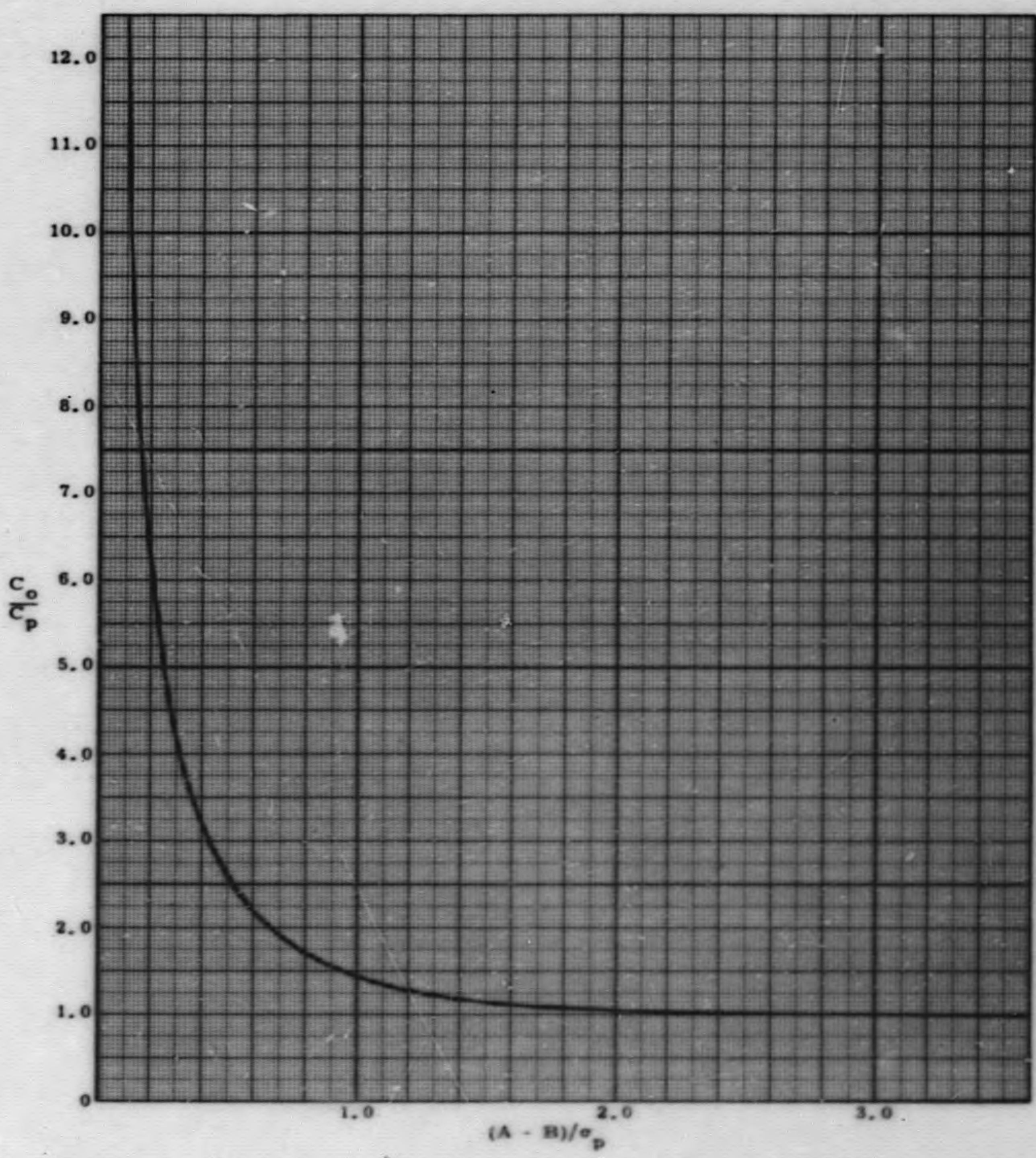


Figure 10

Assuming that the relationships:

$C = f(\sigma_o)$ and

$C = g(\sigma_\Delta)$

have been found, Equation IIIA6 becomes:

$$\sigma_X^2 = \sigma_o^2 + \sigma_\Delta^2 = [f^{-1}(C)]^2 + [g^{-1}(C)]^2 . \qquad \text{(IIIB3)}$$

This equation is quite impractical. In an actual case the σ_o versus cost curves would be estimated from experimental data. Nevertheless, the equation does rather clearly portray the relationships that govern the relationship between the cost and the "use" standard deviation.

Assuming a relationship between cost and standard deviation is largely conjecture at this point. However, in order to briefly discuss the procedure, the following relationship is proposed.

$$\text{Cost}_X = C_X = \frac{K_X}{\sigma_X} . \qquad \text{(IIIB4)}$$

This relationship is at least reasonable in that the cost is infinite to have zero standard deviation, and the cost is zero to have infinite standard deviation.

Perhaps a two-parameter family should be chosen for this relationship; however, to illustrate the method a simple one-parameter family is chosen.

2. Application of the Method

Total system cost (Z) is:

$$Z = \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} \frac{K_i}{\sigma_i} . \quad \text{(IIIB5)}$$

From Section IIIA it was found that the standard deviation of the system can be approximately related to the standard deviation of the components by a relationship of the form:

$$\sigma_T = \sqrt{\sum_{i=1}^{n} (A_i \sigma_i)^2} .$$

or:

$$\sigma_T^2 = \sum_{i=1}^{n} (A_i \sigma_i)^2 . \quad \text{(IIIB6)}$$

To find minimum cost, Lagrange multipliers are again used.

$$f = \sum_{i=1}^{n} \frac{K_i}{\sigma_i} + \lambda \left[\sum_{i=1}^{n} (A_i \sigma_i)^2 - \sigma_T^2 \right] . \quad \text{(IIIB7)}$$

$$\frac{\partial f}{\partial \sigma_i} = - \frac{K_i}{\sigma_i^2} + \lambda 2 A_i^2 \sigma_i = 0 . \quad \text{(IIIB8)}$$

There will be n equations of the form of Equation IIIB8 and Equation IIIB6. To solve this set of equations Equation IIIB8 may be rearranged:

$$2\lambda = \frac{K_i}{A_i^2} \frac{1}{\sigma_i^3} . \quad \text{(IIIB9)}$$

Equating the 1^{st} to the j^{th}:

$$\frac{K_1}{A_1^2}\frac{1}{\sigma_1^3} = \frac{K_j}{A_j^2}\frac{1}{\sigma_j^3};$$

$$\sigma_j^3 = \sigma_1^3\frac{K_j}{K_1}\left(\frac{A_1}{A_j}\right)^2;$$

$$\sigma_j = \sigma_1\sqrt[3]{\frac{K_j}{K_1}\left(\frac{A_1}{A_j}\right)^2} \qquad \text{(IIIB10)}$$

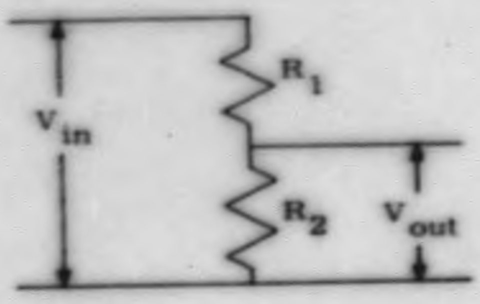
Using this in Equation IIIB6:

$$\sigma_T^2 = \left\{A_1^2 + \sum_{i=2}^{n} A_i^2\left[\frac{K_i}{K_1}\left(\frac{A_1}{A_i}\right)^2\right]^{2/3}\right\}\sigma_1^2 . \qquad \text{(IIIB11)}$$

Thus, σ_1 (the positive square root of σ_1^2) may be found from Equation IIIB11 and σ_i, $i = 2, 3, \ldots, n$ may be found from equations of the form of Equation IIIB10.

3. Example

A voltage divider is needed such that the mean transfer function V_{out}/V_{in} is 1/10 and the allowable variation is from 11/100 to 9/100. The circuit chosen is:



The expression for the transfer function (assuming very small current to the load) is:

$$H = \frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2}.$$

The mean values chosen for R_1 and R_2 are, respectively:

$$\mu_1 = 9 \text{ K ohms };$$

$$\mu_2 = 1 \text{ K ohm }.$$

These mean values are "use" means. That is, if the resistance is expected to change with time, load, temperature, etc., the initial mean should be chosen such that the predicted "use" means will be 9 K and 1 K, respectively.

The Y (cost) versus σ (standard deviation) relationship assumed is that for each resistor:

$$Y_i = \frac{K}{\frac{\sigma_i}{\mu_i}} = \frac{\mu_i K}{\sigma_i}.$$

That is, cost is inversely proportional to the ratio of standard deviation to mean. This seems reasonable in light of the fact that resistance prices are usually quoted in terms of ±5 percent or ±10 percent and the lower the tolerance, the higher the price.

The standard deviation of the transfer function is:

$$\sigma_H^2 = \left[\frac{\partial H}{\partial R_1}(\mu_1, \mu_2)\right]^2 \sigma_1^2 + \left[\frac{\partial H}{\partial R_2}(\mu_1, \mu_2)\right]^2 \sigma_2^2 ;$$

$$\sigma_H^2 = \left[\frac{\mu_2\sigma_1}{(\mu_1 + \mu_2)^2}\right]^2 + \left[\frac{\mu_1\sigma_2}{(\mu_1 + \mu_2)^2}\right]^2 ;$$

$$\sigma_H^2 = \left[\frac{10^3}{10^8}\sigma_1\right]^2 + \left[\frac{9 \times 10^3}{10^8}\sigma_2\right]^2 ;$$

$$\sigma_H = 10^{-5}\sqrt{\sigma_1^2 + 81\sigma_2^2} ;$$

$$\sigma_H^2 = 10^{-10}\left(\sigma_1^2 + 81\sigma_2^2\right) .$$

Four standard deviations of H are set equal to the allowable variation in H. Thus, if H is assumed to be normally distributed, the probability of being outside the four sigma limits is approximately 0.001.

$$4\sigma_H = \frac{1}{100} ;$$

$$\sigma_H = \frac{1}{400} ;$$

$$\sigma_H^2 = \frac{1}{16 \times 10^4} .$$

The total cost is the sum of the resistor costs.

$$\text{Total Cost} = Z = \frac{K_1}{\sigma_1} + \frac{K_2}{\sigma_2} = \frac{\mu_1 K}{\sigma_1} + \frac{\mu_2 K}{\sigma_2} .$$

The cost will be minimized subject to the constraint:

$$\frac{10^{-4}}{16} = 10^{-10}\left(\sigma_1^2 + 81\sigma_2^2\right) .$$

Using Equation IIIB11:

$$\frac{10^{-4}}{16} = 10^{-10}\left[1 + 81\left(\frac{\mu_2}{\mu_1}\frac{1}{81}\right)^{2/3}\right]\sigma_1^2 ;$$

$$\frac{10^6}{16} = \left[1 + 81\left(\frac{1}{9}\frac{1}{81}\right)^{2/3}\right]\sigma_1^2 ;$$

$$\frac{10^6}{16} = (1 + 1)\sigma_1^2 ;$$

$$\sigma_1^2 = \frac{10^6}{(16)2} ;$$

$$\sigma_1 = \frac{10^3}{4\sqrt{2}} \cong 177 \text{ ohms.}$$

Using Equation IIIB10:

$$\sigma_2 = \sigma_1\sqrt[3]{\frac{\mu_2}{\mu_1}\frac{1}{81}} = \sigma_1\frac{1}{9} ;$$

$$\sigma_2 \cong \frac{177}{9} \cong 20 \text{ ohms.}$$

4. Approximations

If the necessary relationships between cost and standard deviation are available, this process can be carried out in the same manner discussed in Section IIIB2. A more practical approach is to define:

1. Standard deviation sensitivity which is:

$$\frac{\partial\sigma_T}{\partial\sigma_{X_i}} .$$

2. Cost sensitivity which is:

$$\frac{\partial Z}{\partial \sigma_{X_i}}.$$

Where σ_T is system standard deviation, σ_{X_i} is the standard deviation of X_i, and Z is system cost.

$$3.\ \frac{\partial \sigma_T / \partial \sigma_{X_i}}{\partial Z / \partial \sigma_{X_i}} = \frac{d\sigma_T}{dZ}\Big|_{\Delta \sigma_{X_i}};$$

which will be called circuit sensitivity of X_i.

Then look for the X_i that makes the absolute value of the circuit sensitivity a maximum in order to reduce the circuit standard deviation and look for the X_i that makes the absolute value of the circuit sensitivity a minimum in order to relax the standard deviation requirement with a large cost savings.

CHAPTER IV -- CONCLUSIONS

A. Summary

Approximate methods for choosing the maximum allowable component catastrophic failure rates, given system catastrophic failure rate specifications; and for choosing component parameter standard deviations, given system output requirements, have been developed within the assumptions stated. Thus far, nothing has been said about how to allocate the total allowed unreliability between catastrophic and out-of-tolerance failure rates. This allocation might be accomplished by introducing system performance as a concept, or by again minimizing cost. It seems, however, that the most realistic approach is to design such that out-of-tolerance failure probability is less than one-tenth of the allowable failure probability and to use the allowable failure probability in catastrophic requirements. The basis for this decision is that, by careful screening, the out-of-tolerance failure rate can be more easily controlled by design and specifications than can the catastrophic failure rate.

B. Recommendations for Further Study

It is proposed that the method developed in this paper be applied to an actual system. This would probably involve a computer program, further investigation of the relationship between cost and catastrophic failure probability for particular modes of failure, and further investigation of the relationship between cost and "use" standard deviation of certain basic components.

APPENDIX -- UNIQUENESS

In Chapter II, the method of Lagrange multipliers was used to find an extreme, subject to the constraint that the properly synthesized event failure probabilities equal the allowable system failure probability. Physical considerations tell us that if only one extreme exists, it should represent a minimum cost. Because every end point results in infinite cost, a minimum cannot exist at an end point, and each end point represents a higher cost than an extreme found at an interior point. Therefore, if one and only one point exists at which all of the partial derivatives equal zero, this must be the minimum sought.

To see that there is a unique solution, consider that if there is a unique solution in the series case and in the parallel case, there is a unique solution to all problems considered. That this is true may be seen by referring to Section IIA, where it is demonstrated how any combination may be represented by successive series and parallel reductions to a simple series or parallel configuration.

Consider the series case. There are n - 1 equations of the form:

$$\exp(-K_{21}X_1 + K_{2i}X_i) = \frac{K_{1i}}{K_{11}}\left(\frac{X_1^2}{1 + K_{21}X_1}\right)\left(\frac{1}{X_i}\right)\left(\frac{1}{X_i} + K_{2i}\right) \; ; \quad \text{(A1)}$$

and there is also the equation,

$$Q = \sum_{i=1}^{n} X_i \, ,$$

relating n unknowns.

When X_1 is chosen, $X_2, X_3, \ldots, X_n$ are determined by equations of the form of Equation A1. Recognizing that all constants (K_{1j}, K_{2j}) are positive, the function on the left of Equation A1 is strictly increasing with X_i, while the function on the right of Equation A1 is strictly decreasing with X_i. There is, however, a possibility that there will not be an intersection in the region $0 \leq X \leq 1$. The left side of Equation A1 starts at $\exp(-K_{21}X_1)$ when $X_i = 0$ and rises to $\exp(-K_{21}X_1 + K_{2i})$ when $X_i = 1$. The function on the right of Equation A1 starts at ∞ when $X_i = 0$ and drops to

$$\frac{K_{1i}}{K_{11}}\left(\frac{X_1^2}{1 + K_{21}X_1}\right)(1 + K_{2i})$$

when $X_i = 1$.

If one can assume that

$$\frac{K_{1i}}{K_{11}}\left(\frac{X_1^2}{1 + K_{21}X_1}\right)(1 + K_{2i})$$

is less than $\exp(-K_{21}X_1 + K_{2i})$, there is a solution, and this solution can be assured by choosing X_1 small enough. This is quite reasonable in view of the fact that if X_1 is chosen too large, no values of X_i, $i = 2, 3, \ldots, n$ in the range $0 \leq X_i \leq$ will satisfy

$$Q = \sum_{i=1}^{n} X_i \,.$$

Thus each X_i can be found uniquely in terms of X_1. Moreover, as X_1 increases, $X_2, X_3, \ldots, X_n$ will also increase as may be seen by considering Equation A1. As X_1 increases, the increasing function (left side of Equation A1) is shifted downward. Similarly as X_1 increases, the decreasing function (right side of Equation A1) is shifted upward. Thus, the solution for X_i, $i = 2, 3, \ldots, n$ will be greater as X_1 increases. Therefore, if the $\hat{Q}$ calculated from a trial is too low, X_1 must be increased. Similarly, if the $\hat{Q}$ calculated from a trial is too high, X_1 must be decreased. This, with the fact X_1 uniquely determines X_i, $i = 2, 3, \ldots, n$, means that the system of equations has a unique solution. An exactly similar argument can be used in the parallel case by using the equation:

$$\exp(K_{2i}X_i - K_{21}X_1) = \frac{K_{1i}}{K_{11}}\left(\frac{X_1}{1 + K_{21}X_1}\right)\left(\frac{1}{X_i} + K_{2i}\right)$$

to show that a unique solution also exists in this case.

In Chapter III, the minimum cost was again sought using the same method, using the relationship between component and system standard deviations as a constraint.

Again a minimum must exist at an interior point if only one extreme exists, because an infinite cost occurs at each end point. The partial derivatives are set equal to zero, resulting in n equations in n unknowns. There is a unique solution to these equations as can be seen by referring to Section II and recognizing that one is only interested in the field of real numbers, and that σ is the positive square root of σ^2.

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