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DYNAMIC LEVITATION OF CHARGED LIQUID DROPLETS

Research Report #3

July 1976

by

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prepared under
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Target Research and Fabrication Group
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DYNAMIC LEVITATION OF CHARGED LIQUID DROPLETS

I. Introduction

The intricacy and required fabrication tolerances of proposed laser fusion targets are generally appreciated. Among the various techniques potentially useful during certain steps of the fabrication procedure is levitation or remote positioning of spherical mandrels, partially completed targets, etc. Considered here is the electrostatic levitation of charged droplets of various insulating liquids potentially useful in target production. Among the liquids are cryogenic hydrogen and its isotopes, molten plastics or glass, and liquid dielectrics.

The levitation scheme considered here is the electrostatic strong-focusing concept from quadrupole mass filter spectrometry. Wuerker, Sheldon and Langmuir [1,2] have successfully levitated and contained solid particles of aluminum, etc. up to $\sim 20 \mu\text{m}$ in diameter with charge-to-mass ratios of $\sim 5 \cdot 10^{-3}$ Coul./kg. using quadrupole electrode structures and relatively modest voltages. Owe Berg, et al. have used essentially the same scheme to levitate charged water droplets [3,4]. Two basic versions of the dynamic confinement structure have been developed: the cylindrically symmetric (three-dimensional) quadrupole which uses single phase ac voltage and the cubic-face multipole which uses three-phase excitation. For particle confinement applications, the primary interest is in dynamically stabilized levitation of the charged particles rather than the highly selective mass filtering capability for which these systems are noted. Hence, there is no discussion of mass filtering in this report.

The distinguishing aspect of electrostatic particle levitation considered here is the levitation of electrically insulating low surface tension liquid droplets. The required conditions for dynamic stabilization of charged droplets are in every way the same as those for solid particles except for the limitations imposed by the electrohydrodynamics of the charged droplets. Two effects, which may couple together, influence the static equilibrium and the stability of the charged liquid droplets. These effects are (i) static elongation and instability of liquid droplets under the influence of an essentially uniform electric field [5], and (ii) the stability of charged liquid droplets (the Rayleigh limit) [6,7]. In this report, the influences of static elongation and charged droplet stability on electrostatic dynamic stabilization and levitation are explored. Based on this analysis, the feasibility of levitating charged droplets of low surface tension liquids such as cryogenic hydrogen for laser fusion target research and fabrication is examined.

II. Three Dimensional Quadrupole Levitation

The more familiar type of quadrupole is the axisymmetric ion trap. This system has been used to levitate charged particles [1], charged water droplets[3,4], etc. By suitable voltage adjustments, this structure can be used to selectively trap ions or charged particles in a certain narrow range of charge-to-mass ratio values.

The system is shown in Figure 1. The electrode structure, shown in sectional view, consists of two end cap electrodes and a ring electrode. Ideally, the electrode surfaces are hyperbolic in shape, but fairly

Figure 1. Three-dimensional quadrupole electrode structure (shown in sectional view) with high voltage circuit. V_g is the gravity compensating potential.

crude approximation of these surfaces does not seriously impair performance, except for extremely high resolution mass filtering [8]. The circuit connections shown in the figure allow imposition of a combined ac/dc quadrupole field (V_{dc} and V_{ac}), plus a dc gravity-compensating field (V_g). In the analysis below, interest is directed to simple particle levitation and not mass filtering. Therefore, it is assumed that $V_{dc} = 0$.

A. Equations of Motion

The electrostatic potential solution for the axisymmetric quadrupole is

$$\phi = \frac{V_{ac}(t)}{2h^2} \left[z^2 - \frac{1}{2} r_c^2 \right], \quad (1)$$

where h is defined in Figure 1 and z, r_c are the cylindrical coordinates.

Since $\vec{E} = -\nabla\phi$,

$$\vec{E} = \frac{V_{ac}(t)}{h^2} \left[\frac{1}{2} r_c \hat{r}_c - z \hat{z} \right]. \quad (2)$$

Now consider the equations of motion for a particle of charge Q and mass m .

$$m \frac{d^2 \xi_{r_c}}{dt^2} = + \frac{QV_{ac}(t)}{2h^2} \xi_{r_c}, \quad (3)$$

$$m \frac{d^2 \xi_z}{dt^2} = - \frac{QV_{ac}(t)}{h^2} \xi_z, \quad (4)$$

where we neglect the influence of the compensating field on the dynamics.* Assume a sinusoidal excitation

$$V_{ac}(t) = V_0 \cos \omega t \quad (5)$$

and the Eq. (3) and (4) reduce to Mathieu's equation [9]

$$\frac{d^2 X_1}{d\tau^2} + (a_1 - 2q_1 \cos 2\tau) X_1 = 0 \quad (6)$$

where $\tau = \frac{1}{2}\omega t$ and

$$a_z = 0, \quad q_z = \frac{-2QV_0}{m\omega^2 h^2}, \quad (7)$$

$$a_{rc} = 0, \quad q_{rc} = + \frac{QV_0}{m\omega^2 h^2}. \quad (8)$$

We desire a condition where both $\xi_z(t)$ and $\xi_{rc}(t)$ are bounded in time.

These conditions are met by solutions to Mathieu's equation for the case of $a_z = a_{rc} = 0$ if $|q_z|$ and $|q_{rc}|$ are both less than .908. The more stringent of these two conditions is

$$|q_z| = \frac{2QV_0}{m\omega^2 h^2} < .908, \quad (9)$$

*The gravity compensating field only enters into the dynamics because of its non-uniformity, that is, $E_g = E_g(z)$. If $|V_g| \ll |V_{ac}(t)|$, which is commonly the case, then the compensating field can legitimately be neglected in the particle dynamics equations.

where, with no loss of generality, we assume that Q and V_0 are positive quantities. To ensure "tight" particle confinement, q_z should be as large as possible, subject to the limit of Eq. (9).

B. Gravity Compensation

The dynamic equations derived above assume that an equilibrium condition exists on the particle. With gravity present, the particle is pulled down somewhat so that the center of motion is not at the center of the electrode structure. If this displacement is small, the stability is not affected substantially, but the particle does develop a gravity dependent trajectory[1,4] which can be troublesome if precision control of the particle position is required. The gravity-compensating field E_g may be used to center the particle and eliminate unwanted gyrations. The condition is

$$QE_g \approx Mg \quad (10)$$

where $g = 9.81 \text{ m/sec}^2$ is the gravitational acceleration. Without the compensating field, the particle gyrations can limit successful particle levitation. With the gravity-compensating field, droplet elongation can occur.

C. Fundamental Electrostatic and Electrohydrodynamic Limits

1. The Rayleigh Limit

When a liquid droplet is charged, the electric field at the interface tends to destabilize the droplet. If the charge is sufficient, the stabilizing tendency of surface tension is overcome and the droplet

fissions. The marginal stability condition for a charged droplet of liquid in a vacuum is called the Rayleigh limit [6]:

$$Q < Q_{\max} = \pi \sqrt{8\epsilon_0 \Sigma D^3} \quad (11)$$

where

$$\epsilon_0 = 8.854 \cdot 10^{-12} \text{ F/m},$$

Σ = surface tension (N/m),

D = droplet diameter (meters).

Note that Eq. (11) puts a limit on the maximum charge-to-mass ratio in terms of the droplet diameter, surface tension, and density:

$$\frac{Q}{m} < \left(\frac{Q}{m}\right)_{\max} = \sqrt{\frac{288\epsilon_0 \Sigma}{\rho^2 D^3}}, \quad (12)$$

where ρ is the liquid density and $\frac{\pi}{6}\rho D^3$ is the particle mass m . Eq. (12) is useful in connection with the dynamic stability because of the choice of Q/m as a parameter.

2. Droplet Elongation and Stability

An uncharged liquid droplet, either conducting or insulating, elongates in the presence of an electric field[5]. The droplet elongates into a prolate spheroid with its major axis parallel to the electric field. See Figure 2 where the semi-major axis a and the semi-minor axis b are defined.

Droplets in the three-dimensional quadrupole levitator are subject to both the effects of the ac quadrupole field and the dc

Figure 2 Definition of the semi-major and semi-minor axes of liquid droplet deformed in an electric field

gravity-compensating field. It is, however, the essentially uniform gravity-compensating field $E_g(V_g)$ which will dominate in any droplet elongation. Since this electric field is not time-varying, charge relaxation will tend to exclude the electric field from the droplet interior. The droplet can therefore be modeled as perfectly conducting and incompressible. The relation of the elongation to the applied electric field is then [5]:

$$E_g = \left[\frac{2\Sigma}{\epsilon_0 D} \right]^{1/2} \left[L_z \right] \left[2\gamma^2 A \right]^{1/2} \quad (13)$$

where $\gamma = \frac{a}{b}$,

$$L_z = \frac{1}{3\gamma^2} \left[1 + \frac{3}{5}(1 - \gamma^{-2}) + \frac{3}{7}(1 - \gamma^{-2})^2 + \dots \right],$$

$$A = 2\gamma - 1 - \gamma^{-2}.$$

For small deformations ($\gamma - 1 < 0.1$), Eq. (13) reduces to [10]

$$\gamma = 1 + \frac{9\epsilon_0 D}{16\Sigma} E_g^2. \quad (14)$$

When a gravity-compensation field is used, then the elongation problem must be contended with. If some elongation of the target is tolerable, the elongation instability must still be avoided. The condition required to prevent the elongation instability is [5]:

$$E_g < E_{crit} = 2.15 \cdot 10^5 \left[\frac{\Sigma}{D} \right]^{1/2}. \quad (15)$$

Combination of Eq. (10) and (15) yields the inequality:

$$\frac{Q}{m} > \frac{g}{2.15 \cdot 10^5} \left[\frac{D}{\epsilon} \right]^{\frac{1}{2}}. \quad (16)$$

3. Breakdown Limit

The net electric field must not exceed the breakdown limit E_b (volts/meter) of the vapor or gaseous medium in which the droplet is supported. The strongest electric field will occur at the surface of the charged droplet, at the point where the field due to the droplet charge and the gravity compensating field add. Assuming a perfectly conducting droplet model for the static field E_g , the breakdown limit becomes:

$$\frac{Q}{\epsilon_0 \pi D^2} + 3E_g < E_b. \quad (17)$$

The first term on the left-hand-side of (17) is the charge produced field and the second term is the gravity compensating field, which is intensified at the surface of the droplet [11]. Using Eq. (10) for E_g , the inequality (17) becomes

$$\frac{cD}{6\epsilon_0} \left(\frac{Q}{m} \right) + 3g \left(\frac{Q}{m} \right)^{-1} < E_b. \quad (18)$$

The inequalities (12), (16) and (18) may be used to plot out limits on the charge-to-mass ratio and the droplet diameter D . Several such plots are shown in Figure 3, for various liquids. In addition to the breakdown, Rayleigh, and elongation instability limits, lines of

constant elongation deformation are shown. Note that if the gravity compensating field E_g is not employed, the elongation curves are not important. In Figure 3a, no vapor breakdown data for H_2 vapor at the triple point is available.

The important result indicated by the Figures 3a, b, c is that fundamental electrostatic and electrohydrodynamic phenomena impose limits on the maximum droplet diameter, even before the matter of levitation is considered. For example, if deformations greater than $\gamma = 1.01$ must be avoided for liquid H_2 droplets and if the gravity compensating field is required (as defined in Eq. (10)), then the maximum droplet diameter is $\sim 3000 \mu m$ ($\sim 3 mm$); whereas if only the elongation instability must be avoided, then the limit is $\sim 6000 \mu m$ ($\sim 6 mm$). Note for Figures 3a, b, c that if no gravity compensating field is imposed, no limit on droplet diameter other than the Rayleigh limit is imposed.

Where the Rayleigh limit and elongation instability curves intersect, the distinct likelihood of coupling exists. Such coupling has been neglected in the analysis presented here, but is likely to have some effect on the limiting curves of Figures 3a, b, c in the region where they intersect.

D. Feasibility Calculations

Parameters and conditions additional to those of the previous section must be utilized in order to consider the feasibility of levitating charged droplets. The condition imposed by Eq. (9) assures dynamic stability of the charged particle orbit. It is convenient here to reduce the inequality (9) to a relationship involving frequency $f = \frac{\omega}{2\pi}$, ac voltage V_0 , and charge-to-mass ratio Q/m ,

Figure 3a. Stability map for droplets of liquid H₂ @ Triple Point (conducting limit).

Figure 3b. Stability map for droplets of Freon-113[®] (conducting limit).

Figure 3c. Stability map for droplets of corn oil (conducting limit).

$$\frac{f^2}{V_o} > \frac{.0558}{h^2} \left(\frac{Q}{m} \right). \quad (19)$$

Another condition on these parameters is imposed if the gravity compensating field E_g is zero. In this case, the center of motion of the levitated particle drops down a distance z_o from the center [1,4]:

$$z_o = \frac{8g}{q_z^2 \omega^2}. \quad (20)$$

For successful levitation, $z_o \ll h$; otherwise, the charged particle strikes the lower electrode and is lost. This condition reduces to

$$\frac{f^2}{V_o^2} << \frac{.0127}{gh^3} \left(\frac{Q}{m} \right)^2. \quad (21)$$

The inequalities (19) and (21) combine to produce a limit on the frequency f

$$f > .50 \sqrt{\frac{g}{h}}. \quad (22)$$

The combined effect of the limits imposed by dynamic stabilization, Eq. (19), and the trajectory limit, Eq. (21), is shown in Figure 4 for various values of (Q/m) with $h = 0.5$ cm. The allowable range of voltages and frequencies is shown for various values of the charge-to-mass ratio. If a gravity compensating field is used, then (i) the inequality (21) does not apply, but (ii) an upper limit on (Q/m) , as illustrated in Figures 3a, b, c, does apply.

Figure 4. Frequency-voltage map for three-dimensional quadrupole ($h = 0.5$ cm).

III. Three Phase Multipole Levitation

An alternative to the three dimensional quadrupole electrode structure is a multipole system excited by three phase ac voltage. The structure considered here is that of Wuerker, Goldenberg and Langmuir [2]. For the injection and observation of large levitated particles, this electrode structure has certain advantages over the 3D quadrupole. The 3ϕ multipole structure has been chosen for use in the present experimental program with charged dielectric fluid droplets.

A. Equations of Motion

Derivation of the equations of motion for the levitation system involves solution for the electric field. The electrode structure is a cubic face arrangement with sides of length a . Opposite faces are connected to the same voltage phase. See Figure 5. The three phase voltages (line-to-neutral) are assumed to be

$$V_a(t) = V_o \cos \omega t, \quad (23a)$$

$$V_b(t) = V_o \cos \left(\omega t + \frac{2\pi}{3} \right), \quad (23b)$$

$$V_c(t) = V_o \cos \left(\omega t + \frac{4\pi}{3} \right). \quad (23c)$$

Wuerker, Goldenberg, and Langmuir use, without derivation, the following expression for the electrostatic potential [2]:

$$\phi = \frac{KV_o}{a^2} \left[z^2 \cos(\omega t) + y^2 \cos\left(\omega t + \frac{2\pi}{3}\right) + x^2 \cos\left(\omega t + \frac{4\pi}{3}\right) \right], \quad (24)$$

a) 3ϕ cubic face
multipole electrode

b) 3ϕ circuit including gravity
compensating potential V_g

Figure 5. 3ϕ cubic face multipole system.

where $K \approx 5.15$. This expression is only considered valid near the center of the cube at $x = y = z = 0$ where the particle is expected to reside. In Appendix A, the potential is derived exactly and then expanded about $x = y = z = 0$. The result is the same as Wuerker, Goldenburg, and Langmuir [2], that is Eq. (24), except $K = 4.2$. The electric fields near the center can be obtained by taking the gradient of the potential function:

$$E_x = - \frac{8.4V_0}{a^2} \cos(\omega t + \frac{4\pi}{3})x, \quad (25a)$$

$$E_y = - \frac{8.4V_0}{a^2} \cos(\omega t + \frac{2\pi}{3})y, \quad (25b)$$

$$E_z = - \frac{8.4V_0}{a^2} \cos(\omega t)z. \quad (25c)$$

where $x, y, z \ll a$ are assumed. The linearized equations of motion are then

$$\frac{d^2\xi_x}{d\tau^2} + 8\left(\frac{Q}{m}\right) \frac{4.2V_0}{\omega^2 a^2} \cos(2\tau + \frac{4\pi}{3})\xi_x = 0, \quad (26a)$$

$$\frac{d^2\xi_y}{d\tau^2} + 8\left(\frac{Q}{m}\right) \frac{4.2V_0}{\omega^2 a^2} \cos(2\tau + \frac{2\pi}{3})\xi_y = 0, \quad (26b)$$

$$\frac{d^2\xi_z}{d\tau^2} + 8\left(\frac{Q}{m}\right) \frac{4.2V_0}{\omega^2 a^2} \cos(2\tau)\xi_z = 0. \quad (26c)$$

where $\tau = \frac{1}{2}\omega t$.

Using Eq. (20) and (28) yields

$$\frac{f^2}{v_o^2} << \frac{.45}{ga^3} \left(\frac{Q}{m}\right)^2. \quad (31)$$

From the inequalities (29) and (31) the limit on the frequency is:

$$f > .70 \sqrt{\frac{E}{a}}. \quad (32)$$

The limits imposed on frequency and voltage by (29) and (31) are illustrated by Figure 6.

Figure 6 shows that the conditions on voltage and frequency required to levitate a particle with charge to mass ratio $Q/m = 10^{-4}$ Coul / kg using no gravity compensating field are reasonable. With a frequency $f = 60$ Hz, the line-to-neutral 3 ϕ voltage magnitude will be ~8 kV (peak) or ~5.5 kV (rms). Reference to Figures 3a, b, c shows that levitation of 1.0mm diameter droplets of liquid H_2 , Freon-113, or other dielectrics with $Q/m = 10^{-4}$ Coul/kg should be possible with relative ease. These figures also show that if the gravity compensating field E_g is used, the voltage restriction will be loosened, but that any kind of stringent limit imposed on the maximum allowed droplet deformation will cut the maximum droplet size fairly substantially.

Figure 6. Frequency-voltage map for 3 ϕ cubic face multipole ($a \approx 1.0$ cm).

IV. Conclusion

Two schemes for the levitation of charged liquid droplets using dynamic stabilization provided by electrostatic fields have been considered. Limits on the maximum charge-to-mass ratio (Q/m) and the diameter D are imposed by the stability of the droplets. The limits are the Rayleigh limit on charge Q , the electrostatic elongation limit on the imposed gravity compensating field E_g , and the electrical breakdown limit. Their combination imposes a limit on the maximum stable droplet diameter, as shown in Figures 3a, b, c. Note that these limits do not seriously impair the prospects for levitation of particles and droplets of suitable size for target fabrication applications. With the limits on allowable droplet size and charge-to-mass ratio defined, the voltage and frequency requirements are discussed and reviewed in Figures 4 and 6. These plots show that if a gravity compensating field is not used, then a lower frequency limit exists. This frequency limit is not considered to be a serious problem. Overall, the results of these feasibility calculations are that charged droplets of low surface tension liquids up to approximately 1 mm in diameter can be successfully levitated dynamically in three-dimensional quadrupoles and in three-phase multipole electrode structures.

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APPENDIX A

CUBIC FACE MULTIPOLE FIELD SOLUTION

The solution of the cubic face multipole electrostatic field problem is complicated by the 3ϕ voltage excitation. However, Eq. (24) can be shown to be entirely consistent with a general solution of Laplace's equation, subject to certain approximations useful for modeling the dynamics of charged particles near the center of the cubic-face multipole. The problem to be solved is shown in Figure 3. The boundary conditions on the electrostatic potential are:

$$\phi(x, y, z = \pm \frac{a}{2}, t) = V_a(t), \quad (A1)$$

$$\phi(x, y = \pm \frac{a}{2}, z, t) = V_b(t), \quad (A2)$$

$$\phi(x = \pm \frac{a}{2}, y, z, t) = V_c(t), \quad (A3)$$

where the 3 phase voltages are given by Eq. (23a, b, c). A complete solution for the electrostatic potential function is [A1]:

$$\begin{aligned} \phi(x, y, z, t) = & \sum_m \sum_n A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{a}\right) \cosh\left(\frac{\sqrt{m^2 + n^2}\pi z}{a}\right) \\ & + \sum_m \sum_n B_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi z}{a}\right) \cosh\left(\frac{\sqrt{m^2 + n^2}\pi y}{a}\right) \\ & + \sum_m \sum_n C_{mn} \cos\left(\frac{m\pi y}{a}\right) \cos\left(\frac{n\pi z}{a}\right) \cosh\left(\frac{\sqrt{m^2 + n^2}\pi x}{a}\right), \end{aligned}$$

$$m = 1, 3, 5, \dots; n = 1, 3, 5, \dots \quad (A4)$$

Due to the cubic symmetry of the electrodes and the boundary conditions, the only terms needed in the above series are the odd terms.

Utilizing the orthogonality condition on the various terms in the series solution, the coefficients may be determined:

$$A_{mn} = a_{mn} V_a(t), \quad (A5)$$

$$B_{mn} = a_{mn} V_b(t), \quad (A6)$$

$$C_{mn} = a_{mn} V_c(t), \quad (A7)$$

$$a_{mn} = \frac{16 \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right)}{mn\pi^2 \cosh\left(\frac{\sqrt{m^2 + n^2}x}{2}\right)}, \quad (A8)$$

With these coefficients known, the complete and exact solution for $\phi(x,y,z,t)$ is obtained.

Approximation of this solution near the center of the cube for $|x|, |y|, |z| \ll a$ is a very convenient way to obtain the linearized equations of motion for a charged dynamically levitated particle. Consider expansions of the type

$$\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)\cosh\left(\frac{\sqrt{m^2+n^2}\pi z}{a}\right) \\ \approx \left[1 - \frac{1}{2}\left(\frac{m\pi x}{a}\right)^2\right] \left[1 - \frac{1}{2}\left(\frac{n\pi y}{a}\right)^2\right] \left[1 + \frac{1}{2}\left(\frac{\sqrt{m^2+n^2}\pi z}{a}\right)^2\right] \quad (A9)$$

$$\approx 1 - \frac{1}{2}\left(\frac{m\pi x}{a}\right)^2 - \frac{1}{2}\left(\frac{n\pi y}{a}\right)^2 + \frac{1}{2}\left(\frac{\sqrt{m^2+n^2}\pi z}{a}\right)^2, \quad (A10)$$

where the approximations

$$\cos \alpha \approx 1 - \frac{\alpha^2}{2}, \quad (A11)$$

$$\cosh \alpha \approx 1 + \frac{\alpha^2}{2}, \quad (A12)$$

have been repeatedly used and where the conditions

$$\left(\frac{m\pi x}{a}\right)^2, \left(\frac{n\pi y}{a}\right)^2, \left(\frac{\sqrt{m^2+n^2}\pi z}{a}\right)^2 \ll 1 \quad (A13)$$

must be maintained.

Only the lower-order terms in the series solution Eq. (A4) are found to be significant, so each of the three series in Eq. (A4) are truncated after the first four terms. Repeated use of the expansion of Eq. (A4) plus convenient grouping of like terms yields:

$$\phi_{11} \approx \frac{5.15}{a^2} \left[v_a(t)z^2 + v_b(t)y^2 + v_c(t)x^2 \right], \quad (A14)$$

$$\phi_{13} + \phi_{31} \approx -\frac{1.11}{a^2} \left[v_a(t)z^2 + v_b(t)y^2 + v_c(t)x^2 \right], \quad (A15)$$

$$\phi_{33} \approx \frac{.122}{a^2} \left[V_a(t)z^2 + V_b(t)y^2 + V_c(t)x^2 \right], \quad (A16)$$

where the condition

$$V_a(t) + V_b(t) + V_c(t) = 0, \quad (A17)$$

which is true for the balanced 3 phase excitation assumed, Eq. (23a, b, c), has been employed. We note that the first term, ϕ_{11} in Eq. (A14), is the result used by Wuerker, Goldenberg, and Langmuir [2]. Evidently, they truncated the series after the first term, thus neglecting a 20% correction factor. Combining the first four terms, defined by Eq. (A14), (A15), (A16) yields

$$\begin{aligned} \phi(x,y,z,t) &\approx \phi_{11} + \phi_{13} + \phi_{31} + \phi_{33} \\ &\approx \frac{4.2}{a^2} \left[V_a(t)z^2 + V_b(t)y^2 + V_c(t)x^2 \right] \quad (A18) \end{aligned}$$

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NOMENCLATURE

a	- semi-major axis of ellipsoid, also size of cubic face multipole
a_i	- coefficient of Mathieu's equation
a_{mn}	- defined in Eq. (A8)
A	- defined on page 9
A_{mn}	- coefficient of cubic face multipole potential solution
B_{mn}	- coefficient of cubic face multipole potential solution
C_{mn}	- coefficient of cubic face multipole potential solution
D	- droplet diameter
$\overline{E}$	- electric field
E_b	- breakdown electric field strength
E_{crit}	- threshold field intensity for droplet elongation instability
E_g	- gravity compensating field intensity
f	- frequency
g	- gravitational acceleration (9.81 m/sec^2)
h	- three-dimensional quadrupole spacing
K	- coefficient defined in Eq. (24)
L_z	- de-polarization factor
m	- droplet mass, also integer index
n	- index
q_i	- coefficient of Mathieu's equation
Q	- particle charge
r_c	- cylindrical radial coordinate
V	- voltage
V_{ac}	- ac potential

NOMENCLATURE (cont.)

V_{dc}	- dc potential
V_g	- gravity compensating potential
V_a, V_b, V_c	- 3 phase voltages
x, y, z	- rectangular coordinates
z	- cylindrical axial coordinate
γ	- semi-major axis ratio
ϵ	- dielectric permittivity
ϵ_0	- permittivity of free space ($8.85 \cdot 10^{-12}$ F/m)
ξ	- particle displacement
ρ	- mass density
Σ	- surface tension
τ	- normalized time
Φ	- potential function
ω	- radian frequency