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# The Numerical Solution of a Parabolic System of Differential Equations Arising in Shallow Water Theory

by

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## ABSTRACT

A finite difference approximation to a non-linear set of parabolic differential equations arising in shallow water theory is given. These difference equations were used to determine the shape and rate of propagation of a hump of fluid down a channel of constant depth. The hump of fluid was found to spread instead of steepen, as is the case in the usual shallow water theory.

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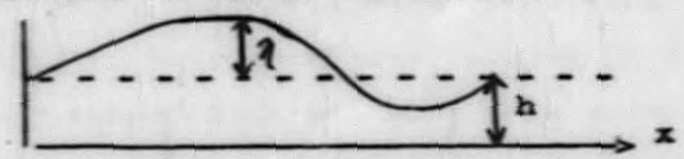
1. Introduction. As is well known the shallow water equations for the flow of fluid down a channel is hyperbolic [5] and the theory suffers from the defect that any hump of fluid traveling down a channel will eventually fall over because the top of the hump moves relatively faster than the trough. Some years ago, Boussinesq [1] suggested an improvement of the shallow water theory which is obtained by taking greater account of the vertical component of the velocity. In this improvement, Boussinesq arrived at the following system of non-hyperbolic equations to describe the flow of fluid in a channel:

$$\frac{\partial(\eta+h)}{\partial t} + \frac{\partial u(\eta+h)}{\partial x} = 0 \qquad \text{(conservation)} \tag{1.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial(\eta+h)}{\partial x} + \frac{h}{3}\frac{\partial^3(\eta+h)}{\partial x\,\partial^2 t} = 0 \qquad \text{(momentum).} \tag{1.2}$$

We take, as usual, $\eta$ to be the height of the fluid measured from the undisturbed level $h$, $u$ to be the horizontal velocity of the fluid, $g$ to be the gravitational constant, $x$ to be the distance downstream and $t$ to be the time.

Fig. 1




This study was suggested by Professor J. J. Stoker who was interested in determining whether the equations (1.1) yield a realistic description of the breaking of a wave. The authors are indebted to Professor J. J. Stoker for his advice and encouragement in carrying out this investigation. Our calculations indicate that no waves break under (1.1) in contrast to the shallow water theory (see [5]) in which all waves break.

The conservation equation (1.1) is identical with the conservation equation of the shallow water theory [1] but the momentum equation has an added term involving a third derivative of $(\eta+h)$ . Recently, equations with a third derivative term have appeared in problems of magneto-hydrodynamics [3]. Because of the interest in equations of this "type" and the lack of analytic solutions of these non-linear equations it is desirable to develop a numerical technique for the solution of such equations.

In this paper we discuss an implicit numerical scheme for the solution of equations of the type (1.1-2). After transforming the equations into dimensionless form, we change the variables and recombine the equations in a manner that would lead to the characteristic form if the third derivative term were not present in the momentum equation. The initial and boundary conditions assumed to be appropriate for this system are those required by the shallow water theory.

Experiments which were performed on an IBM 704 digital computer for an initial boundary-value problem indicated that numerical errors grew from the left boundary, but that the scheme is stable away from the boundary. In order to overcome this instability we modified the difference equation at the boundary using the conservation equation to correct the height in an implicit fashion, i.e. at the advanced time step. The instability from the left boundary then occurred one point to the right i.e. by using the conservation of mass equation at the left boundary we had moved the cause of the perturbation to the right. We then used the implicitly differenced conservation of mass equation at several points adjacent to the left boundary. The start of the instability moved correspondingly to the right. We finally eliminated the instability by using this conservation equation for all of the points. That is, we used an explicit scheme to get a first estimate of the unknown, but then we corrected this estimate with an implicit scheme.

With this method some numerical solutions of the propagation of a hump of fluid into a channel of constant depth were obtained on the IBM 704. These solutions indicate that the hump of fluid does not steepen but spreads in a manner found in an analytic approximation by Gardner and Morikawa [3] for a similar problem.

## 2. The Equations in "characteristic" form.

If the third derivative term were missing from the momentum equation (1.2), we would have the usual hyperbolic system of shallow water theory. Since the third derivative term arises from an attempt to make a small correction to the shallow water theory, and since the characteristic form is very useful in the discussion of the shallow water theory, we will cast the equations (1.1-2) into a form which would be the characteristic form if the third derivative term was not present.

First let us transform to dimensionless variables by the following transformations

$$\frac{1}{h}\ (\eta+h) \longrightarrow h \tag{2.1}$$

$$\frac{1}{\sqrt{gh}}\ u \longrightarrow u \tag{2.2}$$

$$\frac{1}{\sqrt{3}}\ \frac{x}{h} \longrightarrow x \tag{2.3}$$

$$\sqrt{\frac{3g}{h}}\ t \longrightarrow t \quad . \tag{2.4}$$

With these transformations equations (1.1-2) become

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{2.5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} + \frac{\partial^3 h}{\partial x \partial t^2} = 0 \quad . \tag{2.6}$$

By using the conservation equation (2.5) we can change the third derivative term to $-\frac{\partial^3 (uh)}{\partial x^2 \partial t}$, a third derivative term involving only one time differentiation.

We now let

$$c^2 = h \tag{2.7}$$

and obtain from (2.5-6) a form similar to the characteristic form [5]

$$\left\{\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right\}(u+2c) - \frac{\partial^3 (uc^2)}{\partial x^2 \partial t} = 0 \tag{2.8}$$

$$\left\{\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right\}(u-2c) - \frac{\partial^3 (uc^2)}{\partial x^2 \partial t} = 0 \quad . \tag{2.9}$$

Even though the third derivative term may physically give a small correction to the usual shallow water equations, a possible stable difference approximation will have to treat the third derivative term as not being small, and hence we will have to use the third derivative term in solving for the unknown quantities at the advanced time step.

Since both variables appear naturally combined in the third derivative term, we define

(2.10) $$q = u\, c^2$$

called the discharge ($q$ is a measure of the volume of water passing a given point per unit of time). We then have

(2.11) $$\left\{\frac{\partial}{\partial t} + \left(\frac{q}{c^2} + c\right)\frac{\partial}{\partial x}\right\}\left(\frac{q}{c^2} + 2c\right) - \frac{\partial^3 q}{\partial x^2 \partial t} = 0$$

(2.12) $$\left\{\frac{\partial}{\partial t} + \left(\frac{q}{c^2} - c\right)\frac{\partial}{\partial x}\right\}\left(\frac{q}{c^2} - 2c\right) - \frac{\partial^3 q}{\partial x^2 \partial t} = 0$$

If the third derivative term were absent, we would have to prescribe the appropriate initial conditions for a solution in the form

(2.13) (I.C.) $q(x,0) = \text{given}$ , $x(x,0) = \text{given}$ ;

and if we had a semi-infinite channel, $0 \le x < \infty$ we would need only specify one of the two unknowns on the boundary $x = 0$ (under normal circumstances in which $u+c > 0$ , $u-c < 0$ at $x = 0$ cf. [5]). We find it convenient to prescribe the discharge at $x = 0$

(2.14) (B.C) $q(0,t) = \text{given}$ .

Since we have a third derivative term we might need additional conditions. In the form given by (2.11-12) the spatial derivative is raised by one order over that of the usual shallow water theory, while the time derivative is still of first order. This observation suggests that only an extra boundary condition might be required. It seems reasonable to take the extra boundary condition at $x = \infty$. We will be interested in creating a wave motion at $x = 0$ and in observing the motion of the wave towards $x = \infty$. We will not wish any disturbances to arise from $x = \infty$ and hence we will assume that for fixed $t \geq 0$,

(2.15) (B.C.) $\lim_{x \to \infty} q(x,t) = q(\infty, 0)$, and $\lim_{x \to \infty} q_x(x,t) = 0$.

3. The Finite Difference Scheme

We will now consider a finite difference approximation to the "characteristic type" equations (2.11-12) which needs only the initial and boundary conditions given by (2.13-15). Two considerations motivate our choice of difference approximation:

(i) if the third derivative term were not present, the difference approximation should be a known stable approximation to the shallow water theory equations,

(ii) the third derivative term should be used appropriately to advance the $q$.

We consider a mesh such that the spatial and time mesh points are designated as $x_j$, $j = 0,1,2,\ldots$, $t_n$, $n = 0,1,2,\ldots$, with $x_0 = 0$ as the left boundary and $t_0 = 0$ as the initial time. The unknowns $q(x,t)$ and $c(x,t)$ at the mesh points will be written with capitals and subscripts:

(3.1) $$q(x_j, t_n) = Q_{jn}$$

(3.2) $$c(x_j, t_n) = C_{jn} \quad .$$

In order to have the difference approximations of (2.11-12) reduce to the shallow water difference approximations when the third derivative term is absent we will take backward space differences for the $\frac{\partial}{\partial x}\left(\frac{q}{c^2} + 2c\right)$ term in (2.11) and forward space differences for the $\frac{\partial}{\partial x}\left(\frac{q}{c^2} - 2c\right)$ term in (2.12). This difference approximation for the shallow water equations is known to be stable (since $u+c > 0$, $u-c < 0$) cf. [2]. To treat the third derivative term, we difference it forward in time and in a centered fashion in the spatial direction.

Following out this difference approximation, we obtain from (2.11-12)

(3.3) $$\frac{1}{\Delta t}\left(\frac{Q_{j\,n+1}}{C^2_{j\,n+1}} + 2\,C_{j\,n+1} - \frac{Q_{jn}}{C^2_{jn}} - 2\,C_{jn}\right)$$
$$+ \left(\frac{Q_{jn}}{C^2_{jn}} + C_{jn}\right)\frac{1}{\Delta x}\left(\frac{Q_{jn}}{C^2_{jn}} + 2\,C_{jn} - \frac{Q_{j-1n}}{C^2_{j-1n}} - 2\,C_{j-1n}\right)$$
$$- \frac{1}{(\Delta x)^2 \Delta t}\left(Q_{j+1\,n+1} - 2\,Q_{jn+1} + Q_{j-1\,n+1}\right.$$
$$\left. - Q_{j+1n} + 2\,Q_{jn} - Q_{j-1n}\right) = 0 \; .$$

(3.4) $$\frac{1}{\Delta t}\left(\frac{Q_{j\,n+1}}{C^2_{jn}} - 2\,C_{j\,n+1} - \frac{Q_{jn}}{C^2_{jn}} + 2\,C_{jn}\right)$$
$$+ \left(\frac{Q_{jn}}{C^2_{jn}} - C_{jn}\right)\frac{1}{\Delta x}\left(\frac{Q_{j+1n}}{C^2_{j+1n}} - 2\,C_{j+1n} - \frac{Q_{jn}}{C^2_{jn}} + 2\,C_{jn}\right)$$
$$- \frac{1}{(\Delta x)^2 \Delta t}\left(Q_{j+1\,n+1} - 2\,Q_{jn+1} + Q_{j-1\,n+1}\right.$$
$$\left. - Q_{j+1n} + 2\,Q_{jn} - Q_{j-1n}\right) = 0 \; .$$

where $\Delta x = x_{j+1} - x_j$ and $\Delta t = t_{n+1} - t_n$. We will hold $\Delta x$ and $\Delta t$ constant during a given calculation. The equations (3.3-4) hold for all $j \geq 1$ and all $n \geq 0$. For $n = 0$ we know $Q_{j0}$ and $C_{j0}$ from the initial conditions (2.13). For $j = 0$ we know $Q_{0n}$ from the boundary condition (2.14), and for $j$ $(=j_{max})$ taking on its maximum value $j = j_{max}$ we know $Q_{j_{max}n}$ from the boundary condition (2.15). That is, we approximate the infinite channel by a suitably long channel (the length of the channel increases as the calculation proceeds).

Upon subtracting (3.4) from (3.3), we easily solve for $C_{j\,n+1}$ in the form:

$$(3.5)\quad C_{j\,n+1} = C_{jn} + \frac{\Delta t}{4\Delta x}\left\{\frac{Q_{jn}}{C^2_{jn}}\left[\frac{Q_{j+1n}}{C^2_{j+1n}} - 2\frac{Q_{jn}}{C^2_{jn}} + \frac{Q_{j-1n}}{C^2_{jn}} - 2(C_{j+1,n} - C_{j-1,n})\right] + C_{jn}\left[-\left(\frac{Q_{j+1n}}{C^2_{j+1n}} - \frac{Q_{j-1n}}{C^2_{j-1n}}\right) + 2\left(C_{j+1n} - 2C_{jn} + C_{j-1n}\right)\right]\right\} = 0$$

By adding (3.3-4), we find after some rearranging,

$$(3.6)\quad \frac{1}{(\Delta x)^2}Q_{j+1\,n+1} + \left(-\frac{1}{C^2_{j\,n+1}} - \frac{2}{(\Delta x)^2}\right)Q_{j\,n+1} + \frac{1}{(\Delta x)^2}Q_{j-1\,n+1}$$

$$= -\frac{Q_{jn}}{C^2_{jn}} - \frac{\Delta t}{2\Delta x}\frac{Q_{jn}}{C^2_{jn}}\left[-\left(\frac{Q_{j+1n}}{C^2_{j+1n}} - \frac{Q_{j-1n}}{C^2_{j-1n}}\right) + 2(C_{j+1n} - 2C_{jn} + C_{j-1n})\right]$$

$$+ C_{jn}\left[\left(\frac{Q_{j+1n}}{C^2_{j+1n}} - 2\frac{Q_{jn}}{C^2_{jn}} + \frac{Q_{j-1n}}{C^2_{j-1n}}\right) - 2(C_{j+1n} - C_{j-1n})\right]$$

$$+ \frac{1}{(\Delta x)^2}(Q_{j+1n} - 2Q_{jn} + Q_{j-1n}),$$

the equations for $Q_{j\,n+1}$. In equations (3.6) the unknowns

$Q_{j\ n+1}$ (for $j = 1,2,\ldots,(j_{max}-1)$) appear in triplets at the advanced time $t_{n+1}$ . The solution of this tridiagonal system can be easily given by well-known and well-conditioned recursion formulae.

We need one more equation to complete the difference scheme: an equation to determine $C_{0n+1}$ for we have assumed that only $Q_{0n+1}$ is given on the left boundary and the difference equation (3.5) can be solved for the unknowns $C_{j\ n+1}$ for $j = 1,2,\ldots,j_{max}$ . In order to determine $C_{0\ n+1}$ we use the conservation equation (2.5) with the notation (2.7) and take a one sided space difference, to find

$$C_{0\ n+1} = \left\{ C_{0n}^2 - \frac{t}{\Delta x}\,(q_{1n} - q_{0n}) \right\}^{1/2} \qquad (3.7)$$

We recapitulate our algorithm for solving the difference equations (3.5-7) as follows:

(i) We use (3.5) to obtain $C_{jn+1}$ ($j \geq 1$, $n \geq 0$) (which can be done since all the quantities on the right side of (3.5) are presumably known at the previous time $t_n$) . Since the right boundary is at infinity, we solve for $C_{jn+1}$ for $j$ increasing until $C_{jn+1} = C_{jn}$ to within some preset tolerance and define $j_{max}$ to be this last value of $j$ .

(ii) We use (3.7) to obtain $C_{0n+1}$ .

(iii) We solve the coupled system of equations (3.6) for $Q_{jn+1}$ . The boundary conditions on the left and right are just enough for the solution of the coupled equations (3.6). For the semi-infinite channel the right boundary is considered to be that point for which $C_{jn+1} = C_{jn}$ to within the accuracy of the calculation.

Numerical calculations for large discharges at the left boundary indicated an instability near the left boundary and no instability away from the left boundary. Since the instability showed up in $C_{jn}$ with negligible effect on $Q_{jn}$, we were led to a further implicit type correction for $C_{jn}$. The implicit type correction calculation for $C_{jn}$ that we employ uses the values of the discharge at both the advanced and present time step. We accomplish the desired implicitness by differencing the conservation equation (2.5) as follows:

$$(3.8) \quad C_{jn+1} = \left\{ C_{jn}^2 - \frac{\Delta t}{4\Delta x} (Q_{j+1\,n+1} - Q_{j-1\,n+1} + Q_{j+1n} - Q_{j-1n}) \right\}^{1/2} .$$

For the boundary we take a one sided space difference:

$$(3.9) \quad C_{On+1} = \left\{ C_{On} - \frac{\Delta t}{2\Delta x} (Q_{1n+1} - Q_{On+1} + Q_{1n} - Q_{On}) \right\}^{1/2} .$$

Our procedure is as follows: (i) we solve for $C_{jn+1}$ and $Q_{jn+1}$ as before from (3.5-7); (ii) we correct $C_{jn+1}$ using (3.8-9) which requires the discharge at the advanced timestep. We correct $C_{jn+1}$ using (3.8) until we reach the right boundary or until $C_{jn+1} = C_{jn}$ to within the accuracy of the calculation. The correction to $C_{jn+1}$ was carried out far to the right because numerical experiments showed that an instability appeared at that value of $x_j$ which was the last corrected $C_{jn+1}$ obtained by using (3.8). When the correction was carried far to the right, no instability appeared.

## 4. Some Numerical Results.

A series of numerical calculations were performed on an

IBM 704 using the difference scheme described above. In these calculations fluid was introduced on the left hand boundary while the initial flow of fluid was constant. The discharge $Q_{On}$ on the left hand boundary was of the form

(4.10) $$Q_{On} = \alpha - \beta \cos (\pi \omega n \Delta t) \qquad n \leq \frac{1}{\omega \Delta t}$$

$$= \alpha + \beta \qquad n > \frac{1}{\omega \Delta t} \;;$$

the initial conditions were

(4.2) $$Q_{jO} = \alpha - \beta$$

and

(4.3) $$C_{jO} = 1$$

and the right boundary was taken at infinity.

It was the intention of these calculations to see if fluid introduced on the left boundary with a sufficiently large discharge $Q_{On}$ would cause a wave to travel down the channel and break. In all our calculations we did not find any evidence of breaking; even when the free stream discharge was negative (i.e. fluid traveling toward the left) we did not obtain breaking. In each of the cases breaking would have occurred long before our calculations were completed according to the usual shallow water theory (c.f. Stoker [1], pp. 352-357).

In Figure 2 we see the build up of $C_{jn}$ and its development in time for the case in which $\alpha + \beta = 1.5$ and $\alpha - \beta = 1$. As the wave of fluid traveled down stream it spread. After the wave developed, the half way point between the maximum height and the free stream height traveled at a constant speed $\sigma_c$ . The computed speed $\sigma_c$ agreed very closely with the speed of propagation $\sigma$ given by Stoker [cf. Ref. 1, pp. 321-323] for a bore with height equal to the maximum height in back and free stream height in front. In Figure 3 we have plotted the wave profiles against $x - \sigma t$ . These profiles show a decided spread very similar to that found by Gardner and Morikawa [3] for their equation; i.e. $c(x,t) = f(\frac{x-\sigma t}{t^{1/3}})$ . No solutions of this type have been found analytically for equations (2.5-6).

In Figure 4 we show the profiles for a wave arising from a large discharge $Q_{0n}$ at the left. Again no breaking was observed.

In Figure 5 we show the profiles for a wave arising from a large inflow discharge $Q_0n$ at the left when the free stream condition corresponds to fluid flowing to the left. We observed no breaking and the wave continued to grow in height.

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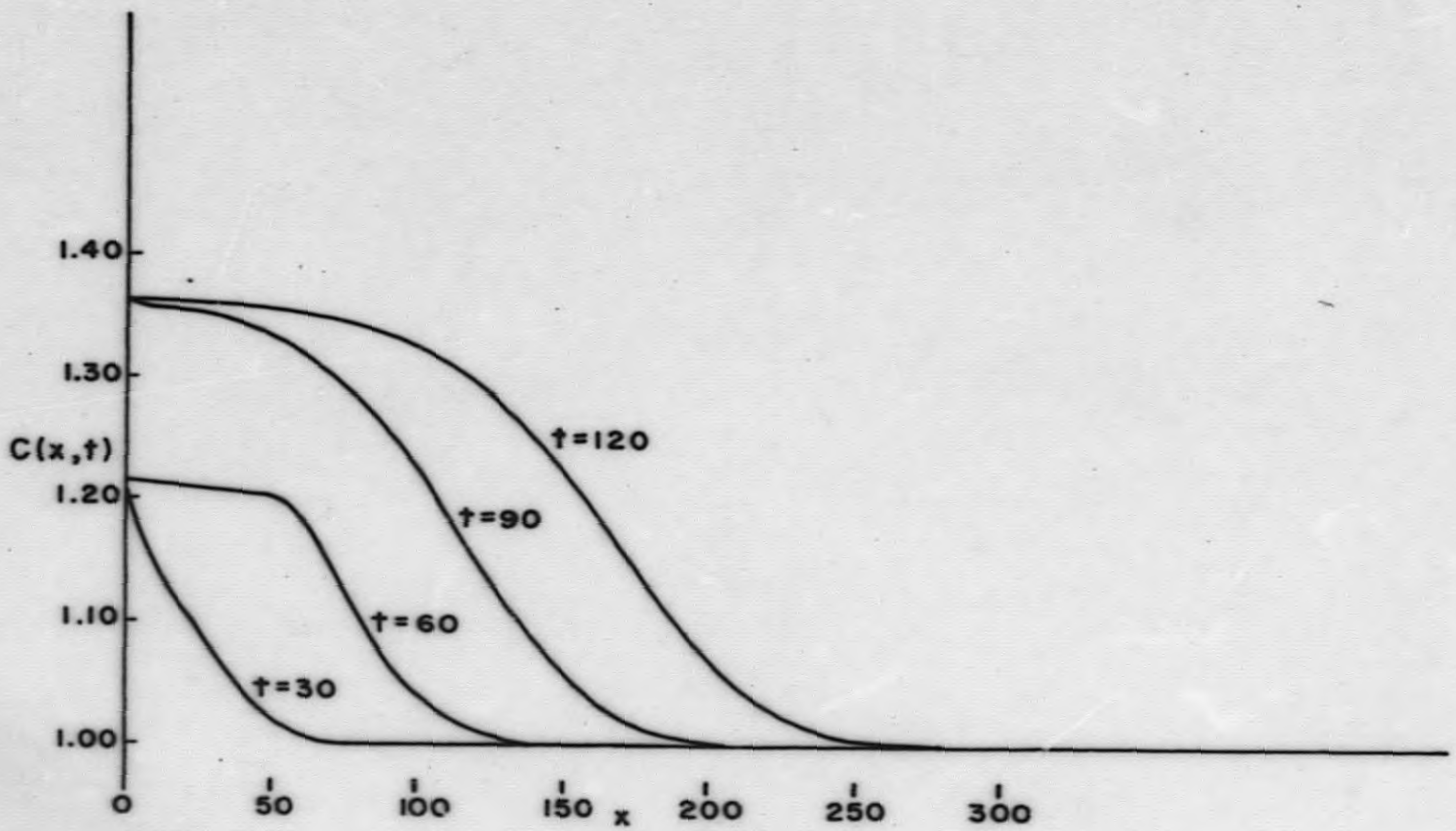
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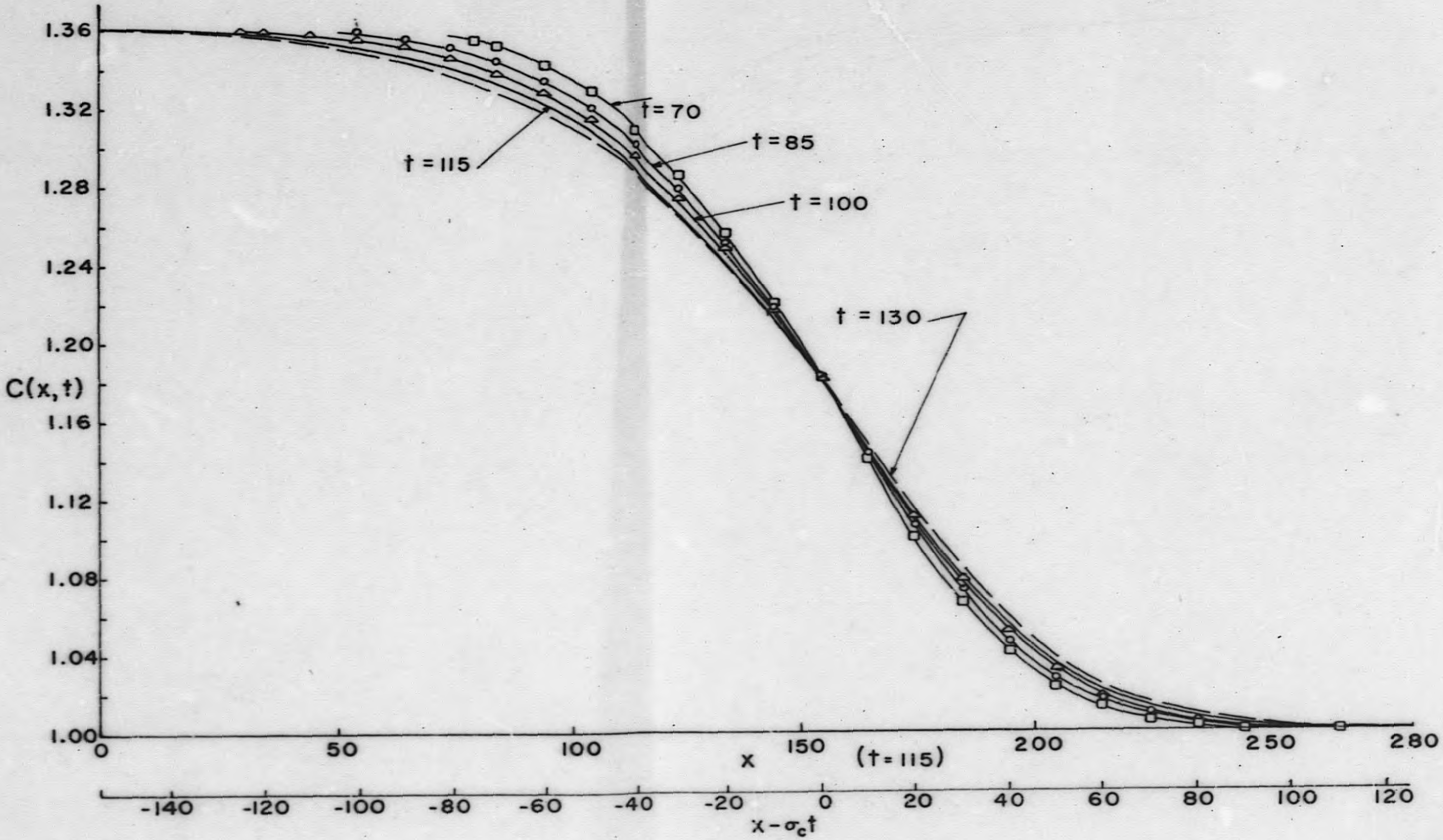
1.30
C(x,t)
1.20
1.10
1.00
t=120
t=90
t=60
t=30
0
50
100
150
x
200
250
300

Figure 2



1.36
1.32
1.28
1.24
1.20
C(x,t)
1.16
1.12
1.08
1.04
1.00
t = 70
t = 85
t = 100
t = 115
t = 130
0
50
100
150
200
250
280
x
(t = 115)
-140
-120
-100
-80
-60
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-20
0
20
40
60
80
100
120
$x - \sigma_c t$


Figure 3

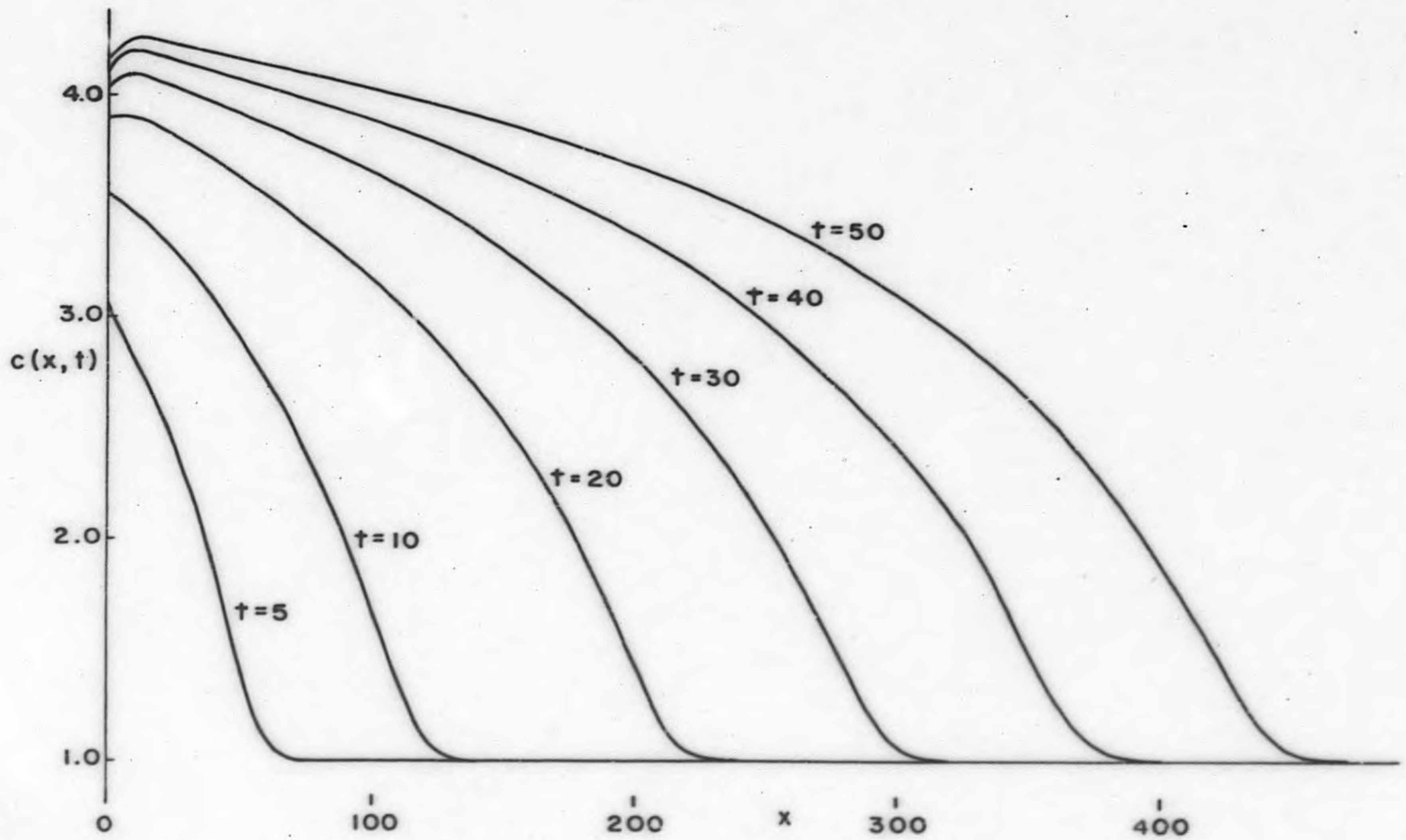
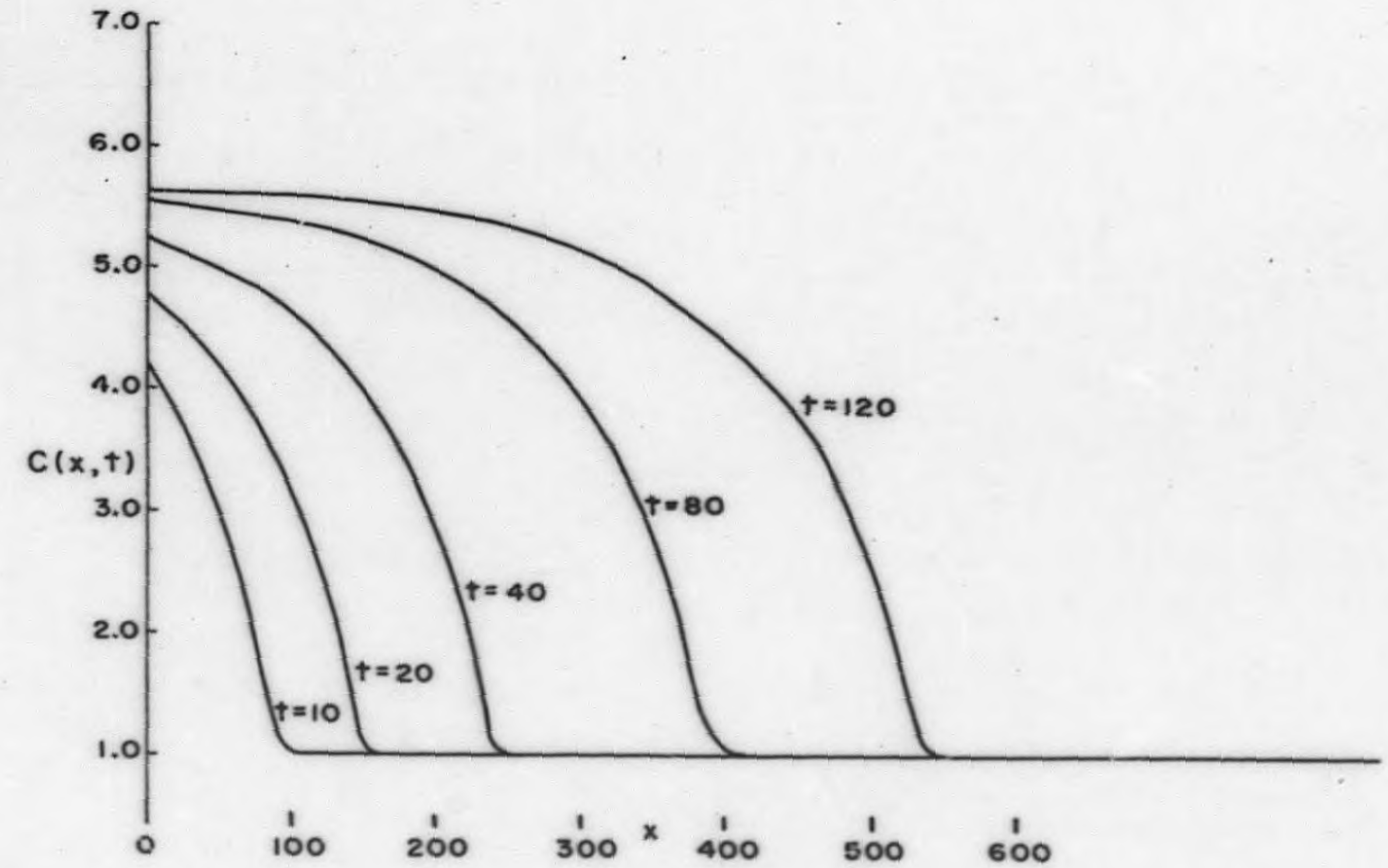



Figure 4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C(x,t)
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500
600
t=10
t=20
t=40
t=80
t=120

Figure 5