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CONICAL REFRACTION IN CRYSTAL OPTICS AND HYDROMAGNETICS

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ABSTRACT

When light propagates with the wave normal in the direction of an optic axis of a biaxial crystal, the usual ray theory breaks down. This phenomenon can be analyzed by means of an asymptotic solution of Maxwell's equations. The intensity is governed by a partial differential equation within the phase surfaces, instead of ordinary differential equations along rays. This example shows that light does not always propagate along rays. A similar phenomenon occurs in hydromagnetics.

TABLE OF CONTENTS

CONICAL REFRACTION IN CRYSTAL OPTICS AND HYDROMAGNETICS

1. Introduction.

The phenomenon of conical refraction is an important exception to the usual laws of geometrical optics. In the case of conical refraction, a single ray, upon striking an interface, splits into a cone of rays (Figure 1). The energy of the corresponding wave is usually thought of as being transported along bundles of rays. When conical refraction takes place, a bundle of rays splits into a conical shell of rays (Figures 2 and 3). Nevertheless, the energy contained within the original bundle is spread over the entire region bounded by the conical shell, not merely the shell itself (as one might believe from the usual principles of geometrical optics).

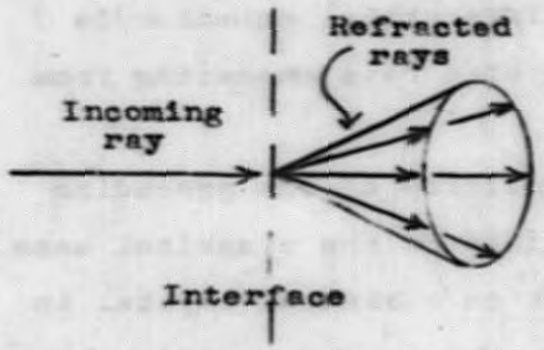

Figure 1

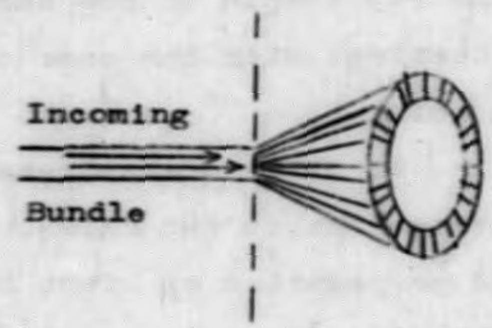

Figure 2

The shaded area is
the locus of
endpoints of
refracted rays.

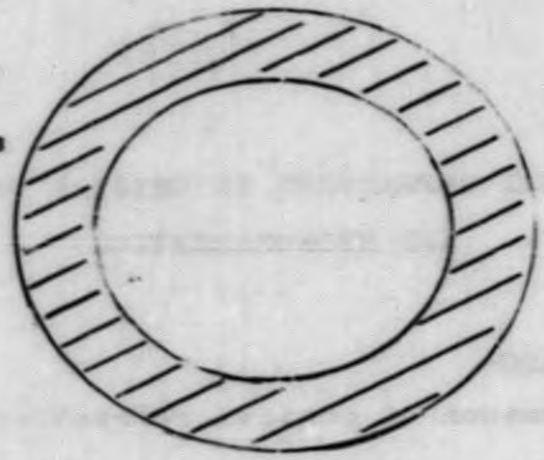

Figure 3

This means that the first term of the asymptotic expansion of geometrical optics is different from zero not only in the shaded annulus shown in Figure 3, but in the entire disc bounded by the outer circle. The mathematical explanation for this phenomenon is that the amplitude of the refracted wave is not governed by ordinary differential equations (as is usual in geometrical optics), but by a hyperbolic partial differential equation within the phase surfaces of the refracted wave. The ray conoid of the partial differential equation is identical with the cone of refracted rays emanating from a point.

We shall demonstrate the validity of the preceding statements in two cases. The first is the classical case of propagation of light incident on a biaxial crystal in the direction of an optic axis. The second case involves the propagation of a small hydromagnetic discontinuity across a plane normal to the magnetic field, in the special case where the Alfven and sound speeds are equal. In harmony with the general rule that the first term in an asymptotic expansion is easier to compute than the exact solution, each of these problems can be reduced to the solution of the wave equation with three independent variables.

An example, given at the end of the following section, shows that if the incoming wave vanishes except on a small bundle of rays, then almost all of the energy is transported along the refracted rays.

It should be pointed out that the following analysis applies only to propagation with the wave normal pointing in a specific direction, i.e. along an optic axis. For neighboring directions of the wave normal, ordinary geometrical optics applies. Thus there is a change in the asymptotic behavior of solutions when the wave normal is near a conical point of the normal surface. This difficulty in the theory has not been satisfactorily resolved, and hinders comparison with experiments.

2. Crystal Optics.

Consider the following situation: we have a biaxial crystal[1] cut in a plane normal to an optic axis. There

1/ The equations of crystal optics are discussed in [2].

is an incoming plane wave which is normal to the surface of the crystal, i.e. the incoming wave propagates in the direction of the optic axis. Let the x-axis be an optic axis, and the y,z plane be the surface of the crystal.

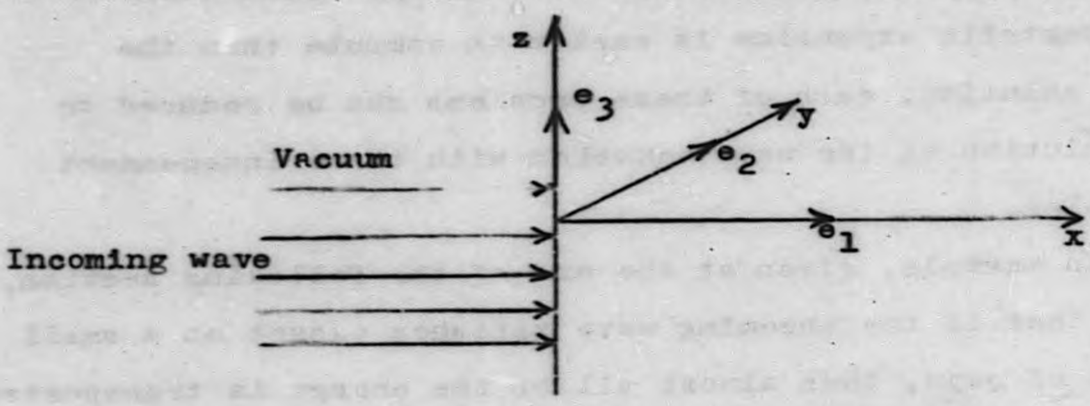

Figure 4

Let e_1', e_2' and e_3' be unit vectors along the principal axes of the dieletric tensor of the crystal, and let ε_1, ε_2 and ε_3 be the corresponding eigenvalues. We assume that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$. Let μ be the magnetic permeability. We shall make use of the following abbreviations:

$$\delta_i = \frac{1}{\mu\ \varepsilon_i} \quad , \qquad (i = 1,2,3) ,$$

$$\phi_1 = \sqrt{\frac{\delta_2-\delta_1}{\delta_3-\delta_1}} \quad ,$$

$$\phi_3 = \sqrt{\frac{\delta_3-\delta_2}{\delta_3-\delta_1}} \quad ,$$

$$\gamma = \frac{1}{2\sqrt{\delta_2}} \sqrt{(\delta_2-\delta_1)(\delta_3-\delta_2)} \quad ,$$

Let e_1, e_2 and e_3 be unit vectors in the direction of the x,y and z axes, given by the following equations:

$$e_1 = -\phi_1 e_1' - \phi_3 e_3' ,$$

$$e_2 = -e_2' ,$$

$$e_3 = -\phi_3 e_1' + \phi_1 e_3' .$$

These equations fix the orientation of the principal axes of the crystal relative to the x,y and z axes.

We assume that the incoming wave has the following form:

$$(1) \qquad \begin{aligned} H &= \bar{H}(y,z)\, e^{i\omega(t-x)} + O(\tfrac{1}{\omega}) \\ E &= \bar{E}(y,z)\, e^{i\omega(t-x)} + O(\tfrac{1}{\omega}) . \end{aligned} \qquad (x < 0)$$

Here the vectors H and E represent the magnetic and electric fields; the polarization and intensity may depend upon y and z, ω represents the frequency, and the terms of order $1/\omega$ indicate that we are interested only in the first term in an asymptotic expansion for large ω. The unit of time is chosen so that the velocity of light in vacuum is unity.

The boundary conditions on the surface of the crystal enable us to prescirbe H and E for $x = 0$.

In order to calculate H and E for $x > 0$, we apply the theory of asymptotic solutions of partial differential equations (outlined in Section 4) to Maxwell's equations. Accordingly, we have

$$
(2) \qquad \begin{aligned} H &= \bar{H}(x,y,z) \exp\left(i\omega\left(t - \frac{x}{\sqrt{\delta_2}}\right)\right) + 0\left(\frac{1}{\omega}\right) \\ E &= \bar{E}(x,y,z) \exp\left(i\omega\left(t - \frac{x}{\sqrt{\delta_2}}\right)\right) + 0\left(\frac{1}{\omega}\right) \quad , \end{aligned}
$$

for $x > 0$. $\bar{H}$ and $\bar{E}$ must each be a linear combination of two fixed vectors:

$$
(3) \qquad \begin{aligned} \bar{H}(x,y,z) &= \sigma^1(x,y,z)\, H^{(1)} + \sigma^2(x,y,z)\, H^{(2)} \\ \bar{E}(x,y,z) &= \sigma^1(x,y,z)\, E^{(1)} + \sigma^2(x,y,z)\, E^{(2)} \ . \end{aligned}
$$

Here

$$
(4) \qquad \begin{aligned} H^{(1)} &= e_3 \ , \\ E^{(1)} &= \mu\sqrt{\delta_2}\; e_2 \ ; \\ H^{(2)} &= e_2 \ , \\ E^{(2)} &= 2\gamma\mu\, e_1 - \mu\sqrt{\delta_2}\; e_3 \ . \end{aligned}
$$

The scalars σ^1 and σ^2 must satisfy the following system of equations:

$$
(5) \qquad \begin{aligned} \sqrt{\delta_2}\,\frac{\partial\sigma^1}{\partial x} - \gamma\,\frac{\partial\sigma^2}{\partial y} &= 0 \\ -\gamma\,\frac{\partial\sigma^1}{\partial y} + \sqrt{\delta_2}\,\frac{\partial\sigma^2}{\partial x} + 2\gamma\,\frac{\partial\sigma^2}{\partial z} &= 0 \ . \end{aligned}
$$

These equations are equivalent to the wave equation with three independent variables. We can see this most easily by introducing new variables ξ and ζ :

$$x = \sqrt{\delta_2}\, \xi$$

$$z = \gamma \xi + \zeta .$$

Then we have

$$(6) \qquad \begin{aligned} &\left(\frac{\partial}{\partial \xi} - \gamma \frac{\partial}{\partial \zeta}\right) \sigma^1 - \gamma \frac{\partial \sigma^2}{\partial y} = 0 , \\ &- \gamma \frac{\partial \sigma^1}{\partial y} + \left(\frac{\partial}{\partial \xi} + \gamma \frac{\partial}{\partial \zeta}\right) \sigma^2 = 0 ; \end{aligned}$$

hence

$$\left[\frac{\partial^2}{\partial \xi^2} - \gamma^2 \left(\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial y^2}\right)\right] \sigma^1 = 0 ,$$

$$\left[\frac{\partial^2}{\partial \xi^2} - \gamma^2 \left(\frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial y^2}\right)\right] \sigma^2 = 0 .$$

Thus we see that the first term in the asymptotic expansion of the field inside the crystal is determined by a solution of the wave equation in three independent variables, where the optic axis takes the place of the time axis. Since Huygens' principle does not hold for this equation, we conclude that propagation along rays cannot take place. For example, if $\overline{H}(0,y,z)$ and $\overline{E}(0,y,z)$ vanish outside a small neighborhood of the origin, then $\overline{H}(x,y,z)$ and $\overline{E}(x,y,z)$ will vanish outside a neighborhood of a certain cone, and in general will be different from zero inside the cone. The cone in question is the ray cone of (5), which has the equation

$$\frac{2\gamma}{\sqrt{\delta_2}} x z = z^2 + y^2 .$$

The optic axis is a generator of this cone, and the intersection of the plane $x = 1$ with the cone is a circle of radius $\gamma/\sqrt{\delta_2}$.

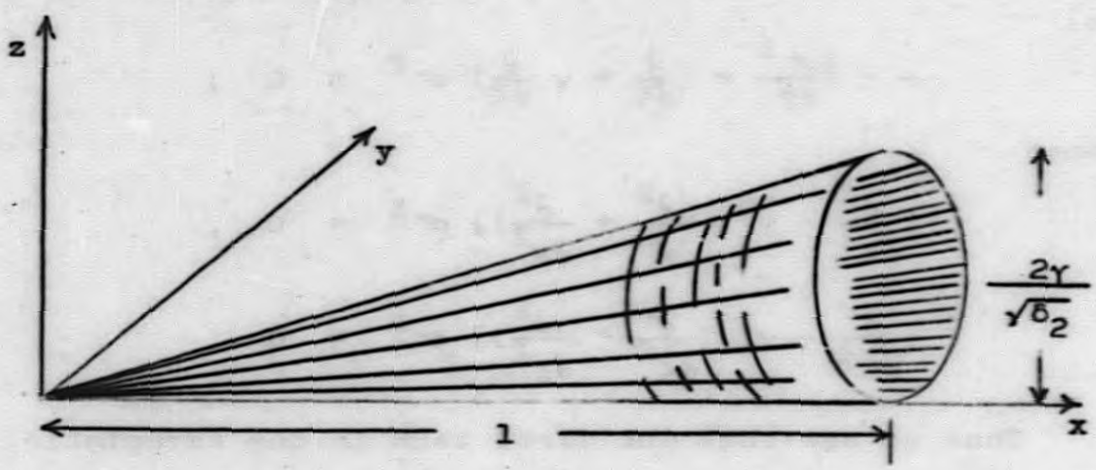

Figure 5

As a (hopefully) physically meaningful example, we shall calculate σ^1 and σ^2 in the interior of the cone of conical refraction, if there is a small source at the origin. The behavior of σ^1 and σ^2 along the refracted rays is more complicated, and depends upon the shape of the source, and the distribution of intensity within the source.

First we consider the corresponding problem for the wave equation.

<u>Lemma</u>: Let $g(a,b)$ be a continuous function such that

$$g(a,b) = 0 \quad \text{if} \quad a^2 + b^2 \geq 1 ,$$

$$\iint g(a,b)\, da\, db = 1 .$$

Let $u(\xi,\eta,\zeta)$ satisfy

$$u_{\xi\xi} = \gamma^2 (u_{\eta\eta} + u_{\zeta\zeta}) ,$$

$$u(0,\eta,\zeta) = 0$$

$$u_\xi(0,\eta,\zeta) = \frac{1}{\varepsilon} g(\frac{\eta}{\varepsilon}, \frac{\zeta}{\varepsilon}) .$$

Then 1. $\iint (u_\xi^2 + u_\eta^2 + u_\zeta^2)\, d\eta\, d\zeta$ is independent of ε and ξ, and

2. If $\gamma^2 \xi^2 - (\eta^2 + \zeta^2) > a > 0$,

then

$$u(\xi,\eta,\zeta) = \frac{\varepsilon}{2\pi\gamma^2} [\xi^2 - (\eta^2+\zeta^2)/\gamma^2]^{-\frac{1}{2}} + O(\varepsilon^2) .$$

<u>Proof</u>: First we note that, for $\xi = 0$,

(7) $$\iint (u_\xi^2+u_\eta^2+u_\zeta^2)\, d\eta\, d\zeta = \iint g^2(a,b)\, da\, db .$$

Thus the energy of the initial data is independent of ε. But, for solutions of the wave equation, the total energy is independent of ξ. This proves our first assertion.

In order to prove the second assertion, we represent u in terms of a fundamental solution:

$$u(\xi,\eta,\zeta) = \frac{1}{2\pi\varepsilon\gamma^2}\iint \frac{g(\frac{y}{\varepsilon},\frac{z}{\varepsilon})\,dy\,dz}{\sqrt{\xi^2-[(\eta-y)^2+(\zeta-z)^2]/\gamma^2}} ,$$

$$u(\xi,\eta,\zeta) = \frac{\varepsilon}{2\pi\gamma^2}\iint \frac{g(a,b)\,da\,db}{[\xi^2-(\eta^2+\zeta^2)\gamma^{-2}+2\varepsilon\gamma^{-2}(\xi a+\eta b)+\varepsilon^2\gamma^{-2}(a^2+b^2)]^{1/2}}$$

Expanding the integrand with respect to ε, we obtain (7).

Now we can use our lemma to obtain approximate solutions of (5), or (6). First let the data for σ^1 and σ^2 have the form

$$\sigma^1(0,\eta,\zeta) = \frac{1}{\varepsilon}\, g\,(\tfrac{\eta}{\varepsilon},\tfrac{\zeta}{\varepsilon}) ,$$

$$\sigma^2(0,\eta,\zeta) = 0 ,$$

where $g(a,b)$ is the function to which the lemma refers, and $\eta = y$. Then we may set

$$\sigma^1 = (\frac{\partial}{\partial\xi} + \gamma\frac{\partial}{\partial\zeta})\, u\,(\xi,\eta,\zeta) ,$$

$$\sigma^2 = \gamma\frac{\partial}{\partial\eta}\, u\,(\xi,\eta,\zeta) .$$

Hence

$$(8)\qquad \begin{cases} \sigma^1 = -\dfrac{\varepsilon}{2\pi}\dfrac{z}{\Gamma^{3/2}} + O(\varepsilon^2) , \\[2ex] \sigma^2 = \dfrac{\varepsilon}{2\pi}\dfrac{y}{\Gamma^{3/2}} + O(\varepsilon^2) , \end{cases}$$

where

$$\Gamma(x,y,z) = \frac{2\gamma}{\sqrt{\delta_2}} xz - z^2 - y^2 .$$

Similarly, if

$$\sigma^1(0,\eta,\zeta) = 0$$

$$\sigma^2(0,\eta,\zeta) = \frac{1}{\varepsilon} g\left(\frac{\eta}{\varepsilon},\frac{\zeta}{\varepsilon}\right) ,$$

then

$$(9) \quad \begin{cases} \sigma^1 = \dfrac{\varepsilon}{2\pi} \dfrac{y}{\Gamma^{3/2}} + O(\varepsilon^2) \\ \sigma^2 = \dfrac{\varepsilon}{2\pi} \dfrac{z-2\gamma\, \delta_2^{-1/2}\, x}{\Gamma^{3/2}} + O(\varepsilon^2) . \end{cases}$$

Combining (8) and (9), we obtain a general solution of (5).

We may interpret these results as follows: Consider a plane wave incident on a crystal in the direction of an optic axis, where the amplitude of the wave vanishes except on a bundle of rays of size ε. We assume that ε is large compared to the wavelength of the light, but small compared to the distance from the interface to the point of observation ($x \gg \varepsilon \gg 1/\omega$). If the intensity of the incoming wave is normalized so that the refracted wave carries unit energy, then the amplitude of the refracted wave is proportional to ε, inside the cone of conical refraction.

On the other hand, since the refracted wave carries unit energy, we conclude that the energy carried along and near the refracted rays is $1-O(\varepsilon^2)$. Thus, although the amplitude of the refracted wave is different from zero inside the cone of refracted rays, almost all of the energy is transported along rays.

3. Hydromagnetics.

In this section we shall apply the technique used for crystal optics to the linearized equations of hydromagnetics, and discuss a phenomenon of conical refraction exactly analogous to conical refraction in crystals.

We shall deal with the equations for small hydromagnetic disturbances[1], where the basic flow is at rest, with constant density ρ_o, and constant magnetic field vector H_o. The equations for the perturbations ρ, u and H are:

$$\frac{\partial \rho}{\partial t} + \rho_o \operatorname{div} u = 0 \ ,$$

$$\rho_o \frac{\partial u}{\partial t} + a^2 \operatorname{grad} \rho - \mu (\operatorname{curl} H) \times H_o = 0 \ ,$$

$$\frac{\partial H}{\partial t} - \operatorname{curl} (u \times H_o) = 0 \ .$$

Here a is the sound speed of the underlying material, and μ is the magnetic permeability. It will be convenient to put our equations in symmetric hyperbolic

[1] These equations are discussed in [1].

form. Let

$$d = \frac{a}{\rho_o} \rho \ ,$$

$$M = \sqrt{\frac{\mu}{\rho_o}}\, H \ ,$$

$$M_o = \sqrt{\frac{\mu}{\rho_o}}\, H_o \ .$$

Then our equations take the symmetric form

$$\begin{aligned} & d_t + a \operatorname{div} u = 0 \ , \\ (10) \qquad & u_t + a \operatorname{grad} d - (\operatorname{curl} M) \times M_o = 0 \ , \\ & M_t - \operatorname{curl} (u \times M_o) = 0 \ . \end{aligned}$$

We look for solutions of (10) of the form

$$(11) \qquad \begin{pmatrix} d \\ u \\ M \end{pmatrix} = h(t - \tfrac{x}{a}) \begin{pmatrix} \bar{d} \\ \bar{u} \\ \bar{M} \end{pmatrix} + \text{continuous function,}$$

where $h(\ )$ is the Heaviside step function. That is, the solution shall have a jump discontinuity which propagates with speed a in the direction of the x-axis. We make the additional assumption that H_o (and hence M_o) points in the direction of the x-axis, and that the Alfven speed is equal to the sound speed, i.e.

$$a = |M_o| \ .$$

The propagation of such discontinuities is covered by the asymptotic theory of partial differential equations outlined in Section 4. According to the theory we may represent the discontinuous part of the solution in the form

$$(12) \qquad \begin{pmatrix} \bar{d} \\ \bar{u} \\ \bar{M} \end{pmatrix} = \sum_{i=1}^{3} \sigma^{i}(x,y,z,t) \begin{pmatrix} d^{(i)} \\ u^{(i)} \\ M^{(i)} \end{pmatrix} .$$

where

$$\begin{pmatrix} d^{(1)} \\ u^{(1)} \\ M^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ e_2 \\ e_2 \end{pmatrix},$$

$$\begin{pmatrix} d^{(2)} \\ u^{(2)} \\ M^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ -e_3 \\ e_3 \end{pmatrix},$$

$$\begin{pmatrix} d^{(3)} \\ u^{(3)} \\ M^{(3)} \end{pmatrix} = \begin{pmatrix} 1 \\ e_1 \\ 0 \end{pmatrix}.$$

Here, as before, e_1, e_2 and e_3 are unit vectors along the x,y and z axes, respectively. The scalars σ^1, σ^2 and σ^3 must satisfy the following set of equations:

$$(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x})\sigma^1 + \frac{a}{2}\frac{\partial}{\partial y}\sigma^3 = 0$$

$$(13) \quad (\frac{\partial}{\partial t} + a\frac{\partial}{\partial x})\sigma^2 - \frac{a}{2}\frac{\partial}{\partial z}\sigma^3 = 0$$

$$\frac{a}{2}\frac{\partial}{\partial y}\sigma^1 - \frac{a}{2}\frac{\partial}{\partial z}\sigma^2 + (\frac{\partial}{\partial t} + a\frac{\partial}{\partial x})\sigma^3 = 0 \ .$$

If we introduce new variables ϕ, $\bar{x}$, y, t by means of the equations

$$t = \phi + \frac{\bar{x}}{a}$$

$$x = \bar{x}$$

$$y = y$$

$$z = z \ ,$$

then our equations take the form

$$\frac{\partial\sigma^1}{\partial\bar{x}} + \frac{1}{2}\frac{\partial\sigma^3}{\partial y} = 0 \ ,$$

$$(14) \quad \frac{\partial\sigma^2}{\partial\bar{x}} - \frac{1}{2}\frac{\partial\sigma^3}{\partial z} = 0 \ ,$$

$$\frac{1}{2}\frac{\partial\sigma^1}{\partial y} - \frac{1}{2}\frac{\partial\sigma^2}{\partial z} + \frac{\partial\sigma^3}{\partial\bar{x}} = 0 \ .$$

Thus all differentiation takes place within the planes ϕ = const. It is easy to see that σ^1, σ^2 and σ^3 satisfy the following equations:

$$\frac{\partial}{\partial \bar{x}} \left(\frac{\partial^2}{\partial \bar{x}^2} - \frac{1}{4}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right) \sigma^1 = 0 ,$$

$$\frac{\partial}{\partial \bar{x}} \left(\frac{\partial^2}{\partial \bar{x}^2} - \frac{1}{4}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right) \sigma^2 = 0 ,$$

$$\left(\frac{\partial^2}{\partial \bar{x}^2} - \frac{1}{4}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right) \sigma^3 = 0 .$$

Thus, as in the case of crystal optics, the first term in the expansion of the solution is essentially governed by the wave equation in three independent variables. The locus of a discontinuity initially localized at the origin will be the interior of a circular cone in x,y,z space.

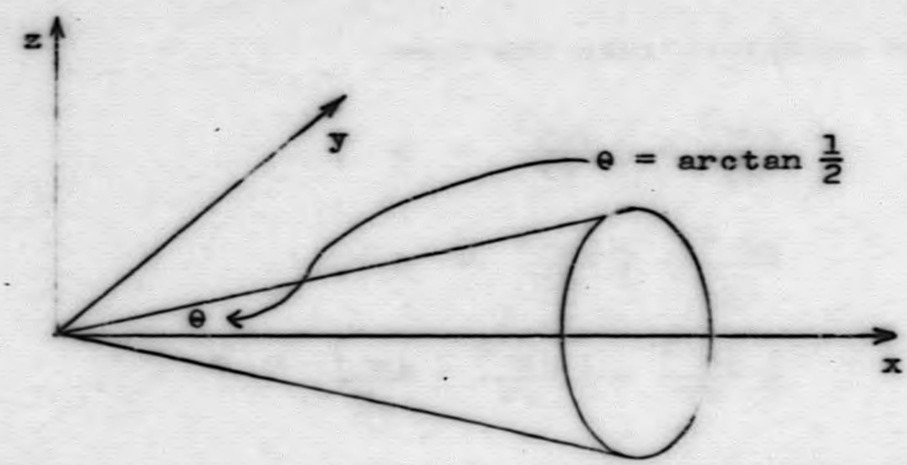

Figure 6

4. The Geometrical Optics Approximation.

In this section, we shall indicate how the systems of equations which govern the first term of an asymptotic expansion are derived. In the interests of brevity and clarity, we consider a general linear hyperbolic system of first order. This theory is presented in greater detail in [3].

Let $u(x^1,x^2,x^3,t)$ be a vector with k components, and consider the linear system

$$\mathcal{L}u \equiv u_t + \sum_{\nu=1}^{3} A^\nu \frac{\partial u}{\partial x^\nu} + Bu = 0 . \tag{15}$$

Here A^ν and B are $k \times k$ matrices. In the case of crystal optics, u represents the pair of vectors (E,H) and $\mathcal{L}u = 0$ corresponds to Maxwell's equations. In the hydromagnetic case, u represents the triple (d,u,M) and $\mathcal{L}u = 0$ corresponds to the system (10). We look for formal solutions of (15) of the form

$$u = \sum_{j=0}^{\infty} \frac{e^{i\omega\phi(x,t)}}{(i\omega)^j} a^j(x,t) . \tag{16}$$

For large values of the frequency ω, the first term in the expansion is a good approximation to the solution u; we write

$$u \sim e^{i\omega\phi(x,t)} a^o(x,t) .$$

Upon substitution of (16) into (15), we see that the following set of equations must be satisfied:

(17) $$A\, a^{o} = 0 \ ,$$

(18) $$A\, a^{j+1} + \mathcal{L}(a^{j}) = 0 \qquad (j = 0,1,\ldots) \ .$$

Here A is the characteristic matrix:

$$A = I\,\phi_t + \sum_{\nu=1}^{3} A^{\nu} \frac{\partial\phi}{\partial x^{\nu}} \ .$$

Equation (17) implies that the determinant of A must vanish; hence $\phi(x,t)$ must satisfy the characteristic equation.

We assume that, for fixed ϕ, there are exactly r linearly independent solutions of equation (17):

$$A\, R^{(i)} = 0 \qquad (i = 1,\ldots,r) \ .$$

Then we must have

(19) $$a^{o}(x,t) = \sum_{i=1}^{r} \sigma^{i}(x,t)\, R^{(i)} \ ,$$

where σ^{i} are scalar factors. There are also r linearly independent vectors $L^{(i)}$ such that

$$L^{(i)} A = 0 \qquad (i = 1,\ldots,r) \ .$$

Now consider equation (18), setting $j = 0$. A necessary and sufficient condition for this equation to have a solution is that

$$L^{(j)} \mathcal{L}(a^{o}) = 0 \qquad (j = 1,\ldots,r) ,$$

or

$$(20) \qquad L^{(j)} \mathcal{L}\left(\sum_{i=1}^{r} \sigma^{i}(x,t)\, R^{(i)} \right) = 0 \quad (j = 1,\ldots,r) .$$

The system (20) corresponds to the system (5) in the case of crystal optics, and the system (13) in the case of hydromagnetics. Thus the leading term in the expansion of u is determined by (20), together with appropriate initial and boundary conditions.

It has been customary, both in the mathematical and physical literature, to treat only those cases in which (20) reduces to a system of ordinary differential equations, or a single equation. In such cases the unknowns σ^{i} are differentiated along rays and the energy is indeed transported along bundles of rays.

When conical refraction takes place, the system (20) is a nontrivial hyperbolic system, and the cone of rays represented in Figure 1 corresponds to the ray conoid of system (20). This fact can be verified directly in the cases of crystal optics and hydromagnetics. The general theorem follows from a lemma of P. D. Lax and R. Courant, quoted in [3], pp. 33-34.

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