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INTRODUCTION

It has been pointed out by Polyakov in Ref (1) that finite action classical solutions to Euclidean field equations may play an important role in quantum field theory. These solutions are called pseudoparticles or instantons. They are localized in space and time and possess some nontrivial topological characteristics. When used to dominate the path integral they can give rise to tunneling processes between different classical vacua⁽²⁾. They can play an important role in developing a mechanism for quark confinement⁽³⁾ and may be relevant in influencing the infrared behavior of some field theories.

The most interesting field theory with pseudoparticle solutions is the SU(2) Yang-Mills gauge theory in four dimensions⁽⁴⁾. For this case it was even possible to derive explicitly a large class of multi-pseudoparticle solutions⁽⁵⁾. Quantum fluctuations about the pseudoparticle solution have been considered in (6).

A theory which is in many respects similar to SU(2) gauge theory is the two-dimensional nonlinear σ -model, also known as the continuum Heisenberg ferromagnet. This theory is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2f} \sum_{a=1}^3 (\partial_\mu \phi_a)^2 \quad (1.1)$$

and the constraint $\sum_{a=1}^3 \phi_a^2 = 1$. General multi-pseudoparticle solutions have been found for this case in Ref (7) by Belavin, and Polyakov. Introducing the angle parametrization for the unit vector ϕ_a and defining

QUANTUM FLUCTUATIONS OF PSEUDOPARTICLES IN THE NONLINEAR σ -MODEL

by

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ABSTRACT

We present an investigation of quantum fluctuations about pseudoparticle solutions in the two-dimensional σ -model. First, for the one pseudoparticle case, we solve the small oscillation problem and compute the first quantum correction to the classical result. In the two-pseudoparticle case, we transform the small oscillation problem into an ordinary Schrödinger equation with an interaction depending on distance between the pseudoparticles. We are then able to solve the limiting cases of zero and infinite separations.

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$$\omega(z) \equiv \omega_1 + i \omega_2 = \operatorname{ctg} \frac{\theta}{2} e^{i\alpha} \quad (1.2)$$

with $z = x_1 + i x_2$, they show that ω satisfies the Cauchy-Riemann conditions. The general solution which satisfies the boundary condition

$$\vec{\phi}(x) \rightarrow (0,0,1) \quad (1.3)$$

$$|x| \rightarrow \infty$$

then reads

$$\omega(z) = \prod_i \left(\frac{z - z_i}{\lambda} \right)^{m_i} \prod_j \left(\frac{\lambda}{z - z_j} \right)^{n_j} \quad (1.4)$$

where $\sum m_i > \sum n_j$.

These finite action field configurations are classified by an integer-valued topological number given by

$$q = \frac{1}{8\pi} \int d^2x \epsilon_{abc} \epsilon_{\mu\nu} \phi_a \partial_\mu \phi_b \partial_\nu \phi_c \quad (1.5)$$

and for (1.4) we have $q = \sum m_i$. The action of this solution is simply

$$A(\phi_q) = \frac{4\pi q}{f} \quad (1.6)$$

so classically the pseudoparticles do not interact.

It is our purpose in this paper to study quantum fluctuations about one and also two-pseudoparticle solutions. We will be explicitly computing the first quantum correction to the classical contribution of this configurations and at this level the path integral expression for the transition amplitude reads

$$e^{-A[\vec{\phi}_{cl}]} \int \mathcal{D}\vec{\phi} \prod_x \delta(\vec{\phi} - \vec{\phi}_{cl}) e^{-A^{(2)}[\vec{\phi}]} \quad (1.7)$$

with

$$A^{(2)}[\vec{\phi}] = \int d^2x \sum_{a=1}^3 \frac{1}{2} \phi_a (-\partial^2 - (\partial \vec{\phi}_{cl})^2) \phi_a \quad (1.8)$$

It is obtained by making a shift $\phi_a = \phi_a^{cl} + \phi_a$ and neglecting higher order terms.

To derive the equation for small oscillations we rewrite the action in the form

$$A^{(2)}[\vec{\phi}] = \int d^2x \frac{1}{2} (\partial \vec{\phi}_{cl})^2 \sum_{a=1}^3 \phi_a \mathcal{K}_a^2 \phi_a \quad (1.9)$$

where

$$\mathcal{K}_a^2 = - \left[\frac{1}{2} (\partial \vec{\phi}_{cl})^2 \right]^{-1} \partial^2 - 2 \quad (1.10)$$

It is the eigenvalue problem associated with this operator that we solve.

First, for the one pseudoparticle case, we determine explicitly the eigenfunctions and the eigenvalues and then evaluate the first quantum correction. Zero-frequency modes are treated according to the collective coordinate method^(8,9) and the renormalization is performed using Pauli-Villars regulators.

For the two-pseudoparticle case we transform the constrained small oscillation problem into an ordinary Schrödinger equation with an interaction term which depends explicitly on distance between the pseudoparticles. Due to the complexity of this potential it is not possible to derive an exact solution.

We are able however to solve the small oscillation problem in two limiting cases, namely that of zero and of infinite separations. This is sufficient for our purpose of demonstrating how interaction between pseudoparticles appears from quantum corrections. What we find is that the eigenvalue spectrum for zero pseudoparticle separation differs drastically from the spectrum in case of infinite separation.

The plan of this paper is as follows: In Section II we solve the small oscillation problem about the one-pseudoparticle solution. Then in Section III we compute the first quantum correction due to this configuration. In Section III we study the two-pseudoparticle case.

II. SMALL OSCILLATIONS ABOUT THE PSEUDOPARTICLE CONFIGURATION

In this section we shall consider the eigenvalue problem for small oscillations about the one-pseudoparticle classical solution^[1]

$$\omega(z) = \frac{10}{\beta} \frac{z - z_1}{z - z_2} \quad (2.1)$$

Here the constant parameters (β , a_0 , z_1 , z_2) appear due to scale, internal $O(3)$ and translational symmetry and some of the associated zero-frequency modes are to be treated according to the collective coordinate method. The eigenvalues do not depend on these parameters and then it is sufficient to consider the eigenvalue problem corresponding to the simplest configuration $\omega(z) = z$.

Introducing the vector notation

$$\vec{\phi}(x) = (\phi_+(x), \phi_-(x), \phi_3(x)) \quad (2.2)$$

$$\vec{\phi}_\pm = \phi_1 \pm i \phi_2$$

the pseudoparticle solution reads

$$\vec{\phi}_{(1)}(x) = \left(\frac{2z}{1 + |z|^2}, \frac{2\bar{z}}{1 + |z|^2}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \quad (2.3)$$

and furthermore it is useful to change variables from (x_1, x_2) to (η, α) which are given by:

$$\eta = \frac{|z|^2 - 1}{|z|^2 + 1}, \quad \eta \in (-1, +1)$$

$$\alpha = \operatorname{tg}^{-1} \left(\frac{x_2}{x_1} \right), \quad \alpha \in (0, 2\pi) \quad (2.4)$$

Then the operator Δ^2 appearing in the quadratic action (1.9) reads

$$\Delta^2 = - \left[\frac{1}{2} (\partial_{\phi(1)}^2)^2 \right]^{-1} \partial^2 - 2 = -L^2 - 2 \quad (2.5)$$

where

$$L^2 = (1 - \eta^2) \frac{\partial^2}{\partial \eta^2} - 2\eta \frac{\partial}{\partial \eta} + \frac{1}{1 - \eta^2} \frac{\partial^2}{\partial \alpha^2} \quad (2.6)$$

represents the Laplacian on the sphere.

Our problem is now to solve the corresponding eigenvalue problem

$$\Delta^2 \hat{\psi}_n(x) = E_n \hat{\psi}_n(x) \quad (2.4)$$

but with the subsidiary condition

$$\hat{\psi}_{(1)} \cdot \hat{\psi}_n = 0 \quad (2.5)$$

which is present in the path integral (1.7). To satisfy this constraint we use two vectors $\hat{e}_+(x)$ and $\hat{e}_-(x)$ which are orthogonal to $\hat{\psi}_{(1)}(x)$. They are readily found from the translational zero-frequency wavefunctions

$$\hat{f}_{0,1}(x) = \hat{e}_1 \hat{\psi}_{(1)}(x) \quad (2.9)$$

through the definition

$$\hat{e}_{\pm} = e^{\mp i\alpha} (\hat{f}_{0,1} \pm i \hat{f}_{0,2}) \quad (2.10)$$

The explicit form of these vectors are

$$\hat{e}_{\pm}(n, \alpha) = (n-1) \left[(n+1)e^{i\alpha}, (n+1)e^{-i\alpha}, -(1-\eta^2)^{\frac{1}{2}} \right] \quad (2.11)$$

and we now search for the solution of the eigenvalue problem in the form

$$\hat{\psi}_n(\pm) = \hat{e}_{\pm} \hat{\phi}_n(\pm) \quad (2.12)$$

But, one can easily see that there exists no nontrivial solution to eigenequation (2.7) which takes this form. However, since in the path integral there is present a subsidiary condition

$$\hat{\psi}_{(1)}(x) \cdot \hat{\psi}(x) = 0 \quad (2.13)$$

we are free to consider the modified eigenvalue problem

$$\Delta^2 \hat{\psi}_n(x) = E_n \hat{\psi}_n(x) + c_n(x) \hat{\psi}_{(1)}(x) \quad (2.14)$$

which allows solutions of the form (2.12). Here c_n 's are certain Lagrange multipliers and in what follows we shall not be interested in explicit form of these functions. Therefore we systematically ignore in the eigenvalue problem terms proportional to $\hat{\psi}_{(1)}$.

Now using the identities

$$\begin{aligned} \partial_{\pm} \vec{f}_{0,\pm} &= (\partial \vec{\phi}_{(1)})^2 \vec{\phi}_{(1)} \\ \partial_{+} \vec{f}_{0,+} &= \frac{-4z}{1+|z|^2} \vec{f}_{0,+} \\ \partial_{-} \vec{f}_{0,-} &= \frac{-4\bar{z}}{1+|z|^2} \vec{f}_{0,-} \end{aligned} \quad (2.15)$$

we derive from (2.14) the eigenvalue equation satisfied for example by $h_n = \frac{1}{|z|} \vec{\phi}_n^{(+)}$ which reads

$$\left[L^2 - 2n \left\{ \frac{\partial}{\partial n} - \frac{1}{1-n^2} \frac{\partial}{\partial \alpha} \right\} + E_n \right] h_n(n, \alpha) = 0 \quad (2.16)$$

Furthermore, separating variables and denoting

$$h_n(n, \alpha) = (1-n^2)^{-\frac{1}{2}} e^{ima} b_{nm}(n) \quad (2.17)$$

we arrive at the equation

$$\left[L^2 - \frac{1+2mn}{1-n^2} + E_n + 2 \right] b_{nm} = 0 \quad (2.18)$$

The solution of this equation is given in terms of associated Legendre functions i.e.

$$\begin{aligned} b_{jm}(n) &= P_{m,-1}^j(n) \\ E_j &= j(j+1) - 2 \\ j &= 1, 2, 3, \dots \end{aligned} \quad (2.19)$$

To summarize, the complete solution of the original

eigenvalue problem with a constraint is given by

$$\vec{\phi}_{jm}^{(+)} = \vec{\phi}_{\pm}(n, \alpha) (1-n)^{-\frac{1}{2}} e^{ima} P_{m,-1}^j(n) \quad (2.20)$$

$$E_j = j(j+1) - 2$$

and the degeneracy is equal to $\delta_j = 2(2j+1)$. We mention that one can rewrite this wavefunction in the form

$$\vec{\phi}_{jm}^{(\pm)} = \vec{\phi}_{\pm} \vec{k}_{\pm} Y_{jm}(\theta, \alpha) \quad (2.21)$$

with $n = \cos \theta$ and

$$\vec{k}_{\pm} = \left(\frac{1+n}{1-n} \right)^{\frac{1}{2}} \left[\frac{\partial}{\partial n} \mp \frac{1}{1-n^2} \frac{\partial}{\partial \alpha} \right] \quad (2.22)$$

This form can equivalently be reached by employing the O(3) formalism, here we used this more direct approach having in mind generalization to multi-pseudoparticle configurations in which case the O(3) formalism is not of much use.

Finally, we list the six zero frequency modes obtained for $j = 1$:

$$\begin{aligned} \vec{\phi}_{1,1}^{(+)} &= \vec{f}_{0,+} & \vec{\phi}_{1,1}^{(-)} &= z^2 \vec{f}_{0,-} \\ \vec{\phi}_{1,0}^{(+)} &= \bar{z} \vec{f}_{0,+} & \vec{\phi}_{1,0}^{(-)} &= z \vec{f}_{0,-} \\ \vec{\phi}_{1,-1}^{(+)} &= \bar{z}^2 \vec{f}_{0,+} & \vec{\phi}_{1,-1}^{(-)} &= \vec{f}_{0,-} \end{aligned} \quad (2.23)$$

It is easy to see that $\vec{\psi}_{1,0}^{(+)}$ represent the zero modes corresponding to scale transformation and internal rotation about the 3-axis, that $\vec{\psi}_{1,1}^{(+)}$, $\vec{\psi}_{11}^{(-)}$ correspond to rotations about the 1 and 2-axis and $\vec{\psi}_{11}^{(+)}$, $\vec{\psi}_{1-1}^{(-)}$ are the translational zero frequency modes.

III. COMPUTATION OF THE QUANTUM CORRECTION

Based on solution to the small oscillation problem given in Sec. II, we will now explicitly compute the first quantum correction due to a pseudoparticle. We are interested in the ratio of the one-pseudoparticle transition amplitude

$$\left[\mathcal{D} \vec{\phi} \prod_x \delta \left(\vec{\phi}_{(1)}(x) - \vec{\phi}(x) \right) \exp \left[-\frac{1}{2} \langle \vec{\phi}, \mathcal{M}^2 \vec{\phi} \rangle \right] \right] \quad (3.1)$$

where

$$\begin{aligned} \langle \vec{\phi}, \mathcal{M}^2 \vec{\phi} \rangle &= \int d\mu(x) \vec{\phi} \cdot \mathcal{M}^2 \vec{\phi} \\ d\mu(x) &\equiv \frac{1}{2} (\partial \vec{\phi}_{(1)}(x))^2 d^2x = \left[\frac{2}{1+r^2} \right] d^2x \end{aligned} \quad (3.2)$$

and the ordinary vacuum transition amplitude

$$\left[\mathcal{D} \vec{\phi} \prod_x \delta(\vec{v} \cdot \vec{\phi}(x)) \exp \left[-\frac{1}{2} \langle \vec{\phi}, \mathcal{M}_0^2 \vec{\phi} \rangle \right] \right] \quad (3.3)$$

Here $\vec{v} = (0,0,1)$ represents the classical vacuum and in the action we have

$$\mathcal{M}_0^2 \equiv - \left[\frac{1}{2} (\partial \vec{\phi}_{(1)})^2 \right]^{-1} \partial^2 \quad (3.4)$$

The basis functions which we will use to expand the field $\vec{\phi}$ are orthonormal with the scalar product

$$(\vec{\phi}_1, \vec{\phi}_2) = \int d\mu(x) \vec{\phi}_1(x) \cdot \vec{\phi}_2(x) \quad (3.5)$$

i.e. with an integration measure different from d^2x .

Now as the first step we separate the contribution to (3.1) and (3.3) which is due to the zero-frequency modes. In the vacuum case there are two zero modes corresponding to internal rotations about the first and second axis. These modes lead to the well-known infrared divergence, characteristic of Goldstone bosons and to control this divergence one may introduce a small mass ϵ^2 . In the pseudoparticle case we have six zero modes written down explicitly in the previous section. Two of these modes lead to similar infrared divergence as in the vacuum case and their contribution cancels out in the ratio. The remaining four modes do not introduce additional divergences since they are to be treated according to the collective coordinate method. These eigenfunctions are given by

$$\begin{aligned}\vec{\psi}_{01}(x) &= \frac{1}{\sqrt{N_1}} \frac{\partial}{\partial a_1} \vec{\phi}_{(1)}(x; \vec{a}) \Big|_{\vec{a}=0} \\ N_1 &= \int d\mu(x) \left(\frac{\partial}{\partial a_1} \vec{\phi}_{(1)} \right)^2\end{aligned}\quad (3.6)$$

where a_i ; $i=1,2,3,4$ denote symmetry parameters corresponding to the translational, scale and $O(2)$ invariance. Then using the collective coordinate method of ref. (9) we derive instead of (3.1) the expression

$$I = \int \prod_{i=1}^4 da_i \int \prod_x d\vec{\phi} \prod_{i=1}^4 \delta(\vec{\phi}_{(i)} \cdot \vec{\phi}) \prod_{i=1}^4 N_i^{\frac{1}{2}} \delta\left(\int d\mu \vec{\psi}_{01} \cdot \vec{\phi}\right) e^{-\frac{1}{2}(\vec{\phi}, \mathcal{M}^2 \vec{\phi})} \quad (3.7)$$

Next we expand the field in terms of the orthonormal set of eigenfunctions found in Sec. II

$$\vec{\phi}(x) = \sum_n C_n \vec{\psi}_n(x) \quad (3.8)$$

and obtain for (3.7) the answer

$$I = J^{-1} \int \prod_{i=1}^4 da_i \left(\frac{N_i}{2} \right)^{\frac{1}{2}} \Pi \quad (3.9)$$

Here Π represents the contribution of all the nonzero modes and from Gaussian integrations we have

$$\Pi = \left(\prod_{\substack{n \\ E_n \neq 0}} E_n \right)^{-\frac{1}{2}} \quad (3.10)$$

J is given by

$$J = \left[\det \left(\int d^2x \vec{\psi}_n \cdot \vec{\psi}_m \right) \right]^{\frac{1}{2}} = \left[\det \left(\int d\mu(x) \vec{\psi}_n \cdot \left[\frac{1}{2} (\partial \vec{\phi}_{(1)})^2 \right]^{-1} \cdot \vec{\psi}_m \right) \right]^{\frac{1}{2}} \quad (3.11)$$

and comes as a Jacobian in a change of variables from $\vec{\phi}(x)$ to $\{C_n\}$ i.e.

$$\int \mathcal{D}\vec{\phi} = \int \prod_x d\vec{\phi}(x) = J \int \prod_n dC_n \quad (3.12)$$

For the vacuum case one derives an equivalent expression to (3.9) but now the Jacobian reads

$$J_0 = \left[\det \left(\int d\mu(x) \vec{\psi}_n^0 \left[\frac{1}{2} (\partial \vec{\phi}_{(1)})^2 \right]^{-1} \cdot \vec{\psi}_m^0 \right) \right]^{\frac{1}{2}} \quad (3.13)$$

We see that both J and J_0 represent determinants of the same operator but in different subspaces of the full Hilbert space. Namely in the pseudoparticle case we have

$$\vec{\phi}_n(x) \cdot \vec{\phi}_{(1)} = 0 \quad (3.14)$$

but for the vacuum

$$\vec{\phi}_n^0(x) \cdot \vec{v} = 0 \quad (3.15)$$

However, $\vec{\phi}_{(1)}(x)$ and $\vec{v}$ are related by a space dependent rotation and consequently one is able to show that $J = J_0$. Therefore these two Jacobians cancel out in the ratio.

After these preliminaries we proceed to explicitly evaluate the contribution of nonzero modes and consider

$$\log \left(\frac{J}{J_0} \right) = -\frac{1}{2} \left[\sum_{j=2}^{\infty} \delta_j E_j - \sum_{j=1}^{\infty} \delta_j E_j^0 \right] \quad (3.16)$$

where the eigenvalues and their degeneracies read

$$\begin{aligned} E_j &= j(j+1) - 2 \\ E_j^0 &= j(j+1) \\ \delta_j &= 2(2j+1) \end{aligned} \quad (3.17)$$

The expression (3.16) as it stands contains an ultraviolet divergence which is to be cancelled by the contribution coming from the renormalization counterterm. Therefore it is necessary to regularize and following 't Hooft (6) we employ Pauli-Villars regulators $\{M_i; i=0,1,\dots,R\}$ with alternating metric e_i such that

$$e_0 = 1, \quad M_0 = 0$$

$$\sum_{i=0}^R e_i = 0, \quad \sum_{i=0}^R e_i M_i = 0, \dots \quad (3.18)$$

and

$$\sum_{i=1}^R e_i \ln M_i = -\ln M \quad (3.19)$$

being finite.

Then taking into account the contribution of regulator fields (3.16) becomes

$$\ln \left(\frac{J}{J_0} \right)_{\text{reg}} = -2 \sum_{i=0}^R e_i \left[A^{M_i}(3/2) - A^{M_i}(1/2) \right] \quad (3.20)$$

where we have denoted

$$A^{M_i}(a) = \sum_{s=1+a}^{\Lambda} s \ln(s^2 - a^2 + M_i^2) \quad (3.21)$$

and the cutoff Λ is to be taken to infinity. Now for the case $M_i \neq 0$ we have by the Euler-Mclaurin formula

$$\begin{aligned} A^{M_i}(a) &= \int_a^{\Lambda} dx \, x \ln(x^2 - a^2 + M_i^2) + \frac{1}{2} \left[x \ln(x^2 - a^2 + M_i^2) \right] \Big|_a^{\Lambda} \\ &\quad + \frac{1}{12} \left[x \ln(x^2 - a^2 + M_i^2) \right]' \Big|_a^{\Lambda} \end{aligned} \quad (3.22)$$

and after some calculation obtain

$$A^{M_i}(a) = -\frac{1}{2} (a^2 \ln a + a \ln M_i^2) + O\left(\frac{1}{\Lambda}\right) + O\left(\frac{1}{M_i^2}\right) \quad (3.23)$$

For the case $l=0$ i.e. when $M_0=0$ we write

$$\begin{aligned} \Lambda^0(a) &= \sum_{s=1}^{\Lambda+a} (s-a) \ln s + \sum_{s=1}^{\Lambda-a} (s+a) \ln s \\ &= \sum_{s=1}^{2a} (a-s) \ln s + \sum_{s=1}^{\Lambda-\frac{1}{2}} 2s \ln s + \sum_{s=\frac{1}{2}+\Lambda}^{\Lambda+a} (s-a) \ln s \\ &\quad - \sum_{s=\frac{1}{2}+\Lambda}^{\Lambda-\frac{1}{2}} (s+a) \ln s \end{aligned} \quad (3.24)$$

for which using again the Euler-McLaurin formula we obtain

$$\Lambda^0(a) = \sum_{s=1}^{2a} (a-s) \ln s + a^2(1 - \ln \Lambda) + O\left(\frac{1}{\Lambda}\right) \quad (3.25)$$

Therefore using (3.23) and (3.25) find for (3.20) the result

$$\ln \left(\frac{\pi}{\mu_0} \right)_{\text{reg}} = -2 \left[2 + \sum_{s=2}^3 \left(\frac{3}{2} - s \right) \ln s + \ln M \right] \quad (3.26)$$

Now due to the fact that in the above calculation we used a regulator with a space dependent mass given by

$$\mu^2(x) = \frac{1}{2} (\partial \phi_{(1)}(x))^2 M^2 = \frac{4M^2}{(1+r^2)^2} \quad (3.27)$$

there exists an additional contribution coming from a finite renormalization effect since we like to define the renormalization procedure with a standard fixed mass regulator μ_0 . For

the nonlinear σ -model in two dimensions the one loop renormalization counterterm has been derived by Polyakov in ref. (10) for purpose of demonstrating the asymptotic freedom of this theory. Using this result we have for the finite counterterm the following expression

$$\begin{aligned} -\Delta A &= \int d^2x \frac{1}{2} (\partial \phi_{(1)}(x))^2 \left[\frac{1}{2\pi} \ln \left(\frac{\mu(x)}{\mu_0} \right) \right] \\ &= \int_0^\infty r dr \frac{1}{(1+r^2)^2} \ln \left[\frac{2M}{\mu_0} \frac{1}{1+r^2} \right] \\ &= 2 \left[\ln \frac{2M}{\mu_0} - 1 \right] \end{aligned} \quad (3.28)$$

Furthermore there is the fact that instead of zero-frequency modes (six for the pseudoparticle and two for the vacuum case) in case of the regulator fields we have the corresponding modes with nonzero eigenvalues. Consequently these modes give an additional contribution equal to μ_0^4 . Therefore taking together (3.26), (3.28) and this factor we obtain for contribution of nonzero modes the expression

$$\ln \left(\frac{\pi}{\mu_0} \right) = \ln \mu_0^2 - 2 \left[3 - \ln 2 + \sum_{s=2}^3 \left(\frac{3}{2} - s \right) \ln s \right] \quad (3.29)$$

Next evaluating the norms of zero-frequency modes which are present in exp.(3.9) and taking into account a factor of 2π coming from the $O(2)$ group integration we find the following total answer

$$4\pi^{-2} \left(\frac{8\pi}{3}\right)^{3/2} \int \frac{d^2 z_0 d\lambda}{\lambda^3} \exp \left\{ -\frac{4\pi}{i\psi_0^2} + \ln(\psi_0^2 \lambda) - 6 \left[1 - \frac{1}{2} \sum_{s=2}^3 \ln s \right] \right\} \quad (3.30)$$

for the contribution of a pseudoparticle at the one loop level. Obviously we have in the exponent the correct renormalization group behavior, namely the one which appears in ordinary perturbation expansion.

IV. CASE OF TWO PSEUDOPARTICLES

We now consider small oscillations about the two pseudoparticle classical solution, which in its most general form reads

$$w(z) = \frac{e^{i\alpha_0}}{g} \frac{(z - z_1)(z - z_2)}{(z - z_3)(z - z_4)} \quad (4.1)$$

Here of the ten independent real parameters eight correspond to scale, translational, conformal (i.e. $1/z \rightarrow 1/z + c$) and internal $O(3)$ invariance of the theory. Since the eigenvalues of small oscillations do not depend on these symmetry parameters we can consider the eigenvalue problem corresponding to the simpler configuration

$$w(z) = (z - a)(z + a) \quad (4.2)$$

$$a = a_1 + i a_2$$

which still contains two arbitrary real parameters (a_1, a_2) representing the separation of pseudoparticles.^[2] The action does not depend on this separation and the pseudoparticles classically do not interact.

However, at the quantum level the eigenvalues of small oscillations about this configuration can still be dependent on distance $|a|$ between the pseudoparticles and in fact by solving the eigenproblem in two limiting cases $|a| = 0$ and $|a| = \infty$ we shall explicitly exhibit this dependence. This then demonstrates how interaction between pseudoparticles arises due to quantum effects.

Therefore, the eigenvalue problem we consider reads

$$\nabla^2 \vec{\psi}_n(x) = \left[-\left(\frac{1}{2} (\vec{\partial}_{(2)})^2 \right)^{-1} \partial^2 - 2 \right] \vec{\psi}_n = E_n \vec{\psi}_n \quad (4.3)$$

and the constraint is

$$\vec{\phi}_{(2)}(x) \cdot \vec{\psi}_n(x) = 0 \quad (4.4)$$

where

$$\vec{\phi}_{(2)}(x) = \left[\frac{2w(z)}{1 + |w(z)|^2}, \frac{2\bar{w}(z)}{1 + |w(z)|^2}, \frac{|w(z)|^2 - 1}{|w(z)|^2 + 1} \right]$$

In comparison with the one pseudoparticle case here $w(z) = z^2 - A$ takes the place of z . One then immediately realizes that a change of variables from (x_1, x_2) to (y, α) given by

$$y = \frac{|w(z)|^2 - 1}{|w(z)|^2 + 1} \quad (4.5)$$

$$\alpha = \arg w(z)$$

can be useful. However, the difficulty is that this mapping does not have a unique inverse and therefore special care must be exercised if we nevertheless want to use it. Namely, we must assure that it is performed only on functions for which the inverse uniquely exists. This is done as follows:

First of all to satisfy the subsidiary condition (4.4) we make use of the zero-frequency modes $\vec{F}_\pm$ corresponding to change of separation between the pseudoparticles

$$\vec{F}_\pm = D_\pm \vec{\phi}_{(2)} \quad (4.6)$$

where $D_+ = 2\partial/\partial\bar{w}$ and $D_- = 2\partial/\partial w$. Now we immediately realize that there exists a series of eigenfunctions identical to those of the one-pseudoparticle case which read

$$e^{i\alpha} \vec{F}_\pm(y, \alpha) (1-y)^{-1} e^{im\alpha} p_{m, \mp 1}^j(y)$$

and are continuous functions of (x_1, x_2) . This constitutes the first branch of the spectrum with the eigenvalues

$$E_j^{(1)} = j(j+1) - 2 \quad (4.7)$$

$$j = 1, 2, \dots$$

which are independent of the distance between the pseudoparticles. The degeneracy is $2(2j+1)$.

But in the present case there also exists an additional series of eigenfunctions. The corresponding eigenvalues depend in a highly nontrivial fashion on the separation of pseudoparticles. To generate this second branch of the spectrum we use the transitional modes which are given by

$$\begin{aligned} \vec{F}_{T, \pm} &= \frac{1}{2} \partial_\pm \vec{\phi}_{(2)}(x) \\ \vec{F}_{T, +} &= \bar{z} \vec{F}_+ \\ \vec{F}_{T, -} &= z \vec{F}_- \end{aligned} \quad (4.8)$$

Then we search for solution^[3] of the eigenvalue problem in the form

$$\begin{aligned} \vec{\phi}_n^{(+)} &= \bar{z} \vec{F}_+ H_n^{(+)} \\ \vec{\phi}_n^{(-)} &= z \vec{F}_- H_n^{(-)} \end{aligned} \quad (4.9)$$

Now we use the identities satisfied by $P_{\pm}$ (they are analogous to those given by Eq. (2.15)) to derive from (4.13) the eigenequations for $H_n^{(\pm)}$. Let us concentrate for example on the equation obtained for $H_n^{(+)}$ which reads [5]

$$\left[L^2 - (1+2y) \left(\frac{\partial}{\partial y} - \frac{1}{1-y^2} \frac{\partial}{\partial \alpha} \right) + E_n^{(2)} \right] H_n^{(+)}(y, \alpha) - \frac{(1-y)^{\frac{1}{2}}}{(1-y)^{\frac{1}{2}} + \frac{(1+y)^{\frac{1}{2}}}{A} e^{-i\alpha}} \left(\frac{\partial}{\partial y} - \frac{1}{1-y^2} \frac{\partial}{\partial \alpha} \right) H_n^{(+)}(y, \alpha) = 0 \quad (4.10)$$

A presence of the A -dependent term in this equation signals that this second series of eigenvalues depends nontrivially on distance between the pseudoparticles i.e. $E_n^{(2)} = E_n^{(2)}(A)$. Due to the complicated form of the interaction in (4.10) we are not able to solve exactly this eigenproblem. We can however solve the limiting cases of infinite and zero separations.

a) Infinite Separation ($A = \infty$)

Letting the limit $A \rightarrow \infty$ in (4.10) we obtain the equation

$$\left[L^2 - 2(1+y) \left(\frac{\partial}{\partial y} - \frac{1}{1-y^2} \frac{\partial}{\partial \alpha} \right) + E_n^{(2)}(\infty) \right] H_n^{(+)} = 0 \quad (4.11)$$

which is solved by

$$H_{jm}^{(+)} = e^{i(m-1)\alpha} (1-y)^{\frac{|m|-3}{2}} (1+y)^{\frac{|m|+1}{2}} p_{j-|m|}^{(|m|-1, |m|+1)}(y) \quad (4.12)$$

$$j = 1, 2, \dots$$

$$m = -j, \dots, j$$

where $P_n^{(a,b)}$ represent standard Jacobi polynomials. [11] A similar solution is obtained for $H_n^{(-)}$ also.

For the eigenvalues of this second branch we find the same result as for the first branch, i.e. $E_j^{(2)}(\infty) = j(j+1) - 2$ and with the same degeneracy equal to $2(2j+1)$. Therefore in comparison with the one-pseudoparticle case we see that for two infinitely separated pseudoparticles we obtain precisely the same eigenvalues but with a doubled degeneracy. This was to be expected.

b) Zero Separation ($A = 0$)

In this case the eigenequation (4.10) reduces to

$$\left[L^2 - (1+2y) \left(\frac{\partial}{\partial y} - \frac{1}{1-y^2} \frac{\partial}{\partial \alpha} \right) + E_n^{(2)}(0) \right] H_n^{(+)} = 0 \quad (4.13)$$

The general solution to this equation can be shown to read

$$H_{jm} = e^{i(m-\frac{1}{2})\alpha} (1-y)^{\frac{|m|+1}{2} - \frac{3}{4}} (1+y)^{\frac{|m|+1}{2} - \frac{1}{4}} p_{j-|m|}^{(|m|+1, |m|+1)}(y) \quad (4.14)$$

$$E_j^{(2)} = j(j+1) - 2, \quad j = 3/2, 5/2, \dots$$

$$3/2 \leq |m| \leq j$$

The reason why we had to exclude the values $m = \pm 1/2$ is due to the fact that for this case the eigenfunctions (4.14) blow up when $y \rightarrow \pm 1$. So for $m = \pm 1/2$ we derive the solution separately and obtain

$$H_{j, \pm \frac{1}{2}} = e^{-i(\frac{1}{2} \mp \frac{1}{2})} P_{j-\frac{1}{2}}^{(\frac{3}{2}, \frac{1}{2})}(\pm y) \quad (4.15)$$

$$E_{j, \frac{1}{2}} = (j + \frac{1}{2})(j + \frac{3}{2}) - 2, \quad j = \frac{1}{2}, \frac{3}{2}, \dots$$

The reader should observe a change in the form of the eigenvalues in comparison with (4.14).

Therefore in case of zero separation between pseudoparticles the eigenvalues of the second branch appear to be much different than for infinite separation. First half integer j 's appear and furthermore the eigenvalues even depend on m 's. To summarize the results for the second branch we have:

$$\begin{aligned} E_{jm}^{(2)}(0) &= j(j+1) - 2, \quad \frac{3}{2} \leq |m| \leq j \\ E_{jm}^{(2)}(0) &= (j + \frac{1}{2})(j + \frac{3}{2}) - 2, \quad |m| = \frac{1}{2} \end{aligned} \quad (4.16)$$

Now at the classical level we have the fact that the action of the two-pseudoparticle solution equals precisely the doubled action for one and so pseudoparticles classically do not interact.^[4] However as we can see from the partial solution to the eigenvalue problem given above the eigenvalues of small oscillations do depend on distance between the pseudoparticles and an interaction energy arises due to quantum effects. One can compute this interaction potential by perturbation expansion in A or $\frac{1}{A}$ using the eigenfunctions and eigenvalues of this section. However, this will not be done in the present paper.

For completeness we now proceed to compute the quantum correction for the case $A=0$ and demonstrate explicitly that the correct renormalization group dependence on the coupling constant

and subtraction point arises. Analogously to the one-pseudo-particle computation given in Section III we now consider

$$\ln \Pi = -\frac{1}{2} \left[\sum_{j=2}^{\infty} \left(2(2j+1) + 4 \right) E_j + \sum_{j=3/2}^{\infty} \left(2(2j+1) - 4 \right) E_j \right] \quad (4.17)$$

and

$$\ln \Pi_0 = -\frac{1}{2} \left[\sum_{j=1}^{\infty} 2(2j+1) E_j^0 + \sum_{j=\frac{1}{2}}^{\infty} 2(2j+1) E_j^0 \right] \quad (4.18)$$

where

$$\begin{aligned} E_j &= j(j+1) - 2 \\ E_j^0 &= j(j+1) \end{aligned} \quad (4.19)$$

Introducing again Pauli-Villars regulators we have the regulated expression

$$\ln \left(\frac{\Pi}{\Pi_0} \right)_{\text{reg}} = (-2) \sum_{i=0}^R e_i \left[A^{M_i(a)} + B^{M_i(a)} \right] \Big|_{a=1/2}^{a=3/2} \quad (4.20)$$

with

$$\begin{aligned} A^{M_i(a)} &= \sum_{s=1+a}^{\Lambda} (s+a-\frac{1}{2}) \ln (s^2 + M_i^2 - a^2) \\ B^{M_i(a)} &= \sum_{s=1+a-\frac{1}{2}}^{\Lambda-\frac{1}{2}} (s-a+\frac{1}{2}) \ln (s^2 + M_i^2 - a^2) \end{aligned} \quad (4.21)$$

After a long calculation we obtain the result

$$\begin{aligned} \sum_{i=0}^R e_i \left[A^{M_i(a)} + B^{M_i(a)} \right] &= 2a \ln M + 2a^2 + \left(\frac{3}{2} - a \right) \sum_{s=1}^{2a} \ln s + \\ &+ \left(\frac{3}{2} + a \right) \sum_{s=\frac{1}{2}}^{2a-\frac{1}{2}} \ln s + 2a \left[\sum_{s=1}^{\Lambda} \ln s - \sum_{s=\frac{1}{2}}^{\Lambda-\frac{1}{2}} \ln s - \frac{1}{2} \ln \Lambda \right] \end{aligned} \quad (4.22)$$

The last term in this expression is finite in the limit $\Lambda \rightarrow \infty$ and we obtain for the regulated contribution of non-zero modes the result

$$\ln \left(\frac{\mu}{\mu_0} \right)_{\text{reg}}^{A=0} = -4 \ln M - \ln 3 + 8 \ln 2 - 6 - 4 \sum_{n=1}^{\infty} \frac{B_n}{2n(2n-1)} \left[\left(\frac{2}{3} \right)^{2n-1} - 1 \right] \quad (4.23)$$

Furthermore the finite counterterm in this case reads

$$-\Delta A = 4 \left(\ln \frac{2M}{\mu_0} - 1 \right) \quad (4.24)$$

and instead of the ten zero-frequency modes in the regulator field case we have nonzero eigenvalues giving a factor μ_0^{10} and also μ_0^2 from the vacuum

Therefore taking all these contributions together we have the following dependence on the coupling constant and the subtraction point μ_0

$$\mu_0^8 \exp \left\{ -\frac{8\pi}{f(\mu_0^2)} - 4 \ln \mu_0 \right\} = \exp \left\{ -\frac{8\pi}{f(\mu_0^2)} + 22 \ln \mu_0^2 \right\} \quad (4.25)$$

This is as it should be

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FOOTNOTES

- [1] Note that this solution satisfies different boundary conditions than those given by (1.3). It is obtained by making internal rotations about the first and second axis and therefore contains two additional symmetry parameters.
- [2] One can understand the origin of these two arbitrary parameters by recalling the well-known fact that massless field theories in two dimensions possess an infinite parameter symmetry group consisting of arbitrary conformal transformations. However, unlike in the case of some simple symmetries (like translations, rotations, etc.) for general conformal transformations one cannot show that the eigenvalues of the small oscillation problem remain invariant. That is the reason why we cannot simply consider the easiest eigenvalue problem corresponding to the configuration $u(z) = z^2$.
- [3] We mention that it is trivial to generalize this method to the n-pseudoparticle case. Namely, then one needs to use a set of n vectors $(\vec{F}_+, \vec{z}, \vec{F}_+, \dots, \vec{z}^{n-1}, \vec{F}_+)$ in order to generate the spectrum which now has n branches.
- [4] At this point it is worth mentioning the possibility that the field given by (4.1) does not represent a most general configuration of two pseudoparticles. Namely, in (4.1) we have only ten parameters while the most general one-pseudoparticle solution given by (2.1) had six. A general configuration for two pseudoparticles which would have twelve

independent parameters cannot be a solution to the equations of motion and would give an interaction at the classical level. It can however appear as the solution of equations of motion with constraints.

[5] Here we choose A to be real.

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