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STUDY OF SPONTANEOUSLY BROKEN CONFORMAL SYMMETRY IN CURVED SPACE-TIMES

MASTER

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This work is dedicated to my father and mother, Robert and Kathleen Janson and to my wife, MaryAnne. They have inspired, sustained and made this effort possible.

I also must thank Brian Patrick Janson; his timely arrival provided the impetus for completing this work.

Slowly the Milky Way revolved, with its fringe of constellations and planets and clouds, and the Sun along with the rest, toward the edge. In all that circling, only the sign remained still, in an ordinary spot, out of all the orbits' reach (to make it, I had leaned over the border of the Galaxy a little, so it would remain outside and all those revolving worlds wouldn't crash into it), in an ordinary point that was no longer ordinary since it was the only point that was surely there, and which could be used as a reference point to distinguish other points.

--- Italo Calvino (Cosmicomics)

There are certain common Privileges of a Writer, the Benefit whereof, I hope, there will be no Reason to doubt; Particularly, that where I am not understood, it shall be concluded, that something very useful and profound is coucht underneath.

---Jonathan Swift

(Tale of a Tub, Preface, 1704)

ABSTRACT

Spontaneous breakdown of Weyl invariance (local scale invariance) in a conformally-invariant extension of a gauge model for weak and electromagnetic interactions is considered. The existence of an asymmetric vacuum for the Higgs field, ϕ , is seen to depend on the space-time structure via the Gürsey-Penrose term, $-\phi^+\phi R$, in the action. (R denotes the scalar curvature.) The effects of a prescribed space-time structure on spontaneously broken Weyl invariance is investigated. In a cosmological space-time, it is found that initially, in the primordial fireball, the symmetry must hold exactly. Spontaneous symmetry breaking (SSB) develops as the universe expands and cools. Consequences of this model include a dependence of G_F , the effective weak interaction coupling strength, on "cosmic time." It is seen to decrease monotonically; in the present epoch $(G_F/G_F)_{\text{TODAY}} \leq 10^{-10} \text{ (year)}^{-1}$.

The effects of the Schwarzschild geometry on SSB are explored. In the interior of a neutron star the Higgs vacuum expectation value, and consequently G_F , is found to have a radial dependence. The magnitude of this variation does not warrant revision of present models of neutron star structures.

Another perspective on the problem considered a theory of gravitation (conformal relativity) to be incorporated in the conformally invariant gauge model of weak and electromagnetic interactions. If SSB develops, the vacuum gravitational field equations

are the Einstein field equations with a cosmological constant. The stability of the asymmetric vacuum solution is investigated to ascertain whether SSB can occur.

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Chapter One Spontaneous Symmetry Breaking

1.1 Introduction

The concept of a spontaneously broken symmetry plays an important role in many recent attempts to better understand elementary particles and their interactions. This notion is a key element in a formulation which unifies the weak and electromagnetic interactions.¹ At present it is an integral component in a large class of theories, — the gauge field theories.

Essential to the idea of a spontaneously broken symmetry is the realization that an invariance of the Hamiltonian of a quantum-mechanical system need not be possessed by the ground state. The Heisenberg model of the ferromagnet is based on this idea. One takes the ferromagnet to be an infinite crystalline array of spin $\frac{1}{2}$ magnetic dipoles. The spin-spin interaction between nearest neighbors is considered the only interaction present. For an infinite array such as this, the Hamiltonian is rotationally invariant. The ground state, however, is not. The configuration of lowest energy has all the dipoles aligned in a single (arbitrary) direction. The symmetry (rotational invariance) is said to be spontaneously broken.

The asymmetry of the ground state makes it difficult to deduce the invariance properties of the Hamiltonian. An experiment conducted within such an array could not verify that the underlying interaction exhibited rotational symmetry. The background field would indicate the existence of a preferred direction. Depending on the

strength of the coupling between the background and the apparatus, one's conclusions could range from

- i) rotational invariance is an "approximate symmetry."
- ii) rotational symmetry is badly broken and of doubtful significance.

The idea of a spontaneously broken symmetry creates a new perspective from which to view physical processes. Symmetries present in a theory at a fundamental (Lagrangian, Hamiltonian) level may be hidden because the ground state does not exhibit the symmetry.

1.2 Simple Models Exhibiting SSB

It will be useful to develop a few simple models involving a spontaneously broken symmetry. This will provide a foundation in which to fix ideas developed later.

Begin by considering a system described by a single real scalar field, $\phi(x)$, — a classical field. The action of the theory is given by

$$W = \int d^4x \left[\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - U(\phi(x)) \right] \quad (1.1)$$

The convention adopted throughout is $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$. The potential, U , is some function of $\phi(x)$ but not of the derivatives of ϕ . The energy density for this system is

$$H = \frac{1}{2} [\dot{\phi}^2 + |\nabla\phi|^2] + U(\phi(x)) \quad (1.2)$$

where the dot denotes the time derivative and

$$|\nabla\phi|^2 = -\eta^{ij} \partial_i \phi(x) \partial_j \phi(x) \quad i, j = 1, 2, 3$$

The notation ∂_i indicates $\frac{\partial}{\partial x^i}$

Since H is positive definite, the state of lowest energy is one for which $\phi = \text{constant}$. Anticipating the transition to a theory of quantized fields, the ground state is referred to as the "vacuum state." The value that ϕ takes in this state is called the "vacuum expectation value" (VEV) and is denoted $\langle\phi\rangle$. This value is determined by the dynamics of the system in the present model. It depends on the location of the minima of the potential, U . Suppose $U(\phi)$ is of the form

$$U(\phi(x)) = \frac{\lambda}{4!} \phi^4(x) + \frac{\mu^2}{2} \phi^2(x) \quad (1.3)$$

where μ^2, λ are real numbers

and $\lambda > 0$ and μ^2 can be either positive or negative. With this potential, it is evident that the action of the system (eq. 1.1) is invariant under the transformation

$$\phi(x) \rightarrow -\phi(x) \quad (1.4)$$

Consider first the case where $\mu^2 > 0$. The general shape of the potential is sketched in Fig. 1.1.

The state of lowest energy is that for which $\langle\phi\rangle = 0$. It is noted that the ground state displays the symmetry of the action (eq. 1.4).

Now consider $\mu^2 < 0$. The potential $U(\phi)$ is now of the form

Fig. 1.1

Fig. 1.2

shown in Fig. 1.2. It is convenient to introduce the quantity

$$a^2 = -6\mu^2/\lambda \quad (1.5)$$

and re-write the potential (up to an irrelevant constant) as

$$U(\phi) = \frac{\lambda}{4!} (\phi^2 - a^2)^2 \quad (1.6)$$

It is now evident that the potential has two minima — at $\phi = \pm a$. Since the action is invariant under $\phi(x) \rightarrow -\phi(x)$, the physics of this system cannot distinguish between these minima. It is noted that neither solution exhibits the symmetry of the action. Choose $\langle \phi \rangle = a$ and define a "shifted" field by

$$\phi' = \phi - a \quad (1.7)$$

In terms of the shifted field, the potential can be written

$$U(\phi') = \frac{\lambda}{4!} (\phi'^2 - 2a\phi')^2 \quad (1.8)$$

$$= \frac{\lambda}{4!} \phi'^4 - \frac{\lambda a}{6} \phi'^3 + \frac{\lambda a^2}{6} \phi'^2 \quad (1.9)$$

The theory has an asymmetric vacuum and describes a scalar field with a "mass" squared $\frac{\lambda a^2}{3}$ and cubic self-interaction. The symmetry (eq. 1.4) of the action is no longer apparent.

While illustrative of some important aspects of SSB, the previous model must be extended. Additional considerations come into play when the action is invariant under a continuous transformation.

Consider a theory involving two real scalar fields, $\phi_1(x)$ and $\phi_2(x)$. The potential is taken to be

$$U = \frac{\lambda}{4!} [\phi_1^2(x) + \phi_2^2(x) - a^2]^2 \quad (1.10)$$

where a is a real number.

Such a theory admits a continuous group of symmetries isomorphic to the two-dimensional rotation group, $SO(2)$. That is, the action is invariant under

$$\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \quad (1.11)$$

where ω is a real number.

The minima of the potential (eq. 1.10) lie on the circle

$$\phi_1^2 + \phi_2^2 = a^2 \quad (1.12)$$

As before, it will not matter which of the vacua is chosen. Regardless of the choice, the vacuum state will not be $SO(2)$ invariant (eq. 1.11)

Choose

$$\begin{aligned} \langle \phi_1 \rangle &= a \\ \langle \phi_2 \rangle &= 0 \end{aligned} \quad (1.13)$$

Again a shift is performed on the fields:

$$\phi'_i = \phi_i - \langle \phi_i \rangle \quad (1.14)$$

The potential may be written in terms of the shifted fields

$$U = \frac{\lambda}{4!} [\phi_1'^4 + 4a^2\phi_1'^2 + 4a\phi_1'^3 + 2\phi_1'^2\phi_2'^2 + \phi_2'^4 + 4a\phi_2'^2\phi_1'] \quad (1.15)$$

The ϕ_1 "meson" has a mass-squared equal to twice that of the ϕ_1 and the object described by the ϕ_2 field is massless. The appearance of such a zero-mass spinless object is the key point here. It has been shown on very general grounds² that such an object necessarily appears whenever a continuous symmetry is spontaneously broken. This is a disastrous blow to the whole idea of SSB because there has been no experimental evidence for the existence of these massless, spinless mesons ("Goldstone Bosons").

Fortunately however there is a loophole in the "Goldstone Theorem" which makes it possible to avoid the appearance of the unwanted Goldstone Boson. A way to circumvent the implications of the theorem was discovered by Higgs.³ Higgs proposed to couple the scalar field (with an asymmetric vacuum) to a gauge field. (The most familiar gauge field is the vector potential, A_μ , in electrodynamics.) After a suitable redefinition of the fields (shifting as before) it can be shown that the Goldstone mode has been absorbed by the gauge field.⁴ For example, if one begins with a massless gauge field, A_μ , (2 degrees of freedom) and two massless scalar fields (Higgs fields), then if the gauge symmetry is spontaneously broken, one has a single massive scalar field and a massive vector field (3 degrees of freedom). What would be the Goldstone mode is now a degree of freedom associated with the gauge field. One may take the point of view that the Goldstone is never really present in a locally gauge invariant theory. It is merely a "gauge phenomenon."⁵

Two principal results of this "Higgs mechanism" are of importance to later work:

- (i) the masses acquired by the gauge fields are proportional to $\langle\phi\rangle$;
- (ii) if massless fermions are originally present in the theory and are coupled to the Higgs field (in a gauge-invariant manner) then they also acquire masses proportional to $\langle\phi\rangle$.

The most important application of the Higgs mechanism to date has been a formulation by Salam, Ward and Weinberg of a unified treatment of the weak and electromagnetic interactions.⁶ This proposal "unifies" these seemingly disparate interactions in the sense that it places them on an equal footing in the Lagrangian. Via the Higgs mechanism, the gauge fields associated with the weak interaction acquire a mass. that associated with electromagnetism (the photon) does not. (See ref. 4 for greater details on how this is arranged.) The theory has been shown to be renormalizable⁷ and as such is enormously attractive to theorists. Its success, particularly with regard to the question of renormalizability, has generated much interest in the entire class of gauge field theories as vehicles for understanding the strong interaction and perhaps gravitation as well.⁸

Finally, we note that the models used here to study SSB have been analyzed at a classical level. As we have indicated, the results can be transcribed into the language of quantized fields. The functional method of quantization is most useful here.⁹ In

this framework it is possible to show that the stable vacua of the theory occur at the minima of an "effective potential." The "effective potential" is represented by an infinite sum of Feynman diagrams, but the first term (in the "loop expansion") can be shown to be just the classical potential.¹⁰ Thus the results of the classical analysis are valid in a quantized version of the theory provided that successive terms in the loop expansion (Feynman diagrams containing one closed loop, two closed loops, etc.) give negligible contributions.

Calculations in subsequent sections of this work take advantage of this situation. We work in the "tree approximation" throughout (no closed loops). The results will be a reliable guide to those expected in a fully quantized calculational program.

Chapter Two Spontaneously Broken Weyl Invariance

2.1 Weyl Invariance

Theories manifesting exact conformal symmetry are notable in two respects: they are the joy of a theorist and the despair of an experimentalist. Such theories are attractive from a theoretical perspective for several reasons. They are more restrictive than theories based on the Poincaré group alone. Experience has generally indicated that an enlargement of the invariance group leads to further structural restrictions on a theory. Conformally invariant theories include strictures on the possible couplings (in the Lagrangian) between fields. For example, a minimal coupling between a Dirac field and a gauge field ($-e\bar{\psi}\gamma^\mu A_\mu\psi$) is permissible but Pauli-type couplings ($-\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi$) are not allowed.

Other physically attractive restrictions imposed by exact conformal symmetry constrain the short-distance expansions of operator products. The "scaling" exhibited by deep inelastic scattering of leptons on hadrons can, in a framework of conformal invariance, be understood in terms of fewer parameters than Poincaré invariance alone provides. Finally, there have been some grounds for believing that conformally invariant theories exhibit a more healthy short-distance behavior than most.¹¹

With all of these features so attractive to a theorist, conformally invariant theories must, nevertheless, remain outrageous from an experimental point of view since they are devoid of a fixed mass (or length) scale. This is in demonstrable conflict with the

experimentally observed scales (~ 1 GeV for strong interactions, a length scale of m_W^{-1} or $\sqrt{G_F} \sim 10^{-16}$ cm for weak interactions). Conformal symmetry, therefore, must not be exact. If the symmetry is to be only an approximate one, recent experience (particularly with gauge theories) indicates that a spontaneous breakdown of the symmetry might be the most promising avenue of approach. We will in fact adopt this point of view. First, however, an examination of some geometrical aspects of conformal symmetry is warranted.

An original point of departure regarding conformal transformations¹² was the view that they constituted a local rescaling of the units of length (equivalently, the units of energy, if one uses the "natural" units, $\hbar = c = 1$). Thus, at any point, P, of space, if one is given an infinitesimal vector $dx^\mu(P)$ (representing, classically, a "measuring rod"), then one changes the units of length locally by

$$dx^\mu(P) \rightarrow S(P) dx^\mu(P) \quad (2.1)$$

S is real

Since one usually characterizes the point P by its coordinates, $S(x)$ is written rather than $S(P)$, etc.

With this change in local units, the length squared of dx^μ becomes

$$g_{\mu\nu}(x) dx^\mu dx^\nu \rightarrow S^2(x) g_{\mu\nu}(x) dx^\mu dx^\nu \quad (2.2)$$

Since functions of points rather than functions of infinitesimal displacements are of primary concern, one reinterprets the transformation via eq. 2.2 as a rescaling of the metric tensor, rather

than of the infinitesimal vector. In infinitesimal form

$$S(x) \equiv 1 + \Lambda(x)$$

(2.3)

$$\delta g_{\mu\nu}(x) \equiv 2\Lambda(x) g_{\mu\nu}(x)$$

with the immediate results

$$\delta g^{\mu\nu}(x) \equiv -2\Lambda(x) g^{\mu\nu}(x)$$

$$\delta \sqrt{-g} \equiv \Lambda(x) \sqrt{-g}$$

(2.4)

where g denotes $\text{Det } g_{\mu\nu}(x)$.

These transformations are called Weyl transformations or conformal rescalings.¹³ It is emphasized that they do not affect the coordinates. Evidently, if $g_{\mu\nu}(x)$ is the metric of some general Riemann space, then (for "sufficiently well-behaved" $S(x)$) so is $S^2(x)g_{\mu\nu}$ and it is a space of the same signature as $g_{\mu\nu}(x)$. Hence, Weyl transformations form an infinite parameter Abelian group, with composition law

$$\delta_1 \delta_2 g_{\mu\nu} = \delta_2 \delta_1 g_{\mu\nu} = 2(\Lambda_1(x) + \Lambda_2(x)) g_{\mu\nu}$$

The group of conformal transformations on Minkowski space is another object entirely. It is the group of coordinate transformations which induce a conformal rescaling on the Minkowski metric and is a 15-parameter non-Abelian group, isomorphic to $SO(4,2)$.¹⁴ This investigation will concern itself exclusively with Weyl transformations.

The representation theory of the group of Weyl transformations, W , appears straightforward. However, W must be represented on a set of quantized fields, both gravitational and "matter" fields, and no one is really certain how to go about this. Since W is Abelian, one expects its irreducible representations to be one-dimensional. Hence if $\phi_{(A)}(x)$ is a physically meaningful (classical) field where (A) stands for a set of spinor indices with respect to a local spinor frame, then ϕ should transform under eq. (2.3) as

$$\delta\phi_{(A)}(x) = w\Lambda(x)\phi_{(A)}(x) \quad (2.5)$$

where w , a real number, is called the "dimension" or "weight" of ϕ .

The covariant metric tensor, $g_{\mu\nu}(x)$, evidently carries conformal weight +2. With the important exception of gauge fields, the weight is equal to the canonical dimension of the field (e.g. Dirac spinors have weight -3/2). However, gauge fields corresponding to (local) internal symmetries carry weight zero, as can be deduced from the gauge-invariant form of the action.

$$W = -\frac{1}{4} \int d^4x \sqrt{-g} g^{\mu\nu} g^{\rho\sigma} \vec{F}_{\mu\rho} \cdot \vec{F}_{\nu\sigma} \quad (2.6)$$

An inner product in the internal symmetry space is conveyed by the dot. The field tensor itself, $\vec{F}_{\mu\nu}$, is the totally anti-symmetric derivative.

$$\vec{F}_{\mu\nu} = [\vec{\nabla}_\mu \times \vec{A}_\nu]$$

where $\vec{\nabla}_\mu$ is the gauge and Riemann covariant derivative and $\vec{A}_\mu$ is

the vector potential.

The assignment of weight zero to $\vec{A}_\mu$ and hence $\vec{F}_{\mu\nu}$ is verified by counting weights. By constructing gauge-invariant couplings between the gauge field and various matter fields (e.g. Dirac spinors), it is possible to verify the consistency of the assignment scheme.

2.2 A Gauge Model: Spontaneously Broken Weyl Invariance

Spontaneous symmetry breaking in a Weyl invariant theory can be investigated in the framework of a gauge model. The salient points of this inquiry are perhaps best illustrated in the context of a Salam-Ward-Weinberg type model in curved space, i.e. together with gravity. The (internal symmetry) gauge group is taken to be $U(1)$; the formalism and conclusions can be easily generalized to include more complicated gauge groups.

In flat space, the action may be written

$$W = \int d^4x \left\{ \frac{1}{2} \eta^{\mu\nu} \phi^* (\partial_\mu + ifA_\mu) (\partial_\nu - ifA_\nu) \phi - \frac{\lambda}{4!} (\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho} \eta^{\nu\sigma} + L(\psi) \right\} \quad (2.7)$$

where ϕ is a massless complex Higgs field, A_μ is the vector potential, and $L(\psi)$ represents the gauge-invariant Lagrangian density for massless spinor fields (e.g. leptons, quarks). The "charge" carried by the Higgs field is f .

In generalizing the action to curved space, the straightforward procedure is

$$\eta^{\mu\nu} + g^{\mu\nu}(x)$$

$$d^4x \rightarrow d^4x \sqrt{-g} \quad (\text{invariant volume element})$$

$$\partial_\mu \rightarrow \nabla_\mu \quad (\text{Riemann covariant derivative})$$

When this procedure is carried out, one finds that part of the action of eq.(2.7) is automatically Weyl invariant. Call this portion of the action W' .

$$W' = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} f^2 \phi^* \phi g^{\mu\nu} A_\mu A_\nu - \frac{\lambda}{4!} (\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \right. \\ \left. - \frac{i}{2} f \phi^* \overleftrightarrow{\partial}_\mu \phi g^{\mu\nu} A_\nu + L(\psi) \right\} \quad (2.8)$$

The assignment of conformal weights is:

$$w_{A_\mu} = 0 \quad ; \quad w_\phi = -1 \quad w_\psi = -3/2$$

However, the "kinetic" part of the action of the Higgs field, call it T ,

$$T = \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi \quad (2.9)$$

is not W-invariant.

Under eq. 2.3, T transforms as

$$\delta T = \frac{1}{2} \int d^4x \sqrt{-g} \phi^* \phi \square \Lambda(x) \quad (2.10)$$

where $\square$ denotes the covariant D'Alembertian, and

$$\square \Lambda(x) = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Lambda(x))$$

In order to have a Weyl-invariant action for the Higgs field,

one must supply a "piece of the action" whose change (under eq. 2.3) exactly cancels δT (2.10). The requisite compensating term was discovered independently by Gürsey and Penrose¹⁵ to be

$$- \frac{1}{12} \int d^4x \sqrt{-g} \phi^* \phi R \quad (2.11)$$

where R is the scalar curvature, $R = g^{\mu\nu} R_{\mu\nu}$. The total Weyl-invariant action then is

$$W = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - \frac{1}{12} \phi^* \phi R \right\} \\ + W' \quad (2.12)$$

The experience of Chapter One suggests that the Gürsey-Penrose term bears some resemblance to a "mass" term in the Higgs Lagrangian. Thus one might expect it to generate, together with the quartic term $(\lambda(\phi^* \phi)^2)$, a non-vanishing vacuum expectation value. Evidently spontaneous symmetry breakdown (SSB) via this mechanism is impossible in flat space-time ($R=0$). It can perhaps be achieved in the curved space-time appropriate for the description of the entire universe. The properties of the matter distribution in the universe, via the space-time curvature it produces, are, in this view, responsible for the breakdown of Weyl invariance. It is a perspective not unlike that advocated by Mach in his explanation of inertia.

2.3 The "Standard" Cosmological Model

With much observational evidence still open to a variety of

interpretations, there is at present no model of the cosmological space-time that could be termed "canonical." What might be called the "standard" cosmological model is one consistent with the Einstein field equations and in agreement with present observational data.¹⁶ In particular, the question of the sign of the spatial curvature and presence of a "cosmological constant" are very much at issue at the moment. The model presently considered has positive spatial curvature and no cosmological constant. (A discussion of these points in greater detail may be found in ref. 16.)

The starting point for any cosmological model is the observation that the universe, when viewed on a large enough scale (cells with edge $\sim 10^8$ light-years), is remarkably spatially homogeneous and isotropic. This is sometimes referred to as the "Cosmological Principle." A coordinate basis may be selected that takes advantage of this symmetry of the spatial portion of cosmological space-time. A convenient choice is the frame at rest relative to the averaged local matter (clusters of galaxies). These are the "comoving" or Robertson-Walker coordinates and are conventionally denoted r, θ, ϕ, t . With positive spatial curvature (closed universe), the metric takes the form

$$ds^2 = dt^2 - a^2(t) \left\{ \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right\} \quad (2.13)$$

$a(t)$ is called the scale factor.

The hypersurfaces of constant t can be regarded as the surface of a 3-sphere embedded in a 4-dimensional Euclidean space with $a(t)$

then the "radius" of the universe. (The proper volume is given by ${}^3V = 2\pi^2 a^3(t)$.)

The components of the (covariant) metric tensor can be read directly from eq. 2.13.

$$\begin{aligned} g_{tt} &= +1 & g_{\theta\theta} &= -a^2(t) r^2 \\ g_{rr} &= \frac{-a^2(t)}{1-r^2} & g_{\phi\phi} &= -a^2(t) r^2 \sin^2 \theta \\ g_{\alpha\beta} &= 0 \quad \alpha \neq \beta \end{aligned}$$

Calculation of R , the scalar curvature, begins with the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \quad (2.17)$$

where G stands for Newton's gravitational constant. Throughout this work, a natural system of units is employed where $\hbar = c = L^* = 1$ (L^* is the Planck length, $L^* = (\hbar G/c^3)^{1/2}$). Hence, Newton's gravitational constant, G , will no longer appear explicitly. Contracting both sides of eq. 2.17 (taking the trace) yields

$$R = 8\pi T^\mu{}_\mu = 8\pi T \quad (2.18)$$

It is via the trace of the energy-momentum tensor, then, that the distribution of the cosmic background matter determines the scalar curvature of the space-time, and also, possibly the spontaneous breakdown of Weyl invariance. A homogeneous and isotropic space-time can only be filled with a "perfect" fluid (no shear stress).

It is re-emphasized here that, in this model, the metric (Robertson-Walker) is prescribed; the influence of the quantized matter fields (present in the gauge model) on this metric are neglected. Hence the energy-momentum tensor of matter in the universe must be of the "perfect fluid" form:

$$T_{\mu\nu} = P g_{\mu\nu} + (P+\rho) u_\mu u_\nu \quad (2.19)$$

where P , ρ , u_ν are the classical pressure, mass-energy density and four-velocity of the cosmic background fluid.

Eq. 2.19 immediately yields

$$T = 3P - \rho \quad (2.20)$$

and

$$R = 8\pi (3P - \rho) \quad (2.21)$$

follows directly.

To determine the dependence of P and ρ , the pressure and mass-energy density of the cosmic fluid, on the scale factor, $a(t)$, one requires conservation of energy.

$$T_{\mu\nu};_\nu = 0 \quad (2.22)$$

yields

$$\dot{\rho} a^3 = \frac{d}{dt} [a^3(P+\rho)] \quad (2.23)$$

or equivalently

$$\frac{d}{da}(\rho a^3) = -3P a^2 \quad (2.24)$$

Once an equation of state $P=P(\rho)$ for the fluid is specified, eq. 2.24 can be used to determine ρ as a function of $a(t)$. The fluid is composed of matter and radiation, originally in thermal equilibrium at the time of the "Big Bang," but cooling (at different rates) and decoupling as the expansion of the universe continues. (The radiative component is presently observed to have a 3°K blackbody spectrum.) The equation of state for the radiation in the cosmic fluid is, under all conditions

$$P_{\text{RAD}} = \rho_{\text{RAD}}/3 \quad (2.25)$$

When combined with eq. 2.24, this yields

$$\rho_{\text{RAD}} \propto a^{-4} \quad (2.26)$$

The equation of state for matter is temperature-dependent. With a range of temperatures from 3°K (today) to the arbitrarily high (according to most "conventional" scenarios¹⁷) temperatures during the primordial fireball, some rough approximations are in order. During the development of the universe, two distinct epochs can be distinguished.¹⁸ The "radiation-dominated", "hot" era begins with the fireball of the Big Bang. During this time, matter and radiation are in thermal equilibrium; the energy density of the radiation ρ_{RAD} is much greater than the density of matter ρ_m . When the temperature falls to about 10⁵°K (about 10¹² sec after the Big Bang) matter and radiation decouple, and cool at different rates.

From this point on,

$$\rho_m \propto a^{-3} \quad (\text{reflecting the constancy of the number of particles}) \quad (2.27)$$

with $\rho_{\text{RAD}} \propto a^{-4}$ as before. (See eq. 2.26.)

At a temperature of about 10^5K , or just prior to the decoupling of matter and radiation, the density of matter exceeds that of the radiation. This signals the onset of the "matter-dominated" regime which will continue either until the radius of the universe reaches maximum expansion (closed universe) or forever (open universe). During this time, the expansion rate of the universe is governed by its non-relativistic matter content. The pressure, P_m , exerted by the matter is negligible in comparison with the mass-energy density ρ_m . Conclusions about the equation of state during the "fireball" stage are less easily drawn. With the radiation and highly relativistic matter maintained at a common temperature during this era, the equation of state for matter approaches that of radiation, eq. 2.25. Summarizing these conclusions

radiation dominated

$$\begin{aligned} T > 10^5 \text{K} \quad P_m &\approx \rho_m/3 ; \quad \rho_m \propto a^{-4} \\ P_R &= \rho_R/3 ; \quad \rho_R \propto a^{-4} \end{aligned} \quad (2.28a)$$

matter dominated

$$\begin{aligned} T < 10^5 \text{K} \quad P_m &\approx 0 ; \quad \rho_m \propto a^{-3} \\ P_R &= \rho_R/3 ; \quad \rho_R \propto a^{-4} \end{aligned} \quad (2.28b)$$

The constants of proportionality can be supplied for eq. 2.28 by knowing the present values of the given quantity (e.g. ρ_{R_0} , the density of radiation as measured today $\approx 10^{-33} \text{g/cm}^3$).

2.4 The Cosmological Background Breaks Weyl Invariance

Spontaneous symmetry breaking develops if the vacuum expectation value of the Higgs field differs from zero. With the action given by eq. 2.12, the Euler-Lagrange equations of motion are

$$\left(\square + \frac{R}{6}\right)\phi - \frac{\lambda}{6}\phi^3 = 0 \quad (2.29)$$

In order to qualify as a physically sensible VEV of a quantized field, ϕ can, at most, depend on the cosmic time, t . That is $\phi = \phi(t)$. With this assumption eq. 2.29 can be written, in Robertson-Walker coordinates (see eq. 2.13), as

$$\frac{1}{a^3(t)} \frac{d}{dt} \left(a^3(t) \frac{d\phi}{dt} \right) + \frac{R}{6}\phi + \frac{\lambda}{6}\phi^3 = 0 \quad (2.30)$$

It is convenient to introduce $a(t)$ as the "time variable." The space-space and time-time components of the field equations (see eq. 2.17) can be combined to yield

$$\dot{a}^2 = \frac{8\pi}{3} \rho a^2 - 1 \quad (2.31)$$

Now one can express

$$\frac{d\phi}{dt} = \frac{d\phi}{da} \sqrt{\frac{8\pi}{3} \rho a^2 - 1} \quad (2.32)$$

With eq. 2.32, and R expressed via eq. 2.21, eq. 2.30 reads

$$\begin{aligned} \frac{1}{a^3} \sqrt{\frac{8\pi}{3} \rho a^2 - 1} \frac{d}{da} \left[a^3 \sqrt{\frac{8\pi}{3} \rho a^2 - 1} \frac{d\phi}{da} \right] + 4\pi (P - \rho/3)\phi \\ + \frac{\lambda}{6} \phi^3 = 0 \end{aligned} \quad (2.33)$$

The equations of state summarized in eq. 2.28 are now employed, and the appropriate dependences of P and ρ on $a(t)$ thus determined.

The VEV of the Higgs' field is now seen to satisfy:

radiation-dominated:

$$T > 10^{50} K \quad \frac{1}{a^3} \sqrt{\frac{8\pi\rho_{R_0} a_0^4}{3a^2} - 1} \frac{d}{da} \left[\sqrt{\frac{8\pi\rho_{R_0} a_0^4}{3a^2} - 1} a^3 \frac{d\phi}{da} \right] + \frac{\lambda}{6} \phi^3 = 0 \quad (2.34a)$$

matter-dominated:

$$T < 10^{50} K \quad \frac{1}{a^3} \sqrt{\frac{8\pi\rho_{m_0} a_0^3}{3a} - 1} \frac{d}{da} \left[\sqrt{\frac{8\pi\rho_{m_0} a_0^3}{3a} - 1} a^3 \frac{d\phi}{da} \right] - \frac{4\pi}{3} \frac{\rho_{m_0} a_0^3}{a^3} \phi + \frac{\lambda}{6} \phi^3 = 0 \quad (2.34b)$$

where the subscript zero is used to denote the present value of a particular quantity. (This follows the usual astronomical convention.) Thus, ρ_{R_0} denotes the energy density of the background radiation as measured today.

It is convenient here to introduce an "expansion parameter," x , as the independent variable.

$$x = a_{\max}/a \quad (2.35)$$

$a_{\max}$, the maximum "radius" of the universe is seen to be, from eq. 2.31

$$a_{\max} = \frac{8\pi}{3} \cdot \rho_{m_0} a_0^3 \quad (2.36)$$

It is possible to "scale out" the dimensionless parameter, λ , by introducing the function $g = (\lambda a_{\max}^2)^{1/2} \phi$. With these changes, eq. 2.34 becomes

$$x < x_d : \left(\frac{1-x^2}{x^2} \right)^{1/2} \frac{d}{dx} \left[x^3 \left(\frac{1-x^2}{x^2} \right)^{1/2} \frac{dg}{dx} \right] + \frac{x^3 g^3}{6} = 0 \quad (2.37a)$$

$$x > x_d : \left(\frac{1-x}{x} \right)^{1/2} \frac{d}{dx} \left[x^3 \left(\frac{1-x}{x} \right)^{1/2} \frac{dg}{dx} \right] - \frac{1}{2} g + \frac{x^3 g^3}{6} = 0 \quad (2.37b)$$

x_d corresponds to $T_d \cong 10^{50} K$.

For $x < x_d$, the vacuum energy of the Higgs field is minimized by the trivial solution $g = \phi = 0$. In order to find the stable vacuum for $x > x_d$, it helps to introduce the variable $y = (1-x)^{1/2}$. Then for $x \approx 1$ (near maximal expansion) eq. 2.37b becomes

$$\frac{d^2 g(y)}{dy^2} - 2[g(y) - \frac{1}{3} g^3(y)] = 0 \quad (2.38)$$

In analogy with the flat space case, the stable solution of eq. 2.38 is found to be $|g| \cong \sqrt{3}$ ($y \approx 0$). Thus, the VEV of the Higgs field is zero as long as the universe is "hot" ($x < x_d$) and then it grows gradually to the limiting value $\phi_{\max} = (\lambda a_{\max}^2/3)^{-1/2}$ as the universe cools down ($x > x_d$). In this model, the value of λ can be adjusted to reproduce the presently observed ratio of weak and electromagnetic couplings.

Eq. 2.37b was integrated numerically. In order to take advantage of certain computational techniques, the first derivative was eliminated. This was achieved by writing

$$g(y) = u(y) v(y) \quad (2.39)$$

Upon substituting this into the differential equation, and requiring that the coefficient of the term in u' vanish, there appears

$$2v' - \frac{5y}{(1-y^2)}v = 0 \quad (2.40)$$

with the solution

$$v(y) = (1-y^2)^{-5/4} \quad (2.41)$$

The constant of integration can be absorbed into the definition of $u(y)$. There now remains a differential equation for $u(y)$ of the form

$$u''(y) = f(u(y), y) \quad (2.42)$$

Explicitly, this equation reads

$$(1-y^2)^2 u''(y) + u(y) \left(\frac{1}{2} - \frac{15}{4}y^2 \right) + u^3(y) \frac{2}{3}(1-y^2)^{3/2} = 0 \quad (2.43)$$

Solution of eq. 2.43 as a boundary value problem was achieved via the numerical methods detailed in Appendix A. A typical solution (with $x_d = 10^{-4}$) is shown in Fig. 2.1. It was found that the solution for $x > 10^{-2}$ is insensitive to the exact choice of x_d within the interval $10^{-6} \leq x_d \leq 10^{-3}$.

One is able to verify the solution achieved by numerical methods by considering the asymptotic behavior of $g(x)$. Since $x_d \ll 1$, the behavior of $g(x)$ near x_d can be estimated on the basis of the asymptotic solution near zero. As $x \rightarrow 0^+$ one finds $g(x) \sim x^\beta$ where $\beta = \frac{\sqrt{17}-3}{4} \approx 0.28$. Thus, one may approximate $g(x)$ near x_d by $(x-x_d)^\beta$. This behavior is in fact borne out by the details of the numerical solution.

Fig. 2.1

2.5 A Preliminary Examination of the Results — Does This Make Sense from a Physical Point of View?

The principal result of the previous section was that the properties of the cosmological background can serve to "spontaneously" break the local conformal (Weyl) invariance of a unified gauge model of weak and electromagnetic interactions. The VEV of the Higgs field was seen to vary with "cosmic" time. It is identically zero (no symmetry breaking) during the fireball era and gradually grows to a limiting value as the universe expands and cools. The gross features, at least, of this temperature-dependent behavior are similar to those exhibited by solid-state phenomena involving an analogous symmetry-breaking mechanism. In the Heisenberg model of the ferromagnet, the "order" parameter gives a measure of the symmetry (rotational) possessed by the ground state; in this sense it is roughly analogous to the VEV in the present work. The order parameter of a ferromagnet is temperature dependent; above a critical temperature it vanishes — the symmetry is then exact.¹⁹ Below this threshold it can assume a non-zero value-SSB. For spontaneously broken Weyl invariance the critical temperature is seen to be $\geq 10^5 \text{K}$ (see eq. 2.34).

As has been emphasized, the notion of a spontaneously broken symmetry has been derived from work in solid state phenomena. It would seem reassuring then that our results exhibit a similar feature in that the symmetry cannot be broken given temperatures exceeding a critical limit. Although this general agreement arises

from entirely different physical principles, it does nevertheless indicate that our approach has captured at least some of the significance of the physical situation.

2.6 Some Implications of these Results

The effective weak interaction coupling constant, G_F , is proportional²⁰ to ϕ^{-2} . During the present epoch, its value changes very slowly

$$\frac{\dot{G}_F}{G_F} = -2\frac{d}{dx} \log g(x) \lesssim 10^{-10} (\text{year})^{-1} \quad (2.44)$$

It is natural to investigate whether a variation of G_F on this order of magnitude can be confronted with experimental data. In fact, such a variation is very close to the upper limit imposed by various pieces of geophysical evidence.

Motivated by numerological and cosmological considerations, a variety of proposals have been advanced (by Gamow, Dirac and others²¹) involving a dependence on cosmic time of fundamental atomic constants. The best evidence against such a possibility comes from intercomparisons of decay rates of the radioactive nuclei which are used to determine the ages of rocks and meteorites. The arguments of Dyson, Wilkinson and others²² use the agreement between diverse radioactive dating methods to place limits on the rate of change of the constants which quantify the decay rates. The "nuclear clocks" commonly used in the determination of sample ages involve the α -decay age (say, uranium-thorium-lead), β -decay

age (typically, rubidium-strontium or rhenium-osmium or potassium-argon) and spontaneous fission (uranium). It has been shown that the U^{238} (α) and Rb-Sr (β) ages of uraninites and cogenetic mica agree to within 5% over 3×10^9 years. The Re-Os (β) age of molybdenites agree with the Rb-Sr (β), U(α) and K-Ar (β) ages of cogenetic minerals to within 10% over 3×10^9 years.

The analysis of Dyson (see ref. 22 for a complete discussion) places the most stringent limits on the rate of change of G_F . Under the hypothesis that the only microscopic "constants" with a time dependence are the fine structure constant, α , and G_F , he obtains

$$\left| \frac{\dot{\alpha}}{\alpha} \right| \leq 2 \times 10^{-14} (\text{year})^{-1} \quad (2.45)$$

$$\left| \frac{\dot{G}_F}{G_F} \right| \leq 10^{-10} (\text{year})^{-1}$$

The radioactive decay rates involved also depend on c , $\hbar$, m_e , m_p , m_n and on the strength and range of the strong interaction. If all of these factors are permitted a time variation, then the above strictures can to some extent be circumvented (providing that compensating changes occur).²³

One may conclude then that the time variation of G_F (eq. 2.44) predicted by our theory is within experimental limits and cannot be confirmed or ruled out by the present geophysical evidence. Our result does have the distinctive feature that it predicts that G_F has always been decreasing (monotonically). Under these conditions, it takes a shorter time for a given amount of daughter nuclei to

accumulate. Hence the actual age of a sample would be less than that ascribed to it by analysis assuming G_F has been constant. If (conventional) β ages are consistently greater than corresponding α ages for the same sample, one may interpret this as an indication that, as predicted here, G_F is decreasing. Considerable caution must be exercised in drawing such conclusions because the analysis of the data involves certain choices and assumptions as to which microscopic constants will be ascribed a time variation.

Another area which has some bearing on our results regarding G_F is the process of "cosmological production" of helium and deuterium. There is a good deal of evidence (though not unequivocal) pointing to a fairly uniform abundance of helium throughout the universe.²⁴ This suggests that the helium was produced not in the diverse observed objects but rather via some universal process taking place before the objects were formed. Nucleosynthesis during the primeval "fireball" that followed the initial singularity seems a likely candidate as a universal mechanism for helium production. A number of calculations²⁵ along these lines have computed the primeval helium abundance to be ~25% by mass — a figure remarkably close to the observed background abundance. These calculations involve a number of assumptions about the conditions during the fireball. In particular, weak interaction processes, which play a key role in the nucleosynthesis scenario, are assumed to be identical to those processes as observed today. The dramatic conclusions of section 2.4, indicating that during the fireball

era, weak and electromagnetic interactions were of comparable strength and range (infinite), are likely to alter significantly a computation of primordial helium abundance.

The conventional calculation proceeds in the following manner. Beginning with equal abundances of neutrons and protons, the equilibrium abundance ratio, n/p , is fixed by reactions like

Above $T \sim 10^{10} \text{K}$, these reactions take place at a rate fast enough to maintain the equilibrium abundance ratio of n/p . As the universe expands and cools below this temperature, the n/p ratio is "frozen in." The residual neutrons can form deuterium via

An appreciable amount of deuterium can accumulate only after the radiation present in the cosmic fluid has cooled below the photodissociation threshold of d .

Remarkably, the expansion (and cooling) rate of the universe dovetails with the weak interaction parameters with the result that nuclear burning commences just as the age of the universe becomes comparable to the free neutron half-life and an appreciable fraction of neutrons are disappearing.

The results of the present work indicate that during the

fireball era the weak interaction parameters (strength and range) differ drastically from their presently observed values. Such differences need not, however, alter the scenario for primordial helium production.²⁶ Although the weak coupling was on the order of electromagnetic coupling, the neutron half-life was not correspondingly decreased. All the masses in the theory are proportional to $\langle \phi \rangle$, the VEV. When the symmetry is exact (above $\sim 10^5 \text{K}$) all the masses are identically zero (no length scale). The neutron cannot decay then until the symmetry is broken. In this way, the general features of the present model of cosmic helium production are retained.

2.7 A Closer Look at the "Fireball Era"

It is evident from the analysis of section 2.4 (see the discussion following eq. 2.37) that R , the Ricci scalar, plays a crucial role in the cosmological mechanism for spontaneous symmetry breakdown. R was computed under assumptions which characterize a large class of cosmological models. It is assumed that the cosmological geometry (homogeneous and isotropic) arises because the cosmological background behaves like a perfect fluid. In this framework it is easy to show that $R \div P = \rho/3$. Observationally, one finds that today the rest-mass density of visible matter far exceeds the matter pressure. For radiation one always has $P_R = \rho_R/3$. Then one has $R_{\text{TODAY}} \div -\rho_m/3$, a condition that has persisted since about $T \sim 10^5 \text{K}$. Above 10^5K the matter and

radiation components of the cosmic fluid are tightly coupled and are in thermal equilibrium. At still higher temperatures (closer to the initial singularity), the matter pressure increases, approaching $\rho_m/3$ as a limit. For the purposes of cosmological model building, one generally takes $R \approx 0$ in this "fireball" phase. In the exploration of a possible cosmological origin for spontaneous symmetry breakdown, a more detailed calculation of R seems warranted.

The universe seems to have originated in a state of infinite compaction comprising a singularity in the space-time manifold. An understanding of this initial state, even at a broad conceptual level, has not yet been formulated. Nevertheless, it seems that a short time after this primordial event all the matter and radiation in the universe reached a state of thermodynamic equilibrium, effectively washing out all the details of the initial state. In addition, thermalization provides a conceptual framework in which we may fix our ideas.

Consider a calculation of R making use of thermal averages.²⁷ The nucleons are taken to comprise the dominant component of "matter."

$$\langle p \rangle = \frac{8\pi}{3} \int_0^\infty \frac{q^3 \left(\frac{\partial E(q)}{\partial q} \right) dq}{e^{[E-\mu(T)]/kT} + 1} \quad (2.48a)$$

$$\langle p \rangle = 8\pi \int_0^\infty \frac{E(q) q^2 dq}{e^{[E-\mu(T)]/kT} + 1} \quad (2.48b)$$

where $E(q) = \sqrt{M_N^2 + q^2}$, M_N is the nucleon rest-mass.

Then

$$R \div - \frac{8\pi}{3} M_N^2 \int_0^\infty \frac{(q^2/E) dq}{e^{[E-\mu(T)]/kT} + 1} \quad (2.49)$$

$\mu(T)$ denotes the chemical potential.

Computation of R at a given temperature requires knowledge of $\mu(T)$. This is obtained in the following manner.

$$\langle n \rangle = 8\pi \int_0^\infty \frac{q^2 dq}{e^{[E-\mu(T)]/kT} + 1} \quad (2.50)$$

n denotes the number density of nucleons.

Conservation of baryon number implies that

$$n \div a^{-3} \quad (2.51)$$

where $a(t)$ is the expansion parameter (see eq. 2.13).

Curing the fireball era

$$a T = \text{constant} \quad (2.52)$$

Thus

$$n \div T^3 \quad (2.53)$$

where the constant of proportionality can be evaluated from

$$n_{\text{TODAY}} \sim 10^{-6} \text{ baryons/cm}^3$$

At a given temperature, T , then $\langle n \rangle$ can be calculated from eq. 2.53 and eq. 2.50 solved (numerically) for $\mu(T)$. One now calculates $R(T)$ from eq. 2.49.

Although such a calculational program does indicate that R decreases at higher temperatures, it cannot be carried out

indefinitely. Above $T \sim 10^{12} \text{K}$ the thermal radiation contains a significant number of pion pairs and the internucleon separation becomes comparable to a Compton wavelength. Details of the internucleon interaction become increasingly important. At still higher temperatures it becomes impossible to use the language of thermal equilibrium. When the characteristic expansion time scale ceases to be much greater than the relaxation time of the system, the notion of achieving equilibrium is no longer meaningful.

It is possible, however, to study the behavior of $R(T)$ without requiring that thermal equilibrium be established. In a study of the general relativistic Boltzman equation,²⁸ Taub has shown the following:

if one writes

$$\rho = m(c^2 + \epsilon)$$

or

$$\rho = m(1 + \epsilon) \text{ in our system of units}$$

where m denotes the rest-mass density and ρ , as before, the total mass-energy density, then it is possible to show that

$$\epsilon \geq \frac{3}{2} \frac{P}{m} + \left[\left(1 + \frac{9}{4} \left(\frac{P}{m} \right)^2 \right)^{1/2} - 1 \right] \quad (2.54)$$

Further, if the equality sign holds then

$$\lim_{y \rightarrow \infty} \rho = 3P \quad (2.55)$$

$$\text{where } y = \frac{P}{m}$$

$$\text{Eq. 2.55 implies that } \lim_{y \rightarrow \infty} R = 0^- \quad (2.56)$$

It is clear that $\frac{P}{m} \rightarrow \infty$ as $T \rightarrow \infty$ in the fireball but without additional assumptions as to the nature of the cosmic fluid, one is unable to assert that the equality sign in 2.54 holds.

More detailed calculation of $R(T)$ lie beyond our capabilities at present. One may take the point of view that $\lim_{T \rightarrow \infty} R(T)$ is among a set of initial data required for a complete cosmological model. It may be taken to vanish. More definite conclusions involve a number of issues at the heart of contemporary physics. At $\sim 10^{-23}$ seconds after the initial event, the "size" of the universe is $\sim 10^{-13} \text{ cm}$ — the spatial extent associated with a single hadron. The conventional space-time description of particle states is inadequate beyond this point. This may indicate that the singularity can be avoided. Alternatively it may signal the need for incorporating a fully quantized version of the gravitational field and perhaps even of space-time.

Chapter Three Spontaneous Symmetry Breaking in the
Interior of a Neutron Star

3.1 Preliminary Remarks

The results of the previous chapter indicate that the spontaneous breakdown of local scale (Weyl) invariance in a gauge model may have a cosmological origin. From this point of view the breakdown of the symmetry occurs via the Gürsey-Penrose term in the Higgs Lagrangian. The properties of the cosmological space-time (as embodied in the scalar (Ricci) curvature) dictate the conditions under which the symmetry may persist or be broken. It is natural to inquire whether other space-time solutions can affect the Weyl invariance of the gauge model. The present chapter investigates what effects the Schwarzschild solution may have on SSB.

For a static, spherically symmetric object such as a (non-rotating) star, the appropriate space-time structure is described by a static, isotropic metric that approaches the Minkowski metric (in spherical coordinates) at spatial infinity. This is the Schwarzschild solution²⁹ and is given by³⁰

$$ds^2 = -\exp(2B(r))dt^2 + \frac{dr^2}{1-2\frac{m(r)}{r}} + r^2 d\Omega^2 \quad \text{for } r < R_s \quad (3.1a)$$

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1-2\frac{M}{r}} + r^2 d\Omega^2 \quad \text{for } r > R_s \quad (3.1b)$$

where M, R_s denote the total mass-energy of the star and its radius respectively. $m(r)$ is defined as the mass-energy within radius r

$$m(r) \equiv \int_0^r 4\pi r'^2 \rho(r') dr' \quad (3.2)$$

with $m(r=0) = 0$.

$P(r)$ and $\rho(r)$ denote the pressure and mass-energy density at radius r . A constraint in the Schwarzschild solution is the condition

$$\frac{dB(r)}{dr} = \frac{m(r) + 4\pi r^3 P(r)}{r(r-2m(r))} \quad (3.3)$$

with $B(R_s) = \frac{1}{2} \ln(1 - \frac{2M}{R_s})$.

As before we will work in a natural system of units where $\hbar = c = L^*$ (Planck length) = 1 so that Newton's gravitational constant, G , will not appear explicitly.

It is evident from eqs. 3.1-3.3 that in order to determine the geometry of a relativistic star, one must know $P(r)$ and $\rho(r)$. Two additional relations furnish this information. The first equation describes the balance between the gravitational force and the pressure gradient.

$$\frac{dP(r)}{dr} = \frac{-(P(r)+\rho(r))(m(r)+4\pi r^3 P(r))}{r(r-2m(r))} \quad (3.4)$$

with $P(r=0) = P_c$ = central pressure.

Eq. 3.4 is the equation of hydrostatic equilibrium; it is sometimes referred to as the Tolman-Oppenheimer-Volkoff equation. The second relation needed is an equation of state, that is, a relation that

provides $P(\rho)$. Eq. 3.1-3.4 together with specification of P_c and an equation of state are sufficient to provide a complete description of the stellar geometry, both inside the object and in the exterior region. It is noted here that the Schwarzschild solution prevents one from having stars where $m(r) > r/2$. Stars with too great a mass are unstable against gravitational collapse.

Evidently the quantity, M/R_s , provides a rough measure of the deviation from flat space-time. Order of magnitude comparisons of M/R_s for objects in various stages of stellar evolution yield

	M/R_s
sun:	$\sim 10^{-6}$
white dwarf:	$\sim 10^{-3}$
neutron star:	$\sim 10^{-1}$

In other words, a typical neutron star may have a radius about twice its Schwarzschild radius ($R_{sc} = 2M$). One may conclude that geometry will have a significant effect on the spontaneous breakdown of Weyl invariance in a collapsed star (if at all).

3.2 A Neutron Star Model

A cross-sectional slice of a neutron star is schematically indicated in Fig. 3.1.³¹ The mass density ranges from about 7.86 g/cm^3 at the surface (Fe^{56}) to $\sim 10^{15} \text{ g/cm}^3$ in the interior core. Typically the atmosphere is several meters thick with a solid crust (composed of nuclei, and free neutrons and electrons) beneath it. The crust is roughly 1 km thick. The interior of the

Fig. 2.1 Neutron star in cross-section

star consists primarily of a fluid of degenerate neutrons with a small contamination of protons and electrons. At densities $>10^{15}$ g/cm³ a variety of hyperons, beginning with Σ^- and Λ^0 , make their appearance. Although unstable in the laboratory, such particles are stable under these conditions because there is no available phase space for their decay products. It is clear that the composition, and consequently macroscopic quantities such as compressibility, depend on the nature of the elementary particle spectrum. For a given (very high) baryon density, the additional hyperonic states make it possible to accommodate more baryons in a given volume at a given Fermi energy than could be achieved if only the neutron and proton states were available. It is generally concluded that the exact details of the elementary particle spectrum are not essential to the description of the neutron star as a whole.³² It is possible though that the exponentially increasing spectrum suggested by the statistical bootstrap and dual resonance models of strong interactions³³ could alter this assumption. The neutron star models considered here will be concerned chiefly with the neutron fluid region and will ignore the hyperon core.

The discussion of section 3.1 indicates that construction of a neutron star model requires an appropriate equation of state. Since P and ρ assume values that vary over a range of about 15 orders of magnitude, a suitable equation of state involves considerable calculation. We begin by developing a somewhat simplified model³¹ with an equation of state derived from a Fermi liquid model of the

degenerate neutrons. This will provide a guideline for calculations involving more realistic equations of state.

In the theory of the quantum liquid, as pioneered by Landau, the Fermi statistics determine many of the properties of the system.³⁴ The Landau theory is applicable when, beginning with a non-interacting system, interparticle interactions are slowly turned on. A gradual change in the properties of the system occurs; these changes can be calculated perturbatively. It is assumed that as the interactions are switched on slowly, the non-interacting ground state goes over smoothly into the interaction ground state. The neutron fluid is taken to be at absolute zero. At a density of 10^{15} g/cm³, the neutron degeneracy temperature, $T_d \sim 10^{12}$ K. As they are thought to originate via a supernova implosion, neutron stars are formed at very high temperatures. They cool very rapidly via neutrino emission. Within several hundred seconds after formation they have cooled below $\sim 10^8$ K.³⁵ To a very good approximation then, the neutron fluid may be considered to be in the ground state. The equation of state for a Fermi liquid at $T=0$ is³⁶

$$\frac{\partial P}{\partial \rho} = \frac{1 + F_0^S}{1 + \frac{1}{3} F_0^S} \frac{k_n^2}{3m_N^2} \quad (3.5)$$

where k_n is the Fermi wavenumber:

$$k_n = (3\pi^2 n_N)^{1/3} \quad (3.6)$$

n_N gives the number density of neutrons, m_N is the neutron rest mass. F_0^S and F_1^S are called the Fermi liquid parameters and contain

the details of the particle interactions. (In S.I. units, a factor of $\hbar^2$ would appear on the r.h.s. of eq. 3.5.)

The equation of state, eq. 3.5, together with the TOV equation of hydrostatic equilibrium, eq. 3.4, can now furnish $P(r)$ and $\rho(r)$. F_0^S and F_1^S are on the order of unity and, for this rough calculation, are set equal to 1. Combining eq. 3.5 and eq. 3.4 we obtain

$$\frac{d\rho(r)}{dr} \cong - \frac{3m_N^2}{k_n^2} \frac{(P(r)+\rho(r))(m(r)+4\pi r^3 P(r))}{r(r-2m(r))} \quad (3.7)$$

Upon specifying P_C and ρ_C , solution of eq. 3.4 and 3.7 can be achieved by so-called "one-step" numerical methods (see Appendix B for more details). The method involves knowing $P(x)$ and $\rho(x)$ and then eq. 3.4 can be used to calculate $P(x+h)$. Then, eq. 3.7 is used to calculate $\rho(x+h)$ and one returns to eq. 3.4 to calculate $P(x+h+h)$. Such an algorithm is easily adapted to machine calculation. The procedure ("stepping out") is continued until ρ falls to about 10 g/cm^3 . This is taken to be the boundary between the neutron core and the crust of ordinary nuclei. Beyond this point the equation of state, eq. 3.5, is no longer appropriate. The results of such a calculation where $\rho_C = 10^{14} \text{ g/cm}^3$, $P_C = 3.0 \times 10^{34} \text{ dynes/cm}^2$ are indicated in Figs. 3.2 and 3.3.

More complicated equations of state that both describe the degenerate neutron core and join smoothly onto relations appropriate

Fig. 3.2

Fig. 3.3

to the ordinary nuclei in the crust have been developed by a number of workers.³⁷ The Harrison-Wheeler equation of state distinguishes four regions in the neutron star interior.³⁸ The expression in each region joins onto that for the next region to about four significant figures at the point of transition. There are no serious discontinuities in the derivatives at each point of overlap. Table 3.1 summarizes the H-W equation of state.

The equation of state in each regime was calculated by determining the most stable configuration of a given number of baryons under the prevailing density and pressure. For example, Regime I begins with Fe⁵⁶ and the volume compressibility of iron. At densities higher than ~15 g/cm³, the specifics of the solid-state forces are overcome and the Thomas-Fermi statistical model is applied to an individual atom located in a cell. At still higher densities (~10⁷ to 10¹² g/cm³) the most stable configuration is separated nuclei in β -equilibrium with a relativistic electron gas. As the electron pressure increases, there develops a mixture of ideal electron, proton and neutron gases. The components are considered to interact only to the extent that charge neutrality and β -equilibrium are maintained.

We now use the Harrison-Wheeler equation of state in conjunction with eq. 3.4 to calculate $P(r)$ and $\rho(r)$. The results of this calculation agreed with those achieved by Harrison and Wakano³⁹ using the H-W equation of state but employing a different numerical algorithm. Figs. 3.4 and 3.5 give these results.

HARRISON-WHEELER EQUATION OF STATE

P in dynes/cm ²
ρ in g/cm ³
Regime I: ρ from 7.86 to 1.00×10^6
$P = 3.0271 \times 10^{12} (\rho^{1/3} - 1.4415)^5 - 1.50319 \times 10^{11}$
Regime II: ρ from 1.00×10^6 to 3.22×10^{11}
$P = 2.4616 \times 10^{15} \rho^{5/4} (1. + 2388.8 \rho^{-1/2})^{-5/6}$
Regime III: ρ from 3.22×10^{11} to 4.63×10^{12}
$P = 3.1134 \times 10^{21} \rho^{2/3} (1. + 0.0031822 \rho^{1/6})^6$
Regime IV: ρ from 4.63×10^{12} to infinity
$P = 2.9959 \times 10^{20} \rho^{5/3} (9.1208 \times 10^8 + \rho^{5/3})^{-6/5}$

TABLE 3.1

Fig. 3.4

Fig. 3.5

3.3 The Higgs VEV in the Interior of a Neutron Star

Having calculated $P(r)$ and $\rho(r)$ via an appropriate equation of state and the Tolman-Oppenheimer-Volkoff equation, it is now possible to completely specify the geometry (Schwarzschild) of the neutron star. (See eqs. 3.1-3.3.) This in turn enables one to investigate the behavior of the Higgs VEV in the interior of a relativistic star.

As we have seen in Chapter Two, spontaneous symmetry break-down develops if the VEV, ϕ , of the Higgs field differs from zero. The Euler-Lagrange equations for ϕ (see eq. 2.29) are

$$\left(\square + \frac{R}{6}\right)\phi + \frac{\lambda}{6}\phi^3 = 0 \quad (3.8)$$

In the Schwarzschild geometry, a non-trivial vacuum solution can have at most a dependence on the radial coordinate. That is $\phi = \phi(r)$. With this assumption and with the components of the covariant metric tensor as given by eqs. 3.1-3.3, eq. 3.8 can be written

$$\frac{d^2\phi}{dr^2} + \frac{d\phi}{dr} \left[B' + \frac{2r-3m(r)}{r(r-2m(r))} - \frac{m'(r)}{r-2m(r)} \right] + \frac{r}{r-2m(r)} \frac{R}{6}\phi + \frac{r}{r-2m(r)} \frac{\lambda}{6}\phi^3 = 0 \quad (3.9)$$

where a prime indicates $\frac{d}{dr}$.

(Recall that $\square \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ where ∇_μ denotes the Riemann covariant derivative. If f is a scalar function,

$$\square f = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu f) \text{ where } g \text{ denotes } \text{Det } g_{\mu\nu}.)$$

It is an excellent approximation to consider the matter within a star to be a "perfect fluid." ("Shear stresses are negligible and energy transport is negligible on a hydrodynamic time scale.")⁴⁰ The trace of the energy-momentum tensor, T , for a perfect fluid is given by

$$T = 3P - \rho \quad (3.10)$$

(see the discussion following eq. 2.19)

Upon taking the trace of the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = -8\pi T_{\mu\nu} \quad (3.11)$$

one obtains

$$R = 8\pi T \quad (3.12)$$

Combining eqs. 3.10 and 3.12 then we have

$$R = 24\pi (P - \rho/3) \quad (3.13)$$

Eq. 3.3 indicates that $dB(r)/dr$ can be calculated once $P(r)$ and $\rho(r)$ are known. The defining relation (eq. 3.2) for $m(r)$ and the expression (eq. 3.13) for the Ricci scalar, R , indicate that a knowledge of $P(r)$ and $\rho(r)$ will be sufficient to solve eq. 3.9. A prerequisite to study of the equation satisfied by the Higgs VEV is then a calculation of $P(r)$ and $\rho(r)$ for the neutron star according to section 3.2.

The following procedure for the study and solution of eq. 3.9 was developed:

- i) the methods of section 3.2 (H-W equation of state together

with TOV equation of hydrostatic balance) furnished $P(r)$ and $\rho(r)$ at various values of r from $r=0$ to $r=R_s$ (radius of the neutron star).

- ii) values of $m(r)$ for various values of r are computed. This requires as input the values obtained in i) for $\rho(r)$. By definition, (eq. 3.2)

$$m(r) \equiv \int_0^r 4\pi r'^2 \rho(r') dr' \quad (3.2)$$

For a given value of r , the definite integral was evaluated using Simpson's Rule. This involved the tabulated values of $\rho(r)$ computed in i).

- iii) the second-order differential equation, eq. 3.9, was reduced to a system of two first-order equations. Define

$$Q(r) \equiv B'(r) + \frac{2r-3m(r)}{r(r-2m(r))} - \frac{m'(r)}{r-2m(r)} \quad (3.14a)$$

$$S(r) \equiv \frac{4\pi r(P(r) - \frac{\rho(r)}{3})}{r - 2m(r)} \quad (3.14b)$$

$$T(r) \equiv \frac{r}{6(r - 2m(r))} \quad (3.14c)$$

It is convenient to "scale out" the dimensionless parameter, λ , by introducing the function $g(r) = \sqrt{\lambda} \phi(r)$ (3.15)

With these changes, eq. 3.9 can now be written

$$\frac{d^2 g}{dr^2} + Q(r) \frac{dg}{dr} + S(r)g(r) + T(r)g^3(r) = 0 \quad (3.16)$$

Now introduce the functions $y^{(1)}(r)$ and $y^{(2)}(r)$, where

$$y^{(1)}(r) = g(r) \quad (3.17a)$$

$$y^{(2)}(r) = \frac{dg(r)}{dr} \quad (3.17b)$$

One now must solve the system of equations:

$$\frac{dy^{(1)}(r)}{dr} = y^{(2)}(r) \quad (3.18a)$$

$$\frac{dy^{(2)}(r)}{dr} = f(y^{(1)}, y^{(2)}, r) \quad (3.18b)$$

where

$$f(y^{(1)}, y^{(2)}, r) = -Q(r)y^{(2)}(r) - S(r)y^{(1)}(r) - T(r)y^{(1)3}(r) \quad (3.19)$$

Initial values are provided by the physical requirement that the solution for the Higgs VEV at the outer boundary of the neutron core must equal and join smoothly onto the cosmological solution achieved in Chapter Two. It is reasonable to assume that no change in the VEV occurs in the crustal layer because ordinary nuclei exist in this region.

The system of eq. 3.18 was solved numerically using a classical four-step Runge-Kutta method.⁴¹ (See Appendix B for greater detail about this method.) It was necessary to coordinate the choice of step-size for this calculation with that used in the computation in step i). This was done to ensure that when an evaluation of say $Q(r)$ was required at $r = \hat{r}$, the tabulated stellar structure from

i) contained $P(\hat{r})$ and $\rho(\hat{r})$.

The calculational scheme as outlined above was carried out for three neutron stars. Table 3.2 and Fig. 3.6 summarize the particulars for each case and the results.

TABLE 3.2

$M(m_\odot)$	$R^S(\text{km})$	$P_c(\text{dynes/cm}^2)$	$\phi(r=0)/\phi_{\text{OUTSIDE}}$
1) 0.67	7.5	1.17×10^{36}	1.03
2) 0.33	26.1	2.49×10^{33}	1.01
3) 0.18	150.	1.65×10^{32}	1.001

3.4 Conclusions

Since the effective weak interaction constant, G_F , is proportional to ϕ^{-2} , its value is seen to decrease slightly toward the center of the star. This small variation present in the central region of the neutron star is not likely to alter significantly our present understanding of its structure. Our conjecture that such an effect is entirely negligible in the interior of a main-sequence or white dwarf star is borne out by the computation.

As noted in section 3.2, this calculation does not take into account the (little-known) nature of the hadronic core of the neutron star. It is possible that dramatic effects on ϕ , the VEV,

Fig. 3.6 $\hat{\phi}$ is VEV normalized to value it takes outside the star

could occur in this region. Such results would depend on the nature of the elementary particle spectrum.

Speculation along these lines had best defer to more experimental data from particle accelerators.

Chapter Four Spontaneously Broken Weyl Invariance — Possible Connection with Conformal Relativity

4.1 A New Perspective on the Problem

The work described in the previous chapters has largely been an exploration of the ways various space-time structures can initiate the spontaneous breakdown of local scale (Weyl) invariance. An essential aspect of this work was the role played by the "Gürsey-Penrose term" in the Higgs Lagrangian. In some recent work on this question of spontaneously broken Weyl invariance,⁴² Domokos has shown that the "Gürsey-Penrose term" is susceptible to a rather different interpretation. The resulting perspective leads to a number of interesting results and intriguing possibilities.

The starting point for this new insight is the gauge model detailed in section 2.2. There it was shown that the total Weyl-invariant action for the theory could be written

$$W = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - \frac{1}{12} \phi^* \phi R - \frac{\lambda}{4!} (\phi^* \phi)^2 \right\} + W' \quad (4.1)$$

where

$$W' = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} f^2 \phi^* \phi g^{\mu\nu} A_\mu A_\nu - \frac{i}{2} f \phi^* \partial_\mu \phi g^{\mu\nu} A_\nu - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + L(\psi) \right\} \quad (4.2)$$

W' is the gauge-invariant action of the massless gauge fields, A_μ , and massless fermions, ψ , in the theory.

The new line of investigation begins with the observation

that the Gürsey-Penrose term, $\frac{1}{12} \phi^* \phi R$, bears some resemblance to the Hilbert-Palatini Lagrangian for the free gravitational field,

$$L_{H-P} = \frac{1}{16\pi G} R \quad (4.3)$$

This similarity suggests that if Newton's gravitational constant, G , were replaced by a field ($\phi^* \phi$), then perhaps a Weyl-invariant extension of a Salam-Ward-Weinberg gauge theory would incorporate a theory of gravitation as well. Evidently this will not be the theory of gravitation according to Einstein, but will be closer to the "conformal relativity" advanced by Brans and Dicke and Hoyle and Narlikar and others.⁴³

Solutions exhibiting spontaneous symmetry breaking are sought under the assumption that only ϕ (and perhaps the gravitational field) have a non-vanishing VEV. As has been done throughout, everything is calculated in the "tree approximation."

One begins by exploiting the gauge invariance of the theory and removing the "phase" of ϕ (the Goldstone boson). A polar decomposition of ϕ is made

$$\phi(x) = \rho(x) U(x) \quad (4.4)$$

where ρ is positive and U is unitary. One may write $U = \exp i\theta$. Then the gauge transformation $A_\mu \rightarrow A_\mu - i f \partial_\mu \ln U$ absorbs U into the longitudinal part of the vector potential, A_μ . This is just the familiar Higgs mechanism wherein the Goldstone boson is "absorbed" by the vector potential.⁴⁴ Now, the effective action for the VEV can be written

$$W_{\text{eff}} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \rho \partial_\nu \rho - \frac{1}{12} \rho^2 R - \frac{\lambda}{4!} \rho^4 \right\} \quad (4.5)$$

The equations satisfied by the VEV are then obtained by varying eq. 4.5 with respect to the field quantities.

i) varying with respect to $g_{\mu\nu}$ yields

$$\begin{aligned} \frac{\rho^2}{6} G_{\mu\nu} &= \partial_\mu \rho \partial_\nu \rho - g_{\mu\nu} \left(\frac{1}{2} g^{\lambda\sigma} \partial_\lambda \rho \partial_\sigma \rho - \frac{\lambda}{4!} \rho^4 \right) \\ &+ \frac{1}{6} g_{\mu\nu} \square \rho^2 - \frac{1}{2} \{ \nabla_\mu, \nabla_\nu \} \rho^2 \end{aligned} \quad (4.6)$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor; $\square$ denotes the covariant D'Alembertian and $\{ \}$ indicate that the symmetrized product is formed. It is noted that the right hand side (r.h.s.) of eq. 4.6 is $\Theta_{\mu\nu}$, the conformal energy-momentum tensor of the scalar field, ρ .

ii) varying with respect to ρ yields

$$\square \rho + \frac{R}{6} \rho + \frac{\lambda}{6} \rho^3 = 0 \quad (4.7)$$

However, eq. 4.7 is not an independent dynamical equation but is already contained in eq. 4.6. The trace of eq. 4.6 yields

$$\rho \left(\square \rho + \frac{R}{6} \rho + \frac{\lambda}{6} \rho^3 \right) = 0 \quad (4.8)$$

This means that either $\rho \equiv 0$ (no SSB) or that ρ is not an independent degree of freedom. As is pointed out in ref. 42, the point here is that the Weyl invariance of the effective action (eq. 4.5) has not been exploited. The gauge invariance of the full action

(eqs. 4.1, 4.2) enabled one to remove the Goldstone boson à la Higgs. In like fashion, the Weyl invariance of W_{eff} permits the removal of the coordinate dependence of ρ via a "second Higgs-Kibble transformation."

Since ρ is a "radial component" in internal space, it is non-negative. One may write

$$\rho(x) = \chi \exp - \frac{1}{2} \sigma(x) \quad (4.9)$$

where χ is a constant of dimension $(\text{length})^{-1}$ and σ is real, $-\infty < \sigma < \infty$.

The effective action now becomes

$$W_{\text{eff}} = \chi^2 \int d^4x \sqrt{-g} e^{\sigma(x)} \left\{ \frac{1}{8} g^{\mu\nu} \partial_\mu \sigma(x) \partial_\nu \sigma(x) - \frac{R}{12} - \frac{\lambda \chi^2}{4!} e^{\sigma(x)} \right\} \quad (4.10)$$

Now one performs the finite Weyl transformation

$$g_{\mu\nu}(x) + g'_{\mu\nu}(x) = e^{\sigma(x)} g_{\mu\nu}(x) \quad (4.11)$$

under which (see section 2.1)

$$\rho(x) + \rho'(x) = e^{\frac{1}{2}\sigma(x)} \rho(x) \quad (4.12)$$

Thus, $\rho'(x) = \chi = \text{constant}$. Under the transformation eq. 4.10, the effective action becomes

$$W_{\text{eff}} = - \frac{\chi^2}{12} \int d^4x \sqrt{-g'} \left(R' + \frac{\lambda}{2} \chi^2 \right) \quad (4.13)$$

where the primes indicate that the quantity is to be calculated with the metric $g'_{\mu\nu}(x)$.

With this choice of conformal "gauge", the VEV, ρ' , is a

constant. The coordinate dependence of the VEV has been absorbed into the metric via what may be called a "second Higgs-Kibble transformation." Upon varying the effective action (eq. 4.13) with respect to g' , one obtains the free gravitational field equations:

$$\frac{\chi^2}{12} (G'_{\mu\nu} - \frac{1}{2} g'_{\mu\nu} \lambda \chi^2) = 0 \quad (4.14)$$

For a non-trivial vacuum ($\chi \neq 0$) these are the Einstein field equations with a "cosmological constant" term. The solution of eq. 4.14 is a space of constant curvature: the Riemann tensor is characterized by a single number, say K . The Einstein tensor is

$$G'_{\mu\nu} = 3K g'_{\mu\nu} \quad (4.15)$$

$$\text{which gives, by eq. 4.14} \quad K = \frac{\lambda}{12} \chi^2 \quad (4.16)$$

One may interpret this result in the following way. The vacuum determines the scale of ρ , χ being of dimension $(\text{length})^{-1}$. A non-trivial VEV generates a curved metric. "The vacuum is 'filled by something', which, not surprisingly generates a non-vanishing curvature."⁴⁵ The following features of the situation are notable:

i) the Einstein field equations are recovered, with the essential addition of the cosmological constant term. A physical basis for this term and for the non-vanishing curvature of the vacuum solution are provided. This is to be contrasted with the conventional perspective on the field equations wherein the presence of the cosmological term is allowed but not required by the theory.

ii) Equanimity between Mach and Einstein is restored. The existence of curved space solutions with no matter present was not anticipated by Einstein and was disturbing to the Mach-Einstein relation: matter $\neq$ curvature.⁴⁶ From the point of view of the present theory, the properties of the particle vacuum state (non-trivial VEV) dictate the structure of the gravitational vacuum solution (non-vanishing curvature).

It is evident from eq. 4.13 that upon choosing the conformal frame where ρ has no coordinate dependence, one may identify

$$\frac{\chi^2}{12} = \frac{1}{16\pi G} \quad (4.17)$$

which means that χ is on the order of the Planck mass, $M^* \sim 10^{19}$ GeV. Since all the mass parameters arising in the gauge theory are proportional⁴⁷ to χ , this presents a problem. How can one get down from the Planck mass to the scale that characterizes strong interactions ~ 1 GeV? It is not clear at this point how one might accomplish this but there have already appeared some preliminary indications that a unified theory of weak, electromagnetic and strong interactions may well require mass scales on the order of M^* .⁴⁸

4.2 Stability of the Vacuum Solution

Before further implications of the theory are pursued, it must be established that the solution exhibiting spontaneous symmetry breakdown is a stable solution. For SSB to occur, the

non-trivial vacuum must be stable. Eq. 4.14 indicates that the solution with a non-zero VEV determines the structure of the gravitational vacuum solution — a space of constant curvature. On the other hand, if the VEV vanishes, no conclusions can be drawn regarding the geometry of the vacuum state.

It is not clear what criteria are appropriate to judge the stability of this solution. A local analysis of perturbations may not be a reliable guide; some disturbances may in fact be only coordinate transformations. Some conclusions about the stability can be drawn from global considerations.

The solution in question is characterized by a non-vanishing VEV and a vacuum space-time of constant curvature. (This is a maximally symmetric space with 10 independent Killing vectors.) It has been established (see the discussion following eq. 4.8) that the VEV is not an independent dynamical degree of freedom. It was also noted that there is one conformal frame in which the geometrical aspects of the situation are most apparent. This is the frame in which the VEV has no coordinate dependence and the vacuum gravitational field equations are

$$G_{\mu\nu} - \frac{1}{2}g_{\mu\nu} \lambda \chi^2 = 0 \quad (4.18)$$

The spatial geometry of the solution is determined by the sign of λ . For $\lambda < 0$ one has a spatially closed geometry which expands forever; $\lambda > 0$ corresponds to a spatially open geometry which oscillates in time. These characteristics are best exhibited by introducing appropriate Robertson-Walker coordinates.⁴⁹

de Sitter:

$$\lambda < 0 \quad ds^2 = dt^2 - a^2 \cosh^2 \frac{t}{a} [d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta d\phi^2)] \quad (4.19a)$$

anti-de Sitter:

$$\lambda > 0 \quad ds^2 = dt^2 - a^2 \cos^2 \frac{t}{a} [d\xi^2 + \sinh^2 \xi (d\theta^2 + \sin^2 \theta d\phi^2)] \quad (4.19b)$$

where $a^2 = |K|^{-1}$ ($K = \frac{\lambda \chi^2}{12}$ see eq. 4.16)

(In the case of $\lambda > 0$, these coordinates do not cover the entire space.)⁵⁰ De Sitter and anti-de Sitter space solutions are conformally flat. That is, the Weyl conformal tensor, $C_{\alpha\beta\gamma\rho}$, vanishes.

$$C_{\alpha\beta\gamma\rho} = R_{\alpha\beta\gamma\rho} + g_{\alpha[\rho} R_{\gamma]\beta} + g_{\beta[\gamma} R_{\rho]\alpha} + \frac{1}{3} R g_{\alpha[\gamma} g_{\rho]\beta} \quad (4.20)$$

where $[\]$ denotes that the antisymmetric combination is to be taken.

The Riemann tensor of a maximally symmetric space, $R_{\mu\nu\alpha\beta}$, is characterized by a single number

$$R_{\mu\nu\alpha\beta} = \frac{R}{12} (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) \quad (4.21)$$

With eq. 4.21 it is a straightforward matter to verify that

$$\tilde{C}_{\alpha\beta\gamma\delta} = 0.$$

The metric for a conformally flat space-time can be cast in the form

$$g_{\mu\nu}(x) = \Omega^2(x) \eta_{\mu\nu} \quad (4.22)$$

where $\eta_{\mu\nu}$ denotes the Minkowski metric (flat space-time).

The question of the stability of de Sitter (anti-de Sitter) solutions is best examined by introducing the Gürsey decomposition of the metric:⁵¹

$$g_{\mu\nu}(x) = f(x)\gamma_{\mu\nu} \quad (4.23)$$

where $f(x) = (-\text{Det } ||g_{\mu\nu}||)^{1/4}$ and $\gamma_{\mu\nu}$ is unimodular.

This separation of the metric tensor into a scalar density, f , and a tensor density, $\gamma_{\mu\nu}$, enables one to distinguish between local and global effects. Information about the cosmological structure is contained in the determinant of the metric and details about the local irregularities of the system are contained in $\gamma_{\mu\nu}$. In a conformally flat empty space-time one may identify $\gamma_{\mu\nu}$ as $\eta_{\mu\nu}$ and

$$\Omega(x) = (-\text{Det } ||g_{\mu\nu}||)^{1/6} \quad (4.24)$$

Global stability then of de Sitter (anti-de Sitter) space can be analyzed in terms of $\Omega(x)$.

Begin by taking the trace of eq. 4.18. This yields

$$R = -\lambda\chi^2 \quad (4.25)$$

Express the Ricci scalar in terms of $\Omega(x)$. This may be done in a straightforward but rather lengthy computation beginning with the definitions:

$$\Gamma_{\alpha\beta}^{\mu} \equiv \frac{1}{2} g^{\mu\nu} (g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) \quad (4.26)$$

Thus, from eq. 4.22

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{\Omega(x)} [\Omega_{,\beta} \delta_{\alpha}^{\mu} + \Omega_{,\alpha} \delta_{\beta}^{\mu} - \Omega^{,\mu} \eta_{\alpha\beta}] \quad (4.27)$$

Then one proceeds in the same way to calculate

$$R_{\mu\nu}^{\lambda} = \Gamma_{\mu\delta,\nu}^{\lambda} - \Gamma_{\mu\nu,\delta}^{\lambda} + \Gamma_{\rho\nu}^{\lambda} \Gamma_{\mu\delta}^{\rho} - \Gamma_{\rho\delta}^{\lambda} \Gamma_{\mu\nu}^{\rho} \quad (4.28a)$$

$$R_{\alpha\beta} = R_{\alpha\lambda\beta}^{\lambda} \quad (4.28b)$$

$$R = g^{\alpha\beta} R_{\alpha\beta} \quad (4.28c)$$

It is easier however to make use of the following relation.⁵²

Under the conformal transformation

$$g_{\alpha\beta} \rightarrow \hat{g}_{\alpha\beta} = \Lambda^2(x) g_{\alpha\beta} \quad (4.29)$$

then $R \rightarrow \hat{R}$ where

$$\hat{R} = \Lambda^{-2}(x) R - 6 \Lambda^{-3}(x) \Lambda(x)_{;\alpha\beta} g^{\alpha\beta} \quad (4.30)$$

where ; denotes the covariant derivative formed with $g_{\alpha\beta}$.

Then the scalar curvature determined by $g_{\mu\nu} = \Omega^2(x) \eta_{\mu\nu}$ is given by

$$R = -6 \Omega^{-3}(x) \eta^{\alpha\beta} \Omega(x)_{;\alpha\beta} \quad (4.31)$$

Eq. 4.25 may now be written

$$-6 \Omega^{-3}(x) \square \Omega(x) = -\lambda\chi^2 \quad (4.32)$$

where $\square$ denotes the flat-space D'Alembertian (see eq. 4.31).

A non-trivial solution to eq. 4.32 is given by

de Sitter:

$$\lambda < 0 \quad \Omega(x) \sim (1 + \frac{x^2}{4a^2})^{-1} \quad (4.33a)$$

anti-de Sitter:

$$\lambda > 0 \quad \Omega(x) \sim (1 - \frac{x^2}{4a^2})^{-1} \quad (4.33b)$$

where, as before, $a^2 = |K|^{-1} = \frac{12}{|\lambda|X^2}$

and where $x^2 = \eta_{\alpha\beta} x^\alpha x^\beta$.

$\Omega(x)$ is a scalar density; the Hamiltonian for a scalar field,

$\phi(x)$, is

$$H = \frac{1}{2} [\dot{\phi}^2 + |\nabla\phi|^2 + m^2\phi^2 + \frac{\lambda}{12}\phi^4] \quad (4.34)$$

where $|\nabla\phi|^2$ denotes $-\eta^{ij}\partial_i\phi\partial_j\phi$ ($i=1,2,3$)

and $\eta^{ij} = -\delta^{ij}$.

H is positive definite. Thus it is the trivial solution ($\Omega=0$) which is the minimum solution. But $\Omega=0$ implies that $g_{\mu\nu}=0$ — the metric also vanishes (see eq. 4.22). In a global sense, the non-trivial vacuum (de Sitter, anti-de Sitter) does not constitute a minimum solution and is therefore an unstable configuration. The disappearance of a vacuum metric in the stable solution is similar to the result of Gürsey (ref. 51) which indicates that if the curvature of the universe is kept fixed as the universe is depleted of matter then the metric vanishes. One may adopt the point of view that ours is a calculation of the minimum curvature of the universe. The solution with a minimum radius of curvature

$$a = \frac{2}{X} \sqrt{\frac{3}{|\lambda|}} \quad (4.35)$$

is seen to be unstable.

Before closing the books on this calculation, a few remarks are in order. A local analysis of the stability of de Sitter and anti-de Sitter spaces was also undertaken.⁵³ The objective was to show that first-order deviations from the de Sitter (anti-de Sitter) solution exhibit oscillatory solutions. That is, gravitational waves rather than unbounded disturbances result from small departures from the initial configuration. The results of this analysis were inconclusive. Indeed local analyses of stability of space-times frequently yield results whose interpretation is not immediately obvious. Even in flat space-time, although all perturbations are stable in a linear approximation, a singularity can arise from certain small perturbations.⁵⁴ (A carefully focused pulse of incoming gravitational radiation can result in a singularity.)

Questions of stability are, in general, not easy to formulate let alone answer decisively. In terms of the criterion developed here, the vacuum solution characterized by a non-vanishing VEV and a space of constant curvature is unstable. This conclusion is tempered by the realization that $g_{\mu\nu}$ is not a primitive observable in the theory (only its derivatives are). A study of the stability of the curvature tensor seems warranted but it is not clear at present how one might achieve this.

Chapter Five Conclusions — Outlook for the Future

The attempts in recent years to understand the physics of elementary particles and their interactions have usually proceeded from the assumption that gravitation can have little bearing on the matter. (The work of S. W. Hawking and others and the formulations of "supergravity" by S. Deser and others represent important exceptions to the general trend.⁵⁵) This assumption appears quite reasonable since the gravitational force between two electrons is $\sim 10^{-40}$ x the electromagnetic force. The very formulation of General Relativity reinforces this sense of dichotomy between gravity and the other interactions. It was precisely this aspect of General Relativity which troubled Einstein; he endeavored to incorporate electromagnetism with gravitation in a single framework. Geometry would provide the unifying principle in the structure.

Though these attempts met with little success, the motivation — to better understand the relation between gravity and the other interactions — remains compelling.

The present work explores a possible connection between gravitation and the unified gauge models for weak and electromagnetic interactions. We find that geometry provides the linking node in the structure. The spontaneous breakdown of Weyl invariance is seen to depend on the scalar curvature, R , of the space-time. The space-time structure is determined by the gravitational interactions of the matter present. This relation between

gravitation and spontaneous symmetry breaking was examined from two distinct points of view. The first approach viewed the space-time structure as fixed — the metric taken to be prescribed. In a cosmological space-time it was found that initially, in the primordial fireball, the local conformal symmetry was exact.⁵⁶ Spontaneous breakdown occurred as the universe expanded and cooled. Implications of this result included a time variation of G_F , the effective weak interaction coupling strength. The magnitude of this variation is just beyond present experimental limits. A study of the conditions in the primordial fireball was undertaken in an attempt to calculate $\lim_{T \rightarrow \infty} R(T)$. The analysis raised some fundamental questions that lie beyond the scope of this inquiry. Any calculation involving "particles" (localized concentration of mass, charge, angular momentum...) founders when the dimensions of the space-time are comparable to those of the "particle." The concept of a nucleon carries meaning then only after the universe has expanded beyond $\sim 10^{-13}$ cm ($\sim 10^{-23}$ sec after the initial event). It is not even clear what notions are available to us before this point.

The spontaneous breakdown of Weyl invariance was also studied in the space-time (Schwarzschild) appropriate to the interior of a non-rotating collapsed star (a neutron star, here).⁵⁷ The study indicated that the effective weak coupling constant, G_F , had a small radial dependence — on the order of a few percent at most. Again, some interesting issues were raised. The details of the

Schwarzschild solution depend on the physics of the elementary particles comprising the stellar matter. The nature of the particle spectrum may be a significant factor in determining the geometry at the center of a collapsed star. As in the study of the cosmological model, the details of SSB are very sensitive to the equation of state. These efforts, studies of spontaneously broken Weyl invariance in cosmological and stellar models constituted the first perspective on the problem.

The second point of view was to regard gravity as an integral component of a Weyl-invariant extension of a gauge model for weak and electromagnetic interactions. It is now possible to perform the usual Higgs-Kibble transformation and follow with a second (exploiting the Weyl-invariance of the Lagrangian) H-K transformation. If the vacuum is non-trivial (SSB), one recovers the usual Einstein field equations with a cosmological constant. The latter term appears in a natural and fundamental way in sharp contrast to its uncertain position in General Relativity. If the VEV does not vanish, the vacuum is filled with "something" which generates a curved metric. The stability of this solution was studied in order to establish that SSB occurs. Global considerations indicated that the non-trivial vacuum solution was not stable. While this seems to be a severe blow to this whole point of view on breaking Weyl invariance spontaneously, it does seem promising to pursue further the relationship between the particle vacuum state and gravitational vacuum solution.

By way of some concluding remarks on this entire investigation we include the following observations. Gravitation, though negligibly weak in comparison to the other known interactions, is apparently the sole determinant of the space-time structure. It is not unreasonable to expect that the physics of elementary particles is dependent on certain aspects of a space-time. The present work indicates that a global characteristic of space-time, the scalar curvature, may have important implications for a gauge model of weak and electromagnetic interactions. We emphasize the global nature of R because we suspect it may be a crucial observation. In the language of two-dimensional surfaces one knows that the difference between a plane and a sphere can be negligible locally (in a small enough patch). In a similar way, locally, any space-time may approach flat (Minkowski) space-time. However, physics on a plane differs greatly from that on a sphere because of the global topological differences between the surfaces (existence of closed geodesics vis à vis boundary conditions). It seems desirable to investigate further situations where global features or topological structures of space-times might affect the physics of quantized matter fields (elementary particles). It is known that a number of widely differing topological structures can share rather simple local metric properties.⁵⁸ One already knows that in flat space there is a difference in vacuum energy between fields in a box and fields in Minkowski space (Casimir effect).⁵⁹ Thus it may turn out that a global or topological perspective may be most useful in

further study of connections between gravitation and the other interactions. Indeed, "there are more things in Heaven and Earth, Horatio, than are dreamt of in your philosophy."

APPENDIX A: Numerical Methods for Solution of a Boundary Value Problem

A.1 The Method

The solution of $u''(y) = f(u(y), y)$ as a boundary value problem is required. The values of the first derivatives are not sought; a method will be used which does not explicitly involve them.⁶⁰ We consider only the case where Dirichlet boundary conditions are specified. One seeks to solve

$$u''(y) = f(u(y), y) \quad (A.1)$$

subject to

$$u(a) = A$$

$$u(b) = B, \quad b > a$$

and $f_u(u, y)$ is continuous and bounded in the strip $a \leq x \leq b$,
 $-\infty < u < \infty$. (Lipschitz condition)

A finite difference scheme is utilized. One introduces the points $y_n = a + nh$ ($n = 0, 1, \dots, N$) where $h = (b-a)/N$ and N is an appropriate integer. (h is referred to as the mesh size.) One then computes the set of values u_n which approximate the values $u(y_n)$ of the true solution at the points y_n . The differential equation

$$u''(y) = f(u(y), y)$$

is replaced by the difference equation

$$-u_{n-1} + 2u_n - u_{n+1} + h^2 f_n = 0 \quad (A.2)$$

where $f_n \equiv f(u_n, y_n)$

The u_n are determined by requiring the difference equation (A.2) to hold at each interior mesh point. u_0 and u_N are determined by the boundary conditions. In dealing with this system of equations it is convenient to introduce a matrix notation.

$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix} \quad \underline{f}(\underline{u}) = \begin{pmatrix} f(u_1, y_1) \\ f(u_2, y_2) \\ \vdots \\ f(u_{N-1}, y_{N-1}) \end{pmatrix} \quad (A.3)$$

$$\underline{A} = \begin{pmatrix} A \\ 0 \\ 0 \\ \vdots \\ 0 \\ B \end{pmatrix} \quad \underline{J} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 2 & -1 \\ \vdots & \vdots & \vdots & \vdots & -1 & 2 \\ \dots & 0 & -1 & 2 \end{pmatrix}$$

(All elements of $\underline{J}$ not on the main diagonal or on the diagonals adjacent to it are zero.)

One seeks the solution of

$$\underline{J} \underline{u} + h^2 \underline{f}(\underline{u}) = \underline{a} \quad (A.4)$$

If $f(u(y), y)$ is non-linear in $u(y)$, an iterative procedure is necessary. A generalization of the Newton-Raphson method was employed. For a single equation this method linearizes the given equation $f(x)=0$ by replacing $f(x)-f(x^{(0)})$ by its differential at

a point $x^{(0)}$ believed to be close to the actual solution. One then solves the linearized equation:

$$f(x^{(0)}) + f'(x^{(0)}) \Delta x^{(0)} = 0 \quad (A.5)$$

The value $x^{(1)} = x^{(0)} + \Delta x^{(0)}$ is then accepted as a better approximation to the true solution. The entire process is repeated until sufficient convergence is achieved.

To adapt the Newton-Raphson procedure to this system of equations, one defines a "residual vector:"

$$\underline{r}(\underline{u}^{(0)}) = \underline{J} \underline{u}^{(0)} + h^2 \underline{f}(\underline{u}^{(0)}) - \underline{a} \quad (A.6)$$

If $\underline{r}(\underline{u}^{(0)})$ is small, one replaces $\underline{r}(\underline{u}) - \underline{r}(\underline{u}^{(0)})$ by the differential at the point $\underline{u} = \underline{u}^{(0)}$ and solves the resulting linear system for the increment $\Delta \underline{u}^{(0)}$ of the vector $\underline{u}$. The linearized system can be written

$$\underline{r}(\underline{u}^{(0)}) + (\underline{J} + h^2 \underline{F}(\underline{u}^{(0)})) \Delta \underline{u}^{(0)} = 0 \quad (A.7)$$

where

$$\underline{F}(\underline{u}) = \begin{pmatrix} f_{,u}(u_1, y_1) & & 0 \\ & f_{,u}(u_2, y_2) & \\ 0 & & \ddots \end{pmatrix} \quad (A.8)$$

Formally one may write the solution of eq. A.7

$$\Delta \underline{u}^{(0)} = - \underline{A}^{-1}(\underline{u}^{(0)}) \underline{r}(\underline{u}^{(0)}) \quad (A.9)$$

provided that the inverse of the matrix $\underline{A}(\underline{u}) = \underline{J} + h^2 \underline{F}(\underline{u})$ exists

for $u = u^{(0)}$.

Once $\Delta u^{(0)}$ is obtained, the vector $u^{(1)} = u^{(0)} + \Delta u^{(0)}$ is a better approximation to the actual solution of eq. A.4. Actually it is not necessary to calculate the inverse of the matrix $A(u^{(0)})$. One only requires the solution, $\Delta u^{(0)}$, to

$$A(u^{(0)}) \Delta u^{(0)} = -r(u^{(0)}) \quad (A.10)$$

This can be rather easily achieved via an adaptation of a Gaussian algorithm⁶¹ since A is tridiagonal (see eq. A.3 and A.8).

It is noted that the method as outlined requires a "start-up" solution; one must supply $u^{(0)}$. The rate of convergence will depend on the proximity of this initial guess to the actual solution.

A.2 The Particulars

We seek the solution of (see eq. 2.43)

$$u''(y) = (1-y^2)^{-2} [u(y)(-\frac{1}{2} + \frac{15}{4}y^2) - \frac{2}{3}u^3(y)\sqrt{1-y^2}] \quad (A.11)$$

subject to

$$u(0) = \sqrt{3} \quad (A.12)$$

$$u(y_d) = 0$$

The boundary conditions on $u(y)$ are obtained from those on

$$\begin{aligned} g(y) &= (1-y^2)^{-5/4} u(y) \\ g(0) &= \sqrt{3} ; \quad g(y_d) = 0 \end{aligned} \quad (A.13)$$

where $y = \sqrt{1-x}$, $y_d = \sqrt{1-x_d}$ and $10^{-6} \leq x_d \leq 10^{-3}$ (see the discussion following eq. 2.37).

A "start-up" solution satisfying the boundary conditions was used:

$$u^{(0)}(y) = \sqrt{3} [1 - \frac{5}{4}y^2(1-(\frac{y}{y_d})^2) - (\frac{y}{y_d})^4] \quad (A.14)$$

The procedure outlined was coded in Fortran and run on a Sigma-7. (An earlier version utilizing a coarser mesh was implemented in BASIC on a PDP-11.) The directly addressable memory field of the Sigma-7 (64 K of 32 bit words versus 8 K of 16 bit words for the PDP-11) made possible computations involving several thousand mesh points. A calculation involving 2000 points done in double-precision arithmetic required slightly less than 21 K of memory and 1-2 minutes of CPU time. Thus it was not necessary to consider space-saving techniques as over-writing arrays.

The accuracy of a difference scheme is dependent on the mesh size. If all arithmetic is done with infinite precision, the results of the finite difference method approach the actual solution of the differential equation as $h \rightarrow 0$, (h is the step-size). A reduction in step-size increases the number of computations required. If these computations are performed with only a finite precision, a round-off error will be introduced at each step of the calculation. In general then, if the step-size is reduced below a certain value, the decrease in global truncation error will be offset by the increase in the round-off error. The optimal step-size for this

particular calculation was not determined. The rate of convergence of the results as a function of the step-size was studied empirically. In double-precision mode, the representation of a real number (floating point) retained 1 part in 10^{15} . With the "initial guess" solution (see eq. A.14) several iterations of the procedure resulted in a residual vector with typical components on the order of 10^{-7} . The results for $x > x_d$ were largely insensitive to the choice of x_d within the range $10^{-6} \leq x_d \leq 10^{-3}$. Throughout this domain, an x_d was selected so that $h = y_d/N$ was of the form 2^{-m} (m some integer). This enabled h to be represented exactly in the machine and helped retain overall accuracy.

A flowchart for the calculational procedure is given in Fig. A.1.

Fig. A.1

APPENDIX B

B.1 Solution of an Initial-Value Problem

We seek the solution to the system of equations of the general form (see eq. 3.18)

$$\begin{aligned}\frac{dy^{(1)}(r)}{dr} &= g(y^{(1)}, y^{(2)}, r) \\ \frac{dy^{(2)}(r)}{dr} &= f(y^{(1)}, y^{(2)}, r)\end{aligned}\quad (B.1)$$

For the problem to be well-posed, f, g must satisfy a Lipschitz condition in $y^{(1)}$ and $y^{(2)}$. That is, $\frac{\partial f}{\partial y^{(1)}}, \frac{\partial f}{\partial y^{(2)}}, \frac{\partial g}{\partial y^{(1)}}, \frac{\partial g}{\partial y^{(2)}}$ must be continuous and bounded.

Initial conditions

$$y^{(1)}(r_0) = A$$

$$y^{(2)}(r_0) = B$$

are prescribed.

In order to achieve a numerical solution to eq. B.1 for $r_0 \leq r \leq r_f$, one introduces the points $r_n = r_0 + nh$ ($n=0, 1, \dots, N$) where $h = (r_f - r_0)/N$ and N is an appropriate integer. One then computes the set of values $y_n^{(i)}$ ($i=1, 2$) which approximate the values $y^{(i)}(r_n)$ of the actual solution at the points r_n . The classical four-step Runge-Kutta method is used to compute $y_n^{(i)}$. This procedure requires that one first calculate

$$k_0^{(1)} = h g(y_n^{(1)}, y_n^{(2)}, r_n)$$

$$k_0^{(2)} = h f(y_n^{(1)}, y_n^{(2)}, r_n)$$

$$k_1^{(1)} = h g(y_n^{(1)} + \frac{1}{2}k_0^{(1)}, y_n^{(2)} + \frac{1}{2}k_0^{(2)}, r_n + \frac{1}{2}h)$$

$$k_1^{(2)} = h f(y_n^{(1)} + \frac{1}{2}k_0^{(1)}, y_n^{(2)} + \frac{1}{2}k_0^{(2)}, r_n + \frac{1}{2}h)$$

$$k_2^{(1)} = h g(y_n^{(1)} + \frac{1}{4}k_0^{(1)} + \frac{3}{4}k_1^{(1)}, y_n^{(2)} + \frac{1}{4}k_0^{(2)} + \frac{3}{4}k_1^{(2)}, r_n + \frac{3}{4}h)$$

$$k_2^{(2)} = h f(y_n^{(1)} + \frac{1}{4}k_0^{(1)} + \frac{3}{4}k_1^{(1)}, y_n^{(2)} + \frac{1}{4}k_0^{(2)} + \frac{3}{4}k_1^{(2)}, r_n + \frac{3}{4}h)$$

$$k_3^{(1)} = h g(y_n^{(1)} + k_2^{(1)}, y_n^{(2)} + k_2^{(2)}, r_n + h)$$

$$k_3^{(2)} = h f(y_n^{(1)} + k_2^{(1)}, y_n^{(2)} + k_2^{(2)}, r_n + h)$$

Then

$$y_{n+1}^{(i)} = y_n^{(i)} + \frac{1}{6} (k_0^{(i)} + 2k_1^{(i)} + 2k_2^{(i)} + k_3^{(i)}) \quad i=1, 2 \quad (B.2)$$

It can be shown that the truncation error introduced by this difference scheme $\sim O(h^4)$

That is, one can write

$$y(r, h) = y(r) + a_1(r)h^4 + a_2(r)h^5 + \dots \quad (B.3)$$

where

$y(r)$ is the actual solution of the differential equation

and

$y(r, h)$ is the solution achieved via eq. B.2 with

step-size h .

Decreasing the step-size will result in a solution more closely approximating the true solution. If the computations are performed with finite precision, however, round-off error will effectively block this limiting process. One way to reduce the truncation error while introducing a minimum of additional round-off error is the Richardson extrapolation (deferred approach to the limit).

Writing

$$y(r, h) = y(r) + a_1(r)h^4 + a_2(r)h^5 + \dots \quad (B.3)$$

then

$$y(r, h/2) = y(r) + a_1(r)\left(\frac{h}{2}\right)^4 + a_2(r)\left(\frac{h}{2}\right)^5 + \dots \quad (B.4)$$

Then

$$y(r) = y(r, \frac{h}{2}) + \frac{y(r, \frac{h}{2}) - y(r, h)}{15} + O(h^5) \quad (B.5)$$

The accuracy of a fifth-order method is achieved in this way. One can continue the extrapolation. The method is simplest when the successive step sizes form a geometric progression: $h_0, q^{-1}h_0, q^{-2}h_0, \dots$. If $y(r, h)$ is given by eq. B.3 then for $m = 0, 1, 2, \dots$

$$\text{set } A_{m,0} = y(r, q^{-m}h_0)$$

and compute

$$\text{for } k = 1, 2, \dots, m$$

$$A_{m,k} = A_{m,k-1} + \frac{A_{m,k-1} - A_{m-1,k-1}}{q^{3+k} - 1} \quad (B.6)$$

The value $A_{m,k+1}$ is accepted as an estimate of the actual solution $y(r)$ when $|A_{m,k} - A_{m-1,k}|$ is less than some permissible error. The error associated with this result is $\sim O(h^{4+k})$.

B.2 Particulars

We seek the solution of (see eq. 3.14-3.19)

$$\frac{dy^{(1)}(r)}{dr} = y^{(2)}(r) \quad (B.7)$$

$$\frac{dy^{(2)}(r)}{dr} = f(y^{(1)}, y^{(2)}, r)$$

where

$$f(y^{(1)}, y^{(2)}, r) = -y^{(2)}(r)Q(r) - y^{(1)}(r)S(r) - y^{(1)}(r)^3 T(r) \quad (B.8)$$

where

$$Q(r) = \frac{-2m(r) + 4\pi r^3 P(r) + 2r}{r(r-2m(r))} - \frac{m'(r)}{(r-2m(r))}$$

$$S(r) = \frac{4\pi r(P(r) - \frac{\rho(r)}{3})}{(r-2m(r))}$$

$$T(r) = \frac{\lambda \phi_0^2}{6} \frac{r}{(r-2m(r))}$$

ϕ_0 denotes the present-day value for the Higgs VEV as obtained in the cosmological model of Chapter Two. Within the neutron star, the VEV, $\phi(r) = y^{(1)}(r)\phi_0$. The boundary conditions on $y^{(1)}(r)$ and $y^{(2)}(r)$ are:

$$y^{(1)}(r=R_C) = 1.$$

$$y^{(2)}(r=R_C) = 0$$

where R_c denotes the outer radius of the degenerate neutron core (see the discussion following eq. 3.19).

The classical Runge-Kutta method outlined in section B.1 was used. Several steps in the Richardson extrapolation were computed.

The solution of eq. B.7 was complicated by the fact that evaluation of the function $f(y^{(1)}, y^{(2)}, r_n)$ required $P(r_n)$ which was only in tabulated form and $m(r)$ which was available only via numerical integration or tabulation. Since the Runge-Kutta method needed only $f(y^{(1)}, y^{(2)}, r_n)$, $f(y^{(1)}, y^{(2)}, r_n + \frac{h}{2})$ it was a straightforward matter to have available arrays containing $P(r_n)$, $P(r_n + \frac{h}{2})$ (similarly for $m(r)$). This was achieved by integrating the TOV equation (eq. 3.4) (also via Runge-Kutta) with a step-size $\hat{h}$ where $\hat{h} = \frac{1}{2}$ step-size used in eq. B.7. The elements of the array were then selected in order when needed for evaluating $f(y^{(1)}, y^{(2)}, r_n)$. Function evaluations were somewhat lengthy (see the Harrison-Wheeler equation of state, Table 3.1).

A typical calculation involved an initial mesh of 500 points ($h_0 \approx 5 \times 10^{-3}$). All routines were tested on differential equations whose analytic solutions were known.

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