217
LA-6126

UC-79p

Reporting Date: October 1975

Issued: February 1976

**The Basics of Bayesian Reliability
Estimation from Attribute Test Data**

by

H. F. Martz, Jr.

R. A. Waller

MASTER

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87545

An Affirmative Action/Equal Opportunity Employer

UNITED STATES
ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION
CONTRACT W-7408-ENG. 28

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22151
Price: Printed Copy \$4.50 Microfiche \$2.25

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights.

THE BASICS OF BAYESIAN RELIABILITY ESTIMATION FROM ATTRIBUTE TEST DATA

by

H. F. Martz, Jr. and R. A. Waller

ABSTRACT

The basic notions of Bayesian reliability estimation from attribute lifetest data are presented in an introductory and expository manner. This report will be useful to persons interested in becoming familiar with the Bayesian approach to reliability estimation. Both Bayesian point and interval estimates of the probability of surviving the lifetest, the reliability, are discussed. The necessary formulas are simply stated, and examples are given to illustrate their use. In particular, a binomial model in conjunction with a beta prior model are considered. Particular attention is given to the procedure for selecting an appropriate prior model in practice. Empirical Bayes point and interval estimates of reliability are discussed and examples are given.

I. INTRODUCTION

Basic statistical notions used in reliability estimation are contained in a companion report by Waller and Martz.¹ The reader unfamiliar with such statistical notions as random variable, probability models, point and interval estimation, and distribution functions should refer to that report, which also provides an introduction to classical, Bayesian, and empirical Bayes reliability estimation for the time-dependent case. Characteristics which serve to identify reliability estimation for the time-dependent case are (1) a probability model for the time-to-failure lifetime of the item under investigation is assumed; (2) the data consist of the observed times-to-failure of a set of items; and (3) the resulting reliability estimate is functionally dependent upon time. In particular, the exponential and Weibull probability models are considered in Ref. 1.

In this report, we discuss the time-independent case. The time-independent case characteristically considers a set of n items placed on lifetest for a specified length of time. At the end of the test, the number of survivors is observed and recorded. Both point and interval estimates of the probability of

failure-free operation (reliability) for the duration of the test must be obtained. Data of this type are sometimes called *attribute test data* because only the survival/nonsurvival, *not* the lifetime, is recorded for each test item. Such a lifetest is called an *attribute test*, whereas a test in which the actual failure times are recorded is called a *variables test*. There are two main advantages of an attribute test over a variables test. First, it is generally more economical to conduct an attribute test than a variables test. Second, a probability model for the times-to-failure variable is unnecessary, and the results are valid regardless of the underlying time-to-failure probability distribution. There are also two major disadvantages when attribute testing is compared to variables testing. First, if the failure times are monitored and recorded and if the probability distribution of these failure times is known, the resulting reliability estimates are generally superior (e.g., shorter interval estimates) to those based on corresponding attribute data. Second, the reliability estimates based on attribute data are not functionally dependent upon time. That is, the estimates pertain only to the reliability

NOTICE
This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

for a time period equal to the test duration, and reliability estimates cannot be given for other time periods. It is important to note that variable test data can be converted to attribute test data but not vice versa.

To illustrate the above and to supply numerical data for use and reference in later sections, consider the following example. Suppose that five independent samples of ten 60-W light bulbs were placed on 1000-h lifetest. The test results are given in Table I.

Using this data we shall calculate the classical, Bayes, and empirical Bayes point and interval estimates of the probability that any bulb in the population from which these samples have been selected will survive 1000 h.

Let the variable X denote the number of survivors of a lifetest of duration t from a sample of size n . It is commonly assumed that the survival/nonsurvival of each item in the sample is independent of the remaining items and that each item has the same underlying probability of surviving the test. Under these assumptions it may be shown that x has a binomial probability distribution given by

$$P(X=x) = P(X=x|R;n) = \frac{n!}{(n-x)!x!} R^x (1-R)^{n-x}, \quad x=0,1,\dots,n, \quad (1)$$

where $n! = n(n-1)\dots(2)(1)$ and R is the unknown probability of surviving the lifetest of duration t . Thus R is the reliability required to be estimated. Also $P(\cdot)$ denotes the probability of the event within the parentheses. For example, in Sample 1, Table I, $n = 10$ and X has the value $x = 5$.

Classical estimators for R are considered in Sec. II; Bayes estimators in Sec. III; and empirical Bayes estimators in Sec. IV.

TABLE I

ATTRIBUTE TEST RESULTS FOR 60-W
LIGHT BULBS IN A 1000-H LIFESTEST

Sample No.	Sample Size	No. of Survivors
1	10	5
2	10	6
3	10	5
4	10	8
5	10	6

II. CLASSICAL ESTIMATION

The classical point estimator for R is simply the observed fraction of items surviving the lifetest; that is,

$$\hat{R} = x/n. \quad (2)$$

Bowker and Lieberman² present the necessary equations for constructing either a $100(1-\alpha)\%$ two-sided or a $100(1-\alpha)\%$ lower one-sided confidence interval estimate of the reliability R . The $100(1-\alpha)\%$ two-sided confidence interval (TCI) is given by

$$100(1-\alpha)\% \text{ TCI: } \frac{x}{x+(n-x+1)F_{\alpha/2; 2n-2x+2, 2x}} \leq R \leq \frac{(x+1)F_{\alpha/2; 2x+2, 2n-2x}}{(n-x+1)F_{\alpha/2; 2x+2, 2n-2x}} \quad (3)$$

where $F_{\alpha/2; 2n-2x+2, 2x}$ is the so-called upper $100(\alpha/2)\%$ percentage point of an F distribution with $2n-2x+2$ numerator degrees of freedom and $2x$ denominator degrees of freedom. Similarly, $F_{\alpha/2; 2x+2, 2n-2x}$ is the upper $\alpha/2$ percentage point of an F distribution with $2x+2$ numerator degrees of freedom and $2n-2x$ denominator degrees of freedom. The Appendix shows the required percentage points for $\alpha/2 = 0.005, 0.01, 0.025, 0.05$, and 0.10 . Thus, 99, 98, 95, 90, and 80% TCI estimates may be constructed from these values. Frequently 95% intervals are desired. To use the Appendix, compute the value of $100(\alpha/2)$ for the α desired. Then in the Appendix find the page heading " $100(\alpha/2)$ Percentage Points of the F Distribution." For example, if a 95% TCI is desired, then $\alpha = 0.05$ and $100(\alpha/2) = 2.5$. The page heading "2.5 Percentage Points of the F Distribution" would then be used.

A $100(1-\alpha)\%$ lower one-sided confidence interval (LCI) is given by

$$100(1-\alpha)\% \text{ LCI: } R \geq \frac{x}{x+(n-x+1)F_{\alpha; 2n-2x+2, 2x}} \quad (4)$$

Here the required percentage points would be obtained from the Appendix under the heading " $100(\alpha)\%$ Percentage Points of the F Distribution," and the 99.5, 99, 97.5, 95, and 90% LCI estimates may be constructed.

Example: For Sample No. 1, Table I, $n = 10$ and $x = 5$. A point estimate of R is $\hat{R} = 5/10 = 0.50$.

Hence we estimate that 50% of all such 60-W light bulbs have a lifetime of at least 1000 h. A 95% TC1 estimate of R becomes

$$95\% \text{ TC1: } \frac{5}{5 + (10 - 5 + 1)(3.62)}$$

$$\leq R \leq \frac{(5 + 1)(3.62)}{(10 - 5) + (5 + 1)(3.62)}$$

or

$$95\% \text{ TC1: } 0.19 \leq R \leq 0.81,$$

since $F_{0.025, 12, 10} = 3.62$. Thus, we are 95% confident that R is contained in the interval $[0.19, 0.81]$.

III. BAYESIAN ESTIMATION

It is important to note that the Bayesian development requires more assumptions than are required for the preceding classical development. As the assumptions are presented, an attempt is made to discuss their relevancy in practical situations. For the classical estimation procedures in Sec. II we assumed that R was an unknown constant and used available sample data to estimate its value. The Bayesian philosophy is to treat R as a variable. That is, we suppose that in any given test of the kind described in Sec. I that the value of R underlying the test is the value of a variable according to a so-called *prior distribution*. Therefore the prior distribution is the probability model for the parameter R . The prior distribution represents the totality of the engineer's knowledge and assumptions about R *before any data are observed* in the test. The prior distribution of R should reflect the engineer's beliefs concerning the likely values of R before he observes the test results. If, on the basis of engineering judgement and experience, he believes that $P(R \geq 0.80) = 0.90$, the prior distribution should exhibit this property.

A. Selecting a Prior Distribution.

Selecting a prior distribution is the engineer's choice and represents an assumptive and subjective step in the analysis. In theory any probability distribution over the interval $[0, 1]$ can serve as a prior model. Further, two engineers will not necessarily choose the same prior model for a given problem owing to different degrees of belief concerning R .

Although there are no rigid rules for selecting prior models, there are analytic advantages in using what Raiffa and Schlaifer³ call *conjugate prior models*. For the parameter R considered here, the family of beta probability density functions defined by

$$g(R) \equiv g(R; x_0, n_0) = \frac{\Gamma(n_0)}{\Gamma(x_0)\Gamma(n_0 - x_0)} R^{x_0-1} (1-R)^{n_0-x_0-1}, 0 \leq R \leq 1, n_0 > x_0 > 0, \\ = 0, \text{ otherwise.} \quad (5)$$

yield conjugate prior models. For reasons discussed later, the prior parameters x_0 and n_0 are looked upon as "pseudo number of survivors" and "pseudo sample size," respectively, of a lifestest of duration t . The modifier "pseudo" should be thought of as meaning "pretended." Information about x_0 and n_0 may have been gained from one or more previous tests of a similar nature (see Sec. IV). Also, the gamma function, Γ , appearing in Eq. (5) is defined as

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt.$$

When a is an integer $\Gamma(a) = (a-1)!$. The prior mean and variance of Eq. (5) are given by $E(R) = x_0/n_0$ and $V(R) = [x_0(n_0 - x_0)] / [n_0^2(n_0 + 1)]$, respectively.

Different prior models may be chosen by specifying different values for x_0 and n_0 . Various prior models with means of 0.5 are shown in Fig. 1. Notice that the prior models are unimodal and bell-shaped for relatively large values of x_0 and n_0 . For small values of x_0 and n_0 , the models tend to become U-shaped with the probability tending to "pile-up" at the extremes of R . Notice also that the uniform model which assumes that all permissible values of R are equally likely is a special case of Eq. (5) whenever $x_0 = 1$ and $n_0 = 2$.

Fig. 1.
Typical beta prior models with mean 0.5

A similar variety of shapes exist for larger values of the prior mean, although the curves become asymmetric for small values of n_0 . Figure 2 illustrates some typical beta models with means of 0.9. Generally, for a fixed mean, the larger the value of n_0 the more the beta prior model is concentrated around the mean. For example, from Fig. 2 $x_0 = 0.90$ and $n_0 = 1$ yields a beta model with a mean of 0.90 and standard deviation 0.21, whereas $x_0 = 90$ and $n_0 = 100$ yields a beta model with the same mean but with a standard deviation of 0.03.

We repeat that the most important consideration in selecting a prior model for R is that the model selected must represent the engineer's knowledge and experience concerning R . That is, the model reflects his beliefs about R before observing the test results. Each of the curves in Figs. 1 and 2 convey different prior beliefs about R . For example, in Fig. 2 the model (c) conveys the essential beliefs that (1) the prior mean of R is 0.90; (2) the "effective" range of R values is $[0.8, 0.975]$; and (3) values of R in the immediate vicinity of 0.9 are quite likely. Thus, in this case, the engineer "strongly believes" in the relative high reliability of the item in question before obtaining the test results. Such a model will be referred to as a "concentrated" beta model. In other words, concentrated beta models are those with small variance. However, consider model (b) in Fig. 2. Although the engineer believes the prior mean to be 0.90 he is uncertain how close R is to this value. This is reflected in the distribution of density over a wider range of R values. Such a prior model will be called a "diffuse" beta model.

There are several existing techniques for determining values of x_0 and n_0 consistent with prior belief in R . This problem is discussed by many authors such as MacFarland,⁴ Schick and Lin,⁵ and Bonis.⁶ However, the most useful treatment appears

Fig. 2
Typical beta prior models with mean 0.9.

to be a paper by Weiler⁷ who considers beta prior distributions in the following situations:

- (1) Past records provide the numbers of defectives found in each of N samples of size n from N previously inspected batches.
- (2) The expected reliability in a batch and the probability of a batch exceeding twice the expected reliability are known approximately.
- (3) The probabilities of R exceeding a certain value R_1 and of R falling below another value R_2 are approximately known.

Situation (1) is essentially an empirical Bayes approach and will be further considered in Sec. IV.A. Situations (2) and (3) are likely not to be as useful to reliability engineers as the following situation:

Suppose that a reliability engineer can specify any two of the following three quantities.

- The prior mean reliability R_1 [$R_1 = E(R)$]
- The value of reliability R_2 such that, before the test results, there is only a 5% chance that the reliability R will exceed R_2 [$P(R \geq R_2) = 0.05$].
- The value of reliability R_3 such that, before the test results, there is only a 5% chance that the reliability R will not exceed R_3 [$P(R \leq R_3) = 0.05$].

Any two of these are sufficient to identify a pair of values x_0 and n_0 , the required parameters (sometimes called "hyperparameters") of the prior beta distribution. Some values are given in Table II for two values of R_1 and various values of R_2 .

An inspection of Table II shows that n_0 and x_0 tend to be large when R_1 and R_2 are close to each other. To illustrate the use of the table, assume that the prior mean reliability 0.96 is desired and that the desired prior 95th percentile is 0.9999. Then this prior belief is consistent with a beta distribution having $n_0 = 11.43755$ and $x_0 = 10.98004$.

B. The Posterior Model.

Suppose that a satisfactory beta prior model has been selected according to the preceding discussion, that is, that the values for x_0 and n_0 have been identified. Further suppose that attribute test data subsequently become available in which it is known that x items from a sample of n items have survived a lifetest of duration t . By means of Bayes theorem,¹ we can obtain the probability distribution

TABLE II
VALUES OF x_0 AND n_0 CORRESPONDING TO
SELECTED
VALUES OF R_1 AND R_2

R_1	R_2	n_0	x_0
0.99000	0.99500	797.71876	789.74157
0.99000	0.99550	631.03126	624.72095
0.99000	0.99600	506.01563	500.95548
0.99000	0.99650	408.78126	404.69345
0.99000	0.99700	332.30282	329.05899
0.99000	0.99750	269.87501	267.17626
0.99000	0.99800	218.28126	216.09845
0.99000	0.99850	174.62501	172.87876
0.99000	0.99900	135.91970	134.57040
0.99000	0.99950	98.22657	97.24430
0.99000	0.99960	90.28907	89.38618
0.99000	0.99970	81.70314	80.88611
0.99000	0.99980	72.42970	71.70540
0.99000	0.99990	60.52345	59.91821
0.99000	0.99991	59.06252	58.47189
0.99000	0.99992	57.54688	56.97141
0.99000	0.99993	56.10939	55.54830
0.99000	0.99994	54.14064	53.59923
0.99000	0.99995	52.17189	51.65017
0.99000	0.99996	50.20314	49.70111
0.99000	0.99997	47.25002	46.77752
0.99000	0.99998	44.29689	43.85392
0.99000	0.99999	39.68751	39.29063
0.98000	0.97000	917.77345	881.06251
0.96000	0.98000	195.46095	187.64251
0.96000	0.99000	66.47657	63.81751
0.96000	0.99500	38.39064	36.85501
0.96000	0.99550	35.71876	34.29601
0.96000	0.99600	33.46877	32.13001
0.96000	0.99650	31.50002	30.24001
0.96000	0.99700	29.06253	27.90003
0.96000	0.99750	26.57814	25.51501
0.96000	0.99800	24.21878	23.25003
0.96000	0.99850	21.82813	20.95501
0.96000	0.99900	19.06255	18.30004
0.96000	0.99950	15.75002	15.12001
0.96000	0.99960	15.00006	14.40006
0.96000	0.99970	14.06256	13.50006
0.96000	0.99980	12.89845	12.38251
0.96000	0.99990	11.43755	10.98004
0.96000	0.99991	11.25006	10.80006
0.96000	0.99992	10.91407	10.47751
0.96000	0.99993	10.65628	10.23003
0.96000	0.99994	10.48442	10.06504
0.96000	0.99995	10.14070	9.73507
0.96000	0.99996	9.84377	9.45001
0.96000	0.99997	9.57506	9.00006
0.96000	0.99998	8.92970	8.57251
0.96000	0.99999	8.15634	7.83009

of R conditional on the test results. This is the so-called *posterior distribution* (or posterior model) of R . For the case of a binomial distribution of the test results and a beta prior distribution, Bayes theorem yields the posterior distribution of R given by

$$h(R|x, n) \equiv h(R|x, n; x_0, n_0)$$

$$= \frac{\Gamma(n + n_0)}{\Gamma(x + x_0) \Gamma(n + n_0 - x - x_0)} R^{x + x_0 - 1} (1 - R)^{n + n_0 - x - x_0 - 1}$$

$$0 \leq R \leq 1 \quad (6)$$

Note that the posterior distribution is also a beta model with x_0 and n_0 in Eq. (5) replaced by $(x + x_0)$ and $(n + n_0)$, respectively. Thus the posterior mean and variance are $(x + x_0)/(n + n_0)$ and $\{(x + x_0)(n + n_0 - x - x_0) / [(n + n_0)^2(n + n_0 + 1)]\}$, respectively. A prior model that yields a posterior model of the same family is called a *conjugate prior model*.³

Example: Consider the data in Sample 1, Table I. Further assume that the prior model selected for R is a beta model with $x_0 = 3$ and $n_0 = 5$. Since $x = 5$ and $n = 10$ for Sample 1, Table I, we substitute these values into (6) to find the posterior model for R given by

$$h(R|5, 10) = \frac{\Gamma(15)}{\Gamma(8)\Gamma(7)} R^7 (1 - R)^6, \quad 0 \leq R \leq 1,$$

whereas the prior model for R is

$$g(R) = \frac{\Gamma(5)}{\Gamma(3)\Gamma(2)} R^2 (1 - R)^1, \quad 0 \leq R \leq 1.$$

Figure 3 gives a graphic comparison between $g(R)$ and $h(R|5, 10)$. We observe from Fig. 3 that the posterior mean of 0.53 has shifted from the prior mean of 0.6 toward the sample result $R = 0.5$. This is a significant change ascribed to using a "diffuse" prior beta model. Figure 4 illustrates the corresponding prior and posterior models for a "concentrated" prior beta model in which $x_0 = 60$ and $n_0 = 100$. Note that the posterior mean of 0.59 represents a smaller shift toward the sample result $R = 0.5$ than that in Fig. 3. Generally, posterior models corresponding to diffuse prior models tend to be more sensitive to the sample results than those based upon concentrated prior models. This important feature has led to increasing use of diffuse prior models in practice.

Fig. 3.

The prior and posterior models for the data in Sample 1, Table I ($x_o = 3$ and $n_o = 5$).

C. Bayesian Estimation of Reliability

Before the attribute test data are obtained, all Bayes estimates are based on the prior model. After the test data become available, the posterior model is used. The posterior model is a function of both the prior model and its parameters and the attribute test results. The "no-data" estimates based on the prior model are referred to by Ref. 4 as "soft" Bayes estimates, and those based on the posterior model are called "hard" or "mixed" Bayes estimates.

In this section we present and illustrate both Bayesian point estimates and Bayesian probability interval estimators for the reliability R.

Fig. 4.

The prior and posterior models for the data in Sample 1, Table I ($x_o = 60$ and $n_o = 100$).

A commonly used Bayesian point estimator for R is the mean of either the prior model (no-data situation) or the posterior model (data situation). Likewise, the Bayesian probability interval estimators are obtained from either the prior model (no-data situation) or the posterior model (data situation). The Bayesian point estimator of R for the no-data situation and a beta prior model is thus the fraction of "pseudo" items surviving a life test of duration t, or

$$\bar{R}_0 = \frac{\text{pseudo number of survivors}}{\text{pseudo sample size}} = \frac{x_o}{n_o} \quad (7)$$

Reference 8 presents a procedure which, in a slightly modified form, can be used here for constructing Bayes probability interval estimators of R. Using their procedure and simplifying, the 100(1- α)% two-sided Bayes probability interval (TBPI) estimator for R is given by

$$100(1-\alpha)\% \text{TBPI}(0):$$

$$\frac{x_o}{x_o + (n_o - x_o) F_{\alpha/2; 2n_o - 2x_o, 2x_o}} \leq R \leq \frac{x_o F_{\alpha/2; 2x_o, 2n_o - 2x_o}}{n_o - x_o + x_o F_{\alpha/2; 2x_o, 2n_o - 2x_o}} \quad (8)$$

where $F_{\alpha/2; 2n_o - 2x_o, 2x_o}$ is the upper 100($\alpha/2$) percentage point of an F distribution with $(2n_o - 2x_o)$ numerator and $2x_o$ denominator degrees of freedom, respectively. The quantity $F_{\alpha/2; 2x_o, 2n_o - 2x_o}$ is defined similarly, except that the numerator and denominator degrees of freedom are reversed. These F values are found in the Appendix (see Sec. II) from which 99, 98, 95, 90, and 80% TBPI estimates may be constructed. Frequently, 95% ($\alpha = 0.05$) interval estimates are required in practice.

A 100(1- α)% lower one-sided Bayes probability interval (LBPI) estimator for R for the no-data situation is

$$100(1-\alpha)\% \text{LBPI}(0): \quad (9)$$

$$R \geq \frac{x_o}{x_o + (n_o - x_o) F_{\alpha; 2n_o - 2x_o, 2x_o}}$$

By means of the F values in Table A, 99.5, 99, 97.5, 95, and 90% LBPI estimates may be constructed.

It is remarked here that the TBPI estimator given above is a so-called "symmetric" Bayes interval estimator in which the probability that R exceeds the upper limit is equal to the probability that R does not exceed the lower limit. For asymmetric posterior models this may not be the shortest Bayes $100(1-\alpha)\%$ interval estimator.

After the test data become available, the Bayesian point estimator of R for a beta prior model is the fraction of "combined" items surviving a lifestest of duration t . That is

$$\begin{aligned} \bar{R} &= \frac{\text{combined number of survivors}}{\text{combined sample size}} \\ &= \frac{x + x_0}{n + n_0} \end{aligned} \quad (10)$$

Several things are noted. When $n = 0$ (and thus $x = 0$), Eq. (10) reduces to the mean of the prior model as expected. Also, as n increases with n_0 held constant, the Bayes point estimate approaches x/n , the classical point estimate. This is sometimes paraphrased by saying that the influence of the prior model, through x_0 and n_0 , on the Bayes estimate decreases as the quantity of test data increases.

For the situation where test data are available, the $100(1-\alpha)\%$ TBPI estimator for R becomes

$$\begin{aligned} 100(1-\alpha)\% \text{TBPI}(n) : & \frac{x + x_0}{x + x_0 + (n + n_0 - x - x_0) F_{\alpha/2; 2n+2n_0-2x-2x_0, 2x+2x_0}} \leq R \\ & \leq \frac{(x + x_0) F_{\alpha/2; 2x+2x_0, 2n+2n_0-2x-2x_0}}{n + n_0 - x - x_0 + (x + x_0) F_{\alpha/2; 2x+2x_0, 2n+2n_0-2x-2x_0}} \end{aligned} \quad (11)$$

where the F values are interpreted as before. Likewise, the corresponding $100(1-\alpha)\%$ LBPI estimator for R is

$$R \geq \frac{x + x_0}{x + x_0 + (n + n_0 - x - x_0) F_{\alpha/2; 2n+2n_0-2x-2x_0, 2x+2x_0}} \quad (12)$$

Example: To illustrate the foregoing results, we suppose that a beta prior model with $x_0 = 3$ and $n_0 = 5$ is appropriate for the light bulb test data recorded in Sample 1, Table I. The prior and posterior models are provided in the preceding example and are graphed in Fig. 3. Before observing the test

results, the no-data point estimate of reliability R is

$$\tilde{R}_0 = 3/5 = 0.6$$

The no-data 95% TBPI is

$$95\% \text{TBPI}(0) : \frac{3}{3 + (5-3)(6.23)} \leq R \leq \frac{3(9.20)}{5-3+3(9.20)},$$

because $F_{0.025; 6, 4} = 9.20$ and $F_{0.025; 4, 6} = 6.23$, or 95% TBPI(0): $0.19 \leq R \leq 0.93$.

Similarly, the Bayes point estimate, once the test data in the sample become available, is

$$\bar{R} = \frac{5 + 3}{10 + 5} = 0.53$$

The corresponding 95% TBPI estimate is

$$95\% \text{TBPI}(10) : \frac{5+3}{5+3+(10+5-5-3)(2.82)}$$

$$\leq R \leq \frac{(5+3)(2.93)}{10+5-5-3+(5+3)(2.93)},$$

where $F_{0.025; 16, 14} = 2.93$ (by interpolation) and $F_{0.025; 14, 16} = 2.82$ (by interpolation), or

$$95\% \text{TBPI}(10) : 0.29 \leq R \leq 0.77.$$

This 95% Bayes probability interval estimate may be compared to the 95% classical confidence interval estimate [0.19, 0.81] in Sec. II. The Bayes interval estimate is shorter than the classical interval estimate for the same data. This gain is the result of the additional assumptions required in the Bayesian analysis. If the assumptions are reasonable, the resulting gain is *beneficial*.

The development in this section requires that the prior model be fully specified; that is, the values of x_0 and n_0 must be stated to use the preceding methods. Thus, the Bayesian methods presented are subjective in that we assume both a family of prior models, such as the beta family, as well as values for the parameters x_0 and n_0 in the beta family. By specifying these values a particular member of the

family is selected for use. In an effort to remove some of this subjectivity, *empirical Bayes* methods use empirical test data to either estimate the parameters of a given prior model or estimate the prior model itself. Some of these empirical Bayes techniques are presented in the next section.

IV. Empirical Bayes Estimation

By empirical Bayes estimation we mean either (1) the procedure and resulting estimates of reliability for a beta prior model in which x_0 and n_0 are both unknown and are to be estimated from past test data, or (2) the procedure and resulting estimates of reliability for a completely unknown and unspecified prior model which must be estimated. In simple terms, empirical Bayes denotes the techniques used to empirically estimate Bayes estimators in the absence of complete knowledge about the prior model. Krutchkoff⁹ provides a brief introductory exposition on empirical Bayes. Martz¹⁰ and Lemon¹¹ discuss the use of empirical Bayes in reliability.

In order to construct empirical Bayes estimates, past test data must be available which reasonably satisfy certain conditions. These conditions will be discussed later.

In Sec. IV.A, we consider the case in which a beta prior model, but with unknown parameters, is assumed. The case in which the prior model is completely unknown and unspecified is considered in Sec. IV.B.

A. Beta Prior Model with Unknown Parameters. Suppose that an engineer has decided that a beta prior model [Eq. (5)] adequately reflects his prior beliefs about the reliability R which is to be estimated. That is, he believes that the variety of curves within the beta family are sufficient to express his prior belief. His immediate problem is to identify values for x_0 and n_0 . Further, suppose that he wishes to calculate values from the results of previous attribute tests that have been conducted on the same or similar items. Let us be more explicit. Consider the following situation: A sequence of N attribute lifetests has previously been conducted in which, for the j^{th} lifetest, x_j items survived a lifetest of duration t out of a sample of size n_j . Further let R_j denote the unknown reliability in the j^{th} lifetest. Note that all previous lifetests are for the same duration, although the sample sizes may be different in each experiment. We assume that (1) the lifetests are conditionally statistically independent of each other and (2) the underlying sequence of reliability values $R_1, \dots, R_N$ are statistically independent

realizations of a beta variable with constant (unchanging) x_0 and n_0 values over the sequence of tests. Such a situation might arise when conducting routine lifetests for purposes of acceptance qualification on successive production lots of the same item. It could also occur whenever test data exists pursuant to previous usage of the same or similar items in other project programs.

Recall that the classical estimator of R_j in the j^{th} lifetest is $\hat{R}_j = x_j/n_j$. Weiler⁷ presents simple moment estimators for x_0 and n_0 under the assumption that the sample size n_j is the same in each lifetest. Under the assumption that the sample sizes do not vary too greatly from lifetest to lifetest, Copas¹² presents simple moment estimators for the prior mean and variance. Upon simplification these estimators yield the estimators for x_0 and n_0 given by

$$\hat{n}_0 = \frac{N^2 (\sum \hat{R}_j - \sum \hat{R}_j^2)}{N (\sum \hat{R}_j^2 - K \sum \hat{R}_j) - (N-K) (\sum \hat{R}_j)^2} \quad (13)$$

and

$$\hat{x}_0 = \hat{n}_0 (\sum \hat{R}_j) / N. \quad (14)$$

where the summations range from 1 to N and where $K = \sum n_j^{-1}$. These estimators reduce to those given by Weiler for the case of a constant sample size, and they provide good estimates of the prior beta parameters x_0 and n_0 when N is large.

When N is small, sampling error may cause Eq. (13) to yield negative estimates. If this occurs, we estimate n_0 from the equation

$$\hat{n}_0 = \left(\frac{N-1}{N} \right) \frac{N \sum \hat{R}_j - (\sum \hat{R}_j)^2}{N \sum \hat{R}_j^2 - (\sum \hat{R}_j)^2} - 1 \quad (15)$$

Example: Suppose that the set of five samples in Table I satisfies the assumptions stated at the beginning of Sec. IV.A. Here $N = 5$, $n = 10$, and $K = 0.5$. Based on these data we desire to estimate x_0 and n_0 in the assumed beta model underlying this data. Now $\hat{R}_1 = 0.5$, $\hat{R}_2 = 0.6$, $\hat{R}_3 = 0.5$, $\hat{R}_4 = 0.8$, and $\hat{R}_5 = 0.6$. Thus

$$\sum \hat{R}_j = 3.0, \sum \hat{R}_j^2 = 1.86,$$

and using Eq. (13) we have

$$\begin{aligned} \hat{n}_0 &= \frac{25(3.0 - 1.86)}{5[5(1.86) - 0.5(3.0)] - (5 - 0.5)(3.0)^2} \\ &= -19.0. \end{aligned}$$

with $\hat{n}_o < 0$, we use Eq. (15) and find that

$$\hat{n}_o = \left(\frac{5-1}{5} \right) \frac{5(3.0) - (3.0)^2}{5(1.86) - (3.0)^2} - 1 = 15.0,$$

and thus

$$\hat{x}_o = (15.00)(3.0)/5 = 9.0.$$

Figure 5 shows a graph of the fitted prior beta model from Eq. (15). The corresponding fitted posterior model for the data in Sample 1, Table I, is also graphed in Fig. 5. The fitted prior model in Fig. 5 has a prior mean of 0.60 and a prior standard deviation of 0.13, whereas the fitted posterior model has a mean of 0.56 and a standard deviation of 0.10.

For $x_o = 9$ and $n_o = 15$, the empirical Bayes point and 95% two-sided empirical probability interval estimates of R for Sample 1, Table I, are 0.56 and [0.37, 0.74], respectively. These estimates may be directly compared to those in the example in Sec. III.C. The previous Bayes point estimate was 0.52 which agrees closely with the estimate here. The previous Bayes probability interval estimate was [0.29, 0.77] which is considerably wider than the empirical Bayes interval estimate here. This is due to the fitted beta model being more concentrated than the beta model assumed in the example in Sec. III.C.

B. Unknown and Unspecified Prior Model. The engineer faced with the decision of selecting a prior

Fig. 5.

The fitted beta prior and posterior models for the data in Sample 1, Table I ($\hat{x}_o = 9$ and $\hat{n}_o = 15$).

model may be uncertain about which prior model he should use. Suppose that he desires Bayesian estimates but wishes to use a method that does not require any decision regarding the family to which the prior model belongs, such as the beta family. Thus not only is the engineer unwilling to assign values to the parameters, such as x_o and n_o , he is unwilling to assume even a prior family, such as the beta family. If past test data are available, these data can be used to construct an empirical estimate of the prior model itself. This is the situation usually referred to as *empirical Bayes*.

Consider the following situation. A sequence of N attribute lifetests has previously been conducted in which for the j^{th} lifetest, x_j items survived a lifetest of duration t out of a sample of size n_j . Suppose that these test results are available to the engineer. Further, let R_j denote the unknown reliability in the j^{th} lifetest. We assume that (1) the lifetests are conditionally statistically independent of each other, and (2) the underlying sequence of reliability values $R_1, \dots, R_N$ are statistically independent realizations of a variable from a completely unknown and unassumed common prior density.

Several investigators have considered this situation. Copas¹² presents two point estimators for R_j and Griffin and Krutchkoff¹³ develop an estimator similar to one of those proposed by Copas. Lemon and Krutchkoff¹⁴ present a general method for constructing estimators of R_j . This technique is used by Lemon¹¹ to develop a simple point estimator of R_j . Martz and Lian^{15,16} survey and compare several of these estimators and also develop their own estimators for R_j . The simplest and most direct estimator is that suggested by Lemon¹¹. This estimator is presented here, although it is emphasized that point estimators exist that generally possess smaller mean squared errors (see Refs. 15 and 16).

Lemon's procedure is basically to approximate the unknown and unspecified prior model by the discrete probability model given by

$$P(R = \hat{R}_j) = 1/N, j = 1, 2, \dots, N, \quad (16)$$

where $\hat{R}_j = x_j/n_j$. It is also assumed that the sample sizes do not vary too greatly for the sequence of lifetest experiments. This estimated prior model places a probability of $1/N$ at each of the observed sample estimates in the sequence of lifetest experiments. Using Bayes theorem in conjunction with Eq. (1), the corresponding empirical posterior model becomes

$$P(R=\hat{R}_j | x, n) \equiv P(R=\hat{R}_j | x, n; \hat{R}_1, \dots, \hat{R}_n)$$

$$= \begin{cases} \frac{\hat{R}_j^x (1-\hat{R}_j)^{n-x}}{\sum_{j=1}^N \hat{R}_j^x (1-\hat{R}_j)^{n-x}}, & j=1, 2, \dots, N \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

Note that the posterior model also places probability only at the observed sample estimates. Therefore, interval estimates based on such a posterior model cannot be expected to be "smooth" in the sense that the probability of interval coverage will always be close to the desired value. Smoother posterior models can be empirically obtained from empirical prior models which assign probabilities to those values of R adjacent to the observed estimates $\hat{R}_j$. Such an approach, which is a more complicated refinement of the simple approach presented here, has been considered by Martz¹⁰ and Martz and Lian.^{15,16} In general, the more refined approach yields superior interval estimates, and also point estimates which have a smaller mean squared error. If two or more values $\hat{R}_j$ coincide, both the empirical prior and posterior models accumulate the probability for the coincident values.

Example: Suppose that the set of five samples in Table I satisfies the assumptions stated at the beginning of Sec. IV. B. Now $\hat{R}_1 = 0.5$, $\hat{R}_2 = 0.6$, $\hat{R}_3 = 0.5$, $\hat{R}_4 = 0.8$, and $\hat{R}_5 = 0.6$. The empirical prior model becomes

$$P(R = 0.5) = 1/5 + 1/5 = 0.4$$

$$P(R = 0.6) = 1/5 + 1/5 = 0.4$$

$$P(R = 0.8) = 1/5 = 0.2$$

Figure 6 shows a graph of this prior probability model. After the data in Sample 1, Table I, have been observed, the corresponding empirical posterior model is obtained using Eq. (17)

$$P(R = 0.5 | 5, 10) = 0.27 + 0.27 = 0.54,$$

$$P(R = 0.6 | 5, 10) = 0.22 + 0.22 = 0.44,$$

$$P(R = 0.8 | 5, 10) = 0.02,$$

Fig. 6.

The empirical prior model fitted to the data in Table I.

The posterior model is presented in Fig. 7. It is interesting to compare the empirical prior and posterior models. Because the sample test data support the prior value 0.5, the posterior model significantly shrinks the prior probability that $R = 0.8$. At the same time, the posterior probability that $R = 0.6$ is slightly increased from its prior value because 0.6 is close to the test results. However, the posterior probability that $R = 0.5$ is larger than the posterior probability that $R = 0.6$, because the test results coincided with 0.5. Thus, the empirical posterior model exhibits the desired behavior of altering prior belief based on the observed lifetest results.

Now let us consider empirical Bayes estimators for R_k , the reliability in the k^{th} experiment. The empirical Bayes point estimator of R_k becomes

$$\tilde{R}_k = \frac{\sum_{j=1}^N \hat{R}_j^{x_k+1} (1-\hat{R}_j)^{n_k-x_k}}{\sum_{j=1}^N \hat{R}_j^{x_k} (1-\hat{R}_j)^{n_k-x_k}}, \quad k=1, 2, \dots, N, \quad (18)$$

where $\hat{R}_j = x_j/n_j$. If the posterior model has been calculated, an alternative form for computing $\tilde{R}_k$ is given by

$$\tilde{R}_k = \sum_{j=1}^N \hat{R}_j P(R=\hat{R}_j | x, n). \quad (19)$$

A $100(1-\alpha)\%$ or larger two-sided symmetric empirical Bayes probability interval (TEBPI) estimator of the reliability R_k in the k^{th} lifetest experiment is given by

$$\text{at least } 100(1-\alpha)\% \text{ TEBPI: } \hat{R}_{(j^+)} \leq R_k \leq \hat{R}_{(j^-)}, k = 1, 2, \dots, N, \quad (20)$$

where $\hat{R}_{(1)}, \dots, \hat{R}_{(n)}$ denotes the ordered (smallest to largest) sequence of classical reliability estimates,

Example: Suppose that the set of five samples in Table 1 satisfies the assumptions stated at the beginning of Sec. IV.B. Further, suppose we desire both point and 95% or larger two-sided interval estimates of the reliability R_1 corresponding to Sample 1 in Table 1. Thus $k = 1$. Now $\hat{R}_{(1)} = 0.5$, $\hat{R}_{(2)} = 0.5$, $\hat{R}_{(3)} = 0.6$, $\hat{R}_{(4)} = 0.6$, and $\hat{R}_{(5)} = 0.8$. Also $x_k = x_1 = 5$ and $n_k = n_1 = 10$. Substituting these values into (18) yields the empirical Bayes point estimate of R_1 given by

$$\tilde{R}_1 = \frac{2(0.5)^6(1-0.5)^5 + 2(0.6)^6(1-0.5)^5 + (0.8)^6(1-0.8)^5}{2(0.5)^5(1-0.5)^5 + 2(0.6)^5(1-0.5)^5 + (0.8)^5(1-0.8)^5} = 0.55$$

j^+ is the smallest integer which satisfies the relation

$$\sum_{j=1}^{j^+} P(R = \hat{R}_{(j)} | x_k, n_k) \geq 1 - \alpha/2,$$

and j^- is the largest integer satisfying

$$\sum_{j=1}^{j^-} P(R = \hat{R}_{(j)} | x_k, n_k) \leq \alpha/2.$$

If no such integer j^- exists between 1 and N , then j^- is set equal to 0. The lower end point of the interval estimate will thus be $\hat{R}_{(1)}$. In the event that $j^- = 0$ and $j^+ = n$, the required interval estimate will be the observed range of the classical sequence of reliability estimates. Also, the empirical probability that R_k lies within the interval is 1. Similarly, the empirical probability that R_k lies within any interval [Eq. (20)] can be calculated by summing the probabilities in the posterior model [Eq. (17)] corresponding to the values $\hat{R}_i$ lying within the interval.

A corresponding $100(1-\alpha)\%$ or larger lower one-sided empirical Bayes probability interval (LEBPI) estimator of the reliability R_k in the k^{th} lifetest experiment is given by

$$\text{at least } 100(1-\alpha)\% \text{ LEBPI: } \hat{R}_{(j^*)} \leq R_k, k = 1, 2, \dots, N, \quad (21)$$

where j^* is the largest integer satisfying the relation

$$\sum_{j=1}^{j^*} P(R = \hat{R}_{(j)} | x_k, n_k) \leq \alpha.$$

Again, j^* is set equal to 0 if no such integer exists between 1 and N .

The alternative form [Eq. (19)] and the posterior model in the preceding example also yield

$$\tilde{R}_1 = 0.5(0.54) + 0.6(0.44) + 0.8(0.02) = 0.55.$$

This estimate closely agrees with the Bayes estimate of 0.56 based on the fitted beta prior model. Now consider the required interval estimate in which $\alpha = 0.05$. From the posterior model in Fig. 7, $j^+ = 4$ and $j^- = 0$. Thus the interval estimate is

at least 95% TEBPI: $0.5 \leq R_1 \leq 0.6$

The empirical probability that this interval contains R_1 is 0.98. Recall that the corresponding classical, Bayes, and fitted Bayes interval estimates were previously found to be [0.19, 0.81], [0.29, 0.77], and [0.37, 0.74], respectively. The interval estimate here is shorter than any of these other estimates.

Fig. 7.

The empirical posterior model fitted to the data in Sample 1, Table I.

REFERENCES

1. R. A. Waller and H. F. Martz, Jr., "Bayesian Reliability Estimation: The State of the Art and Future Needs." Los Alamos Scientific Laboratory, report LA-6003 (October 1975).
2. A. H. Bowker and G. J. Lieberman, *Engineering Statistics* (Prentice Hall, New York, 1972), 2nd Ed.
3. H. Raiffa and R. Schlaifer, *Applied Statistical Decision Theory* (Harvard University Press, Boston, MA, 1961).
4. W. J. MacFarland, "Bayes' Equation, Reliability, and Multiple Hypothesis Testing," *IEEE Trans. Rel.* **R-21**, 136-147 (1972).
5. G. J. Schick and C. Y. Lin, "Determination of Subjective Prior Distributions for Bayesian Analysis of Maintainability Problems," Proceedings of the 1974 Annual Reliability and Maintainability Symposium, Los Angeles, CA (January 1974), pp. 21-26.
6. A. J. Bonis, "Why Bayes Is Better," Proceedings of the 1975 Annual Reliability and Maintainability Symposium, Washington, DC, January 1975, pp. 340-348.
7. H. Weiler, "The Use of Incomplete Beta Functions for Prior Distributions in Binomial Sampling," *Technometrics* **7**, 335-347 (August 1965).
8. N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, *Methods for Statistical Analysis of Reliability and Life Data* (John Wiley and Sons, New York, 1974).
9. R. G. Krutchkoff, "Empirical Bayes Estimation," *The American Statistician* **26**, 14-16 (December 1972).
10. H. F. Martz, Jr., "Pooling Life-Test Data by Means of the Empirical Bayes Method," *IEEE Trans. Rel.* **R-24**, 27-30 (April 1975).
11. G. H. Lemon, "An Empirical Bayes Approach to Reliability," *IEEE Trans. Rel.* **R-21**, 155-158 (August 1972).
12. J. B. Copas, "Empirical Bayes Methods and the Repeated Use of a Standard," *Biometrika* **59**, 349-360 (1972).
13. B. S. Griffin and R. G. Krutchkoff, "Optimal Linear Estimators: An Empirical Bayes Version with Application to the Binomial Distribution," *Biometrika* **58**, 195-201 (1969).
14. G. H. Lemon and R. G. Krutchkoff, "An Empirical Bayes Smoothing Technique," *Biometrika* **56**, 361-365 (1969).
15. H. F. Martz, Jr., and M. G. Lian, "A Survey and Comparison of Several Empirical Bayes Estimators for the Binomial Parameter," *J. Statis. Comput. Simul.* **3**, 165-178 (1974).
16. H. F. Martz, Jr., and M. G. Lian, "Empirical Bayes Estimation of the Binomial Parameter," *Biometrika* **61**, 517-523 (1974).

APPENDIX

0.5 PERCENTAGE POINTS OF THE F DISTRIBUTION

TABLE OF $F_{0.005; \nu_1, \nu_2}$

$\nu_1 \backslash \nu_2$		Degrees of freedom for the numerator (ν_1)																Degrees of freedom for the denominator (ν_2)													
		1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120												
1	16211	20000	21615	22500	23356	23937	24375	23925	24091	24224	24426	24630	24836	24960	25044	25148	25253	25359	25463												
2	198.5	199.0	199.2	199.3	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5												
3	55.55	49.80	47.27	46.17	45.39	44.84	44.43	44.13	43.88	43.69	43.59	43.48	43.38	43.28	43.18	43.08	42.98	42.88	42.78												
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.87	20.77	20.67	20.57	20.47	20.37	20.27	20.17	20.07												
5	22.78	18.31	16.53	15.50	14.94	14.51	14.20	13.96	13.77	13.62	13.52	13.42	13.32	13.22	13.12	13.02	12.92	12.82	12.72												
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.13	10.03	9.93	9.83	9.73	9.63	9.53	9.43	9.33												
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.28	8.18	8.08	7.97	7.87	7.77	7.67	7.57	7.47												
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.10	7.01	6.91	6.81	6.71	6.61	6.51	6.41	6.31												
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.32	6.23	6.13	6.03	5.93	5.83	5.73	5.63	5.53												
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.75	5.66	5.57	5.47	5.37	5.27	5.17	5.07	4.97												
11	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.32	5.24	5.15	5.05	4.95	4.85	4.75	4.65	4.55												
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.99	4.91	4.82	4.72	4.62	4.53	4.43	4.33	4.23												
13	11.37	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.72	4.64	4.56	4.47	4.37	4.27	4.17	4.07	3.97												
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.50	4.43	4.35	4.26	4.17	4.07	3.97	3.87	3.77												
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.32	4.25	4.17	4.07	3.97	3.87	3.77	3.67	3.57												
16	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.17	4.10	4.02	3.92	3.82	3.72	3.62	3.52	3.42												
17	10.38	7.35	6.16	5.50	5.07	4.77	4.56	4.39	4.25	4.14	4.04	3.97	3.89	3.80	3.70	3.60	3.50	3.40	3.30												
18	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.93	3.86	3.78	3.69	3.59	3.49	3.39	3.29	3.19												
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.83	3.76	3.68	3.59	3.49	3.39	3.29	3.19	3.10												
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.75	3.68	3.60	3.51	3.41	3.31	3.21	3.11	3.01												
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.68	3.60	3.51	3.41	3.31	3.21	3.11	3.01	2.91												
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.61	3.53	3.44	3.34	3.24	3.13	3.03	2.93	2.83												
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.55	3.47	3.38	3.28	3.17	3.07	2.97	2.87	2.77												
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.50	3.42	3.33	3.23	3.12	3.02	2.92	2.82	2.72												
25	9.48	6.60	5.46	4.84	4.44	4.15	3.94	3.78	3.64	3.54	3.45	3.37	3.28	3.18	3.06	2.95	2.85	2.75	2.65												
26	9.41	6.54	5.41	4.79	4.39	4.10	3.89	3.73	3.60	3.49	3.40	3.32	3.23	3.13	3.02	2.91	2.81	2.71	2.61												
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.36	3.28	3.19	3.08	2.97	2.86	2.76	2.66	2.56												
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.32	3.24	3.15	3.04	2.93	2.82	2.72	2.62	2.52												
29	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	3.48	3.38	3.29	3.21	3.12	3.01	2.90	2.79	2.69	2.59	2.49												
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.35	3.26	3.18	3.09	2.98	2.87	2.76	2.66	2.56	2.46												
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	3.03	2.95	2.86	2.75	2.64	2.53	2.43	2.33	2.23												
60	8.49	5.73	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.91	2.82	2.74	2.65	2.54	2.43	2.32	2.22	2.12	2.02												
120	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.62	2.54	2.45	2.34	2.23	2.12	2.01	1.91	1.81												
∞	7.88	5.30	4.28	3.72	3.35	3.08	2.90	2.75	2.62	2.52	2.43	2.36	2.27	2.16	2.05	1.94	1.83	1.72	1.61												

Example: $P(F > F_{0.005; 9, 15}) = P(F > 4.54) = 0.005$.

Degrees of freedom for the denominator (ν_2)

APPENDIX (cont)

1 PERCENTAGE POINTS OF THE F DISTRIBUTION

TABLE OF F

0.01; v_1, v_2

$v_2 \backslash v_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	∞
1	4052	4999	5403	5625	5764	5859	5928	5981	6022	6056	6106	6157	6209	6240	6261	6287	6313	6339	6366
2	99.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.44	99.46	99.47	99.47	99.48	99.49	99.50
3	36.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.58	26.50	26.41	26.32	26.22	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.91	13.84	13.75	13.65	13.56	13.46
5	10.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.45	9.38	9.29	9.20	9.11	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.44	7.30	7.23	7.14	7.06	6.97	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.98	6.84	6.72	6.62	6.47	6.31	6.16	6.06	5.99	5.91	5.82	5.74	5.65
8	11.26	8.55	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.26	5.20	5.12	5.03	4.95	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.71	4.65	4.57	4.48	4.40	4.31
10	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.01	3.94	3.86	3.78	3.69	3.60
11	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.57	3.51	3.43	3.34	3.25	3.17
12	8.66	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.41	3.35	3.27	3.18	3.09	3.01
13	8.33	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.16	3.10	3.02	2.93	2.84	2.75
14	8.08	6.01	5.09	4.57	4.24	4.00	3.83	3.70	3.59	3.49	3.35	3.21	3.06	2.96	2.90	2.82	2.73	2.64	2.55
15	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.73	2.67	2.58	2.50	2.40	2.31
16	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.69	2.62	2.54	2.45	2.35	2.26
17	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.64	2.58	2.49	2.40	2.31	2.21
18	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.60	2.54	2.45	2.36	2.27	2.17
19	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.81	2.66	2.57	2.50	2.42	2.33	2.23	2.13
20	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.54	2.47	2.38	2.29	2.20	2.10
21	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.51	2.44	2.35	2.26	2.17	2.07
22	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.72	2.57	2.48	2.41	2.32	2.23	2.14	2.04
23	7.56	5.39	4.51	4.02	3.70	3.48	3.31	3.17	3.07	2.98	2.84	2.70	2.55	2.45	2.39	2.30	2.21	2.11	2.01
24	7.51	5.35	4.47	3.98	3.67	3.45	3.28	3.15	3.04	2.95	2.82	2.67	2.52	2.43	2.36	2.27	2.18	2.09	1.99
25	7.47	5.31	4.43	3.94	3.63	3.41	3.24	3.11	3.00	2.91	2.78	2.63	2.48	2.39	2.32	2.23	2.14	2.05	1.95
26	7.43	5.27	4.39	3.90	3.59	3.37	3.20	3.07	2.96	2.87	2.74	2.59	2.44	2.35	2.28	2.19	2.10	2.01	1.91
27	7.39	5.23	4.35	3.86	3.55	3.33	3.16	3.03	2.92	2.83	2.70	2.55	2.40	2.31	2.24	2.15	2.06	1.97	1.87
28	7.35	5.19	4.31	3.82	3.51	3.29	3.12	2.99	2.88	2.80	2.66	2.51	2.36	2.27	2.20	2.11	2.02	1.93	1.83
29	7.31	5.15	4.27	3.78	3.47	3.25	3.08	2.95	2.84	2.75	2.62	2.47	2.32	2.23	2.16	2.07	1.98	1.89	1.79
30	7.27	5.11	4.23	3.74	3.43	3.21	3.04	2.91	2.80	2.71	2.58	2.43	2.28	2.19	2.12	2.03	1.94	1.85	1.75
40	7.08	4.98	4.11	3.62	3.31	3.09	2.92	2.79	2.68	2.59	2.46	2.31	2.16	2.07	2.00	1.91	1.82	1.73	1.63
60	6.85	4.79	3.95	3.46	3.15	2.93	2.76	2.63	2.52	2.43	2.30	2.15	2.00	1.91	1.84	1.75	1.66	1.57	1.47
120	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.19	2.04	1.89	1.77	1.70	1.59	1.48	1.39	1.30

Example: $P\{F > F_{0.01; 9, 15}\} = P\{F > 3.09\} = 0.01$.Degrees of freedom for the denominator (v_2)

APPENDIX (cont)

2.5 PERCENTAGE POINTS OF THE F DISTRIBUTION

TABLE OF $F_{0.025; v_1, v_2}$

$v_1 \backslash v_2$	Degrees of freedom for the numerator (v_1)																										
	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	∞								
1	447.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	976.7	984.9	993.1	998.1	1001	1006	1010	1014	1018								
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50								
3	17.44	18.04	18.44	18.76	19.00	19.18	19.32	19.44	19.54	19.62	19.69	19.75	19.80	19.84	19.87	19.90	19.92	19.94	19.96								
4	12.22	12.65	12.98	13.26	13.49	13.67	13.82	13.95	14.07	14.18	14.28	14.36	14.43	14.49	14.54	14.58	14.61	14.64	14.66								
5	10.01	10.43	10.76	11.03	11.26	11.45	11.61	11.75	11.87	11.98	12.08	12.16	12.23	12.29	12.34	12.38	12.41	12.44	12.46								
6	8.01	8.43	8.76	9.03	9.26	9.45	9.61	9.75	9.87	9.98	10.08	10.16	10.23	10.29	10.34	10.38	10.41	10.44	10.46								
7	6.01	6.43	6.76	7.03	7.26	7.45	7.61	7.75	7.87	7.98	8.08	8.16	8.23	8.29	8.34	8.38	8.41	8.44	8.46								
8	4.57	4.99	5.32	5.59	5.82	6.01	6.17	6.31	6.43	6.54	6.64	6.72	6.79	6.85	6.89	6.92	6.95	6.97	6.99								
9	3.57	3.99	4.32	4.59	4.82	5.01	5.17	5.31	5.43	5.54	5.64	5.72	5.79	5.85	5.89	5.92	5.95	5.97	5.99								
10	2.84	3.26	3.59	3.86	4.09	4.28	4.44	4.58	4.70	4.81	4.91	4.99	5.06	5.12	5.16	5.19	5.22	5.24	5.26								
11	2.34	2.76	3.09	3.36	3.59	3.78	3.94	4.08	4.20	4.31	4.41	4.49	4.56	4.62	4.66	4.69	4.72	4.74	4.76								
12	1.97	2.39	2.72	2.99	3.22	3.41	3.57	3.71	3.83	3.94	4.04	4.12	4.19	4.25	4.29	4.32	4.35	4.37	4.39								
13	1.68	2.10	2.43	2.70	2.93	3.12	3.28	3.42	3.54	3.65	3.75	3.83	3.90	3.96	3.99	4.02	4.05	4.07	4.09								
14	1.45	1.87	2.20	2.47	2.70	2.89	3.05	3.19	3.31	3.42	3.52	3.60	3.67	3.73	3.76	3.79	3.82	3.84	3.86								
15	1.28	1.70	2.03	2.30	2.53	2.72	2.88	3.03	3.15	3.26	3.36	3.44	3.51	3.57	3.60	3.63	3.66	3.68	3.70								
16	1.15	1.57	1.90	2.17	2.40	2.59	2.75	2.90	3.02	3.13	3.23	3.31	3.38	3.44	3.47	3.50	3.53	3.55	3.57								
17	1.04	1.46	1.79	2.06	2.29	2.48	2.64	2.79	2.91	3.02	3.12	3.20	3.27	3.33	3.36	3.39	3.42	3.44	3.46								
18	0.94	1.36	1.69	1.96	2.19	2.38	2.54	2.69	2.81	2.92	3.02	3.10	3.17	3.23	3.26	3.29	3.32	3.34	3.36								
19	0.86	1.28	1.61	1.88	2.11	2.30	2.46	2.61	2.73	2.84	2.94	3.02	3.09	3.15	3.18	3.21	3.24	3.26	3.28								
20	0.79	1.21	1.54	1.81	2.04	2.23	2.39	2.54	2.66	2.77	2.87	2.95	3.02	3.08	3.11	3.14	3.17	3.19	3.21								
21	0.73	1.15	1.48	1.75	1.98	2.17	2.33	2.48	2.60	2.71	2.81	2.89	2.96	3.02	3.05	3.08	3.11	3.13	3.15								
22	0.68	1.10	1.43	1.70	1.93	2.12	2.28	2.43	2.55	2.66	2.76	2.84	2.91	2.97	3.00	3.03	3.06	3.08	3.10								
23	0.63	1.05	1.38	1.65	1.88	2.07	2.23	2.38	2.50	2.61	2.71	2.79	2.86	2.92	2.95	2.98	3.01	3.03	3.05								
24	0.59	1.01	1.34	1.61	1.84	2.03	2.19	2.34	2.46	2.57	2.67	2.75	2.82	2.88	2.91	2.94	2.97	2.99	3.01								
25	0.56	0.98	1.31	1.58	1.81	1.99	2.15	2.30	2.42	2.53	2.63	2.71	2.78	2.84	2.87	2.90	2.93	2.95	2.97								
26	0.53	0.95	1.28	1.55	1.78	1.96	2.12	2.27	2.39	2.50	2.60	2.68	2.75	2.81	2.84	2.87	2.90	2.92	2.94								
27	0.51	0.93	1.26	1.53	1.76	1.94	2.10	2.25	2.37	2.48	2.58	2.66	2.73	2.79	2.82	2.85	2.88	2.90	2.92								
28	0.49	0.91	1.24	1.51	1.74	1.92	2.08	2.23	2.35	2.46	2.56	2.64	2.71	2.77	2.80	2.83	2.86	2.88	2.90								
29	0.47	0.89	1.22	1.49	1.72	1.90	2.06	2.21	2.33	2.44	2.54	2.62	2.69	2.75	2.78	2.81	2.84	2.86	2.88								
30	0.45	0.87	1.20	1.47	1.70	1.88	2.04	2.19	2.31	2.42	2.52	2.60	2.67	2.73	2.76	2.79	2.82	2.84	2.86								
40	0.39	0.81	1.14	1.41	1.64	1.82	1.98	2.13	2.25	2.36	2.46	2.54	2.61	2.67	2.70	2.73	2.76	2.78	2.80								
60	0.34	0.76	1.09	1.36	1.59	1.77	1.93	2.08	2.20	2.31	2.41	2.49	2.56	2.62	2.65	2.68	2.71	2.73	2.75								
120	0.30	0.72	1.05	1.32	1.55	1.73	1.89	2.04	2.16	2.27	2.37	2.45	2.52	2.58	2.61	2.64	2.67	2.69	2.71								
∞	0.25	0.67	1.00	1.27	1.50	1.68	1.84	1.99	2.11	2.22	2.32	2.40	2.47	2.53	2.56	2.59	2.62	2.64	2.66								

Example: $P(F_{0.025; 9, 15}) = P(F > 3.12) = 0.025$.

APPENDIX (cont)

5 PERCENTAGE POINTS OF THE F DISTRIBUTION

TABLE OF $F_{0.05; v_1, v_2}$

v_1	Degrees of freedom for the numerator (v_1)																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.3	250.1	251.1	252.2	253.3	254.3
2	10.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.46	19.47	19.48	19.49	19.50	19.51
3	10.13	9.55	9.58	9.62	9.64	9.66	9.68	9.69	9.70	9.71	9.72	9.73	9.74	9.75	9.76	9.77	9.78	9.79	9.80
4	7.71	6.94	6.96	6.99	7.01	7.02	7.03	7.04	7.05	7.06	7.07	7.08	7.09	7.10	7.11	7.12	7.13	7.14	7.15
5	6.61	5.79	5.81	5.83	5.85	5.86	5.87	5.88	5.89	5.90	5.91	5.92	5.93	5.94	5.95	5.96	5.97	5.98	5.99
6	5.99	5.14	5.16	5.18	5.20	5.21	5.22	5.23	5.24	5.25	5.26	5.27	5.28	5.29	5.30	5.31	5.32	5.33	5.34
7	5.59	4.74	4.76	4.78	4.79	4.80	4.81	4.82	4.83	4.84	4.85	4.86	4.87	4.88	4.89	4.90	4.91	4.92	4.93
8	5.32	4.46	4.48	4.50	4.51	4.52	4.53	4.54	4.55	4.56	4.57	4.58	4.59	4.60	4.61	4.62	4.63	4.64	4.65
9	5.12	4.26	4.28	4.30	4.31	4.32	4.33	4.34	4.35	4.36	4.37	4.38	4.39	4.40	4.41	4.42	4.43	4.44	4.45
10	4.96	4.10	4.12	4.14	4.15	4.16	4.17	4.18	4.19	4.20	4.21	4.22	4.23	4.24	4.25	4.26	4.27	4.28	4.29
11	4.84	3.98	3.99	4.01	4.02	4.03	4.04	4.05	4.06	4.07	4.08	4.09	4.10	4.11	4.12	4.13	4.14	4.15	4.16
12	4.75	3.89	3.90	3.92	3.93	3.94	3.95	3.96	3.97	3.98	3.99	4.00	4.01	4.02	4.03	4.04	4.05	4.06	4.07
13	4.67	3.81	3.82	3.84	3.85	3.86	3.87	3.88	3.89	3.90	3.91	3.92	3.93	3.94	3.95	3.96	3.97	3.98	3.99
14	4.60	3.74	3.75	3.77	3.78	3.79	3.80	3.81	3.82	3.83	3.84	3.85	3.86	3.87	3.88	3.89	3.90	3.91	3.92
15	4.54	3.68	3.69	3.71	3.72	3.73	3.74	3.75	3.76	3.77	3.78	3.79	3.80	3.81	3.82	3.83	3.84	3.85	3.86
16	4.49	3.63	3.64	3.66	3.67	3.68	3.69	3.70	3.71	3.72	3.73	3.74	3.75	3.76	3.77	3.78	3.79	3.80	3.81
17	4.45	3.59	3.60	3.62	3.63	3.64	3.65	3.66	3.67	3.68	3.69	3.70	3.71	3.72	3.73	3.74	3.75	3.76	3.77
18	4.41	3.55	3.56	3.58	3.59	3.60	3.61	3.62	3.63	3.64	3.65	3.66	3.67	3.68	3.69	3.70	3.71	3.72	3.73
19	4.38	3.52	3.53	3.55	3.56	3.57	3.58	3.59	3.60	3.61	3.62	3.63	3.64	3.65	3.66	3.67	3.68	3.69	3.70
20	4.35	3.49	3.50	3.52	3.53	3.54	3.55	3.56	3.57	3.58	3.59	3.60	3.61	3.62	3.63	3.64	3.65	3.66	3.67
21	4.32	3.47	3.48	3.50	3.51	3.52	3.53	3.54	3.55	3.56	3.57	3.58	3.59	3.60	3.61	3.62	3.63	3.64	3.65
22	4.30	3.44	3.45	3.47	3.48	3.49	3.50	3.51	3.52	3.53	3.54	3.55	3.56	3.57	3.58	3.59	3.60	3.61	3.62
23	4.28	3.42	3.43	3.45	3.46	3.47	3.48	3.49	3.50	3.51	3.52	3.53	3.54	3.55	3.56	3.57	3.58	3.59	3.60
24	4.26	3.40	3.41	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50	3.51	3.52	3.53	3.54	3.55	3.56	3.57	3.58
25	4.24	3.39	3.40	3.42	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50	3.51	3.52	3.53	3.54	3.55	3.56	3.57
26	4.23	3.37	3.38	3.40	3.41	3.42	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50	3.51	3.52	3.53	3.54	3.55
27	4.21	3.35	3.36	3.38	3.39	3.40	3.41	3.42	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50	3.51	3.52	3.53
28	4.20	3.33	3.34	3.36	3.37	3.38	3.39	3.40	3.41	3.42	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50	3.51
29	4.18	3.33	3.34	3.36	3.37	3.38	3.39	3.40	3.41	3.42	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50	3.51
30	4.17	3.32	3.33	3.35	3.36	3.37	3.38	3.39	3.40	3.41	3.42	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50
40	4.08	3.23	3.24	3.26	3.27	3.28	3.29	3.30	3.31	3.32	3.33	3.34	3.35	3.36	3.37	3.38	3.39	3.40	3.41
60	4.00	3.15	3.16	3.18	3.19	3.20	3.21	3.22	3.23	3.24	3.25	3.26	3.27	3.28	3.29	3.30	3.31	3.32	3.33
120	3.92	3.07	3.08	3.10	3.11	3.12	3.13	3.14	3.15	3.16	3.17	3.18	3.19	3.20	3.21	3.22	3.23	3.24	3.25
∞	3.84	3.00	3.01	3.03	3.04	3.05	3.06	3.07	3.08	3.09	3.10	3.11	3.12	3.13	3.14	3.15	3.16	3.17	3.18

Example: $P(F > F_{0.05; 9, 15}) = P(F > 2.59) = 0.05$.

APPENDIX (cont)

10 PERCENTAGE POINTS OF THE F DISTRIBUTION

TABLE OF $F_{0.10, v_1, v_2}$

$v_1 \backslash v_2$		Degrees of freedom for the numerator (v_1)																	
		1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120
1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	60.71	61.22	61.74	62.05	62.26	62.53	62.79	63.06	63.32
2	8.53	9.00	9.16	9.34	9.49	9.59	9.67	9.73	9.78	9.83	9.87	9.91	9.94	9.97	9.99	10.01	10.03	10.05	10.07
3	5.54	5.66	5.74	5.81	5.87	5.91	5.94	5.97	5.99	6.01	6.03	6.05	6.07	6.09	6.10	6.12	6.14	6.16	6.18
4	4.54	4.64	4.71	4.77	4.82	4.86	4.89	4.92	4.94	4.96	4.98	5.00	5.02	5.04	5.05	5.07	5.09	5.11	5.13
5	4.04	4.14	4.20	4.25	4.29	4.33	4.36	4.38	4.40	4.42	4.44	4.46	4.48	4.50	4.51	4.53	4.55	4.57	4.59
6	3.74	3.84	3.89	3.93	3.97	3.99	4.01	4.03	4.05	4.07	4.09	4.11	4.13	4.15	4.16	4.18	4.20	4.22	4.24
7	3.59	3.68	3.73	3.77	3.80	3.83	3.85	3.87	3.89	3.91	3.93	3.95	3.97	3.99	4.00	4.02	4.04	4.06	4.08
8	3.46	3.54	3.59	3.62	3.65	3.67	3.69	3.71	3.73	3.75	3.77	3.79	3.81	3.83	3.84	3.86	3.88	3.90	3.92
9	3.36	3.43	3.47	3.50	3.53	3.55	3.57	3.59	3.61	3.63	3.65	3.67	3.69	3.71	3.72	3.74	3.76	3.78	3.80
10	3.29	3.35	3.38	3.41	3.44	3.46	3.48	3.50	3.52	3.54	3.56	3.58	3.60	3.62	3.63	3.65	3.67	3.69	3.71
11	3.23	3.28	3.31	3.34	3.37	3.39	3.41	3.43	3.45	3.47	3.49	3.51	3.53	3.55	3.56	3.58	3.60	3.62	3.64
12	3.18	3.22	3.25	3.28	3.31	3.33	3.35	3.37	3.39	3.41	3.43	3.45	3.47	3.49	3.50	3.52	3.54	3.56	3.58
13	3.14	3.17	3.20	3.23	3.25	3.27	3.29	3.31	3.33	3.35	3.37	3.39	3.41	3.43	3.44	3.46	3.48	3.50	3.52
14	3.10	3.13	3.15	3.18	3.20	3.22	3.24	3.26	3.28	3.30	3.32	3.34	3.36	3.38	3.39	3.41	3.43	3.45	3.47
15	3.07	3.09	3.11	3.13	3.15	3.17	3.19	3.21	3.23	3.25	3.27	3.29	3.31	3.33	3.34	3.36	3.38	3.40	3.42
16	3.05	3.06	3.08	3.10	3.12	3.14	3.16	3.18	3.20	3.22	3.24	3.26	3.28	3.30	3.31	3.33	3.35	3.37	3.39
17	3.03	3.04	3.05	3.07	3.09	3.11	3.13	3.15	3.17	3.19	3.21	3.23	3.25	3.27	3.28	3.30	3.32	3.34	3.36
18	3.01	3.02	3.03	3.05	3.07	3.09	3.11	3.13	3.15	3.17	3.19	3.21	3.23	3.25	3.26	3.28	3.30	3.32	3.34
19	2.99	2.99	3.00	3.02	3.04	3.06	3.08	3.10	3.12	3.14	3.16	3.18	3.20	3.22	3.23	3.25	3.27	3.29	3.31
20	2.97	2.97	2.98	2.99	3.01	3.03	3.05	3.07	3.09	3.11	3.13	3.15	3.17	3.19	3.20	3.22	3.24	3.26	3.28
21	2.96	2.95	2.96	2.97	2.98	2.99	3.01	3.03	3.05	3.07	3.09	3.11	3.13	3.15	3.16	3.18	3.20	3.22	3.24
22	2.95	2.94	2.95	2.96	2.97	2.98	2.99	3.01	3.03	3.05	3.07	3.09	3.11	3.13	3.14	3.16	3.18	3.20	3.22
23	2.94	2.93	2.94	2.95	2.96	2.97	2.98	2.99	3.01	3.03	3.05	3.07	3.09	3.11	3.12	3.14	3.16	3.18	3.20
24	2.93	2.92	2.93	2.94	2.95	2.96	2.97	2.98	2.99	3.01	3.03	3.05	3.07	3.09	3.10	3.12	3.14	3.16	3.18
25	2.92	2.91	2.92	2.93	2.94	2.95	2.96	2.97	2.98	2.99	3.01	3.03	3.05	3.07	3.08	3.10	3.12	3.14	3.16
26	2.91	2.90	2.91	2.92	2.93	2.94	2.95	2.96	2.97	2.98	2.99	3.01	3.03	3.05	3.06	3.08	3.10	3.12	3.14
27	2.90	2.89	2.90	2.91	2.92	2.93	2.94	2.95	2.96	2.97	2.98	2.99	3.01	3.03	3.04	3.06	3.08	3.10	3.12
28	2.89	2.88	2.89	2.90	2.91	2.92	2.93	2.94	2.95	2.96	2.97	2.98	2.99	3.01	3.02	3.04	3.06	3.08	3.10
29	2.88	2.87	2.88	2.89	2.90	2.91	2.92	2.93	2.94	2.95	2.96	2.97	2.98	3.00	3.01	3.03	3.05	3.07	3.09
30	2.88	2.87	2.88	2.89	2.90	2.91	2.92	2.93	2.94	2.95	2.96	2.97	2.98	3.00	3.01	3.03	3.05	3.07	3.09
40	2.84	2.83	2.84	2.85	2.86	2.87	2.88	2.89	2.90	2.91	2.92	2.93	2.94	2.96	2.97	2.99	3.01	3.03	3.05
60	2.79	2.79	2.80	2.81	2.82	2.83	2.84	2.85	2.86	2.87	2.88	2.89	2.90	2.92	2.93	2.95	2.97	2.99	3.01
120	2.75	2.75	2.76	2.77	2.78	2.79	2.80	2.81	2.82	2.83	2.84	2.85	2.86	2.88	2.89	2.91	2.93	2.95	2.97
	2.71	2.71	2.72	2.73	2.74	2.75	2.76	2.77	2.78	2.79	2.80	2.81	2.82	2.84	2.85	2.87	2.89	2.91	2.93

Example: $P\{F > F_{0.10; 9, 15}\} = P\{F > 2.09\} = 0.10$.