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SOUND PROPAGATION ACCORDING TO KINETIC MODELS

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Abstract

Sound propagation and the initial value problem in kinetic theory are investigated. Linearized forms of certain kinetic models suggested by Gross and Jackson are employed. The most general model contains three relaxation times and is capable of producing Euler, Navier-Stokes, Burnett and thirteen moments equations for smooth phenomena. A study of dispersion relations is made and some novel features are uncovered. One finds that kinetic models are unable to describe phenomena of higher than a certain wave number, the latter depending on the model chosen. It compensates for this by introducing an interesting but unphysical dispersion picture for higher wave numbers. It is further suggested by one of the models that the phase speed of sound waves achieves a maximum value.

Existence, uniqueness, and boundedness of solutions of the initial value problem are shown for any model equations. It is further shown that asymptotically the solutions become hydrodynamical.

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Sound Propagation According to Kinetic Models

Introduction.

Sound propagation by its very nature is a molecular effect. Yet it was not until the work of Wang Chang and Uhlenbeck[1] that sound propagation was studied from a molecular viewpoint. Prior analytical attempts were essentially hydrodynamical and its extensions. By the latter we shall mean the equations obtained by means of the Chapman-Enskog procedure. In a sense, the work of Wang Chang and Uhlenbeck is also macroscopic. They, in fact, considered their method a Chapman-Enskog process and labelled their calculations as Euler, Navier-Stokes, Burnett, etc. These appellations can be more appropriately given in terms of the moments equations of H. Grad.[2,3] This was pointed out in a recent study by one of the authors.[4] From the results of the latter report it is clear that a complete dispersion theory of sound propagation using the Boltzmann equation is made unfeasible by the difficulty in obtaining results for large wave number phenomena. It is our goal to obtain a description valid throughout the wave number range.

To accomplish this the Boltzmann equation is abandoned in favor of certain kinetic models suggested by Gross and Jackson.[5] We will neither obtain these equations from the Boltzmann equation nor relate it to that equation. The former discussion may be found in reference 5 and the latter in reference 4. We will be content with knowing that our most general equation, given in Section 2, shares many common features with the Boltzmann equation.[5] Although we do not relate the model equations to the Boltzmann equation, we can look in the other direction and relate them to macroscopic equations. We will find that our general equation, which contains three constants (and is referred to as the three relaxation time model) is capable of producing the Euler, Navier-Stokes, Burnett, and thirteen moments theories. In the course of our investigations we will introduce several approximate equations which will be referred to as isosteric (mass preserving), isothermal (constant temperature), and the single relaxation[5] cases. In no case will we have the latitude of our general equation; in fact the latter approximation, which is the most general, is only capable of furnishing the Euler theory. It, therefore, seems necessary to choose a case as general as the three relaxation model if more than qualitative agreement with physics is sought.

In their well-known paper, Bhatnager, Gross and Krook[6], among other things, consider sound propagation. Their equation, which is now referred to as the single relaxation model mentioned above, is considered. Actually, they only carry out the analysis for what we have called the isothermal case. Our methods and results are sufficiently different to warrant a reconsideration of this case. The most notable difference is the exhibition of a part of the spectrum which was not previously discussed. It can be legitimately referred to as the analytic continuation of the dispersion relation but, nevertheless, is still part of the spectrum. During the course of the discussion an argument is given which shows that this analytical continuation of the spectrum has no physical basis. This spurious part of the spectrum arises because a model equation regards, in a sense, phenomena past a certain wave number as free-flow. The dispersion relation of a model equation, therefore, may only be used with confidence for phenomena of low enough wave number. The threshold value of wave number depends on the model chosen.

In spite of the approximate nature of the model equations, actual results from the dispersion relation are difficult to obtain except for small wave numbers. However, the dispersion relation does appear in closed form. Although the dispersion relations are amenable to numerical calculation, only a few calculations were made. A report now in progress will contain an extensive numerical study of the dispersion relations. The few calculations do show the possibility of a maximum phase speed -- it is implied by the single relaxation model. This will not be explored but must await the future numerical work.

The entire report is attuned to solving an initial value problem in the absence of boundaries. The boundary value problem is physically more desirable but unfortunately more difficult to deal with. Inasmuch as the dispersion relation is concerned, the boundary value problem requires solving for wave number in terms of the frequency, whereas the pure initial value problem requires the inverse relation. In the forthcoming numerical work the former case will also be considered.

Since the physical basis of kinetic model equations is somewhat obscure, a mathematical exploration of linearized model equations is made in the last section. The model equations like the Boltzmann equation, are integro-differential equations and lend themselves very nicely to the standard procedures used in showing the existence and uniqueness of solutions. The latter properties are shown for all time under rather innocuous restrictions on the initial data. It is in fact shown that the solutions are exponentially bounded in time, which validates the use of Laplace transforms. Further, using the results of the dispersion theory, one can show that the solutions are uniformly bounded in time and behave asymptotically as hydrodynamics.

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1. Hydrodynamic Preliminaries.

Before developing the kinetic theory description, it is advantageous to first consider the hydrodynamic description of sound propagation. This will be helpful for later comparison. A convenient starting point is the linearized one-dimensional Navier-Stokes equations,

$$(1.1) \qquad \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0$$

$$(1.2) \qquad \frac{\partial u}{\partial t} + \frac{\partial \rho}{\partial x} + \frac{\partial T}{\partial x} = \varepsilon \frac{\partial^2}{\partial x^2} u$$

$$(1.3) \qquad \frac{\partial T}{\partial t} + \frac{2}{3} \frac{\partial u}{\partial x} = \delta \frac{\partial^2}{\partial x^2} T \quad .$$

All quantities (which are perturbations from equilibrium) occuring in (1.1-3) and the temporal coordinate t, and the spatial coordinate x have been made dimensionless with respect to the mean-free-path L and a collision time defined by

$$(1.4) \qquad \tau = \frac{L}{\sqrt{RT_o}} \quad .$$

R is the gas constant and T_o the unperturbed temperature. Denoting all the unperturbed quantities by zero subscripts, and their dimensional perturbations by circumflexes, the normalization is given by

$$(1.5) \qquad \rho = \hat{\rho}/\rho_o \,, \quad \text{density}$$

$$(1.6) \qquad p = \hat{p}/p_o \,, \quad \text{pressure}$$

$$(1.7) \qquad T = \hat{T}/T_o \,, \quad \text{temperature}$$

$$(1.8) \qquad u = \hat{u}/\sqrt{RT_o} \,, \text{ velocity.}$$

Space has been normalized by L and time by τ. The two dimensionless quantities ε and δ are given by

$$(1.9) \qquad \varepsilon = \frac{4\nu}{3L\sqrt{RT_o}}$$

$$(1.10) \qquad \delta = \frac{\kappa}{L\sqrt{RT}}$$

where ν is the kinematic viscosity and κ the thermometric conductivity. Further, the ratio of specific heats $(\gamma = c_p/c_v)$ has been taken as 5/3. Under the linearization and normalization the "gas law" is given by,

$$(1.11) \qquad p = \rho + T \,.$$

Equation (1.11) will be used to eliminate the pressure in the sequel.

It will also be of interest for us to now consider the physically unreasonable but simpler case of isothermal propagation gotten by taking $T = 0$,

$$(1.12) \qquad \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0$$

$$(1.13) \qquad \frac{\partial u}{\partial t} + \frac{\partial \rho}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2} \quad .$$

This case may be thought of as coming from equations (1.11-3) by making the heat conductivity unbounded.

We now consider the initial value problem for (1.12-13) in the absense of boundaries. This will be the only problem considered in this report and we take this initial value problem and sound propagation as being synonymous. On taking a Laplace transform in time and a Fourier transform in space and using matrix notation the isothermal system is

$$(1.14) \quad \begin{pmatrix} \sigma & -ik \\ -ik & \sigma + \epsilon k^2 \end{pmatrix} \begin{pmatrix} \rho \\ T \end{pmatrix} = \begin{pmatrix} \text{Initial} \\ \text{Conditions} \end{pmatrix}$$

σ and k denote the Laplace and Fourier variables respectively. Since we will not be interested in solving any particular initial value problem the right hand side of equation (1.14) is left unspecified. We are interested in the roots of the determinant of the matrix of (1.14), i.e., in the dispersion relation. Strictly speaking, therefore, we shall be investigating the class of possible plane wave solutions..

The roots of the isothermal dispersion relation are easily computed and given by

$$(1.15) \qquad \sigma_{\pm} = \frac{-\varepsilon k^2 \pm \sqrt{\varepsilon^2 k^4 - 4k^2}}{2} \quad .$$

If this is plotted in the complex σ-plane with k^2 as a parameter, we get the sketch of Figure 1. There are two propagating branches starting at the origin, one to the right and the other to the left. For a sufficiently large value of $k^2 (k^2 = 4/\varepsilon^2)$ the two branches return to the negative real axis, as is indicated in Figure 1, and then continue as two non-propagating diffusion modes. One stops at a finite point while the other continues to negative infinity. The details of the latter effect do not really concern us since from the point of view of kinetic theory, hydrodynamical equations can only be considered for small wave number k. In this limit

$$(1.16) \qquad \sigma_+ = ik - \frac{\varepsilon k^2}{2} - \frac{i\varepsilon^2 k^3}{8} + O(k^4)$$

$$(1.17) \qquad \sigma_- = -ik - \frac{\varepsilon k^2}{2} + \frac{i\varepsilon^2 k^3}{8} + O(k^4)$$

As will always be the case, the roots occur in conjugate pairs. Splitting σ_+, for instance, into real and imaginary parts,

$$(1.18) \qquad \sigma_+ = \sigma_+^r + i\sigma_+^i \; ,$$

we get

$$(1.19) \qquad \sigma_+^r = -\frac{\varepsilon k^2}{2} + O(k^4)$$

$$(1.20) \qquad \frac{\sigma_+^i}{ik} = v_p = 1 - \frac{\varepsilon^2 k^2}{8} + O(k^4) .$$

The quantity (1.19) gives the attenuation rate of a plane wave of wave number k, while the second quantity is the phase velocity. It states that the speed of a wave decreases with wave number from the customary value of unity (in our normalization).

The same analysis is now applied to the full Navier-Stokes equations, (1.1-3). Taking transforms we obtain

$$(1.21) \qquad \begin{pmatrix} \sigma & -ik & 0 \\ -ik & \sigma + \varepsilon k^2 & -ik \\ 0 & -\frac{2}{3}\, ik & \sigma + \delta k^2 \end{pmatrix} \begin{pmatrix} \rho \\ u \\ T \end{pmatrix} = \begin{pmatrix} \text{Initial} \\ \text{Conditions} \end{pmatrix}$$

Again we are only interested in the dispersion relation, now given by

$$(1.22) \qquad \sigma^3 + (\varepsilon + \delta)\sigma^2 k^2 + (\varepsilon\delta k^4 + \tfrac{5}{3} k^2)\sigma + \delta k^4 .$$

In keeping with the view that the hydrodynamical equations are valid for only small wave numbers, we obtain the roots of equation (1.22) directly for small k, (1.23) $$\sigma_o \sim -\frac{3}{5}\,\delta k^2 + O(k^4)$$

(1.24) $$\sigma_\pm \sim \pm\, i\sqrt{5/3}\,k - \left(\frac{\delta}{5} + \frac{\varepsilon}{2}\right)k^2 \pm ik^3\sqrt{3/5}\left[\left(\frac{\delta}{5} + \frac{\varepsilon}{2}\right)^2 + \delta\varepsilon\right] + O(k^4).$$

The mode σ_o given by (1.23) is strictly diffusing, and through δ is identified with heat conduction. Equation (1.24) gives two propagating modes, and considering σ_+, for instance,

(1.25) $$\sigma_+^r = -\left(\frac{\delta}{5} + \frac{\varepsilon}{2}\right)k^2 + O(k^4)$$

(1.26) $$\frac{\sigma_+^i}{k} = v_p = \sqrt{5/3}\left(1 + \frac{3}{5}\left[\left(\frac{\delta}{5} + \frac{\varepsilon}{2}\right)^2 + \delta\varepsilon\right]k^2\right) + O(k^4)\ .$$

The first quantity, σ_+^r, as before, gives the attenuation rate, while (1.26) gives the phase velocity. We now notice however, that the lowest order sound speed is the adiabatic value (in our normalization) and furthermore that the phase velocity increases with wave number.

More accurate dispersion relations may be found by choosing higher approximations of the Chapman-Enskog procedure. For instance, the Burnett equations supply three constants besides viscosity and heat conductivity and give the roots correctly to $O(k^3)$. Each higher approximation furnishes the next coefficient in a series started by (1.25-6).[4]

Such investigations are to be found, for instance, in the book on ultrasonics by Herzfeld and Litovitz.[7] As we have mentioned, we are interested in obtaining a dispersion relation valid over the full range of wavelengths and not just an expansion which furnishes information for large wavelength phenomena. We start this study in the next section.

2. Formulation of One-Dimensional Sound Propagation According to Kinetic Models.

Instead of employing the Boltzmann equation to describe the molecular distribution function, f, we shall use a kinetic equation with a mock collision term. The most general such equation to be considered in this paper in,

$$(2.1)\qquad \left(\frac{\partial}{\partial t'} + \hat{\xi}_1 \frac{\partial}{\partial x'}\right) f = \rho' \frac{\nu'}{m} (f_o - f) + \frac{\rho' u \hat{p}_{11} f_o}{m p'} \left(\frac{3\hat{\xi}_1^2 - \hat{\xi}^2}{2RT}\right) + \frac{\rho' \gamma' \hat{S}_1 f_o}{m p' \sqrt{RT'}} \frac{\xi_1}{\sqrt{RT}} \left(\frac{\hat{\xi}^2}{5RT'} - 1\right)$$

This is the one-dimensional (x'), time varying (t') form of one of a class of equations suggested by Gross and Jackson[5], which can be extracted from the Boltzmann equation. For reasons which will become clear, we refer to this as the triple relaxation model. The following nomenclature is used:

$f = f(x',t',\hat{\xi}_1,\hat{\xi}_2,\hat{\xi}_3)$, the molecular distribution function,

$\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3)$, the molecular velocity,

R, the gas constant,

ρ_o, $u_o=0$, p_o, equilibrium mass density, velocity and pressure,

$$\begin{bmatrix} \rho' \\ u' \\ p' \end{bmatrix} = \int f \begin{bmatrix} 1 \\ \hat{\xi}_1/\rho' \\ \frac{(\hat{\xi}-u')^2}{3} \end{bmatrix} d\hat{\xi} ,$$ the corresponding non-equilibrium values

$\hat{p}_{11} = \int (\hat{\xi}_1-u)^2 f \, d\hat{\xi} - p'$, the stress in the x-direction,

$\hat{S}_1 = \int \frac{(\hat{\xi}_1-u')}{2} (\hat{\xi}-u')^2 f \, d\hat{\xi}_1$, the heat conduction in the x'-direction,

$f_o = \frac{\rho'}{(2\pi RT')^{3/2}} e^{-(\hat{\xi}-u')^2/2RT}$, the local Maxwellian,

$f^o = \frac{\rho_o}{(2\pi RT_o)^{3/2}} e^{-\hat{\xi}^2/2RT_o}$, the absolute Maxwellian,

m , the molecular mass .

The three symbols $\mu', \nu', \gamma' > 0$ represent quantities which may be obtained in the determination of (2.1). As given in reference 5, they are constants which are obtained from the molecular force law. In the cited reference equation (2.1) is obtained for the case of Maxwell molecules,* for which the eigentheory of the Boltzmann equation is known.[1] In fact, these constants are merely related to the eigenvalues of the linearized collision operator.

For the case of Maxwell molecules equation (2.1) has sufficient detail in it to give correctly the Navier-Stokes, Burnett, and thirteen moments equations.[4] We shall see evidence of this in later calculations. For other molecular models, equation (2.1) gives the values of transport coefficients correctly to what Chapman and Cowling[8] call the first approximation. Reversing our viewpoint, we can prescribe μ', ν', γ' so as to give transport coefficients or combinations of them correctly. For instance, we can prescribe μ' and γ' to furnish viscosity and heat conductivity correctly (for a particular gas) and then by a judicious choice of ν' approximate the Burnett dispersion law.

For the case of sound propagation we write

$$f = f^{0} + \hat{g} \tag{2.2}$$

and linearize about the equilibrium given by f^{0}. Linearizing the local Maxwellian we get

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* Molecules which interact with an inverse fifth power force law.

(2.3) $$f_o \sim f^o[1 + \frac{\hat{\rho}}{\rho_o} + \frac{\hat{T}}{T_o}(\frac{\hat{\xi}^2}{2RT_o} - \frac{3}{2}) + \frac{\hat{\xi}_1\hat{u}}{RT_o}$$

where

(2.4) $$\begin{cases} \hat{\rho} = \rho' - \rho_o \\ \hat{T} = T' - T_o \\ \hat{u} = u' - u_o \ . \end{cases}$$

We use the following normalization consistent with Section 1:

(2.5) $$\rho = \frac{\hat{\rho}}{\rho_o}\ , \quad T = \frac{\hat{T}}{T_o}\ , \quad u = \frac{\hat{u}}{\sqrt{RT_o}}$$

and in addition

(2.6) $$\begin{cases} p_{11} = \frac{\hat{p}_{11}}{p_o} \\ S_1 = \frac{\hat{S}_1}{p_o\sqrt{RT_o}} \\ \xi = \frac{\hat{\xi}}{\sqrt{RT_o}} \\ \nu = \frac{\rho\nu'}{m}\ , \quad \mu = \frac{\rho_o u'}{m\nu'}\ , \quad \gamma = \frac{\rho_o\gamma'}{m\nu'} \\ g = \frac{(RT_o)^{3/2}\ \hat{g}}{\rho_o} \end{cases}$$

(2.7) $$\omega = \frac{e^{-\xi^2/2}}{(2\pi)^{3/2}}$$

$$(2.8)^{*} \quad \begin{cases} \nu = \frac{1}{\tau} \\ \frac{\nu}{\sqrt{RT_o}} = \frac{1}{L} \\ \frac{x'}{L} = x \,, \qquad \frac{t'}{\tau} = t \end{cases}$$

When all this is applied to equation (2.1) we get, finally,

$$(2.9) \qquad \left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + 1\right) g = \omega\left[\rho + \left(\frac{\xi^2}{2} - \frac{3}{2}\right)T + \xi_1 u + \mu p_{11}\left(\frac{3}{2}\xi_1^2 - \frac{1}{2}\xi^2\right) + \gamma S\, \xi_1\left(\frac{1}{5}\xi^2 - 1\right)\right]$$

The relation of the "flow field" to the perturbed distribution function is given by

$$(2.10) \qquad \int \begin{pmatrix} 1 \\ \xi_1 \\ \left(\frac{1}{3}\xi^2 - 1\right) \\ \left(\xi_1^2 - \xi^2/3\right) \\ \frac{(\xi^2 - 5)\xi_1}{2} \end{pmatrix} g\,d\xi = \begin{pmatrix} \rho \\ u \\ T \\ p_{11} \\ S \end{pmatrix}$$

* Our normalizing constants, τ and L, are no longer the mean-free-time and the mean free path. The procedure given in reference 5 dictates the choice of ν, which for equation (2.1) is considerably larger than the collision frequency of the gas.

Note as in Section, we have eliminated the pressure with the identity

(2.11) $\quad p = \rho + T$.

We again consider a problem for which there are no boundaries and consider the initial value problem. In Section 6 it is shown that with suitable initial conditions we can rigorously apply transforms to equation (2.9). Operating on equation (2.9) with $e^{-\sigma t + ikx}\,dtdx$, i.e., taking a Laplace transform in time and Fourier transform in space, we get after solving for the transformed perturbed distribution function,

$$(2.12) \qquad g = \frac{\omega}{1+\sigma-ik_1}\,[\rho + (\frac{\xi^2}{2} - \frac{3}{2})T + \xi_1 \cdot u \quad \mu p_{11}(\frac{3}{2}\,\xi_1^2 - \frac{1}{2}\,\xi^2)$$

$$+ \gamma S_1 \xi_1 (\frac{1}{5}\,\xi^2 - 1] + \frac{g_o}{1+\sigma-ik\xi_1}$$

where we have denoted the transformed variable by the same letter, and given the initial data a zero subscript. Performing the moment integrations indicated in equation (2.10), and integrating out the transverse velocities ξ_2, ξ_3, we get

(2.13) $\quad V = \bar{\bar{C}}\,\overline{V} + \overline{V}_o$.

with

(2.14) $$\bar{V} = \begin{pmatrix} \rho \\ u \\ T \\ p_{11} \\ S_1 \end{pmatrix}$$

(2.15) $$\bar{V}_o = \int \frac{g_o}{1 + \sigma - ik\xi_1} \begin{bmatrix} 1 \\ \xi_1 \\ (\frac{1}{3}\,\xi^2 - 1) \\ (\xi_1^2 - \xi^2/3) \\ \dfrac{(\xi^2 - 5)\xi_1}{2} \end{bmatrix} \overline{d\xi}$$

(2.16) $$\bar{\bar{C}} = \int \frac{e^{-\xi_1^2/2}}{(2\pi)^{1/2}(1+\sigma-ik\xi_1)}\, d\xi_1 \quad \times$$

$$\begin{bmatrix} 1 & \xi_1 & \xi_1^2/2 - \frac{1}{2} & \mu(\xi_1^2-1) & \gamma\xi_1(\frac{\xi_1^2}{5} - \frac{3}{5}) \\ \xi_1 & \xi_1^2 & \xi_1^3/2 - \xi_1/2 & \mu(\xi_1^3-\xi_1) & \frac{\gamma}{5}(\xi_1^4-3\xi_1^2) \\ \xi_1^2/3 - \frac{1}{3} & \xi_1^3/3 - \xi_1/3 & \xi_1^4/6 - \xi_1^2/3 + \frac{5}{6} & \frac{\mu}{3}(\xi_1^4-2\xi_1^2-1) & \frac{\gamma}{15}(\xi_1^5-4\xi_1^3+7\xi_1) \\ \frac{2}{3}(\xi_1^2-1) & \frac{2}{3}(\xi_1^3-\xi_1) & \xi_1^4/3 - \frac{2}{3}\xi_1^2 - \frac{1}{3} & \frac{2\mu}{3}(\xi_1^4-2\xi_1^2+2) & \frac{2\gamma}{15}(\xi_1^5-4\xi_1^3+\xi_1) \\ \frac{\xi_1}{2}(\xi_1^2-3) & \frac{\xi_1^2}{2}(\xi_1^2-3)\frac{1}{2} & \xi_1^5/2 - 2\xi_1^3 + \frac{7}{2}\xi_1 & \frac{\mu}{2}(\xi_1^5-4\xi_1^3+\xi_1) & \frac{\gamma}{10}(\xi_1^6-6\xi_1^4+13\xi_1) \end{bmatrix}$$

Each element of $\bar{\bar{C}}$ may be evaluated in terms of a single integral (see Appendix II, Lemma 1),

$$(2.17) \qquad M = \lambda \int_{-\infty}^{\infty} \frac{\omega}{\lambda - i\xi_1} \, d\xi_1$$

with

$$(2.18) \qquad \lambda = \frac{1 + \sigma}{k} .$$

As is well-known[9], an integral of the type given by equation (2.17) has the real axis as a natural cut. Depending on whether the real part of λ is positive or negative the integral of (2.17) defines two different functions which are in fact entire analytic. Moreover, the analytic continuation of each of these functions across the cut does not lead to the other function.

At this point of the analysis σ must be considered as having a positive real part, it having bean so chosen to define the Laplace transform. Hence λ will have either a positive or negative real part depending on the sign of k. Equation (2.17) thus defines two functions M^+ and M^- for $k > 0$ and $k < 0$, respectively. The precise forms of these functions are developed and given in Appendix I. It is interesting to consider the jump of M across the cut. This is immediately given by a formula of Hermite[9], and is

$$(2.19) \qquad [M/\lambda] = \sqrt{2\pi}\, e^{-\lambda_1^2/2}$$

where the bracket denotes the jump and λ_1 the imaginary part of λ. In the important limit when $\lambda_1 \longrightarrow \infty$ (corresponding to k passing through zero), the jump is asymptotically small and the transition between functions is smooth.

At this point, with all the elements of the matrix computed, the dispersion relation is obtained by taking

$$(2.20) \qquad \det(\overline{U}-C) = 0$$

where U is the unit matrix. A study of the complete dispersion relation does not seem feasible. We will, instead, make use of expansions, asymptotics and approximations. In this way we will map out the dispersion relation and at worst gain only qualitative results. In the next section we will be concerned with the possible assumptions that can be made, and their physical basis.

3. Approximate Formulations.

The work in the dispersion relation is considerably reduced by the use of the conservation equations. To obtain these, consider the transform of (2.9).

$$(3.1) \qquad (\sigma - ik\xi_1 + 1)g = \omega[\rho + (\frac{\xi^2}{2} - \frac{3}{2})T + \xi_1 u$$

$$+ \; p(\frac{3}{2}\,\xi_1^2 - \frac{1}{2}\,\xi^2) + \gamma S\xi_1(\frac{1}{5}\,\xi^2 - 1)].$$

On taking the first three moments indicated by equation (2.10) we get the transformed conservation equations,

$$(3.2) \qquad \sigma\rho - iku = p_1$$

$$(3.3) \qquad \sigma u - ik(\rho + T) - ikp_{11} = u_1$$

$$(3.4) \qquad \sigma T - \frac{2}{3}\, iku - \frac{2}{3}\, ikS = T_1 \ .$$

The subscript i refers to the comparable moment of the initial data. With the use of equations (3.2-4), we can study the dispersion relation given by

$$(3.5) \qquad \begin{vmatrix} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & -ik & -ik & 0 \\ 0 & -2ik\ 3 & \sigma & 0 & -2ik/3 \\ C_{41} & C_{42} & C_{43} & C_{44}-1 & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55}-1 \end{vmatrix} = D_5(k,\sigma) = 0$$

where the C's are those given in Table 1. As is clear by comparison of (3.5) and (2.20), the determinant (3.5) is much simpler to deal with. We will therefore consider (3.5) in place of (2.20). One will notice that the initial data problem as dealt with in connection with (3.5) appears differently than in connection with (2.20). Later in this section we will remark on the equivalence of the two formulations.

The dispersion relation as given by (3.5) will be discussed in Section 5. A more complete discussion can be given for systems which are simpler than (3.5). Instead of taking (2.16) as the case to study we can, for instance, consider the equations for (ρ, u, T) only, gotten by considering the upper left hand three by three matrix of equation (2.16), i.e.,

$$(3.6) \qquad \begin{bmatrix} \rho \\ u \\ T \end{bmatrix} = \frac{1}{(2\pi)^{1/2}} \int \frac{e^{-\xi_1^2/2}\, d\xi_1}{(1+\sigma-ik\xi_1)} \quad \times$$

$$\begin{bmatrix} 1 & \xi_1 & \xi_1^2/2 - \frac{1}{2} \\ \xi_1 & \xi_1^2 & \xi_1^3/2 - \xi_1/2 \\ \xi_1^2/3 - \frac{1}{3} & \xi_1^3/3 - \xi_1/3 & \xi_1/6 - \xi_1^2/3 + \frac{5}{6} \end{bmatrix} \begin{bmatrix} \rho \\ u \\ T \end{bmatrix} + \int \frac{g_0\, d\xi}{1+\sigma-ik\xi_1} \begin{bmatrix} 1 \\ \xi_1 \\ (\frac{1}{3}\xi^2 - 1) \end{bmatrix}$$

For reasons which will become clear in Section 5, this can be referred to as the adiabatic case. This system could be obtained by setting $\mu = \gamma = 0$, or equivelently, by considering instead of equation (2.9), the equation

$$(3.7) \qquad \left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + 1\right) g = \omega\left(\rho + \left(\frac{\xi^2}{2} - \frac{3}{2}\right) T + \xi_1 \hat{u}\right) ,$$

or on transforming

$$(3.8) \qquad (\sigma - ik\xi_1 + 1) g = \omega\left(\rho + \left(\frac{\xi^2}{2} - \frac{3}{2}\right) T + \xi_1 \hat{u}\right) + g_0 .$$

Equation leads to the dispersion relation

$$(3.9) \qquad D_3' = \begin{vmatrix} c_{11}-1 & c_{12} & c_{13} \\ c_{21} & c_{22}-1 & c_{23} \\ c_{31} & c_{32} & c_{33}-1 \end{vmatrix} = 0 \ .$$

A simplification similar to (3.5) can be made now. From equation (3.8) we see that on taking the mass moment we get equation (3.2) the continuity equation. This, as before, can be used to replace the mass moment in (3.6). On doing this we get

$$(3.10) \qquad \begin{vmatrix} \sigma & -ik & 0 \\ c_{21} & c_{22}-1 & c_{23} \\ c_{31} & c_{32} & c_{33}-1 \end{vmatrix} = D_3 = 0$$

which is equivalent to considering (3.9). Equation (3.7) is referred to in the literature as the single relaxation model.*

---

* This is the name given to equation (3.7) by Gross and Jackson in reference 5. A relaxation theory obviously has all non-hydrodynamic moments decaying with a single time inversely proportional to ν of equation (2.8).

We have been somewhat cavalier in setting μ and γ equal to zero, since (3.7) still incorrectly contains ν in its normalization. The correct value as given by Gross and Jackson is related to the collision frequency (see footnote, page 18), which is smaller than ν of (2.8). We, therefore, think of equation (3.7) as having been obtained by a process in which μ and γ vanish and ν goes to the collision frequency. This distinction plays an important role in an argument given at the close of Section 4.

It first appears in the well-known paper of Bhatnager, Gross and Krook.[6] In that paper the authors, among other problems, consider sound propagation. Although they consider (3.7), they restrict their calculations to a simpler model which we will consider in the next paragraph. Since our methods are different and since new features are brought to light we will reconsider the case taken up in reference 4 in Section 4.

By further truncation of the system (2.16), simpler systems can be derived. On truncating the system to a two by two, we can consider

$$(3.11)\qquad \begin{pmatrix} \rho \\ u \end{pmatrix} = \frac{1}{(2\pi)^{1/2}} \int \frac{e^{-\xi_1^2/2}\, d\xi_1}{1+\sigma-ik\xi_1} \begin{pmatrix} 1 & \xi_1 \\ \xi_1 & \xi_1^2 \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix} + \int \frac{g_o d\xi}{1+\sigma-ik\xi_1}$$

which is equivalent to choosing the model equation,

$$(3.12)\qquad \left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + 1\right)g = \omega(\rho + \xi_1 u)$$

or the equivalent transform

$$(3.13)\qquad (\sigma - ik\xi_1 + 1)g = \omega(\rho + \xi_1 u) + g_o \quad .$$

As in the previous cases, one is led to the dispersion relation given by

$$(3.14)\quad \begin{vmatrix} 1 - \int \frac{\omega d\xi_1}{1+\sigma-ik\xi_1} & - \int \frac{\xi_1 \omega \, d\xi_1}{1+\sigma-ik\xi_1} \\ - \int \frac{\omega\xi_1 d\xi_1}{1+\sigma-ik\xi_1} & 1 - \int \frac{\omega\xi_1^2 d\xi_1}{1+\sigma-ik\xi} \end{vmatrix} = D_2' = 0$$

Instead of the first row of (3.14), once again we can introduce the continuity equation which still holds, to get

$$(3.15)\quad \begin{vmatrix} \sigma & -ik \\ - \int \frac{\omega\xi_1 d\xi_1}{1+\sigma-ik\xi_1} & 1 - \int \frac{\omega\xi_1^2 d\xi_1}{1+\sigma-ik\xi_1} \end{vmatrix} = D_2 = 0$$

as the dispersion relation. Using the forms given in Table 1 it is straightforward to see that (3.14) is $-1/1+\sigma$ times (3.15). The previous two cases go in the same way. For reasons which will become clear shortly this will be referred to as the isothermal case.

The most drastic truncation which can be made, is to consider the one by one system, i.e.,

$$(3.16)\quad \rho = \frac{1}{(2\pi)^{1/2}} \int \frac{e^{-\xi_1^2/2} \, d\xi_1}{1+\sigma-ik\xi_1} \rho + \int \frac{g_o d\xi}{1+\sigma-ik\xi_1} ,$$

which is equivalent to first considering

$$(3.17)\quad \left(\frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + 1\right) g = \omega\rho ,$$

or equivalently,

$$(3.18) \qquad (\sigma - ik\xi_1 + 1)g = \omega\rho + g_o \; .$$

The dispersion relation in this is simply

$$(3.19) \qquad 1 - \frac{1}{2\pi}\int \frac{e^{-\xi_1^2/2}\, d\xi_1}{1+\sigma-ik\xi_1} = 1 - C_{11} = D_1 = 0 \; .$$

This will be referred to as the isosteric case.

Some light can be shed on the meaning of each of the cases in question by studying the moment equations comparable to the conservation equations. Taking the mass, velocity and temperature moments of (3.7), (3.11) and (3.17), we get, respectively,

$$(3.20) \qquad \begin{cases} \frac{\partial\rho}{\partial t} + \frac{\partial u}{\partial t} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\rho+T) + \frac{\partial p_{11}}{\partial x} = 0 \\ \frac{\partial T}{\partial t} + \frac{2}{3}\frac{\partial u}{\partial x} + \frac{2}{3}\frac{\partial}{\partial x}S_1 = 0 \end{cases}$$

$$(3.21) \qquad \begin{cases} \frac{\partial\rho}{\partial t} + \frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\rho+T) + \frac{\partial p_{11}}{\partial x} = 0 \\ \frac{\partial T}{\partial t} + \frac{2}{3}\frac{\partial u}{\partial x} + \frac{2}{3}\frac{\partial}{\partial x}S_1 = -T \end{cases}$$

$$
(3.22)\quad \begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial u}{\partial x} = 0 \\[2ex] \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(\rho + T) + \dfrac{\partial p_{11}}{\partial x} = -u \\[2ex] \dfrac{\partial T}{\partial t} + \dfrac{2}{3}\dfrac{\partial u}{\partial x} + \dfrac{2}{3}\dfrac{\partial}{\partial x} S_1 = -T \end{cases} .
$$

So we see that the adiabatic case preserves the conservation equations, the isothermal case only preserves the continuity and the momentum equations, and the isosteric preserves only the continuity equation (hence the appellation, mass-preserving). The conservation equations (3.20) should be well understood. If the stress, p_{11}, and heat conduction S_1 are dropped, the inviscid Euler equations result. This leads to unattenuated propagation with the adiabatic speed ($= \sqrt{5/3}$ in our normalization). On the other hand if the Navier-Stokes relations for S_1 and p_{11} are substituted there results the propagation of a diffusing wave at the adiabatic speed and in addition a purely diffusing mode.* A similar analysis can be made for the systems (3.21-23). The inviscid theory is obtained by dropping quantities without a time derivative. For (3.21) this yields propagation with speed one (the isothermal speed, in our normalization), and for (3.22) we get a zero propagation speed. In order to obtain an analogous viscous description we must develop relations allied to the Navier-Stokes relations. To do this,

* See reference 10. Section 6 contains the analysis for the isothermal case.

we use the method of interpolation which is akin to the Chapman-Enskog theory, and was invented by H. Grad.[11] Essentially, one considers space derivatives as being smooth with respect to time derivatives. Under this assumption the energy equation of (3.21) or (3.22) can be integrated (considering the space derivatives as constants) and after sufficient time the solution is,

$$(3.23) \qquad T \sim -\frac{2}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial}{\partial x} S_1$$

On taking higher moments of equation (3.7) we can obtain equations in p_{11} and S_1 (i.e., containing the time derivative of these quantities), and again perform an interpolation (see references 4 and 9 for more details of this method.) Neglecting second derivatives we find that

$$(3.24) \qquad \begin{cases} p_{11} \sim -\frac{4}{3}\frac{\partial u}{\partial x} \\ S_1 \sim -\frac{5}{2}\frac{\partial T}{\partial x} \end{cases}$$

and (3.23) becomes

$$(3.25) \qquad T \sim -\frac{2}{3}\frac{\partial u}{\partial x} \; .$$

When substituted into (3.21) this gives

$$(3.26) \qquad \begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0 \\ &\frac{\partial u}{\partial t} + \frac{\partial \rho}{\partial x} - 2\,\frac{\partial^2 u}{\partial x^2} = 0 \end{aligned}$$

An asymptotic study of the fundamental solution of the system (3.24) (these are the circumstances under which (3.24) can be presumed valid) shows that a diffusing wave propagates at isothermal propagation speed.[11]

When the same procedure is applied to the isosteric system one gets

$$(3.27) \qquad \frac{\partial}{\partial t}\rho - \frac{\partial^2}{\partial x^2}\rho = 0 \; .$$

In this case, then, we have diffusion and no propagation.

The properties of the various systems which have been sketched above will, in the sequel, be found in a more straightforward manner. As might be expected, the isosteric and isothermal cases allow the most analytical work and are the most easily understood. In the following section we deal with these two cases somewhat exhaustively. Thereafter, the more tedious parts of the analysis will be avoided and only the results stated.

4. Isosteric and Isothermal Propagation.

In this section we deal with the two most degenerate models discussed in the previous section. Both of these models are so devoid of physical content that they are useless from this viewpoint. However, many of the properties of the dispersion relation found in this section will carry over to the more general cases found in the later sections. Because of the nature of the function M, an exact discussion of the dispersion law does not seem possible. Instead we will resort to a whole gamut of approximate devices which will give us a reasonably good description of the dispersion law.

Isosteric case.

Referring to (3.19) we see that the dispersion relation in this case is given by

$$k - \frac{M}{\lambda} = F(k, \lambda) = 0 \ . \tag{4.1}$$

As is shown in Appendix II, it is sufficient to consider (4.1) for $k \geq 0$ and $\operatorname{Im} \lambda \geq 0$. We will, therefore, restrict attention to the upper half of the λ-plane and use M^+ of Appendix I. Substituting (I.8) for M^+ into (4.1) gives us

$$k = \left(\frac{\pi}{2}\right)^{1/2} e^{\lambda^2/2} \left[1 - \left(\frac{2}{\pi}\right)^{1/2} \int_0^\lambda e^{-t^2/2}\, dt\right] \tag{4.2}$$

Also from Appendix I, we have the asymptotic expansions

(4.3) $$\frac{M}{\lambda} \sim \frac{1}{\lambda} - \frac{1}{\lambda^3} + \frac{1\cdot 3}{\lambda^5} - \frac{1\cdot 3\cdot 5}{\lambda^7} + \cdots ,$$

$$-\frac{3\pi}{4} < \arg \lambda < +\frac{3\pi}{4} ,$$

and

(4.4) $$\frac{M}{\lambda} \sim (2\pi)^{1/2} e^{\lambda^2/2} + \frac{1}{\lambda} - \frac{1}{\lambda^3} + \frac{1\cdot 3}{\lambda^5} + \cdots ,$$

$$\frac{\pi}{4} < \arg \lambda < \frac{7\pi}{4} .$$

It is immediate from (4.3) and (4.4) that the ray $e^{13\pi/4}$ is a Stokes line of M across which the asymptotic behavior jumps. This will play an important role in what follows. Before investigating the asymptotics we examine (4.2) for real roots.

For real λ the right hand side of (4.2) is real. From (4.3) we see that $k \to 0$ as $\lambda \to \infty$, and from (4.4) $k \to \infty$ ad $\lambda \to -\infty$. By direct substitution $k = (\pi/2)^{1/2}$ for $\lambda = 0$. We see therefore that $0 \leq k \leq \infty$ goes into the entire real λ line. If we examine the roots in the plane of the Laplace variable σ, we see that

(4.5) $$\begin{cases} \sigma \sim -k^2 & \text{for } k \sim 0 \\ \sigma = -1 & \text{for } k = (\pi/2)^{1/2} \\ \sigma \to -\infty & \text{for } k \to \infty . \end{cases}$$

In what follows we shall refer to the strip $-1 \leq R\sigma \leq 0$, and the half-plane $R\lambda \geq 0$ as the hydrodynamic proportions of the respective planes.

We now estimate the number of roots to (4.1) in the λ-plane. To do this we use the method of winding numbers (see [9], p. 102). We will consider closed regions only in the upper half plane by virtue of the remarks in Appendix II. Since our expression (4.1) is entire the winding number will only count zeros, and since it is not identically zero we always choose a closed path no point of which is a zero. Because of the latter and Rouche's theorem[12] we can with confidence use the asymptotic evaluations for the determination of the zeros.

Let $|\lambda|$ be large and consider the closed path of the diagram, Figure 4.1. For λ sufficiently large, the leading terms of the dispersion relation are

$$(4.6) \qquad 0 = k - (2\pi)^{1/2} e^{\lambda^2/2} - \frac{1}{\lambda} + O(1/\lambda^3) \qquad \frac{\pi}{2} \leq \arg \lambda \leq \pi$$

$$(4.7) \qquad 0 = k - \frac{1}{\lambda} + O(1/\lambda^3) \qquad 0 \leq \arg \lambda \leq \pi/2$$

The angular splitting which is somewhat arbitrary is made merely for convenience in reference. We see that neither (4.6) nor (4.7) (or for that matter (4.1)) can have roots in R as

$k \to \infty$.

Next consider the region of Figure 2. To consider this region we make use of the fact that if $R_e\{F(\lambda,k)\}$ vanishes at 2n distinct points on a + b + c, F has at most n zeros in R.[12] For any fixed value of k as $|\lambda| \to \infty$, we see directly from (4.2) that $R_e[F]$ has at most one zero on a + b, and from (4.3) that it has no zero on c. Hence, with the possible exception of the zero on real line, already discussed, $F(k,\lambda)$ can have no zero in the right half of the λ-plane.

We now consider the region of $\frac{3\pi}{4} > \arg \lambda \geq \frac{\pi}{2}$. From the asymptotic expansion given by (4.4), we see that for any $|k| > 0$, no matter how small, (4.2) will not have a solution for sufficiently large distances from the origin along a ray. This is true for any ray in the region as close as we please to the Stokes line. Of course, the closer such a ray is to the stokes line, the further out we must go to guarantee non-solvability. One must not conclude that the solutions are on the Stokes line, but rather that they are closer than any ray (for instance a parabola is closer than a ray). We now further explore this region, i.e., region I given in Figure 3. We again make use of the theorm on the $R_e\{F(k,\lambda)\}$. From (4.1) and (4.6) for a fixed k and $|\lambda|$ sufficiently large, $R_e F$ has no zero on b, and at most one zero on c. Along the Stokes one may easily estimate the magnitude of the right hand side of (4.2) to be $< (\pi/2)^{1/2} + \sqrt{2}$ (asymptotically the estimate is $\sqrt{2\pi}$ which is slightly smaller). Therefore, the $R_\lambda\{F\}$ will not have zeros along the Stokes line if $k > (\pi/2)^{1/2} + \sqrt{2}$. And with the proceeding information, F will not have zeros in I for $k > (\pi/2)^{1/2} + \sqrt{2}$. We will later asymptotically locate these roots.

To estimate the number of zeros in region II of Figure 3, we compute the winding number in traversing the path a + d + e. The only contribution will come from a + d. First we consider the arc d. The winding number is obtained from

$$(4.8) \qquad \frac{1}{2\pi} \tan^{-1} \left[\frac{-(2\pi)^{1/2}\, e^{\mu^2-\nu^2/2} \sin \mu\nu}{k-(2\pi)^{1/2}\, e^{\mu^2-\nu^2/2} \cos \mu\nu} \right]$$

where

$$(4.9) \qquad \begin{aligned} \lambda &= \mu + i\nu \\ &= Re^{i\theta} \quad \text{on d.} \end{aligned}$$

On the arc a the product $\mu\nu$ decreases from $R^2/2$ to 0. Using only asymptotics on the path a, the winding number is gotten from

$$(4.10) \qquad \frac{1}{2\pi} \tan^{-1} \left[\frac{(2\pi)^{1/2} \sin \rho^2}{k-(2\pi)^{1/2} \cos \rho^2} \right]$$

where

$$(4.11) \qquad \lambda = \rho(-1 + i) .$$

The important thing to note is that if

$$(4.12) \qquad k < (2\pi)^{1/2}$$

the contribution from (4.10) "unwinds" the contribution from (4.8). In this case the number of roots is $O(1)$. If, on the other hand,

$$(4.13) \qquad k > (2\pi)^{1/2} ,$$

(4.10) does not contribute and it is easily seen from (4.8) that the number of roots in II is $O(R^2)$. One notices in (4.8) that the denominator will not, for a fixed large k, vanish unless $\mu^2 - \nu^2 = O(\ell nk^2)$. This is in agreement with our previous remark on the decrease of the number of zeros in a bounded region as k grows. We can, in fact, very easily calculate from (4.8) the "exact" number of zeros in II for $k > (2\pi)^{1/2}$. We will not do this but rather go on to the location of the zeros.

We first asymptotically locate these roots which lie on the Stokes line. On the Stokes line we may write, on using (4.11) in (4.6), and splitting into real and imaginary parts

$$(4.14) \qquad k = (2\pi)^{1/2} \cos \rho^2 + \frac{1}{\rho\sqrt{2}}$$

(4.15) $$(2\pi)^{1/2}\sin\rho^2 = -\frac{1}{\rho\sqrt{2}} \quad .$$

For

(4.16) $$\begin{cases} \rho^2 \sim (2\pi N) - \varepsilon \\ |\lambda| \sim 2(\pi N)^{1/2} \end{cases}$$

equation (4.15) gives

(4.17) $$\varepsilon \sim \frac{1}{2\pi\sqrt{N2}}$$

and from (4.14)

$$k \sim (2\pi)^{1/2} \ .$$

For further reference it is worth noting that

(4.18) $$\begin{cases} \rho^2 \sim (2N+1)\pi + \dfrac{1}{2\pi\sqrt{2N+1}} \ , \qquad |\lambda| \sim \{2(2N+1)\pi\}^{1/2} \\ k \sim -(2\pi)^{1/2} \ . \end{cases}$$

On returning to (4.2) we get on differentiation that

(4.19) $$\frac{d\lambda}{dk} = \frac{1}{\lambda k - 1} \quad .$$

Using the notation of (4.9) we find the following equations for the real and imaginary parts of (4.19).

(4.20) $$\frac{d\mu}{dk} = \frac{\mu k - 1}{k^2(\mu^2 + \nu^2) - 2\mu k + 1}$$

(4.21) $$\frac{d\nu}{dk} = \frac{-\nu k}{k^2(\mu^2 + \nu^2) - 2\mu k + 1} .$$

Taking the ratio of the two equations,

(4.22) $$\frac{d\nu}{d\mu} = \frac{-\nu k}{\mu k - 1} .$$

From our previous estimates we know that in the neighborhood of the Stokes line and to the left of it $\mu k >> 1$. Under this assumption,

(4.23) $$\frac{d\nu}{d\mu} = \frac{-\nu}{\mu}$$

and on integrating

(4.24) $$\nu = \frac{-\pi N2}{\mu} .$$

The information along the Stokes line given by (4.16) and (4.18) has been used as initial data to get the solution (4.24). This information along the Stokes line also states, by virtue of (4.23), that the branches of the dispersion relation are all normal to the Stokes line. Further, from (4.20) and (4.21) we have that

(4.25) $$\begin{cases} \dfrac{d\nu}{dk} = 0 \\ \\ \dfrac{d\mu}{dk} = -1 \end{cases} \qquad \text{for } k = 0$$

and

$$(4.26) \quad \begin{cases} \frac{d\mu}{dk} = 0 \\ \frac{d\nu}{dk} = \frac{-1}{\nu k} \end{cases} \quad \text{for } k = \frac{1}{\mu}, \quad \text{note} \quad k < 0 \; .$$

With the information gathered in the preceeding, we may, with reasonable accuracy, sketch the dispersion curves in the λ-plane. A typical curve is plotted in Figure 4. Actually, of course, an infinite number of curves like the one shown should be plotted. The dotted curve in the diagram indicates the value for $k < 0$ which does not belong to the dispersion relation, and is plotted only for completeness. One should also notice that the real axis is also part of the dispersion relation. In fact, the contribution in the hydrodynamic half-plane is asymptotically the most important part of the spectrum. In connection with this plot we make one last calculation. For the region to the left of the Stokes line we can neglect ν compared to μ and if (4.20) is solved under this assumption, we get

$$(4.25) \qquad \mu \sim -\sqrt{\ln \frac{k^2}{2\pi} + 2\pi N}$$

and from (4.24)

$$(4.26) \qquad \nu \sim \frac{\pi N}{\sqrt{\ln \frac{k^2}{2\pi} + 2\pi N}} \quad .$$

Since

(4.27) $$\sigma = k\lambda - 1 ,$$

we can give the sketch (Figure 5) of the spectrum in the σ-plane.

Isothermal case

It is not our intention to go through the same for this case as in the isosteric case just considered. With the exception of the hydrodynamic part of the plane, the same features of the dispersion equation are found. For this reason, only a sketchy analysis will be given. The hydrodynamic part of the plane, on the other hand, will receive more care.

The determinant in this case, D_2, is given by (3.15), and in a slightly different form is

(4.27) $$D_2 = \begin{vmatrix} k\lambda - 1 & -ik \\ \frac{i}{k}(1-M) & \frac{\lambda}{k}(1-M)-1 \end{vmatrix} = 0$$

On expansion, the dispersion relation is given by

(4.28) $$0 = D_2 = (M-1) + \lambda^2(1-M) + 1 - \frac{\lambda}{k}(1-M) - k\lambda .$$

We first restrict attention to the second quadrant of the

λ-plane. In this region the asymptotic expansion is given in Appendix I as

$$(4.29) \qquad M \sim (2\pi)^{1/2} \lambda e^{\lambda^2/2} + 1 - \frac{1}{\lambda^2} + \frac{1\cdot 3}{\lambda^4} - \dots .$$

With the experience gained in the isosteric case, a few short calculations give us the dispersion law in this region. First, we consider the asymptotic roots for $k = 0$. The dispersion relation then is

$$(4.30) \qquad (2\pi)^{1/2} e^{\lambda^2/2} \sim \frac{1}{\lambda^3} + O\left(\frac{1}{\lambda^5}\right) .$$

This implies that these roots all lie to the right of the Stokes line. To see this we recall that along the Stokes line $e^{\lambda^2/2}$ is $O(1)$ whereas along any ray, to the right is exponentially decreasing and any ray to the left, exponentially increasing. Next it is useful to find the roots along the Stokes line as $\lambda \rightarrow \infty$. It is simple to see that the only possible k's for which there are roots are those for which $k \rightarrow \infty$. In this case, then, the dispersion relation is approximately given by

$$(4.31) \qquad D_2 \sim \lambda(1-M) - k \sim 0 .$$

Introducing (4.11) we may write the dispersion relation as, (4.32) $$2i\rho^2 e^{-i\rho^2} = \frac{k}{(2\pi)^{1/2}} + O(\frac{1}{\rho}) .$$

This leads to

(4.33) $$\rho^2 \sim \frac{(2N-1)}{2} \pi$$

and

(4.34) $$\frac{k}{(2\pi)^{1/2}} \sim \pm (2N-1)\pi .$$

Finally, we consider the asymptotics in the region $\frac{3\pi}{4} < \arg \lambda < \pi$. Looking at (4.28) the same terms are dominant, i.e.,

(4.35) $$D_2 \sim \lambda^2(1-M) - k\lambda \sim 0 .$$

This being the case, each branch may be associated with the point at which it crosses the Stokes line. We will be content with just the asymptotic estimate of λ which a simple calculation shows to be

(4.36) $$\lambda \sim \sqrt{\ln \frac{k^2}{2\pi} - (N-1)i\pi}$$

The three short calculations just made tell us that the left half of the λ-plane would have a form comparable to that given in Diagram 4. We will sketch this shortly, after having considered the hydrodynamic region.

Rather than perform the appropriate asymptotics with the expansion of D_2, it is more revealing to perform the asymptotics directly in (4.27). In this form the asymptotics give

$$(4.37) \qquad D_2 \sim \begin{vmatrix} \sigma & -ik \\ \frac{ik}{(1+\sigma)^2} & \frac{1+\sigma}{k^2}\left(\frac{k^2}{(1+\sigma)^2} - \frac{3k^4}{(1+\sigma)^4}\right) - 1 \end{vmatrix} = 0$$

For convenience we have returned to the Laplace variable σ. Performing row operations this can be put into the form

$$(4.38) \qquad \begin{vmatrix} \sigma & -ik \\ \frac{ik}{1+\sigma} & \frac{3k^2}{(1+\sigma)^2} - \sigma \end{vmatrix} = 0 .$$

Inspection shows that the only roots of (4.38) consistent with $\lambda \longrightarrow \infty$ are of the form $\sigma = O(k)$. Imposing this condition and again making use of row operation, we get

$$(4.39) \qquad \begin{vmatrix} \sigma & -ik \\ -ik & \sigma + 2k^2 \end{vmatrix} = 0 .$$

On comparing this with the system (3.26) we see that it is precisely the Fourier-Laplace transform of that system. The Chapman-Enskog procedure performed on that system is equivalent to an expansion in small wave number. To $O(k^2)$, (4.39) gives

$$\sigma = \pm ik - k^2 . \tag{4.40}$$

Therefore, to this order we get two attenuating waves, traveling with the isothermal speed, to the right and left. Translating (4.40) into the λ-plane we have

$$\lim_{k \to 0} \lambda \to +\infty \pm i . \tag{4.41}$$

Collecting together the calculations made for the isothermal case we have Figure 6, the λ-plane.

Representing the dispersion relation in the σ-plane we get Figure 7. On comparison with Figure 5 we see that the primary difference is the propagating modes which occur in the hydrodynamic portions of the plane, for the isothermal case. Using existing tables[13], a numerical calculation shows that the hydrodynamical branches leave the hydrodynamical portion of the plane at

$$\left\{ \begin{array}{l} \lambda = 1.1\, i \\ k = 1.34 \\ \sigma = -1 + 1.47\, i \\ \operatorname{Im} \sigma / k = 1.1 \end{array} \right. \tag{4.42}$$

Therefore, when the absorption time is equal to the collision time the sound speed is approximately 10% over its isothermal value.

In both cases studied in this section we encountered a peculiar behavior of the dispersion relation in the region $R_x \sigma < -1$. Since the frequencies in this region exceed the collision frequency, we shall refer to this as the Knudsen spectrum. It is indeed very closely related to free-flow. Referring to equation (3.7) for instance, we see that if the dispersion law is plotted in a dimensional σ-plane and $\nu \rightarrow 0$, that the hydromagnetic region narrows to a thinner and thinner slit. As we approach the limit, the "Knudsen" spectrum becomes the left half-plane. The limit $\nu = 0$, however, cannot be assumed since the dispersion relation is wiped out identically. This state of affairs is directly due to a certain property of the model equations we are studying. It is shown in reference 4 (see also Section 6) that solutions of such equations evolve in time according to a finite number of moments. For instance, the solution to equation (3.7) is time-dependent only through its functional dependence on ρ, T, and u. For free flow there is no basis for such a functional dependence. This accounts for the singular behavior of the dispersion relation when $\nu = 0$.

On referring back to equation (2.17) defining M, we see that the Knudsen spectrum occurs in connection with the analytic continuation of the basic integral, M. This same effect, i.e., of a fountain of lines from $\sigma = -1$, occurs for the single relaxation model (3.7), for our canonical five moments case (2.1), and indeed for all generalizations of (2.1) given by Gross and Jackson. The "fountain" occurs specifically at $\sigma = -1$ only because of our normalization, and in a dimensional framework would occur at $\sigma = -\nu$. Recalling the remarks made in connection with footnote 9, we see that the location of the fountain of times for the single relaxation model occurs to the right of $\sigma = -\nu$, say at $\sigma = -\nu_s$. In the slit region of the μ-plane given by $-\nu_s \leq R\,\sigma \leq -\nu$ the five moment system gives something entirely different than the fountain predicted by the single relaxation model in this case (see next section). A better approximation than the five moments case would, in turn, move the location of the fountain further to the left in the σ-plane. It is, in fact, the case that the exact dispersion relation does not contain the fountain nor anything resembling it.[4]

The fountain is only a peculiarity of our equation and has no basis in physical reality. For this reason it will not be further considered. On the other hand, the hydrodynamic spectrum has many reassuring features.

5. Single and Triple Relaxation Equations.

Single Relaxation Dispersion Law

Continuing the program outlined in Section 3, we now look into the single relaxation model given by equation (3.7). The dispersion relation for this case is given by (3.10), which upon simple row and column operations becomes,

$$(5.1)\qquad D_3 = \begin{vmatrix} k\lambda - 1 - \frac{k}{\lambda} & -1k & 0 \\ \frac{1}{\lambda} & C_{22}-1 & C_{23} \\ 0 & C_{32} & C_{33}-1 \end{vmatrix} = 0$$

On expanding the determinant we may write

$$(5.2)\qquad k^3 - Z_1k^2 + Z_2k - Z_3 = 0$$

where

$$(5.3)\qquad Z_1 = \frac{\lambda^3(M-1)}{6} + \frac{11M}{6\lambda} - \frac{2\lambda M}{3} + \frac{5\lambda}{6}$$

$$(5.4)\qquad Z_2 = \frac{2M^2}{3\lambda^2} - \frac{2M(M-1)}{3} + \frac{\lambda^2(M-1)}{3} - \frac{2M}{3} + 1$$

$$(5.5)\qquad Z_3 = \frac{-2M(M-1)}{3\lambda} + \frac{\lambda(M-1)}{6} + \frac{M}{6\lambda}$$

We are first interested in the asymptotic roots corresponding to $k \sim 0$. This will correspond to $|\arg \lambda| < \frac{3\pi}{4}$. Rather than performing the asymptotics directly in equation (5.2) it is more revealing to perform the asymptotics directly in the determinant, (5.1). On returning to the (σ, k) notation this is,

$$(5.6) \qquad \begin{vmatrix} \sigma & -ik & 0 \\ -ik & \sigma + \frac{4}{3} k^2 & -ik \\ 0 & -\frac{2}{3} ik & \sigma + \frac{5}{3} k^2 \end{vmatrix} = 0$$

In order to obtain this we have made use of the fact that all roots are at least such that $\sigma = O(k)$. Furthermore, the entries have been calculated to $O(k^3)$. Comparison of (5.6) with eauations (3.20) and (3.24) shows that (5.6) can be identified with the Fourier-Laplace transform of that system. This, as mentioned before, is formally the linearized Navier-Stokes equations. Equations (3.24) show that the ratio of viscosity to heat conductivity is off a factor of 2/3 from the first approximation of this ratio as given by Chapman and Cowling[8]. This has already been noted in the literature.[14,15] That this should be so is evident from the fact that the single relaxation model contains only a single constant (which we used to normalize the space and time coordinates). As pointed out previously, this single constant can be adjusted to give either the viscosity or heat conductivity correctly, but not both, in general. The presence of only one constant prevents the results of the single relaxation model from being anything but qualitatively correct. For, in order to have agreement for small wave number phenomena with the Navier-Stokes theory, two constants are necessary. The three relaxation model, taken up later in this section, does fulfill this minimum requirement.

The determination of the roots of (5.1) in the asymptotic limit is gotten in a straightforward manner and one obtains,

(5.7) $$\sigma = -k^2 + O(k^4)$$

(5.8) $$\sigma = \pm \sqrt{5/3}\, ik \left[1 + \frac{7k^2}{30}\right] - k^2 + O(k^4) \quad .$$

The first root (5.7) is pure diffusion and is associated with hydrodynamic heat conduction. The two roots given by (5.8) involve propagation to the right and left as well as diffusion. One notices that to O(k) the propagation speed is given by $\sqrt{5/3}$, which is the adiabatic speed in our normalization. (For this reason we sometimes refer to this as the adiabatic model.) To $O(k^2)$ diffusion enters and to $O(k^3)$ we obtain an increase in the phase speed.

Using the tables of reference 13, a calculation of the phase speed when $R_k \sigma = -1$ was made. The results are

$$(5.9) \quad \begin{cases} \operatorname{Im} \frac{\sigma}{k} \sim 1.307 \\ k \sim 1.35 \, . \end{cases}$$

One sees that the phase speed is only slightly larger than the adiabatic speed (~ 1.307), of the order of one per cent. In view of the relation for $k \sim 0$ given by (5.8), this indicates the interesting possibility of the phase speed achieving a finite maximum for $0 < k < 1.35$. In a report to follow, a full numerical analysis of this situation will be given.

Using the above calculations we get Figure 5.1. A sketch of the "fountain" has been included since, as inspection shows, it is slightly different than the isothermal case given in Figure 4.7.

Triple Relaxation Dispersion Law.

As shown in Section 3, rather than consider equation (2.20) we can study the three relaxation time system with the conservation equations already included.

$$(5.10) \quad D_5 = \begin{vmatrix} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & -ik & -ik & 0 \\ 0 & -2/3\ ik & \sigma & 0 & -2/3\ ik \\ c_{41} & c_{42} & c_{43} & c_{44}-1 & c_{45} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55}-1 \end{vmatrix} = 0$$

As usual the last two rows are to be evaluated from Table I. If the determinant is expanded and the entries substituted from Table I, we get the following dispersion relation:

$$
\begin{aligned}
(5.11) \qquad 0 = {} & k^5 - k^4[Z_1 + \frac{\gamma\lambda A}{10} + E\mu] + k^3[Z_2 - \mu\Big\{\frac{C}{3}(1 - \frac{1}{\lambda^2}) \\
& + 2B(M + 1 + \frac{1}{\lambda^2})\Big\} + \frac{\gamma}{5}\Big\{A + D(\frac{\lambda^2}{2} - \frac{5}{6})\Big\} \\
& + \mu\gamma\Big\{\frac{C}{5}(\lambda^2 + \frac{5}{3})\Big\}] - k^2[Z_3 + \mu\Big\{\frac{C}{3\lambda} + \frac{MB}{\lambda}\Big\} \\
& + \gamma\Big\{\frac{3\lambda D}{10} + \frac{2C}{15\lambda}\Big\} + \frac{\mu\gamma C}{15}\Big\{9\lambda + \frac{5}{\lambda}\Big\}] + k[\frac{-\mu MB}{\lambda^2} + \frac{\gamma D}{10} + \frac{\mu\gamma C3}{5}] \\
& - \frac{\mu\gamma C}{5\lambda}.
\end{aligned}
$$

where

$$
(5.12) \quad \begin{cases}
A = \dfrac{10k\,C_{55}}{\gamma\lambda} \\
B = \frac{3}{2}k\lambda C_{41} \\
C = -\lambda[-2\lambda^2 + 5\lambda^2 M - M^2(5 + 3\lambda^2) \\
D = 4M^2 - (\lambda^2 + 1)M + \lambda^2 + 6 \\
E = \dfrac{C_{44}k}{\mu}
\end{cases}
$$

The calculation of (5.11) is tedious and of a type that one does not like to often repeat. It is, furthermore, virtually impossible to analyse in its entirety. However, a report to follow will contain a numerical analysis of (5.11).

The remainder of this section will be devoted to the asymptotic study of the triple relaxation time model. As one can see, this relation is very cumbersome and difficult to analyse. We will only consider the asymptotics in this section. We will also make plausible conjectures on the remainder of the spectrum based on what we have found in the previous sections. In a report to follow we will give a full discussion of D_5 from numerical data. As before, rather than considering the expansion of D_5, we will perform the asymptotics directly in (5.10). Using the forms of Table I, we get .

$$
(5.13) \quad 0 = D_5 \sim \begin{vmatrix} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & -ik & -ik & 0 \\ 0 & \frac{2}{3}\, ik & \sigma & 0 & -\frac{2}{3}\, ik \\ -\frac{4}{3k\lambda^3} & \frac{4i}{3k\lambda^2} & -\frac{8}{3k\lambda^3} & \frac{\mu 2}{k\lambda} - 1 & \frac{8i\gamma}{15k\lambda^2} \\ 0 & -\frac{3}{k\lambda^3} & \frac{5i}{2k\lambda^2} & \frac{2i\mu}{k\lambda^2} & \frac{\gamma}{k\lambda} - 1 \end{vmatrix}
$$

The only roots of (6.2) consistent with the asymptotic expansion are those for which $k \rightarrow 0$. Further, $O(1/\lambda^3)$ is only retained in the first three columns since expansion shows that those appearing in the fourth and fifth columns contribute orders which have already been neglected. Another equivalent way of seeing this is in noticing that if (5.13) is considered as the matrix of the system, then the fourth and fifth columns multiply the stress and heat conduction, respectively. In the limit of small k, both these quantities are $O(k)$ whereas the density, temperature and velocity are $O(1)$. Replacing λ in favor of σ and k, we reduce (5.13) to,

$$(5.14) \quad 0 = D_5 \sim \begin{vmatrix} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & -ik & -ik & 0 \\ 0 & -\frac{2}{3}ik & \sigma & 0 & -\frac{2}{3}ik \\ 0 & -\frac{8ik}{3(1+\sigma)}+\frac{4ik}{3(1+\sigma)^2} & 0 & \sigma+1-2\mu & -\frac{8ik}{15} \\ 0 & 0 & \frac{5ik}{2(1+\sigma)^2}-\frac{5ik}{(1+\sigma)} & -ik & \sigma+1-\gamma \end{vmatrix}$$

Reduction to (5.14) is tedious to explain. In short, it is gotten by multiplying a row by the proper coefficient and adding it to another row, always neglecting higher orders. Setting $k = 0$ in (5.14), we get the roots

$$(5.15) \qquad \sigma = 0,\ -1+2\mu,\ -1+\gamma$$

where the zero is counted thrice. The triple root corresponds to the three branches found in the single relaxation model. The other two branches implied by the non-zero roots will, as we shall see shortly, introduce some novel effects. We first deal with the triple branch emanating from the origin, and which we will call the hydrodynamic branch.

In this case as in the previous treatment of the hydrodynamic part of the plane we may write

$$(5.16) \qquad \sigma = \alpha k + \beta k^2 + \gamma k^3 + \dots ,$$

and then evaluate the constants $\alpha, \beta, \gamma, \dots$. Before doing this we observe that for these roots, since $\sigma = O(k)$, we may put (5.14) in the following form

$$(5.17) \qquad 0 = D_5 \sim \begin{vmatrix} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & -ik & -ik & 0 \\ 0 & -\frac{2}{3}ik & \sigma & 0 & -\frac{2}{3}ik \\ 0 & -\frac{4}{3}ik & 0 & \sigma+1-2\mu & -\frac{8ik}{15} \\ 0 & 0 & -\frac{5}{2}ik & -ik & \sigma+1-\gamma \end{vmatrix} ,$$

This is precisely the form of the dispersion relation which is gotten from the thirteen moment equations in our normalization, (see reference 4). However, it can only be applied to the three branches which have the form given by (5.16). As we will see shortly the remaining roots behave quite differently than what is predicted by the thirteen moments equations:

Returning to equation (5.16) we have, on substituting into (5.17), the three branches

$$(5.18) \qquad \sigma = \frac{k^2}{\gamma-1} + O(k^4)$$

$$(5.19) \qquad \sigma = \pm\, i\sqrt{5/3}\,\Big[k + k^3\Big(\frac{-2}{5(\gamma-1)(2\mu-1)} + \frac{1}{10(\gamma-1)^2} + \frac{8}{15(2\mu-1)^2}\Big)\Big] + k^2\Big(\frac{1}{3(\gamma-1)} + \frac{2}{3(2\mu-1)}\Big) + O(k^4) .$$

Comparison with the similar results found for the single relaxation model shows that the expansions disagree at the $O(k^2)$ term and onward. Of course, as was pointed out in the discussion of the single relaxation model, only one constant is given in that theory. Thus, for instance, we can only give either viscosity or heat conductivity correctly. On the other hand, since the three relaxation time theory has three constants built in, it is capable of more latitude. A study of the exact linearized Boltzmann equation made in reference 4 shows that (5.18) and (5.19) give the correct result to $O(k^3)$.

In examining the other two branches indicated by equation (5.15) we can no longer avail ourselves of (5.17) since the expansion used in deriving it is no longer valid. This then tells us that the behavior in the neighborhood of the two remaining branches will differ from that predicted by the thirteen moments dispersion relation. A straightforward calculation gives for these two branches

$$(5.20) \qquad \sigma = -1 + \gamma + k^2\left[\frac{8}{15(2\mu-\gamma)} + \frac{2}{3}\left(\frac{5/2\gamma^2 - 5/\gamma}{(\gamma-1)}\right)\right] + O(k^4)$$

$$(5.21) \qquad \sigma = -1 + 2\mu + k^2\left[\frac{(2/3\mu^2 - 4/3\mu)}{(2\mu-1)} + \frac{8}{15(\gamma-2\mu)}\right] + O(k^4)$$

If the values for a Maxwell gas for μ and γ are used, one gets the sketch given in Figure 5.2. By slightly varying the parameters μ and γ, we can get the picture sketched in Figure 5.3.

A detailed analysis of the exact Boltzmann equation, however, shows that the circled regions of both diagrams of this section are incorrect. For a detailed discussion of this point the reader is referred to [4]. The latter report also shows that the hydrodynamic branch is given exactly to $O(k^3)$ in the neighborhood of origin. We further expect it to give at least a good qualitative description in the remainder of the region.

6. General Properties of Model Equations.

The model equations which we have been considering have been systematically developed from the Boltzmann equation by Gross and Jackson [5]. Although their procedure is quite elegant, it is without rigorous foundation and from the mathematical viewpoint is strictly ad hoc. Further, the physical basis for any model equation is at best obscure. This being the case, one would like some assurance of the reasonableness of problems formulated with the model equations. This section, which stands apart from the rest of the report in spirit (although it depends upon the previous results), will demonstrate some essential features of the initial value problem for the model equations, namely that solutions exist and are unique and that they asymptotically approach hydrodynamics.

Existence and Uniqueness of Solutions

We investigate the following equation

$$\frac{\partial g}{\partial t} + \xi \cdot \frac{\partial g}{\partial x} + g = \omega \sum_{n=0}^{N} a_n \lambda_n \phi_n \tag{6.1}$$

where

$$\left\{ \begin{array}{l} \delta mn = \int_{-\infty}^{\infty} \omega \phi_n \phi_m d\xi \\ an = \int_{-\infty}^{\infty} g \phi_n d\psi \; : \end{array} \right. \tag{6.2}$$

The λ_n are constants and the $\phi_n = \phi_n(\xi)$ polynomials in ξ. For the following considerations the sign of λ_n is immaterial. As before we denote the Gaussian by

$$\omega = \frac{e^{-\xi^2/2}}{(2\pi)^{3/2}} \quad . \tag{6.3}$$

One should further note that we are no longer restricting attention to one dimensional phenomena. It is clear that all models mentioned earlier are of the form of equation (6.1). We denote the initial conditions of (6.1) by

$$g(x,\xi,0) = g_o \quad . \tag{6.4}$$

Rather than consider the quality g, we equivalently consider

$$G = \frac{e^t g}{\omega} \, . \tag{6.5}$$

Using an obvious notation we have

$$\frac{\partial G}{\partial t} + \xi \cdot \frac{\partial G}{\partial x} = \sum_{n=1}^{N} A_n \lambda_n \phi_n \tag{6.6}$$

$$G(t=0) = G_o \tag{6.7}$$

$$A_n = \int \omega G \phi_n d\xi \; . \tag{6.8}$$

Regarding the $A_n (= A_n(x,t))$ as known functions, we may integrate (6.6) to obtain,

$$G = G_o(x-\xi t,\ \xi) + \int_0^t \sum_{n=1}^{N} A_n(x^*,\xi,s)\lambda_n \phi_n ds \tag{6.9}$$

with

$$x^* = x-\xi[t-s] \ . \tag{6.10}$$

It is easily seen that solving (6.9) is equivalent to solving (6.6). We shall show existence and uniqueness of solutions to (6.9) in the usual way, i.e., by showing that the following iteration

$$G^{(n+1)} = G_o(x-\xi t,\xi) + \int_0^t \sum_{k=1}^{N} A_k^{(n)}(x^*)\phi_k \lambda_k ds \tag{6.11}$$

defines a contraction mapping. To show this we will assume certain conditions on a finite number of moments of the initial data. We assume the existence of

$$A_{no} = \int \omega\phi_n \max_x G_o(x,\xi)d\xi \ . \tag{6.12}$$

Actually instead of (6.11) we consider equivalently,

---

* It is, in fact, shown in the cited reference that any finer model will not alter the location of the roots to $O(k^4)$.

$$(6.13) \qquad A_\mu^{(n+1)} = \int_\xi G_o(x-\xi t,\xi)\phi_\mu\omega \, d\xi + \int_o^t\int \sum_{k=1}^{N} A_k^{(n)}(x^*)\lambda_k\phi_\mu\phi_k\omega \, d\xi \, ds$$

This should be regarded as a matrix equation

$$(6.14) \qquad \bar{A}_\mu^{(n+1)} = \int_o^t \bar{\bar{M}}_{\mu k} A_k^{(n)} ds + \bar{A}_{o\mu}$$

where

$$(6.15) \qquad \bar{\bar{M}}_{\mu k} = [\int d\xi \phi_\mu \phi_k \omega \, \lambda_k]$$

and is an operator, and

$$(6.16) \qquad A_k = [\int \omega G \phi_k d\xi] \, , \qquad k = 1, \ldots, N$$

and similarly for A_{ok}. As a norm we take

$$(6.17) \qquad \| A(x,t) \| = \max_{x,t}[A_1(x,t), A_2(x,t), \ldots, A_N(x,t)],$$

i.e., we take the maximal element. For the norm of the matrix we write

$$(6.18) \qquad \| M \| = \| \sum_{k=1}^{N} \int_\xi \phi_\mu \phi_k \omega \, d\xi \| = M, \text{ say.}$$

With this notation out of the way we may proceed to show the contraction mapping. We only note that the operator of (6.14) maps uniformly continuous and bounded functions into the same space. Further, the choice of an initial iterate is relatively unimportant and for definiteness we take it to be identically zero. Without further ado we have,

$$\|A^{n+1} - A^{n}\| < tM \|A^{n} - A^{n-1}\| \tag{6.19}$$

which, for $t < T < \frac{1}{M}$ establishes the existence and uniqueness to solutions of our equation. Further, because of the norm which we have chosen, the iteration converges itself to a function which is uniformly continuous. Furthermore, since M is independent of t we have existence and uniqueness for all time. For, by taking the solution obtained at $t = T$ as the initial data we can go through the same argument to extend the solution to 2T, and so on. We can use this agrument to estimate the behavior of the solution in time. We choose M so that it is also a bound for the initial data given by (6.12), then from (6.19)

$$\|A^{(n+1)} - A_{o}\| < MR , \quad \text{for } 0 \leq t \leq T \tag{6.20}$$

where

$$R = \frac{MT}{1-MT} . \tag{6.21}$$

On passing to the limit in (6.20)

$$(6.21) \qquad ||A-A_o|| < MR \ , \qquad 0 \leq t \leq T \ .$$

Reapplying this argument we have, in general,

$$(6.22) \qquad ||A-A_o|| < MR^n \qquad (n-1)T \leq t \leq nT$$

or

$$(6.23) \qquad ||A-A_o|| < (M/R)R^{t/T} \quad ,$$

so that the A_k are exponentially bounded.

So far our remarks have been restricted to the moments A_k. To summarize, we have shown that they exist for all time and are unique in the class of uniformly continuous and bounded functions, and that they are exponentially bounded in time. If these solutions are now substituted into (6.9) it furnishes us with a unique solution for the quantity G. Further, because the initial conditions in (6.9) have the argument $(x-\xi t)$ and because of the integral, G has the continuous derivative $\frac{\partial}{\partial t} + \xi \cdot \frac{\partial}{\partial x}$. In order to obtain any further information on the differentiability of G similar conditions must be demanded of the initial conditions. We have been able to obtain all of these results under the mild boundedness restrictions imposed on the initial conditions by the remarks in connection with (6.12). In essence only the boundedness of a finite number of velocity space moments were required. However, as N increases (and presumably equation (6.1) more accurately describes the Boltzmann equation), more boundedness conditions are required of G_o.

To obtain any further information on solutions of model equations, we will need a finer analysis. The basis for this is furnished by the dispersion relation analysis which was carried out earlier. We, therefore, now restrict our attention to one-dimensional unsteady equations. Using the results of the previous sections we can show the uniform boundedness of solutions and, moreover, display the asymptotic character of the solution.

In order to accomplish the above plan we must further specify the model equation (6.1). It is first clear from the conservation of mass, momentum, and energy in a two particle collision that

$$\lambda_o = \lambda_1 = \lambda_2 = 1 \tag{6.25}$$

(when $\phi \sim 1$, $\phi_1 \sim \xi_1$, $\phi_2 \sim \xi^2$). Further, from the construction of kinetic models given in reference 8 and from our normalization the remainder of the λ's form a sequence of non-negative numbers which approach zero.

We propose to solve (6.1) (actually the one-dimensional form of (6.1)) by means of transforms. Denoting the initial data of g by g_o, the transformed system can be written as

$$g = \frac{g_o}{(1+\sigma-ik\xi_1)} + \frac{\omega \sum_{n=0}^{N} a_n \lambda_n \phi_n}{1+\sigma-ik\xi_1} \tag{6.26}$$

where, as usual, we use the same letter to denote a transformed quantity. As earlier we "reduce" the problem by taking moments of (6.26) to obtain

$$a_n = A_n + \sum_{m=0}^{N} a_m \lambda_m C_{mn} , \qquad n = 0, \ldots, N \tag{6.27}$$

where

$$A_n = \int_{-\infty}^{\infty} \frac{\omega \phi_m \phi_n d\xi}{1+\sigma-ik\xi_1} \qquad 0 \leq n,m \leq N \tag{6.28}$$

and

$$C_{mn} = \int_{-\infty}^{\infty} \frac{\phi_n \phi_m d\xi}{1+\sigma-ik\xi_1} \tag{6.29}$$

C_{mn} is the type of expression found in Table I. An immediate property of C_{mn} which will prove important later is

$$C_{mn} = O(\tfrac{1}{\sigma}) , \qquad \text{for } \operatorname{Re} \sigma > -1 \text{ and } |\sigma| \to \infty . \tag{6.30}$$

This is true for any bounded k. As was discussed at the end of Section 2, integrals of the type (6.28, 29) define two different functions depending on whether k is greater or less than zero. We shall not indicate the change of functional form and make the agreement that only the correct form is under discussion. The change of functional form plays no role in the sequel and the only pertinent remark in this vein is that both functional forms have the same asymptotic expansion (see Appendix I).

The linear equation given by equation (6.27) may be inverted to give a_n,

$$a_n = \frac{\sum_{m=0}^{N} F_{nm} A_m}{D} \tag{6.31}$$

where

$$D = \det(\delta_{mn} - \lambda_m C_{mn}), \qquad 0 \leq m,n \leq N \tag{6.32}$$

and F_{nm} denotes an element of the transposed cofactor matrix of $(\delta_{mn} - \lambda_m C_{mn})$. We shall shortly show that $D \neq 0$ for $\Re\, \sigma > 0$ (which at this stage is a requirement on σ), permitting us to perform the operation of (6.31). The vanishing of (6.32) defines the dispersion relation. We shall now give a different but equivalent representation of the dispersion relation.

Consider the following relations:

$$q = \frac{\omega \sum_{n=0}^{N} b_n \lambda_n \phi_n}{1+\sigma-ik\xi} \tag{6.33}$$

$$b_n = \int \phi_n q \, d\xi \qquad n = 0, \ldots, N. \tag{6.34}$$

To avoid confusion with equation (6.26) we have replaced g and a_n by q and b_n, respectively. Upon taking the moments (6.34) of (6.33) and taking $\sum_{n=0}^{N} |b_n|^2 \neq 0$ we obtain an eigenvector b_n, $n = 0, 1, \ldots, N$. Equation (6.32) set to zero defines σ as a function of k and so the b_n's can be considered as functions of k alone. Next substituting these expressions for b_n into (6.33) we obtain q as a function of k and ξ. Alternately using this definition of q we automatically satisfy (6.34). Thus (6.33,34) are equivalent to considering σ as a function of k as given by the dispersion relation of (6.32).

We now take q as having been obtained by the above process, and further we take

$$(6.35) \qquad q = \omega \sum_{n=0}^{\infty} b_n \phi_n \quad .$$

Further, letting

$$(6.36) \qquad Q = q/\omega$$

we have from (6.33)

$$(6.37) \qquad \sigma \int \omega Q \overline{Q} \, d\xi - ik \int \omega \xi_1 Q \overline{Q} \, d\xi = - \int \omega (Q - \sum_{n=0}^{N} b_n \lambda_n \phi_n) \overline{Q} d\xi$$

where the bar denotes the complex conjugate. Taking the real part of (6.37)

$$(6.38) \qquad R\, \sigma = \frac{- \int \omega (Q - \sum_{n=0}^{N} b_n \lambda_n \phi_n) \overline{Q} \, d\xi}{\int \omega Q \overline{Q} \, d\xi}$$

From the restrictions placed on the λ_n at the outset, we have (see also reference 4 where this technique was applied to get similar results)

$$(6.39) \qquad -1 \leq R_\ell\,\sigma \leq 0 \,.$$

From the orthonormality of the ϕ_n we have the condition

$$(6.40) \qquad \begin{cases} Q = \sum_{n=0}^{2} b_n \phi_n & \text{for } R_\ell\,\sigma = 0 \\ \text{or} & \\ Q \neq \sum_{n=0}^{2} b_n \phi_n & \text{for } R_\ell\,\sigma \neq 0 \,. \end{cases}$$

It therefore follows from (6.33) that if $\sigma = 0$ then $k = 0$.

We now wish to show that for $|k| \geq k_o > 0$, that $R_\ell\,\sigma \leq \sigma_o(k_o) < 0$. Otherwise we would need a sequence k_i such that, as $k_i \rightarrow k^*$, $R_\ell\,\sigma(k_i) \rightarrow 0$, where k^* is finite or infinite, which is a contradiction to the above results.

We now conclude our preparatory work by some crude estimates on D and F_{mn}. From (6.30) and under the same restrictions we have immediately

$$(6.41) \qquad D \sim 1 + O(1/\sigma)$$

and

$$(6.42) \qquad F_{mn} \sim 1 + O(1/\sigma) \,.$$

In the latter, unity may be absent.

Returning to our problem as given by (6.27-29), we rewrite the initial conditions as

$$g_o = e^{-\xi^2}[\mathring{G}_\varepsilon(k,\xi) + G_\varepsilon(k,\xi)] \tag{6.43}$$

such that

$$e^{-\xi^2}\,\mathring{G}_\varepsilon = 0 \quad , \quad |k| \geq \varepsilon$$
$$= g_o \quad , \quad |k| < \varepsilon \tag{6.44}$$

To avoid tedious arguments we assume that $\mathring{G}_\varepsilon$, G_ε are uniformly bounded in ξ and k, and further that G_ε vanishes for $|k|$ sufficiently large. These assumptions, especially the latter, are far too strong; however, their assumption eliminates messy estimates. The quantity in (6.44) is still left open.

We first show that for

$$g_o = e^{-\xi^2}G_\varepsilon(\xi,k) \tag{6.45}$$

the solution to our problem is exponentially damped in time. The solution for a_n is gotten by inverting a_n. Rather than considering the entire series (6.31), we can consider a typical term, which we denote by a tilda,

$$(6.46) \qquad \tilde{a}_n = \frac{1}{(2\pi)^2 i}\int_{-\infty}^{\infty} dk \int_B \frac{e^{-\sigma t}e^{-ik}\, F_{mn}(\sigma,k)}{D(\sigma,k)} \int_{-\infty}^{\infty} \frac{e^{-\xi^2/2}G_\varepsilon(k,\xi)d\xi}{1+\sigma-ik\xi}$$

The B denotes a Bromwich path in the right half of the complex σ-plane. For $\varepsilon > 0$ sufficiently small, we can by an earlier argument, choose a $\delta > 0$ such that D will not vanish on $R\cdot\sigma = -\delta$. We distort our original path B into this path, $\mathcal{L}$ of Figure 6.1. Further, from the asymptotic cases studied in Sections 4 and 5, we know that we can take

$$(6.47) \qquad \delta = \varepsilon^2/2 \; .$$

Moreover, it can be shown (see reference 4) that (6.47) is sufficient for any model of the type (6.1). From (6.41,42) we may choose a σ_m on $\mathcal{L}_1$ such that

$$(6.48) \qquad |\tilde{a}_n| < e^{-\delta t}\int dk \int_{\mathcal{L}_1,|\sigma_i|<\sigma_m} \frac{|F_{mn}|}{|D|\cdot|\sigma_r|}\int e^{-\xi^2/2}|G_\varepsilon(k,\xi)|d\xi$$

$$+ \left| e^{-\delta t}\int dk \int_{\mathcal{L}_1,|\sigma_i|>\sigma_m} \frac{e^{-i\sigma_i t}[1+O(\frac{1}{\sigma})]}{(1+\sigma)[1+O(\frac{1}{\sigma})]}\int \frac{e^{-\xi^2/2}G_\varepsilon(k,\xi)d\xi}{1-\frac{ik}{1+\sigma}\xi}\right|$$

where we have written

$$(6.49) \qquad \sigma = -\delta + i\sigma_i \; .$$

The integration of the first term, being finite, can be majorized by a constant M_1 say. In the second term the ξ and k integrations when carried out are seen to lead to a bounded function of σ. Hence, by Dirichlet's test the integration converges and is majorized by M_2, say. We have then

$$(6.50) \qquad |\tilde{a}_n| < e^{-\delta t}(M_1 + M_2) = e^{-\delta t}M \quad .$$

It is important to note that M is at worst an inverse power of ε. The value of δ is still open and we shall fix it in the next phase of the argument.

We now go through the same type of analysis for initial conditions

$$(6.51) \qquad g_o = e^{-\xi^2}\hat{G}_\varepsilon(k,\xi)$$

as given by (6.44). Again it is only necessary to consider a typical term. Under the present initial conditions

$$(6.52) \qquad \tilde{a}_n = \frac{1}{(2\pi)^2 i}\int_{-\varepsilon}^{\varepsilon} dk \int_B \frac{e^{-\sigma t - ikx}P_{mn}(\sigma,k)}{D(\sigma,k)} \int_{-\infty}^{\infty} \frac{e^{-\xi^2/2}G_\varepsilon(\xi,k)d\xi}{1+\sigma-ik\xi}$$

As before, B denotes a path in the right half of the complex σ-plane. Since ε, which is still to be chosen, is arbitrarily small we can avail ourselves of the asymptotics developed in the earlier sections. We first exhibit the asymptotic expansion of the last integral of (6.52). Using the identity

(6.53) $$\frac{1}{1-x} = 1 + x + x^2 + \dots + n^{n-1} + \frac{x^n}{1-x}$$

we can write

(6.54) $$\int_{-\infty}^{\infty} \frac{e^{-\xi^2/2}\mathring{G}_\varepsilon}{1-\frac{ik}{1+\sigma}\xi}\,d\xi = \int_{-\infty}^{\infty} e^{-\xi^2}\mathring{G}_\varepsilon d\xi + \frac{ik}{1+\sigma}\int_{-\infty}^{\infty} e^{-\xi^2/2}\mathring{G}_\varepsilon \xi\, d\xi +$$

$$+ \dots + \left(\frac{ik}{1+\sigma}\right)^n \int_{-\infty}^{\infty} \frac{e^{-\xi^2/2}\xi^n\mathring{G}_\varepsilon d\xi}{1-\frac{ik}{1+\sigma}\xi}$$

$$= \mathring{a}_o + \frac{\mathring{a}_1 k}{1+\sigma} + \dots + O\left[\left(\frac{k}{1+\sigma}\right)^n\right]$$

which by the usual arguments shows that (6.54) is an asymptotic development.

In considering the three relaxation model we showed from the asymptotic analysis that for $k \to 0$, there are three roots to D in the neighborhood of the origin in the complex σ-plane. Each of these roots is of the form (see (5.18, 19)),

(6.55) $$\sigma = i\alpha k - \beta k^2 + O(k^2) \quad , \qquad \beta > 0$$

where α and β are real. It is furthermore shown in reference 4, that any more detailed model (given in general by (6.1)) has three roots in the neighborhood of the origin.* Since the values of k are restricted by the integration in (6.52) we may enclose the three roots in a small circle about the origin. Further, we distort the path B so that it is now composed of the circle $\mathcal{L}_2$ and the imaginary path $\mathcal{L}_1$ in the left half plane, as is indicated in Figure 6.2. Using the estimates used previously we can easily show

$$(6.56) \quad a_n = \frac{1}{2\pi i}\int_{-\varepsilon}^{\varepsilon} dk \int_{\mathcal{L}_2} e^{-\sigma t - ikx} d\sigma \, \frac{F_m(\sigma,k)}{D(\sigma,k)(1+\sigma)} \left[a_o + \frac{a_1 k}{1+\sigma} + \dots + O\left(\frac{k^n}{(1+\sigma)^n}\right)\right] + e^{-\delta t} M$$

where the latter term represents the contribution of the path $\mathcal{L}_1$. The contribution from the path $\mathcal{L}_2$ can be evaluated by means of residues to given a contribution from each of the roots of D. Using the representation of a typical root (6.55), we get a sum of three terms each of which may be written as

$$(6.57) \quad I = \int_{-\varepsilon}^{\varepsilon} e^{-ikx + i\alpha kt - \beta k^2 t + O(k^3)t} [A_o + A_1 k + A_2 k^2 + \dots + O(k^n)] dk$$

---

* It is, in fact, shown in the cited reference that any finer model will not alter the location of the roots to $O(k^4)$.

where A_n's are constants which can be found.* Further, by expanding the $O(k^3)$ term of the exponential we have

$$(6.58) \qquad I = \int_{-\varepsilon}^{\varepsilon} e^{-ik(x-\alpha t)-\beta k^2 t}[(1+O(k^3)t+\dots)(A_o+\dots O(k^n)]dk$$

We now choose

$$(6.59) \qquad \varepsilon = O(\frac{1}{t^\alpha}) \, , \qquad \frac{1}{3} < \alpha < \frac{1}{2} \, .$$

With this choice of ε we have from (6.47) that the term $e^{-\delta t}M$ which we obtained in connection with (6.46) and (6.52) is exponentially decaying in time. Estimating the remainder terms we get

$$(6.60) \qquad \int_{-\varepsilon}^{\varepsilon} e^{-\beta k^2 t}|O(k^3)|t \; dk < \frac{c}{t^{3\alpha-1}} \int_{-\varepsilon}^{\varepsilon} e^{-\beta k^2 t} \; dk$$

and

$$(6.61) \qquad \int_{-\varepsilon}^{\varepsilon} e^{-\beta k^2 t}|O(k^n)|dk < \frac{c}{t^{n\alpha}} \int_{-\varepsilon}^{\varepsilon} e^{-\beta k^2 t} \; dk$$

where the c is a suitable constant. An expansion of the $O(k^3)$ term shows that the coefficient of the integral in (6.60)

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* It is possible for a k to occur in the denominator of (6.57); this is easily taken care of by a principal parts integration.

can be taken as $t^{-1/2}$. To complete the argument which shows that I of (6.57) is an asymptotic expansion, we observe,

$$\frac{\int_{-\varepsilon}^{\varepsilon} e^{-\beta k^2 t} k^n dk}{\int_{-\varepsilon}^{\varepsilon} e^{-\beta k^2 t} k^{n+1} dk} \sim O\left(\frac{1}{t^{1/2}}\right) . \tag{6.62}$$

Finally we see that extanding the integration to infinite limits in (6.58) only adds an asymptotically negligible quantity ($\sim$ exponentially decaying).

For the purposes of this study we have, in essence, proved our assertion that the solutions to model equations are asymptotically hydrodynamical. In Section 5 we showed that the dispersion relation of the three relaxation time model asymptotically became the hydrodynamical dispersion relation. More generally it can be shown (see reference 4) that for $k \sim 0$ and σ in the neighborhood of the origin, (6.33) can be reduced to the transform of the Navier-Stokes equations or, more generally, to some higher approximation of the Chapman-Enskog procedure. This alone is not enough to give us the assertion stated at the beginning of the paragraph. This is so on two counts. First (6.33) involves no initial data and secondly we have no license to invert this system. The above analysis removes these difficulties. First we have shown that the asymptotic solution only involves $k \sim 0$, $\sigma \sim 0$ which by the previous discussion gives the Chapman-Enskog terms. Secondly, the initial data is taken care of by

(6.54) which uses only moments of the initial distribution function.* There is one last point. Hydrodynamics, although it is given by an expansion in $k \sim 0$ in kinetic theory, regards itself as exact. It then requires an integration over all k in the inversion. However, by the last paragraph this offers no difficulty.

Integrals of the type (6.58) lead to diffusing waves in general. We do not consider them further in this study.**

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* Any hydrodynamical theory only uses the initial data of density, temperature, and velocity. It is interesting to note from (6.54) that, depending on the asymptotic order of a solution, any number of moments of the initial data will enter. This will be taken up at another time.

** One can find this in reference 10 where the fundamental solutions of the linearized Navier-Stokes equations are considered.

Appendix I: Evaluation of the Fundamental Integral.

Each element of the matrix C of Section 2 may be expressed in terms of the following integral (see Lemma 1 of Appendix II):

$$(I.1) \qquad M = (1+\sigma)\int_{-\infty}^{\infty} \frac{\omega d\xi_1}{1+\sigma-ik\xi_1} = \lambda\int_{-\infty}^{\infty} \frac{\omega d\xi_1}{\lambda-i\xi_1} = M(\lambda)$$

where

$$(I.2) \qquad \lambda = \frac{1+\sigma}{k} \quad .$$

As mentioned in Section 2, the functional form of (I.1) depends on the sign of the real part of λ, in any case for $\mathrm{Re}\,\lambda \neq 0$ we have

$$(I.3) \qquad M' = \frac{dM}{d\lambda} = \int_{-\infty}^{\infty} \frac{\omega d\xi_1}{\lambda-i\xi_1} - \lambda\int_{-\infty}^{\infty} \frac{\omega d\xi}{(\lambda-i\xi_1)^2}$$

From this it is easy to show that

$$(I.4) \qquad \frac{d}{d\lambda}\left(\frac{e^{-\lambda^2/2}M}{\lambda}\right) = -e^{-\lambda^2/2} \quad .$$

Denoting by $0^{\pm}$ the limit as we approach the origin with $\mathrm{Re}\,\lambda > 0$ or $\mathrm{Re}\,\lambda < 0$, we may integrate (I.4) to get,

$$(I.5) \qquad \frac{e^{-\lambda^2/2}\, M^{\pm}}{\lambda} = -\int_{0^{\pm}}^{\lambda} e^{-\mu^2/2}\, d\mu + \text{const.}$$

the path of integral lying in the appropriate half-plane. In order to evaluate the constant of equation (I.5) we return to (I.1) from which we get

$$\text{(I.6)} \qquad \left.\frac{M^{\pm}}{\lambda}\right|_{\lambda=0^{\pm}} = \pm\sqrt{\pi/2}$$

since the principal part of

$$\int_{-\infty}^{\infty} \frac{\omega d\xi_1}{\xi_1}$$

vanishes. Making use of the error function,

$$\text{(I.7)} \qquad \Phi(x) = \sqrt{2/\pi} \int_0^{\sqrt{2}\,x} e^{-t^2/2}\, dt \quad \text{(error function)}$$

we may write

$$\text{(I.8)} \qquad M = \begin{cases} M^+(\lambda) = \sqrt{\pi/2}\, \lambda\, e^{\lambda^2/2}\, (1 - \Phi(\lambda/\sqrt{2})) & \text{Re } \lambda > 0 \\ M^-(\lambda) = -\sqrt{\pi/2}\, \lambda\, e^{\lambda^2/2}\, (1 + \Phi(\lambda/\sqrt{2})) & \text{Re } \lambda < 0 \end{cases}$$

Using the asymptotic properties of the error function we now list the asymptotic properties of $M^{\pm}$ that will be of interest to us.

(I.9) $$M^{\pm}(\lambda) = 1 - \frac{1}{\lambda^2} + \frac{1\cdot 3}{\lambda^4} - \frac{1\cdot 3\cdot 5}{\lambda^6} + \ldots + (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-3)}{\lambda^{2n-2}} + O(\frac{1}{\lambda^{2n}})$$

for $|\arg \lambda| \leq \frac{3\pi}{4} - \varepsilon$, where $\varepsilon > 0$ is small.

(I.10) $$M^{\pm}(\lambda) = (-1)^{\frac{1}{2} \pm \frac{1}{2}} \sqrt{2\pi}\, \lambda\, e^{\lambda^2/2} + 1 - \frac{1}{\lambda^2} + \frac{1\cdot 3}{\lambda^4} - \frac{1\cdot 3\cdot 5}{\lambda^6} + \ldots + (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdots(2n-3)}{\lambda^{2n-2}} + O(\frac{1}{\lambda^{2n}})$$

for $\frac{\pi}{4} + \varepsilon < |\arg \lambda| < \frac{5\pi}{4} - \varepsilon$. The expansion for small values of λ is immediately found using the error function expansion[16]

(I.11) $$\Phi(\lambda/\sqrt{2}) = \sqrt{2/\pi}\, (\lambda - \frac{\lambda^3}{2\cdot 3} + \frac{\lambda^5}{2!\cdot 2^2\cdot 5} - \frac{\lambda^7}{3!\cdot 2^3\cdot 7} \pm \ldots) .$$

Appendix II: Properties of the Dispersion Relation.

We will now prove several general properties of the dispersion relation of Section 2. These will be proven in a series of lemmas given below. Actually, the properties exhibited below are quite general and apply to the generalizations of equation (2.1) given in reference 5. Several of the arguments are for convenience repeated from reference 4.

Lemma 1. Each element C_{ij} of the matrix $\bar{\bar{C}}$ of Section 2 can be written as

$$\text{(II.1)} \qquad \frac{i^n F(\lambda)}{k}$$

where F is a real function of λ which is either odd or even, in which case n is either zero or one.

We see from Section 2 that any matrix element may be written as

$$\text{(II.2)} \qquad \int \frac{\omega g(\xi_1) d\xi_1}{1+\sigma-ik\xi_1}$$

where $g(\xi_1)$ is either an odd or even polynomial. It suffices, therefore, to consider

$$\text{(II.3)} \qquad \int \frac{\omega \xi_1^n \, d\xi_1}{1+\sigma-ik\xi_1} = \frac{i}{k} \int \frac{\omega \xi_1^n \, d\xi_1}{i\lambda + \xi_1} \quad .$$

Using the binomial expansion we may write

(II.4) $$\xi_1^n = (\xi_1 + i\lambda)^n - g\xi_1^{n-1}\, i\lambda \ \ldots \ -(i\lambda)^n \quad .$$

Now consider the first term of (3) when (4) is substituted; this gives

(II.5) $$\int_{-\infty}^{\infty} \omega(\xi_1 + i\lambda)^{n-1}\, d\xi$$

which, since $\omega \propto e^{-\xi_1^2/2}$, is an even polynomial in λ if n is odd, and i times an odd polynomial in λ if n is even. By induction each term of (5) contributes the same type of term. And hence, since $g(\xi_1)$ is either an odd or even polynomial in ξ_1, we have proven the lemma.

In addition we know that if g is an even polynomial, F is odd and vice versa. An immediate consequence of this is that the trace elements of $\bar{\bar{C}}$ are real functions.

The above lemma states that each element of the determinant (2.20) has either real or imaginary parity.

Lemma 2. A determinant for which the parities of any row may be gotten by multiplying the preceding row by i (or any determinant which may be put into this form, say, on extracting i from a row) has itself a single parity. That is, the determinant expanded is a real function times i or 1.

The lemma is clearly true for a two by two determinant. Expanding an $n \times n$ determinant (which satisfies the hypothesis) along the first column we get $n(n-1)$ $(n-1)$ determinants. The first determinant of the expansion satisfies the hypothesis, the second determinant can be given the same form by extracting an i from the first row. The third determinant can be given the same form by interchanging the first and second rows. In fact, each determinant of the expansion can be put into the form of the first determinant by interchanging rows and extracting an i when appropriate. Imposing the inductive hypothesis we prove the lemma.

From the form of equation I2.16) and Lemma 1, we see that $D(k,\lambda)$ satisfies the hypothesis of the last lemma. Therefore, we have

Theorem 1: If $D(k,\lambda) = 0$, then

$$D(k,\lambda^*) = 0. \tag{II.6}$$

This, since $\lambda = \frac{1+\sigma}{k}$, states that roots of the dispersion relation occur in conjugate pairs. With the use of Lemma 2, still more properties of the determinant may be shown.

<u>Lemma 3</u>. D is of even degree in k and λ.

The only elements of $\bar{\bar{C}}$ odd in k and λ are those of imaginary parity. But D is a real function, hence the terms of imaginary parity must occur to an even power, and the lemma is proven.

Finally we prove

Theorem 2: If $D(-k,-\lambda) = 0$, then $D(k,\lambda) = 0$.

It is to be remembered that positive and negative k lead to different functional forms for M, denoted by M^+ and M^-. From Appendix I we have

$$\text{(II.7)} \qquad M^-(-k,-\lambda) = M^+(k,\lambda).$$

Therefore, denoting the corresponding dispersion relation by the same subscripts, if $k > 0$

$$\text{(II.8)} \qquad D_+(k,\lambda) = D(k,\lambda;\, M^+(k,\lambda)) = 0$$

$$= D(-k,-\lambda;\, M^+(-k,-\lambda))$$

which by (7) proves the theorem. For the purpose of this paper this says that we can restrict attention to $k > 0$ or $k < 0$.

Again, since $\lambda = \frac{1+\sigma}{k}$, the latter theorem states that $\pm k$ lead to the same value of σ.

TABLE I

$C_{11} = \frac{M}{k\lambda} \qquad \sim \; 1/\lambda k - 1/k\lambda^3$

$C_{12} = \frac{i}{k}[1-M] \qquad \sim \; i/k\lambda^2$

$C_{13} = \frac{1}{2k}[\lambda - \lambda M - \frac{M}{\lambda}] \qquad \sim \; -1/k\lambda^3$

$C_{14} = \frac{\mu}{k}[\lambda - \lambda M - \frac{M}{\lambda}] \qquad \sim \; -\mu/k\lambda^3$

$C_{15} = \frac{i\gamma}{5k}[-\lambda^2 + \lambda^2 M - 2 + 3M] \qquad \sim \; 0$

$C_{21} = \frac{i}{k}[1-M] \qquad \sim \; i/k\lambda^2$

$C_{22} = \frac{1}{k}(\lambda - \lambda M) \qquad \sim \; 1/k\lambda - 3/k\lambda^3$

$C_{23} = \frac{i}{2k}[-\lambda^2 + \lambda^2 M + M] \qquad \sim \; i/k\lambda^2$

$C_{24} = \frac{i\mu}{k}[-\lambda^2 + \lambda^2 M + M] \qquad \sim \; 2\mu i/k\lambda^2$

$C_{25} = \frac{\gamma}{5k}[-2\lambda - \lambda^3 + \lambda^3 M + 3\lambda M] \qquad \sim \; -6\gamma/5k\lambda^3$

$C_{31} = \frac{1}{3k}[\lambda - \lambda M - \frac{M}{\lambda}] \qquad \sim \; -2/3k\lambda 3$

$C_{32} = \frac{i}{3k}[-\lambda^2 + \lambda^2 M + M] \qquad \sim \; 2i/3k\lambda^2$

$C_{33} = \frac{1}{6k}[-\lambda - \lambda^3 + \frac{5M}{\lambda} + 2\lambda M + \lambda^3 M] \qquad \sim \; 1/k\lambda - 7/3k\lambda^3$

$C_{34} = \frac{\mu}{3k}[-\lambda - \lambda^3 + \lambda^3 M + 2\lambda M - \frac{M}{\lambda}] \qquad \sim \; -8\mu/3k\lambda^3$

$C_{35} = \frac{i\gamma}{15k}[6 + 3\lambda^2 + \lambda^4 - 7M - 4\lambda^2 M - \lambda^4 M] \qquad \sim \; 2i\gamma/3k\lambda^2$

$$c_{41} = \frac{2}{3k}\,[\lambda - \frac{M}{\lambda} - \lambda M] \qquad \sim -4/3k\lambda^3$$

$$c_{42} = \frac{2i}{3k}\,[-\lambda^2 + \lambda^2 M + M] \qquad \sim 4i/3k\lambda^2$$

$$c_{43} = \frac{1}{3k}\,[-\lambda - \lambda^3 + 2\lambda M - \frac{M}{\lambda} + \lambda^3 M] \qquad \sim -8/3k\lambda^3$$

$$c_{44} = \frac{2\mu}{3k}\,[-\lambda - \lambda^3 + \frac{2M}{\lambda} + 2\lambda M + \lambda^3 M] \qquad \sim 2\mu/k\lambda - 22\mu/3\lambda^3 k$$

$$c_{45} = \frac{2\gamma i}{15k}\,[3\lambda^2 + \lambda^4 - M - 4\lambda^2 M - \lambda^4 M] \qquad \sim 8i\gamma/15k\lambda^2$$

$$c_{51} = \frac{i}{2k}\,[-2 - \lambda^2 + 3M + \lambda^2 M] \qquad \sim 0$$

$$c_{52} = \frac{1}{2k}\,[-2\lambda - \lambda^3 + 3\lambda M + \lambda^3 M] \qquad \sim -3/k\lambda^3$$

$$c_{53} = \frac{i}{4k}\,[6 + 3\lambda^2 + \lambda^4 - 7M - 4\lambda^2 M - \lambda^4 M] \qquad \sim 5i/2k\lambda^2$$

$$c_{54} = \frac{i\mu}{2k}\,[3\lambda^2 + \lambda^4 - M - 4\lambda^2 M - \lambda^4 M] \qquad \sim 2i\mu/k\lambda^2$$

$$c_{55} = \frac{\gamma}{10k}\,[10\lambda + 5\lambda^3 + \lambda^5 - 13\lambda M - 6\lambda^3 M - \lambda^5 M] \qquad \sim \gamma/k\lambda - 27\gamma/5k\lambda^3$$

Footnotes

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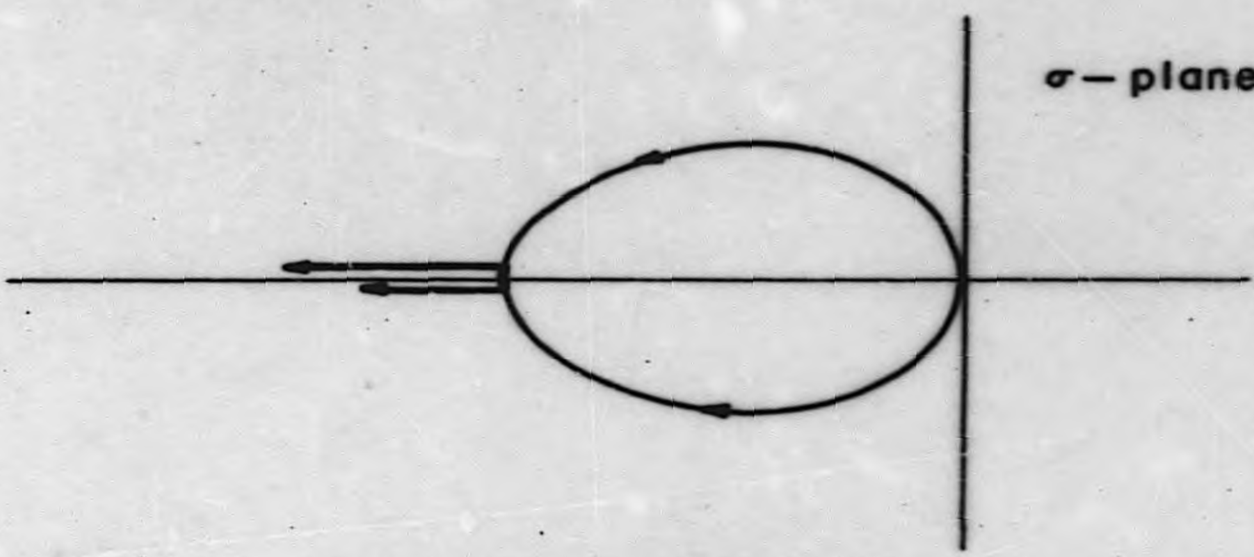

Figure I.1 ISOTHERMAL PROPAGATION

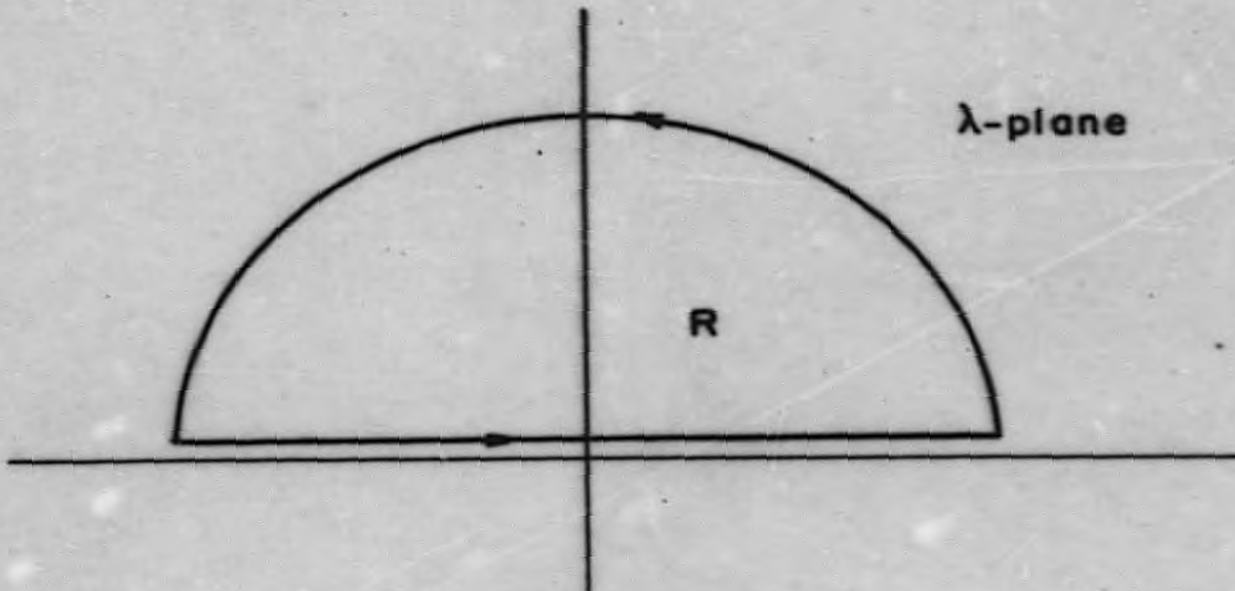

Figure 4.1

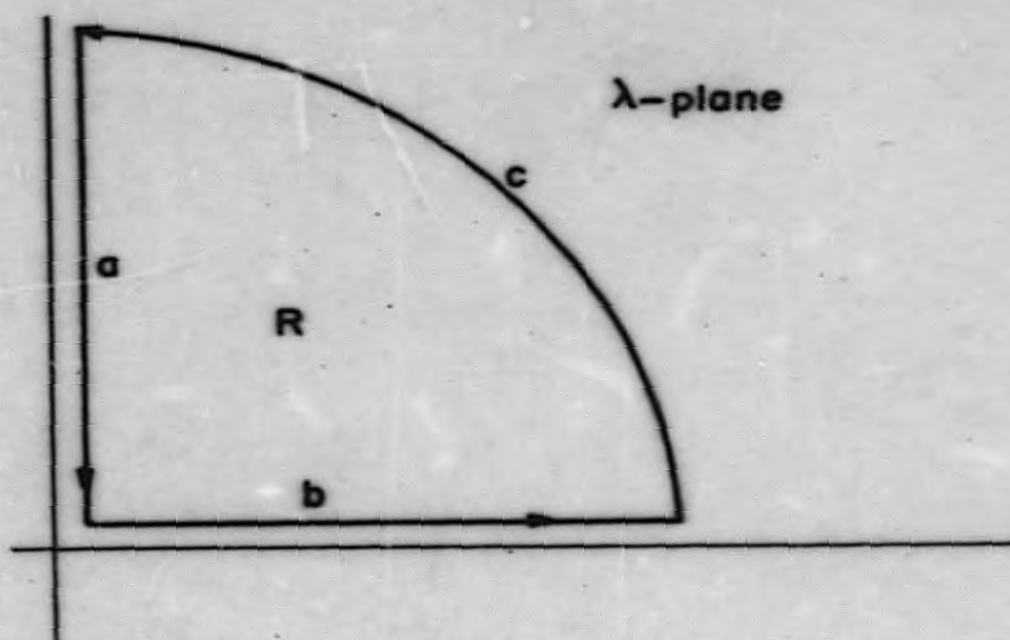

Figure 4.2

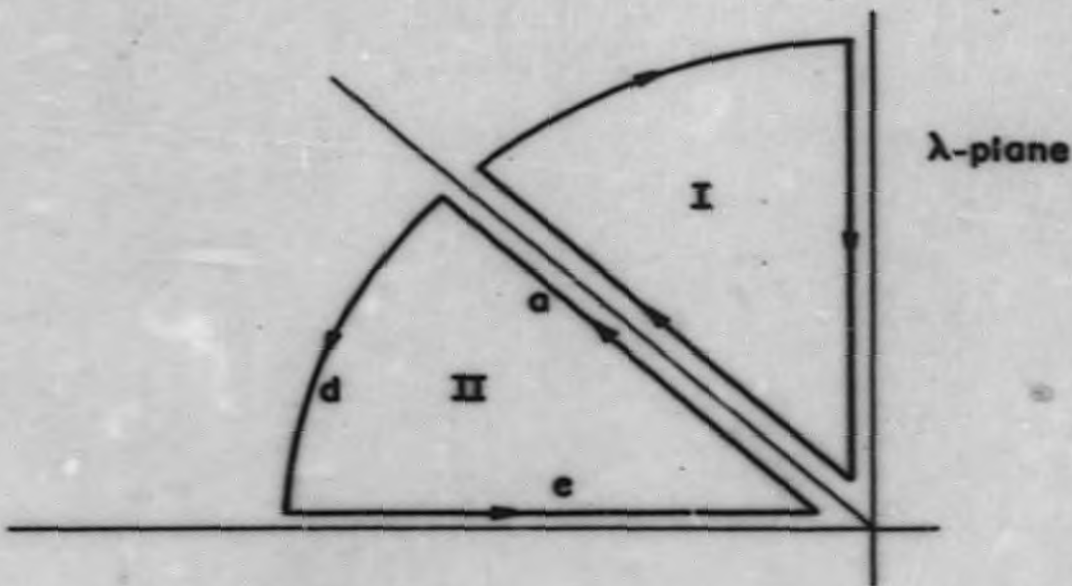

Figure 4.3

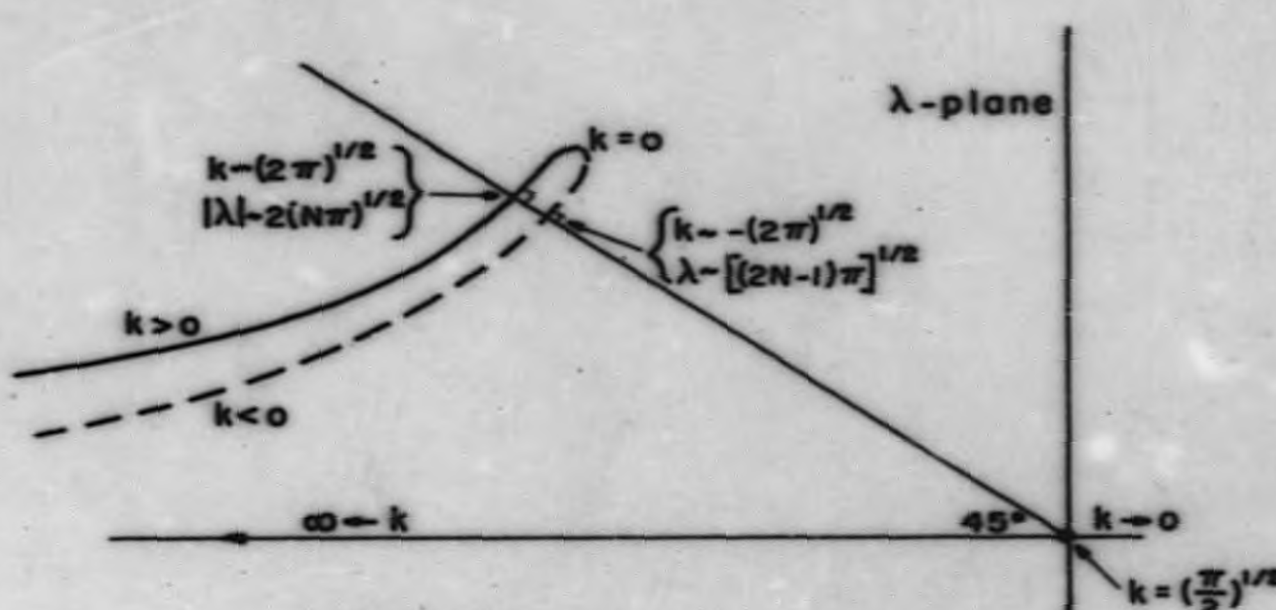

Figure 4.4

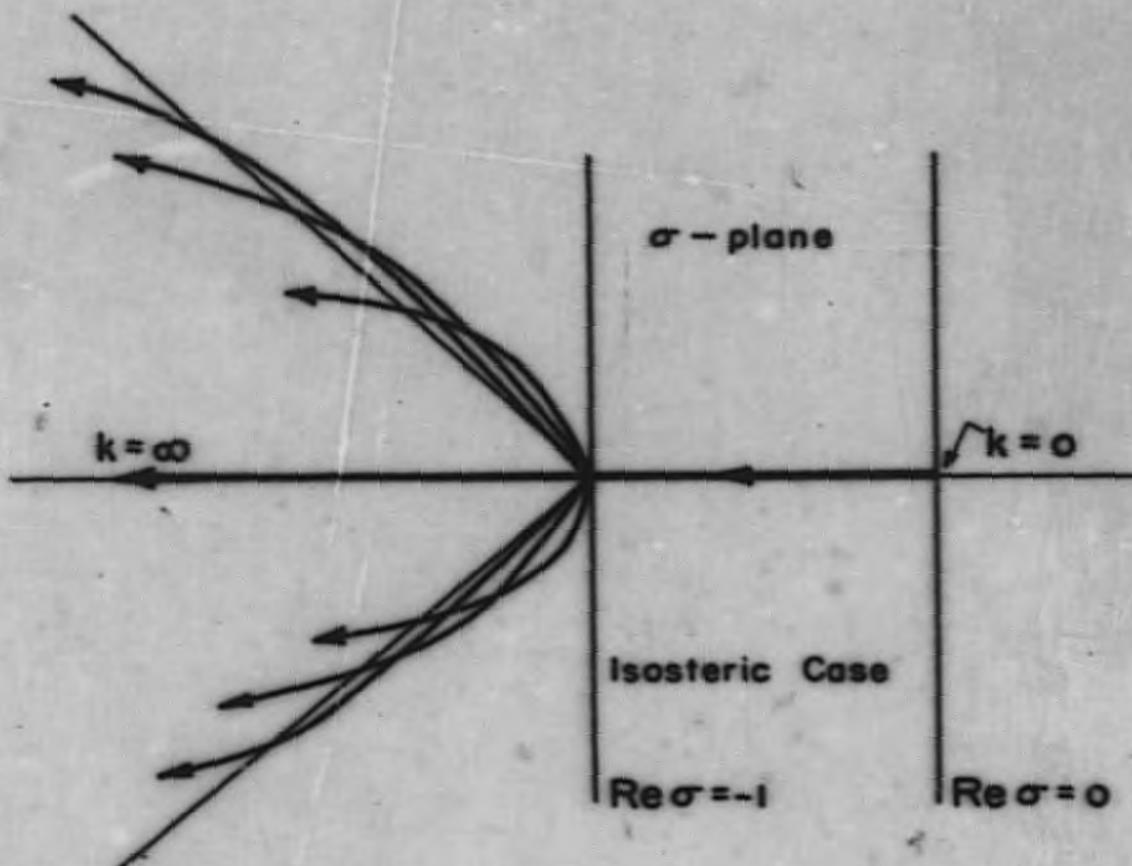

Figure 4.5

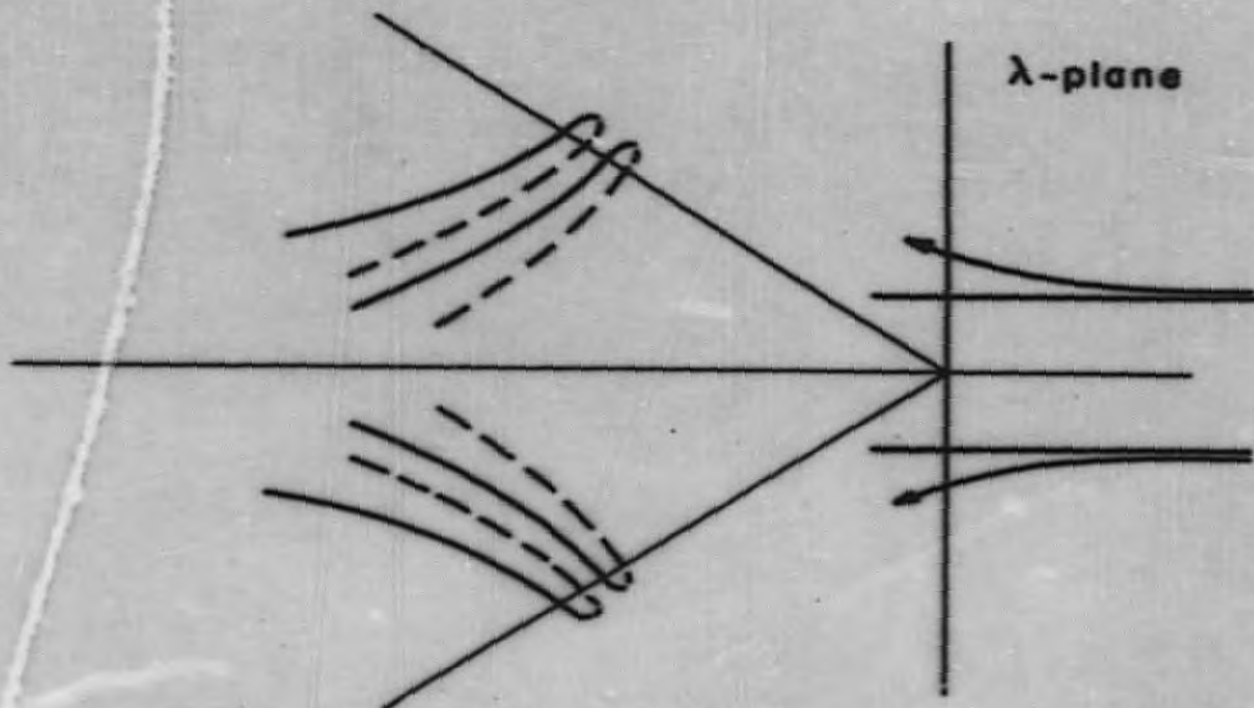

Figure 4.6

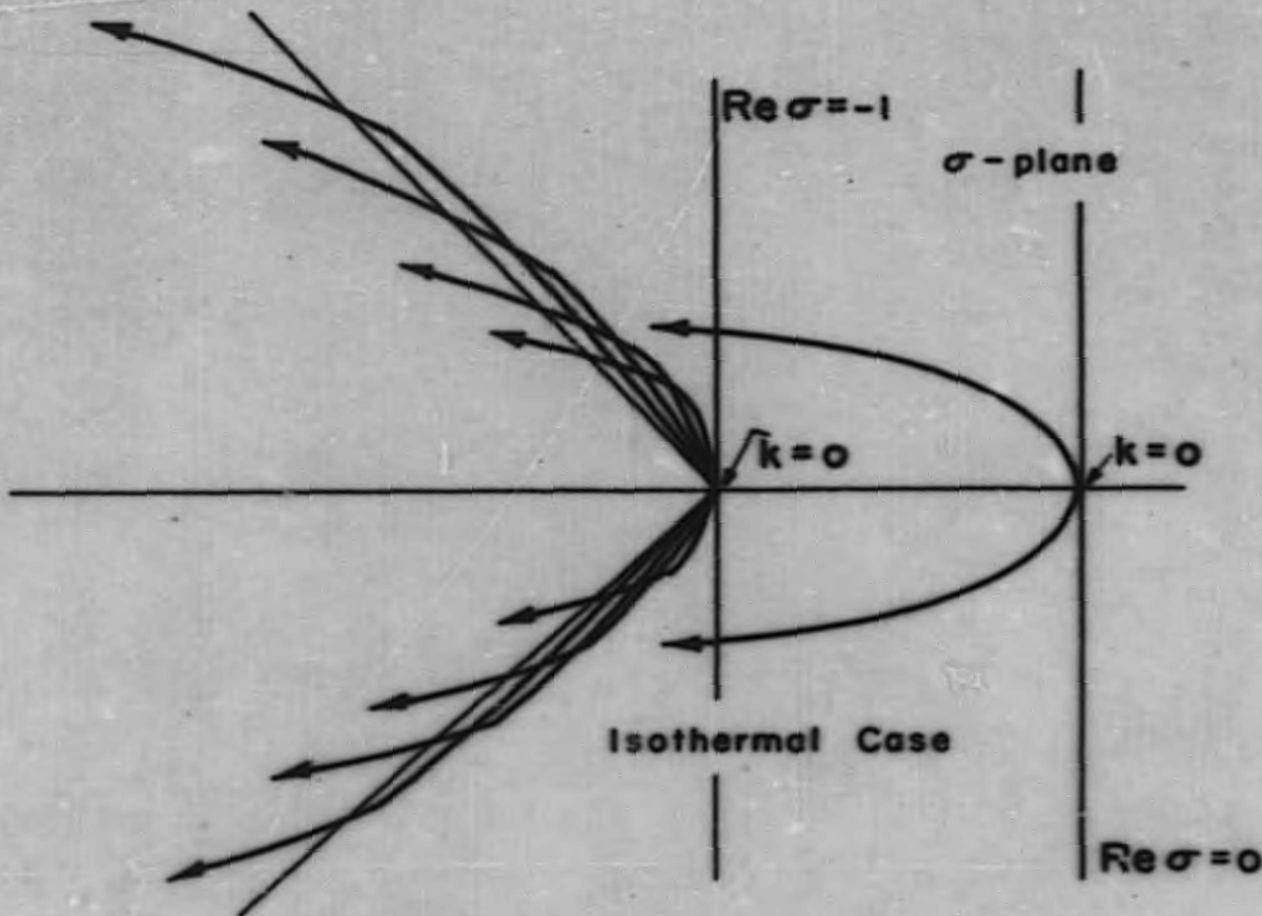

Figure 4.7

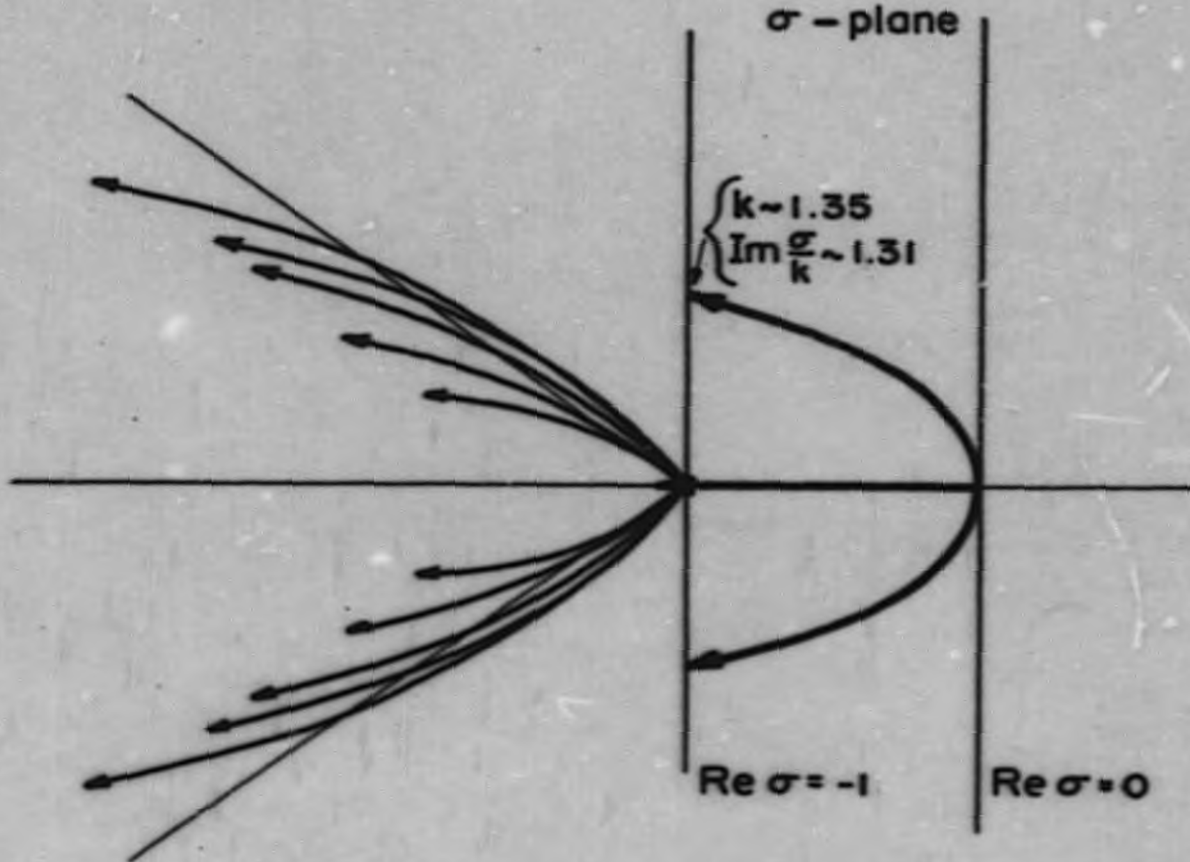

Figure 5.1 SINGLE RELAXATION MODEL

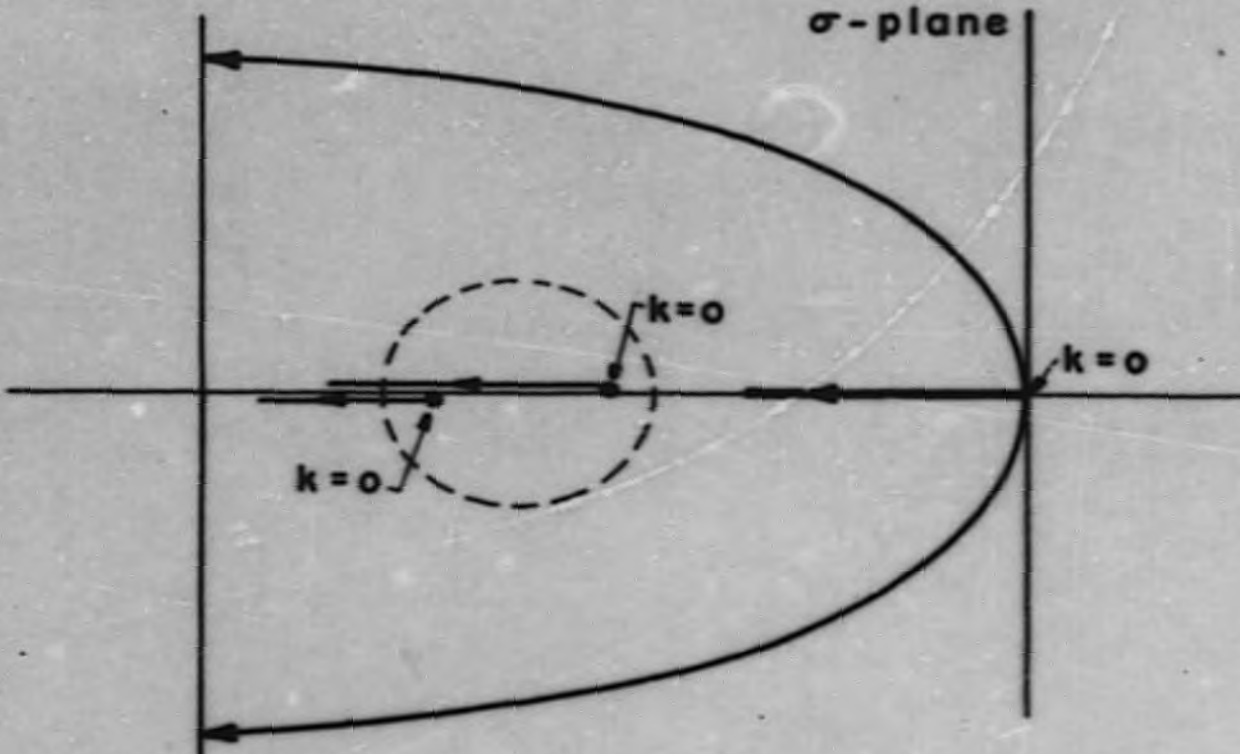

Figure 5.2 THREE RELAXATION MODEL

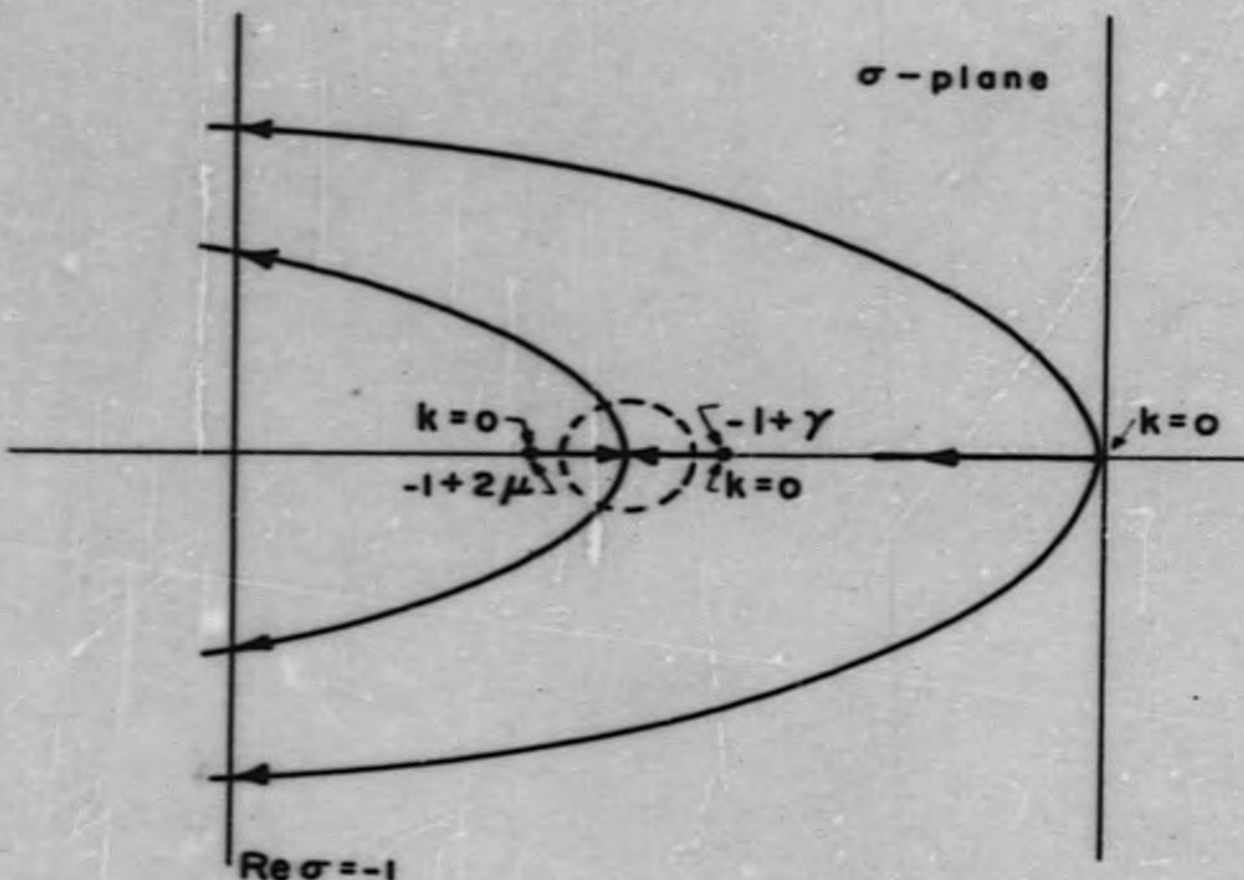

Figure 5.3 THREE RELAXATION MODEL