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EXPANDED FORM OF MAGNETIC FIELD WITH MEDIAN PLANE

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The expanded form of a magnetic field with a median plane has been worked out by many people; however, a convenient, comprehensible, and complete reference is not always easily available. In this report, we try to supply an easily applicable formulation.

For the magnetic field in vacuum, we can express it in terms of a scalar potential ϕ by $\vec{B} = \vec{\nabla} \phi$. The scalar potential will be expanded from a plane reference curve lying in the median plane $y = 0$. The coordinates are defined in Fig. 1 where it is shown that x = outward normal

Fig. 1

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distance in median plane away from the reference curve. y = perpendicular distance from the median plane, z = distance along the reference curve, and $k = k(z)$ is the curvature of the reference curve. These coordinates

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(x, y, and z) form a right-handed orthogonal system.

The existence of the median plane requires that ϕ be an odd function of y. The most general expanded form of ϕ is, therefore

$$\begin{aligned} \phi(x, y, z) &= \left(A_{10} + A_{11}x + A_{12}\frac{x^2}{2!} + A_{13}\frac{x^3}{3!} + \dots \right) y \\ &\quad + \left(A_{30} + A_{31}x + A_{32}\frac{x^2}{2!} + \dots \right) \frac{y^3}{3!} \\ &\quad + \dots \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1, n} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!} \end{aligned} \quad (1)$$

where the coefficients $A_{2m+1, n}$ are functions of z.

In this coordinate system, the differential line element ds is given by

$$ds^2 = dx^2 + dy^2 + (1+kx)^2 dz^2 \quad (2)$$

The metric tensors are, therefore

$$\left\{ \begin{aligned} g_{ij} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1+kx)^2 \end{pmatrix} \\ \sqrt{g} &= 1+kx \end{aligned} \right. \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1+kx)^{-2} \end{pmatrix} \quad (3)$$

The Laplace equation has the form

$$\nabla^2 \phi = \frac{1}{1+kx} \frac{\partial}{\partial x} \left[(1+kx) \frac{\partial \phi}{\partial x} \right] + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{1+kx} \frac{\partial}{\partial z} \left[\frac{1}{1+kx} \frac{\partial \phi}{\partial z} \right] = 0 \quad (4)$$

substitution of (1) in (4) gives the recursion formula

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$$\begin{aligned}
-A_{2m+3,n} = & A'_{2m+1,n} + nkA'_{2m+1,n-1} - nk'A'_{2m+1,n-1} + A_{2m+1,n+2} \\
& + (3n+1)kA_{2m+1,n+1} + n(3n-1)k^2A_{2m+1,n} + n(n-1)^2k^3A_{2m+1,n-1} \\
& + 3nkA_{2m+3,n-1} + 3n(n-1)k^2A_{2m+3,n-2} + n(n-1)(n-2)k^3A_{2m+3,n-3} \quad (5)
\end{aligned}$$

where prime means $\frac{d}{dz}$, and where it is understood that all coefficients A with negative subscripts are zero. This recursion formula expresses all the coefficients in terms of

$$A_{l,n} = \left(\frac{\partial^n B_y}{\partial x^n} \right)_{\substack{x=0 \\ y=0}} = \text{functions of } z$$

Since ϕ is an odd function of y , on the median plane we have $B_x = B_z = 0$. The normal (in x direction) derivatives of B_y on the reference curve defines B_y over the entire median plane, hence the magnetic field $\vec{B}$ over the whole space. The components of the field are expressed in terms of ϕ explicitly

$$\begin{cases}
B_x = \frac{\partial \phi}{\partial x} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1,n+1} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!} \\
B_y = \frac{\partial \phi}{\partial y} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1,n} \frac{x^n}{n!} \frac{y^{2m}}{(2m)!} \\
B_z = \frac{1}{1+kx} \frac{\partial \phi}{\partial z} = \frac{1}{1+kx} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A'_{2m+1,n} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!}
\end{cases} \quad (6)$$

where we have not given B_z in a pure power expansion form. This form can be obtained straightforwardly by expanding $\frac{1}{1+kx}$ in a power series of kx and multiplying out the two series; however, the result is so messy that there does not seem to be any point in doing so.

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The coefficients up to the 6th degree terms in x and y are given explicitly below from Eq. (5)

$$\left\{ \begin{array}{l} A_{30} = -A''_{10} - A_{12} - kA_{11} \\ A_{31} = -A''_{11} + 2kA''_{10} + k'A'_{10} - A_{13} - kA_{12} + k^2A_{11} \\ A_{32} = -A''_{12} + 4kA''_{11} + 2k'A'_{11} - 6k^2A''_{10} - 6kk'A'_{10} - A_{14} - kA_{13} + 2k^2A_{12} - 2k^3A_{11} \\ A_{33} = -A''_{13} + 6kA''_{12} + 3k'A'_{12} - 18k^2A''_{11} - 18kk'A'_{11} + 24k^3A''_{10} + 36k^2k'A'_{10} \\ \quad - A_{15} - kA_{14} + 3k^2A_{13} - 6k^3A_{12} + 6k^4A_{11} \end{array} \right. \quad (7)$$

$$\left\{ \begin{array}{l} A_{50} = A''''_{10} + 2A''_{12} - 2kA''_{11} + k''A_{11} + 4k^2A''_{10} + 5kk'A'_{10} + A_{14} + 2kA_{13} - k^2A_{12} + k^3A_{11} \\ A_{51} = A''''_{11} - 4kA''''_{10} - 6k'A''_{10} - 4k''A''_{10} - k'''A'_{10} + 2A''_{13} - 6kA''_{12} - 2k'A'_{12} \\ \quad + k''A_{12} + 10k^2A''_{11} + 7kk'A'_{11} - 4kk''A_{11} - 3k'^2A_{11} - 16k^3A''_{10} - 29k^2k'A'_{10} \\ \quad + A_{15} + 2kA_{14} - 3k^2A_{13} + 3k^3A_{12} - 3k^4A_{11} \end{array} \right. \quad (8)$$

In the special case when the field has cylindrical symmetry, we can choose a circle with radius $\frac{1}{k}$ for the reference curve. The coefficients $A_{2m+1,n}$ in Eq. (1) and the curvature k of the reference curve are then all independent of z . Eqs. (7) and (8) are greatly simplified by putting all terms with primed quantities equal to zero.

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