CONDITIONS FOR SHEAR LOCALIZATION IN THE DUCTILE FRACTURE OF VOID-CONTAINING MATERIALS*

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CONTENTS

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ABSTRACT

This paper investigates the possibility that ductile fracture occurs by the McClintock-Berg mechanism of localization of deformation within a narrow shear band, owing to the progressive softening of the material by increasing porosity due to void growth. The ductility predicted for a macroscopically homogeneous sample of a voided material is shown to be unrealistically large and hence an initial inhomogeneity of properties is considered, in the sense of an analysis by Marciniak and Kuczynski in the related problem of local necking in sheet metals. General conditions for a localization bifurcation with an initial inhomogeneity (imperfection), concentrating deformation to allow localization within it, are derived. The initial imperfection is taken in the form of a void-containing, thin slice of a material and is assumed to have a void volume fraction slightly larger than the outside of the imperfection. Elastic-plastic constitutive rate relations for void-containing materials proposed by Gurson are adopted to the conditions for the localization bifurcation. The critical conditions are analyzed numerically to discuss the sensitivity of localization conditions to an initial imperfection, in consideration of the implications for the theory of ductile fracture. The results suggest that the existence of an initial imperfection makes it possible for localization to occur at a reasonable strain, and the predictions from this analysis seem broadly consistent with reported experimental observations.

1. Introduction

When a ductile metal is deformed into plastic range a zone of localized deformation sometimes appears in the form of a narrow shear band in highly stressed regions. Non-uniform deformation within it leads to ductile fracture by growth and coalescence of voids [1,2,3]. These voids generally arise from cracking of second-phase particles such as inclusions, precipitates and dispersions or from decohesion at particle-matrix interfaces [4,5,6], although other processes may contribute in some cases. It remains an open question (see, e.g., Rice [7]) whether the localization occurs because of the progressive softening of the material by void growth [8,9] or because some other instability of the plastic flow process first occurs.

In the present paper the first possibility is explored. We adopt the elastic-plastic constitutive rate relations of Gurson [10,11] for void-containing materials, and derive conditions for a localization bifurcation to occur due to the progressive softening of the material by the porosity increase due to void growth. Indeed, Rudnicki and Rice [12] have considered a localization bifurcation of this type for a class of elastic-plastic constitutive laws which is a simple generalization of the Prandtl-Reuss equations. Their viewpoint is that the macroscopic constitutive relations may permit the homogeneous deformation of an initially uniform material to give way to an incipient non-uniform deformation field, concentrated within a localized band (shear band) but uniform outside it. Using these considerations, Rudnicki and Rice [12] have discussed conditions for the localization bifurcation in plastically dilatant materials with pressure sensitive yielding but, as is relevant to our present considerations, with no provision for initial spatial non-uniformities of material properties (imperfections). Then, they have shown that materials seem to be unusually resistant to the localization bifurcation in the case of axially-symmetric extension or compression.

In order to assess the effects of non-uniformities of properties, Rice [7] has proposed a formulation in the spirit of the Marciniak and Kuczynski [13] analysis of initial thickness non-uniformities in local necking of thin metal sheets. This idea can be explained in the sense that a part of a material may have slightly different properties from the remaining portion and that continuing concentrated deformation within this inhomogeneity (imperfection) leads to failure at a strain smaller, sometimes dramatically so, than a value required for a perfectly homogeneous body.

Since the mode of the localization bifurcation has the form of a planar band in which voids are concentrated more than outside it [1,3], we take the imperfection in the form of a void-containing, planar band with a non-uniform distribution of voids. Also, we assume the void volume fraction to be slightly larger within the imperfection than outside it. This imperfection should appear within the material by the process of void nucleation during plastic deformation [1,3], but we presume it as a pre-existing one (initial imperfection).

Possible elastic-plastic constitutive rate relations for void-containing materials are derived from yield surface equations given by Gurson [10,11]. He has considered a simple cell model which has a single spherical void centered in a spherical cell. This model certainly does not reflect with precision the response of actual materials with a random distribution of voids. But the Gurson [10,11] constitutive model is simple and enables an estimate of the effect of progressively growing voids in leading to a localization bifurcation.

Thus, in this paper, conditions for a localization bifurcation with an initial, finite imperfection will be studied by developing the localization theory given by Rice [7] in a general theoretical framework for localization of plastic deformation. Specifically, the localization bifurcation presented here is understood in the following sense. Before localization, the macroscopic constitutive relations allow homogeneous deformation within and outside an imperfection, with finite discontinuities of the deformation fields between them. Then, at the inception of localization, the macroscopic constitutive relations cause the deformation rate fields to be infinite within the imperfection but to remain finite outside it.

In the following sections, we first develop the elastic-plastic constitutive rate relations to be adopted for void-containing materials. Next, localization conditions for the case with an initial imperfection are developed in the sense described above. This is followed with the analyses of the localization bifurcation with no imperfection, developed by applying the general results of Rudnicki and Rice [12], and with the specific calculations for localization conditions with an initial imperfection in the cases of tensile extension under axially-symmetric and plane-strain conditions. The sensitivity of localization conditions to an initial imperfection is then discussed in relation to implications of the work for the theory of ductile fracture.

2. Constitutive rate relations for void-containing materials

In this section, we discuss a possible formulation of elastic-plastic constitutive rate relations for void-containing materials, based on studies by Gurson [10,11]. He modelled a randomly voided material as a spherical cell with a single spherical void centered in the cell. This model is outlined here. Throughout the section, the adjective "macroscopic" refers to average values of physical quantities (stress, strain, etc.) which represent the behavior of the total voided aggregate, and "microscopic" refers to their pointwise values in the matrix surrounding the void, which is a homogeneous, incompressible von Mises material.

Consider a body with void volume fraction f. Let σ_{ij}' be the macroscopic deviatoric true stress components defined as $\sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ where σ_{ij} are the macroscopic true stress components and δ_{ij} is the Kronecker delta. Also, let $\bar{\sigma}_m$ be the microscopic equivalent tensile flow strength. The Gurson [10,11] approximation to the yield surface, based on his cell model, is given by

$$\phi = \frac{3}{2}\,\frac{\sigma_{ij}'\sigma_{ij}'}{\bar{\sigma}_m^{\,2}} + 2f\cosh\left(\frac{1}{2}\frac{\sigma_{kk}}{\bar{\sigma}_m}\right) - (1+f^2) = 0 \quad . \tag{2.1}$$

We observe that this reduces to the isotropically hardening Mises form when there are no voids $(f = 0)$.

In formulating the elastic-plastic constitutive rate relations for isotropically hardening materials with voids, we assume that the total macroscopic rate of deformation may be written as the sum of the elastic term D_{ij}^{e} and plastic term D_{ij}^{p}. When the spin-invariant Jaumann stress rate, which is a stress rate observed in a reference system that rotates with the spin tensor Ω_{ij} of the particle, is adopted, the elastic rate of deformation - Jaumann stress rate relation for the macroscopic field is taken to be [12]

$$D^{e}_{ij} = \frac{1}{2G}\overset{\nabla}{\sigma}_{ij} + \frac{1}{3}\left(\frac{1}{3K} - \frac{1}{2G}\right)\delta_{ij}\overset{\nabla}{\sigma}_{kk} \quad , \tag{2.2}$$

where G and K are the elastic shear and bulk moduli respectively.

$$\overset{\nabla}{\sigma}_{ij} = \dot{\sigma}_{ij} - \Omega_{ik}\sigma_{kj} - \sigma_{ik}\Omega_{jk} \tag{2.3}$$

are the Jaumann stress rate components and $\dot{\sigma}_{ij}$ are the true stress rate components. Also, the expressions

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad , \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{2.4}$$

give the rate of deformation and spin tensor components respectively, where v_i are the velocity components.

Next, the plastic rate of deformation-Jaumann stress rate relation for the macroscopic field is obtained in the following way. The macroscopic dissipation during plastic deformation is defined as

$$\sigma_{ij}D^{p}_{ij} = \frac{V_m\bar{\sigma}_m d\bar{\varepsilon}^{p}_{m}/dt}{V} \quad , \tag{2.5}$$

where V is the volume of the total body, V_m is the volume of the matrix in the body, and the time derivative term involves the microscopic plastic equivalent strain $\bar{\varepsilon}^{p}_{m}$. Eq. (2.5) gives the time derivative of the microscopic equivalent tensile flow strength as [10]

$$\dot{\bar{\sigma}}_m = h_m \frac{\sigma_{ij} D^p_{ij}}{(1-f)\bar{\sigma}_m} \quad , \tag{2.6}$$

where $h_m = d\bar{\sigma}_m/d\bar{\varepsilon}^p_m$ is the microscopic hardening modulus of the equivalent tensile flow strength-plastic strain curve. The matrix is assumed to satisfy the plastic incompressibility condition but the macroscopic response of the body does not because of the existence of voids. Thus, the rate of change of the total volume is related to the time derivative of the void volume fraction as [10]

$$\dot{f} = (1-f)D^p_{kk} \quad . \tag{2.7}$$

This shows that the void volume fraction increases during plastic deformation.

Now, normality of the plastic rate of deformation at a point of the smooth yield surface requires

$$D^p_{ij} = \Lambda \frac{\partial\Phi}{\partial\sigma_{ij}} \quad , \tag{2.8}$$

where Λ is a function of stress, stress rate, deformation history and void volume fraction. Furthermore, as Berg [9] and Gurson [10] have commented, following an argument by Bishop and Hill [14], the validity of normality locally within the matrix implies macroscopic normality. Taking the time derivative of the yield surface equation, Eq. (2.1), gives the "consistency condition",

$$\dot{\Phi} = \frac{\partial\Phi}{\partial\sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial\Phi}{\partial\bar{\sigma}_m} \dot{\bar{\sigma}}_m + \frac{\partial\Phi}{\partial f} \dot{f} = 0 \quad . \tag{2.9}$$

Substituting Eq. (2.6) and Eq. (2.7) into Eq. (2.9) and solving Eq. (2.9) for

Λ yields [10]

$$\Lambda = \frac{1}{F}\frac{\partial\Phi}{\partial\sigma_{k\ell}}\dot{\sigma}_{k\ell} \quad , \tag{2.10}$$

where

$$F = -\left\{\frac{\partial\Phi}{\partial\bar{\sigma}_m}\frac{h_m\sigma_{ij}}{(1-f)\bar{\sigma}_m}\frac{\partial\Phi}{\partial\sigma_{ij}} + \frac{\partial\Phi}{\partial f}(1-f)\frac{\partial\Phi}{\partial\sigma_{kk}}\right\} .$$

By considering the spin-invariant Jaumann stress rate, Eq. (2.8) can be written with the use of Eq. (2.10) as

$$D^P_{ij} = \frac{1}{H}\left(\frac{3}{2}\frac{\sigma_{ij}'}{\bar{\sigma}_m} + \alpha\delta_{ij}\right)\left(\frac{3}{2}\frac{\sigma_{k\ell}'}{\bar{\sigma}_m} + \alpha\delta_{k\ell}\right)\overset{\nabla}{\sigma}_{k\ell} \quad , \tag{2.11}$$

where

$$H = \left\{\frac{h_m}{(1-f)}\left(\omega + \frac{\sigma_{kk}}{\bar{\sigma}_m}\alpha\right)^2 - 3\bar{\sigma}_m(1-f)\alpha\left[\cosh\left(\frac{1}{2}\frac{\sigma_{kk}}{\bar{\sigma}_m}\right) - f\right]\right\} \quad ,$$

$$\alpha = \frac{1}{2}f\sinh\left(\frac{1}{2}\frac{\sigma_{kk}}{\bar{\sigma}_m}\right) .$$

Here, $\omega = [(1+f^2) - 2f\cosh(\sigma_{kk}/2\bar{\sigma}_m)]$ is the square of the ratio of the macroscopic to microscopic equivalent flow strength in shear. (See the yield surface equation.)

Finally, the total macroscopic rate of deformation is obtained by adding the elastic term of Eq. (2.2) to the plastic term of Eq. (2.11). The equations are

$$D_{ij} = \frac{1}{2G}\overset{\nabla}{\sigma}_{ij} + \frac{1}{3}\left(\frac{1}{3K} - \frac{1}{2G}\right)\delta_{ij}\overset{\nabla}{\sigma}_{kk} \tag{2.12}$$

for elastic loading or any unloading and

$$D_{ij} = \frac{1}{2G}\overset{\nabla}{\sigma}_{ij} + \frac{1}{3}\left(\frac{1}{3K} - \frac{1}{2G}\right)\delta_{ij}\overset{\nabla}{\sigma}_{kk} + \frac{1}{H}\left(\frac{3}{2}\frac{\sigma_{ij}'}{\bar{\sigma}_m} + \alpha\delta_{ij}\right)\left(\frac{3}{2}\frac{\sigma_{k\ell}'}{\bar{\sigma}_m} + \alpha\delta_{k\ell}\right)\overset{\nabla}{\sigma}_{k\ell} \tag{2.13}$$

for loading at yield. The inverse form of Eq. (2.13) is given by

$$\overset{\nabla}{\sigma}_{ij} = 2GD_{ij} + (K - \frac{2}{3}G)\delta_{ij}D_{kk} - \frac{\left(G\frac{\sigma_{ij}'}{\bar{\sigma}_m} + K\alpha\delta_{ij}\right)\left(G\frac{\sigma_{k\ell}'}{\bar{\sigma}_m} + K\alpha\delta_{k\ell}\right)}{\frac{1}{9}H + \frac{1}{3}\omega G + \alpha^2 K} D_{k\ell} \ . \tag{2.14}$$

Rudnicki and Rice [12] have proposed a simple generalization of the Prandtl-Reuss equations, in the same general form as Eq. (2.2) and Eq. (2.11), to describe the elastic-plastic behavior of rock and soil masses under compressive principal stresses. Taking β to be the "dilatancy" factor and μ to be the "internal friction" coefficient, they write the equation analogous to Eq. (2.11) as

$$D^p_{ij} = \frac{1}{h}\left(\frac{\sigma_{ij}'}{2\bar{\tau}} + \frac{\beta}{3}\delta_{ij}\right)\left(\frac{\sigma_{k\ell}'}{2\bar{\tau}} + \frac{\mu}{3}\delta_{k\ell}\right)\overset{\nabla}{\sigma}_{k\ell} \ . \tag{2.15}$$

Here, $\bar{\tau} = \bar{\sigma}/\sqrt{3} = \sqrt{(\sigma_{ij}'\sigma_{ij}')/2}$ is the equivalent shear stress, while $\bar{\sigma}$ is the equivalent stress. h is the hardening modulus of the equivalent shear stress-plastic shear strain curve. Eq. (2.15) gives the constitutive rate relations analogous to Eq. (2.14),

$$\overset{\nabla}{\sigma}_{ij} = 2GD_{ij} + (K - \frac{2}{3}G)\delta_{ij}D_{kk} - \frac{\left(G\frac{\sigma_{ij}'}{\bar{\tau}} + K\beta\delta_{ij}\right)\left(G\frac{\sigma_{k\ell}'}{\bar{\tau}} + K\mu\delta_{k\ell}\right)}{h + G + \beta\mu K} D_{k\ell} \quad . \tag{2.16}$$

Comparing Eq. (2.14) with Eq. (2.16), we find the following relations.

Eq. (2.16)		Eq. (2.14)
β	$\longleftrightarrow$	$\sqrt{\frac{3}{\omega}}\,\alpha$
μ	$\longleftrightarrow$	$\sqrt{\frac{3}{\omega}}\,\alpha$
$\bar{\tau}$	$\longleftrightarrow$	$\sqrt{\frac{\omega}{3}}\,\bar{\sigma}_m$
h	$\longleftrightarrow$	$\frac{1}{3\omega}H$

(2.17)

The dilatancy factor (or α in the present case) depends on the void volume fraction which changes by void growth and on the ratio of the macroscopic hydrostatic stress to the microscopic equivalent tensile flow strength. Especially, in the case of no voids $(f=0)$, the dilatancy factor becomes zero and Eq. (2.13) and Eq. (2.14) reduce to the Prandtl-Reuss elastic-plastic constitutive rate relations which satisfy the plastic incompressibility condition.

With the substitutions of Eqs. (2.17), the expression for the critical value of h for localization in an initially uniform body, as derived by Rudnicki and Rice [12], may be applied directly to localization in the present case. Such results are referred to subsequently.

3. Conditions for a localization bifurcation with an initial imperfection

In this section, we present the method for obtaining deformation fields within an imperfection in terms of the fields outside it and discuss conditions for a localization bifurcation with an initial imperfection.

Consider a body with an initial, thin imperfection [7] and assume that the material outside it is subjected to uniform quasi-static deformation. Assume that both the imperfection and the material surrounding it are homogeneous but that the imperfection has initial properties which differ slightly from those of the surrounding material [13]. We introduce rectangular cartesian coordinate $x_i (i=1,2,3)$, so that the x_2 - axis is normal to the planar band of the imperfection as in Fig. 1. Take the unit normal to the planar band to be $\underset{\sim}{n}$. Since deformation fields of the imperfection and surrounding material are homogeneous and the velocity field is to be continuous throughout the body, the velocity can vary only in the direction perpendicular to the band. Thus, the difference in velcoity between the band layer and the surrounding material is given by

$$v_i - v_i^o = f_i(n_k x_k) \quad , \tag{3.1}$$

where the superscripted quantities, $(\text{---})^o$, apply for the uniform fields outside the imperfection and f_i is a function of $n_k x_k$. The difference in velocity gradient is given by $v_{i,j} - v^o_{i,j} = n_j f'_i(n_k x_k)$, where $f'_i(n_k x_k)$ is the derivative of $f_i(n_k x_k)$. We choose f'_i to be zero outside the imperfection and to have the spatially uniform values q_i within it. Therefore, the velocity gradients within and outside the imperfection are related by the kinematical condition.

$$v_{i,j} = v^o_{i,j} + q_i n_j \quad , \quad \text{or} \quad \Delta v_{i,j} = q_i n_j \quad , \tag{3.2}$$

where, hereafter, the non-superscripted quantities, (---), denote the uniform fields within the imperfection and Δ denotes the difference between the uniform fields within and outside the imperfection. Eq. (3.2) gives the rate of deformation and spin tensors within the imperfection.

$$D_{ij} = D^{o}_{ij} + \frac{1}{2}(n_i q_j + n_j q_i) \quad , \tag{3.3}$$

$$\Omega_{ij} = \Omega^{o}_{ij} - \frac{1}{2}(n_i q_j - n_j q_i) \quad . \tag{3.4}$$

From Eq. (3.3), Eq. (3.4) and the relation between the true stress rate and Jaumann stress rate given by Eq. (2.3), the true stress rate tensor within the imperfection is expressed in terms of the rate of deformation and spin tensors outside it and the q_ℓ's .

$$\dot{\sigma}_{ij} = \mathscr{L}_{ijk\ell}D^{o}_{k\ell} + \sigma_{ik}\Omega^{o}_{jk} + \Omega^{o}_{ik}\sigma_{kj} + \mathscr{L}_{ijk\ell}n_k q_\ell - \frac{1}{2}[\sigma_{ik}(n_j q_k - n_k q_j) + \sigma_{kj}(n_i q_k - n_k q_i)] \quad , \tag{3.5}$$

where $\mathscr{L}_{ijk\ell}$ is the modulus tensor, evaluated as appropriate for the fields within the imperfection, in the constitutive rate relations given by

$$\overset{\nabla}{\sigma}_{ij} = \mathscr{L}_{ijk\ell}D_{k\ell} \quad . \tag{3.6}$$

Also, from the assumption of homogeneous deformation fields within and outside the imperfection, the continuity condition in stress exists.

$$n_i\sigma_{ij} = n_i\sigma^{o}_{ij} \quad , \quad \text{or} \quad n_i\Delta\sigma_{ij} = 0 \quad . \tag{3.7}$$

Here, the uniform stress fields within and outside the imperfection satisfy automatically the requirement that stress equilibrium must continue to be satisfied. Taking the time derivative of Eq. (3.7) in consideration of $\dot{n}_j = n_k(n_j n_\ell v_{k,\ell} - v_{k,j})$ (see Appendix) and Eq. (3.7), we can obtain the continuity condition in stress rate,

$$n_i \Delta\dot{\sigma}_{ij} - n_i v_{i,k} \Delta\sigma_{kj} = 0 \quad . \tag{3.8}$$

By using the Jaumann stress rate, Eq. (3.8) becomes

$$n_i \Delta\overset{\nabla}{\sigma}_{ij} + n_i R_{ijk\ell} n_\ell q_k - n_i v_{i,k} \Delta\sigma_{kj} = 0 \quad , \tag{3.9}$$

where

$$R_{ijk\ell} = \frac{1}{2} (\sigma_{ij}\delta_{k\ell} + \sigma_{i\ell}\delta_{jk} - \sigma_{ik}\delta_{j\ell} - \sigma_{kj}\delta_{i\ell}) \quad .$$

Substituting Eq. (3.2) and Eq. (3.6) into Eq. (3.9) yields the following relation,

$$n_i(\mathcal{L}_{ijk\ell} + R_{ijk\ell}) n_\ell q_k = n_i [(\mathcal{L}^o_{ijk\ell} - \mathcal{L}_{ijk\ell}) + \delta_{ik}\Delta\sigma_{\ell j}] D^o_{k\ell} \quad . \tag{3.10}$$

Here, if the body has no initial imperfection and if at the inception of localization, all material properties are the same within and outside a localization band in which non-uniform deformation occurs on localizing, as discussed by Rudnicki and Rice [12], the right-hand side of Eq. (3.10) becomes zero. Then, Eq. (3.10) reduces to the relation obtained by them,

$$n_i(\mathcal{L}_{ijk\ell} + R_{ijk\ell}) n_\ell q_k = 0 \quad , \tag{3.11}$$

and the localization condition for a perfect system is met when this equation first allows a non-zero solution for the q_k's.

If we express Eq. (3.10) in matrix representation, it can be written as

$$[a_{jk}][q_k] = [b_j] \quad , \tag{3.12}$$

where

$$[a_{jk}] = n_i(\mathcal{L}_{ijk\ell} + R_{ijk\ell}) n_\ell \quad ,$$

$$[b_j] = n_i[(\mathcal{L}^o_{ijk\ell} - \mathcal{L}_{ijk\ell}) + \delta_{ik}\Delta\sigma_{\ell j}] D^o_{k\ell} \quad .$$

Since Eq. (3.12) is a set of linear, homogeneous equations in the q_k's, this has a solution if and only if the determinant of $[a_{jk}]$ is not zero. Then, the q_k's are obtained as

$$[q_k] = [a_{jk}]^{-1}[b_j] \quad , \tag{3.13}$$

where

$$[a_{jk}]^{-1} = \frac{[A_{kj}]}{A} \quad , \quad A \neq 0 \quad .$$

Here, $[A_{kj}]$ is the cofactor of $[a_{jk}]$ and A is the determinant of $[a_{jk}]$.

Now, the condition is given for which the deformation rate fields within the imperfection become infinite, corresponding to a localization bifurcation. This condition is

$$A = \det[n_i(\mathscr{L}_{ijk\ell} + R_{ijk\ell})n_\ell] = 0 \quad . \tag{3.14}$$

Eq. (3.14) relates the hardening modulus at localization to the prevailing stress state and to the constitutive parameters (G,K, etc.). Rudnicki and Rice [12] have obtained the hardening modulus h which just allows the localization condition to be met on a surface with the normal x_2, and their result is

$$h = \frac{1}{K + \frac{4}{3}G}\left(G\frac{\sigma_{22}'}{\bar{\tau}} + K\beta\right)\left(G\frac{\sigma_{22}'}{\bar{\tau}} + K\mu\right) + G\left[\left(\frac{\sigma_{12}'}{\bar{\tau}}\right)^2 + \left(\frac{\sigma_{23}'}{\bar{\tau}}\right)^2\right] - (G+K\beta\mu)$$

$$+ \frac{2G}{2G+(\sigma_{22}'-\sigma_{11}')}\left(\frac{\sigma_{12}'}{\bar{\tau}}\right)^2\left[\frac{\bar{\tau}}{K + \frac{4}{3}G}\left(G\frac{\sigma_{22}'}{\bar{\tau}} + K\beta\right) - \frac{\sigma_{22}' - \sigma_{11}'}{2}\right] . \tag{3.15}$$

The localization condition of Eq. (3.14) is the same as the one with no initial imperfection, derived from Eq. (3.11) by Rudnicki and Rice [12], but the following things are different.

In the case with no initial imperfection, from the assumption that all material properties remain the same within and outside a localization band, the rate of deformation and spin tensors within and outside the band remain the same until the onset of localization. However, at the inception of localization, some of those tensor components become infinite to cause discontinuities in deformation rate within the band. On the other hand, in the case with an initial imperfection, some of the rate of deformation and spin tensor components within the imperfection continuously increase (or decrease) by comparison to those outside the imperfection by the factors q_i . Then, at the inception of localization, the rates within the imperfection become infinite, whereas those outside the imperfection remain finite.

4. Analyses and numerical results

In this section, with the use of the elastic-plastic constitutive rate relations for void-containing materials, the localization conditions with no imperfection, studied by Rudnicki and Rice [12], and with an initial imperfection, studied in section 3, are analyzed numerically in the cases of tensile extension under axially-symmetric and plane-strain conditions. Then, the results of these numerical analyses are discussed in connection with implications of the work for the theory of ductile fracture.

4.1. *Analysis for localization conditions with no imperfection*

Consider a void-containing body with void volume fraction f. Suppose the unit normal to the prospective band of localization is $\underline{n}$. Rudnicki and Rice [12] have shown that if the dilatancy factor (or α in the present case) is not large, the normal to the localization band becomes perpendicular to the σ_{II}-direction under most stress states of $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$. Here, σ_I, σ_{II} and σ_{III} are the principal true stress components. Thus, take the reference coordinate system to be (X_1, X_2, X_3), corresponding to the principal stress axes $(\sigma_I, \sigma_{II}, \sigma_{III})$ which are assumed to remain fixed throughout the pre-localized deformation, so that the normal to the band is perpendicular to the X_2-direction as in Fig. 1. Then, the unit normal to the band becomes

$$n_1 = \cos\theta \quad , \quad n_2 = 0 \quad , \quad n_3 = -\sin\theta \quad , \tag{4.1}$$

where θ is the angle between the X_1-axis and the normal.

The microscopic hardening modulus h_m, for void-containing materials, which just allows a localization bifurcation on a surface with the normal

x_2 can be obtained from Eq. (3.15), in consideration of the relations of Eqs. (2.17), as

$$h_m = \frac{3(1-f)}{\left(\omega + \frac{\sigma_{kk}}{\bar{\sigma}_m}\alpha\right)^2} \left\{ \frac{3}{\left(K + \frac{4}{3}G\right)}\left(G\frac{\sigma_{22}'}{\bar{\sigma}_m} + K\alpha\right)^2 + 3G\left[\left(\frac{\sigma_{12}'}{\bar{\sigma}_m}\right)^2 + \left(\frac{\sigma_{23}'}{\bar{\sigma}_m}\right)^2\right]\right.$$

$$+ \frac{6G}{2G+\left(\sigma_{22}'-\sigma_{11}'\right)}\left(\frac{\sigma_{12}'}{\bar{\sigma}_m}\right)\left[\frac{\bar{\sigma}_m}{\left(K+\frac{4}{3}G\right)}\left(G\frac{\sigma_{22}'}{\bar{\sigma}_m} + K\alpha\right) - \frac{\sigma_{22}'-\sigma_{11}'}{2}\right]$$

$$\left. - (G\omega + 3K\alpha^2) + \bar{\sigma}_m(1-f)\alpha\left[\cosh\left(\frac{1}{2}\frac{\sigma_{kk}}{\bar{\sigma}_m}\right) - f\right]\right\} . \tag{4.2}$$

Since all stress components, here referred to the (x_1, x_2, x_3) in Fig. 1, are functions of θ and the principal stresses, this equation gives the critical microscopic hardening modulus, h_{mcr}, at which the onset of localization is first possible, as a function of G, K, f, $\bar{\sigma}_m$ and the principal stresses. The critical microscopic hardening modulus is obtained as the maximum of h_m over all angles θ, because the value of h_m is a decreasing function of the microscopic equivalent tensile strain. The corresponding angle θ is the angle between the X_1-axis and the normal to the band in Fig. 1. By considering Eq. (4.1) and following the details of Rudnicki and Rice [12] (Eq. (24) in their paper), the critical microscopic hardening modulus is given to the neglect of the term $(\bar{\sigma}_m/G)^2$ and smaller by

$$h_{mcr} = \frac{3(1-f)}{(\omega + \frac{\sigma_{kk}}{\bar{\sigma}_m}\alpha)^2}\left\{- \frac{3G}{2}(1+\nu)\left(\frac{\sigma_{II}'}{\bar{\sigma}_m} + \frac{2\alpha}{3}\right)^2\right.$$

$$\left. + \bar{\sigma}_m (1-f)\alpha\left[\cosh\left(\frac{1}{2}\frac{\sigma_{kk}}{\bar{\sigma}_m}\right) - f\right]\right\} , \tag{4.3}$$

where

$$\cos 2\theta_{cr} = \frac{-(1-2\nu)\left(\frac{\sigma_{II}'}{\bar{\sigma}_m}\right) + \frac{4(1+\nu)}{3}\alpha}{\left(\frac{\sigma_I' - \sigma_{III}'}{\bar{\sigma}_m}\right)} .$$

Here, σ_I' , σ_{II}' and σ_{III}' are the deviatoric principal true stress components, θ_{cr} is the critical angle which maximizes h_m , and ν is Poisson's ratio.

When analyzing problems of the localization bifurcation, we take the relation between true shear stress and total equivalent shear strain (elastic plus plastic) for the material of the matrix surrounding the voids to be the so-called power hardening law,

$$\frac{\tau}{\tau_y} = \begin{cases} \frac{\gamma}{\gamma_y} & \text{for} \quad \bar{\sigma} < \sigma_y \\ \left(\frac{\gamma}{\gamma_y}\right)^N & \text{for} \quad \bar{\sigma} \geq \sigma_y \end{cases} , \tag{4.4}$$

where $\tau = \bar{\sigma}/\sqrt{3}$, $\gamma = \bar{\varepsilon}/\sqrt{3}$, $\tau_y = \sigma_y/\sqrt{3}$ and $\gamma_y = 2(1+\nu)\varepsilon_y/\sqrt{3}$. N is a strain-hardening coefficient and σ_y and ε_y are the true stress and strain at yield for a material with no voids in the uniaxial tensile test. This relation is plotted in Fig. 2 for three sets of $N = 0.05$, 0.1 and 0.2. From Eq. (4.4), the microscopic hardening modulus h_m is calculated as

$$h_m = \frac{3G}{\frac{1}{N}\left(\frac{\bar{\sigma}_m}{\sigma_y}\right)^{\frac{1-N}{N}} - 1} . \tag{4.5}$$

Thus, by seeking the point at which the value of h_m in Eq. (4.5) becomes equal, with continuing plastic deformation, to that of h_{mcr} in Eq. (4.3), we can determine the critical strain which gives the first bifurcation point in the body.

In the special case of $f = 0$, the dilatancy factor becomes zero and the value of the critical microscopic hardening modulus becomes non-positive to the neglect of the term $(\bar{\sigma}_m/G)^2$ and smaller.

$$h_{mcr} = -\frac{9G}{2}(1+\nu)\left(\frac{\sigma_{II}'}{\bar{\sigma}_m}\right)^2 \leq 0 \quad . \tag{4.6}$$

This suggests that localization never occurs at a finite strain because the value of the microscopic hardening modulus is expected to be positive. However, in the case of axially-symmetric extension with $f > 0$, localization can be expected to occur in a homogeneous body only at a very large strain, at least in the absence of significant stress triaxiality. Especially, in the case of uniaxial extension, even if the value of f becomes 0.1, α is approximately of the magnitude 0.01 while $|\sigma_{II}'/\bar{\sigma}_m|$ is about 20 times larger than the value of α around realistic values of strain. Thus, for a not large value of the strain-hardening coefficient, h_{mcr} becomes a strongly negative value and localization is expected to occur at a tremendously large strain.

In the case of plane-strain extension with $f > 0$, the value of the critical hardening modulus can become positive for realistic values of f and has the possibility of allowing localization at a finite strain. For example, Fig. 3 shows curves of the ratio of critical strain in the tensile direction to yield strain, $\varepsilon_I^{cr}/\varepsilon_y$, versus initial void volume fraction f_o for the case of uniaxial plane-strain extension with $\dot{\sigma}_I > \dot{\sigma}_{II}(D_{II} = 0) > \dot{\sigma}_{III} = 0$. Here, $\dot{\sigma}_I$, $\dot{\sigma}_{II}$ and $\dot{\sigma}_{III}$ are the principal true stress rates and D_{II} is the principal rate of deformation in the X_2-direction. The material modelled has Poisson's ratio of 0.3 and ϵ_y of 0.003. Also, curves of the critical angle Θ_{cr} versus initial void volume fraction f_o is shown in Fig. 4.

The critical strain depends strongly on the strain-hardening coefficient N and on the initial void volume fraction f_o, and it decreases with the decrease of N or with the increase of f_o. If a material has a large value of the initial void volume fraction, or if a material has a very small value of the strain-hardening coefficient like a non-hardening material, localization may be expected to occur at a reasonably finite strain as an incipient point of ductile fracture. However, if the initial void volume fraction is not large, even with a small value of N, or if the strain-hardening coefficient is not small, even with a large value of f_o, the predicted strains to localization seem rather large.

Thus, the case with no initial imperfection does not necessarily seem to allow localization at reasonable strains. This suggests a possible significance of initial imperfections, which are considered next.

4.2. *Analysis for localization conditions with an initial imperfection*

Consider a body with an initial, finite imperfection with a sufficiently thin slice, so that its strain does not affect the total strain. Assume the initial imperfection to be a void-containing material with void volume fraction f and with the same elastic shear and bulk moduli as the material surrounding the imperfection has, and suppose that material also to be a void-containing material with void volume fraction f^o which is different from the value of f. Take the reference coordinate axes to be (X_1, X_2, X_3), corresponding to the principal stress axes $(\sigma^o_I, \sigma^o_{II}, \sigma^o_{III})$ which are assumed to remain fixed throughout the pre-localized deformation. Also, for the same reason described in section 4.1, take the current coordinate axes to be (x_1, x_2, x_3), so that the x_2-axis is normal to the planar band of the imperfection and the x_3-axis coincides with the X_2-axis as in Fig. 1.

If we choose the coordinate systems as in Fig. 1, we can take q_3 to be zero. Then, with the use of the constitutive rate relations given by Eq. (2.14), Eq. (3.13) can be expressed in terms of the rate of deformation tensor outside the imperfection as

$$
\begin{aligned}
q_1 = \frac{1}{J}\Bigg\{ &(\sigma_{12}')\left(K + \frac{4}{3}G + \sigma_{22}' + \frac{K}{G}\bar{\sigma}_m\alpha\right)\left(G\sigma_{k\ell}' + K\bar{\sigma}_m\alpha\delta_{k\ell}\right) \\
&- (\sigma_{12}')\Bigg[\frac{I}{I^o}\left(\frac{\bar{\sigma}_m}{\bar{\sigma}_m^o}\right)^2\left(K + \frac{4}{3}G + \sigma_{22}^{o\prime} + \frac{K}{G}\bar{\sigma}_m^o\alpha^o\right) \\
&- \frac{3(G\sigma_{22}' + K\bar{\sigma}_m\alpha)}{I^o\bar{\sigma}_m^{o2}}\left(G(\sigma_{22}' - \sigma_{22}^{o\prime}) + K(\bar{\sigma}_m\alpha - \bar{\sigma}_m^o\alpha^o)\right)\Bigg]\left(G\sigma_{k\ell}^{o\prime} + K\bar{\sigma}_m^o\alpha^o\delta_{k\ell}\right) \\
&+ \left[(K + \frac{4}{3}G)\frac{I\,\bar{\sigma}_m^2}{3} - (G\sigma_{22}' + K\bar{\sigma}_m\alpha)^2\right]\frac{(\sigma_{\ell 1} - \sigma_{\ell 1}^o)}{G}\delta_{2k}\Bigg\} D_{k\ell}^o \quad ,
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
q_2 = \frac{1}{J}\Bigg\{ &\left(1 + \frac{\sigma_{22}' - \sigma_{11}'}{2G}\right)\left(G\,\sigma_{22}' + K\,\bar{\sigma}_m\alpha\right)\left(G\,\sigma_{k\ell}' + K\,\bar{\sigma}_m\alpha\delta_{k\ell}\right) \\
&- \Bigg[\frac{I}{I^o}\left(\frac{\bar{\sigma}_m}{\bar{\sigma}_m^o}\right)^2\left(1 + \frac{\sigma_{22}' - \sigma_{11}'}{2G}\right)\left(G\sigma_{22}^{o\prime} + K\bar{\sigma}_m^o\alpha\right) \\
&+ \frac{3G(\sigma_{12}')^2}{I^o\bar{\sigma}_m^{o2}}\left(G(\sigma_{22}' - \sigma_{22}^{o\prime}) + K(\bar{\sigma}_m\alpha - \bar{\sigma}_m^o\alpha^o)\right)\Bigg]\left(G\sigma_{k\ell}^{o\prime} + K\bar{\sigma}_m^o\alpha^o\delta_{k\ell}\right) \\
&+ (G\sigma_{12}')(G\sigma_{22}' + K\bar{\sigma}_m\alpha)\frac{(\sigma_{\ell 1} - \sigma_{\ell 1}^o)}{G}\delta_{2k}\Bigg\} D_{k\ell}^o \quad ,
\end{aligned}
$$

$$q_3 = 0 \quad ,$$

where

$$I = \frac{1}{3} H + \omega G + 3\alpha^2 K \quad ,$$

$$H = \left\{ \frac{h_m}{(1-f)} \left(\omega + \frac{\sigma_{kk}}{\bar{\sigma}_m} \alpha \right)^2 - 3\bar{\sigma}_m (1-f) \alpha \left[\cosh\left(\frac{1}{2} \frac{\sigma_{kk}}{\bar{\sigma}_m} \right) - f \right] \right\} ,$$

$$J = \left\{ \left[\frac{1}{3} I \bar{\sigma}_m^{\,2} \left(1 + \frac{\sigma'_{22} - \sigma'_{11}}{2G} \right) - G(\sigma'_{12})^2 \right] (K + \frac{4}{3} G) \right.$$

$$\left. - \left(1 + \frac{\sigma'_{22} - \sigma'_{11}}{2G} \right) (G\sigma'_{22} + K\bar{\sigma}_m \alpha)^2 - (\sigma'_{12})^2 (G\sigma'_{22} + K\bar{\sigma}_m \alpha) \right\} .$$

The rate of deformation and spin tensors within the imperfection are given in terms of the rate of deformation tensor outside it by Eq. (3.3), Eq. (3.4) and Eq. (4.7). These show that there are discontinuities between the inside and outside of the imperfection in the rate of deformation components, D_{12} , D_{21} and D_{22} , and in the spin tensor components, Ω_{12} and Ω_{21} . Also, the true stress rate tensor within the imperfection is expressed in terms of the rate of deformation and spin tensors outside it by Eq. (3.5) and Eq. (4.7).

The rate of deformation and spin tensors outside the imperfection are given by

$$D^o_{ij} = \mathscr{M}^o_{ijk\ell} \overset{\nabla o}{\sigma}_{ij} \quad , \tag{4.8}$$

$$\Omega^o_{12} = - \Omega^o_{21} = \dot{\theta} \quad , \quad \Omega^o_{ij} = 0 \quad \text{(otherwise)},$$

where

$$\dot{\theta} = \left(\frac{\tan\theta}{1+\tan^2\theta} \right) \left[\frac{1}{2G} (\dot{\sigma}^o_I - \dot{\sigma}^o_{II}) \right]$$

for elastic loading or any unloading and

$$\dot{\theta} = \left(\frac{\tan\theta}{1+\tan^2\theta}\right)\left\{\frac{1}{2G}(\dot{\sigma}^o_I - \dot{\sigma}^o_{II})\right.$$

$$\left. + \frac{9}{4H^o}\ \frac{(\sigma^{o'}_I - \sigma^{o'}_{III})}{\bar{\sigma}^{o2}_m}\left[\sigma^{o'}_I\dot{\sigma}^o_I + \sigma^{o'}_{II}\dot{\sigma}^o_{II} + \sigma^{o'}_{III}\dot{\sigma}^o_{III} + \frac{2\bar{\sigma}^o_m\alpha^o}{3}\left(\dot{\sigma}^o_I + \dot{\sigma}^o_{II} + \dot{\sigma}^o_{III}\right)\right]\right\}$$

for loading at yield. $\mathscr{M}_{ijk\ell}$ is given in Eq. (2.13) and θ is the angle between the x_2-axis and X_1-axis in Fig. 1. If the principal stress rate in the reference system $(\dot{\sigma}^o_I\ ,\ \dot{\sigma}^o_{II}\ ,\ \dot{\sigma}^o_{III})$ are specified, the Jaumann stress rates in Eq. (4.8) and the true stress rates outside the imperfection are expressed in terms of the principal stress rates in the reference system as

$$\begin{aligned}
\overset{\nabla}{\sigma}{}^o_{11} &= \dot{\sigma}^o_{11} - \sin2\theta\,(\sigma^o_I - \sigma^o_{III})\dot{\theta} = \frac{1}{2}\left[(1-\cos2\theta)\dot{\sigma}^o_I + (1+\cos2\theta)\dot{\sigma}^o_{III}\right] \quad , \\
\overset{\nabla}{\sigma}{}^o_{22} &= \dot{\sigma}^o_{22} + \sin2\theta\,(\sigma^o_I - \sigma^o_{III})\dot{\theta} = \frac{1}{2}\left[(1+\cos2\theta)\dot{\sigma}^o_I + (1-\cos2\theta)\dot{\sigma}^o_{III}\right] \quad , \\
\overset{\nabla}{\sigma}{}^o_{33} &= \dot{\sigma}^o_{33} = \dot{\sigma}^o_{II} \quad , \\
\overset{\nabla}{\sigma}{}^o_{12} &= \dot{\sigma}^o_{12} - \cos2\theta\,(\sigma^o_I - \sigma^o_{III})\dot{\theta} = \frac{1}{2}\sin2\theta\,(\dot{\sigma}^o_I - \dot{\sigma}^o_{III}) \quad , \\
\overset{\nabla}{\sigma}{}^o_{21} &= \overset{\nabla}{\sigma}{}^o_{12} \quad , \quad \dot{\sigma}^o_{21} = \dot{\sigma}^o_{12} \quad , \\
\overset{\nabla}{\sigma}{}^o_{13} &= \overset{\nabla}{\sigma}{}^o_{31} = \overset{\nabla}{\sigma}{}^o_{23} = \overset{\nabla}{\sigma}{}^o_{32} = 0 \quad , \quad \dot{\sigma}^o_{13} = \dot{\sigma}^o_{31} = \dot{\sigma}^o_{23} = \dot{\sigma}^o_{32} = 0 \quad .
\end{aligned} \tag{4.9}$$

Take the particular stress-strain relation for the material of the matrix surrounding the voids to be the power hardening law given by Eq. (4.4). Then, the microscopic hardening modulus h_m is obtained as Eq. (4.5).

Thus, if the initial angle θ_o , initial void volume fraction f_o within the imperfection and f^o_o outside the imperfection, and principal true stress increments $(\Delta\sigma^o_I\ ,\ \Delta\sigma^o_{II}\ ,\ \Delta\sigma^o_{III})$ are specified, the increments of deformation, spin and stress tensors outside and within the imperfection can be calculated step by step, following the given stress-strain relation.

Then, a bifurcation point at which the imperfection is driven to a localization bifurcation can be obtained by seeking a point which causes the deformation rate fields within the imperfection to be infinite. Here, define the localization strain as the value of strain which brings to a localization bifurcation an imperfect slice having the initial angle θ_o. Also, the critical strain, over all orientations of an initial imperfection of given amplitude, can be determined by seeking the minimum of localization strains over all initial angles θ_o.

Now, we analyze the localization conditions in two cases, axially-symmetric extension with $\dot{\sigma}^o_I > \dot{\sigma}^o_{II} = \dot{\sigma}^o_{III} = 0$ and plane-strain extension with $\dot{\sigma}^o_I > \dot{\sigma}^o_{II}$ $(D^o_{II} = 0) > \dot{\sigma}^o_{III} = 0$. The material properties are the same as those with no imperfection. Fig. 5 shows an example for curves of the ratio of a rate of deformation component within an imperfection, D_{12}, to outside it, D^o_{12}, versus true strain in the tensile direction, ε_I. As the strain approaches a localization point, the ratio increases rapidly and at the localization point, it goes to infinity. This means that the onset of localization corresponds to the point at which enormous growth of an imperfection starts to cause the deformation rate fields to be infinite within the imperfection while the steady deformation continues outside it. Fig. 6 shows an example for curves of the localization strain ε^{loc}_I versus initial angle θ_o. The minimum of the localization strains gives the critical strain at which the inception of localization is first possible. Curves of the ratio of critical strain to yield strain, $\varepsilon^{cr}_I/\varepsilon_y$, versus difference of initial void volume fractions between the inside and outside of an imperfection, $\Delta f = f_o - f^o_o$, are shown in Fig. 7 in the case of $N = 0.1$ and 0.2 with $f^o_o = 0$ and 0.01. Also, curves of the critical angle θ_{cr} versus difference of initial void volume fractions, Δf_o , are shown in Fig. 8.

In the case of uniaxial extension under axially-symmetric conditions, a material with no initial imperfection can localize only at an unrealistically large strain, whereas a small initial imperfection allows localization at a reasonably finite strain. Also, in the case of uniaxial extension under plane-strain conditions, although localization is possible to occur at a finite strain in both cases, the strain at localization with an initial imperfection can be much smaller than with no initial imperfection. For example, the critical strains with $f_o^o = 0$, shown in Fig. 7, are two to five times smaller than those with no imperfection, shown in Fig. 3. Thus, the existence of an initial imperfection promotes a localization bifurcation strongly in comparison to the case with no imperfection, and makes it possible to allow localization at a reasonable strain in both uniaxial extension cases under axially-symmetric and plane-strain conditions.

The critical strain with an initial imperfection depends on the initial void volume fractions within and outside an imperfection and on the strain-hardening coefficient as shown in Fig. 7. These correlations are almost analogous to those with no imperfection and furthermore, as the initial void volume fraction outside an imperfection, f_o^o , increases, localization seems to become more imperfection sensitive. The strain-hardening dependence of critical strain is consistent with the suggestion from the hole growth analysis of McClintock [8] for ductile fracture that the increase in fracture strain is expected from increased strain-hardening, and the void volume fraction dependence is also compatible with the experimental results of several copper dispersion alloys tested by Edelson and Baldwin [15].

Fig. 9 shows curves of the ratio of plane-strain critical strain to axially-symmetric critical strain, $\left(\varepsilon_I^{cr}\right)_{plane}/\left(\varepsilon_I^{cr}\right)_{axi}$, versus reciprocal of strain-hardening coefficient, $1/N$. The ratio, $\left(\varepsilon_I^{cr}\right)_{plane}/\left(\varepsilon_I^{cr}\right)_{axi}$, decreases with the decrease of N in all cases presented there, and the slope of their curves becomes steep with the increase of the difference of initial void volume fractions, Δf_o , under constant f_o^o or with the increase of the initial void volume fraction outside an imperfection, f_o^o , under constant Δf_o . Rosenfield and Hahn [16] have investigated the relation between the yield strength and the strain-hardening coefficient for plain carbon steels to show that the reciprocal of strain-hardening coefficient is approximately proportional to the yield strength. In consideration of this relation, the strain-hardening dependence of the ratio, $\left(\varepsilon_I^{cr}\right)_{plane}/\left(\varepsilon_I^{cr}\right)_{axi}$, seems to be consistent with the experimental results reported by Clausing [17] that the ratio of plane-strain tension ductility to axially-symmetric tension ductility could vary from 72 percent to 17 percent for a range of steels, generally decreasing with increasing yield strength level. However, as a whole, the values of its ratio in Fig. 9 look a little smaller than those obtained by Clausing [17]. This may be due to the following factors. First, the necking effect has been neglected. Necking, which will generally occur before localization, is known to induce significant stress triaxiality to exert a strong influence on void growth, which is seen by the equations of Section 2. This is expected to reduce the critical strain obtained in Fig. 7 more in the case of uniaxial extension under axially-symmetric conditions than under plane-strain conditions. Second, the imperfection has been assumed as an initial one in this analysis. However, since void nucleation occurs after some plastic deformation [3,5,18], there is an incubation period depending on the type of a material, stress state and so on until the initiation of void nucleation. This would increase the predicted strain levels in both uniaxial extension cases, and thus increase somewhat the ratios shown in Fig. 9.

So far, we have analyzed conditions for a localization bifurcation with an initial imperfection. This analysis is valid only for such a single imperfection or parallel imperfections that allow the deformation fields in the material surrounding imperfections to remain uniform. However, this analysis might be helpful to estimate an incipient point of ductile fracture in an actual material with a random distribution of initial imperfections. Thus, by considering the necking effect in addition to adequate choice of a strain-hardening coefficient and initial void volume fractions within and outside an imperfection, according to the type of a material, it may be possible to explain experimental results more sufficiently in the present context.

5. Conclusion

Conditions for a localization bifurcation with an initial imperfection have been derived from the viewpoint that at the inception of localization, the macroscopic constitutive relations allow the deformation rate fields to become unbounded within an imperfection but to remain finite outside it. The critical conditions have been analyzed numerically with the use of the constitutive rate relations for void-containing materials in the cases of tensile extension under axially-symmetric and plane-strain conditions to compare with the case with no initial imperfection.

The critical strain with an initial imperfection depends on the initial void volume fractions within and outside an imperfection and on the strain-hardening coefficient. It decreases with the decrease of the strain-hardening coefficient or with the increase of the initial void volume fraction within an imperfection and also, localization becomes more imperfection sensitive with the increase of the initial void volume fraction outside an imperfection.

The existence of an initial imperfection greatly reduces the strain to localization in both cases. Especially, in the case of uniaxial extension under axially-symmetric conditions, its existence makes it possible to cause localization at a reasonable strain, whereas a material with no initial imperfection can localize only at an unrealistically large strain. Also, the case with an initial imperfection makes it clear quantitatively that the critical strain is smaller in the case of plane-strain extension than in the case of axially-symmetric extension. This is consistent with the experimental results that the tensile ductility is smaller in the former case than in the latter case. However, as necking seems to have a strong influence on void growth, the effect of necking should be taken into account.

Appendix: derivation of the time rate of the unit normal to the planar band of an imperfection.

Let $\underline{N}$ and $\underline{n}$ be the unit normal vectors to the planar band of an imperfection in the reference and current coordinate systems respectively. $\underline{n}$ can be expressed in terms of $\underline{N}$ as follows [19].

$$n_j = \frac{N_k F^{-1}_{kj}}{R} , \qquad \text{(A.1)}$$

where

$$R = |N_k F^{-1}_{kj}| .$$

Taking the time derivative of Eq. (A.1) gives

$$\dot{R} n_j + R \dot{n}_j = N_k \dot{F}^{-1}_{kj} .$$

Here,

$$N_k \dot{F}^{-1}_{kj} = - R n_k v_{k,j} ,$$

$$\dot{R} n_j = - R n_j n_k n_\ell v_{k,\ell} .$$

Thus,

$$\dot{n}_j = n_k (n_j n_\ell v_{k,\ell} - v_{k,j}) . \qquad \text{(A.2)}$$

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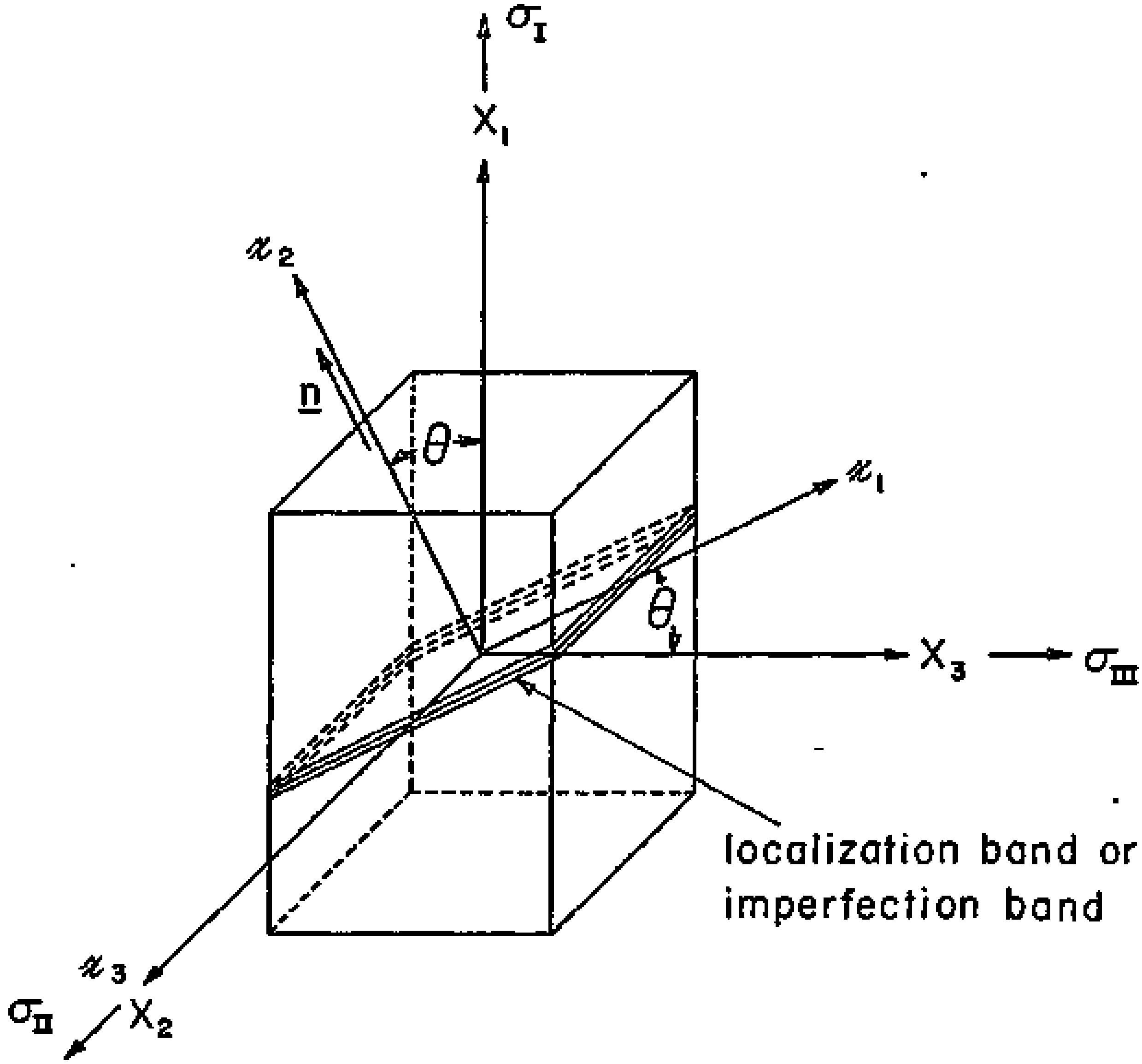

Figure 1. Coordinate systems for the localization band or imperfection band. (X_1, X_2, X_3) is the reference coordinate for the localization band or imperfection band. (x_1, x_2, x_3) is the current coordinate for the imperfection band.

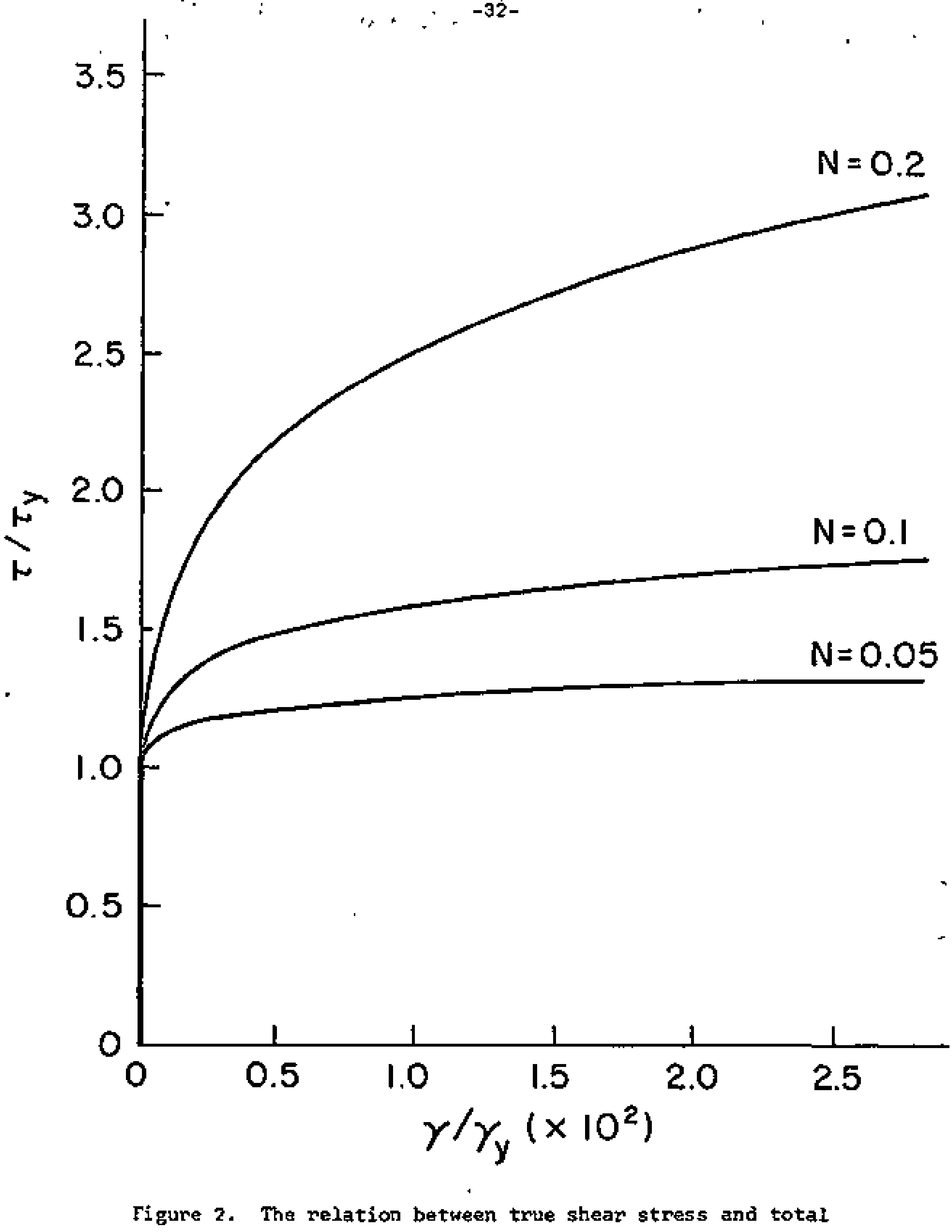

Figure 2. The relation between true shear stress and total equivalent shear strain for the material of the matrix surrounding the voids.

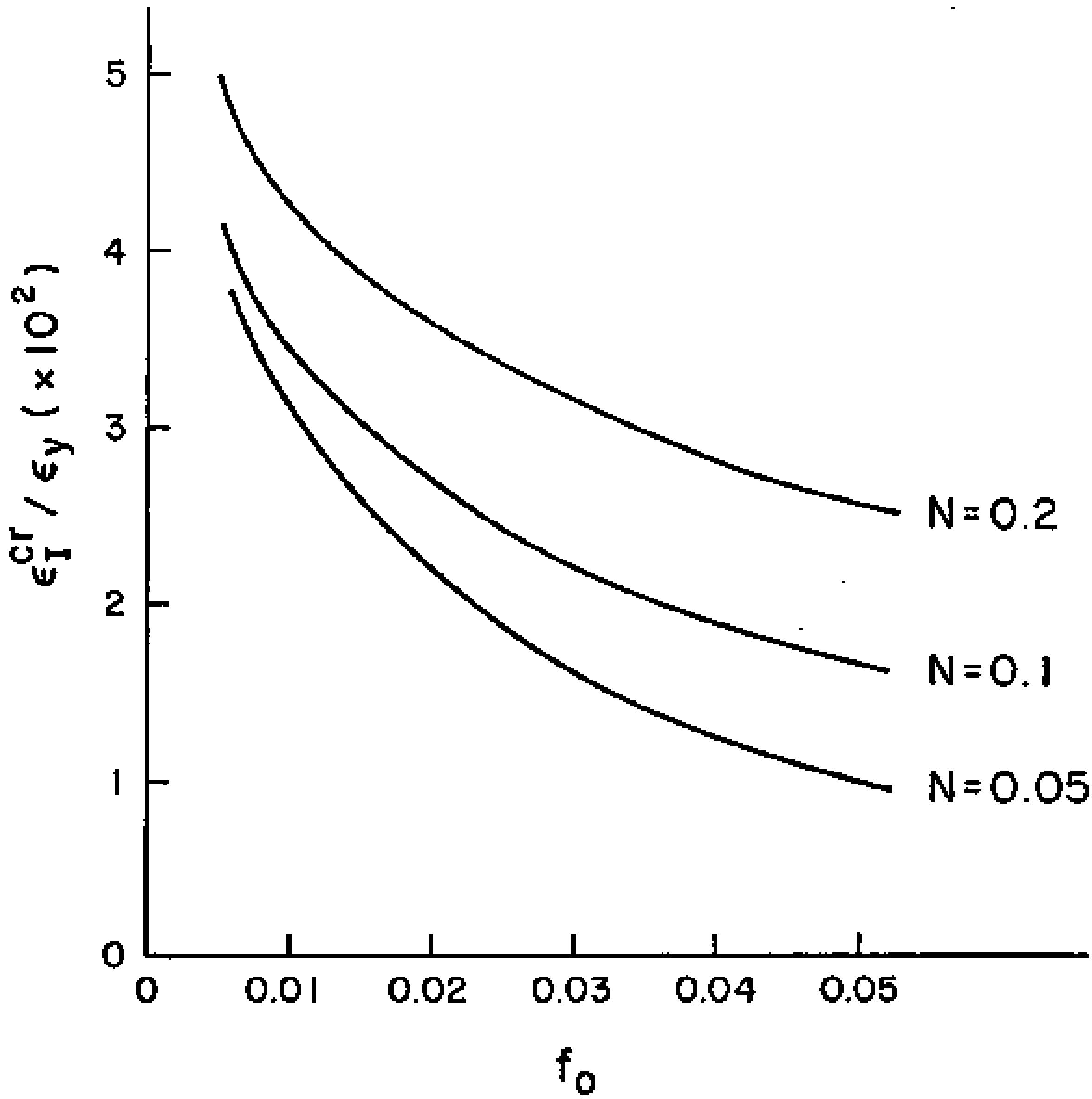

Figure 3. Curves of the ratio of critical strain to yield strain, $\epsilon_I^{cr}/\epsilon_y$, vs. initial void volume fraction f_o with no imperfection for the case of uniaxial plane-strain extension (for ϵ_y = 0.003).

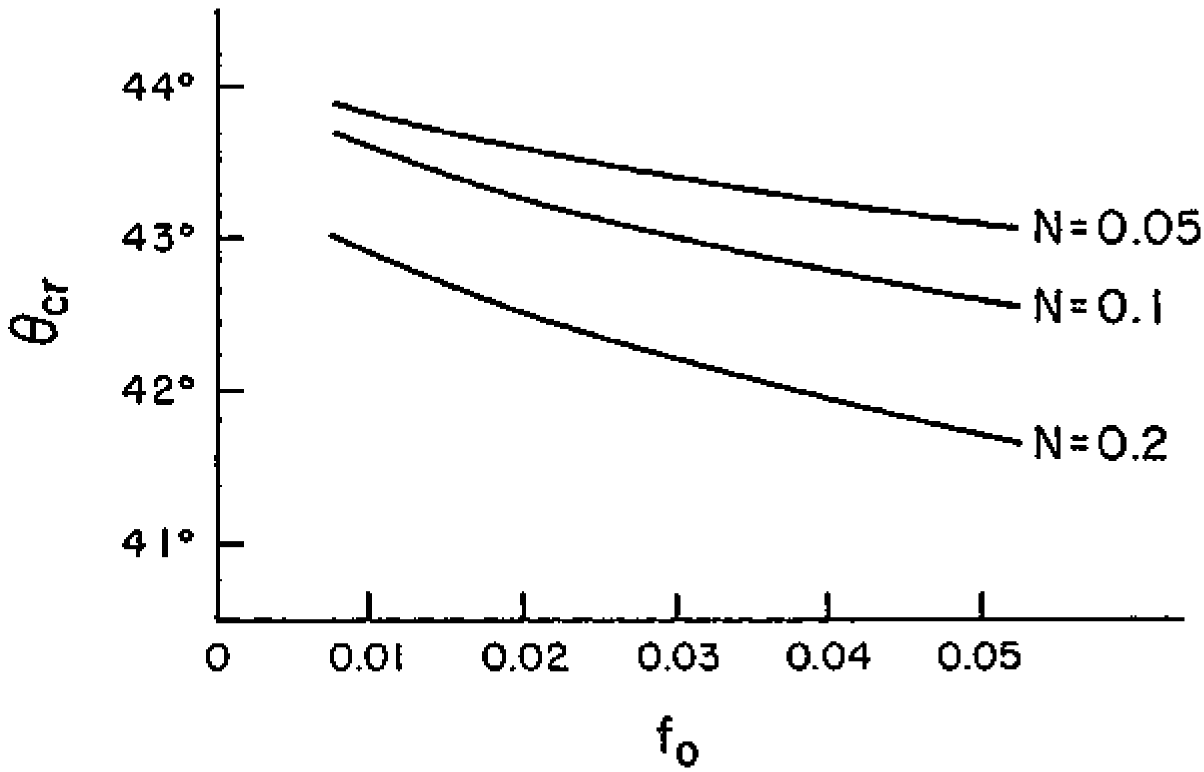

Figure 4. Curves of the critical angle Θ_{cr} vs. initial void volume fraction f_o with no imperfection for the case of uniaxial plane-strain extension.

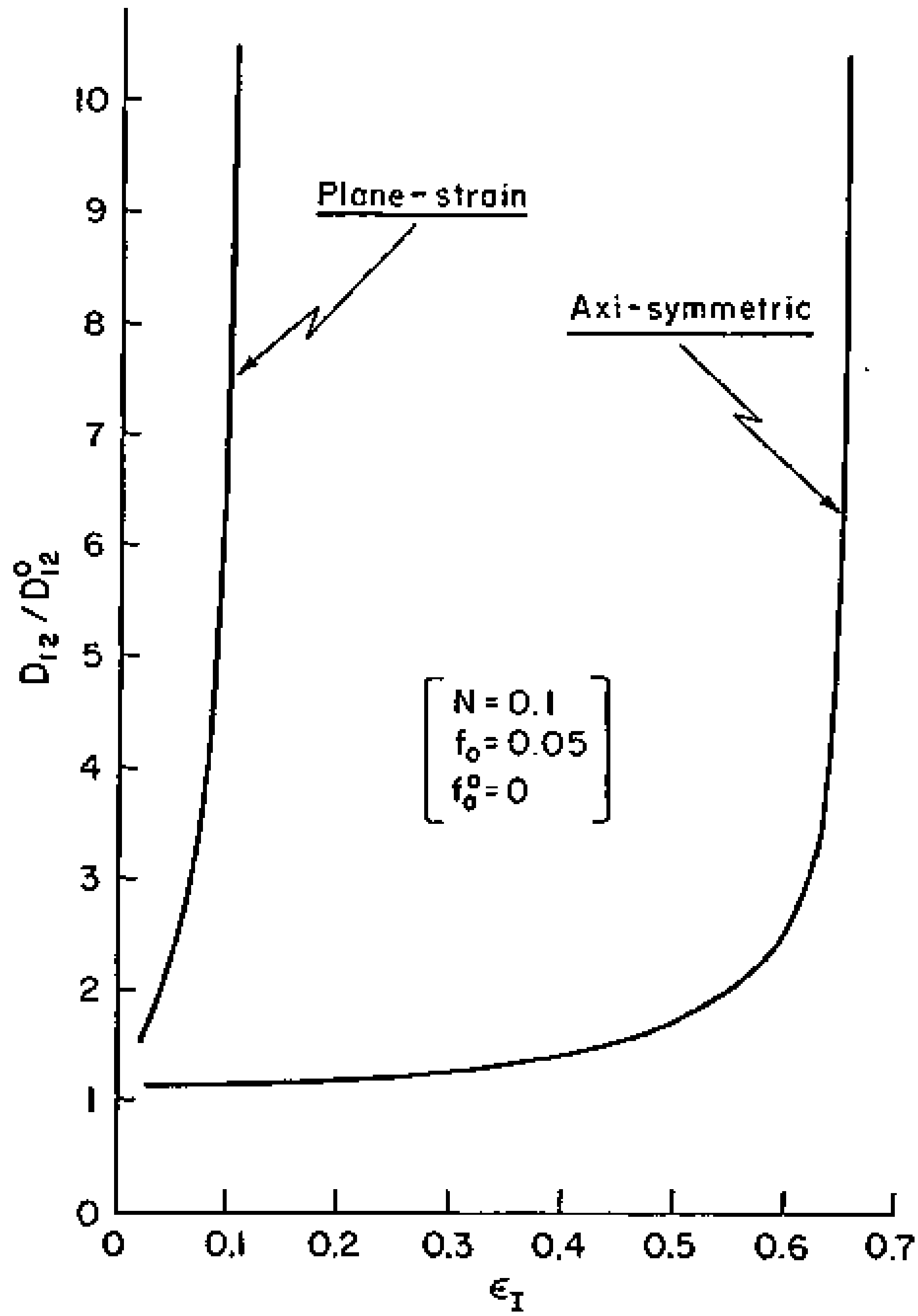

Figure 5. Curves of the ratio of a rate of deformation within an imperfection, D_{12}, to outside it, D_{12}^{o}, vs. true strain ϵ_I at the angle which gives the critical strain.

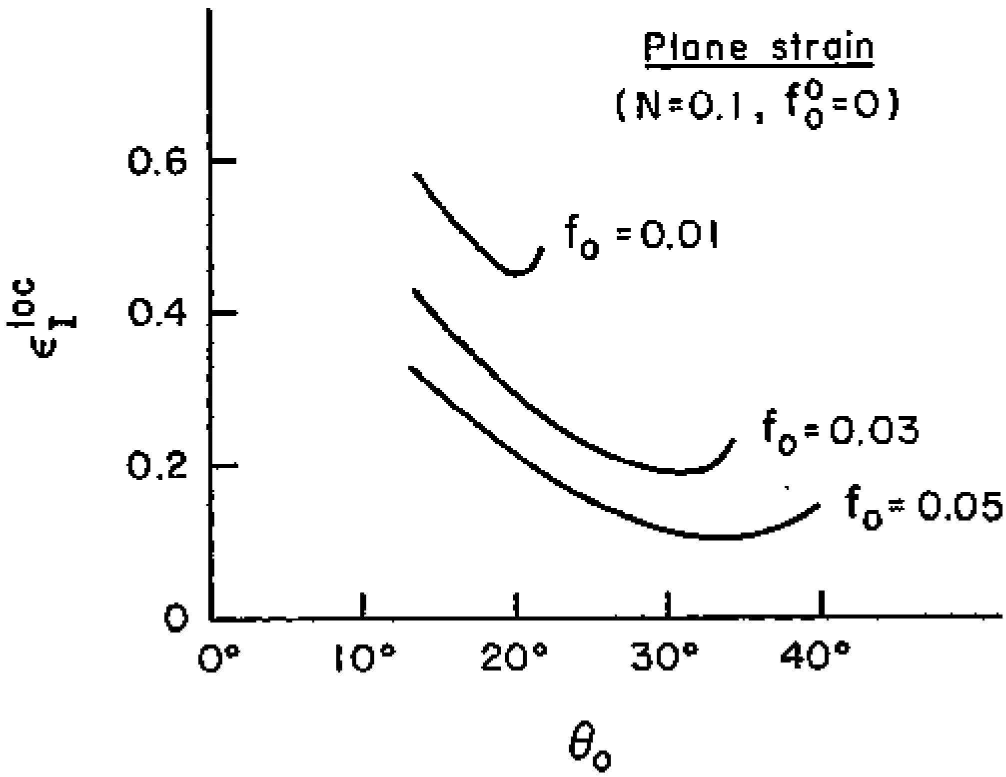

Figure 6. Curves of the localization strain ϵ_I^{loc} vs. initial angle θ_o. The minimum of ϵ_I^{loc} gives the critical strain ϵ_I^{cr}.

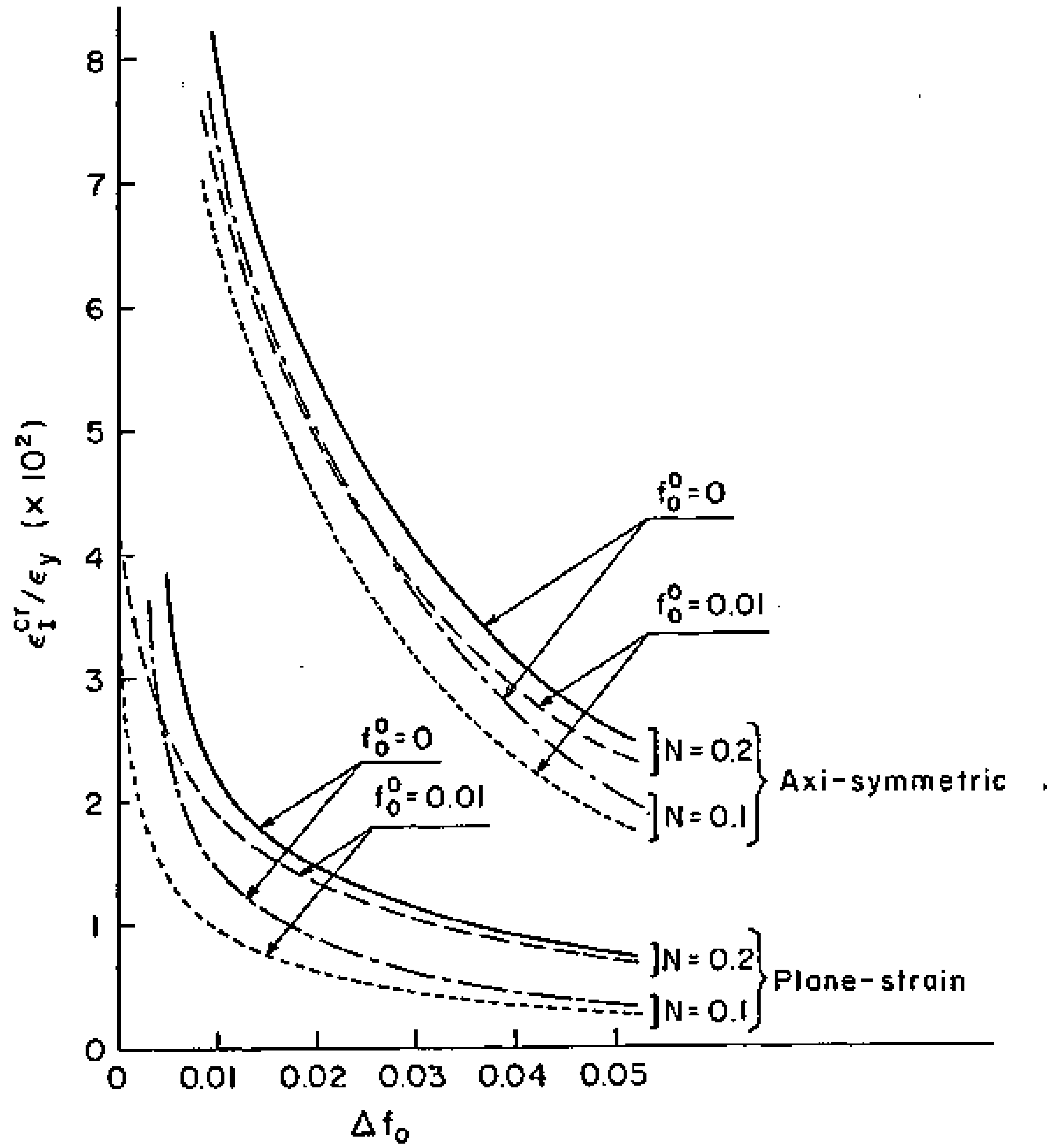

Figure 7. Curves of the ratio of critical strain to yield strain, $\epsilon_I^{cr}/\epsilon_y$, vs. difference of initial void volume fractions between the inside and outside of an imperfection, Δf_o (for $\epsilon_y = 0.003$).

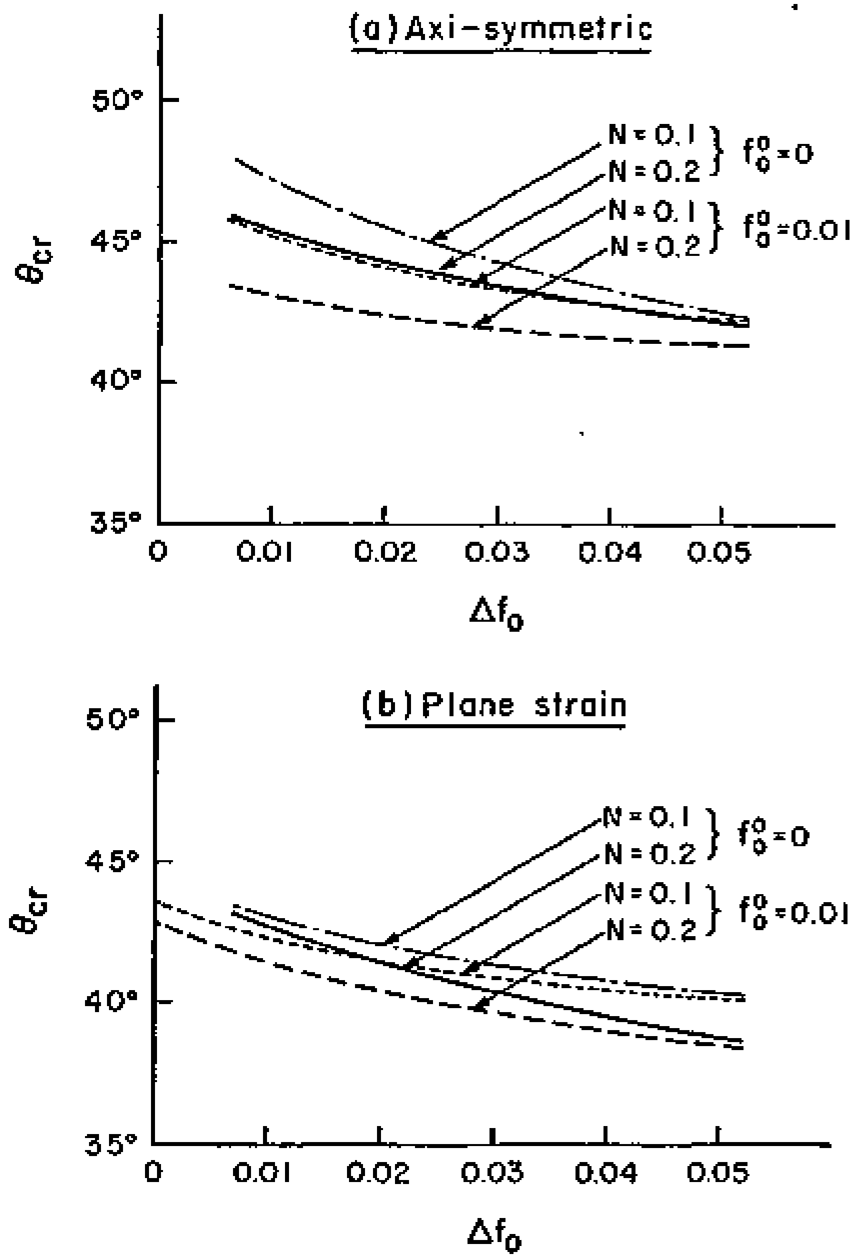

Figure 8. Curves of the critical angle θ_{cr} vs. difference of initial void volume fractions between the inside and outside of an imperfection, Δf_o .

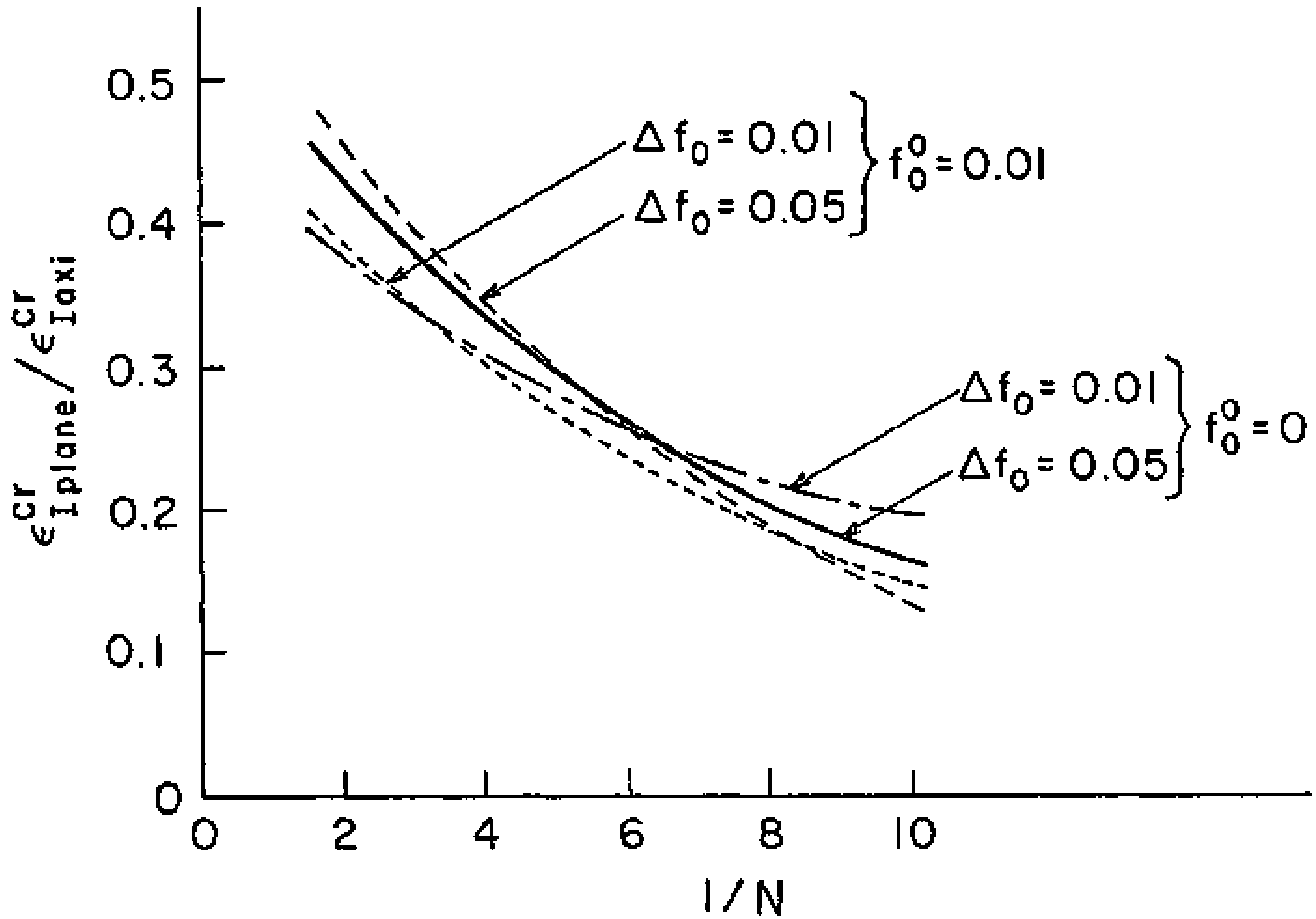

Figure 9. Curves of the ratio of plane-strain critical strain to axially-symmetric critical strain, $\left(\epsilon_I^{cr}\right)_{plane}/\left(\epsilon_I^{cr}\right)_{axi}$, vs. reciprocal of strain-hardening coefficient, $1/N$.