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THE CANONICAL THEORY OF MOTION OF A CHARGED PARTICLE IN A SLOWLY VARYING ELECTROMAGNETIC FIELD

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Abstract

This paper formulates the canonical theory of motion of a charged particle in a slowly varying, static electromagnetic field. The paper is not, however, concerned with the mathematically rigorous proof of adiabatic invariance, or of the convergence of the perturbation method, but with an attempt to write down explicitly the Hamiltonian in terms of the coordinates of the gyration and the drift.

The method of approach is closely analogous to that of the canonical formalism with subsidiary condition, as used in the theories of collective motion in many body systems, such as the motion of the center of gravity. In the lowest order of the perturbation it is shown that the Hamiltonian for the motion of drift averaged over the phase of gyration is given by adding to the original Hamiltonian of a particle, a potential term equal to the product of the magnetic moment and the magnetic field strength.

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The Canonical Theory of Motion of a Charged Particle in a Slowly-Varying Electromagnetic Field

Introduction

The motion of a charged particle in a slowly varying electromagnetic field has been investigated by many authors [1], [2]. As is well known, the characteristic features of this system are the adiabatic invariance of the magnetic moment of the gyrating particle and the fact that the motion can be separated into two parts, the drift and the gyration. The canonical theory of this system has been established by Gardner [3]. In his theory, invariance is proved to an arbitrarily high order in a slowness parameter, on the basis of the canonical transformation. The present paper is concerned with a different approach which attempts to introduce the coordinates of the guiding center as independent variables and to write down explicitly the Hamiltonian in terms of the coordinates of gyration and of the guiding center. In this attempt one encounters a difficulty that the degrees of freedom of motion are increased due to the introduction of new coordinates, namely those of the guiding center. In order to reduce the apparent increase of the degrees of freedom we are led to introduce subsidiary conditions.

We have, however, encountered a similar situation in various collective motions in many body problems, such as plasma oscillations, the vibration of the nucleus, and the motion of the center of gravity. For instance, we easily find the same difficulty of increased degrees of freedom if we attempt to introduce as independent variables the coordinates of the center of gravity of a many particle system and to write down the Hamiltonian in terms of these coordinates and the coordinates of the relative motion referred to at the center of gravity. In connection with these problems the canonical theory with subsidiary conditions has been investigated extensively by many authors, [5], [6]. The method of solution established there was applied by the present author [4] to a special case of our system, namely one in which the magnetic field is in one direction and the motion of particles is in the plane perpendicular to the magnetic field.

The present paper is essentially an extension of the previous theory to the general case and does not deal with the mathematically rigorous proof of the adiabatic invariance or with the existence of the solution.

In Section 1 we will introduce curvilinear coordinates, one axis of which is in the direction of the magnetic field. The representation of the magnetic field vector B and the vector potential A in this system of coordinates will be given. In Section 2, the coordinates of the guiding center will be introduced in a way similar to that in the previous paper [4] and the Hamiltonian will be given in a form separated into three parts, the drift part, the gyration part and the part expressing the interaction between drift and gyration which is renormalized as a drift.

In the last section it will be shown that in the lowest order of the perturbation the Hamiltonian for the motion of the guiding center, averaged over the phase of gyration is given in the form obtained by adding to the starting Hamiltonian a potential $\mu|B|$.

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1. The Electromagnetic Field and the Curvilinear Coordinates.

The static electromagnetic field is governed by the equations,

$$\text{div}\, \mathbb{B} = 0 \tag{1.1}$$

$$\text{curl}\, \mathbb{B} = \frac{4\pi}{c} \mathfrak{j} \tag{1.2}$$

$$\text{div}\, \mathbb{E} = 4\pi\rho \tag{1.3}$$

$$\text{curl}\, \mathbb{E} = 0 \tag{1.4}$$

in which $\mathbb{B}$ is the magnetic and $\mathbb{E}$ the electric field vector, $\mathfrak{j}$ is the current density and ρ is the charge density. All these quantities are independent of time. We assume throughout this paper that $\mathbb{B}$ and $\mathbb{E}$ are slowly-varying functions of space variables. Let the representative magnitude of B be $\{B\}$, then we can introduce the representative cyclotron frequency, $\{\omega_B\}$, given by the equation

$$\{\omega_B\} = \frac{e\{\mathbb{B}\}}{mc}\,, \tag{1.5*}$$

in which e and m are the charge and the mass of a particle, and are assumed to be given.

In the subsequent chapter we shall consider the motion of the particle in the electromagnetic field under consideration. The representative velocity component in a plane perpendicular to the magnetic field will be designated by $\{v_\perp\}$. We then

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* Hereafter the representative value of a quantity A will be designated by $\{A\}$.

introduce the representative cyclotron radius $\{\gamma_B\}$,

$$\{\gamma_B\} = \{v_\perp\}/\{\omega_B\} = \frac{cm\{v_\perp\}}{e\{\mathbb{B}\}} \quad , \tag{1.6}$$

and assume the following conditions among these quantities $\{\gamma_B\}$, $\{\omega_B\}$ and the field strength:

$$\frac{|\{\gamma_B\}\nabla\cdot|\mathbb{B}||}{|\mathbb{B}|} \ll 1 \tag{1.7}$$

$$\frac{|\{\gamma_B\}\nabla\cdot|\mathbb{E}||}{\{\mathbb{E}\}} \ll 1 \tag{1.8}$$

$$\frac{|(1/\{\omega_B\})\ d\mathbb{B}/dt|}{\{\mathbb{B}\}} \ll 1. \tag{1.9}$$

In Eq. (1.9) the substantial derivative $d\mathbb{B}/dt$ is used for the variation of the magnetic field experienced by the charged particle. Since, by virtue of Eq. (1.7), the variation of the magnetic field due to the gyration around an instantaneous guiding center is small, Eq. (1.9) means that the drift distance over one period of gyration is small compared with the space variation of the magnetic field. Let us assume that the space variation of the electric and the magnetic fields are of the same order and denote them by $\{L\}$; then the conditions (1.7)-(1.9) are equivalent to the equations

$$\varepsilon \equiv \{\gamma_B\}/\{L\} \ll 1 \tag{1.10a}$$

$$\nu \equiv \{u\}/\{L\}\{\omega_B\} \ll 1 \tag{1.10b}$$

in which $\{u\}$ is the representative drift velocity. If the drift velocities parallel to and normal to the magnetic field, $\mathbf{u}_{\parallel}$ and $\mathbf{u}_{\perp}$ respectively, differ greatly, $\{u\}$ denotes the order of the larger one. Recalling the well-known relation

$$\mathbb{E} \approx \frac{1}{c}\,[\mathbf{u} \times \mathbb{B}],$$

we have from Eq. (1.10b) the conditions for $\{\mathbb{E}\}$,

$$\{\mathbb{E}\} << \frac{1}{c}\,\{L\}\{\omega_B\}\{\mathbb{B}\}. \tag{1.10c}$$

We now discuss the representation of the magnetic field. As is well known, the magnetic field $\mathbb{B}$ is given in terms of the vector potential $\mathbb{A}$, as follows:

$$\mathbb{B} = \operatorname{curl} \mathbb{A}\,. \tag{1.11}$$

Another representation of the magnetic field is given by the equation [3]

$$\mathbb{B} = \nabla x^1 \times \nabla x^2 \tag{1.12}$$

in which x^1 and x^2 are scalar functions of the cartesian coordinates X^o, X^1, and X^2; namely they are given in a form,

$$x^1 = x^1(X^o,\, X^1,\, X^2) \tag{1.13a}$$

$$x^2 = x^2(X^c,\, X^1,\, X^2). \tag{1.13b}$$

Using an appropriate gauge transformation for the vector potential A, we may always derive the following expression for A (see [3]),

$$A = x^1 \nabla x^2 . \tag{1.14}$$

Introducing a scalar function x^0 which is independent of x^1 and x^2, otherwise arbitrary, we construct curvilinear coordinates x^0, x^1, and x^2. Then an arbitrary displacement $d\mathbf{r}$ is given by the equation*,

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^k}\, dx^k = \mathbf{a}_k dx^k , \tag{1.15}$$

in which the base vector $\mathbf{a}_k$ is defined by

$$\mathbf{a}_k = \frac{\partial \mathbf{r}}{\partial x^k} \qquad (k = 0, 1, 2). \tag{1.16}$$

The adjoint base vector $\mathbf{a}^0$ is given by the equation

$$\mathbf{a}^0 = \mathbf{a}_1 \times \mathbf{a}_2/\sqrt{g} , \quad \text{etc.} \tag{1.17}$$

in which g is the determinant of the metric tensor g_{ik} given by the equation

$$g_{ik} = \mathbf{a}_i \cdot \mathbf{a}_k . \tag{1.18}$$

We can, alternatively, introduce the tensor g^{ik} by the equation

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* Hereafter, the usual summation convention will be used for the Latin suffices assuming the values 0, 1, and 2.

(1.19) $$g^{ik} = a^i \cdot a^k \ .$$

We shall often use the identity

(1.20a) $$a^i \cdot a_k = \delta^i_k$$

(1.20b) $$g_{ik}\, g^{kj} = \delta^j_i \ .$$

The contravariant or covariant component of a vector is defined in the usual manner. Since the gradient of a scalar function ϕ is given by the formula

$$\nabla\phi = \frac{\partial\phi}{\partial x^i}\, a^i \ ,$$

we have, from (1.14),

(1.21a) $$\mathbb{A} = x^1 a^2$$

or

(1.21b) $$A_k = x^1 \delta_{i1} \delta_{2k} \ .$$

On the other hand Eq. (1.12) reduces to

(1.22) $$\mathbb{B} = a^1 \times a^2 = \frac{1}{\sqrt{g}}\, a_o$$

or we have

(1.23) $$|\mathbb{B}| \equiv B = \sqrt{g_{oo}/g} \ .$$

Since ∇x^1 and ∇x^2 are normal to the surfaces in which the magnetic lines of force are embedded, we have

(1.24a) $$\frac{|\{\gamma_B\}\, \nabla \cdot |a^k||}{\{a^k\}} \ll 1$$

(1.24b) $$\frac{|\{\gamma_B\}\, \nabla \cdot |a_k||}{\{a_k\}} \ll 1 \ .$$

or, equivalently, we have

$$(1.25a) \qquad \frac{|\{\gamma_B\}\, \nabla \cdot g_{ik}|}{\{\sqrt[3]{g}\}} \ll 1$$

$$(1.25b) \qquad \frac{|\{\gamma_B\}\, \nabla \cdot g^{ik}|}{\{\sqrt[3]{g}\}} \ll 1.$$

When the electric field E is expressed by a scalar potential ϕ, namely

$$\mathbb{E} = -\nabla\phi ,$$

we have for ϕ the similar relation,

$$(1.26) \qquad \frac{|\{\gamma_B\}\nabla \cdot\ |\nabla\phi||}{\{\nabla\phi\}} \ll 1$$

2. The Canonical Transformations

We consider the motion of a charged particle, of mass m and charge e, in the external field specified in Section 1. Let us begin with the Lagrangian L given by the equation

$$(2.1) \qquad L = \frac{m}{2}\, g_{ik}\, \dot{x}^i \dot{x}^k - e\phi + \frac{e}{c}\, \dot{x}^k A_k \quad ,$$

in which we refer to the curvilinear coordinates introduced in Section 1. From (2.1) we have the Hamiltonian H:

$$(2.2) \qquad H = \frac{1}{2m}\, g^{ik}\, (p_k - \frac{e}{c} A_k)(p_i - \frac{e}{c} A_i) + e\phi,$$

where the p_k's are the canonical momenta conjugate to the x^k's and are given by the formulae,

$$p_k = \frac{\partial L}{\partial \dot{x}^k} = mg_{ik}\,\dot{x}^i + \frac{e}{c}\,A_k \quad . \tag{2.3}$$

Let us introduce the canonical variables ξ^k and their conjugate momenta π_k $(k = 0, 1, 2)$, and assume the following subsidiary conditions:*

$$\pi_k = 0, \quad (k = 0, 1, 2). \tag{2.4}$$

We now introduce a canonical transformation through a generating function Ω_1, given by the equation

$$\Omega_1 = (\pi_k' - p_k')\xi^k + x^k \cdot p_k' \quad . \tag{2.5}$$

The equations connecting the original variables and the transformed ones take the form:

$$\left\{ \begin{array}{ll} p_k' = p_k \quad , & x^{k'} = x^k - \xi^k \\ \pi_k' = \pi_k + p_k \quad , & \xi^{k'} = \xi^k \;. \end{array} \right. \tag{2.6}$$

The subsidiary conditions (2.4) are transformed into

$$\pi_k' = p_k' \quad . \tag{2.4'}$$

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* The way of introduction of these redundant variables is the same as that of the collective coordinate in the theory of many body problems, such as the theory of the center of gravity in many particle systems, the theory of plasma oscillations, the theory of nuclear vibration and so on, [5], [6]. In the following discussions, the variable ξ^k will be transformed so that it will be identified with the coordinates of the guiding center.

Introducing the above relations (2.6) into (2.2) and using the condition (2.4'), we have the following new Hamiltonian H', equivalent to H,

$$(2.7)\qquad H' = \frac{1}{2m}\bar{g}^{ik}(p_i' - \frac{e}{c}DA_i)(p_k' - \frac{e}{c}DA_k) - \frac{e}{mc}\bar{g}^{ik}\pi_i' A_k$$

$$+ \frac{e^2}{mc^2}\bar{g}^{ik} DA_i \cdot A_k + (\frac{e^2}{2mc^2})\bar{g}^{ik} A_i A_k + e\bar{\phi} ,$$

in which $\bar{g}^{ik}$ and $\bar{\phi}$ are defined by the equations

$$(2.8a)\qquad \bar{g}^{ik} = g^{ik}(\xi^1 + x^{1'},\ \xi^2 + x^{2'},\ \xi^o + x^{o'}) ,$$

$$(2.8b)\qquad \bar{\phi} = \phi(\xi^1 + x^{1'},\ \xi^2 + x^{2'},\ \xi^o + x^{o'}).$$

On the other hand the A_k's are functions of ξ^o, ξ^1, and ξ^2, namely

$$(2.9)^*\qquad A_k = A_k(\xi^o, \xi^1, \xi^2) = \xi^1 \delta_{i1}\delta_{k2} \cdot$$

The operator D is given by the equation

$$(2.10)\qquad D = x^{j'} \frac{\partial}{\partial \xi^j} ;$$

consequently we have

$$(2.11)\qquad DA_k = x^{1'}\delta_{i1}\delta_{k2} \cdot$$

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* In the following discussions, the field quantities without the upper bar, such as A_k and g^{ik}, etc., designate functions of the ξ^i, independent of x^i and should be distinguished from the quantities with upper bar, such as $\bar{A}_k$ and $\bar{g}^{ik}$ etc., defined by Eq. (2.8).

Let us introduce the second canonical transformation through a generating function Ω_2, given by the equation

$$(2.12) \qquad \Omega_2 = (p_k'' + mU_k + \frac{e}{c} A_k)x^{k'} + \pi_k' \cdot \xi^{k''} ,$$

in which the U_k's are functions of the ξ^1's, the functional forms of which will be specified later, so that the ξ^k's become the coordinates of the guiding center. The canonical variables p_k' and π_k' etc., are transformed into the new variables p_k'' and π_k'' etc., as follows,

$$(2.13) \qquad \begin{cases} p_k'' = p_k' - mU_k - (e/c)A_k \quad , \qquad x^{k''} = x^{k'} , \\ \pi_k'' = \pi_k' - \frac{\partial}{\partial \xi^{k'}} (mU_1 + \frac{e}{c} A_1)\cdot x^{1''} , \\ \xi^{k''} = \xi^{k'} . \end{cases}$$

The subsidiary condition (2.4') becomes

$$(2.14) \qquad \pi_k'' = p_k'' + mU_k + (e/c)A_k - m \frac{\partial}{\partial \xi^{k''}} (U_1 \cdot x^{1''}) - (e/c) \frac{\partial}{\partial \xi^{k''}} (A_1 \cdot x^{1''}).$$

Using the relations (2.13) and the condition (2.14), and omitting all the double primes of the new canonical variables, such as p_k'' and $x^{k''}$ etc., we have the following Hamiltonian $\bar{H}$ equivalent to H,

$$(2.15) \qquad \bar{H} = \bar{H}_o + \bar{g}^{1k}(\Lambda_{1k} + \Xi_{1k}) + e\phi ,$$

in which $\bar{H}_o$, Λ_{ik} and Ξ_{ik} are given by the following equations:

$$\bar{H}_o = g^{ik}(p_i - \frac{e}{c} DA_i)(p_k - \frac{e}{c} DA_k) \ , \tag{2.16}$$

$$\Lambda_{ik} = U_i(p_k - m \frac{\partial U_j}{\partial \xi^k} x^j - \frac{e}{c} \frac{\partial A_j}{\partial \xi^k} x^j) + \frac{m}{2} U_i U_k \tag{2.17}$$

$$= U_i \pi_k - \frac{m}{2} U_i U_k - \frac{e}{c} U_i A_k \ ,$$

$$\Xi_{ik} = \left\{ mU_i \frac{\partial U_j}{\partial \xi^k} + \frac{e}{c} U_i \left(\frac{\partial A_j}{\partial \xi^k} - \frac{\partial A_k}{\partial \xi^j} \right) \right\} x^j . \tag{2.18}$$

In the subsequent discussions we shall show that the Hamiltonian $\bar{H}$ is modified further and the U_i's are determined appropriately so that the ξ^i's become the coordinates of the guiding center and the x^i's are identified with the coordinates of the particle in the system of coordinates moving with the guiding center. Let us assume this for the time being and note that it then turns out that the distance $ds \equiv \sqrt{g_{ik} dx^i dx^k}$ is of the order of $\{\gamma_B\}$ and that the quantities $\bar{g}^{ik}$ and $\bar{\phi}$ can be expanded in power series in an increasing order of ε; namely, we have

$$\bar{g}^{ik} = g^{ik} + Dg^{ik} + \frac{1}{2} D^2 g^{ik} + \dots \ , \tag{2.19}$$

$$\bar{\phi} = \phi + D\phi + \frac{1}{2} D^2 \phi + \dots \ , \tag{2.20}$$

in which we have that $|D^n g^{ik}| / |D^{n-1} g^{ik}| \sim \varepsilon$, for $n = 1, 2, \dots$, and $|D^n \phi| / |D^{n-1} \phi| \sim \varepsilon$, for $n = 2, 3, \dots$. Introducing equations (2.19) and (2.20) into $\bar{H}$ and using Eqs. (2.16)-(2.18), we have the final form of the Hamiltonian given by the equation,

$$(2.21) \qquad \bar{\bar{H}} = H_{oo} + H_{o1} + H_d + H_I + H_{II} + R$$

in which H_{oo} etc. are given as follows:

$$(2.22) \qquad H_{oo} = \frac{1}{2m} g^{ik}(p_i - \frac{e}{c} DA_i)(p_k - \frac{e}{c} DA_k)$$

$$(2.23) \qquad H_{o1} = \frac{1}{2m} Dg^{ik}(p_i - \frac{e}{c} DA_i)(p_k - \frac{e}{c} DA_k)$$

$$(2.24) \qquad H_I = F_j x^j = \left\{ mU^k \frac{\partial U_j}{\partial \xi^k} + \frac{m}{2} \frac{\partial g^{ik}}{\partial \xi^j} U_i U_k + \frac{e}{c} U^i \left(\frac{\partial A_j}{\partial \xi^i} - \frac{\partial A_i}{\partial \xi^j} \right) + e \frac{\partial \phi}{\partial \xi^j} \right\} x^j$$

$$(2.25) \qquad H_d = g^{ik} \Lambda_{ik} + e\phi$$

$$(2.26) \qquad H_{II} = \frac{\partial g^{ik}}{\partial \xi^j} U_i (p_k - \frac{e}{c} DA_k) x^j + \frac{e}{2} D^2 \phi$$

$$(2.27) \qquad R = (\tfrac{1}{2} D^2 g^{ik} + \ldots\ldots)(\Lambda_{ik} + \Xi_{ik}) + e(\tfrac{1}{3!} D^3 \ldots)\phi.$$

As is obvious in the above expression of $\bar{\bar{H}}$, H_d depends on the ξ^i's only and may be considered as the drift Hamiltonian; H_{oo} is a Hamiltonian of a particle gyrating around a slowly varying magnetic field. It can be easily said that the remaining terms in (2.21) are zero and the Hamiltonian is separated completely in two parts, the gyration and the drift if E and B are constant. However, the remaining terms give, in general, an interaction between these two systems and lead to a further drift. In order to renormalize the effect of this interaction as a drift, we introduce a vector $\mathbb{G}(\xi^o, \xi^1, \xi^2)$ and rewrite $\bar{\bar{H}}$ in the following form,

$$\bar{\bar{H}} = H_{oo} + H_d + H_{o1} + H_{II} + G_j x^j + R \tag{2.28}$$

and moreover impose a subsidiary condition,

$$F_j - G_j = 0. \tag{2.29a}$$

The above equation (2.29a) may also be written in a vector form,

$$m \frac{d\mathbb{U}}{dt} - e\left\{\mathbb{E} + \frac{1}{c}[\mathbb{U} \times \mathbb{B}]\right\} + \mathbb{G} = 0 \tag{2.29b}$$

in which the symbol d/dt denotes the derivative, $\mathbb{U}\cdot\nabla \equiv U^k \frac{\partial}{\partial\xi^k}$. The vector $\mathbb{G}$ will be determined as a function of the ξ^k's so that the drift resulting from the interaction between H_{oo} and H_d is cancelled out by $\mathbb{G}$.

The adiabatic invariance of the magnetic moment can be seen easily from the form of the unperturbed Hamiltonian H_{oo}. On the other hand, the condition (2.29), with G thus determined, serves as an equation for U, and the canonical equation of $\bar{\bar{H}}$,

$$\frac{d\xi^i}{dt} = \frac{\partial\bar{\bar{H}}}{\partial\pi_i} = U^i \tag{2.30}$$

shows that U^i is the contravariant component of the drift velocity d/dt.

3. The Determination of the Vector G.

In order to perform the scheme in Section 2 to determine G, we now write down the canonical equation for x^i. However, considerable effort is needed if the exact Hamiltonian is used and so we apply a perturbation method in terms of ε and v. To this end, we introduce a differential operator $\hat{D}$ through the following equations

$$D^n g^{ik} = \varepsilon^n \hat{D}^n g^{ik} \quad ; \tag{3.1a}$$

consequently we also have for ϕ,

$$D^n \nabla\phi = \varepsilon^n \hat{D}^n \nabla\phi \ . \tag{3.1b}$$

Introducing Eqs. (3.1) into (2.28) and retaining in $\bar{\bar{H}}$ only the zero'th and the first order terms in ε or v, we get the equations of motion for the x^k's,

$$g_{ij}\dot{v}^j + (e/mc)v^j \left(\frac{\partial A_i}{\partial \xi^j} - \frac{\partial A_j}{\partial \xi^i}\right) + K_i + M_i + \frac{1}{m} G_i + L_i = 0, \tag{3.2}$$

$$(i = 0,\ 1,\ \text{and}\ 2),$$

in which v^j, K_i, L_i and M_i are given as follows:

$$v^j = dx^j/dt$$

$$K_i = \varepsilon(\hat{D}g_{ij})\dot{v}^j + \frac{1}{2}\frac{\partial g^{jk}}{\partial \xi^i} g_{jr}\, g_{ks}\, v^r v^s + \frac{\partial g_{ij}}{\partial \xi^k} v^k v^j \tag{3.3}$$

$$(3.4)\qquad M_i = -\varepsilon(\hat{D}g_{ij})\frac{\partial g^{jk}}{\partial \xi^r}U_k v^r - \varepsilon(\hat{D}g^{kj})\frac{\partial g_{ij}}{\partial \xi^r}U_k v^r$$

$$+ \varepsilon(\hat{D}\frac{\partial g_{ij}}{\partial \xi^k})U^k v^j + \varepsilon(\hat{D}g_{jr})\frac{\partial g^{jk}}{\partial \xi^i}U^r g_{ks}v^s$$

$$- \varepsilon(\hat{D}g^{rs})\frac{\partial g^{jk}}{\partial \xi^i}g_{kr}U_s v_j - \varepsilon(\hat{D}g_{kj})\frac{\partial g^{rk}}{\partial \xi^i}U_r v^j .$$

$$L_i = U^k\left(\frac{\partial g_{ji}}{\partial \xi^k} + \frac{\partial g_{ik}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^i}\right)v^j + \varepsilon\ell_i$$

with ℓ_i a linear function of $\hat{D}$.

We now, at an arbitrary instant $t = \bar{t}$, refer to locally orthogonal coordinates at the point $(\bar{\xi}^0(\bar{t}), \bar{\xi}^1(\bar{t}), \bar{\xi}^2(\bar{t}))$, and investigate the motion during one period of gyration. Denoting $g_{ii}(\bar{\xi})$ by $g_i(\bar{\xi})$ and $eB(\bar{\xi})/mc$ by $\bar{\omega}$, we have for $t < \bar{t} + 1/\bar{\omega}$,

$$(3.5)\qquad B(\xi) = B(\bar{\xi}) + D_\xi B + \dots$$

$$(3.6)\qquad g_{ik}(\xi) = g_{ik}(\bar{\xi}) + D_\xi g_{ik} + \dots$$

in which D_ξ is an operator given by

$$(3.7)\qquad D_\xi = (\xi^j - \bar{\xi}^j)\frac{\partial}{\partial \xi^j} .$$

Since $\xi_j - \bar{\xi}_j$ is the component of the displacement of the guiding center during the time interval $t-\bar{t} < 1/\bar{\omega}$, it is convenient to introduce an operator $\hat{D}_\xi$ through the equation,

$$D_\xi = \nu\hat{D}_\xi \ . \tag{3.8}$$

Then Eqs. (3.5) and (3.6) become the power series in ν,

$$B(\xi) = B(\bar{\xi}) + \nu\hat{D}_\xi B + \dots \tag{3.9}$$

$$g_{ik}(\xi) = g_{ik}(\bar{\xi}) + \nu\hat{D}_\xi g_{ik} + \dots \ . \tag{3.10}$$

Introducing Eq.(3.10) into the first term in Eq. (3.2), we get

$$g_L(\bar{\xi})\dot{v}^L + \frac{e}{mc} v^i(\delta^{il}\delta_{L2} - \delta_{i2}\delta^{Ll}) + K_L(\xi) + M_L(\xi) + \frac{1}{m} G_L(\xi) + \tilde{L}_L = 0 \tag{3.11}$$

in which $\tilde{L}_L$ is linear with respect to x^i and v^i and is given by the equation

$$\tilde{L}_L = L_L + \nu\hat{D}_\xi g_{Li}\dot{v}^i + \frac{1}{2}\nu^2\hat{D}_\xi^2 g_{Li}\dot{v}^i + \dots \ . \tag{3.12}$$

We now introduce the quantities having the dimension of length, $\hat{x}^L$'s and $\hat{\xi}^L$'s given by the equations

$$\hat{x}^L = \sqrt{g_L(\bar{\xi})}\ x^L \ , \qquad \hat{\xi}^L = \sqrt{g_L(\bar{\xi})}\ \xi^L$$

Eq.(3.11) can then be expressed in terms of these quantities, namely noting the relation

$$B(\bar{\xi}) = g_{oo}(\bar{\xi})/g(\bar{\xi}) = (g_1(\bar{\xi})g_2(\bar{\xi}))^{-1}$$

we have

$$\text{(3.13)} \qquad \frac{d}{dt}\,\hat{v}^L + \bar{\omega}\hat{v}^1(\delta_{2L}\delta^{11} - \delta^{12}\delta_{L1}) + \frac{1}{\sqrt{g_L(\bar{\xi})}}\Big\{K_L(\xi) + M_L(\xi) + \frac{1}{m}\,G_L(\xi) + \tilde{L}_L(\xi)\Big\} = 0.$$

Since $G_L(\xi)$ contains, at lowest, first order terms in ε and ν, we have from (3.13) the following zero-order equations:

$$\text{(3.16a)} \qquad \dot{\hat{v}}^1 - \bar{\omega}\,\hat{v}^2 = 0$$

$$\text{(3.16b)} \qquad \dot{\hat{v}}^2 + \bar{\omega}\,\hat{v}^1 = 0$$

$$\text{(3.16c)} \qquad \dot{\hat{v}}^o = 0.$$

From (3.16c) we may put

$$\text{(3.17)} \qquad \hat{x}^o = \hat{v}^o = 0.$$

The solutions of (3.16a, b) are given by

$$\text{(3.18a)} \qquad \hat{v}^1 = -V(\bar{\xi})\cos\left\{\bar{\omega}(t-\bar{t}) + \phi\right\}$$

$$\text{(3.18b)} \qquad \hat{v}^2 = V(\bar{\xi}).\sin\left\{\bar{\omega}(t-\bar{t}) + \phi\right\}$$

in which $V(\bar{\xi})$ is the absolute value of the velocity $(\hat{v}^1(\bar{\xi}),\ \hat{v}^2(\bar{\xi}))$, and ϕ is a constant phase. As a result, $\hat{x}^1$ and $\hat{x}^2$ become

(3.19a) $$\dot{x}^1 = -\frac{V(\bar{\xi})}{\omega}\sin\left\{\omega(t-\bar{t}) + \phi\right\}$$

(3.19b) $$\dot{x}^2 = \frac{V(\bar{\xi})}{\bar{\omega}}\cos\left\{\omega(t-\bar{t}) + \phi\right\}.$$

Introducing Eq. (3.18) and (3.19) and (3.17) into the expressions for $K_L(\xi)$ and $M_L(\xi)$, we can see that the order of magnitude of $K_L(\xi)/\bar{\omega}$ is, at the lowest, $\varepsilon\, V(\bar{\xi})$; on the other hand, that of $M_L(\xi)/\bar{\omega}$ is, at the lowest, $\varepsilon^2 V(\bar{\xi})$, $\varepsilon v V(\bar{\xi})$. Hence $M_L(\xi)$ can be neglected in comparison with $K_L(\xi)$.

Now we are ready to determine $G_L(\xi)$. Since the drift should be independent of the constant phase of gyration, we determine $G_L(\xi)$ in such a way that the perturbation forces, independent of ϕ disappear when we introduce the zero-order solutions (3.17)-(3.19) into $K_L(\xi)$ and $\tilde{L}_L(\xi)$. Namely, we have

(3.20) $$G_L(\xi) = -m\,\langle K_L(\xi)\rangle_{av} - m\,\langle \tilde{L}_L(\xi)\rangle_{av} \quad ,$$

in which $\langle\ \rangle_{av}$ means the average over the phase ϕ. Since $\langle \tilde{L}_L(\xi)\rangle_{av} = 0$ and the first and the last term in Eq. (3.3) cancel with each other in average, neglecting the higher order terms with respect to v in Eq. (3.20), we obtain finally

(3.21) $$G_L(\bar{\xi}) = -\bar{\mu}\,\frac{\partial B}{\partial \bar{\xi}^L}$$

where $\bar{\mu}$ is the magnetic moment given by $\frac{1}{2} mV(\bar{\xi})^2/B(\bar{\xi})$. Since $\bar{\xi}$ is arbitrary we have, for any ξ

(3.22) $$\mathbf{G} = -\mu\,\nabla B(\xi).$$

Eq. (3.21) becomes a differential equation for U,

(3.23) $$m \frac{dU}{dt} - e \left\{ E + \frac{1}{c} U \times B \right\} + \mu \nabla B = 0,$$

which determines completely the drift velocity. However, it should be noted that Eq. (3.23) has to admit only solutions which vary slowly. For instance, from (2.32) and (3.23), we may express the drift velocity transverse to the magnetic field $u_\perp$ as follows:

(3.24) $$u_\perp = (\frac{c}{B^2}) E \times B + (\frac{c}{e}) \mu B^2 \times \nabla B + (\frac{mc}{eB^2}) B \times \frac{dU}{dt} .$$

If, moreover, E is a quantity of order ε and $j = 0$, then the last term reduces to

(3.25) $$(2c/e)\mu_{\parallel}\, e_0 \times (e_0 \nabla) e_0$$

in which e_0 is the unit vector in the direction of the magnetic field and $\mu_{\parallel}$ is given by

$$\mu_{\parallel} = m(U \cdot e_0)^2/2B.$$

In this approximation, the equation along the magnetic lines of force reduces to

(3.26) $$m \frac{du_{\parallel}}{dt} = -\mu \frac{\partial B}{\partial \xi_{\parallel}} - e \frac{\partial \phi}{\partial \xi_{\parallel}}$$

in which $d\xi_{\parallel}$ is the infinitesimal arc length of the magnetic lines of force and $u_{\parallel}$ is equal to $d\xi_{\parallel}/dt$.

In any case, introducing the solutions of Eq. (3.23) into $\bar{\bar{H}}$, we can get the explicit form of Hamiltonian, or of the canonical equations of motion. As for the remaining canonical variables π_k's, instead of integrating the canonical equation for them, we can determine them more easily from the subsidiary condition (2.14).* We can, thus, obtain the complete solutions of our dynamical system. However, in most physical situations, the constant phase of gyration, ϕ, is not determined from initial conditions. Hence it seems to be worthwhile to consider the motion averaged over phase ϕ. Let an averaged quantity $\langle q\rangle$, be defined by

$$\langle q\rangle = \frac{1}{2\pi}\int_0^{2\pi} q \, d\phi,$$

then it satisfies the equation

$$\frac{d}{dt}\langle q\rangle = \langle f\rangle ,$$

if q satisfies the equation

$$\frac{dq}{dt} = f.$$

Hence it can be seen from Eq. (3.2) that the $\langle x^k\rangle$'s and the $\langle v^k\rangle$'s obviously satisfy the following equations:

$$d\langle x^1\rangle/dt = \langle v^1\rangle ,$$

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* Of course, it can be shown that the differentiation of Eq. (2.14) with respect to time gives the canonical equation for π_k. This guarantees the consistency of the subsidiary condition.

$$g_{ij}\frac{d\langle v_i\rangle}{dt}+\frac{e}{mc}\langle v^j\rangle\,(\delta_{i2}\delta^{j1}-\delta_{j2}\delta^{i1})$$

$$+\langle K_i\rangle+\frac{1}{m}G_i+\langle L_i\rangle=0,$$

in which the higher order term $\langle M_i\rangle$ is neglected. However, as is obvious from the definition of G_i and the property of L_i, $\langle K_i\rangle+\frac{1}{m}G_i+\langle L_i\rangle=0$. Therefore we obtain

(3.27) $$\langle x^k\rangle=\langle p_k\rangle\equiv 0.$$

Introducing Eq. (3.27) into Eq. (2.14), we have

(3.28) $$\langle \pi_k\rangle-\frac{e}{c}A_K=m\,U_K\,.$$

Eqs. (3.27), (3.28) and (3.23) determine completely the canonical system of the averaged variables $\langle x^k\rangle$, $\langle p_k\rangle$, ξ^k, and $\langle \pi_k\rangle$; and it is easy to see that this system can be characterized by the Hamiltonian,

(3.29) $$\langle H\rangle=\frac{1}{2m}\left(\langle \pi^k\rangle-\frac{e}{c}A_k\right)\left(\langle \pi_k\rangle-\frac{e}{c}A_k\right)+e\phi+\mu B.$$

Therefore the Hamiltonian for the averaged motion of the guiding center can be obtained by simply adding to the original Hamiltonian (2.2) the potential $\mu|B|$. It should, however, be noted that the solutions of the canonical equation of motion have to be restricted to only those solutions which vary slowly.

If the drift in the direction perpendicular to the magnetic field is negligible, Eq. (3.28) obviously gives an adiabatic invariant quantity for a motion along the line of force.

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