THE CHARGE RADIUS OF THE PION

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MASTER

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The charge radius of the pion

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1. INTRODUCTION

A recent experiment performed at Serpukhov, U.S.S.R. by a Russian-American collaboration (1) has for the first time directly measured the charge radius of the pion. They find

$$\langle r_{\pi}^2 \rangle = 0.61 \pm 0.15 F^2,$$

a considerably larger result than would be predicted on the basis of a naive vector dominance model with a form factor

$$f_{\text{VD}}^{e\pi}(q^2) = (q^2 + m_{\rho}^2)^{-1} m_{\rho}^2$$

$$\langle r_{\pi}^2 \rangle = -6 f'^{e\pi}(0)$$

$$\langle r_{\pi}^2 \rangle_{\text{VD}} = 6 m_{\rho}^{-2} = 0.39 F^2.$$

The π^+ electroproduction data of C.M. Brown, et al. (2) is consistent with the smaller pion charge radius $\langle r_{\pi}^2 \rangle_{\text{VD}}$ but the data is scattered for small values of q^2 as may be seen in Fig. 1.

G.J. Gounaris and J.J. Sakurai (3) have also calculated the pion form factor but with the correct analytic properties as obtained from the p-wave pion-pion scattering amplitude. If their form factor is used to evaluate the pion charge radius one obtains

$$\langle r_{\pi}^2 \rangle = 6 m_{\rho}^{-2} \left[1 - \frac{f_1}{\pi m_{\rho}} \left(\frac{8}{3} + 2 \ln \frac{m_{\rho}}{m_{\pi}} \right) \right],$$

a result smaller than $\langle r_{\pi}^2 \rangle_{\text{VD}}$. Gounaris and Sakurai obtain values for the

Fig. 1. Space-like momentum pion form factor data.

p-wave pion-pion phase shift, given by

$$\delta_{\pi\pi}(q^2) = \arctan \frac{\text{Im} f^{\pi\pi}(q^2)}{\text{Re} f^{\pi\pi}(q^2)}, \quad (1.1)$$

which are consistent with the existing phase shift data (4). It therefore becomes apparent that a discrepancy exists between the large experimental values of the pion radius and the existing p-wave pion-pion phase shift data.

This problem has been investigated by D.N. Levin and S. Okubo (5), who have obtained upper and lower bounds on the pion charge radius. The analytic properties of the form factor are used to obtain, from an upper limit on $|f^{\pi\pi}|$ in the time-like momentum region, both an upper and lower limit on $|f^{\pi\pi}|$ in the space-like momentum region. By fixing some upper bound on the time-like form factor they arrive at upper and lower bounds on $\langle r_\pi^2 \rangle$.

It should be emphasized that the work of Levin and Okubo does not make an attempt to fit the time-like form factor data. Their results point to the fact that a large enhancement of $\delta_{\pi\pi}(q^2)$ near the $4m_\pi$ threshold is required in order to obtain a large value for the pion radius. The various forms they assumed for the phase shift are shown in Fig. 2. Forms C and D are required in order to obtain a large radius. Note that $\delta_{\pi\pi}(16 m_\pi^2)$ is normalized at 12.5° in all of these forms and would give nonsensical results for q^2 less than $16 m_\pi^2$.

Recently Hammer and Weber (6) have expressed the pion form factor in a way which emphasizes the role of the ρ -propagator. They find

$$f^{\pi\pi}(q^2) = 1 - m_\rho^2 \frac{g^{\rho\pi}}{2\gamma_\rho} G_\rho(q^2), \quad (1.2)$$

Fig. 2. Forms of $\delta_{\pi\pi}(t)$ used by Levin and Okubo (5).

where

$$G_p(q^2) = K_p'(0) + \frac{1}{q^2} [K_p(0) - K_p(q^2)] \quad (1.3)$$

$$K_p(q^2) = \frac{m_{op}^2}{2iq} [D^{-1}(iq, m_p, m_p^*) - D^{-1}(-iq, m_p, m_p^*)] \quad (1.4)$$

for space-like momentum transfer q (7). Here

$$D(q, m_p, m_p^*) = q - m_p - R(q) \quad (1.5)$$

where

$$R(q) = D(q) - iI(q) \quad (1.6)$$

is the level shift. The quantity $g^{\rho\pi}$ is the strength of the $\rho\pi\pi$ vertex and γ_ρ is the γ - ρ coupling constant. The complex mass of the ρ -meson is given by

$$m_\rho = m_{op} - i\frac{\Gamma_\rho}{2}$$

This calculation differs from the others in that the formal expression for the ρ -propagator is exact.

A calculation of $\langle r_\pi^2 \rangle$ using the form of Hammer, et al. gives

$$\begin{aligned} \langle r_\pi^2 \rangle &= -6 m_{op}^4 \frac{g^{\rho\pi}}{2\gamma_\rho} \frac{[D^{-1}(0)]^{(5)}}{5!} \\ &= 0.59 F^2 \end{aligned} \quad (1.7)$$

where

$$[D^{-1}(q)]^{(5)} = \frac{d^5}{dq^5} D^{-1}(q, m_\rho, m_\rho^*) \Big|_{q=0}$$

The values $m_{op} = 0.7669$ and $\Gamma_\rho = 0.1550$ were used to obtain this value of $\langle r_\pi^2 \rangle$.

An investigation (Section II) shows that the high order derivative of $D^{-1}(q)$ is responsible for the large enhancement to the pion radius.

While this form for the electromagnetic form factor produces excellent fits (8) to the time-like and space-like pion form factor data, it also gives an extremely large value of $\delta_{\pi\pi}(16 m_\pi^2)$ (25° for the values of the parameters used to obtain Eq.(1.7)). However the general shape of $\delta_{\pi\pi}(q^2)$ predicted by this form factor is in agreement with the p-wave pion-pion phase shift data (4). It differs from the data by a multiplicative constant indicating a bias between the form factor data and the phase shift data. On the other hand, the forms of $\delta_{\pi\pi}(q^2)$ used by Levin and Okubo to obtain large values of $\langle r_\pi^2 \rangle$ do not agree with the shape or the values of the phase shift curve.

The form factor of Hammer, et al. is a good starting point to continue an investigation of the discrepancy between phase shifts on $\langle r_\pi^2 \rangle$ because of the fact that it not only gives an excellent fit to the form factor data but also calculates a curve for $\delta_{\pi\pi}(q^2)$ which has the same shape as the data. The appropriate normalization to the phase shift data can be made by modifying the imaginary part of the level shift with an overall factor

$$f(q,a) = \frac{q^2}{m_\rho^2} \left[(m_\rho^2 + a^2)(q^2 + a_\rho^2)^{-1} \right]. \quad (1.8)$$

This parameterization, normalized so $f(q,0) = 1$, is a phenomenological way of introducing structure into the $\rho\pi\pi$ vertex.

The parameter a_ρ is treated as an adjustable parameter whose value is to be determined by a least squares fit to the time-like pion form factor data and simultaneously to the phase shift data. This parameterization will introduce a new degree of freedom that hopefully will allow the phase

shift data to be fit by normalizing it at threshold. Then comparison between a calculation of $\langle r_\pi^2 \rangle$ using this value of a_ρ as well as $a_\rho = 0$ will give a measure of the increase in the radius values due to the threshold enhancement of the phase shift.

The purpose of the present work is to develop the computer programs used in the analysis of this problem and to discuss the computational techniques. Final interpretation of the results will be given in a forthcoming paper.

Section II gives an introduction to the equations to be used and describes another method for determining the experimental bias among different data sets. Section III gives the results of an attempt to achieve the stated objective using only the ρ and ω -mesons in the form factor - phase shift fits. The effect of the ρ' -meson is included in the work of Section IV. Here also the problem of the ρ' -branching ratio is discussed. Section V gives a brief discussion of the conclusions to be drawn. Appendix A lists some of the relevant equations used, Appendix B describes and lists the computer program used in this analysis, and Appendix C discusses some future work.

II. THEORY

To use Eqs. (1.2-6) to derive the electromagnetic form factor one must assume a model for the imaginary part of the level shift. Take

$$I(q^2) = \frac{\Gamma_p}{2m_p} \left(1 - \frac{\lambda_p^2}{m_p^2}\right)^{-3/2} q \left(1 - \frac{\lambda_p^2}{q^2}\right)^{3/2} f(q), \quad q \gg \lambda_p$$

$$= 0, \quad q \leq \lambda_p \quad (2.1)$$

where $f(q)$ is defined by Eq. (1.8). Then, following Ref. (8),

$$\frac{\pi D(q, m_p, m_p^*)}{\alpha \left(1 + \frac{q^2}{m_p^2}\right)} = 2 \operatorname{Re} \left\{ \left[\pi i + \ln \frac{1+z_{m_p^*}}{1-z_{m_p^*}} \right] \frac{(m_p^{*2} - \lambda_p^2)^{3/2} |q - m_p|}{(m_p^{*2} + a^2)(m_p^* - m_p)} \right\}$$

$$+ \left[\pi i - \ln \frac{1+z_q}{1-z_q} \right] \frac{(q^2 - \lambda_p^2)^{3/2}}{q^2 + a^2}$$

$$+ \frac{(\lambda_p^2 + a^2)^{3/2} (q - m_p)^2}{(q^2 + a^2)(m_p^2 + a^2)(m_p^{*2} + a^2)} \left\{ -\frac{\pi}{2} \left[(|m_p|^2 - a^2) + q(m_p + m_p^*) \right] \right.$$

$$\left. - \frac{1}{a} \left[(|m_p|^2 - a^2) q - a^2 (m_p + m_p^*) \right] \ln \frac{a + \sqrt{a^2 + \lambda_p^2}}{\lambda_p} \right\},$$

$$\frac{\pi D(-q, m_p, m_p^*)}{\alpha \left(1 + \frac{q^2}{m_p^2}\right)} = 2 \operatorname{Re} \left\{ \left[\pi i + \ln \frac{1+z_{m_p^*}}{1-z_{m_p^*}} \right] \frac{(m_p^{*2} - \lambda_p^2)^{3/2} |-q - m_p|}{(m_p^{*2} + a^2)(m_p^* - m_p)} \right\}$$

$$+ \left[\ln \frac{1+z_q}{1-z_q} \right] \frac{(q^2 - \lambda_p^2)^{3/2}}{(q^2 + a^2)}$$

$$+ \frac{(\lambda_p^2 + a^2)^{3/2} (-q - m_p)^2}{(q^2 + a^2)(m_p^2 + a^2)(m_p^{*2} + a^2)} \left\{ -\frac{\pi}{2} \left[(|m_p|^2 - a^2) - q(m_p + m_p^*) \right] \right.$$

$$\left. - \frac{1}{a} \left[-(|m_p|^2 - a^2) q - a^2 (m_p + m_p^*) \right] \ln \frac{a + \sqrt{a^2 + \lambda_p^2}}{\lambda_p} \right\}$$

Here

$$\alpha = \frac{\Gamma_p}{2m_p} \left(1 - \frac{\lambda_p^2}{m_p^2}\right)^{-3/2}$$

Referring to Eq.(1.7) it is now possible to write an explicit formula for the pion radius,

$$\langle r_{\pi}^2 \rangle_{\text{HWZ}} = \langle r_{\pi}^2 \rangle_{\text{VD}} C \left\{ A^5 + 4A^3B + 3AB^2 + \left[A^2 + \frac{2B}{3} \right] m_{\rho}^2 \frac{8D^{(3)}(0)}{D^{(1)}(0)} + \frac{ABm_{\rho}^2 D^{(1)}(0)}{6D^{(1)}(0)} + \frac{Bm_{\rho}^2 D^{(3)}(0)}{60D^{(1)}(0)} \right\}. \quad (2.4)$$

A, B, and C, defined in Appendix A, are positive real numbers because of the dispersion definition of $D(q, m_{\rho}, m_{\rho}^*)$ (6). The remaining terms are also positive. It is apparent, then, that $\langle r_{\pi}^2 \rangle_{\text{HWZ}}$ must be larger than $\langle r_{\pi}^2 \rangle_{\text{VD}}$ because of the bracketed terms in Eq.(2.4).

A quantity which will be of interest throughout this work is the so-called sum-rule. If one assumes that no hard core exists in the charge distribution, i.e. that $f^{\pi}(q^2) = 0$ for infinite q^2 , then there follows the result that

$$1 = m_{\rho}^2 \int_0^{\infty} \frac{K'_{\rho}(0)}{2\lambda} d\lambda. \quad (2.5)$$

The right-hand side of Eq.(2.5) will frequently be referred to as SR for sum-rule.

A final quantity requiring definition is the pion scattering length. In the usual manner, the scattering length is defined as

$$a_{\pi} m_{\pi}^3 = \lim_{q \rightarrow \lambda_{\rho}} q \lambda_{\rho}^2 (q^2 - \lambda_{\rho}^2)^{-3/2} \frac{\text{Im } f^{\pi}(q^2)}{\text{Re } f^{\pi}(q^2)}.$$

Some algebra shows that for the propagator under consideration

$$a_{\pi} m_{\pi}^3 = \frac{-\alpha \lambda_{\rho} (1 + \frac{\alpha^2}{m_{\rho}^2})}{(D(\lambda_{\rho}))^2 (1 + \frac{\alpha^2}{\lambda_{\rho}^2})} \left[D^{-1}(\lambda_{\rho}) - D^{-1}(-\lambda_{\rho}) - \frac{2\lambda_{\rho}}{m_{\rho}^2} K_{\rho}(0) \right]^{-1} \quad (2.6)$$

Here use is made of the fact that $\text{Im } D(\lambda_\rho) = 0$. The experimental value of this quantity is fairly well determined, and is usually taken to be 0.11 ± 0.01 F. (9).

The time-like pion form factor data used in this work is taken from Refs. (10-13). Table 1 lists this data. Table 2 contains the p-wave pion-pion phase shift data of Ref. (4). A simultaneous fit to these two sets of data is to be made.

Since the time-like form factor data is obtained from four separate sources, it is expected that there will be some experimental bias among the different sets of data. To investigate this possibility a χ^2 of the form

$$\chi^2 = \sum_{n,q^2} \left[\frac{\lambda_n \text{Theory}_n(q^2) - \text{Exp.}_n(q^2)}{\sigma_n(q^2)} \right]^2 + \sum_n \left(\frac{\lambda_n - 1}{\Delta_n} \right)^2 \quad (2.7)$$

is used. The sum on n is over all data sets used. The quantity λ_n is a bias parameter for the n th data set and Δ_n is the experimental normalization of the n th data set. In this study λ_n is treated as an adjustable parameter in the fit and Δ_n is given the same fixed value for each data set, so that each set of data has the same probability for being in error.

This device may also be used to determine the bias between the phase shift data and the form factor data. In this case, for a given value of a_ρ , the overall bias between the phase shift data and the form factor data may be determined, but it is found that Δ_n for the phase shift must be adjusted in such a way that the minimization procedure will not stop when the χ^2 contribution from the phase shift matches that from the form factor. Essentially this points out that a larger probability for

Table 1. Time-like pion form factor data

$q^2 \text{ (GeV/c)}^2$	$ f_Y^\pi(q^2) ^2$	Error
-0.087	1.3	± 0.32
-0.112	1.69	0.42
-0.497	22.3	2.5
-0.575	39.0	3.5
-0.593	40.5	3.5
-0.604	44.7	3.6
-0.613	35.6	2.5
-0.624	24.8	1.7
-0.637	23.9	3.0
-0.656	22.3	2.5
-0.697	15.0	2.0
-0.98	2.9	0.5
-1.04	2.03	0.07
-1.3924	1.5	0.8
-1.44	1.28	0.34
-1.69	0.72	0.24
-1.96	0.67	0.19
-2.56	0.13	0.07
-3.61	0.135	0.06
-3.76	0.068	0.07
-4.41	0.034	0.023
-9.0	0.026	0.027

Table 2. P-wave $\pi\pi$ phase shift data

$q^2 \text{ (GeV/c)}^2$	$\delta_{\pi\pi}(q^2) \text{ (degrees)}$	Error
-0.325	12.5	± 0.5
-0.303	11.5	0.5
-0.281	10.0	0.6
-0.260	9.3	0.5
-0.240	7.9	0.4
-0.221	6.2	0.6
-0.203	4.8	0.9

the phase shift data to be in error exists.

The first term in Eq.(2.6) with $\lambda_n = 1$ serves to define the term χ^2 .

In performing a fit to the data the computer program adjusts the parameters in Theory (q^2) in such a way as to minimize the χ^2 .

III. FITS TO THE DATA WITH THE ρ AND ω -MESONS

Preliminary work indicates that the ρ -meson alone is not sufficient to fit the data in Table I. The difficulty is due to an enhancement in the data near the ρ -peak at $q^2 \simeq -0.60 \text{ (GeV/C)}^2$ and again at $q^2 \simeq -2.0 \text{ (GeV/C)}^2$. The first effect is the result of an $\omega \rightarrow 2\pi$ decay mode. This decay is a violation of G-parity and therefore produces a small effect, but which is nonetheless essential to fit the data well.

The second enhancement is generally attributed to the presence of a ρ' -meson. This effect is nebulous at best because of the enormous errors in the data in this region.

To accommodate these difficulties, a form factor which utilizes ω and ρ' -mesons must be used. This is done by writing

$$f^{\pi\pi} = f_{\rho}^{\pi\pi} + f_{\omega}^{\pi\pi} + f_{\rho'}^{\pi\pi}.$$

Assuming ρ -dominance this becomes

$$f^{\pi\pi}(q^2) = 1 - \frac{m_{\rho}^2 g^{\rho\pi}}{2\beta} G_{\rho}(q^2) - \frac{m_{\omega}^2 g^{\omega\pi}}{2\beta_{\omega}} G_{\omega}(q^2) \\ - \frac{m_{\rho'}^2 g^{\rho'\pi}}{2\beta_{\rho'}} G_{\rho'}(q^2).$$

It is not possible to calculate the $\omega\pi\pi$ vertex so the strength of that vertex, $g^{\omega\pi}$, is introduced in a phenomenological way by writing

$$g^{\omega\pi} = g_1^{\omega\pi} + i g_2^{\omega\pi}$$

and allowing fits to the data to determine values for $g_{1,2}^{\omega\pi}$. Since the ω -mass and width are quite well known they are kept at fixed values and

are not adjusted in the fit. In calculating the ω and ρ' contributions to the form factor it is necessary, for computer programming reasons, to introduce the parameter a in the level shift of these mesons also. It is not possible to set $a = 0$ in the computer program so it is given the value 0.07, which is the smallest practical value. The effect of a_ω and $a_{\rho'}$ is then negligible.

As a preliminary test of the equations and computer programs the ρ' effect is ignored by fitting the form factor to only the first 13 data points in Table 1. Thus, the first series of fits will take into account only the ρ and ω contributions. Also, attention will be given only to the first two phase shift points of Table 2, as normalization at threshold is of primary interest here.

The equation for the form factor contains the ρ -mass, width, and coupling strength. These three quantities, together with $g^{\omega\pi}$ and a_ρ are taken as adjustable parameters in the fits to the form factor and phase shift data. The χ^2 -space (Appendix B) for this fit is quite complex with the result that several minima in the χ^2 seem to exist. To fully comprehend the effect of the various parameters on the fit, it was found to be beneficial to minimize the χ^2 for a matrix of the two parameters most sensitive to the phase shift. These are a_ρ and Γ_ρ . Table 3 shows this matrix for a reasonable range of both parameters.

Attempts to produce good fits to the form factor and phase shift simultaneously failed. This is apparent from Table 3 where the reported value of the χ^2 is for the 13 form factor data points. The indicated fits are to the form factor data only, with the phase shift value calcu-

Table 3. Fit to the form factor data with ρ and ω mesons.

Γ_ρ a_ρ	.1200	.1300	.1400	.1500
0.10	$\chi^2 = 46.14$	17.32	5.99	6.70
	$m_\rho = 0.7662$	0.7675	0.7676	0.7669
	$g_\rho = 5.6674$	5.7564	5.8451	5.9349
	$\text{Reg}_\omega = -0.2860$	-0.1960	-0.1037	-0.0185
	$\text{Im}g_\omega = 0.1988$	0.2584	0.2940	0.3183
	$\delta_{\pi\pi} = 21.48$	23.04	24.65	26.24
0.40	$\chi^2 = 55.19$	21.61	6.01	3.09
	$m_\rho = 0.7661$	0.7675	0.7677	0.7670
	$g_\rho = 5.8021$	5.9144	6.0265	6.1434
	$\text{Reg}_\omega = -0.2985$	-0.2090	-0.1160	-0.0292
	$\text{Im}g_\omega = 0.2010$	0.2597	0.2932	0.3134
	$\delta_{\pi\pi} = 18.31$	19.64	20.96	22.31
0.70	$\chi^2 = 57.98$	22.93	6.04	2.09
	$m_\rho = 0.7649$	0.7670	0.7673	0.7666
	$g_\rho = 5.9159$	6.0461	6.1768	6.3137
	$\text{Reg}_\omega = -0.2947$	-0.2045	-0.1103	-0.0217
	$\text{Im}g_\omega = 0.2046$	0.2628	0.2947	0.3124
	$\delta_{\pi\pi} = 15.88$	17.01	18.18	19.35
1.00	$\chi^2 = 56.83$	22.11	5.52	1.85
	$m_\rho = 0.7650$	0.7666	0.7669	0.7663
	$g_\rho = 5.9802$	6.1206	6.2613	6.4097
	$\text{Reg}_\omega = -0.2872$	-0.1961	-0.1011	-0.0118
	$\text{Im}g_\omega = 0.2072$	0.2645	0.2957	0.3123
	$\delta_{\pi\pi} = 14.77$	15.86	16.95	18.08

$$m_\omega = 0.7827 \text{ and } \Gamma_\omega = 0.10.$$

Table 4. Fit to the form factor data with the ρ and ω mesons

Γ_ρ a_ρ	.1200	.1300	.1400	.1500
0.10	$r_\pi^2 = 0.5549$	0.5692	0.5838	0.5982
	SR = 1.0716	1.0837	1.0944	1.1044
0.40	$r_\pi^2 = 0.5054$	0.5158	0.5257	0.5363
	SR = 1.0396	1.0502	1.0599	1.0691
0.70	$r_\pi^2 = 0.4749$	0.4818	0.4897	0.4980
	SR = 1.0188	1.0287	1.0368	1.0445
1.00	$r_\pi^2 = 0.4616$	0.4681	0.4752	0.4830
	SR = 1.0121	1.0212	1.0286	1.0356

$m_\omega = 0.7827$ and $\Gamma_\omega = 0.10$.

lated concurrently. Fits to both sets of data simultaneously produced χ^2 s and phase shifts comparable to those in the central region of the matrix. Apparently the effect of a_ρ alone is not sufficient to reduce the phase shift from 26° to 12.5° .

The extent to which the form factor obeys the sum-rule is indicated with each run. The quantity called SR is to be compared with the left side of Eq.(2.5).

To determine the biasing among the various data sets the physical parameters were maintained at fixed values while the λ_n s of Eq.(2.6) were treated as adjustable parameters. This was done for the parameters corresponding to the fits in the 1-4 and 4-1 positions in the matrix of Table 3. The results are listed in Table 5.

Table 5. Results of data biasing

	χ^2	λ_n	Δ_n
$a_p = 0.10$	3.30	$\lambda_{BR} = 0.8968$	$\Delta_{BR} = 0.1$
$\Gamma_p = 0.1500$		$\lambda_{OR} = 1.0191$	$\Delta_{OR} = 0.1$
		$\lambda_{NO} = 0.9882$	$\Delta_{NO} = 0.1$
		$\lambda_{PS} = 0.4853$	$\Delta_{PS} = 0.3$
$a_p = 1.00$	19.84	$\lambda_{BR} = 0.9739$	$\Delta_{BR} = 0.1$
$\Gamma_p = 0.1200$		$\lambda_{OR} = 0.8881$	$\Delta_{OR} = 0.1$
		$\lambda_{NO} = 1.1404$	$\Delta_{NO} = 0.1$
		$\lambda_{PS} = 0.8699$	$\Delta_{PS} = 0.3$

BR $\equiv$ Berezhnev (Ref (10))NO $\equiv$ Novosibirsk (Ref (12))OR $\equiv$ Orsay (Ref (11))PS $\equiv$ Phase Shift

IV. FITS TO THE DATA WITH THE ρ , ω , and ρ' -MESONS

The form factor data listed in Table 1 exhibits a small enhancement near $q^2 = -2.00 \text{ (GeV/c)}^2$. Even though this enhancement is generally attributed to a ρ' -meson, such a meson need not be included from a goodness of fit point of view. Attempts to fit all but the first two data points of Table 1 give a χ^2 value of 13.5. No enhancement is produced at $q^2 \approx -2.00 \text{ (GeV/c)}^2$, as may be seen in Fig. 3, but the error bars on the data are so large that the χ^2 is relatively unaffected. Table 6 lists the parameters, phase shift and radius values for this fit.

However, the data easily accommodates a ρ' -meson. Previous work (8) suggests that this meson should have a mass of 1.51 (GeV/c)^2 and a width of 0.25 (GeV/c)^2 . Ref. (8) also assumes a 4-pion decay as the dominant decay mode for the ρ' . The $\rho'\pi\pi\rho$ vertex strength is found, from the fits of Ref. (8), to have a negative value. It would therefore appear that the inclusion of the ρ' in the form factor might have the effect of reducing the phase shift, especially if both a 2-pion and a 4-pion decay mode are allowed for. This follows from reasoning that the imaginary part of $f^{e\pi}$ will undergo a subtraction when one takes

$$f^{e\pi} = f_{\rho}^{e\pi} + f_{\omega}^{e\pi} + f_{\rho'}^{e\pi}$$

This will reduce the phase shift,

$$\delta_{\pi\pi}(q^2) = \arctan \frac{\text{Im} f^{e\pi}(q^2)}{\text{Re} f^{e\pi}(q^2)}$$

provided the change due to $\text{Im} f_{\rho'}^{e\pi}$ is greater than that due to $\text{Re} f_{\rho'}^{e\pi}$.

Fig. 3. Fit to the pion form factor data without the ρ' .

Table 6. Fit to the form factor data without the ρ' meson.

	$a_\rho = 0.10$	$a_\rho = 1.00$
m_ρ	0.7663	0.7627
Γ_ρ	0.1463	0.1513
g_ρ	5.9062	6.5182
$\text{Re}g_\omega$	-0.0139	0.0251
$\text{Im}g_\omega$	0.2947	0.2872
$\delta_{\pi\pi}$	25.7	18.5
$\langle r_\pi^2 \rangle$	0.5936	0.4935
SR	1.1013	1.0483

In order to learn which parameters for the ρ' -meson are best suited to the data, fits are made which allow for a 100% 2-pion, a 100% 4-pion, and a 50% decay mode for each of the two modes. This is accomplished by incorporating a double cut in the ρ' -form factor,

$$D_{\rho'}(q, m_{\rho'}, m_{\rho'}^*) = \alpha_1 D_1(q, m_{\rho'}, m_{\rho'}^*) + \alpha_2 D_2(q, m_{\rho'}, m_{\rho'}^*),$$

where

$$\alpha_i = \frac{\Gamma_i}{2m_{\rho'}} \left(1 - \frac{\lambda_i^2}{m_{\rho'}^2}\right)^{-3/2}$$

$$\lambda_1 = 2m_{\pi}, \lambda_2 = 4m_{\pi}.$$

This can be rewritten as

$$D_{\rho'}(q, m_{\rho'}, m_{\rho'}^*) = \alpha \left[R_1 D_1(q, m_{\rho'}, m_{\rho'}^*) + R_2 D_2(q, m_{\rho'}, m_{\rho'}^*) \right]$$

where

$$R_i = \alpha_i \alpha^{-1}$$

Now since

$$\Gamma_{\rho'} = \Gamma_1 + \Gamma_2$$

it follows that

$$\frac{\Gamma_{\rho'}}{2m_{\rho'}} = \alpha_1 \left(1 - \frac{\lambda_1^2}{m_{\rho'}^2}\right)^{3/2} + \alpha_2 \left(1 - \frac{\lambda_2^2}{m_{\rho'}^2}\right)^{3/2}$$

or

$$\frac{\Gamma_{\rho'}}{2m_{\rho'}} = \alpha \left[R_1 \left(1 - \frac{\lambda_1^2}{m_{\rho'}^2}\right)^{3/2} + R_2 \left(1 - \frac{\lambda_2^2}{m_{\rho'}^2}\right)^{3/2} \right]$$

which implies that

$$\alpha = \frac{\Gamma_{\rho'}}{2m_{\rho'}} \left[R_1 \left(1 - \frac{\lambda_1^2}{m_{\rho'}^2}\right)^{3/2} + R_2 \left(1 - \frac{\lambda_2^2}{m_{\rho'}^2}\right)^{3/2} \right]^{-1}.$$

Thus, choosing values for R_1 and R_2 between 0 and 1 allows one to fix the ρ' branching ratio. Note that the actual branching ratio B_i is given by

$$B_i = \frac{P_i}{\Gamma_p} = R_i \left(1 - \frac{\lambda_i^2}{m_{\rho'}^2}\right)^{3/2} \left[R_1 \left(1 - \frac{\lambda_1^2}{m_{\rho'}^2}\right)^{3/2} + R_2 \left(1 - \frac{\lambda_2^2}{m_{\rho'}^2}\right)^{3/2} \right]^{-1}$$

With the inclusion of the ρ' -meson the formulas for the sum-rule and the charge radius must be modified. The sum-rule becomes

$$1 = m_{\rho'}^2 \frac{g^{\rho\pi}}{2\lambda_p} K_p'(0) + m_{\rho'}^2 \frac{g^{\rho'\pi}}{2\lambda_{p'}} K_{p'}'(0)$$

and the radius is simply

$$\langle r_{\pi}^2 \rangle = \langle r_{\pi}^2 \rangle_p + \langle r_{\pi}^2 \rangle_{p'}$$

The scattering length undergoes a similar modification,

$$SL = \left[\frac{m_{\rho'}^4 g_p'' (1 + \frac{\alpha^2}{m_{\rho'}^2})}{2\lambda_1^2 (1 + \frac{\alpha^2}{\lambda_1^2}) D_p^2(\lambda_1)} + \frac{m_{\rho'}^4 g_{p'}'' R_1}{2\lambda_1^2 D_{p'}^2(\lambda_1)} \right] \times \left[1 - m_{\rho'}^2 g_p'' \operatorname{Re} G_p(\lambda_1) - m_{\rho'}^2 g_{p'}'' \operatorname{Re} G_{p'}(\lambda_1) \right]^{-1}$$

The ω contribution is found to be negligible in all cases.

With the inclusion of the ρ' parameters and the branching ratios, the number of parameters in the fit again becomes so large that the χ^2 -space is quite complex. The number of parameters to which the phase shift is sensitive becomes so large that a matrix display of the fits is also out of question. In this section, therefore, all values of a_p will not be considered. It is apparent from the work of Section III that a large

value of a_ρ most nearly produces the desired result of fitting the form factor data as well as the phase shift. Figs. 4-7 show the a_ρ -dependence of several critical quantities. Fig. 4 shows that this propagator does not predict the experimental value of the scattering length, but $a_\rho = 1$ will produce the best agreement. Fig. 5 indicates that a value of a_ρ near unity gives a minimum in the phase shift, for fixed values of the ρ parameters. The pion radius apparently becomes insensitive to a_ρ , once that parameter exceeds 0.8, as may be seen in Fig. 6. Finally, Fig. 7 shows the sum-rule will be best satisfied by $g^{\rho\pi} = 6.5$. These figures will be taken as argument to use $a_\rho = 1$, and allowing the data to adjust the ρ and ρ' parameters. Three distinct cases for the branching ratio are considered: $R_1 = 1, R_2 = 0$; $R_1 = 1, R_2 = 1$; $R_1 = 0, R_2 = 1$.

A first attempt to fit the additional form factor data together with the phase shift seems, at first glance, to be successful. Figs. 8, 9, and 10 show the results of these fits. The parameters for Fits 1, 2, and 3 are listed in Table 7. Fits 1-3 are fitting only the form factor data with the phase shift calculated concurrently. The fact that this phase shift is quite close to the experimental value leads one to believe that the inclusion of the ρ' -meson should allow one to fit both data sets simultaneously.

Fits 4, 5, and 6 fit both sets of data simultaneously. Figs. 11, 12, and 13 show the form factor plots while Table 8 lists the parameters for these three fits. Table 9 shows the sum-rule, radius, and scattering length for Fits 1-6. Note that in all but Fits 2 and 3 the radius has values consistent with vector dominance and smaller.

Fig. 4. Scattering Length as a function of a .

Fig. 5. Pion-pion phase shift as a function of a_ρ .

Fig. 6. Pion radius as a function of a_ρ .

Fig. 7. The sum rule as a function of a.

Fig. 8. Fit to the pion form factor data with $\rho^1 \rightarrow 2\pi$.

Fig. 9. Fit to the pion form factor data with $\rho^1 \rightarrow 2\pi$ and 4π .

Fig. 10. Fit to the pion form factor data with $\rho^1 \rightarrow 4\pi$.

Table 7. Fit to form factor data with ρ'

	Fit 1	Fit 2	Fit 3
m_ρ	0.7646	0.7654	0.7652
$m_{\rho'}$	1.3884	1.5220	1.3977
Γ_ρ	0.1419	0.1433	0.1450
$\Gamma_{\rho'}$	0.3091	0.0428	0.1130
Reg_ω	0.0141	0.0157	0.0144
$\text{Im}g_\omega$	0.2833	0.2877	0.2937
g_ρ	6.2183	6.1349	6.1943
$g_{\rho'}$	-1.9862	-1.6593	-0.6417
R_1	1.00	1.00	0.00
R_2	0.00	1.00	1.00
χ^2	5.2	16.56	19.13
$\delta_{\pi\pi}$	14.64	17.2	17.54

These runs do not fit the phase shift data.

The given χ^2 is for 22 data points.

The sum-rule is not enforced in these fits.

21a

Fig. 11. Fit to the pion form factor and phase shift data with $p' \rightarrow 2\pi$.

21b

Fig. 12. Fit to the pion form factor and phase shift data
with $\rho' \rightarrow 2\pi$ and 4π .

Fig. 13. Fit to the pion form factor and phase shift data with $\rho' - 4\pi$.

Table 8. Fit to the form factor and the phase shift with the ρ'

	Fit 4	Fit 5	Fit 6
m_ρ	0.7693	0.7697	0.7732
$m_{\rho'}$	1.3541	1.4173	1.2330
Γ_ρ	0.1375	0.1340	0.1183
$\Gamma_{\rho'}$	0.3268	0.4965	0.2297
Reg_ω	-0.0103	-0.0430	-0.0513
$\text{Im}g_\omega$	0.2856	0.2736	0.2793
g_ρ	6.0755	5.9950	5.1664
$g_{\rho'}$	-2.1849	-5.6064	-2.2450
R_1	1.00	1.00	0.00
R_2	0.00	1.00	1.00
χ^2	7.97	21.16	22.5
$\delta_{\pi\pi}$	13.3	13.50	13.76

The given χ^2 is for 22 data points in Fit 4.

The given χ^2 is for 20 data points in Fits 5 and 6.

The sum-rule is not enforced in these fits.

Table 9. Sum-rule, $\langle r_\pi^2 \rangle$, and scattering length for Fits 1-6

	Fit 1	Fit 2	Fit 3	Fit 4	Fit 5	Fit 6
sum-rule	0.8181	0.9262	0.9478	0.7873	0.7274	0.6890
$\langle r_\pi^2 \rangle$	0.3865	0.4528	0.4474	0.3664	0.3112	0.2584
scattering length	-0.0787	0.1808	0.2058	-0.1263	-0.2860	0.2280

A study of Fits 1-6 shows that those fits which obey the sum-rule reasonably well (Fits 2 and 3) have fairly large radii associated with them. The remaining fits, which violate the sum-rule badly, have small values for $\langle r_\pi^2 \rangle$. The explanation for this correlation lies in the fact that the coupling strength of the ρ' is negative, and the fit to the phase shift data incorporates the ρ' -meson in such a way that the ρ' strength becomes large. This results in a substantial subtraction from the radius, scattering length and sum-rule. Note that in Fits 1, 4, and 5 the subtraction of the ρ' from the scattering length is so large as to give nonsensical values for that quantity.

Since the sum-rule obtained from Fits 1-6 is less than unity, the indication is that another meson is required to fit the data. There is little further justification for including another meson in the pion form factor, and since its effect would only be to counter the benefit of lowering the phase shift, this possibility will not be considered. Rather, an attempt will be to fit the two data sets with the existing mesons in such a way that the sum-rule is not violated.

This can be accomplished by including the sum-rule as a data point to be fit by the calculated form factor (Appendix B). The ρ and ρ' parameters are then adjusted in such a way that the calculated value of the sum-rule is unity within certain limits, usually taken as ± 0.01 . Fits 7, 8, and 9 (Fig. 14, 15, and 16) fit the form factor data together with the sum-rule. The phase shift, which is not included in these fits, is shown together with the adjustable parameters in Table 10. Most notable of these results is that the phase shift is again driven up to the vicinity of 18° . Apparently if the ρ' -meson is included in such a way as to not

Fig. 14. Fit to the pion form factor data and the sum rule with $\rho \rightarrow 2\pi$.

Fig. 15. Fit to the pion form factor data and the sum rule with $\rho^1 \rightarrow 2\pi$ and 4π .

Fig. 16. Fit to the pion form factor data and the sum rule with $\rho' \rightarrow 4\pi$.

Table 10. Parameters, $\delta_{\pi\pi}$, for Fits 7, 8, 9

	Fit 7	Fit 8	Fit 9
m_ρ	0.7628	0.7639	0.7614
$m_{\rho'}$	1.5321	0.5246	1.2931
Γ_ρ	0.1473	0.1466	0.1484
$\Gamma_{\rho'}$	0.0018	0.0034	0.0224
Reg_ω	0.0188	0.0195	0.0149
$\text{Im}g_\omega$	0.2851	0.2873	0.2805
g_ρ	6.3490	6.2963	6.4241
$g_{\rho'}$	-0.7408	-1.4630	-0.3311
R_1	1.0000	1.0000	0.0
R_2	0.0000	1.0000	1.0
χ^2	13.5	13.53	34
$\delta_{\pi\pi}$	18.06	17.85	18.3

Fits 7-9 do not fit the phase shift.

All fits obey the sum-rule to within 4%.

$m_\omega = 0.7820$, $\Gamma_\omega = 0.0090$, and $a_\rho = 1.000$ for all fits.

Table 11. Sum-rule, $\langle r_\pi^2 \rangle$, and scattering length for Fits 7-9

	Fit 7	Fit 8	Fit 9
sum-rule	0.9674	0.9567	1.0066
$\langle r_\pi^2 \rangle$	0.4761	0.4708	0.4709
scattering length	0.1985	0.1961	0.2043

violate the sum-rule it does not significantly lower the phase shift.

All attempts to obtain a simultaneous fit to the form factor, phase shift, and sum-rule failed. The best attempt, which contained the 4-pion decay mode for the ρ' , produced a χ^2 of 53 for 22 form factor data points while lowering the phase shift to 15.7° . The sum-rule value was 0.9845.

Very conspicuous to all fits which obey the sum-rule is a large enhancement at $q^2 \simeq -2.00 \text{ (GeV/c)}^2$. This enhancement falls precisely between data points, so there is no effect on the χ^2 except for Fit 9. The large χ^2 reported for Fit 9 is due to only one data point.

Finally, in order to determine the bias among the different data sets, fits were performed in which the ρ , ω , and ρ' parameters were held at the fixed values of Fits 7, 8, and 9, and a minimum χ^2 was calculated by allowing the λ_n s to be adjusted. The deviation of each λ from unity is a measure of the bias of that particular data set. Table 12 lists these results.

Table 12. Biasing results for Fits 7-9.

	Fit 7	Fit 8	Fit 9
λ_{BR}	0.9635	0.9635	0.9555
λ_{OR}	0.9996	0.9947	1.0047
λ_{NO}	0.9738	0.9118	0.9656
λ_{BE}	0.9233	0.1846	0.6493
λ_{PS}	0.7031	0.7079	0.6927
Δ_{BR}	0.1	0.1	0.1
Δ_{OR}	0.1	0.1	0.1
Δ_{NO}	0.1	0.1	0.1
Δ_{BE}	0.1	0.1	0.1
Δ_{PS}	0.3	0.3	0.3

V. DISCUSSION AND CONCLUSIONS

The mass parameter a_ρ is introduced with the expectation that it will properly normalize the phase shift at threshold. The resulting effect on the pion radius will serve to determine to what extent the large value of $\langle r_\pi^2 \rangle$ predicted by this propagator is due to the threshold enhancement on the phase shift.

When fits to the data are made with only the ρ and ω -mesons it is found that a_ρ does indeed depress the pion radius. Tables 4 and 5 show that in lowering the phase shift from 24 to 17, the pion (radius)² decreases correspondingly from .58 to .47. This new value is still considerably larger than $\langle r_\pi^2 \rangle_{VD}$ however. The difficulty encountered in this procedure is that the form factor data cannot be fit when values of Γ_ρ and a_ρ which give small phase shifts are used. Furthermore, as may be seen in Table 4, the effect of a_ρ is insufficient to depress the phase shift to 12.5° at the inelastic threshold.

When attempts are made to fit all the form factor data without the ρ' -meson, it is found that, from a goodness of fit point of view, the ρ' is essentially unnecessary. The large error on the data in the vicinity of $q^2 = -2.00 \text{ (GeV/c)}^2$ make it possible to obtain a satisfactory χ^2 for a fit which produces no enhancement in this region.

However, the negative vertex strength of the ρ' (8) may be used to lower the phase shift. When this investigation is carried out it is found that the phase shift is depressed to a value of approximately 13°, while $\langle r_\pi^2 \rangle$ drops to 0.3 (fermi)². When the charge sum-rule is calculated

it is found that the form factor and phase shift in conjunction use the ρ' in such a way that the sum-rule is badly violated. Fits 4-6 illustrate this point. When the ρ' effect is then incorporated into the radius and scattering length calculations, large subtractions from these quantities occur. Apparently the conclusion to be drawn is that, unless one wishes to relinquish the charge sum-rule requirement, the ρ' cannot be incorporated in a way which will reduce the phase shift effectively.

Fits 7-9 enforce the sum-rule while fitting the form factor data and calculating the phase shift. The phase shift is in the vicinity of 18° so the point made above is confirmed. The radius values for these fits are significantly larger than vector dominance predictions.

A curious phenomenon is encountered in all fits which obey the sum-rule. There exists, particularly in Fits 7-9, a sharp enhancement in the data near $q^2 = -2.00 \text{ (GeV/c)}^2$. The strength and width of this peak are particularly unusual when one considers that it falls precisely between data points. The appearance of this enhancement occurs in fits which fit only the form factor data and the sum-rule. Attempts to fit the phase shift as well do not give good fits to the form factor data.

Finally a biasing method is used to determine a measure of the discrepancy among the various data sets and the phase shift. No large bias among the data sets themselves is found, but the phase shift seems to have a bias of 30 per cent. Fig. 17 emphasizes how closely the phase shift calculated by this form factor follows the data. The upper curve is the phase shift of Fit 3, while the lower shows the effect of the biasing parameter on Fit 9.

Fig. 17. The p-wave $\pi-\pi$ phase as a function of q^2 with and without biasing.

VI. APPENDIX A: SOME RELEVANT EQUATIONS

In calculating the pion charge radius, the first five derivatives of Eq.(2.2) are required. To simplify the algebra required for this task one makes use of the expansions

$$\left(1 - \frac{q^2}{\lambda^2}\right)^{3/2} \approx 1 - \frac{3}{2} \frac{q^2}{\lambda^2} + \frac{3}{8} \frac{q^4}{\lambda^4}$$

$$-i \ln \frac{1+z_q}{1-z_q} \approx \frac{\pi}{2} - \frac{q}{\lambda} - \frac{1}{6} \frac{q^3}{\lambda^3} - \frac{3}{40} \frac{q^5}{\lambda^5}$$

$$\left(1 + \frac{q^2}{a^2}\right)^{-1} \approx 1 - \frac{q^2}{a^2} + \frac{q^4}{a^4}$$

Then there follows

$$\begin{aligned} \frac{\pi D(q)}{\alpha(1 + \frac{q^2}{m_0^2})} &\approx 2 \operatorname{Re} \left[\left(\pi i + \ln \frac{1+z_{m^*}}{1-z_{m^*}} \right) \left(\frac{q-m}{m^*-m} \right) \left(\frac{m^2-\lambda^2}{m^{*2}+a^2} \right)^{3/2} \right] \\ &+ \frac{\lambda^3}{a^2} \left[\frac{\pi}{2} + \frac{q}{\lambda} + \frac{1}{6} \frac{q^3}{\lambda^3} + \frac{3}{40} \frac{q^5}{\lambda^5} \right] \left[1 - \frac{3}{2} \frac{q^2}{\lambda^2} + \frac{3}{8} \frac{q^4}{\lambda^4} \right] \left[1 - \frac{q^2}{a^2} + \frac{q^4}{a^4} \right] \\ &+ \frac{(\lambda^2+a^2)^{3/2}}{\alpha^2(m^2+a^2)(m^{*2}+a^2)} \left[|m|^2 - q(m+m^*) + q^2 \right] \left\{ -\frac{\pi}{2} \left[|m|^2 - a^2 + q(m+m^*) \right] \right. \\ &\left. - \frac{1}{a} \left[q(|m|^2 - a^2) - a^2(m+m^*) \right] \ln \frac{a+\sqrt{a^2+\lambda^2}}{\lambda} \right\} \left(1 - \frac{q^2}{a^2} + \frac{q^4}{a^4} \right) \end{aligned}$$

Here $D(q) \equiv D(q, m, m^*)$. The first five derivatives of $D(q)$ with $q = 0$ are then given as follows. A subscript "V", denoting vector meson, is implied on all m 's and λ 's.

$$\frac{\pi D^{(0)}(0)}{\alpha(1 + \frac{a^2}{m_0^2})} = -2 \operatorname{Re}(\eta f) + \frac{\pi \lambda^3}{2a^2} - \frac{|m|^2 (\lambda^2 + a^2)^{3/2}}{(m^2 + a^2)(m^{*2} + a^2)} \left[\frac{\pi}{2a^2} (|m|^2 - a^2) - \frac{2m_0}{a} \ln \frac{a + \sqrt{a^2 + \lambda^2}}{\lambda} \right]$$

$$\frac{\pi D^{(1)}(0)}{\alpha(1 + \frac{a^2}{m_0^2})} = 2 \operatorname{Re}(f) + \frac{\lambda^2}{a^2} - \frac{(\lambda^2 + a^2)^{3/2}}{(m^2 + a^2)(m^{*2} + a^2)} \left\{ \pi m_0 + \frac{1}{a^3} \left[|m|^2 + a^2 (m^2 + m m^* + m^{*2}) \ln \frac{a + \sqrt{a^2 + \lambda^2}}{\lambda} \right] \right\}$$

$$\frac{\pi D^{(2)}(0)}{\alpha(1 + \frac{a^2}{m_0^2})} = \frac{\pi \lambda^3}{a^4} \left[\left(1 + \frac{a^2}{\lambda^2}\right)^{3/2} - \left(1 + \frac{3}{2} \frac{a^2}{\lambda^2}\right) \right]$$

$$\frac{\pi D^{(3)}(0)}{\alpha(1 + \frac{a^2}{m_0^2})} = \frac{6\lambda^3}{a^5} \left[\left(1 + \frac{a^2}{\lambda^2}\right)^{3/2} \ln \frac{a + \sqrt{a^2 + \lambda^2}}{\lambda} - \frac{a}{\lambda} \left(1 + \frac{4}{3} \frac{a^2}{\lambda^2}\right) \right]$$

$$\frac{\pi D^{(4)}(0)}{\alpha(1 + \frac{a^2}{m_0^2})} = \frac{-12 \pi D^{(2)}(0)}{a^2 \alpha(1 + \frac{a^2}{m_0^2})} + \frac{9}{2a^2 \lambda}$$

$$\frac{\pi D^{(5)}(0)}{\alpha(1 + \frac{a^2}{m_0^2})} = \frac{-10 \pi D^{(3)}(0)}{a^2 \alpha(1 + \frac{a^2}{m_0^2})} + \frac{24}{a^2 \lambda^2}$$

$$f = \left[\pi i + \ln \frac{1 + z_{m^*}}{1 - z_{m^*}} \right] \frac{(m^{*2} - \lambda^2)^{3/2}}{i \Gamma(m^{*2} + a^2)}$$

$$z_{m^*} = \sqrt{\frac{m^* - \lambda}{m^* + \lambda}}$$

One also wishes to evaluate the function

$$G(q^2) = I'(0) - \frac{m_0^2}{q^2} \left[I(0) - I(q^2) \right].$$

Writing

$$D(q) = \left[-\frac{m_0}{\epsilon} + \frac{A}{\epsilon} q + \frac{B}{\epsilon} \frac{q^2}{m_0^2} (1 + \dots) \right]$$

serves to define the constants A, B, and C

$$A = -m_0 \frac{D^{(1)}(0)}{D(0)}$$

$$B = -\frac{m_0^2}{2} \frac{D^{(2)}(0)}{D(0)}$$

$$C = -\frac{m_0}{D(0)}$$

Then some algebra shows that

$$D^{-1}(0)^{(1)} = -\frac{AC}{m_0^2}$$

$$D^{-1}(0)^{(3)} = -\frac{6C}{m_0^4} \left[A^3 + 2AB + \frac{m_0 B}{3} \frac{D^{(3)}(0)}{D^{(2)}(0)} \right]$$

$$D^{-1}(0)^{(5)} = -\frac{120C}{m_0^6} \left[A^5 + 4A^3B + 3AB^2 + (A^2Bm_0 + \frac{2}{3}B^2m_0) \frac{D^{(3)}(0)}{D^{(2)}(0)} \right. \\ \left. + \frac{1}{6}AB^2m_0^2 \frac{D^{(4)}(0)}{D^{(2)}(0)} + \frac{1}{60}Bm_0^3 \frac{D^{(5)}(0)}{D^{(2)}(0)} \right].$$

By expanding the function

$$I(q^2) = \frac{m_0^2}{2iq} \left[D^{-1}(iq) - D^{-1}(-iq) \right]$$

one can show that

$$I(0) = m_p^2 D^{-1}(0)^{(1)}$$

$$I'(0) = \frac{m_p^2}{6} D^{-1}(0)^{(3)}$$

$$I''(0) = \frac{m_p^2}{60} D^{-1}(0)^{(5)}$$

Figs. (18-23) show the a_p dependence of some quantities which are critical to this and future work.

Figs. 18-23. The a dependence of some critical quantities.

VII. APPENDIX B: THE COMPUTER PROGRAM

The computer program utilized in the analysis of the form factors is based on the gradient expansion algorithm of Marquardt. A brief description of the theory underlying this program follows.

Let the data points be (x_i, y_i) and let σ_i represent the standard deviation for each measurement, which is assumed to be equal to the experimental error. Ignore errors in x_i . Let the theory function be represented by $y(x, a_k)$, where a_k represents a set of NTERM parameters to be determined by fitting the data; $k=1,2,3,\dots,NTERM$.

The theory function may be expanded about its value at the initial parameters, a_{0k} , as follows:

$$y(x, a) = y(x, a_0) + \sum_{j=1}^{NTERM} (\partial y / \partial a_j)_{a_{0j}} \delta a_{0j}, \quad (B.1)$$

where $\delta a_j = a_j - a_{0j}$. The deviation from the i th experimental value is

$$(DEV)_i = y_i - y(x_i, a) = y_i - y(x_i, a_0) - \sum_j (\partial y(x_i) / \partial a_j)_{a_{0j}} \delta a_j$$

The chi-square function is now defined as the sum of the squares of the deviations from the observed values weighted by the mean standard deviations; that is,

$$\chi^2 = \sum_i \frac{(DEV)_i^2}{\sigma_i^2}, \quad (B.2)$$

and is a measure of the goodness of the fit of the theory function to the data. The goal is to determine the set of parameters which minimizes the difference between theory and experiment.

```

C      CURFIT:  A MINIMIZATION PROGRAM UTILIZING THE GRADIENT
C      EXPANSION ALGORITHM OF MARQUARDT TO FIT THE TIME-LIKE
C      PION FORM FACTOR
C
C      DESCRIPTION OF PARAMETERS
C      X:  ARRAY OF DATA POINTS FOR INDEPENDENT VARIABLE
C      Y:  ARRAY OF DATA POINTS FOR DEPENDENT VARIABLE
C      SIGMA:  ARRAY OF STANDARD DEVIATIONS FOR Y DATA POINTS
C      NDP:  NUMBER OF PAIRS OF DATA POINTS
C      NTERMS:  NUMBER OF PARAMETERS
C      NVPAR:  NUMBER OF VARIABLE PARAMETERS
C      A:  ARRAY OF PARAMETERS
C      THEORY:  THE THEORY FUNCTION; CALCULATED VALUES OF Y
C      CHISQ1:  CHI SQUARE
C      COMMENTS:
C      DIMENSION STATEMENTS ALLOW NTERMS UP TO 8
C      FLAMDA=0.001 AT BEGINNING OF SEARCH
C
      REAL*4 MU(3)
      REAL BETA(20),ALPHA(20,20),FLAMDA,A(20),B(20)
      REAL BET(20),WT(20),ARRAY(20,20),ALPH(20,20),C(20)
      INTEGER COUNT(25),C1(20),C2(20)
      REAL*4 X(4,22),Y(4,22)
      INTEGER Z
      INTEGER N(4)
      COMMON/AAA/N,X,Y
      COMMON /BBB/ NDP,NVPAR
      COMMON/CCC/ WT
      COMMON/EEE/NQ
      DATA MU/.2792,.4141,.5584/
      DATA PI/3.14159/
      DEFALT=.0000000000000001
C
C      DEFALT SHOULD BE <<(EPS)**2
C
      5      READ(5,111) EPS,NTERMS,NSTEPS
      111     FORMAT(F10.3,8X,I2,8X,I2)
      WRITE(6,111) EPS,NTERMS,NSTEPS
      NS=NSTEPS
      DO 12 I=1,NSTEPS
      12     COUNT(I)=0
      NSTEPS=0
      FLAMDA=.001
C
C      INPUT DATA:  INITIAL PARAMETER VALUES AND WEIGHTS
C
      READ(5,10,END=90)(C1(I),C2(I),A(I),WT(I),I=1,NTERMS)
      10     FORMAT(9X,A4,6X,A4,5X,F12.7,F10.4)
      WRITE(6,11)
      11     FORMAT(5X,'INPUT PARAMETERS:')

```

```

WRITE(6,13)(I,C1(I),C2(I),A(I),WT(I),I=1,NTERMS)
13  FORMAT(4X,I2,3X,A4,2X,A4,4X,F10.4,F7.1)
    READ(5,321) NQ
321  FORMAT(9X,I1)
C
    NVPAR=0
    DO 17 I=1,NTERMS
      C(I)=A(I)
    17 NVPAR=NVPAR+WT(I)
C
C  CALCULATE CHISQ, ALPHA, EETA
C
    CALL FCN(NTERMS, CHISQ1,A,1,ALPHA,BETA)
    CHISQ3=CHISQ1/(NDP-NVPAR)
    WRITE(6,104) CHISQ1,CHISQ3
104  FORMAT('0',/,1X,9X,'CHISQUARE',5X,'CHI SQUARE/(NDP-NVPAR)',
      C,3X,'NDF',3X,'NVPAR',2X,'STEPS',3X,'COUNT',/,1X,2F20.10)
    IF(NQ .EQ. 0) GO TO 91
C
C  COUNT THE ITERATIONS
C
    70 NSTEPS=NSTEPS+1
C
C  DEFINE ARRAY ACCORDING TO FORMULA, NORMALIZING IT TO
C  THE DIAGCNAL
C
    55 DO 30 J=1,NTERMS
      DO 34 K=1,J
    35  ARRAY(J,K)=ALPHA(J,K)/ SQRT(ABS(ALPHA(J,J)*ALPHA(K,K))+
      CDEFAULT)*WT(K)*WT(J)
    34  ARRAY(K,J)=ARRAY(J,K)
      ARRAY(J,J)=(1. + FLAMDA )*WT(J)
    30 CONTINUE
C
C  INVERT ARRAY
C
    CALL MATINV (ARRAY,NTERMS,DET)
C
C  INCREMENT A, AND CALCULATE NEW VALUES OF CHISQ, ALPHA,
C  AND BETA
C
    DO 40 J=1,NTERMS
      B(J)=A(J)
      DO 40 K=1,NTERMS
    40  B(J)=B(J)+ BETA(K)*ARRAY(J,K)/SQRT(ABS(ALPHA(J,J)*
      1 ALPHA(K,K))+DEFAULT)
    CALL FCN(NTERMS,CHISQ2,B,4,ALPH,BET)
    WRITE(6,18)
    DO 27 J=1,4
      Z=N(J)

```

```

DO 27 K=1,Z
D=THEORY(X,B,J,K)
E=Y(J,K)
WRITE(6,21)D,E
27 CONTINUE
IF(CHISQ1-CHISQ2) 45,50,50
45 FLAMDA=10*FLAMDA
COUNT(NSTEPS)=COUNT(NSTEPS)+1
C
C IF SEQUENCE IS NOT CONVERGING, ABORT
C
IF(FLAMDA.GT.10000) GO TO 61
GO TO 55
C
C CHECK TO SEE IF DELTA CHISQ IS LESS THAN EPS
C
50 WRITE(6,76) CHISQ2
76 FORMAT('0',/,10X,'INTERMEDIATE CHISQ=',F20.10)
WRITE(6,1131)
1131 FORMAT('0',T13,'PARAMETER',T25,'WEIGHT',T36,'INITIAL'
1,'VALUE',T56,'FINAL VALUE')
WRITE(6,1121)(I,C1(I),C2(I),WT(I),C(I),B(I),I=1,NTERMS)
1121 FORMAT(1X,3X,I2,5X,A5,A5,5X,F2.0,2F20.10)
IF(CHISQ1-CHISQ2-EPS ) 65,60,60
65 CHISQ1=CHISQ2
C
C IF SEQUENCE IS NOT CONVERGING, ABORT
C
IF(NSTEPS.GT.NS) GO TO 61
DO 75 J=1,NTERMS
BETA(J)=BET(J)
A(J)=B(J)
DO 75 I=1,NTERMS
75 ALPHA(I,J)= ALPH(I,J)
FLAMDA=FLAMDA/10.
C
C SINCE DELTA CHISQ WAS TOO LARGE, BUT CHISQ DECREASED,
C REITERATE
C
GO TO 70
61 WRITE(6,153)
153 FORMAT(' THE SEQUENCE IS NOT CONVERGING')
DO 699 J=1,NTERMS
699 BETA(J)=B(J)-A(J)
WRITE(6,651)(A(J),BETA(J),J=1,NTERMS)
651 FORMAT(1X,'THE COMPARISON SET OF PARAMETERS AND THE LAST'
C,'SET OF INCREMENTS WERE',/,10(2F20.10,/))
WRITE(6,124) CHISQ1
124 FORMAT(1X,'THE COMPARISON CHISQUARE WAS ',F20.10)
60 CHISQ3=CHISQ2/(NDP-NVPR)

```

```

      WRITE(6,102) CHISQ2,CHISQ3,NDP,NVPAR,NSTEPS,(COUNT(I),I=1
1,NSTEPS)
102 FORMAT(      2F20.10,7X,3(I3,4X),20(I2,1X))
C
C      OUTPUT INITIAL AND FINAL VALUES OF PARAMETERS
C
      WRITE(6,113)
113 FORMAT(T13,'PARAMETER',T25,'WEIGHT',T36,'INITIAL VALUE',
1T58,'FINAL VALUE')
      WRITE(6,112)(I,C1(I),C2(I),WT(I),C(I),B(I),I=1,NTERMS)
112 FORMAT(1X,3X,I2,5X,A5,A5,5X,F20.0,2F20.10)
C CALL FCN FOR THE LAST TIME, IFLAG = 3
C
      CALL FCN(NTERMS, CHISQ2,E,3, ALPHA, BETA)
91 CONTINUE
C
C      THE NEXT 2 CARDS ARE NECESSARY WHEN NQ=0
C
      DO 19 J=1,NTERMS
19 B(J)=C(J)
16 WRITE(6,18)
18 FORMAT('0',5X,'THEORY',8X,'EXPERIMENT',/)
      DO 20 J=1,4
      Z=N(J)
      DO 20 K=1,2
      D=THEORY(X,B,J,K)
      E=Y(J,K)
      WRITE(6,21)D,E
21 FORMAT(2F13.4)
20 CONTINUE
C
C      GO ON TO THE NEXT DATA SET
C
      GO TO 5
90 STOP
END
      SUBROUTINE MATINV(ARRAY,NCRDER,DET)
      REAL ARRAY,DET
      DIMENSION ARRAY(20,20),IK(20),JK(20)
C THE FOLLOWING CARD DOES NOT APPEAR IN BEVINGTON,S PROGRAM
      NSTEPS=0
      DET=1.
      DO 100 K=1,NORDER
      AMAX=0.
21 DO 30 I=K,NCRDER
      DO 30 J=K,NORDER
      IF(ABS(AMAX)-ABS(ARRAY(I,J))) 24,24,30
24 AMAX=ARRAY(I,J)
      IK(K)=I
      JK(K)=J

```

```

30 CONTINUE
   IF(AMAX) 41,32,41
C   A CARD IN BEVINGTON,S PROGRAM HAS BEEN OMITTED AT THIS
C   POINT
41  I=IK(K)
   IF(I-K) 21,51,43
43  DO 50 J=1,NORDER
     SAVE=ARRAY(K,J)
     ARRAY(K,J)=ARRAY(I,J)
50  ARRAY(I,J)=-SAVE
51  J=JK(K)
   IF(J-K) 21,61,53
53  DO 60 I=1,NORDER
     SAVE=ARRAY(I,K)
     ARRAY(I,K)=ARRAY(I,J)
60  ARRAY(I,J)=-SAVE
61  DO 70 I=1,NORDER
     IF(I-K) 63,70,63
63  ARRAY(I,K)=-ARRAY(I,K)/AMAX
70  CONTINUE
     DO 80 I=1,NORDER
       DO 80 J=1,NORDER
         IF(I-K) 74,80,74
74  IF(J-K) 75,80,75
75  ARRAY(I,J)=ARRAY(I,J)+ARRAY(I,K)*ARRAY(K,J)
80  CONTINUE
     DO 90 J=1,NORDER
       IF(J-K) 83,90,83
83  ARRAY(K,J)=ARRAY(K,J)/AMAX
90  CONTINUE
     ARRAY(K,K)=1./AMAX
C   THE FOLLOWING CARD DOES NOT APPEAR IN BEVINGTON,S PROGRAM
     NSTEPS=NSTEPS+1
100 DET=DET*AMAX
C   THE NEXT CARD IS CHANGED FROM MATINV OF BEVINGTON
32 DO 130 L=1,NSTEPS
C   THE NEXT CARD IS CHANGED FROM MATINV OF BEVINGTON
     K=NSTEPS+1-L
     J=IK(K)
     IF(J-K) 111,111,105
105 DO 110 I=1,NORDER
     SAVE=ARRAY(I,K)
     ARRAY(I,K)=-ARRAY(I,J)
110 ARRAY(I,J)=SAVE
111 I=JK(K)
     IF(I-K) 130,130,113
113 DO 120 J=1,NORDER
     SAVE=ARRAY(K,J)
     ARRAY(K,J)=-ARRAY(I,J)
120 ARRAY(I,J)=SAVE

```

```

130 CONTINUE
140 RETURN
END
SUBROUTINE FDTH(NTERMS,X,A,I,J)
REAL X(4,22),A(20),WT(20),DERIV(20)
COMMON/CCC/WT
COMMON/DCD/DERIV
DO 18 K=1,NTERMS
  A(K)=A(K)+.0005
  F2=THEORY(X,A,I,J)
  A(K)=A(K)-.001
  F1=THEORY(X,A,I,J)
  DERIV(K)=(F2-F1)/.001
18  A(K)=A(K)+.0005
RETURN
END
SUBROUTINE FCN(NTERMS,CHISQ1,A,IFLAG,ALPHA,BETA)
REAL ALPHA(20,20),EETA(20),DERIV(20),A(20),WT(20)
REAL X(4,22),Y(4,22),SUM(4),SIGMAY(4,22)
INTEGER N(4),Z
COMMON/AAA/N,X,Y
COMMON/BBB/NDP,NVPAR
COMMON/CCC/WT
COMMON/DDD/DERIV
COMMON/EEE/NQ
GO TO (10,20,30,40),IFLAG
10  CONTINUE
    READ(5,1400) N1,N2,N3,N4
1400 FORMAT(8X,I2,8X,I2,8X,I2,8X,I2)
C    READ IN PION DATA
    READ(5,1301)(X(1,K),Y(1,K),SIGMAY(1,K),K=1,N1)
1301 FORMAT(3F13.4)
    WRITE(6,1360)
1360 FORMAT('0',5X,' TIME-LIKE PION DATA: ',/,T4,'MOMENTUM**2',
1T18,'FORM FACTOR**2',T34,'ERROR')
    WRITE(6,1301)(X(1,K),Y(1,K),SIGMAY(1,K),K=1,N1)
    READ(5,1301)(X(2,K),Y(2,K),SIGMAY(2,K),K=1,N2)
    WRITE(6,1301)(X(2,K),Y(2,K),SIGMAY(2,K),K=1,N2)
    READ(5,1301)(X(3,K),Y(3,K),SIGMAY(3,K),K=1,N3)
    WRITE(6,1301)(X(3,K),Y(3,K),SIGMAY(3,K),K=1,N3)
    READ(5,1301)(X(4,K),Y(4,K),SIGMAY(4,K),K=1,N4)
    WRITE(6,1301)(X(4,K),Y(4,K),SIGMAY(4,K),K=1,N4)
    N(1)=N1
    N(2)=N2
    N(3)=N3
    N(4)=N4
    NDP=N(1)+N(2)
40  CONTINUE
    DO 12 K=1,4
12  SUM(K)=0.0

```

```

      DO 11 J=1,4
      Z=N(J)
      DO 11 K=1,Z
11    SUM(J)=SUM(J)+(((THEORY(X,A,J,K)-Y(J,K))/(SIGMAY(J,K)))**2
      CHISQ1=SUM(1)+SUM(2)+SUM(3)+SUM(4)
      WRITE(6,1001)SUM(1),SUM(2),SUM(3),SUM(4)
1001  FORMAT(' SUMS=',4F13.4)
      IF(NQ .EQ. 0) GO TO 30
C
C    INITIALIZE CURVATURE MATRIX
      DO 34 J=1,NTERMS
      BETA(J)=0.
      DO 34 K=1,J
34    ALPHA(J,K)=0.
C
C    DEFINE UPPER ECHELON OF CURVATURE MATRIX
      DO 50 M=1,4
      Z=N(M)
      DO 50 I=1,Z
      CALL FDTH(NTERMS,X,A,M,I)
      DO 46 J=1,NTERMS
      BETA(J)=BETA(J)+(((Y(M,I)-THEORY(X,A,M,I))/(SIGMAY(M,I))
1**2))*DERIV(J)*WT(J)
      DO 46 K=1,J
46    ALPHA(J,K)=ALPHA(J,K)+((DERIV(J)*DERIV(K))/(SIGMAY(M,I))
1**2))*WT(J)*WT(K)
50    CONTINUE
C
C    DEFINE LOWER ECHELON OF CURVATURE MATRIX
      DO 53 J=1,NTERMS
      DO 53 K=1,J
53    ALPHA(K,J)=ALPHA(J,K)
      GO TO 20
30    CONTINUE
20    CONTINUE
      RETURN
      END
      FUNCTION THEORY(T,X,J,K)
      REAL*8 DBLE,DATAN
      REAL*4 T(4,22) , GAMA(5) ,LC
      REAL*4 SIN,COS
      REAL*4 MU(5),P1,ALPHR(5),NU(5), R(3)
      REAL*4 ALOG,SQRT,ABS,ATAN,R1(5),ALPHA(5),AIMAG
      REAL*4 X(20),WT(20),I2(5)
      COMPLEX FP1,FP2,FP3
      COMPLEX D1,D2,D3,D4,D5,D10,D20,D30,D40,D50,D0,D00
      COMPLEX C11,C12,C13
      COMPLEX DV1,DV2 ,DV10,DV20,E,B,C,I0,I11,FPI
      COMPLEX C6,C9,I20,G2,G3,GV(5),DV11,IV,I1(3)
      INTEGER C1(20),C2(20)

```



```

COMPLEX M1,M2, L1,S1,RS,CMPLX,CONJG,CEXP,      A1,A2
COMPLEX A3,A4,M5,A5,M3,M4
REAL*4 REAL,CABS,PHI,SNGL
DATA    NU/0.2792,0.4141,0.2792,0.9877,0.5584/
DATA GAMA/2.6,7.7,5.2,6.2,5.2/
DATA PI/3.141593/
DATA ALPHR/0.08824,0.0064,0.0022,0.1094,0.1094/
GO TO (30,30,31,30),J
31 THEORY=X(15+K)
GO TO 50
30 ALPHA(2)=((X(5)*X(2)**2)/2.)*(1./((SQRT((X(2)**2)-
1 (NU(2)**2)**3))
R(1)=1.
R(2)=ALPHA(2)
R(3)=SQRT(X(12)**2)
DO 9 I=1,3
A=X(12+I)
C5=(1.+(A/X(I))**2)/PI
L=1
DV1=0.
DV2=0.
D0=0.
D1=0.
D2=0.
D3=0.
D4=0.
D5=0.
MU(I)=NU(I)
GO TO 40
41 MU(3)=2.*NU(3)
L=-1
R(3)=SQRT(X(7)**2)
40 CONTINUE
XI3=SQRT(X(I+3)**2)
M1=CMPLX(X(I), XI3 /2.)
M2=CONJG(M1)
THETA=SNGL(DATAN(XI3 /(2.D0*(DBLE(X(I))-DBLE(MU(I)))))
PHI =SNGL(DATAN(XI3 /(2.D0*(DBLE(X(I))+DBLE(MU(I)))))
B1=CABS(M1-CMPLX(MU(I),0.0))
B2=CABS(M1+CMPLX(MU(I),0.0))
A1=CMPLX(0.0,THETA)
A3=CMPLX(0.0,PHI)
A4=(A1-A3)/2.
ANGLE=(THETA-PHI)/2.
M3=B1*CEXP(A1)
M5=B2*CEXP(A3)
Z=SQRT(B1/B2)
BETA=ATAN((2.*Z*SIN(ANGLE))/(1.-(Z**2)))
A5=CMPLX(0.0,BETA)
B3=1.+(Z**2)-(2.*Z*CCS(ANGLE))

```

```

B4=1.-(Z**2)
B5=2.*Z*SIN(ANGLE)
M4=CMPLX(B4/B3,B5/B3)
L1=CMPLX(ALOG(CABS(M4)),EETA)
GAMMA=3.*(THETA+PHI)/2.
S1=((SQRT(B1*B2))**3)*CEXP(CMPLX(0.0,GAMMA))
RS=((CMPLX(0.0,PI)+L1)*S1)/((M1-M2)*((M1**2)+(A**2)))
G1=SQRT(((MU(I)**2)+(A**2))**3)
G2=((M1**2)+(A**2))*((M2**2)+(A**2))
G3=G1/G2
C3=(CABS(M1))**2
C4=ALOG((A+SQRT((A**2)+(MU(I)**2)))/MU(I))
C6=C3+(M1**2)+(M2**2)+((C3/A)**2)
C7=(1./(A**2))+(3./(2.*MU(I)**2))
C8=(3./(8.*MU(I)**4))+(C7/(A**2))
C9=(M1**2)+(M2**2)+(A**2)+((C3/A)**2)
D00=((M1+M2)/A)*C4
D00=D00-(PI/(2.*A**2))*C3
D00=D00+((FI*MU(I)**3)/(2.*A**2))
D00=D00-2.*REAL(RS*M2)
D00=D00*C5
D0=D00*R(I)+D0
D10=C6*C4/A
D10=D10+(PI*(M1+M2)/2.)
D10=-D10*G3
D10=D10+((MU(I)/A)**2)
D10=D10+2.*REAL(RS)
D10=D10*C5
D1=D10*R(I)+D1
D20=-((PI*MU(I)**3)/(A**2))*C7
D20=D20+((C9*G3*PI)/(A**2))
D20=D20*C5
D2=D20*R(I)+D2
D30=(1./(6.*MU(I)**3))-(C7/MU(I))
D30=((6.*MU(I)**3)/(A**2))*D30
D30=D30+((C9*C4*G3*6.)/(A**3))
D30=D30*C5
D3=D30*R(I)+D3
D40=((12.*PI*MU(I)**3)/(A**2))*C8
D40=D40-(C9/A**2)*G3*(12.*PI/(A**2))
D40=D40*C5
D4=D40*R(I)+D4
D50=3./(40.*MU(I)**5)
D50=D50-(C7/(6.*MU(I)**3))
D50=D50+(C8/MU(I))
D50=((120.*MU(I)**3)/(A**2))*D50
D50=D50-((120.*G3*C4*C9)/(A**5))
D50=D50*C5
D5=D50*R(I)+D5

```

```

      IF(I .LT. 3) GO TO 48
      IF(L .GT. 0) GO TO 47
48    C=-X(I)/D0
      E=D1*C
      B=((X(I)*C)/2.)*D2
      I0=-E*C
      I11=(E**3)+(2.*E*B)
      I11=I11+((E*X(I)/3.)*(D3/D2))
      I11=(C/(X(I)**2))*I11
      I1(I)=I11
47    Q=SQRT(-T(J,K))
      SQ=ABS(Q-MU(I))*ABS(Q+MU(I))
      RQ=SQRT(SQ)
      LQ=ALOG((Q+RQ)/MU(I))
100   CONTINUE
      C10=(CABS(M2)**2)-(A**2)+Q**2.*X(I)
      C11=Q*((CABS(M2)**2)-(A**2))-2.*X(I)*(A**2)
      DV10=(-(PI/2.)*C10)-(C4*C11/A)
      DV10=((CABS(Q-M2))**2)*G3*DV10/((Q**2)+(A**2))
      DV10=2.*REAL(RS*(Q-M2))+DV10
      IF(Q .LT. 0) GO TO 101
      DV11=DV10+((CMPLX(0.0,PI)-LC)*RQ**3)/((Q**2)+(A**2))
      Q=-Q
      GO TO 100
101   CONTINUE
      DV20=DV10+((LQ*SQRT(SQ)**3)/((Q**2)+(A**2)))
      DV11=DV11*C5
      DV1=DV11*R(I)+DV1
      DV22=DV20*C5
      DV2=DV22*R(I)+DV2
      IF(L .LT. 0) GO TO 42
      IF(I .EQ. 3) GO TO 41
42    IV=((X(I)**2)/(2.*G))*((1./DV2)-(1./DV1))
      GV(I)=I1(I)+((I0-IV)/T(J,K))
      9   CCNTINUE
      IF(J.EQ. 4)GO TO 51
      FP1=1.-((X(8)      *GV(1)*X(1)**2))
      FP2=-CMPLX(X(9),X(10 ))*GV(2)*((X(2)**2)/(2.*GAMA(2)))
      FP3=-X(11)*GV(3)*(X(3)**2)
      FPI=FP1+FP2+FP3
      GO TO (10,20),J
10    THEORY=(CABS(FPI))**2
49    IF(K .GT. 2) GO TO 21
      THEORY      =X(16)*THEORY
      GO TO 50
21    IF(K .GT. 12) GO TO 22
      THEORY      =X(17)*THEORY
      GO TO 50
22    IF(K .GT. 14) GO TO 23
      THEORY      =X(18)*THEORY

```

```
GO TO 50
23 THEORY          =X(19)*THEORY
GO TO 50
20 DELTA=ATAN(AIMAG(FPI)/REAL(FPI))
DELTA=(180./PI)*DELTA
THEORY=DELTA
THEORY          =X(20)*THEORY
GO TO 50
51 SR1=I1(1)*X(8)*(X(1)**2)
SR3=I1(3)*X(11)*(X(3)**2)
SR=SR1+SR3
THEORY=SR
50 CONTINUE
RETURN
END
```

VIII. APPENDIX C: RESEARCH IN PROGRESS

It has been found that $D(q, m, m^*)$ contains a zero for large values of a_ρ . While it is not obvious what interpretation to give such a pole in the propagator, or just how it occurs, its effect on the present work is negligible. The zero occurs in the vicinity of $a_\rho = 10$ while the largest value of a_ρ used here is unity.

The position of the pole is at the origin in momentum space, so if the singularity is reabsorbed in the form factor, one obtains

$$f^{\pi\pi}(q^2) = \frac{g^{\rho\pi}}{m_\rho^2} q^2 K(q^2)$$

with $g^{\rho\pi}$ chosen so that $f^{\pi\pi}(q^2)$ is normalized to unity at $q=0$. This form of the ρ form factor, together with the ω and ρ' contributions used previously is found to fit the data quite well. χ^2 values of 13 for 22 data points have been obtained.

The method for determining $g^{\rho\pi}$ is to write (6)

$$D(q) = q Z^{-1}(0) + D^{(1)}(0) q + \dots$$

Differentiation gives

$$\begin{aligned} D'(q)|_{q=0} &= Z^{-1}(0) \\ &= D^{(1)}(0) \\ &= g^{\rho\pi} \end{aligned}$$

Fits in which $g^{\rho\pi}$ is treated as an adjustable parameter obtain minimum

χ^2 s at values of $g^{\rho\pi}$ which are comparable to $D^{(1)}(0)$.

The pion radius, given by $-Gf^{e\pi}(0)$, is calculated to be

$$\langle r_\pi^2 \rangle = \left[\left(\frac{8}{A\omega_0} \right)^2 - \frac{CD^3}{GA} \right]$$

For best fits to the data, it is found that the radius is

$$\langle r_\pi^2 \rangle = 0.8 \langle r_\pi^2 \rangle_{VD}$$

while the phase shift is quite small

$$\delta_{\pi\pi}(q^2) = 9^\circ$$

As would be expected, the scattering length, given by the modified formula

$$SL = - \lambda \left(1 + \frac{g^2}{\omega_0^2} \right) \left(1 + \frac{g^2}{\lambda^2} \right)^{-1} D^{-1}(\lambda) \left[1 - \frac{D(\lambda)}{D(-\lambda)} \right]^{-1}$$

is found to be small as well,

$$SL = 0.033$$

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