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PARTICLE SIMULATION MODELS
AND
THEIR APPLICATION TO CONTROLLED FUSION

BY

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Particle Simulation Models and
Their Application to Controlled Fusion*

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- * Presented at the American Nuclear Society, Tucson, Arizona, March 28-30, 1977.

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Plasma simulation models using particles are described which have been developed for studying the microscopic behavior of a confined plasma in a magnetic field. The first model is developed to investigate the anomalous diffusion of particles and energy due to low-frequency electrostatic microinstabilities in cylindrical and toroidal systems. The model makes use of the combination of eigenfunction expansion in one direction and the multipole expansion on a two-dimensional spatial grid for solving the Maxwell equations and for pushing particles. The second model is developed to study the neutral beam injection heating of a tokamak plasma taking into account the spatial variation of plasma parameters and the finite ion beam banana orbit. The self-consistent electric and magnetic fields are totally ignored in this model and the Fokker-Planck collisions on the beam ions due to background ions and electrons are built in through the Monte Carlo method.

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I. INTRODUCTION

One of the key problems for the successful achievement of controlled nuclear fusion is to understand the mechanisms of anomalous diffusion of particles and energy of plasmas in a confining magnetic field and its scaling laws with respect to various plasma parameters so that the anomalous diffusion may be controlled in the experiments. While it is generally accepted that the most likely candidate for the anomalous diffusion is plasma turbulence generated by the low-frequency microinstabilities driven by the plasma density and temperature gradients, the theoretical analysis of the nonlinear behavior of the instabilities and the estimate of the resultant anomalous diffusion is an extremely difficult problem as proved by the history of plasma physics and controlled fusion research in the past several decades. There are reasons for this: first, the basic equation, Boltzmann transport equation, is highly nonlinear due to the presence of long-range electromagnetic fields; secondly, high-temperature fusion plasmas are often collisionless so that the particle velocity distribution may depart significantly from Maxwell distribution. This means that the plasma problems are often in six-dimensional phase space, not just in three dimensions. Furthermore, the enormous variance of the experimental parameters makes any analysis highly limited. The complexities

of the geometry of the actual toroidal devices, the inaccessibility and the uncertainties of the experimental measurements make it often impossible to compare the theoretical predictions with the experimental results. Furthermore, there are even more complex atomic processes and plasma wall interactions.

Since the plasma problem is so complex in real experimental devices, it is now apparent that various numerical approaches are often the most powerful tools for obtaining satisfactory and useful solutions to plasma problems when reliable answers are required for the interpretation of the laboratory experiment and for the design and the construction of large devices for controlled fusion. Section II is devoted to the description of one of such numerical models using particle simulation techniques developed for the study of anomalous diffusion due to low-frequency microinstabilities.

Heating of a confined plasma to the ignition temperature is another key problem for the controlled fusion. Here we will describe another particle simulation model specifically developed for studying the plasma heating associated with the injection of high-energy ion beams into a plasma. This method of neutral injection has been proved to be very successful for the existing tokamak experiments. Here the injected high-energy ions lose their energy and heat the background ions and electrons through the collisional process.

In order to understand the heating process as well as the diffusion in detail, it is necessary to take into account the spatial variation of plasma density and temperature in addition to the finite banana orbit in the beam dynamics. In this simulation model, the high-energy ions are modeled by the discrete particles while the background plasmas are represented by the average fluid quantities such as density and temperature. The interaction between the beam ions and the background plasma is through the Fokker-Planck collisions modeled by the Monte Carlo method. The model may be used to estimate the heating and beam ion spatial diffusion for existing tokamaks as well as for the planned large tokamaks.

II. CYLINDRICAL MODEL FOR MICROINSTABILITIES

Let us first consider the cylindrical model, shown in Fig. 1, which is developed to study the anomalous diffusion due to plasma microinstabilities in a magnetic field. The basic problem here is to follow the motions of ions and electrons in a confining magnetic field and in their self-consistent electric field due to microinstabilities and to deduce the mechanisms of anomalous plasma diffusion. In the particle model simulation, which is going to be described here, the electrons and ions are represented in terms of discrete finite-size particles called macro- or super-particles. To describe the motion of such

particles, the well-known classical Newton-Lorentz equation is used.

$$\frac{d\tilde{v}_j}{dt} = \frac{q_j}{m_j} \left(\tilde{E}(\tilde{x}_j) + \frac{1}{c} \tilde{v}_j \times \tilde{B}(\tilde{x}_j) \right) \quad (1)$$

$$\frac{d\tilde{x}_j}{dt} = \tilde{v}_j$$

together with the Maxwell equations to determine $\tilde{E}$ and $\tilde{B}$ fields. For simplicity we consider in this paper only the electrostatic field which is valid for low-density plasmas in a strong external magnetic field. Then the equation for the electrostatic potential is given by the Poisson equation

$$\nabla^2 \phi = -4\pi\rho$$

$$\tilde{E} = -\nabla\phi \quad (2)$$

and

$$\rho = \sum_j q_j S(\tilde{x} - \tilde{x}_j)$$

where $S(\tilde{x} - \tilde{x}_j)$ is the charge density of a finite-size particle centered at $\tilde{x} = \tilde{x}_j$.

In principle, one should be able to follow the evolution of a plasma by solving Eqs. (1) and (2) simultaneously iterating in time as long as it is necessary. When one is trying to simulate a realistic plasma device such as a tokamak, it becomes

immediately obvious that without drastic approximations no available computers can handle the problems due to their limited capacity and speed. It is therefore clear that several approximations have to be made while keeping the fundamental physics one is trying to follow unmodified. For low-frequency micro-instabilities, $\omega < \Omega_i$, Ω_i being the ion gyrofrequency; the wavelength along the confining magnetic field is very long comparable to the machine size while the wavelength perpendicular to it is much shorter; it can vary from the radius of a machine to the size of ion gyroradius where high resolution for short-wavelength fluctuations is necessary. Under such conditions, one may solve the Poisson equation in the following manner. First, let us Fourier analyse the potential with respect to z . We consider the cylindrical model first.

$$\Phi(x, y, z) = \sum_{n=-N}^N \phi_n(x, y) \exp(i2\pi n z / L_z) \quad (3)$$

where L_z is the length of the model and N is the number of modes kept in the z direction. Typically, $N = 5 \sim 10$ is satisfactory for the problems considered here. The system is assumed periodic in z simulating a toroidal device.

Similarly the charge density $\rho(x, y, z)$ is expanded in the same manner,

$$\rho(x, y, z) = \sum_{n=-N}^N \rho_n(x, y) \exp(i2\pi n z / L_z)$$

where

$$\rho_n(x, y) = \frac{1}{L_z} \int_0^{L_z} \rho(x, y, z) \exp(-i2\pi n z / L_z) dz$$

The charge density is given from the particle position which may be written as

$$\rho(x, y, z) = \frac{1}{(2\pi)^{3/2} a_x a_y a_z} \sum_j q_j \exp \left[- \frac{(x - x_j)^2}{2a_x^2} - \frac{(y - y_j)^2}{2a_y^2} - \frac{(z - z_j)^2}{2a_z^2} \right]$$

where instead of classical delta function distribution, a gaussian shape, finite-size particle is employed. The reason for using such particles in the simulation is to reduce the noise and collisional effects for close encounters which are exaggerated due to the small number of simulation particles compared with real plasma. This replacement certainly changes the physics at the wavelength shorter than the particle size. However, we are interested in the long wavelength phenomena which are not modified by the use of finite-size particles. By adopting a finite-size particle model, one can avoid the divergence of the interaction among particles at the very short distance. Then the Poisson equation becomes

$$\left[\nabla_{xy}^2 - (2\pi n/L_z)^2 \right] \phi_n(x, y) = -4\pi \rho_n(x, y) \quad (4)$$

$$\rho_n(x, y) = \frac{1}{2\pi a_x a_y L_z} \exp(-k_z^2 a_z^2/2) \sum_j q_j \exp$$

$$\left[- (x - x_j)^2/2a_x^2 - (y - y_j)^2/2a_y^2 \right] \exp(-ik_z z_j).$$

Equation (4) is basically the Poisson equation in two dimensions, which can be solved numerically by introducing a two-dimensional spatial grid using the fast Fourier transform. To do so, let us first Fourier transform Eq. (4) with respect to x and y . Assuming a periodic boundary condition in x and y , we find

$$- \left[(k_x^2 + k_y^2) + (2\pi n/L_z)^2 \right] \bar{\phi}_n(k_x, k_y) = -4\pi \bar{\rho}_n(k_x, k_y)$$

where

$$\begin{aligned} \bar{\rho}_n(k_x, k_y) &= \frac{1}{L_x L_y} \int_0^{L_x} \int_0^{L_y} \rho_n(x, y) \exp[-i(k_x x + k_y y)] dx dy \\ &= \frac{1}{L_x L_y L_z} \exp \left[- (k_x^2 a_x^2 + k_y^2 a_y^2 + k_z^2 a_z^2)/2 \right] \sum_j q_j \\ &\quad \exp[-i(k_z z_j + k_x x_j + k_y y_j)] \end{aligned}$$

Introducing a two-dimensional spatial grid in the (x, y) plane, the j -th particle location (x_j, y_j) is expanded around its nearest grid point, i.e., $x_j = n_j^x \Delta + \delta x_j$ and $y_j = n_j^y \Delta + \delta y_j$ where (n_j^x, n_j^y) is the nearest grid point and $(\delta x_j, \delta y_j)$ is the displacement from the nearest grid point.

In order to use the fast Fourier transform technique, we now expand $\exp(-ik_x x_j) = \exp(-ik_x n_j^x \Delta) (1 - ik_x \delta x_j)$ and $\exp(-ik_y y_j) = \exp(-ik_y n_j^y \Delta) (1 - ik_y \delta y_j)$ keeping up to the first order in $(\delta x_j, \delta y_j)$. Then

$$\begin{aligned} \bar{\rho}_n(k_x, k_y, k_z) &= \frac{1}{L_x L_y L_z} \exp \left[- (k_x^2 a_x^2 + k_y^2 a_y^2 + k_z^2 a_z^2) / 2 \right] \\ &\cdot \sum_{s=1}^{L_x} \sum_{s'=1}^{L_y} \left(\rho_{ss'} - ik_x \delta \rho_{ss'}^x - ik_y \delta \rho_{ss'}^y \right) \\ &\exp \left[- i 2\pi (s\ell/L_x + s'm/L_y) \right] \end{aligned} \quad (5)$$

where $\rho_{ss'}$ and $\delta \rho_{ss'}^x, \delta \rho_{ss'}^y$ are the monopole and the dipole charge densities at the (s, s') grid point weighted by the $\exp(-ik_z z_j)$. They are defined by

$$\begin{aligned} \rho_{ss'} &= \sum_j q_j \exp(-ik_z z_j) U(s\Delta - x_j) U(s'\Delta - y_j), \\ \delta \rho_{ss'}^x &= \sum_j q_j \exp(-ik_z z_j) (x_j - s\Delta) U(s\Delta - x_j) U(s'\Delta - y_j) \end{aligned}$$

and

$$\delta \rho_{ss'}^Y = \sum_j q_j \exp(-ik_z z_j) (y_j - s'\Delta) U(s\Delta - x_j) U(s'\Delta - y_j)$$

where

$$U(x - x_j) = \begin{cases} 1 & \text{for } |x - x_j| < \Delta/2 \\ 0 & \text{for } |x - x_j| > \Delta/2 \end{cases}$$

and the summation j is over all the particles. Here L_x and L_y are the length of the system in x and y in terms of the grid size Δ . It is clear that the right-hand side of Eq. (5) is the two-dimensional fast Fourier transform. Then the potential $\phi_n(x, y)$ can be found by the inverse fast Fourier transform, i.e.,

$$\phi_n(s\Delta, s'\Delta) = \sum_{\ell=1}^{L_x} \sum_{\ell'=1}^{L_y} \bar{\phi}_n(k_x, k_y) \exp[i2\pi (s\ell/L_x + s'\ell'/L_y)]$$

and the total potential is given by Eq. (3).

Since the potential is calculated, let us now calculate the force on a finite-size particle in the equation of motion. The electric field force on the j -th particle will be given by

$$F(x_j, y_j, z_j) = q_j \int E(x, y, z) (2\pi)^{-3/2} (a_x a_y a_z)^{-1} \exp \left[- \left((x - x_j)^2 / 2a_x^2 + (y - y_j)^2 / 2a_y^2 + (z - z_j)^2 / 2a_z^2 \right) \right]$$

$dx dy dz$

where $\tilde{E}$ is calculated from ϕ and is assumed to take the form of

$$\tilde{E}(x, y, z) = \sum_{n=-N}^N \tilde{E}_n(x, y) \exp(-i2\pi n z/L_z).$$

The z integration is carried out for the force in the x -direction

$$F_x(x_j, y_j, z_j) = q_j (2\pi a_x a_y)^{-1} \sum_{n=-N}^N \exp(-k_z^2 a_z^2/2) \exp(ik_z z_j) \cdot$$

$$\iint \tilde{E}_{nx}(x, y) \exp\left[-\left((x - x_j)^2/2a_x^2 + (y - y_j)^2/2a_y^2\right)\right] dx dy.$$

We now Fourier transform F_x with respect to x_j and y_j in order to use fast Fourier transform technique again.

$$F_x(k_x, k_y, k_z) = q_j \sum_{n=-N}^N \exp(-k_z^2 a_z^2/2) \exp(ik_z z_j) \tilde{E}_{nx}(k_x, k_y) \cdot \exp\left[-(k_x^2 a_x^2 + k_y^2 a_y^2)/2\right]. \quad (6)$$

$F_x(x_j, y_j, z_j)$ is calculated from the inverse fast Fourier transform of Eq. (6) with the expansion around the nearest grid point,

$$F_x(x_j, y_j, z_j) = F_x(n_j^x \Delta, n_j^y \Delta, z_j) + \delta x_j \frac{\delta F_x}{\delta x} + \delta y_j \frac{\delta F_x}{\delta y}.$$

Equation (1) is now solved using the electric field calculated. In the simulation, Eq. (1) is written in the form of finite difference with respect to t , called a leap-frog scheme, accurate up to $(\Delta t)^2$ and time reversible. They are

$$\frac{v_j^{t+\Delta t/2} - v_j^{t-\Delta t/2}}{\Delta t} = \frac{q_j}{m_j} \left(E(x_j^t) + \frac{v_j^{t+\Delta t/2} + v_j^{t-\Delta t/2}}{2c} \times B(x_j^t) \right)$$

$$\frac{x_j^{t+\Delta t} - x_j^t}{\Delta t} = v_j^{t+\Delta t/2} \quad (7)$$

In the actual simulations, the size of the two-dimensional grid is typically 64×64 and the number of Fourier modes is $N = 5 \sim 10$. The number of ions and electrons are $1/4 \sim 1/2$ million. Ordinarily grid quantities are stored in the fast core, while the particle positions and velocities are stored in the external disks and only a small fraction of the particles are in the fast core. It is necessary to iterate several hundred to a thousand time steps for each run to obtain the desired results. This easily takes several hours using the largest scientific computers.

The model described here has been tested and its validity confirmed.¹ The model is now used to study the anomalous diffusion due to low-frequency drift instabilities.² Several very interesting results have been discovered, which indicate the power of numerical simulation in plasma physics where the pure

analytic method is rather limited for strongly nonlinear plasma turbulence and anomalous diffusion.

III. TOROIDAL MODEL FOR MICROINSTABILITIES

Let us briefly describe the simulation model for a toroidal system where essentially the same technique can be used.¹ We now discuss how to solve the Poisson equation in a toroidal system shown in Fig. 2. Using a cylindrical coordinate, the Poisson equation takes the form of

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \phi^2} + \frac{\partial^2 \phi}{\partial z^2} = -4\pi\rho(r, \phi, z) \quad (8)$$

where $\phi(r, \phi, z)$ is the electrostatic potential.

Now expanding ϕ and ρ in the toroidal direction,

$$\begin{aligned} \phi(r, \phi, z) &= \sum_{n=-N}^N \phi_n(r, z) \exp(in\phi) \\ \rho(r, \phi, z) &= \sum_{n=-N}^N \rho_n(r, z) \exp(in\phi) \end{aligned}$$

Eq. (8) is reduced to

$$\frac{\partial^2 \phi_n}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_n}{\partial r} - \frac{n^2}{r^2} \phi_n + \frac{\partial^2 \phi_n}{\partial z^2} = -4\pi\rho_n$$

To see how the multipole expansion method works for the toroidal system, let us calculate $\rho_n(r, z)$ first. Assuming

a particle has a gaussian charge density in the cylindrical coordinate,

$$\rho_j(r, \phi, z) = \frac{q_j}{(2\pi)^{3/2} a_z a_r r_j a_\phi} \exp \left[- \left(\frac{(r-r_j)^2}{2a_r^2} + \frac{(\phi-\phi_j)^2}{2a_\phi^2} + \frac{(z-z_j)^2}{2a_z^2} \right) \right]$$

$$\begin{aligned} \rho_n(r, z) &= \frac{1}{2\pi} \int_0^{2\pi} \rho(r, \phi, z) e^{-in\phi} d\phi \\ &= \frac{\exp(-n^2 a_\phi^2 / 2)}{(2\pi)^2 a_r a_z} \sum_j \frac{q_j}{r_j} \exp \left[- \left(\frac{(r-r_j)^2}{2a_r^2} + \frac{(z-z_j)^2}{2a_z^2} + in\phi_j \right) \right] \end{aligned}$$

where a_r , a_ϕ and a_z are the particle size in the (r, ϕ, z) coordinate.

Fourier transforming $\rho_n(r, z)$ with respect to z , we find

$$\begin{aligned} \rho_n(r, k_z) &= \frac{1}{L_z} \int_0^{L_z} \rho_n(r, z) \exp(-ik_z z) dz \\ &= \frac{\exp(-k_z^2 a_z^2 / 2 - n^2 a_\phi^2 / 2)}{(2\pi)^{3/2} a_r L_z} \sum_j \frac{q_j}{r_j} \left[-i(n\phi_j + k_z z) \right] \exp \left[-(r-r_j)^2 / 2a_r^2 \right] \\ &= \frac{\exp(-k_z^2 a_z^2 / 2 - n^2 a_\phi^2 / 2)}{(2\pi)^{3/2} a_r L_z} \sum_j q_j \exp \left[-i(n\phi_j + k_z n_j^z \Delta z) \right] (1 - ik_z \delta z_j) \\ &\quad \cdot \exp \left[-(r-r_j)^2 / 2a_r^2 \right] \frac{1}{r_j} . \end{aligned}$$

While the trigonometric functions are not the eigenfunctions in the radial direction, let us calculate $\rho_n(k_r, k_z)$ to determine the charge density on the radial grid in terms of the multipole

expansion method.

$$\begin{aligned}
 \bar{\rho}_n(k_r, k_z) &= \frac{1}{L_R} \int_0^{L_R} \rho_n(R_1+s, k_t) \exp(-ik_r s) ds \\
 &= \frac{\exp(-k_z^2 a_z^2/2 - n^2 a_\phi^2/2 - k_r^2 a_r^2/2)}{2\pi L_R L_z r_j} \sum_j q_j \exp[-i(n\phi_j + \\
 &\quad k_z n_j^z \Delta z)] (1 - ik_z \delta z_j) \exp[-ik_r(r_j - R_1)] \\
 &= \frac{\exp[-(k_z^2 a_z^2 + k_r^2 a_r^2 + n^2 a_\phi^2)/2]}{2\pi L_R L_z} \sum_{s=1}^{L_z} \sum_{s'=1}^{L_R} \frac{(\rho_{ss'}, -ik_r \delta \rho_{ss'}^r, -ik_z \delta_{ss'}^z)}{r_{s'}} \\
 &\quad \exp(-i2\pi s \ell / L_z) \exp(-i2\pi s' \ell' / L_R)
 \end{aligned}$$

where $z_j = n_j^z \Delta z + \delta z_j$, $r_j = R_1 + n_j^r \Delta r + \delta r_j$, $k_z = 2\pi \ell / L_z$, $k_r = 2\pi \ell' / L_R$, and $\rho_{ss'}$, and $\delta \rho_{ss'}^{z,r}$ are the nearest grid point and the dipole charge contributions at the grid point (s, s') weighted by the $\exp(-in\phi_j)$ defined by

$$\rho_{ss'} = \sum_j q_j \exp(-in\phi_j) U(s\Delta z - z_j) U(R_1 + s'\Delta r - r_j),$$

$$\delta \rho_{ss'}^z = \sum_j q_j \exp(-in\phi_j) (z_j - s\Delta z) U(s\Delta z - z_j) U(R_1 + s'\Delta r - r_j)$$

and

$$\delta \rho_{ss'}^r = \sum_j q_j \exp(-in\phi_j) (r_j - R_1 - s'\Delta r) U(s\Delta r - z_j) U(R_1 + s'\Delta r - r_j)$$

Inverting the finite Fourier transform of $\rho_n(k_r, k_z)$ with respect to k_r , the Poisson equation finally becomes

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left(\frac{n^2}{r^2} + k_z^2 \right) \right] \phi_n(r, k_z) = -4\pi \rho_n(r, k_z) \quad (9)$$

This equation (9) has been solved by using the cubic spline technique which is accurate up to $(\Delta r)^3$.

Let us calculate the force on a finite-size particle. The electric field force on a particle at (r_j, ϕ_j, z_j) is given by

$$F(r_j, \phi_j, z_j) = q_j \int \underline{E}(r, \phi, z) (2\pi)^{-3/2} (a_z a_r r_j a_\phi)^{-1} \exp \left[-(r-r_j)^2/2a_r^2 - (z-z_j)^2/2a_z^2 - (\phi-\phi_j)^2/2a_\phi^2 \right] r dr d\phi dz$$

where $\underline{E}$ is calculated from ϕ and is given by, for example,

$$E_z(r, \phi, z) = \sum_{n=-N}^N E_n(r, z) \exp(in\phi),$$

then

$$F_z(r_j, \phi_j, z_j) = \sum_{n=-N}^N q_j \exp(-n^2 a_\phi^2/2) \exp(in\phi_j) (2\pi a_r a_z)^{-1} \int E_{nz}(r, z) \cdot \exp \left[-(z-z_j)^2/2a_z^2 - (r-r_j)^2/2a_r^2 \right] dr dz.$$

$F_z(r_j, \phi_j, z_j)$ is Fourier transformed with respect to (r_j, z_j)

$$F_z(k_r, \phi_j, k_z) = \sum_{n=-N}^N q_j \exp \left[-(k_z^2 a_z^2 + k_r^2 a_r^2 + n^2 a_\phi^2) / 2 \right] \sum_{n=-N}^N q_j \exp(in\phi_j) \cdot E_{nz}(k_r, k_z) \quad (10)$$

The force on the j -th particle is calculated from the inverse transform of Eq. (10) and then expanded around the nearest grid point. Writing $r_j = n_j^r \Delta r + \delta r_j$ and $z_j = n_j^z \Delta z + \delta z_j$,

$$F_z(r_j, \phi_j, z_j) = F_z(n_j^r \Delta r, \phi_j, n_j^z \Delta z) + \delta r_j \frac{\partial F_z}{\partial r} + \delta z_j \frac{\partial F_z}{\partial z}.$$

The evolution of the system is obtained by pushing ions with the Lorentz force (1) and integrating the guiding center drift equations for electrons. The electron guiding center equations are given by

$$\frac{d\tilde{r}}{dt} = \frac{(\tilde{v} \cdot \tilde{B})}{B^2} \tilde{B} + \frac{c \tilde{E} \times \tilde{B}}{B^2} - \frac{1}{\Omega_e B^2} \left(\frac{\mu B}{m_e} + v_{||}^2 \right) \tilde{B} \times \nabla B$$

$$\frac{dv_{||}}{dt} = - \frac{q}{m_e} \frac{\tilde{E} \cdot \tilde{B}}{B} - \frac{1}{m_e B} \tilde{B} \cdot \nabla (\mu B) + \frac{c v_{||}}{B^3} (\tilde{E} \times \tilde{B}) \cdot \nabla B \equiv F_{||}$$

$$\mu = \frac{mv_{\perp}^2}{2B} = \text{constant}$$

where

$$v_{||} = \frac{\tilde{v} \cdot \tilde{B}}{B}.$$

The integration scheme used is a two-step predictor-corrector scheme because the standard leap-frog scheme is not applicable.

Symbolically,

$$\bar{r}_{n+1} = r_{n-1} + 2\Delta t \, v(r_n, t_n)$$

$$\bar{v}_{||n+1} = v_{||n-1} + 2\Delta t \, F_{||}(r_n, t_n)$$

$$r_{n+1} = r_n + \left(v(r_n, t_n) + v(\bar{r}_{n+1}, t_{n+1}) \right) \Delta t/2$$

$$v_{||n+1} = v_{||n} + \left(F_{||}(r_n, t_n) + F_{||}(\bar{r}_{n+1}, t_{n+1}) \right) \Delta t/2$$

Note that this scheme requires two solutions of the Poisson equation per one time step. The test of the toroidal model gives satisfactory results and is now used for the study of trapped particle instabilities in tokamaks.

IV. NEUTRAL BEAM INJECTION INTO A TOKAMAK

Neutral beam injection is expected to play a major role in heating large scale tokamaks together with the Ohmic heating and various RF plasma heating.³ The heating of tokamak plasmas using the neutral beams has been successful for the existing tokamaks and appears to be promising in larger tokamaks. The physics involved in neutral beam injection is highly complex;

it includes the long-range Coulomb collisions, various atomic processes including charge exchange and impurities, and beam-induced microinstabilities. So far, there has been no experimental evidence for the anomalous effects induced by the injection of neutral beams. Therefore the basic underlying process is classical Coulomb collisions described by the Fokker-Planck equation. The predictions based on the Fokker-Planck equation, coupled with the neoclassical plasma transport equations, are found in good agreement with the experimental results.⁴ The analytical treatment of this problem is, however, so complicated due to the complex experimental device and inherent plasma inhomogeneity that a numerical approach is necessary when detailed information is required for the design and the interpretations of the tokamak experiments.

In this section, we describe an approach to the neutral injection problem which makes use of a particle simulation model. In the model, the orbits of beam ions are followed in a toroidal magnetic field with a concentric magnetic surfaces. As the fast ions are injected into a tokamak, they interact with the background ions and electrons and transfer the energy and momentum through the Coulomb collisions.

In order to simulate this process, we assume that : (1) the background ions and electrons remain Maxwellian velocity distributions defined in terms of macroscopic densities and temperatures, (2) the interaction among the fast ions is negli-

gible because the beam particle density is low, (3) the velocity space is two-dimensional in speed v and pitch angle θ , and (4) the typical fast ion speed, v_f , satisfies $v_i \ll v_f \ll v_e$, where v_i and v_e are the ion and electron thermal speeds of the background plasma. Then the evolution of the fast ion distribution function, $f(v, \theta, t)$ after the injection is given by⁵,

$$\begin{aligned} \frac{1}{\Gamma} \frac{\partial f}{\partial t} = & \frac{1}{v^2} \frac{\partial}{\partial v} v^2 D(v) \frac{\partial f}{\partial v} + \frac{1}{v^2} \frac{\partial}{\partial v} v^2 A(v) f \\ & + \frac{1}{v \sin \theta} \frac{\partial}{\partial \theta} \sin \theta P(v) \frac{\partial f}{\partial \theta} \end{aligned} \quad (11)$$

where the three terms on the RHS represent the energy diffusion, slowing down due to the background plasma, and the pitch-angle scattering due to the background ions, respectively. For $v_i \ll v_f \ll v_e$,

$$D(v) \approx \frac{1}{2} \frac{v_i^2}{v^3} + \frac{2}{3\sqrt{\pi}} \frac{1}{v_e}$$

$$A(v) \approx \frac{m_f}{m_i} \frac{1}{v^2} + \frac{4}{3\sqrt{\pi}} \frac{m_f}{m_e} \frac{v}{v_e^3}$$

$$P(v) \approx \frac{1}{2v^2}$$

$$\Gamma = \frac{4\pi e^4 n}{m_f^2} \ell n \Lambda$$

where m_e , m_i and m_f are the electron, ion, and fast ion masses, n is the background plasma density, and $\ln \Lambda$ is the Coulomb logarithm. Since the energy diffusion is small compared with the slowing down and the pitch angle scattering under $v_i \ll v_f \ll v_e$, we shall ignore this term in the present calculation.

Taking the first and second moments of Eq. (11), we obtain

$$\frac{1}{\Gamma} \frac{d\langle v \rangle}{dt} \approx - \langle A(v) \rangle$$

$$\frac{1}{\Gamma} \frac{d\langle (\Delta\theta)^2 \rangle}{dt} \approx \langle \frac{2P(v)}{v} \rangle$$

where $\langle \rangle$ denote an average over the fast-ion distribution. We incorporate these two effects into the simulation model by adding two terms to the particle velocity calculated using the guiding center drift equations as shown later. For the slowing down, the change in speed, Δv , given by $\Delta v = -\Gamma A(v)\Delta t$ is added at every time step. The pitch-angle scattering is provided by using the Monte Carlo method. The change in pitch angle for a random scattering is given by $\Delta\theta = [4\Gamma P(v) v^{-1}\Delta t \ln(1-\alpha)^{-1}]^{1/2}$ at every time step where α is a random number uniformly distributed between 0 and 1.⁶ The energy transferred from the fast ions to the electrons and ions of the background plasma is assumed to take place instantaneously and is deposited to each species.

The collisionless orbits of the fast beam ions between collisions are described by the guiding center drift equation

given by

$$\frac{d\tilde{r}}{dt} = v_{||} \frac{\tilde{B}}{B} + \frac{1}{\Omega_f B} \left(\frac{\mu B}{m_f} + v_{||}^2 \right) \frac{\tilde{B}}{B} \times \nabla B$$

$$\frac{dv_{||}}{dt} = - \frac{\mu}{m_f} \frac{\tilde{B}}{B} \cdot \nabla B$$

where $\tilde{r}$ is the radial vector in the minor cross section. Ω_f , μ and $v_{||}$ are the fast ion gyrofrequency, magnetic moment and the speed along the local magnetic field. The above equations have been solved using a predictor-corrector two-time step method during Δt and the change in y due to collisions is added to the collisionless orbits. The model has been tested to ensure its validity by comparing with the numerical solution of the Fokker-Planck equation.⁶

V. CONCLUDING REMARKS

We have described three particle simulation models which have been developed for studying microscopic processes such as anomalous diffusion and neutral beam heating of a tokamak plasma. These codes have been tested for different parameters to check the validity of the codes and they are now in heavy use for production runs. The three-dimensional codes for micro-instabilities described here are general so that they may be used for other purposes such as RF heating of tokamak plasmas. Furthermore, the self-consistent magnetic field may be added to

produce magnetic field oscillations important for high density and high temperature tokamaks. With the advent of larger and faster computers together with the development of our techniques in numerical modeling of plasmas, it is clear that the numerical simulation method becomes essential both in plasma theory and fusion applications.

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FIG. 1

A sketch of the three-dimensional cylindrical model with the rectangular cross section. Spatial grid is used in x - y plane while the mode expansion is employed in z -direction.

A sketch of the toroidal model with the rectangular cross section.

FIG. 2

762140